

On General Notions of Depth for Regression

Yijun Zuo

Abstract. Depth notions in location have generated tremendous attention in the literature. In fact, data depth and its applications remain as one of the most active research topics in statistics over the last three decades. Most favored notions of depth in location include Tukey (In *Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974)*, Vol. 2 (1975) 523–531) half-space depth (HD), Liu (*Ann. Statist.* **18** (1990) 405–414) simplicial depth and projection depth (PD) (Stahel (1981) and Donoho (1982), Liu (In *L₁-Statistical Analysis and Related Methods (Neuchâtel, 1992)* (1992) 279–294 North-Holland), Zuo and Serfling (*Ann. Statist.* **28** (2000) 461–482) and (ZS00) and Zuo (*Ann. Statist.* **31** (2003) 1460–1490)), among others.

Depth notions in regression have also been proposed sporadically, nevertheless. The regression depth (RD) of Rousseeuw and Hubert (*J. Amer. Statist. Assoc.* **94** (1999) 388–433) (RH99), the most famous, exemplifies a direct extension of Tukey HD to regression. Other notions include Carrizosa (*J. Multivariate Anal.* **58** (1996) 21–26) and the ones proposed in this article via modifying a functional in Maronna and Yohai (*Ann. Statist.* **21** (1993) 965–990) (MY93). Is there any relationship between Carrizosa depth and the RD of RH99? Do these depth notions possess desirable properties? What are the desirable properties? Can existing notions serve well as depth notions in regression? These questions remain open.

The major objectives of the article include (i) revealing the connection between Carrizosa depth and RD of RH99; (ii) expanding location depth evaluating criteria in ZS00 for regression depth notions; (iii) examining the existing regression notions with respect to the gauges; and (iv) proposing the regression counterpart of the eminent location projection depth.

Key words and phrases: Depth, unfitness, linear regression, maximum depth regression estimating functionals, robustness.

1. INTRODUCTION

Notion of depth in location has attracted vast attention and has been increasingly pursued as a powerful tool for multidimensional nonparametric data analysis and inference.

Prevailing location depth notions include Tukey (1975) half-space depth (HD) (popularized by Donoho and Gasko, 1992), Liu (1990) simplicial depth, the spatial depth (Vardi and Zhang, 2000), projection depth (PD) (Stahel, 1981 and Donoho, 1982, Liu, 1992, Zuo and Serfling, 2000 and Zuo, 2003), and zonoid depth (Koshevoy and Mosler, 1997, Mosler, 2002). Applications of data depth in multivariate statistics include:

(i) construction of multivariate inference procedures, such as depth-based tests, rank tests, multivariate quantiles, control charts, and confidence regions (Liu, 1992, Liu, 1995, Liu and Singh, 1993, Liu, Parelius and Singh, 1999, Li and Liu, 2004, Hallin, Paindaveine and Šiman, 2010, Chakraborty and Chaudhuri, 2014, Chernozhukov, Galichon, Hallin and Henry, 2017, Yeh and Singh, 1997, Zuo, 2009, 2010);

(ii) multivariate exploratory data analysis, such as in geophysical, hydrological and physio meteorological research (Liu, Parelius and Singh, 1999, Chebana and Ouarda, 2008, 2011);

(iii) outlier detection, such as in environmental studies (Dang and Serfling, 2010, Serfling and Wang, 2014, Wang and Serfling, 2015, 2018, Febrero, Galeano and González-Manteiga, 2008);

(iv) clustering and classification, especially in microarray gene expression data analysis, discriminant anal-

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ysis, and supervised learning (Hoberg, 2000, Jörnsten, 2004, Ghosh and Chaudhuri, 2005, Mosler and Hoberg, 2006, Cui, Lin and Yang, 2008, Li, Cuesta-Albertos and Liu, 2012, Lange, Mosler and Mozharovskiy, 2014, Paindaveine and Van Bever, 2015);

(v) multivariate risk measurement in financial engineering (Casco and Molchanov, 2007);

(vi) remote sensing and signal processing (Velasco-Forero and Angulo, 2011, 2012); robust linear programming (Mosler and Bazovkin, 2014), and

(vii) econometric and social studies (Caplin and Nalebuff, 1988, 1991a, 1991b),

among others. In fact, data depth and its applications remain as one of the most active research topics in statistics over the last three decades.

The notion of location depth has been extended to local depth (Agostinelli and Romanazzi, 2011, Paindaveine and Van Bever, 2013); to depth for concentration, scatter and shape matrices (Chen et al., 2018, Paindaveine and van Bever, 2018); and to depth for functional data (e.g., López-Pintado and Romo, 2009, Claeskens, Hubert, Slaets and Vakili, 2014, Hubert, Rousseeuw and Segaut, 2015, Nieto-Reyes and Battey, 2016, Gijbels and Nagy, 2017, among others).

Mizera (2002) introduced a scheme as a calculus technique/tool to derive depth functions for different statistical models, and Mizera and Müller (2004) extended the tangent depth for location and regression to the location-scale setting. Data depth has also been employed as penalties in penalized regression (Majumdar and Chatterjee, 2018).

Depth notions in regression have been inevitably proposed, yet sporadically. Regression depth (RD) by Rousseeuw and Hubert (1999) (RH99), the most famous, exemplifies a direct extension of Tukey location depth in regression. Others include pioneer Carrizosa depth (Carrizosa, 1996) and the ones proposed here which are induced from Maronna and Yohai (1993) (MY93). Attention paid to regression depth has been disproportionately light, compared with its location counterpart. One of the reasons for this might be that there exist no clear fundamental principles to evaluate or measure proposed regression depth notions. Lack of the evaluation criteria not only prohibits any further advance and theoretical development of the depth notions in regression but also impedes their applications in practice. One major objective of the current article is to extend the set of criteria (or desired axiomatic properties) for depth notions in location in ZS00 to regression and to examine the existing regression notions of depth with respect to the proposed criteria.

There exist a variety of robust methods including M-estimate approach and ad hoc ones (see Rousseeuw and Leroy, 1987 (RL87); Maronna, Martin and Yohai, 2006

(MMY06)) for estimating the parameters in a linear regression model. In this article, regression depth is utilized to introduce the median-type deepest estimating functionals for regression parameters, manifesting one of the prominent advantages of notions of depth. The functionals are the minimizers of the maximum of unfitness of regression parameters and recover in the empirical case the classical least squares, least absolute deviations and other existing leading estimators. Under a general framework, depth notions induced from projection-pursuit approach include the RD_{RH} and projection regression depth (PRD) (induced from MY93) as special cases. The latter is an extension of the eminent PD in location to regression.

The rest of this article is organized as follows. Section 2 presents a general definition for notions of unfitness and depth in regression and puts forward four general approaches for introducing the notions of unfitness or depth and the maximum (deepest) depth functionals while examining three special examples. It is found that Carrizosa (1996) depth, D_C , known of recovering the HD in location, is not identical but closed related to RD_{RH} in regression. Section 3 provides a rigorous definition of depth (or unfitness) notion in regression based on four axiomatic properties which then are employed for the evaluation of four types of special depth notions. Section 4 ends the article with brief concluding remarks. The Appendix collects some major proofs and derivations and auxiliary lemmas.

2. DEFINITIONS, APPROACHES AND EXAMPLES

2.1 Regression Model

Consider a general linear regression model:

$$(1) \quad y = \mathbf{x}'\boldsymbol{\beta} + e,$$

where $'$ denotes the transpose of a vector; random variable y and e are in \mathbb{R}^1 ; and random vector $\mathbf{x} = (x_1, \dots, x_p)'$ and parameter vector $\boldsymbol{\beta}$ are in \mathbb{R}^p . Note that this general model includes the special case with an intercept term. For example, if $\boldsymbol{\beta} = (\beta_1, \boldsymbol{\beta}_2)'$ and $x_1 = 1$, then one has $y = \beta_1 + \mathbf{x}_2'\boldsymbol{\beta}_2 + e$, where $\mathbf{x}_2 = (x_2, \dots, x_p)'$ $\in \mathbb{R}^{p-1}$. Denote $\mathbf{w} = (1, \mathbf{x}_2)'$, then $y = \mathbf{w}'\boldsymbol{\beta} + e$. We use this model or (1) interchangeably, depending on the context. Denote by $F_{(y, \mathbf{x})}$ the joint probability distribution of y and \mathbf{x} under the model (1).

In the following sections, we discuss the notions of *unfitness* or *depth* and *general approaches* to introduce regression depth and induced deepest estimating functionals.

2.2 Notions of Unfitness and Depth in Regression

Unfitness of a candidate parameter $\boldsymbol{\beta}$: $UF(\boldsymbol{\beta})$, is a function of the residual $r(\boldsymbol{\beta}) := (y - \mathbf{x}'\boldsymbol{\beta})$. Namely, $UF(\boldsymbol{\beta}) = f(r(\boldsymbol{\beta}))$. Examples of $f(x)$ include, x^2 and $|x|$. Generally speaking, an even, monotonic in $|x|$, and convex $f(\cdot)$ with its minimum value 0 at 0 will serve the purpose.

Depth of β then can be defined as a bounded reciprocal (reverse) function of $\phi(F_R)$ (e.g., $1/(1+x)$), say on $[0, 1]$, where ϕ is a functional on the distribution of $R := UF(\beta)$. A typical example of ϕ is the expectation or quantile functional, $\phi(F_R)$ could also just be R . Likewise, given its depth, one can define the unfitness of β to be a reciprocal function of the depth.

A minimizer β^* of unfitness function of β over all $\beta \in \mathbb{R}^p$ can serve as a regression estimating functional for β . Similarly, a maximizer of depth function plays the same role.

2.3 Four Approaches for Notions of Unfitness and Depth

2.3.1 Classical objective function approach. Directly employing the scheme above, one can recover many classical regression estimators in the empirical distribution case (i.e., $r_i(\beta) = y_i - \mathbf{x}'_i \beta$, $i = 1, \dots, n$). Here, the classical objective function in regression serves as the unfitness function, that is, $UF(\beta) = f_{\text{Obj}}(r(\beta))$. Maximizing depth of β is equivalent to the minimization of $\phi(F_R)$ and then the minimizer denoted by β^* could serve as an estimator for $\beta \in \mathbb{R}^p$. In the sequel, consider examples of ϕ : (i) the expectation functional μ , and (ii) quantile functional q_τ , $\tau \in (0, 1)$.

EXAMPLE 2.1.

(I) If $\phi = \mu$ and $f(x) = x^2$, then $\beta^*(F_{(y,x)}) = \arg \min_{\beta \in \mathbb{R}^p} \int (t - \mathbf{s}'\beta)^2 dF_{(y,x)}(t, \mathbf{s})$, which induces the least squares (LS) estimator when $F_{(y,x)}$ is the empirical distribution.

(II) If $\phi = \mu$ and $f(x) = |x|$, then the approach above leads to the least absolute deviations (LAD) estimator;

(III) If $\phi = \mu$ and $f_\tau(x) = x(\tau - \mathbf{I}(x < 0))$, $\tau \in (0, 1)$, where \mathbf{I} is the indicator function, then the approach above results in the quantile regression estimator (Koenker and Bassett, 1978). When $\tau = 1/2$, it recovers the L_1 regression estimator, for related discussions on quantile regression; see Portnoy (2003, 2012);

(IV) If $\phi = q_{0.5}$ and $f(x) = x^2$, then the approach above yields the least median squares (LMS) estimator (Rousseeuw, 1984); and

(V) If $\phi = \mu$, coupled with an appropriately chosen function $f(x)$, one can actually recover the M-estimators (Huber, 1973 (including the famous (a) Huber's proposal 2 (Huber, 1964), (b) Hampel's three-parts (Hampel, 1974), and (c) Tukey bisquare (Beaton and Tukey, 1974) ones), the L-estimators (Ruppert and Carroll, 1980) and the R-estimators (Koul, 1970, 1971); Jurečková, 1971, and Jaeckel, 1972).

