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Exponential tightness of a family of Skorohod integrals*

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Abstract

Exponential tightness of a family of Skorohod integrals is studied in this paper. We first provide a counterexample to illustrate that in general the exponential tightness with speed ε similar to Itô integral does not hold, even for any speed ε^α with $\alpha > 0$. Then, some characterizations of this subject are given. Application is also provided to illustrate our results.

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1 Introduction

Large deviations principle (see e.g., [3] for a full discussion of this subject) plays an important role in both theory and application such as averaging principle of fast-slow systems, equilibrium and non-equilibrium statistical mechanics, multi-fractals, and thermodynamic formulation of chaotic systems; see [3, 15] and the reference therein. In Polish space, the exponential tightness is a necessary condition for the large deviations principle (with inf-compact rate functions) and implies the large deviations relative compactness (i.e., every sub-sequence contains another sub-sequence satisfying the large deviations principle with some rate function); see e.g., [3]. Hence, this property has a crucial role in the large deviations theory, for example, see [4, 8, 9, 13]. Moreover, since it provides a kind of “exponential tail estimates for the tightness”, it is also very interesting in its own right in stochastic analysis. Recall that we say a family of random variables $\{F_\varepsilon\}_{\varepsilon>0}$ in $\mathbb{R}^d$ is exponentially tight with speed $v(\varepsilon)$ satisfying $v(\varepsilon) \to 0$ as $\varepsilon \to 0$, if

$$\lim_{L\to\infty} \limsup_{\varepsilon\to 0} v(\varepsilon) \log \mathbb{P}(|F_\varepsilon| > L) = -\infty.$$

It is noted that if $v_1(\varepsilon) < v_2(\varepsilon)$ (as $\varepsilon \to 0$) then the exponential tightness with the speed $v_1(\varepsilon)$ implies the exponential tightness with the speed $v_2(\varepsilon)$.

Given a d-dimensional standard Brownian motion $W(t)$ defined on the canonical probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and the Hilbert space $H = L^2([0,1], \mathbb{R}^d)$, consider an isonormal Gaussian process $W = \{W(h) : h \in H\}$ defined by $W(h) = \int_0^1 h(t)dW(t)$. Denote by D the Malliavin derivative operator and by δ its adjoint, the divergence

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operator, which is called Skorohod integral in our setting; see [11]. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of Skorohod integrable processes and we consider the family of random variables

$$F_\varepsilon = \sqrt{\varepsilon}\delta(u_\varepsilon).$$

As a special case, it is well-known in the classical Itô stochastic calculus that if u_ε is a non-anticipating process, $\delta(u_\varepsilon)$ can be understood in the Itô sense. If we assume further that u_ε are bounded (uniformly in ε) by K, i.e., for all $\varepsilon > 0$, $|u_\varepsilon(t)| \leq K, \forall t \in [0,1]$ a.s., then $\{F_\varepsilon\}_{\varepsilon>0}$ is exponentially tight with the speed $v(\varepsilon) = \varepsilon$. To be more precise, by Schilder's theorem [3, Lemma 5.2.2] one has

$$\mathbb{P}(|F_\varepsilon| > L) \leq 4d \exp\Big\{ - \frac{L^2}{2d\varepsilon K^2}\Big\}.$$

What about the exponential tightness of the family $\{F_\varepsilon\}_{\varepsilon>0}$ in the general case? This work aims to address such a question.

First, we are wondering if one relaxes the measurability and allows $\{u_\varepsilon\}_{\varepsilon>0}$ to be anticipating processes and keeps the (uniform) boundness, is the family $\{F_\varepsilon\}_{\varepsilon>0}$ still exponentially tight with the speed $v(\varepsilon) = \varepsilon$ or at least with some speed $v(\varepsilon) = \varepsilon^\alpha$ (for some $\alpha \in (0,1)$)? Unfortunately, it is not true in general. We first provide a counterexample as follows.

Theorem 1.1. *Assume $W(t)$ is a one-dimensional Brownian motion. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = (0 \vee x \wedge 1)^{3/4}$. Define $X = f(W(1))$ and*

$$F_\varepsilon = \sqrt{\varepsilon}\delta\Big(X\mathbf{1}_{[0,1]}(t)\Big).$$

The family $\{F_\varepsilon\}_{\varepsilon>0}$ is not exponentially tight with any speed $v(\varepsilon) = \varepsilon^\alpha$, $\alpha > 0$. Moreover, for any $\alpha > 0$ one has

$$\limsup_{L\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}(|F_\varepsilon| > L) = 0. \tag{1.1}$$

Therefore, it is very natural to ask the question: under what conditions on u_ε is the family $\{F_\varepsilon\}_{\varepsilon>0}$ exponentially tight (with some suitable speed)? We aim to provide some sufficient conditions for the exponential tightness of this family of Skorohod integrals.

It will be seen from the above counterexample that the family $\{F_\varepsilon\}_{\varepsilon>0}$ may not be exponentially tight because the relationship between the integrands and the whole paths of Brownian motion is somewhat uncontrollable. This relation is often described by the derivative of the integrands with respect to the Brownian motion. Therefore, if we can control the moments of the derivative of the integrands, we can have the exponential tightness with some suitable speed. To be precise, we have the following theorem. [In this paper, $\otimes$ denotes the tensor product; and for a normed space V endowed with the norm $\|\cdot\|_V$, and V-valued random variable v, $\|v\|_{L^p(\Omega,V)} := \big(\mathbb{E}(\|v\|_V^p)\big)^{1/p}$.]

Theorem 1.2. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of Skorohod integrable stochastic processes. Assume that there are $\kappa_1 > -1/2$ and $\kappa_2 \geq 0$ such that*

$$\|u_\varepsilon\|_{L^p(\Omega,H)} + \|Du_\varepsilon\|_{L^p(\Omega,H\otimes H)} \leq c\varepsilon^{\kappa_1} p^{\kappa_2},\ \forall \varepsilon > 0, p \geq 1, \tag{1.2}$$

for some universal constant c, independent of ε, p. Then the family $\{F_\varepsilon = \sqrt{\varepsilon}\delta(u_\varepsilon)\}_{\varepsilon>0}$ is exponentially tight with the speed $v(\varepsilon) = \varepsilon^\alpha$ for any α satisfying

$$\alpha < \frac{1/2 + \kappa_1}{5 + \kappa_2}. \tag{1.3}$$

This theorem is proved by using Meyer's inequality after revisiting its proof with the use of best constants and the estimate of the exponential moment. As seen in the above theorem, compared with the speed in the Itô case, we have the exponential tightness in slower speeds due to the non-adaptedness.