2.3.2 Facility location approach. In addition to the general approach mentioned in the Section 2.3.1, there exist other approaches for the introduction of notions of

depth or unfitness. The classical one is the *facility location approach*, prevailing in location analysis and operations research.

Let $\mathbf{F}_1 \in \mathbb{R}^2$ be a candidate for a facility location, and P be the probability distribution of a random vector $\mathbf{X} \in \mathbb{R}^2$ (or of consumers' locations), and $d(\mathbf{F}_1, \mathbf{X})$ (defined in Remarks 2.1 below) measures in some sense the closeness of \mathbf{F}_1 to the distribution P of \mathbf{X} (or the coverage of consumers). Let \mathbf{F}_2 be a candidate for the facility location of another competitive company. Similarly, $d(\mathbf{F}_2, \mathbf{X})$ measures the coverage of the consumers in the vicinity of the facility at \mathbf{F}_2 . With respect to the \mathbf{F}_1 , the maximum market share which can be captured by any other facility is $\sup_{\mathbf{F}_2 \in \mathbb{R}^2} P(\omega : d(\mathbf{F}_2, \mathbf{X}(\omega)) < d(\mathbf{F}_1, \mathbf{X}(\omega)))$. Also, the \mathbf{F}_1 should be chosen to maximize its market share:

$$\begin{aligned} \mathbf{F}_1^* &= \arg \max_{\mathbf{F}_1 \in \mathbb{R}^2} \left(1 - \sup_{\mathbf{F}_2 \in \mathbb{R}^2} P(\omega : d(\mathbf{F}_2, \mathbf{X}(\omega)) \right. \\ &\quad \left. < d(\mathbf{F}_1, \mathbf{X}(\omega))) \right) \\ (2) \quad &= \arg \max_{\mathbf{F}_1 \in \mathbb{R}^2} \inf_{\mathbf{F}_2 \in \mathbb{R}^2} P(\omega : d(\mathbf{F}_1, \mathbf{X}(\omega)) \\ &\quad \leq d(\mathbf{F}_2, \mathbf{X}(\omega))). \end{aligned}$$

Carrizosa (1996) extended \mathbb{R}^2 above to \mathbb{R}^p ($p \geq 2$) and introduced a depth notion (normalized depth, see Definition 2.1 below). Let us use a generic term, *Carrizosa depth*, hereafter. The Carrizosa depth of \mathbf{x} w.r.t. $P: D_C(\mathbf{x}; P)$ (P and $F_{\mathbf{X}}$ are used interchangeably), is defined as

$$(3) \quad D_C(\mathbf{x}; P) = \inf_{\mathbf{y} \in \mathbb{R}^p} P(\omega : d(\mathbf{x}, X(\omega)) \leq d(\mathbf{y}, X(\omega))).$$

Then \mathbf{F}_1^* in (2) is the maximum depth solution (functional) for the facility location problem.

REMARKS 2.1.

(I) Note that the distance measure d above includes a class of possible choices. For example, d could be an L_p norm or weighted L_p norm (see Zuo, 2004) ($p \geq 1$).

(II) When $d(x, y) = \|x - y\|$, where “ $\|\cdot\|$ ” stands for the Euclidean (or L_2) norm, (3) recovers the *normalized depth* $ND(x; P)$ of Carrizosa (1996) that is quoted below:

DEFINITION 2.1 (Carrizosa, 1996). The normalized depth of $ND(x; P)$ of a point $\mathbf{x} \in \mathbb{R}^p$ in P is defined as

$$(4) \quad ND(\mathbf{x}; P) = \inf_{\mathbf{y} \in \mathbb{R}^p} P(\{a : \|y - a\| \geq \|x - a\|\}).$$

Henceforth, we focus on L_2 norm for distance measure d , unless stated otherwise.

In the location setting, Donoho and Gasko (1992) first addressed the notion of depth proposed in Tukey (1975), their empirical depth is some integer among $\{1, \dots, n\}$.

In the following, we invoke a slightly different characterization for Tukey's depth given in Zuo (1998) (called half-space depth (HD)):

$$(5) \quad \text{HD}(x; P) = \inf_H \{P(H) : H \text{ is a closed half-space and } x \in H\}, \quad x \in \mathbb{R}^p.$$

It turns out that $D_C(x; P)$ can actually recover $\text{HD}(x; P)$ as shown in Carrizosa (1996).

PROPOSITION 2.1. *If d in (3) is the L_2 norm, then $D_C(x; P)$, equivalently $\text{ND}(\mathbf{x}; P)$ in (4), is identical to $\text{HD}(x; P)$ in (5).*

PROOF. See the proof of Proposition 1 of Carrizosa (1996). \square

Although the depth $D_C(x; P)$ above is introduced initially for the location problem, it can be extended for the regression problem, as done in Carrizosa (1996) with the L_1 norm.

Indeed, given a probability measure P (or equivalently $F_{(y, \mathbf{x})}$) in \mathbb{R}^p , one could define the $D_C(\boldsymbol{\beta}; P)$ of $\boldsymbol{\beta} = (\beta_1, \boldsymbol{\beta}'_2)' \in \mathbb{R}^p$ as follows: for $|\beta_1| < \infty$,

$$(6) \quad D_C(\boldsymbol{\beta}; P) = \inf_{\boldsymbol{\alpha} \in \mathbb{R}^p} P(d(y, (1, \mathbf{x}')'\boldsymbol{\beta}) \leq d(y, (1, \mathbf{x}')'\boldsymbol{\alpha})), \\ \mathbf{x} \in \mathbb{R}^{p-1} \quad (p \geq 2),$$

where $\boldsymbol{\alpha} = (\alpha_1, \boldsymbol{\alpha}'_2)'$ and $\alpha_2, \boldsymbol{\beta}_2 \in \mathbb{R}^{p-1}$; if $|\beta_1| \rightarrow \infty$, then define $D_C(\boldsymbol{\beta}; P) \rightarrow 0$. When $d(x, y) = |x - y|$, (6) recovers the *depth in regression* in Carrizosa (1996). The latter seems to be the pioneer notion of depth in regression in the literature. Does it have anything to do with the RD_{RH} of RH99? Let us first quote the original definition of RH99.

DEFINITION 2.2 (RH99). The regression depth of $\boldsymbol{\beta}$ is the minimum probability mass that needs to be passed when tilting $\boldsymbol{\beta}$ in any way until it is vertical.

Since D_C recovers HD in location and RD_{RH} is an extension of HD in regression, naturally, one wonders whether D_C can recover RD_{RH} in regression. The two are closely connected but not identical as revealed in Proposition 2.2 below.

The same idea of Carrizosa (1996) was proposed again in Adrover, Maronna and Yohai (2002) (AMY02). AMY02 first flawlessly defined RD_{RH} above to be

$$(7) \quad \text{RD}_{\text{RH}}(\boldsymbol{\beta}, P) = \inf_{\boldsymbol{\lambda} \neq \mathbf{0}} P\left(\frac{r(\boldsymbol{\beta})}{\boldsymbol{\lambda}'\mathbf{x}} < 0, \boldsymbol{\lambda}'\mathbf{x} \neq 0\right),$$

under assumptions (a) and (b) below, where $\boldsymbol{\lambda} \in \mathbb{R}^p$, $r(\boldsymbol{\beta}) = y - \mathbf{x}'\boldsymbol{\beta}$. They then proposed the depth:

$$(8) \quad D(\boldsymbol{\beta}, P) = \inf_{\boldsymbol{\gamma} \in \mathbb{R}^p} P(|r(\boldsymbol{\beta})| \leq |r(\boldsymbol{\gamma})|).$$

If the first coordinate x_1 of \mathbf{x} in (8) is 1 and $d(x, y) = |x - y|$ in (6), then (8) and (6) coincides.

Under the assumptions (a) $P(\mathbf{x}'\mathbf{v} = 0) = 0$ for all $\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^p$ and (b) $P(r(\boldsymbol{\beta}) = 0) = 0$ for all $\boldsymbol{\beta} \in \mathbb{R}^p$, AMY02 showed that (8) is equivalent to (7) (the last step of the proof is debatable though). (a) and (b) exclude *any* discrete distribution cases of (y, \mathbf{x}) , nevertheless. The following result characterizes $D_C(\boldsymbol{\beta}; P)$ and reveals its connection with $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$.

Write $\mathbf{w} = (1, \mathbf{x}')'$ and $r(\boldsymbol{\beta}) = y - \mathbf{w}'\boldsymbol{\beta}$. If $d(x, y) = |x - y|$, then (6) is equivalent to

$$(9) \quad D_C(\boldsymbol{\beta}, P) = \inf_{\boldsymbol{\alpha} \in \mathbb{R}^p} P(|r(\boldsymbol{\beta})| \leq |r(\boldsymbol{\alpha})|).$$

For a given $\boldsymbol{\beta} \in \mathbb{R}^p$ with $\|\boldsymbol{\beta}\| < \infty$, denote by $H_{\boldsymbol{\beta}}$ the unique hyperplane determined by $y = \mathbf{w}'\boldsymbol{\beta}$. Likewise, a given nonvertical hyperplane H uniquely identifies an $\boldsymbol{\alpha} \in \mathbb{R}^p$ through $y = \mathbf{w}'\boldsymbol{\alpha}$. Define $S(\boldsymbol{\beta}) := \{\boldsymbol{\alpha} \in \mathbb{R}^p : H_{\boldsymbol{\alpha}} \text{ intersects with } H_{\boldsymbol{\beta}}\}$ for the given $\boldsymbol{\beta}$.

PROPOSITION 2.2. *If $d(x, y) = |x - y|$ in (6), then (i) $D_C(\boldsymbol{\beta}; P) = P(r(\boldsymbol{\beta}) = 0)$, and (ii) $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) = \inf_{\boldsymbol{\alpha} \in S(\boldsymbol{\beta})} P(|r(\boldsymbol{\beta})| \leq |r(\boldsymbol{\alpha})|)$.*

PROOF. See the Appendix. \square

$D_C(\boldsymbol{\beta}; P)$ in (9) is not identical to original RD_{RH} of RH99, but is closely related to the latter. In fact, if the infimum in RHS of (9) performs over $S(\boldsymbol{\beta})$, then they are identical. This is another characterization of RD_{RH} . $D_C(\boldsymbol{\beta}; P)$ is no greater than $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$.

Based on the depth functional in (6), we can introduce the maximum regression depth estimating functional for $\boldsymbol{\beta}$, which is defined, for $d(x, y) = |x - y|$, as

$$(10) \quad \boldsymbol{\beta}^*(P) = \arg \max_{\boldsymbol{\beta} \in \mathbb{R}^p} D_C(\boldsymbol{\beta}; P),$$

$\boldsymbol{\beta}^*(P)$ above is well defined. That is, the maximum on the RHS of (10) is attained at a bounded $\boldsymbol{\beta}$. The latter is safeguarded by the result below under the assumption:

(A): $P(H_v) = 0$ for any vertical hyperplane H_v .

PROPOSITION 2.3. *Under (A), (i) $\lim_{\|\boldsymbol{\beta}\| \rightarrow \infty} D_C(\boldsymbol{\beta}; P) = 0$, and (ii) the maximum on the RHS of (10) exists and is attained at a bounded $\boldsymbol{\beta}$.*

PROOF. See the Appendix. \square

2.3.3 Projection-pursuit approach. There is another approach based on the projection-pursuit (PP) scheme to induce the regression estimating functional for parameter $\boldsymbol{\beta}$. One starts with a univariate regression estimating functional w.r.t. the univariate variable $\mathbf{u}'\mathbf{x} \in \mathbb{R}$ and $r(\boldsymbol{\beta})$ along each direction $\mathbf{u} \in \mathbb{S}^{p-1} := \{\mathbf{v}, \|\mathbf{v}\| = 1, \mathbf{v} \in \mathbb{R}^p\}$ and calculates the $\text{UF}_{\mathbf{u}}(\boldsymbol{\beta})$ (the unfitness along \mathbf{u}) (see Section 2.2 for the definition of unfitness). Then one obtains $\text{UF}(\boldsymbol{\beta})$, the supremum of $\text{UF}_{\mathbf{u}}(\boldsymbol{\beta})$ over all $\mathbf{u} \in \mathbb{S}^{p-1}$. Finally, one minimizes $\text{UF}(\boldsymbol{\beta})$ over all $\boldsymbol{\beta} \in \mathbb{R}^p$ to obtain a regression estimating functional $\boldsymbol{\beta}^*$ for $\boldsymbol{\beta}$ via the min-max scheme.