If we can control the moments of the second-order Malliavin derivative (denoted by D^2) of the integrands, we can combine Itô's formula for Skorohod integral and Meyer's inequality to estimate the pth-moment and obtain the following result, which can be used to improve the speed in certain cases.

Theorem 1.3. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of Skorohod integrable stochastic processes. Assume that there are $\overline{\kappa}_1, \overline{\kappa}_3 > -1/2$, and $\overline{\kappa}_2, \overline{\kappa}_4 \geq 0$ such that*

$$\|Du_\varepsilon\|_{L^p(\Omega\times H^{\otimes 2})} + \|D^2 u_\varepsilon\|_{L^p(\Omega\times H^{\otimes 3})} \leq c\varepsilon^{\overline{\kappa}_1} p^{\overline{\kappa}_2}, \quad \|u_\varepsilon\|_{L^p(\Omega);H} \leq c'\varepsilon^{\overline{\kappa}_3} p^{\overline{\kappa}_4}, \ \forall \varepsilon > 0, p \geq 1, \tag{1.4}$$

for some universal constants c, c', independent of ε, p. In the above,

$$\|Du_\varepsilon\|_{L^p(\Omega\times H^{\otimes 2})} := \Big(\int_0^1 \int_0^1 \mathbb{E}|D_s u_\varepsilon(r)|^p dr ds \Big)^{1/p},$$

$$\|D^2 u_\varepsilon\|_{L^p(\Omega\times H^{\otimes 3})} := \Big(\int_0^1 \int_0^1 \int_0^1 \mathbb{E}|D_t D_s u_\varepsilon(r)|^p dr dt ds \Big)^{1/p},$$

and

$$\|u_\varepsilon\|_{L^p(\Omega);H} := \Big(\int_0^1 (\mathbb{E}|u_\varepsilon(s)|^p)^{2/p} ds \Big)^{1/2}.$$

Then the family $\{F_\varepsilon = \sqrt{\varepsilon}\delta(u_\varepsilon)\}_{\varepsilon>0}$ is exponentially tight with the speed $v(\varepsilon) = \varepsilon^\alpha$ for any α satisfying

$$\alpha < \frac{1/2 + \widehat{\kappa}_1}{\widehat{\kappa}_2},$$

where $2\widehat{\kappa}_1 = \min\{\overline{\kappa}_1 + \overline{\kappa}_3, 2\overline{\kappa}_3\}$, $2\widehat{\kappa}_2 = \max\{6 + \overline{\kappa}_2 + \overline{\kappa}_4, 1 + 2\overline{\kappa}_4\}$.

The rest of the paper is organized as follows. Section 2 recalls briefly the Malliavin calculus emphasizing the Malliavin derivative operator, its adjoint operator, and Meyer's inequality. The proofs of our main results are given in Section 3. An application to mathematical physics is given in Section 4.

2 Malliavin calculus

2.1 Malliavin derivative and Skorohod integral

We recall briefly some basic definitions in Malliavin calculus and refer the readers to [11] for a full construction. Let $W(t)$ be a d-dimensional standard Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $H = L^2([0,1], \mathbb{R}^d)$. Consider $W = \{W(h), h \in H\}$, the space of Gaussian isonormal processes defined by $W(h) = \int_0^1 h(t) dW(t)$. Denote by

$$\mathcal{S} = \Big\{ F = f(W(h_1), \ldots, W(h_n)) | f \in \mathcal{C}_p^\infty(\mathbb{R}^n), h_i \in H, n \geq 1 \Big\},$$

the class of smooth random variables, where $\mathcal{C}_p^\infty(\mathbb{R}^n)$ is the space of smooth function $f \in \mathcal{C}^\infty$ with polynomial growth on derivatives; and by $\mathcal{P}$ the class of all random variables F of the form $F = f(W(h_1), \ldots, W(h_n))$ such that f is a polynomial.

Definition 2.1. (*See [11, Definition 1.2.1]*) *The derivative of a smooth random variable $F \in \mathcal{S}$ is the H-valued random variable given by*

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i.$$

Define the norm

$$\|F\|_{1,2}^2 = \mathbb{E}\big(|F|^2 + \|DF\|_H^2\big).$$

Let $\mathbb{D}^{1,2}$ be the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2}$. One can extend D on $\mathcal{S}$ as a closed operator on $\mathbb{D}^{1,2}$. We call this operator D the Malliavin derivative operator. Moreover, we can also define the iteration of the operator D in such a way that for a smooth random variable F, the iterated derivative D^kF is a random variable with values in $H^{\otimes k}$, then we denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$ defined by

$$\|F\|_{k,p} = \left(\mathbb{E}(|F|^p) + \sum_{j=1}^{k} \mathbb{E}(\|D^jF\|_{H^{\otimes j}}^p)\right)^{\frac{1}{p}}.$$

For $k = 0$, we use the convention $\|\cdot\|_{0,p} = \|\cdot\|_p$ and $\mathbb{D}^{0,p} = L^p(\Omega)$.

Definition 2.2. (*See [11, Definition 1.3.1]*) *Denote by δ the adjoint of the operator D, i.e., δ is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ such that:*

(i) *The domain of δ, denoted by* $\mathrm{Dom}\delta$*, is the set of H-valued square integrable random variables $u \in L^2(\Omega; H)$ such that*

$$|\mathbb{E}(\langle DF, u\rangle_H)| \le c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

(ii) *If u belongs to* $\mathrm{Dom}\delta$*, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following expression*

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_H) \text{ for any } F \in \mathbb{D}^{1,2}.$$

The operator δ is called the divergence operator and is closed since D is an unbounded and densely defined operator. In our case, H is an L^2 space, the elements of $\mathrm{Dom}\delta$ are square integrable processes, and $\delta(u)$ is called the Skorohod stochastic integral. One says u is Skorohod integrable if $\delta(u)$ is well-defined. We will often use the notation

$$\int_0^1 u(s)\delta W(s) := \delta(u) \text{ and } \int_0^t u(s)\delta W(s) := \delta(u\mathbf{1}_{[0,t]}).$$

2.2 Meyer's inequality and the best constants

Meyer's inequality gives us an effective way to bound the moments of the Skorohod integrals by the moments of the integrands and its derivatives. Denote by $\mathbb{L}^{1,2}$ the class of processes $u \in L^2([0,1]\times\Omega)$ such that $u(t) \in \mathbb{D}^{1,2}$ for almost all t, and there exists a measurable version of the two-parameter process $D_su(t)$ verifying $\mathbb{E}\int_0^1\int_0^1 (D_su(t))^2 dsdt < \infty$. [Here, it is noted that for each t, $u(t)$ is a random variable; and $Du(t)$ is a H-valued random variable and thus, can be parameterized as $D_su(t)$.]

We first recall Meyer's inequality; see e.g., [11, Theorem 1.5.1].

Proposition 2.3. *For $1 < p < \infty$ there are constants $\overline{K}_p$ such that*

$$\|DF\|_{L^p(\Omega, H\otimes H)} \le \overline{K}_p\|CF\|_p, \tag{2.1}$$

for any random variable $F \in \mathbb{D}^{1,p}$, where $C := -\sqrt{-L}$ and $L = -\delta D$ is the Ornstein-Uhlenbeck operator (*see [11, Section 1.4] for the definition*).

Therefore, we can obtain from Proposition 2.3 the following estimate; see e.g., [11, Proposition 1.5.4].

Proposition 2.4. *Let u be a stochastic process in $\mathbb{L}^{1,2}$, and let $p>1$. Then we have*

$$\Big(\mathbb{E}|\delta(u)|^p\Big)^{\frac{1}{p}} \le K_p\Big(\|u\|_{L^p(\Omega,H)} + \|Du\|_{L^p(\Omega,H\otimes H)}\Big). \tag{2.2}$$

To estimate the speed of the exponential tightness, we want to use the best constants, and have precise estimates for the constants in the estimate (2.2). We have the following result.

Proposition 2.5. *The constants K_p, $p\ge 2$ in* (2.2) *can be chosen smaller than a positive constant multiple of p^5, i.e., there is a universal constant c such that $K_p \le cp^5$ for all $p\ge 2$.*

Proof. We prove this proposition by revisiting the proof of Proposition 1.5.4 in [11] with the use of the best constants for Meyer's inequality and taking care of the norms of multiplier operators used in the arguments. In the below, we use the letter c to represent universal constants (independent of ε, p, q), whose values may change for different usage.

We first revisit the generalized Meyer's inequality [11, Theorem 1.5.1] and have the following lemma.

Lemma 2.6. *For $1<q<\infty$, there are constants $\overline{K}_{2,q}$ satisfying that for all random variable $G\in\mathcal{P}$,*

$$\|D^2G\|_{L^q(\Omega,H\otimes H)} \le \overline{K}_{2,q}\|C^2G\|_q, \tag{2.3}$$

and that $\overline{K}_{2,q} \le \frac{c}{(q-1)^3}$, $\forall 1<q<2$.

Proof. The proof of (2.3) is given in [11, Proof of Theorem 1.5.1, page 73]. Now, we provide a precise estimate for the constant $\overline{K}_{2,q}$. From the computations in [11, page 73], we obtain that

$$\begin{aligned}\|D^2G\|^q_{L^q(\Omega,H\otimes H)} &\le cA_q\overline{K}_q^q\|DCRG\|^q_{L^q(\Omega,H)} \quad \text{(see [11, page 73])}\\ &\le cA_q\overline{K}_q^{2q}\|C^2RG\|_q^q \quad \text{(applying Proposition 2.3)}\\ &\le cA_q\overline{K}_q^{2q}\|R\|^q_{L^q(\Omega)\to L^q(\Omega)}\|C^2G\|_q^q,\end{aligned} \tag{2.4}$$

where $\overline{K}_q$ is the constant in Meyer's inequality (Proposition 2.3), A_q is the q-th moment of a Gaussian variable (see [11, Appendix A.1]) and $R=\sum_{n=1}^\infty\sqrt{1-\frac{1}{n}}J_n$, J_n is the projection operator onto n-th Wiener chaos (see e.g., [11, Section 1.2]). The operator R is used to exchange the derivative operator (by using the commutativity relationship [11, Lemma 1.4.2]). The last estimate in (2.4) follows from the fact that $C^2=-L=\sum_{n=1}^\infty nJ_n$. As a result, $\overline{K}_{2,q}$ can be chosen such that

$$\overline{K}_{2,q} \le cA_q^{1/q}\overline{K}_q^2\|R\|_{L^q(\Omega)\to L^q(\Omega)}.$$

It follows from [6] that $\overline{K}_q \le \frac{c}{q-1}$, $\forall 1<q<2$. Now, we estimate $\|R\|_{L^q(\Omega)\to L^q(\Omega)}$. Let T_t be the Ornstein-Uhlenbeck semigroup (see e.g., [11, Definition 1.4.1]). It is shown in [11, Lemma 1.4.1] that: for any $q>1$, integer $N\ge 1$, there exists a constant $\widehat{K}$ such that

$$\|T_t(I-J_0-J_1-\cdots-J_{N-1})(G)\|_q \le \widehat{K}e^{-Nt}\|G\|_q, \text{ for all } t>0.$$

Moreover, as shown in the proof of [11, Lemma 1.4.1], if $q\ge 2$, $q=e^{2t_0}+1$ (for some t_0), then $\widehat{K}$ can be chosen such that $\widehat{K}=Ne^{2Nt_0}+e^{Nt_0}$, i.e., $\widehat{K}\le 2Nq^N$. If $1<q<2$, by the duality (of $L^q(\Omega)$ and $L^{\frac{q}{q-1}}(\Omega)$) and the fact that T_t is symmetric (i.e., $\mathbb{E}(GT_t(G'))=\mathbb{E}(G'T_t(G))$ for all random variables G,G'), $\widehat{K}$ can be chosen such that $\widehat{K}\le\frac{2Nq^N}{(q-1)^N}$. Therefore, the multiplier theorem [11, Theorem 1.4.2] states that: if we let $\{\phi(n), n\ge 0\}$

be a sequence of real numbers such that $\phi(0) = 0$ and $\phi(n) = \sum_{k=0}^{\infty} a_k n^{-k}, \forall n \geq N$ for some integer $N \geq 1$, $a_k \in \mathbb{R}$, and such that $\sum_{k=0}^{\infty} |a_k| N^{-k} < \infty$, then

$$\| \sum_{n=N}^{\infty} \phi(n) J_n G\|_q \leq \widehat{K} \sum_{k=0}^{\infty} |a_k| N^{-k} \|G\|_q, \text{ for any } 1 < q < \infty,$$

where $\widehat{K}$ can be chosen such that $\widehat{K} \leq 2Nq^N$ if $q \geq 2$, and such that $\widehat{K} \leq \frac{2Nq^N}{(q-1)^N}$ if $1 < q < 2$. As a result, by applying the multiplier theorem [11, Theorem 1.4.2] and [11, Lemma 1.4.1] for $R = \sum_{n=1}^{\infty} \sqrt{1 - \frac{1}{n}} J_n$, $N = 1$, one can obtain that R is a bounded operator and $\|R\|_{L^q(\Omega) \to L^q(\Omega)} \leq \frac{c}{q-1}$, $\forall 1 < q < 2$. Therefore, we can complete the proof of this lemma. □