REMARKS 2.2.

(I) The approach above actually can recover the maximum regression depth functional in RH99 and induce a maximum projection depth functional that is closely related to P1-estimate in MY93. We elaborate the two special cases in the following:

(II) A related PP approach was discussed in RL87 (page 144). In the empirical case, to obtain the estimator β^* , it minimizes an objective dispersion function $s(r_1(\beta), \dots, r_n(\beta))$, where $r_i(\beta) = y_i - \mathbf{x}'_i \beta$, s is just scale equivariant (not translation invariant), and $r_i(\beta)$ are regarded as a projection of the point $(y_i, \mathbf{x}'_i)'$ onto $(1, -\beta)'$. By varying s , this approach covers a very large family of estimators (LS, LAD, LTS, LMS and S, etc.).

EXAMPLE 2.2 (Regression depth and maximum regression depth functional). Consider the linear model: $y = \beta_1 + \mathbf{x}'\beta_2 + e$, where $\mathbf{x}, \beta_2 \in \mathbb{R}^{p-1}$. Denote $\mathbf{w} = (1, \mathbf{x}')'$, $\beta = (\beta_1, \beta_2)'$. Then the model is $y = \mathbf{w}'\beta + e$. That is, \mathbf{w} here corresponds to \mathbf{x} in general model (1) and vice versa.

When $p = 2$, we define $F(\beta) = E(\mathbf{I}((y - \mathbf{w}'\beta) * \mathbf{v}'\mathbf{w} \geq 0))$, where $F(\beta)$ stands for “fitness” of β , \mathbf{I} for the indicator function, and $\mathbf{v} = (-v_1, v_2)$, $v_1 \in \mathbb{R}$, $|v_2| = 1$. When $v_2 = 1$, it represents the total probability mass touched (covered) by tilting the line $y = \beta_1 + \beta_2 x$ counter-clockwise around the point $(v_1, \beta_1 + \beta_2 v_1)$ to the vertical position (note that the point is the intersection point of the line with the vertical line $x = v_1$). By considering the clockwise tilting ($v_2 = -1$), it is seen that the closer to 1/2 the total mass is, the better (more balanced) the candidate parameter β is.

When $p > 2$, with the same $F(\beta)$ as defined above, it can be shown that in the empirical case, minimizing $F(\beta)$ over all $v_1 \in \mathbb{R}$ and \mathbf{v}_2 with $\|\mathbf{v}_2\| = 1$ ($\mathbf{v} = (-v_1, \mathbf{v}'_2)'$) leads essentially to the regression depth RD_{RH} of β in RH99 (see the derivations in the Appendix for the general case where it is shown that the approach here is equivalent to (12) below). That is,

$$(11) \quad \inf_{v_1 \in \mathbb{R}, \|\mathbf{v}_2\|=1} E(\mathbf{I}((y - \mathbf{w}'\beta) * \mathbf{v}'\mathbf{w} \geq 0)) \\ = \text{RD}_{\text{RH}}(\beta; P).$$

Equivalent definitions (or characterizations) of RD_{RH} in Definition 2.2 exist in the literature (see Remarks A.1). The one given in Rousseeuw and Struyf (2004) (RS04) is

$$(12) \quad \text{RD}_{\text{RH}}(\beta; P) = \inf_{D \in \mathcal{D}} \{P((r(\beta) \geq 0) \cap D) \\ + P((r(\beta) \leq 0) \cap D^c)\},$$

where \mathcal{D} is the set of all vertical closed half-spaces D .

Now, we can define $\text{UF}(\beta)$ as a simple reciprocal function of $F(\beta)$ (e.g., $f(x) = a(1-x)/x$, $a > 0$) such that it equals ∞ if the latter equals zero and equals zero if the

latter is 1. Maximizing $\text{UF}(\beta)$ leads to the RD_{RH} of β . Furthermore, minimizing the maximum of $\text{UF}(\beta)$ over all $\beta \in \mathbb{R}^p$ leads to the maximum regression depth functional β^* .

EXAMPLE 2.3 (Projection regression depth and maximum depth functional). Hereafter, assume that R is a univariate regression estimating functional which satisfies

(A1) regression, scale and affine equivariant, that is,

$$R(F_{(y+xb, x)}) = R(F_{(y, x)}) + b, \forall b \in \mathbb{R}; \\ R(F_{(sy, x)}) = sR(F_{(y, x)}), \forall s \in \mathbb{R}; \text{ and} \\ R(F_{(y, ax)}) = a^{-1}R(F_{(y, x)}), \forall a \in \mathbb{R} \text{ and } a \neq 0,$$

respectively, where $x, y \in \mathbb{R}$ are random variables.

(A2) $\sup_{\|\mathbf{v}\|=1} |R(F_{(y, \mathbf{x}'\mathbf{v})})| \leq \infty$.

(A3) $R(F_{(y - \mathbf{x}'\beta, \mathbf{x}'\mathbf{v})})$ is quasi-convex and continuous in $\beta \in \mathbb{R}^p$ for any fixed $\mathbf{v} \in \mathbb{S}^{p-1}$.

Assume that S is a positive scale estimating functional such that $S(F_{(sz+b)}) = |s|S(F_z)$ for random variable $z \in \mathbb{R}$ and scalar $b, s \in \mathbb{R}$; that is, S is *scale equivariant* and *location invariant*.

REMARKS 2.3.

(I) Note that, the R above applies for the regression models that do not contain an intercept term (regression through the origin). The latter situation is required in certain applications (see page 62 in RL87) or is generally applicable by some simple treatments of original data (see Eisenhauer, 2003).

(II) In the sequel, R will be restricted to the form $R(F_{(y - \mathbf{x}'\beta, \mathbf{x}'\mathbf{v})}) = T(F_{(y - \mathbf{x}'\beta)/\mathbf{x}'\mathbf{v}})$, $\mathbf{x}'\mathbf{v} \neq 0$. T could be a univariate location functional that is location, scale and affine equivariant (see pages 158–159 of RL87 for definitions). Examples of T include mean and quantile functionals, among others. An example of R is $R(F_{(y - \mathbf{x}'\beta, \mathbf{x}'\mathbf{v})}) = \text{Med}((y - \mathbf{x}'\beta)/\mathbf{x}'\mathbf{v})$, where Med stands for the median functional and $\text{Med}(Z)$ for $\text{Med}(F_Z)$.

(III) (A2) holds trivially if T is a quantile-type functional (such as median functional) or mean-type functional if the moments of the underlying distribution exist. (A3) holds for those T as long as integrands involved are quasi-convex and continuous.

Pairs of T and S induce a class of projection regression estimating functionals. Define

$$(13) \quad \text{UF}_{\mathbf{v}}(\beta; F_{(y, x)}, T) := |T(F_{(y - \mathbf{x}'\beta)/\mathbf{x}'\mathbf{v}})|/S(F_y),$$

which represents unfitness of β at $F_{(y, x)}$ w.r.t. T along the direction $\mathbf{v} \in \mathbb{S}^{p-1}$. Note that if T is a Fisher consistent estimating functional, then $T(F_{(y - \mathbf{x}'\beta)/\mathbf{x}'\mathbf{v}}) = 0$ under the assumption $E(e|\mathbf{x}) = 0$ in model (1) for some β_0 (the true parameter of the model) and the classical model assumption that 0 is some kind of center of the error distribution,

and \mathbf{x} and e are independent. Note that $S(F_y)$ does not depend on \mathbf{v} and $\boldsymbol{\beta}$.

That is, overall one expects $|T|$ to be small and close to zero for a candidate $\boldsymbol{\beta}$, independent of the choice of \mathbf{v} and $\mathbf{x}'\mathbf{v}$. The magnitude of $|T|$ measures the unfitness of $\boldsymbol{\beta}$ along the \mathbf{v} . Dividing here by $S(F_y)$ is simply to guarantee the scale invariance of $\text{UF}_{\mathbf{v}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}, T)$. Taking the supremum over all $\mathbf{v} \in \mathbb{S}^{p-1}$ and suppressing T , yields

$$(14) \quad \text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) = \sup_{\|\mathbf{v}\|=1} \text{UF}_{\mathbf{v}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}, T),$$

the *unfitness* of $\boldsymbol{\beta}$ at $F_{(y,\mathbf{x})}$ w.r.t. T . Now applying the min–max scheme, we obtain the *projection regression estimating functional*

$$(15) \quad \boldsymbol{\beta}^*(F_{(y,\mathbf{x})}) = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^p} \text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}).$$

REMARKS 2.4.

(I) $\text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ corresponds to the *outlyingness* $O(x, F_X)$, and $\boldsymbol{\beta}^*(F_{(y,\mathbf{x})})$ corresponds to the *projection median functional* $\text{PM}(F_X)$ in the location setting (see Zuo, 2003). In (13), (14) and (15), we have suppressed S since it does not involve \mathbf{v} and is nominal (besides achieving the scale invariance). T in (14) and (15) is also suppressed for convenience.

(II) A similar $\boldsymbol{\beta}^*$ was first studied in MY93, where it was called P1-estimate (denote it by T_{P1} , see (16)). However, they are different. First, MY93 did not talk about the “unfitness” (or “depth”). Second, the definition of $\boldsymbol{\beta}^*$ here is *different from* T_{P1} of MY93, the latter multiplies by $S(F_{\mathbf{v}'\mathbf{x}})$ instead of dividing by $S(F_y)$ in (13). They instead defined the following:

$$A(\boldsymbol{\beta}, \mathbf{v}) = |R(F_{(y-\boldsymbol{\beta}'\mathbf{x}, \mathbf{v}'\mathbf{x})})| S(F_{\mathbf{v}'\mathbf{x}}),$$

where $\mathbf{v}, \boldsymbol{\beta} \in \mathbb{R}^p$. Their P1-estimate is defined as

$$(16) \quad T_{P1} = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^p} \sup_{\|\mathbf{v}\|=1} A(\boldsymbol{\beta}, \mathbf{v}).$$

Later we will revisit T_{P1} and explain why we divide by $S(F_y)$ in (13) instead of multiplying $S(F_{\mathbf{v}'\mathbf{x}})$. Note that $S(F_y)$ here could also be replaced by $S(F_{y-\boldsymbol{\beta}'\mathbf{x}})$.

(III) The projection-pursuit idea here was first employed in a multivariate location setting by Stahel (1981) and Donoho (1982) independently.