Now, let $q = \frac{p}{p-1}$ be the conjugate of p and $G \in \mathcal{P}$ be any polynomial random variable with $\mathbb{E}(G) = 0$. Under the convention ($\mathbb{E}u = 0$) for simplicity as in [11, Proof of Proposition 1.5.4], we have from [11, Proof of Proposition 1.5.4], Lemma 2.6, the commutativity relationship [11, Lemma 1.4.2] that

$$\begin{aligned} |\mathbb{E}(\delta(u)G)| \leq & \|Du\|_{L^p(\Omega; H \otimes H)} \|DC^{-2}DG\|_{L^q(\Omega, H \otimes H)} \\ = & \|Du\|_{L^p(\Omega; H \otimes H)} \|D^2 C^{-2} RG\|_{L^q(\Omega, H \otimes H)} \\ \leq & \overline{K}_{2,q} \|Du\|_{L^p(\Omega; H \otimes H)} \|RG\|_q \\ \leq & \overline{K}_{2,q} \|R\|_{L^q(\Omega) \to L^q(\Omega)} \|Du\|_{L^p(\Omega; H \otimes H)} \|G\|_q. \end{aligned} \tag{2.5}$$

In the above, $R = \sum_{n=2}^{\infty} \frac{n}{n-1} J_n$, which is used to exchange the derivative operator by using the commutativity relationship. By applying the multiplier theorem [11, Theorem 1.4.2] and [11, Lemma 1.4.1] for $R = \sum_{n=2}^{\infty} \frac{n}{n-1} J_n$, $N = 2$, one can obtain that R is a bounded operator and $\|R\|_{L^q(\Omega) \to L^q(\Omega)} \leq \frac{c}{(q-1)^2}$, $\forall 1 < q < 2$, and thus $\|R\|_{L^q(\Omega) \to L^q(\Omega)} \leq cp^2$, $\forall p \geq 2$. Moreover, it follows from Lemma 2.6 that $\overline{K}_{2,q} \leq \frac{c}{(q-1)^3}$, $\forall 1 < q < 2$; and thus, $\overline{K}_{2,q} \leq cp^3$, $\forall p \geq 2$. Therefore, we obtain from (2.5) that there is a universal constant c, which is independent of $p \geq 2$ such that

$$|\mathbb{E}(\delta(u)G)| \leq cp^5 \|Du\|_{L^p(\Omega; H \otimes H)} \|G\|_q. \tag{2.6}$$

The proposition follows from (2.6) after taking the supremum with respect to polynomial random variable G as in the standard duality argument [11]. □

3 Proof of main results

Proof of Theorem 1.1. Direct calculation shows that

$$f'(x) = \begin{cases} 0 \text{ if } x < 0 \text{ or } x > 1, \\ \frac{3x^{-1/4}}{4} \text{ if } 0 < x < 1. \end{cases}$$

Therefore, $f'(W(1)) \in L^2(\Omega)$. Applying the chain rule (see e.g., [11], [5, Proposition 2.3.1] or [12, Theorem 5.7]), we have that $DX = Df(W(1)) = f'(W(1))\mathbf{1}_{[0,1]}(t) \in L^2(\Omega, L^2([0,1]))$. Moreover, it is readily seen that $X\mathbf{1}_{[0,1]}(t)$ is Skorohod integrable. Using integration by parts (see [11, Proposition 1.3.3]), we have that

$$\begin{aligned} F_\varepsilon &= \sqrt{\varepsilon}\delta\Big(X\mathbf{1}_{[0,1]}(t)\Big) \\ &= \sqrt{\varepsilon}X\delta(1_{[0,1]}(t)) - \sqrt{\varepsilon}\int_0^1 D_r X dr \\ &= \sqrt{\varepsilon}XW(1) - \sqrt{\varepsilon}f'(W(1)). \end{aligned} \tag{3.1}$$

To prove Theorem 1.1, it suffices to prove (1.1) for any $\alpha \in (0,1)$. We aim to use a contradiction argument by assuming that

$$\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}(|F_\varepsilon|>L)\le\eta\in(-\infty,0). \tag{3.2}$$

Since X is bounded almost surely and $W(1)$ is Gaussian, we can obtain that the family $\left\{\sqrt{\varepsilon}XW(1)\right\}_{\varepsilon>0}$ is exponentially tight with speed $v(\varepsilon)=\varepsilon$. Thus, one has

$$\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}\Big(\Big|\sqrt{\varepsilon}XW(1)\Big|>L\Big)=-\infty.$$

As a result, the family $\left\{F_\varepsilon-\sqrt{\varepsilon}XW(1)\right\}_{\varepsilon>0}$ satisfies that

$$\begin{aligned}&\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}\Big(\Big|F_\varepsilon-\sqrt{\varepsilon}XW(1)\Big|>L\Big)\\ &\quad\le\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\Big\{\mathbb{P}\Big(|F_\varepsilon|>\frac{L}{2}\Big)+\mathbb{P}\Big(\Big|\sqrt{\varepsilon}XW(1)\Big|>\frac{L}{2}\Big)\Big\}\\ &\quad=\max\Big\{\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}\Big(|F_\varepsilon|>\frac{L}{2}\Big),\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}\Big(\Big|\sqrt{\varepsilon}XW(1)\Big|>\frac{L}{2}\Big)\Big\}\\ &\quad\le\eta<0.\end{aligned} \tag{3.3}$$

A consequence of (3.1) and (3.3) is that

$$\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}\Big(\Big|\sqrt{\varepsilon}f'(W(1))\Big|>L\Big)\le\eta,$$

and then one gets

$$\limsup_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{\alpha}\log\mathbb{P}\Big(\Big|f'(W(1))\Big|>L\varepsilon^{-1/2}\Big)\le\eta. \tag{3.4}$$

Now, denote $Y=f'(W(1))$. From (3.4), there are $L_0=L_0(\eta)>1$ and $\varepsilon_0=\varepsilon_0(L_0,\eta)<1$ such that for all $L>L_0$, $\varepsilon<\varepsilon_0$

$$\mathbb{P}(|Y|>L\varepsilon^{-1/2})\le\exp\big\{\frac{\eta\varepsilon^{-\alpha}}{2}\big\}.$$

Particularly, let $\varepsilon=\frac{\varepsilon_0L_0}{L^2}$, one has for all $L>L_0$

$$\mathbb{P}\Big(|Y|>L^2(\varepsilon_0L_0)^{-1/2}\Big)\le\exp\Big\{\frac{\eta(\varepsilon_0L_0)^{-\alpha}L^{2\alpha}}{2}\Big\},$$

which implies that

$$\mathbb{P}\Big(|Y|^{\alpha/2}>L^{\alpha}(\varepsilon_0L_0)^{-\alpha/4}\Big)\le\exp\Big\{\frac{\eta(\varepsilon_0L_0)^{-\alpha}L^{2\alpha}}{2}\Big\},\ \forall L>L_0. \tag{3.5}$$

We obtain from (3.5) that for all $t>L_0^{3\alpha/4}\varepsilon_0^{-\alpha/4}$

$$\mathbb{P}\Big(|Y|^{\alpha/2}>t\Big)\le\exp\{-c_0t^2\}, \tag{3.6}$$

where $c_0=\frac{-\eta(\varepsilon_0L_0)^{-\alpha/2}}{2}>0$. From (3.6), which is a kind of tail estimates of a sub-Gaussian random variable, we can get a kind of "moments control property" for $|Y|^{\alpha/2}$ (see e.g., [14, Lemma 1.4]), i.e., for all $p>0$

$$\mathbb{E}|Y|^{\alpha p/2}\le c_1c_2^pp^{p/2}, \tag{3.7}$$

for some constants c_1, c_2 depending only on $\eta, \varepsilon_0, L_0, \alpha$ and being independent of p.

On the other hand, we have

$$\begin{aligned}\mathbb{E}|Y|^p = &\int_{-\infty}^{\infty} |f'(x)|^p \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \geq \int_0^1 |f'(x)|^p \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ \geq &\frac{(3/4)^p}{\sqrt{2e\pi}} \int_0^1 x^{-p/4} dx \\ = &\infty \text{ if } p \geq 4.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) leads to a contradiction. So, we obtain (1.1) and complete the proof. □

Remark 3.1. As was seen in the above counterexample, although the process $u_\varepsilon(t) = f(W(1))\mathbf{1}_{[0,1]}(t)$ is bounded uniformly, the corresponding family of Skorohod integrals is not exponentially tight with any speed ε^α, $\alpha > 0$. The reason is that the relationship between the non-adapted intergrand and the whole paths of the Brownian motion is "uncontrollable", which is illustrated by (3.8). Since all moments of the Malliavin derivative do not exist, the assumptions in Theorems 1.2 and 1.3 are violated.

Proof of Theorem 1.2. Denote $\alpha = (1/2 + \kappa_1)\beta$ for some β satisfying $\beta < \frac{1}{5+\kappa_2} < 1$. We have from Meyer's inequality with precise constants (Proposition 2.4 and 2.5) and (1.2) that

$$\begin{aligned}\mathbb{E}\exp\{|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)|^\beta\} &= \sum_{n=0}^{\infty} \frac{\mathbb{E}|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)|^{n\beta}}{n!} \leq \sum_{n=0}^{\infty} \frac{\left(\mathbb{E}|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)|^n\right)^\beta}{n!} \\ &\leq c_3 + \sum_{n=2}^{\infty} \frac{\varepsilon^{-n\kappa_1\beta} K_n^{n\beta} \|u_\varepsilon\|_{1,n}^{n\beta}}{n!} \\ &\leq c_3 + c_4\Big(\sum_{n=0}^{\infty} \frac{n^{(5+\kappa_2)n\beta}}{n!}\Big) \leq c_5,\end{aligned}$$

for some constants c_3, c_4, c_5, independent of n, ε. In the above, the last estimate follows from the fact that $\sum_{n=0}^{\infty} \frac{n^{(5+\kappa_2)n\beta}}{n!} < \infty$, which is implied by the fact that $(5+\kappa_2)\beta < 1$ and the ratio test.

By Markov's inequality, one has that for any $L > 0$,

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}(|F_\varepsilon| > L) &= \limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}(\varepsilon^{1/2+\kappa_1}|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)| > L) \\ &= \limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}(\varepsilon^{(1/2+\kappa_1)\beta}|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)|^\beta > L^\beta) \\ &= \limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}\Big(\exp\{|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)|^\beta\} > \exp\{L^\beta \varepsilon^{-(1/2+\kappa_1)\beta}\}\Big) \\ &\leq \limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \frac{\mathbb{E}\exp\{|\varepsilon^{-\kappa_1}\delta(u_\varepsilon)|^\beta\}}{\exp\{L^\beta\varepsilon^{-(1/2+\kappa_1)\beta}\}} \\ &\leq \limsup_{\varepsilon\to 0} \varepsilon^\alpha \log \frac{c_5}{\exp\{L^\beta\varepsilon^{-(1/2+\kappa_1)\beta}\}} \\ &= -\limsup_{\varepsilon\to 0} \varepsilon^{\alpha-(1/2+\kappa_1)\beta} L^\beta \\ &= -L^\beta.\end{aligned}$$

Therefore, the exponential tightness with the speed $v(\varepsilon) = \varepsilon^\alpha$ follows immediately. □

Remark 3.2. Actually, the constant $5 + \kappa_2$ in (1.3) comes from the order needed to control the p-th moment of $\delta(u_\varepsilon)$. Moreover, let us come back to the non-anticipating stochastic integral case and let u_ε be constant for simplicity. In that case $\kappa_1 = \kappa_2 = 0$ and $\delta(u_\varepsilon)$ is Gaussian. It is well-known that the p-th moment of a Gaussian random variable is controlled by $p^{p/2}$ only and thus, replacing 5 in the denominator of the right hand side in (1.3) by $1/2$ will bring us back to the results in the classical case (the case of Itô integrals) as given by Schilder's theorem [3, Lemma 5.2.2]. The details are left to the reader.

Proof of Theorem 1.3. Let $Z(t) = \int_0^t u_\varepsilon(t)\delta W(t) := \delta(u_\varepsilon \mathbf{1}_{[0,t]})$. For the simplicity of notation, let us assume $W(t)$ and $u_\varepsilon(t)$ have real values, i.e., the dimension $d = 1$. [The general case ($d > 1$) is the same by understanding appropriate calculations in their corresponding vector operations.]