PROJECTION REGRESSION DEPTH (PRD). One can also introduce the notion of projection depth in regression using the $\text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$. For example, to make the depth between 0 and 1, define a *projection regression depth* (PRD) functional of $\boldsymbol{\beta}$ at $F_{(y,\mathbf{x})}$ w.r.t. a pair (T, S) as

$$(17) \quad \text{PRD}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) = (1 + \text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}))^{-1}.$$

It is readily seen that the LHS of (15) is also a *maximizer* of projection regression depth functional. For the specific pair of T and S such as

$$T(F_{(y-\mathbf{x}'\boldsymbol{\beta})/(\mathbf{x}'\mathbf{v})}) = \text{Med}_{\mathbf{x}'\mathbf{v} \neq 0} \left\{ \frac{y - \mathbf{x}'\boldsymbol{\beta}}{\mathbf{x}'\mathbf{v}} \right\},$$

$$S(F_y) = \text{MAD}(F_y),$$

we have

$$(18) \quad \begin{aligned} & \text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) \\ &= \sup_{\|\mathbf{v}\|=1} \left| \text{Med}_{\mathbf{x}'\mathbf{v} \neq 0} \left\{ \frac{y - \mathbf{x}'\boldsymbol{\beta}}{\mathbf{x}'\mathbf{v}} \right\} \right| / \text{MAD}(F_y), \end{aligned}$$

and

$$(19) \quad \begin{aligned} & \text{PRD}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) \\ &= \inf_{\|\mathbf{v}\|=1, \mathbf{x}'\mathbf{v} \neq 0} \frac{\text{MAD}(F_y)}{\text{MAD}(F_y) + |\text{Med}\{\frac{y-\mathbf{x}'\boldsymbol{\beta}}{\mathbf{x}'\mathbf{v}}\}|}. \end{aligned}$$

The empirical case of PRD above is closely related to the so-called “centrality” in Hubert, Rousseeuw and Van Aelst (2001) (HRVA01). In the definition of the latter, all the terms of “ $\text{MAD}(\cdot)$ ” on the RHS of (19) are divided by $\text{Med}|\mathbf{x}'\mathbf{v}|$.

2.3.4 *Other approaches.* Besides the three approaches above, there are certainly other approaches (including ad hoc ones). Among them, Mizera (2002) (M02) is a famous one.

In extending the idea of RD_{RH} of RH99, M02, with a decision-theoretic flavor and under the vector optimization framework (vector differential approach), introduced the notions of global, local and tangent depth rigorously. The former two are based on the so-called “critical” function. The latter (the tangent depth) is based on the vector differential approach and includes local depth as a special case. The local depth in turn includes the global depth as its special case. They are identical under certain conditions.

With mainly Euclidean norm (and/or L_1 norm) of $X - \theta$ (in location) and of $y - \mathbf{x}'\boldsymbol{\beta}$ (in regression) as *the typical critical functions*, M02 applied the notions of depth to location and (linear, nonlinear and orthogonal) regression models and obtained specific depth functions in those models *recovering mainly* both the HD of Tukey (1975) in location and RD_{RH} in linear regression under a single unified notion of depth (the tangent depth). It is not difficult to see that the critical function could be regarded as a form of unfitness measure of the underlying parameter (note that the words “unfitness,” “nonfit,” “critical function,” “objective function” and “loss function” are interrelated in some sense. Different people have different preferences.) For the linear regression model, the critical function in M02 can be summarized as follows:

$$(20) \quad \text{CF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) = c_p \|y - \mathbf{x}'\boldsymbol{\beta}\|_p,$$

where $\|\cdot\|_p$ is the absolute value or squared value w.r.t. $p = 1$ or 2, respectively, and $c_p = 1/p$. The global depth of this leads to RD_{RH} of RH99.

Based on the definitions of M02, one can introduce notions of depth in regression models with appropriate chosen critical functions. The key issue is how to construct “reasonable” or “optimal” critical functions besides the

L_1 norm and the L_2 norm approaches given in M02. With the depth functions obtained via M02 approach, one can introduce the maximum (deepest) regression depth estimating functionals via the min–max scheme.

In addition to the approaches, we have discussed so far, there are certainly other ones which introduce notions of unfitness or depth. Can all these notions really serve as depth notions in regression? Gauging or evaluating those notions naturally becomes an issue. Namely, all the unfitness or depth notions must satisfy some basic desired axiomatic properties or possess some desirable and intrinsic features and meet some criteria. What are the criteria?

In the following, we will propose and discuss four axiomatic properties that are deemed necessary for any notion of regression depth or unfitness, thereby providing a systematic basis for the selection and evaluation of a depth notion in regression.

3. AXIOMATIC PROPERTIES FOR DEPTH AND UNFITNESS

3.1 Four Axiomatic Properties

DEFINITION 3.1 (A depth notion in regression). A nonnegative functional G defined on space $\mathbb{R}^p \times \mathcal{P} \rightarrow [0, \infty)$ is called a depth functional in regression, where \mathcal{P} is the collection of distribution functions on \mathbb{R}^{p+1} , if it satisfies the following four properties:

(P1) Invariance (regression, scale, affine invariance) The functional G is regression, scale and affine invariant w.r.t. a given $F_{(y,x)}$ iff, respectively,

$$\begin{aligned} G(\boldsymbol{\beta} + \mathbf{b}; F_{(y+\mathbf{x}'\mathbf{b},\mathbf{x})}) &= G(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) \quad \forall \mathbf{b} \in \mathbb{R}^p, \\ G(s\boldsymbol{\beta}; F_{(sy,\mathbf{x})}) &= G(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) \quad \forall s (\neq 0) \in \mathbb{R}, \\ G(A^{-1}\boldsymbol{\beta}; F_{(y,A'\mathbf{x})}) &= G(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) \end{aligned}$$

\forall nonsingular p by p matrix A .

(P2) Maximality at center The functional G possesses its maximum over $\boldsymbol{\beta} \in \mathbb{R}^p$ w.r.t. a given $F_{(y,\mathbf{x})}$. That is, $\max_{\boldsymbol{\beta} \in \mathbb{R}^p} G(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ exists. Furthermore, it is attained at $\boldsymbol{\beta}^*$ if $\boldsymbol{\beta}^*$ is the center of symmetry of $F_{(y,\mathbf{x})}$ w.r.t. some notion of symmetry in regression.

(P3) Monotonicity relative to deepest point With respect to a maximum depth point $\boldsymbol{\beta}^*$ of the functional G , for any $\boldsymbol{\beta} \in \mathbb{R}^p$ and $\lambda \in [0, 1]$,

$$G(\lambda\boldsymbol{\beta}^* + (1 - \lambda)\boldsymbol{\beta}; F_{(y,\mathbf{x})}) \geq G(\boldsymbol{\beta}; F_{(y,\mathbf{x})}).$$

(P4) Vanishing at infinity The functional G is vanishing when $\|\boldsymbol{\beta}\| \rightarrow \infty$. That is, $\lim_{\|\boldsymbol{\beta}\| \rightarrow \infty} G(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) = 0$.

Note that due to the reverse relationship, if the depth notion above changes to a unfitness notion, then the above four properties need obvious changes except the **(P1)**. Maximum in **(P2)** becomes the minimum. **(P3)** changes

maximum to minimum and reverses the direction of the inequality. **(P4)** becomes $\lim_{\|\boldsymbol{\beta}\| \rightarrow \infty} \text{UF}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}) = \infty$.

The four properties above were first investigated for the simplicial depth function in Liu (1990) and formulated for general depth functions in location in ZS00. They have been adopted and extended for depth notions in other settings, especially for the functional data in Nieto-Reyes and Battey (2016) (NRB16) from the topological validity point of view, for general functional data in Gijbels and Nagy (2017) (GN17), and for the relevance of half-space depths in scatter, concentration and shape matrices in Paindaveine and Van Bever (2018).

Sophisticated discussions on the adaptations and the replacements of the four properties and the appropriateness have been given in Dyckerhoff (2004) and Serfling (2006, 2019), and in NRB16 and GN17 for functional data. Here, for the sake of consistency and simplicity, we keep focusing on the four core axiomatic properties and make some remarks below.

REMARKS 3.1.

(I) **(P1)** guarantees that the notion of depth in regression does not depend on the underlying coordinate system or measurement scale. This provides an advantage in the study of the depth induced functionals (estimators) by just dealing with an easily manageable special case (e.g., a spherically symmetric distribution) to cover a large class of cases (e.g., all elliptically symmetric distributions) without loss of generality (see, e.g., Van Aelst and Rousseeuw (2000) (VAR00)).

(II) **(P2)** says that the maximum of G always exists, and it is attained at the center of symmetry w.r.t. some notion of symmetry in regression, when there is such a center. This allows one to discuss the maximum regression depth estimating functional (or estimator in the empirical case). Note that the supremum of bounded G always exists but not necessarily for the maximum. If **(P4)** holds, one then can just focus on bounded $\boldsymbol{\beta}$, however, since G is not necessarily continuous in $\boldsymbol{\beta}$, the maximum of G is not guaranteed to exist. In the empirical distribution case, however, if there are only finitely many hyperplanes that need to be concerned for a given depth functional, then the maximum always exists.

(III) **(P3)** guarantees that $G(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ is monotonically decreasing in $\boldsymbol{\beta}$ along any ray stemming from a deepest point. This is equivalent to the quasi-convexity of the depth functional under **(P2)**, which further implies that the set of all $\boldsymbol{\beta}$ that has depth at least α (≥ 0) is convex (which will be useful when studying the depth induced contours in the parameter space of $\boldsymbol{\beta} \in \mathbb{R}^p$), and fewer ties in depth computations of $\boldsymbol{\beta}$ (in the strictly decreasing case) will be yielded.

(IV) **(P4)** dictates that when the hyperplane $H_{\boldsymbol{\beta}}$ determined by $y = \mathbf{x}'\boldsymbol{\beta}$ becomes vertical, its depth should be

vanishing. This makes sense since when the hyperplane is vertical, it can no longer serve as an estimating functional for a linear regression parameter. It is obviously no longer useful for the prediction of future responses as well. Note that $\|\boldsymbol{\beta}\| \rightarrow \infty$ could mean (i) $|\beta_1| \rightarrow \infty$ and or (ii) $\|\boldsymbol{\beta}_2\| \rightarrow \infty$. (ii) just means the hyperplane $H_{\boldsymbol{\beta}}$ turns out to be vertical. When (i) happens, the intercept of the hyperplane $H_{\boldsymbol{\beta}}$ becomes unbounded (assume that $x_1 = 1$), the hyperplane becomes useless and its depth logically should be vanishing.

3.2 Examining Depth Notions

Now that four axiomatic properties have been presented, a natural question is: do the regression depth functions induced from the four approaches in Sections 2.3.1 to 2.3.4 satisfy all the desired properties? That is, are they really notions of depth w.r.t. (P1)–(P4)? First, let us summarize the depth functionals from these sections.

The approach in Section 2.3.1 based on the classical objective functions induces a class of regression depth functionals, defined by

$$(21) \quad D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}, \phi, f) = (1 + \phi(F_R))^{-1},$$

where $R = f(r(\boldsymbol{\beta})/S(F_y))$, ϕ and f (objective function) are given in Section 2.2 or Example 2.1, and $S(\cdot)$ is a scale functional that is translation invariant and scale equivalent; dividing by it achieves scale invariance of the depth functions; it is suppressed in D_{Obj} .

Facility location approach in Section 2.3.2 induces $D_C(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ (or $D_C(\boldsymbol{\beta}; P)$) that is closely related to $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ when the distance d in (6) is the L_1 norm. We will only consider $D_C(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ with $d(x, y) = |x - y|$ as the representative for this approach.

Typical depth functionals from Section 2.3.3 (the PP approach) are $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ and $\text{PRD}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$.

Mizera's approach in Section 2.3.4 can recover Tukey HD in location and $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ in regression, and its critical function (see (20)) could be regarded an objective function. Its general version of tangent depth in linear regression (on page 1694) essentially recovers $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$. No distinct depth function in linear regression from this approach will be discussed here.

Consequently, in the sequel we will investigate (i) $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})}, \phi, f)$, (ii) $D_C(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$, (iii) $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$ and (iv) $\text{PRD}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$.

PROPOSITION 3.1. *Regression depth functional (i), (ii), (iii) and (iv) satisfy (P1).*

PROOF. See the [Appendix](#). \square

REMARKS 3.2.

(I) Without modifying the original function $A(\boldsymbol{\beta}; \mathbf{v})$ of MY93 (see (II) of [Remarks 2.4](#)), the induced depth functional (iv), $\text{PRD}(\boldsymbol{\beta}; F_{(y,\mathbf{x})})$, can never satisfy (P1).

(II) As a by-product of (P1), maximum regression depth functionals induced from regression depth notions in [Proposition 3.1](#) are *equivariant* as declared in [Corollary 3.1](#) below.