By Itô's formula for Skorohod integral [11, Theorem 3.2.2], we have for $n \geq 2$

$$\begin{aligned} \big|Z(t)\big|^n = &\int_0^t n\big|Z(s)\big|^{n-1} u_\varepsilon(s)\delta W(s) \\ &+ \int_0^t n(n-1)\big|Z(s)\big|^{n-2}\Big(\frac{|u_\varepsilon(s)|^2}{2} + u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big)ds. \end{aligned} \tag{3.9}$$

Therefore, it follows from (3.9) and Hölder's inequality that for $n > 2$

$$\begin{aligned} \mathbb{E}\big|Z(t)\big|^n =& n(n-1)\mathbb{E}\int_0^t \big|Z(s)\big|^{n-2}\Big(\frac{u_\varepsilon^2(s)}{2} + u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big)ds \\ \leq& n(n-1)\int_0^t \Big(\mathbb{E}\big|Z(s)\big|^n\Big)^{\frac{n-2}{n}}\Bigg(\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{\frac{2}{n}} \\ &+ 2\bigg(\mathbb{E}\Big|u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big|^{n/2}\bigg)^{\frac{2}{n}}\Bigg)ds. \end{aligned} \tag{3.10}$$

It is known from Bihari-LaSalle inequality [7] that if

$$v(t) \leq a\int_0^t k(s)(v(s))^{\frac{n-2}{n}}ds, \quad \forall t \in [0,1],$$

then we have

$$v(t) \leq \left(\frac{2a\int_0^t k(s)ds}{n}\right)^{n/2}, \quad \forall t \in [0,1].$$

Applying this fact and (3.10), we deduce that

$$\begin{aligned} \mathbb{E}|Z(1)|^n \leq& \Bigg((n-1)\int_0^1 \Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{2/n}ds \\ &+ 2(n-1)\int_0^1 \bigg(\mathbb{E}\Big|u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big|^{n/2}\bigg)^{2/n}ds\Bigg)^{n/2} \\ \leq& \Bigg(n\|u_\varepsilon\|^2_{L^n(\Omega);H} + 2n\int_0^1 \bigg(\mathbb{E}\Big|u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big|^{n/2}\bigg)^{2/n}ds\Bigg)^{n/2}. \end{aligned} \tag{3.11}$$

On the other hand, one has from Hölder's inequality and Proposition 2.4 that for $n > 2$

$$\begin{aligned}
&\mathbb{E}\Big|u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big|^{n/2}\\
\leq&\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{1/2}\Big(\mathbb{E}\Big|\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big|^n\Big)^{1/2}\\
\leq&\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{1/2}K_n^{n/2}\bigg(\Big(\mathbb{E}\|D_s u_\varepsilon\|^n_{L^2([0,1])}\Big)^{1/n}+\Big(\mathbb{E}\|DD_s u_\varepsilon\|^n_{L^2((0,1)^2)}\Big)^{1/n}\bigg)^{n/2}\\
\leq&K_n^{n/2}\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{1/2}\bigg(\bigg(\mathbb{E}\Big(\int_0^1|D_s u_\varepsilon(r)|^2dr\Big)^{n/2}\bigg)^{1/n}\\
&+\bigg(\mathbb{E}\Big(\int_0^1\int_0^1|D_tD_s u_\varepsilon(r)|^2drdt\Big)^{n/2}\bigg)^{1/n}\bigg)^{n/2}\\
\leq&K_n^{n/2}\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{1/2}\bigg(\Big(\mathbb{E}\int_0^1|D_s u_\varepsilon(r)|^ndr\Big)^{1/n}+\Big(\mathbb{E}\int_0^1\int_0^1|D_tD_s u_\varepsilon(r)|^ndrdt\Big)^{1/n}\bigg)^{n/2}\\
\leq&2^{n/2}K_n^{n/2}\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{1/2}\bigg(\Big(\int_0^1\mathbb{E}|D_s u_\varepsilon(r)|^ndr\Big)^{1/2}+\Big(\int_0^1\int_0^1\mathbb{E}|D_tD_s u_\varepsilon(r)|^ndrdt\Big)^{1/2}\bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\bigg(\mathbb{E}\Big|u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big|^{n/2}\bigg)^{2/n}\\
&\qquad\leq 2K_n\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{1/n}\bigg(\Big(\int_0^1\mathbb{E}|D_s u_\varepsilon(r)|^ndr\Big)^{1/n}+\Big(\int_0^1\int_0^1\mathbb{E}|D_tD_s u_\varepsilon(r)|^ndrdt\Big)^{1/n}\bigg),
\end{aligned} \tag{3.12}$$

Combining (3.12), Hölder's inequality, Proposition 2.5 and (1.4), we have for $n > 2$

$$\begin{aligned}
&\int_0^t\bigg(\mathbb{E}\Big(u_\varepsilon(s)\int_0^s D_s u_\varepsilon(r)\delta W(r)\Big)^{n/2}\bigg)^{2/n}ds\\
\leq&4K_n\bigg(\int_0^1\Big(\mathbb{E}|u_\varepsilon(s)|^n\Big)^{2/n}ds\bigg)^{1/2}\bigg(\Big(\int_0^1\int_0^1\mathbb{E}|D_s u_\varepsilon(r)|^ndrds\Big)^{1/n}\\
&+\Big(\int_0^1\int_0^1\int_0^1\mathbb{E}|D_tD_s u_\varepsilon(r)|^ndrdtds\Big)^{1/n}\bigg)\\
\leq&c_6n^{5+\overline{\kappa}_2+\overline{\kappa}_4}\varepsilon^{\overline{\kappa}_1+\overline{\kappa}_3},
\end{aligned} \tag{3.13}$$

for some constant c_6, independent of n, ε. Combining (3.11) and (3.13), we have

$$\begin{aligned}
\mathbb{E}|Z(1)|^n\leq&c_7^{n/2}\Big(n^{1+2\overline{\kappa}_4}\varepsilon^{2\overline{\kappa}_3}+n^{6+\overline{\kappa}_2+\overline{\kappa}_4}\varepsilon^{\overline{\kappa}_1+\overline{\kappa}_3}\Big)^{n/2}\\
\leq&c_8^{n/2}n^{\widehat{\kappa}_2n}\varepsilon^{\widehat{\kappa}_1n},
\end{aligned} \tag{3.14}$$

for some constants c_7, c_8, independent of n, ε.

Now, denote $\alpha = (1/2 + \widehat{\kappa}_1)\beta$ for some β satisfying $\beta\widehat{\kappa}_2 < 1$. We have from (3.14) that

$$\begin{aligned}
\mathbb{E}\exp\{|\varepsilon^{-\widehat{\kappa}_1}\delta(u_\varepsilon)|^\beta\}=\mathbb{E}\exp\{|\varepsilon^{-\widehat{\kappa}_1}Z(1)|^\beta\}&=\sum_{n=0}^\infty\frac{\mathbb{E}|\varepsilon^{-\widehat{\kappa}_1}Z(1)|^{n\beta}}{n!}\\
&\leq c_8+\sum_{n=2}^\infty\frac{c_9^{n\beta/2}n^{\widehat{\kappa}_2n\beta}}{n!}\leq c_{10},
\end{aligned}$$

for some constants c_8, c_9, c_{10}, independent of n, ε. In the above, the last estimate follows from the fact that $\sum_{n=2}^{\infty} \frac{c_9^{n\beta/2} n^{\widehat{\kappa}_2 n\beta}}{n!} < \infty$, which is implied by the fact $\widehat{\kappa}_2\beta < 1$ and the ratio test. As in the proof of Theorem 1.2, by Markov's inequality one has that for any $L > 0$

$$\limsup_{\varepsilon\to 0} \varepsilon^{\alpha} \log \mathbb{P}(|F_\varepsilon| > L) \le -L^{\beta}.$$

Therefore, the exponential tightness with the speed $v(\varepsilon) = \varepsilon^{\alpha}$ follows immediately. □

Remark 3.3. It is seen from Theorem 1.2 and 1.3 that when $\kappa_1, \widehat{\kappa}_1$ are small and $\overline{\kappa}_4$ is not too large, the exponential tightness in Theorem 1.3 is stronger than that in Theorem 1.2.

Remark 3.4. One can reduce the moment needed in the conditions (1.2) and (1.4). For example, we can replace the term $\|u\|_{L^p(\Omega,H)} = \left(\mathbb{E}\|u_\varepsilon\|_H^p\right)^{1/p}$ by a smaller term $\|\mathbb{E}|u_\varepsilon|\|_H$ in (1.2) and (1.4) by using the argument as in, for example, [11, Proposition 1.5.8] based on the use of operator $(I-L)^{\frac{1}{2}}$ and a bound (in $L^p(\Omega)$) on the operator R which is used to exchange the derivative operator. However, one may need higher order term than p^2 to bound the constant K_p in (2.2). It is worth noted that one can modify the results in the paper and use weaker norms, which may be more suitable for their own problems. However, in those cases, the order needed to bound the constant K_p would be higher.

4 An application

This section is devoted to an application of our main results. Let $\{\xi_\varepsilon(t)\}_{\varepsilon>0}$ be a family of stochastic processes depending on a Brownian motion $W(t)$. We are concerned with the exponential tightness of the following family of random variables

$$F_\varepsilon = \sqrt{\varepsilon} e^{-\frac{1}{\varepsilon^2}\int_0^1 \lambda(\xi_\varepsilon(r))dr} \int_0^1 e^{\frac{1}{\varepsilon^2}\int_0^s \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))dW(s), \tag{4.1}$$

where λ, g are smooth functions, bounded together with their derivatives. Moreover, $\lambda(x) \ge \kappa_0 > 0, \forall x$. In many problems in mathematical physics such as Langevin equations, stochastic acceleration, we need to deal with this family and establish its tightness (to obtain the limit behavior, the large deviations principle, the averaging principle, etc); see e.g., [1, 2, 9] and references therein. In general, such a term is often related to the solution of a second-order stochastic differential equations in random environment or in the setting of fast-slow second-order system; see e.g., [10]. To be self-contained, we write down a simple Langevin equation with strong damping after scaling the time (see e.g., [2]) in random environment as follows:

$$\varepsilon^2\ddot{x}(t) = f(x(t), \xi_\varepsilon(t)) - \lambda(\xi_\varepsilon(t))\dot{x}(t) + \sqrt{\varepsilon} g(\xi_\varepsilon(t))\dot{W}(t).$$

By using the variation of parameter formula (see e.g., [2]), we can obtain explicitly the diffusion part of x_ε. Dealing with this part requires the treatment of the family $\{F_\varepsilon\}_{\varepsilon>0}$ defined as in (4.1); see e.g., [2, 9, 10]. The non-adaptedness of $e^{-\frac{1}{\varepsilon^2}\int_0^1 \lambda(\xi_\varepsilon(r))dr}$ is a main challenge here because we cannot move it inside the stochastic integral in Itô's sense and estimates for martingales are no longer valid. Meanwhile, we really need such variable to balance the large factor $e^{\frac{1}{\varepsilon^2}\int_0^s \lambda(\xi_\varepsilon(r))dr}$ inside the stochastic integral. In the literature, much effort has been devoted to overcoming this challenge. For example, if we consider the case where $\xi_\varepsilon(t)$ is continuously differentiable (or piecewise continuously differentiable), Cerrai and Freidlin [2] have tried to interpret $\int_0^1 e^{\frac{1}{\varepsilon^2}\int_0^s \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))dW(s)$ in the pathwise sense. But this approach is no longer valid without the regularity of the random environment and also we cannot cancel out effectively the large factor $\frac{1}{\varepsilon^2}$

to provide estimates in probability; see the details in [1, 2, 9]. Another approach in [10] is to decompose $\lambda(\xi_\varepsilon(s))$ into two parts, one of them is adapted and the other is "controllable". However, this approach requires the decay of the derivative of λ to control such a decomposition.

With the results developed in this work, we propose a new approach for such problems. Using Malliavin calculus and integration by parts (see [11, Proposition 1.3.3]), we can write

$$\begin{aligned} F_\varepsilon =& \sqrt{\varepsilon}\int_0^1 e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))\delta W(s) \\ &+ \sqrt{\varepsilon}\int_0^1 e^{\frac{1}{\varepsilon^2}\int_0^s \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s)) D_s e^{-\frac{1}{\varepsilon^2}\int_0^1 \lambda(\xi_\varepsilon(r))dr} ds \\ =&: F_\varepsilon^{(1)} + F_\varepsilon^{(2)}. \end{aligned}$$

In fact, if $\{F_\varepsilon^{(1)}\}_{\varepsilon>0}$ is exponentially tight with the speed $v_1(\varepsilon)$ and $\{F_\varepsilon^{(2)}\}_{\varepsilon>0}$ is exponentially tight with the speed $v_2(\varepsilon)$, we will obtain that the family $\{F_\varepsilon\}_{\varepsilon>0}$ is exponentially tight with the speed $v(\varepsilon) = \max\{v_1(\varepsilon), v_2(\varepsilon)\}$. In the below, we use the letter c to represent universal constants (independent of ε, p), whose values may change for different usage.