(III) Note that in (16), $T_{P1}(F_{(sy,\mathbf{x})}) = s^2 T_{P1}(F_{(y,\mathbf{x})})$. That is, by definition below, T_{P1} is *not scale equivariant*, contrary to the popular belief in the literature.

COROLLARY 3.1. *The maximum regression depth functionals $\boldsymbol{\beta}^*(F_{(y,\mathbf{x})})$ induced from (i), (ii), (iii) and (iv) are regression, scale and affine equivariant. That is, respectively,*

$$\boldsymbol{\beta}^*(F_{(y+\mathbf{x}'\mathbf{b},\mathbf{x})}) = \boldsymbol{\beta}^*(F_{(y,\mathbf{x})}) + \mathbf{b} \quad \forall \mathbf{b} \in \mathbb{R}^p;$$

$$\boldsymbol{\beta}^*(F_{(sy,\mathbf{x})}) = s\boldsymbol{\beta}^*(F_{(y,\mathbf{x})}) \quad \forall \text{ scalar } s (\neq 0) \in \mathbb{R};$$

$$\boldsymbol{\beta}^*(F_{(y,A'\mathbf{x})}) = A^{-1}\boldsymbol{\beta}^*(F_{(y,\mathbf{x})})$$

$$\forall \text{ nonsingular } A \in \mathbb{R}^{p \times p}.$$

PROOF. It is trivial. \square

If a maximum regression depth estimating functional $\boldsymbol{\beta}^*(F_{(y,\mathbf{x})})$ is equivariant, then it is symmetric w.r.t. (y, \mathbf{x}) in the sense that $\boldsymbol{\beta}^*(F_{(y,\mathbf{x})}) = \boldsymbol{\beta}^*(F_{(-y,-\mathbf{x})})$. By virtue of the corollary, one can assume (w.l.o.g.) that $\boldsymbol{\beta}^*(F_{(y,\mathbf{x})})$ equals $\mathbf{0}$.

For the joint distribution $F_{(y,\mathbf{x})}$ and the univariate location estimating functional T given in [Example 2.3](#), $F_{(y,\mathbf{x})}$ is said to be *T-symmetric* about a $\boldsymbol{\beta}_0$ iff for any $\mathbf{v} \in \mathbb{S}^{p-1}$

$$(22) \quad (\mathbf{C0}): \quad T(F_{(y-\mathbf{x}'\boldsymbol{\beta}_0, \mathbf{x}'\mathbf{v})}) = 0.$$

REMARKS 3.3.

(I) *T-symmetric* $F_{(y,\mathbf{x})}$ includes a wide range of distributions. For example, if the univariate functional T is the mean functional, then this becomes the classical assumption in regression when $\boldsymbol{\beta}_0$ is the true parameter of the model: the conditional expectation of the error term e (that is assumed to be independent of \mathbf{x}) given \mathbf{x} is zero, that is,

$$(\mathbf{C1}): \quad T(F_{(y-\mathbf{x}'\boldsymbol{\beta}_0, \mathbf{x}'\mathbf{v})}) = E(F_{(y-\mathbf{x}'\boldsymbol{\beta}_0, \mathbf{x}'\mathbf{v})} | \mathbf{x}=\mathbf{x}_0) \\ = 0 \quad \forall \mathbf{x}_0 \in \mathbb{R}^p,$$

(II) When T is the second most popular choice, the quantile functional, especially the median (Med) functional, the *T-symmetric* of $F_{(y,\mathbf{x})}$ about $\boldsymbol{\beta}_0$ is closely related to a weaker version (when $\mathbf{v} = (1, 0, \dots, 0)$) of the so-called *regression symmetry* in RS04. Or precisely,

$$(\mathbf{C2}): \quad T(F_{(y-\mathbf{x}'\boldsymbol{\beta}_0, \mathbf{x}'\mathbf{v})}) = \text{Med}(F_{y-\mathbf{x}'\boldsymbol{\beta}_0} | \mathbf{x}=\mathbf{x}_0) \\ = 0 \quad \forall \mathbf{x}_0 \in \mathbb{R}^p.$$

For a thorough discussion of this type of symmetry, refer to RS04.

In the following, for $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f)$, we consider only the combinations $\phi = \boldsymbol{\mu}$, (a) $f(x) = x^2$ (the case (I) of Example 2.1) and (b) $f(x) = |x|$ (the case (II) of Example 2.1).

PROPOSITION 3.2. *Regression depth function (i), (ii), (iii) and (iv) satisfy (P2) in the following sense:*

(a) *The maximum of regression depth (i) (i.e., $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f)$) exists and is attained at $\boldsymbol{\beta}_0 \in \mathbb{R}^p$ if $\phi = \boldsymbol{\mu}$, $f(x) = x^2$ and (C1) holds or if $\phi = \boldsymbol{\mu}$, $f(x) = |x|$ and (C2) holds.*

(b) *The maximum of regression depth (ii) (i.e., $D_C(\boldsymbol{\beta}; F_{(y,x)})$) exists if (A) holds and is attained at a bounded $\boldsymbol{\beta}_0 \in \mathbb{R}^p$.*

(c) *The maximum of regression depth (iii) (i.e., $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; F_{(y,x)})$) exists if (A) holds and is attained at $\boldsymbol{\beta}_0 \in \mathbb{R}^p$ if (C2) holds.*

(d) *The maximum of regression depth (iii) (i.e., $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)}, T)$) exists and is attained at $\boldsymbol{\beta}_0 \in \mathbb{R}^p$ if (C0) holds.*

PROOF. See the Appendix. \square

REMARKS 3.4.

(I) Part (a) of the proposition could be extended to cover more cases. If functional ϕ has the ‘‘monotonicity’’ property ($\phi(F_{R1}) \leq \phi(F_{R2})$ if $R1 \leq R2$) and $f(x)$ has the unique minimum value, then existence is guaranteed. When ϕ is the expectation or quantile functional, then it has monotonicity, and if $f(x)$ is even, monotonic in $|x|$ and convex, then $f(x)$ has a unique minimum value. This covers a large class of combinations of ϕ and f .

(II) Existence of maximum for D_C and RD_{RH} in the proposition is established under (A). The latter *sufficient condition* excludes the discrete distributions. In the empirical case, existence always holds true for both, nevertheless.

PROPOSITION 3.3. *Regression depth function (i), (iii) and (iv) satisfy (P3) in the following sense.*

(a) *The regression depth (i) (i.e., $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f)$) monotonically decreases along any ray stemming from a deepest point if ϕ has the monotonicity property (i.e., $\phi(F_{R1}) \leq \phi(F_{R2})$ if $R1 \leq R2$), and f is quasi-convex and has a unique minimum.*

(b) *The regression depth (iii) (i.e., $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$) monotonically decreases along any ray stemming from a deepest point if (A) holds.*

(c) *The regression depth (iv) (i.e., $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)})$) monotonically decreases along any ray stemming from a deepest point.*

(d) *The regression depth (ii) (i.e., $D_C(\boldsymbol{\beta}; P)$) violates (P3) generally.*

PROOF. See the Appendix. \square

REMARKS 3.5.

(I) When ϕ is the expectation or quantile functional in (a) of the Proposition, then it has the monotonicity property, and when f is x^2 or $|x|$ or even the check function in (III) of Example 2.1, then it again meets all the requirements in (a) of the proposition.

(II) For RD_{RH} to meet (P3) (or (P2)), we have to ask for (A) to hold. (P3) always holds for $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)})$ with T in Example 2.3.

PROPOSITION 3.4. *Regression depth functional (i), (ii), (iii) and (iv) satisfy (P4) in the following sense.*

(a) *The regression depth (i): $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f) \rightarrow 0$ when $\|\boldsymbol{\beta}_1\| \rightarrow \infty$ and $\|\boldsymbol{\beta}_2\| < \infty$ if $\phi(F_R) \rightarrow \infty$ as $|R| \rightarrow \infty$ and $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.*

(b) *The regression depth (ii): $D_C(\boldsymbol{\beta}; F_{(y,x)}) \rightarrow 0$ when $\|\boldsymbol{\beta}\| \rightarrow \infty$ if (A) holds.*

(c) *The regression depth (iii): $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) \rightarrow 0$ when $\|\boldsymbol{\beta}\| \rightarrow \infty$ if (A) holds.*

(d) *The regression depth (iv): $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)}, T) \rightarrow 0$ as $\|\boldsymbol{\beta}\| \rightarrow \infty$ for T in Example 2.3.*

PROOF. See the Appendix. \square

REMARKS 3.6.

(I) (a) is established under some assumptions on ϕ and $\boldsymbol{\beta}$. If ϕ is the expectation or quantile functional and $f(x)$ is even, monotonic in $|x|$ and convex, then they satisfy the assumptions. (a) only treats one case of $\|\boldsymbol{\beta}\| \rightarrow \infty$. This is, the intercept becomes unbounded while $\|\boldsymbol{\beta}_2\| < \infty$ (as argued in (IV) of Remarks 3.1, in this case, the depth function ought to vanish). The other case of $\|\boldsymbol{\beta}\| \rightarrow \infty$ remains untouched.

(II) (b) and (c) are established under the assumption (A). (d) holds for $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)}, T)$ with T in Example 2.3 without any extra assumption. This T could be the median or quantile functional or the weighted mean functional in Wu and Zuo (2009) (WZ09).

4. CONCLUDING REMARKS

This article extends four axiomatic properties (evaluation criteria) for location depth notions in ZS00 to depth notions in regression and discusses four general approaches for introducing notions of depth or unfitness in regression. The latter leads to four representative depth notions: (i) $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f)$, (ii) $D_C(\boldsymbol{\beta}; P)$, (iii) $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ and (iv) $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)})$.

It characterizes (ii) and reveals that this depth notion in regression is not identical yet closely related to the RD_{RH} of RH99. The latter is contrary to a claim in the literature.

It further investigates the leading regression depth notions (i), (ii), (iii) and (iv) w.r.t. the evaluation criteria and shows that (a) $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f)$ satisfy all the four properties under some conditions on ϕ and f , with (P4)

proved under just one special case of $\|\beta\| \rightarrow \infty$; (b) under **(A)**, D_C satisfy all (but **P3**) properties; (c) $\text{RD}_{\text{RH}}(\beta; P)$ satisfy all the four axiomatic properties if **(A)** holds; (d) $\text{PRD}(\beta; F_{(y,x)})$ satisfy all four properties.

Therefore, all but Carrizosa depth (ii) are real regression depth notions w.r.t. the four properties under those assumptions. Moreover, depth functions induced from PRD are representative extensions of eminent projection depth in location to regression.

As by-product of this article, two new characterizations of RD_{RH} are obtained. One is in Proposition 2.2 and the other in Example 2.2. The latter one turns out to be extremely helpful in studying the asymptotics of the deepest regression estimator $\beta_{\text{RD}_{\text{RH}}}^*$ (see Zuo, 2019a).

One of the primary advantages of the notions of depth is that it can be employed directly to define median-type deepest (or maximum depth) estimating functionals (estimators in the empirical distribution case) for parameters in regression or location models. The most outstanding feature of the univariate median is its exceptional robustness. Do the deepest regression estimating functionals induced from real regression depth notions here inherit this robustness property? Answers to this for most cases of (i) and for (iii) have been given in the literature (e.g., VAR00). Encouraging answers to (iv) have been established in Zuo (2018).

Besides **(P1)–(P4)**, in evaluating and comparing the overall performance of various regression depth notions, one certainly has to further take into account the robustness and efficiency of their induced maximum depth estimators and their computability. Taking all these factors into consideration, preliminary results (see Zuo, 2019b) indicate that projection regression depth, just as its location counterpart, is competitive among leading competitors.

APPENDIX

PROOF OF PROPOSITION 2.2. *Proof of part (ii).* Assume that $\|\beta\| < \infty$, we need to show that

$$(23) \quad \text{RD}_{\text{RH}}(\beta; P) = \inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|).$$

Denote the angle between the hyperplane H_β (determined by $y = \mathbf{w}'\beta$) and the horizontal hyperplane plane H_h (determined by $y = 0$) by θ_β (consider the acute one only, hereafter). That is, θ_β is the angle between the normal vector $(-\beta_2', 1)'$ and the normal vector $(\mathbf{0}', 1)'$ in the $(\mathbf{x}', y)'$ -space. Therefore, it is easy to see that $|\tan(\theta_\beta)| = \|\beta_2\|$. For any $\alpha = (\alpha_1, \alpha_2') \in S(\beta)$ ($\|\alpha\| < \infty$) define similarly (hereafter) H_α and θ_α .

First, we show that the LHS of (23) is no less than its RHS. Tilting β to a vertical position in Definition 2.2 means tilting H_β along a hyperline $l_v(\beta)$ which is the intersection line of H_β with some vertical hyperplane H_v .

Let $P(l_v(\beta))$ be the minimum probability mass touched by tilting H_β in the definition of RD_{RH} to a vertical position along $l_v(\beta)$ in two ways. Then it is readily seen that

$$(24) \quad \text{RD}_{\text{RH}}(\beta; P) = \inf_{l_v(\beta)} P(l_v(\beta)).$$

Let H_γ be the hyperplane with $\theta_\gamma = \arctan((\|\alpha_2\| + \|\beta_2\|)/2)$ which contains the hyperline $l_v(\beta)$. Then it is seen that H_γ is in-between H_β and H_α (consider again the situation that the angle formed between H_β and H_α is acute, w.l.o.g.). Furthermore, points on H_γ have the same vertical distances to H_β and H_α . That is, H_γ bisects the double wedge formed by H_β and H_α (i.e., it bisects the vertical distance between the two hyperplanes).

Now it is not difficult to see that $P(|r(\beta)| \leq |r(\alpha)|)$ equals the probability mass touched by tilting H_γ (toward H_β initially) along the hyperline $l_v(\beta)$ to the vertical position. In order to reach the infimum over $S(\beta)$, we need to seek α 's such that the probability mass above becomes smaller.

Consider $\alpha_m \in S(\beta)$ that approach β (or let $\theta_{\alpha_m} \rightarrow \theta_\beta$) while H_{α_m} and H_β still intercept at $l_v(\beta)$ (that is, tilting H_α toward H_β along $l_v(\beta)$ yields α_m). As $m \rightarrow \infty$, the probability mass contained in the interior of the double wedge formed between H_β and H_{α_m} approaches zero and $P(|r(\beta)| \leq |r(\alpha_m)|)$ decreases to the probability mass touched by tilting H_β to the vertical position along the hyperline $l_v(\beta)$ in one of two ways (the other way is described below).

Consider $\alpha_n \in S(\beta)$ that approach β (or let $\theta_{\alpha_n} \rightarrow \theta_\beta$) with H_{α_n} being on the *other side* of H_β and still intercept at $l_v(\beta)$ (i.e. if previously $\theta_{\alpha_m} < \theta_\beta$, then $\theta_{\alpha_n} \geq \theta_\beta$ now, vice versa). Using the same hyperline $l_v(\beta)$ above, one can conclude similarly that $P(|r(\beta)| \leq |r(\alpha_n)|)$ decreases to the probability mass touched by tilting H_β to the vertical position along the hyperline $l_v(\beta)$ in the other way, as $n \rightarrow \infty$.

The above results imply that $\inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|) \leq P(l_v(\beta))$. The arbitrariness of $l_v(\beta)$ (which can be any hyperline that is the intersection line of H_β and any vertical hyperplane H_v), in conjunction with (24) implies that $\text{RD}_{\text{RH}}(\beta; P) \geq \inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|)$.

Now we show that the LHS of (23) is no greater than its RHS. For a given $\alpha \in S(\beta)$, H_β intersects H_α at a hyperline, say $l(\beta, \alpha)$. Replace $l_v(\beta)$ with this line in the above proof, it is readily seen that for the given $\beta, \alpha \in S(\beta)$, and $l(\beta, \alpha)$, $P(|r(\beta)| \leq |r(\alpha)|)$ equals the probability mass touched by tilting H_γ (toward H_β initially) along the hyperline $l(\beta, \alpha)$ to the vertical position, which implies that $P(|r(\beta)| \leq |r(\alpha)|) \geq P(l(\beta, \alpha))$, where $P(l(\beta, \alpha))$ is again the minimum probability mass touched by tilting H_β along the hyperline $l(\beta, \alpha)$ to the vertical position in two ways in the Definition of 2.2. Hence, the $\text{RD}_{\text{RH}}(\beta; P) \leq \inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|)$ in light of (24). This completes the proof of (ii).

Proof of part (i). Consider only the α that does not belong to $S(\beta)$. Hence, H_α is parallel to H_β . Let H_γ be the hyperplane in the middle of the hyperstripe with H_α and H_β as its two boundaries (i.e., $\theta_\gamma = \arctan((\|\alpha\| + \|\beta_2\|)/2)$). Then it is readily seen that $P(|r(\beta)| \leq |r(\alpha)|)$ equals to the probability mass carried by the closed half of the hyperstripe with H_γ and H_β as its two boundaries.

Consider $\alpha_n \notin S(\beta)$ that approach β (or let $\alpha_n \rightarrow \beta$), it is readily seen that the probability mass contained in the interior of the half-hyperstripe formed between H_β and H_{γ_n} approaches zero, and $P(|r(\beta)| \leq |r(\alpha_n)|)$ decreases to $P(H_\beta) = P(r(\beta) = 0)$ as $n \rightarrow \infty$. Similar to the proof of part (ii) above, it is readily shown that $\inf_{\alpha \notin S(\beta)} P(|r(\beta)| \leq |r(\alpha)|) = P(r(\beta) = 0)$. On the other hand, by the proof of part (ii) above, it is readily seen that $\inf_{\alpha \in S(\beta)} P(|r(\beta)| \leq |r(\alpha)|) \geq P(r(\beta) = 0)$. This completes the proof of (i). \square

PROOF OF PROPOSITION 2.3. (i) For any given $\beta = (\beta_1, \beta_2)'$, let the angle between the hyperplane H_β (determined by $y = \mathbf{w}'\beta$) and the horizontal hyperplane H_h (determined by $y = 0$) be θ . That is, θ is the angle between the normal vector $(-\beta_2', 1)'$ and the normal vector $(0', 1)'$. Therefore, it is easy to see that $|\tan(\theta)| = \|\beta_2\|$. When $\|\beta\| = (\|\beta_1\|^2 + \|\beta_2\|^2)^{1/2} \rightarrow \infty$, assume w.o.l.g. that $|\beta_1| < \infty$ (otherwise $D_C(\beta; P) \rightarrow 0$ by definition (6)), then $|\tan(\theta)| \rightarrow \infty$, H_β turns to be vertical, which further implies by Proposition 2.2 that $D_C(\beta; P) \rightarrow 0$ since the closed double wedge formed by H_β and its eventual vertical hyperplane H_v becomes smaller and smaller (in Lebesgue measure sense), and H_β approaches its eventual vertical hyperplane H_v .

(ii) Part (i) implies that when $\|\beta\|$ becomes unbounded, the RHS of (10) cannot reach its maximum value at such β . For a fixed α , $f(\beta; \alpha) = P(|r(\beta)| \leq |r(\alpha)|)$ is upper semicontinuous in β , hence the infimum of upper semicontinuous functions $D_C(\beta; P)$ is also upper semicontinuous. The upper semicontinuity of $D_C(\beta; P)$ in β over a bounded set, in conjunction with the extreme value theorem, yields (ii). \square

PROOF OF THE STATEMENT IN EXAMPLE 2.3. Let $\mathbf{v} = (-v_1, \mathbf{v}_2) \in \mathbb{R}^p$, $v_1 \in \mathbb{R}$, $\mathbf{v}_2 \in \mathbb{R}^{p-1}$, and $\|\mathbf{v}_2\| = 1$; $r(\beta) = y - \mathbf{w}'\beta$ and $g(\beta, \mathbf{v}) = r(\beta) * ((\mathbf{v}_2)' \mathbf{x} - v_1) = r(\beta) \mathbf{w}' \mathbf{v}$. Here, we wanted to show that

$$(25) \quad \text{RD}_{\text{RH}}(\beta, P) = \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} E(\mathbf{I}(g(\beta, \mathbf{v}) \geq 0)).$$

That is, the RHS above is equivalent to (12).

(i) Let us just focus on $\mathbf{w}' \mathbf{v} \geq 0$ (the case $\mathbf{w}' \mathbf{v} \leq 0$ can be treated similarly). Since $\mathbf{w}' \mathbf{v} \geq 0$ is equivalent to $\mathbf{x}'(\mathbf{v}_2) - v_1 \geq 0$, the latter represents a closed half-plane $H_{\mathbf{x}}(v_1, \mathbf{v}_2)$ in the \mathbf{x} -hyperplane (horizontal hyperplane $y = 0$ in the $(\mathbf{x}', y)'$ space). From the $(\mathbf{x}', y)'$ space

point of view, it represents a closed vertical half-space D in $(\mathbf{x}', y)'$ space. The intersection of this D with \mathbf{x} -hyperplane (or the vertical projection of D onto \mathbf{x} -hyperplane) results in $H_{\mathbf{x}}(v_1, \mathbf{v}_2)$.

(ii) On the other hand, given a closed vertical half-space D in $(\mathbf{x}', y)'$ space, it intercepts with the $y = 0$ hyperplane (or the \mathbf{x} -hyperplane) at a closed half-plane $H_{\mathbf{x}}$ in \mathbf{x} -hyperplane with its boundary a hyperline $l_{\mathbf{x}}$ in the \mathbf{x} -hyperplane. Call the direction in the \mathbf{x} -hyperplane that is perpendicular to the hyperline $l_{\mathbf{x}}$ and pointing into the half-plane $H_{\mathbf{x}}$ as \mathbf{v}_2 . Denote the distance from the origin to the point on \mathbf{v}_2 and $l_{\mathbf{x}}$ as v_1 ; it then follows that D is equivalent to $\mathbf{x}' \mathbf{v}_2 \geq v_1$ in the $(\mathbf{x}', y)'$ space. That is, $\mathbf{w}' \mathbf{v} \geq 0$.

It is readily seen from (i) and (ii) above that the RHS of (25) is equivalent to (12). Also, it is straightforward to see that it is equivalent to (28) under the assumptions there. Furthermore, it can be shown that

$$(26) \quad \begin{aligned} \text{RD}_{\text{RH}}(\beta, P) &= \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} E(\mathbf{I}(g(\beta, \mathbf{v}) \geq 0)) \\ &= \inf_{\mathbf{v} \in \mathbb{S}^{p-1}} E(\mathbf{I}(r(\beta) * (\mathbf{v}' \mathbf{w}) \geq 0)). \end{aligned}$$

Incidentally, it is seen that

$$\begin{aligned} &\inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} E(\mathbf{I}(g(\beta, \mathbf{v}) \geq 0)) \\ &= \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} E(\mathbf{I}(r(\beta) * (\mathbf{v}' \mathbf{w}) \geq 0)) \\ &= \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} \min\{E(\mathbf{I}(r(\beta) * (\mathbf{v}' \mathbf{w}) \geq 0)), \\ &\quad E(\mathbf{I}(r(\beta) * (-\mathbf{v}' \mathbf{w}) \geq 0))\} \\ &= \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} \min\{E(\mathbf{I}(r(\beta) * (\mathbf{v}' \mathbf{w}) \geq 0)), \\ &\quad E(\mathbf{I}(r(\beta) * (\mathbf{v}' \mathbf{w}) \leq 0))\} \\ &= \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} \min\{E(\mathbf{I}(g(\beta, \mathbf{v}) \geq 0)), \\ &\quad E(\mathbf{I}(g(\beta, \mathbf{v}) \leq 0))\}. \end{aligned}$$

Note that the RHS of the last equality is the quantity used for the empirical regression depth calculation in RH99 (up to a constant factor n). \square

PROOF OF PROPOSITION 3.1. (i) For the $D_{\text{Obj}}(\beta; F_{(y, \mathbf{x})}, \phi, f)$ in Section 2.3.1, notice the facts that

$$\begin{aligned} (y + \mathbf{x}' \mathbf{b}) - \mathbf{x}'(\beta + \mathbf{b}) &= y - \mathbf{x}' \beta, \\ s * y - \mathbf{x}'(s * \beta) &= s * (y - \mathbf{x}' \beta), \quad s \neq 0, \\ y - \mathbf{x}' A(A^{-1} \beta) &= y - \mathbf{x}' \beta. \end{aligned}$$

These, in conjunction with the scale equivalence of S , yield the invariance of $R = f(r(\beta)/S(F_y))$ and of the depth function. **(P1)** follows immediately for the $D_{\text{Obj}}(\beta; F_{(y, \mathbf{x})}, \phi, f)$ in (21).