Because there is no stochastic integral involving $F_\varepsilon^{(2)}$, the family $\{F_\varepsilon^{(2)}\}$ can be handled in the usual methodology in the literature. Indeed, we have that

$$D_s e^{-\frac{1}{\varepsilon^2}\int_0^1 \lambda(\xi_\varepsilon(r))dr} = -\frac{1}{\varepsilon^2} e^{-\frac{1}{\varepsilon^2}\int_0^1 \lambda(\xi_\varepsilon(r))dr} \int_0^1 \lambda'(\xi(r)) D_s \xi_\varepsilon(r) dr.$$

So, one gets

$$F_\varepsilon^{(2)} = \varepsilon^{-3/2}\int_0^1 e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s)) \int_0^1 \lambda'(\xi_\varepsilon(r)) D_s \xi_\varepsilon(r) dr ds,$$

and thus,

$$|F_\varepsilon^{(2)}| \le c\varepsilon^{-3/2}\int_0^1 e^{-\frac{\kappa_0(1-s)}{\varepsilon^2}} \|D_s\xi_\varepsilon\|_{L^1([0,1])} ds.$$

For any $p \in (1,\infty]$, Hölder's inequality yields that

$$\begin{aligned} |F_\varepsilon^{(2)}| \le& c\varepsilon^{-3/2}\Big(\int_0^1 e^{-\frac{p\kappa_0(1-s)}{(p-1)\varepsilon^2}} ds\Big)^{\frac{p-1}{p}} \Big(\int_0^1 \|D_s\xi_\varepsilon\|^p_{L^1([0,1])} ds\Big)^{\frac{1}{p}} \\ \le& c\varepsilon^{\frac{p-4}{2p}} \Big(\int_0^1 \|D_s\xi_\varepsilon\|^p_{L^1([0,1])} ds\Big)^{\frac{1}{p}}. \end{aligned}$$

Therefore, establishing the exponential tightness for $\{F_\varepsilon^{(2)}\}_{\varepsilon>0}$ reduces to establishing the exponential tightness for $\left(\int_0^1 \|D_s\xi_\varepsilon\|^p_{L^1([0,1])} ds\right)^{1/p}$. Thus, under certain conditions on $\xi_\varepsilon(s)$, we can obtain the exponential tightness of $\{F_\varepsilon^{(2)}\}_{\varepsilon>0}$.

The challenge, which we now should focus more on, is to handle the family $\{F_\varepsilon^{(1)}\}_{\varepsilon>0}$, which is in fact a family of Skorohod integrals. By applying our results, we can obtain the exponential tightness of $\{F_\varepsilon^{(1)}\}_{\varepsilon>0}$ under certain conditions. First, it is readily seen that

$$\int_0^1 e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s)) ds \le c\varepsilon^2.$$

On the other hand, we have

$$\begin{aligned}
&D_t e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s)) \\
&= -\frac{1}{\varepsilon^2} e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s)) D_t \int_s^1 \lambda(\xi_\varepsilon(r))dr + e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g'(\xi_\varepsilon(s)) D_t\xi_\varepsilon(s) \\
&= -\varepsilon^{-2} e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s)) \int_s^1 \lambda'(\xi_\varepsilon(r)) D_t\xi_\varepsilon(r)dr + e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g'(\xi_\varepsilon(s)) D_t\xi_\varepsilon(s).
\end{aligned} \tag{4.2}$$

Therefore, by direct computations, one has

$$|D_t e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))|^2 \le c\varepsilon^{-4} e^{-\frac{2\kappa_0(1-s)}{\varepsilon^2}} \int_0^1 |D_t\xi_\varepsilon(r)|^2 dr + c|D_t\xi_\varepsilon(s)|^2,$$

and thus

$$\int_0^1\int_0^1 |D_t e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))|^2 dtds \le c\varepsilon^{-2}\|D\xi_\varepsilon\|^2_{L^2([0,1]^2)}.$$

Therefore, by applying Theorem 1.2, under certain condition on $\|D\xi_\varepsilon\|_{L^2(\Omega,L^2([0,1]^2))}$, we can obtain the exponential tightness for $\{F_\varepsilon^{(1)}\}_{\varepsilon>0}$ with some speed $v_1(\varepsilon)$. To be clear, let us state an explicit result as the following theorem, which follows immediately from Theorem 1.2.

Theorem 4.1. *Assume that the family of random environments ξ_ε is such that there are constants $\kappa_1 > 1/2$, $\kappa_2 \ge 0$ satisfying*

$$\|D\xi_\varepsilon\|_{L^p(\Omega,L^2([0,1]^2))} \le c\varepsilon^{\kappa_1}p^{\kappa_2},\ \forall\varepsilon>0, p\ge 1,$$

for some universal constant c. The family $\{F_\varepsilon^{(1)}\}_{\varepsilon>0}$ is exponentially tight with the speed $v_1(\varepsilon)=\varepsilon^\alpha$ for any α satisfying

$$\alpha < \frac{\kappa_1 - 1/2}{5+\kappa_2}.$$

One may worry about the condition $\kappa_1 > 1/2$ in Theorem 4.1. In particular, if ξ_ε is independent of ε, such a "decaying condition" on ε may be violated. In that case, we can modify the above as follows. We have from (4.2) that

$$|D_t e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))|^2 \le c\varepsilon^{-4} e^{-\frac{2\kappa_0(1-s)}{\varepsilon^2}} (1-s)\sup_{r\in[0,1]} |D_t\xi_\varepsilon(r)|^2 + c|D_t\xi_\varepsilon(s)|^2. \tag{4.3}$$

A change of variable leads to

$$\int_0^1 \exp\left\{\frac{-\kappa_0 s}{\varepsilon^2}\right\}\cdot\frac{s}{\varepsilon^2}ds = \varepsilon^2\int_0^{\frac{1}{\varepsilon^2}} e^{-\kappa_0 r} r dr \le c\varepsilon^2. \tag{4.4}$$

Combining (4.3) and (4.4), one gets that

$$\int_0^1\int_0^1 |D_t e^{-\frac{1}{\varepsilon^2}\int_s^1 \lambda(\xi_\varepsilon(r))dr} g(\xi_\varepsilon(s))|^2 dtds \le c\int_0^1 \sup_{r\in[0,1]} |D_t\xi_\varepsilon(r)|^2 dt.$$

Theorem 4.2. *Assume that the family of random environments ξ_ε is such that there are constants $\kappa_1 > -1/2$, $\kappa_2 \ge 0$ satisfying*

$$\left\|\int_0^1 \sup_{r\in[0,1]} |D_t\xi_\varepsilon(r)|^2 dt\right\|_p \le c\varepsilon^{\kappa_1}p^{\kappa_2},\ \forall\varepsilon>0, p\ge 1,$$

for some universal constant c. The family $\{F_\varepsilon^{(1)}\}_{\varepsilon>0}$ is exponentially tight with the speed $v_1(\varepsilon)=\varepsilon^\alpha$ for any α satisfying

$$\alpha < \frac{\kappa_1 + 1/2}{5+\kappa_2}.$$

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