(ii) By (ii) of Proposition 2.3, $D_C(\boldsymbol{\beta}; P) = P(y - \mathbf{w}'\boldsymbol{\beta} = 0)$, replacing the \mathbf{x} in (i) above verification by \mathbf{w} , it is readily seen the **(P1)** follows immediately for $D_C(\boldsymbol{\beta}; P)$.

(iii) For $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$, in the empirical case, the fact that the latter satisfies it has already been declared in Section 2.1 of RH99. For the general population case, note that a characterization of $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ is (see the proof above)

$$\begin{aligned} \text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) &= \inf_{\|\mathbf{v}_2\|=1, v_1 \in \mathbb{R}} E(I(r(\boldsymbol{\beta}) * (v_1, \mathbf{v}'_2)\mathbf{w} \geq 0)) \\ (27) \quad &= \inf_{\mathbf{v} \in \mathbb{S}^{p-1}} E(I(r(\boldsymbol{\beta}) * \mathbf{v}'\mathbf{w} \geq 0)). \end{aligned}$$

Similar to the proof in (i), **(P1)** follows immediately for $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$.

(iv) For $\text{PRD}(\boldsymbol{\beta}; F_{(y, \mathbf{x})})$ in (17). **(P1)** follows straightforwardly from (13), (14) and (17), coupled with **(A1)** and **(A4)**. \square

REMARKS A.1.

(I) (27) is one of representations of the RD_{RH} . Many other characterizations exist. For example, one is given in RS04 displayed in (12) and another one given in VAR00 is

$$\begin{aligned} \text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) &= \inf_{\mathbf{u} \in \mathbb{R}^{p-1}, v \in \mathbb{R}} \{P(r(\boldsymbol{\beta}) > 0 \cap \mathbf{x}'\mathbf{u} < v) \\ (28) \quad &+ P(r(\boldsymbol{\beta}) < 0 \cap \mathbf{x}'\mathbf{u} > v)\}. \end{aligned}$$

They assumed that $P(\mathbf{x}'\mathbf{u} = v) = 0$ (and implicitly assumed that **(A0)**: $P(r(\boldsymbol{\beta}) = 0) = 0$).

(II) Another representation of the RD_{RH} given in AMY02 is displayed in (7), which is slightly more general than (28) but again also implicitly made the assumptions above. The latter implies that these representations are valid only for regression lines or hyperplanes that do not contain any probability mass. The empirical version of (7) was also given on page 158 of Maronna, Martin and Yohai (2006) (MMY06).

(III) Empirical versions of the regression depth of RH99 and its relationship to the location (half-space) depth were also extensively investigated in Mizera (2002) (pages 1689–1690).

(IV) Another empirical version (which actually is slightly different from RD_{RH}) was given in Bai and He (1999):

$$\begin{aligned} \text{RD}_{\text{RH}}(\boldsymbol{\beta}, \mathbf{Z}_n) \\ &= \inf_{\|\mathbf{u}\|=1, v \in \mathbb{R}} \min \left\{ \sum_{i=1}^n I(r_i(\boldsymbol{\beta})(\mathbf{u}'\mathbf{x}_i - v) > 0), \right. \\ &\quad \left. \sum_{i=1}^n I(r_i(\boldsymbol{\beta})(\mathbf{u}'\mathbf{x}_i - v) < 0) \right\}, \end{aligned}$$

where $y_i = \beta_0 + \mathbf{x}'_i\boldsymbol{\beta}_1 + e_i$, $\boldsymbol{\beta}' = (\beta_0, \boldsymbol{\beta}'_1) \in \mathbb{R}^p$, $\mathbf{x}_i \in \mathbb{R}^{p-1}$, $r_i(\boldsymbol{\beta}) = y_i - (1, \mathbf{x}'_i)\boldsymbol{\beta}$, and $\mathbf{Z}_n = \{(\mathbf{x}_i, y_i), i = 1, \dots, n\}$. They again implicitly assumed that **(A0)** hold and $P(\mathbf{x}'\mathbf{u} = v) = 0$.

PROOF OF PROPOSITION 3.2. (a) In light of (21), the existence of maximizer of D_{Obj} is equivalent to the existence of the minimizer of $\phi(F_R)$, where $R = f(r(\boldsymbol{\beta})/S(F_y))$. The latter holds true by virtue of the property (“monotonicity”) of the given functional ϕ (i.e. $\phi(F_{R1}) \leq \phi(F_{R2})$ if $R1 \leq R2$) and the unique minimizer 0 of the given f with the minimum value 0. Under **(C1)** or **(C2)**, it is readily seen that the maximizer is $\boldsymbol{\beta}_0$, in the respective cases.

(b) For $D_c(\boldsymbol{\beta}; P)$, **P2** follows directly from Proposition 2.3.

(c) Following the proof of Proposition 2.3, it can show similarly that under the given condition $P(H_v) = 0$, $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) \rightarrow 0$ when $\|\boldsymbol{\beta}\| \rightarrow \infty$ for $\boldsymbol{\beta} = (\beta_1, \boldsymbol{\beta}'_2)'$ with bounded β_1 . For $|\beta_1| \rightarrow \infty$ case, we have to adopt the slightly modified definition for RD_{RH} , as done in (6) for D_C . That is, $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) \rightarrow 0$ when $|\beta_1| \rightarrow \infty$. Following the arguments given in (ii) of Proposition 2.3, we see that the maximum of $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ exists and is attained at a bounded $\boldsymbol{\beta}^*$ (note that $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ is upper semi-continuous). Now, if **(C2)** holds, that is, $F_{(y, \mathbf{x})}$ is regression symmetric about the $\boldsymbol{\beta}_0$, then by Theorem 3 of RS04

$$\text{RD}_{\text{RH}}(\boldsymbol{\beta}_0) = \frac{1}{2} + \frac{1}{2}P(y - \mathbf{w}'\boldsymbol{\beta}_0 = 0),$$

which is the maximum possible depth value for all $\boldsymbol{\beta} \in \mathbb{R}^p$ in this case.

(d) We have to show that (a) the depth value cannot be maximized when the norm of $\boldsymbol{\beta} \in \mathbb{R}^p$ becomes unbounded and (b) within the set of bounded $\boldsymbol{\beta} \in \mathbb{R}^p$, there exists a $\boldsymbol{\beta}_0$ which can attain the maximum depth value. For (a), by Lemma A.1 (given below), one immediately sees that $\text{PRD}(\boldsymbol{\beta}) \rightarrow 0$ as $\|\boldsymbol{\beta}\| \rightarrow \infty$. For (b), first, the continuity of $\text{PRD}(F_{(y, \mathbf{x})}; \boldsymbol{\beta})$ in $\boldsymbol{\beta}$ follows directly from the **(A3)** and the property of supremum; second, by the extreme value theorem, the existence of a bounded maximizer $\boldsymbol{\beta}_0$ is guaranteed. When $F_{(y, \mathbf{x})}$ is T -symmetric about $\boldsymbol{\beta}_0 \in \mathbb{R}^p$, (iii) of Lemma A.1 yields the desired result. \square

A function f from $\mathbb{R}^d \rightarrow \mathbb{R}$ is *quasi-concave* if $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$, $\forall \lambda \in [0, 1]$ and x, y in \mathbb{R}^d ($d \geq 1$). For the distribution F_X of any random vector X , denote its empirical version by F_X^n .

LEMMA A.1. *The projection regression depth $\text{PRD}(\boldsymbol{\beta}; F_{(y, \mathbf{x})})$ in (17) is*

- (i) *affine invariant, quasi-concave and continuous in $\boldsymbol{\beta}$,*
- (ii) *vanishing when $\|\boldsymbol{\beta}\| \rightarrow \infty$,*
- (iii) *maximized at the center $\boldsymbol{\beta}_0$ of T -symmetric $F_{(y, \mathbf{x})}$,*

(iv) *continuous in $F_{(y,x)}$ in the sense that $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)}^n) \xrightarrow{m} \text{PRD}(\boldsymbol{\beta}; F_{(y,x)})$ in the same mode as $F_{(y,x)}^n \xrightarrow{m} F_{(y,x)}$ when $n \rightarrow \infty$, provided that (for a given $\boldsymbol{\beta}$)*

$$\begin{aligned} & \text{(a) } T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}}^n) \xrightarrow{m} T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}}) \text{ uniformly} \\ & \text{in } \mathbf{v} \in \mathbb{S}^{p-1} \text{ and } S(F_y^n) \xrightarrow{m} S(F_y), \\ & \text{(b) } \sup_{\|\mathbf{v}\|=1} |T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}})| < M_T, \quad \text{and} \\ & \inf_n S(F_y^n) > M_S > 0, \end{aligned}$$

where convergence mode “ m ” could be in $o_P(1)$, $o(1)$ a.s., or in $O_P(n^{-1/2})$.

PROOF OF LEMMA A.1. (i) This is a straightforward verification, by (A1), (A3), and (13), (14) and (17).

(ii) In light of (A1), it is readily seen that $T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}}) = T(F_{(y/x'\mathbf{v})}) - \|\boldsymbol{\beta}\|$ for $\boldsymbol{\beta} \neq 0$ and $\mathbf{v} = \boldsymbol{\beta}/\|\boldsymbol{\beta}\|$. This, in conjunction with (A2), (13), (14) and (17), yields $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)}) \rightarrow 0$ as $\|\boldsymbol{\beta}\| \rightarrow \infty$.

(iii) In virtue of the definition of T-symmetric about $\boldsymbol{\beta}_0$ in (22), (13), (14) and (17), one sees that $\text{PRD}(\boldsymbol{\beta}_0; F_{(y,x)})$ attains its maximum possible value 1.

(iv) Write G for PRD, and in light of (17), (14) and (13), a simple derivation leads to

$$\begin{aligned} & |G(\boldsymbol{\beta}; F_{(y,x)}^n) - G(\boldsymbol{\beta}; F_{(y,x)})| \\ & \leq \sup_{\|\mathbf{v}\|=1} |\text{UF}_{\mathbf{v}}(\boldsymbol{\beta}; F_{(y,x)}^n, T) - \text{UF}_{\mathbf{v}}(\boldsymbol{\beta}; F_{(y,x)}, T)| \\ & \leq \sup_{\|\mathbf{v}\|=1} \frac{D(T_n)S(F_y) + D(S_n)|T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}})|}{S(F_y^n)S(F_y)} \\ & \leq \frac{1}{M_S} \sup_{\|\mathbf{v}\|=1} D(T_n) + \frac{M_T}{M_S S(F_y)} D(S_n) \end{aligned}$$

by the given (b), where $D(T_n) = |T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}}^n) - T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}})|$ and $D(S_n) = |S(F_y^n) - S(F_y)|$. This, in conjunction with the given (a), leads immediately to (iv). \square

REMARKS A.2.

(I) The assumption (a) in (iv) of the lemma holds for classical location estimating functionals T such as Med functional or trimmed and winsorized mean functionals of WZ09 and for S such as MAD or trimmed and winsorized standard deviations functionals (Wu and Zuo, 2008). Uniformity in $\mathbf{v} \in \mathbb{S}^{p-1}$ can usually be established via the II.4. and II.5. of Pollard (1984).

(II) The assumption (b) in (iv) of the lemma holds true for T and S above, as long as $S(F_y) > \delta > 0$ and $|T(F_{(y-x'\boldsymbol{\beta})/x'\mathbf{v}})| < M_{\mathbf{v}} < \infty$ for any $\mathbf{v} \in \mathbb{S}^{p-1}$.

PROOF OF PROPOSITION 3.3. It is readily seen that to prove (P3), it suffices to show that (1) there exists a deepest point $\boldsymbol{\beta}_0$ of $G(\boldsymbol{\beta}; P)$, and (2) the regression depth function $G(\boldsymbol{\beta}; P)$ is quasi-concave in $\boldsymbol{\beta}$.

(a) For regression depth (i) (i.e., $D_{\text{Obj}}(\boldsymbol{\beta}; F_{(y,x)}, \phi, f)$), (1) follows from the given condition and (I) of Remarks 3.4. For (2), in light of (21), we only need to show that $\phi(F_R)$ is quasi-convex in $\boldsymbol{\beta}$ with $R = f(r(y - \mathbf{x}'\boldsymbol{\beta})/S(F_y))$. The latter follows from the quasi-convexity of f and the “monotonicity” of ϕ .

(b) For regression depth (iii) (i.e., RD_{RH}), (1) follows from (ii) of Proposition 3.2 and (2) follows directly from (iii) of Lemma A.2 given below.

(c) For the projection regression depth functional, (1) follows from the (i) and (ii) of Lemma A.1. For (2), in virtue of definition (17), it is seen that it suffices to show that $\text{UF}(F_{(y,x)}; \boldsymbol{\beta}, T)$ is quasi-convex in $\boldsymbol{\beta}$. By (A3), coupled with (13) and (14), it is readily seen that for any $\boldsymbol{\beta} = \lambda\boldsymbol{\beta}_1 + (1 - \lambda)\boldsymbol{\beta}_2$, $\lambda \in [0, 1]$

$$\begin{aligned} & \text{UF}(\boldsymbol{\beta}; F_{(y,x)}, T) \\ & \leq \max\{\text{UF}(\boldsymbol{\beta}_1; F_{(y,x)}, T); \text{UF}(\boldsymbol{\beta}_2; F_{(y,x)}, T)\}. \end{aligned}$$

This completes the proof of (P3) for $\text{PRD}(\boldsymbol{\beta}; F_{(y,x)}, T)$.

(d) Let $\lambda \in (0, 1)$ is fixed. For given $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ ($\neq \boldsymbol{\beta}_1$) in \mathbb{R}^p , assume that $P(r(\boldsymbol{\beta}_i) = 0) = 1/2$, $i = 1, 2$. Then we have that $D_C(\lambda\boldsymbol{\beta}_1 + (1 - \lambda)\boldsymbol{\beta}_2; P) = 0$ and $D_C(\boldsymbol{\beta}_i; P) = 1/2$ by Proposition 2.2. This implies that

$$\begin{aligned} & D_C(\lambda\boldsymbol{\beta}_1 + (1 - \lambda)\boldsymbol{\beta}_2; P) \\ & < \min\{D_C(\boldsymbol{\beta}_1; P); D_C(\boldsymbol{\beta}_2; P)\} \\ & = D_C(\boldsymbol{\beta}_2; P). \end{aligned}$$

That $D_C(\boldsymbol{\beta}; P)$ attains its maximum value at $\boldsymbol{\beta}_1$ yields the desired result. \square

LEMMA A.2. $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ of RH99 is

(i) *upper semicontinuous in $\boldsymbol{\beta}$, and continuous in $\boldsymbol{\beta}$ if the density of P exists and discontinuous in $\boldsymbol{\beta}$ generally;*

(ii) *continuous in P in the sense that $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; Q_n)$ converges to $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ in the same mode (in distribution, in probability, with probability one) as Q_n converges to P ; If Q_n is the empirical version P_n of P , then $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P_n)$ converges to $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ almost surely and uniformly in $\boldsymbol{\beta} \in \mathbb{R}^p$.*

(iii) *quasi-concave in $\boldsymbol{\beta} \in \mathbb{R}^p$.*

PROOF OF LEMMA A.2. (i) For fixed \mathbf{v} , $f(\boldsymbol{\beta}; \mathbf{v}) = P(r(\boldsymbol{\beta}) * \mathbf{v}'\mathbf{w})$ in (27) is upper semicontinuous in $\boldsymbol{\beta}$, hence the infimum of upper semicontinuous functions $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ is also upper semicontinuous. If a density of P exists, then $f(\boldsymbol{\beta}; \mathbf{v})$ is continuous in $\boldsymbol{\beta}$ and so is the infimum of continuous functions $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$. We focus on the discontinuity part. Suppose that the distribution of (y, \mathbf{x}) has its entire probability mass on the hyperplane determined by $y = \mathbf{w}'\boldsymbol{\beta}_0$ for some $\boldsymbol{\beta}_0 \in \mathbb{R}^p$, and any hyperline contains zero probability mass; then $\text{RD}_{\text{RH}}(\boldsymbol{\beta}_0; P) = 1$, and $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P) = 0$ for any $\boldsymbol{\beta} (\neq \boldsymbol{\beta}_0) \in \mathbb{R}^p$. Thus, when $\boldsymbol{\beta}$ approaches $\boldsymbol{\beta}_0$, $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ can never approach $\text{RD}_{\text{RH}}(\boldsymbol{\beta}_0; P)$.

(ii) First part follows directly from the characterization of $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ given in (27) and the continuity of the infimum function; the second part follows from standard empirical process theory, such as Pollard (1984).

(iii) Let $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathbb{R}^p$ and $\lambda \in [0, 1]$, and $\boldsymbol{\beta} := \lambda\boldsymbol{\beta}_1 + (1 - \lambda)\boldsymbol{\beta}_2$. Let $H_{\boldsymbol{\beta}}$ be the hyperplane determined by $y = \mathbf{w}'\boldsymbol{\beta}$, and $a = \min\{\mathbf{w}'\boldsymbol{\beta}_1, \mathbf{w}'\boldsymbol{\beta}_2\}$, $b = \max\{\mathbf{w}'\boldsymbol{\beta}_1, \mathbf{w}'\boldsymbol{\beta}_2\}$. Denote by $W(H_{\boldsymbol{\beta}_1}, H_{\boldsymbol{\beta}_2}) = \{(\mathbf{x}', y) : \mathbf{x} \in \mathbb{R}^{p-1}, y \in [a, b]\}$ the closed double wedge formed by two hyperplanes $H_{\boldsymbol{\beta}_1}$ and $H_{\boldsymbol{\beta}_2}$ (assume w.l.o.g. that $H_{\boldsymbol{\beta}_1}$ is not parallel to $H_{\boldsymbol{\beta}_2}$).

By Definition 2.2, $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ is the minimum probability mass that needs to pass when $H_{\boldsymbol{\beta}}$ is tilted into a vertical position. Notice that the position of $H_{\boldsymbol{\beta}}$ is in-between that of $H_{\boldsymbol{\beta}_1}$ and $H_{\boldsymbol{\beta}_2}$, it is readily seen that

$$\begin{aligned} & \text{RD}_{\text{RH}}(\lambda\boldsymbol{\beta}_1 + (1 - \lambda)\boldsymbol{\beta}_2; P) \\ & \geq \min\{\text{RD}_{\text{RH}}(\boldsymbol{\beta}_1; P), \text{RD}_{\text{RH}}(\boldsymbol{\beta}_2; P)\} \\ & \quad + \min\{P(W(H_{\boldsymbol{\beta}}, H_{\boldsymbol{\beta}_1})), P(W(H_{\boldsymbol{\beta}}, H_{\boldsymbol{\beta}_2}))\} \\ & \geq \min\{\text{RD}_{\text{RH}}(\boldsymbol{\beta}_1; P), \text{RD}_{\text{RH}}(\boldsymbol{\beta}_2; P)\}. \end{aligned}$$

This completes the proof of part (iii). \square

PROOF OF PROPOSITION 3.4. (a) By virtue of its definition (21), it suffices to show that $\phi(F_R) \rightarrow \infty$ when $\|\boldsymbol{\beta}\| \rightarrow \infty$ with $R = f(r(\boldsymbol{\beta})/S(F_y))$. The latter, in light of given conditions, follows if we can show that $|r(\boldsymbol{\beta})| \rightarrow \infty$ when $|\beta_1| \rightarrow \infty$. Note that $S(F_y)$ is a fixed positive number and $|r(\boldsymbol{\beta})| \geq |\beta_1| - |y - \boldsymbol{\beta}'_2\mathbf{x}| \geq |\beta_1| - |\boldsymbol{\beta}'_2\mathbf{x}| - |y| \geq |\beta_1| - |y| - \|\boldsymbol{\beta}_2\|\|\mathbf{x}\| \rightarrow \infty$ with probability one. This implies that $|R| \rightarrow \infty$ by virtue of the given condition on f , which in turn implies that $\phi(F_R) \rightarrow \infty$.

(b) $D_C(\boldsymbol{\beta}, P)$ satisfies **P4** follows directly from the Proposition 2.3.

(c) $\text{RD}_{\text{RH}}(\boldsymbol{\beta}; P)$ satisfies **P4** has been proved in (c) of the proof of Proposition 3.2.

(d) This part was given in (ii) of Lemma A.1. \square

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REFERENCES

- ADROVER, J. G., MARONNA, R. A. and YOHAI, V. J. (2002). Relationships between maximum depth and projection regression estimates. *J. Statist. Plann. Inference* **105** 363–375. [MR1910058](https://doi.org/10.1016/S0378-3758(01)00264-6) [https://doi.org/10.1016/S0378-3758\(01\)00264-6](https://doi.org/10.1016/S0378-3758(01)00264-6)
- AGOSTINELLI, C. and ROMANAZZI, M. (2011). Local depth. *J. Statist. Plann. Inference* **141** 817–830. [MR2732952](https://doi.org/10.1016/j.jspi.2010.08.001) <https://doi.org/10.1016/j.jspi.2010.08.001>

- BAI, Z.-D. and HE, X. (1999). Asymptotic distributions of the maximal depth estimators for regression and multivariate location. *Ann. Statist.* **27** 1616–1637. [MR1742502](https://doi.org/10.1214/aos/1017939144) <https://doi.org/10.1214/aos/1017939144>
- BEATON, A. E. and TUKEY, J. W. (1974). The fitting of power series, meaning polynomials, illustrated on band-spectroscopic data. *Technometrics* **16** 147–185.
- CAPLIN, A. and NALEBUFF, B. (1988). On 64%-majority rule. *Econometrica* **56** 787–814. [MR0951758](https://doi.org/10.2307/1912699) <https://doi.org/10.2307/1912699>
- CAPLIN, A. and NALEBUFF, B. (1991a). Aggregation and social choice: A mean voter theorem. *Econometrica* **59** 1–23. [MR1085582](https://doi.org/10.2307/2938238) <https://doi.org/10.2307/2938238>
- CAPLIN, A. and NALEBUFF, B. (1991b). Aggregation and imperfect competition: On the existence of equilibrium. *Econometrica* **59** 25–59. [MR1085583](https://doi.org/10.2307/2938239) <https://doi.org/10.2307/2938239>
- CASCOS, I. and MOLCHANOV, I. (2007). Multivariate risks and depth-trimmed regions. *Finance and Stochastics* **11** 373–397. [MR2322918](https://doi.org/10.1007/s00780-007-0043-7) <https://doi.org/10.1007/s00780-007-0043-7>
- CARRIZOSA, E. (1996). A characterization of halfspace depth. *J. Multivariate Anal.* **58** 21–26. [MR1408126](https://doi.org/10.1006/jmva.1996.0037) <https://doi.org/10.1006/jmva.1996.0037>
- CHAKRABORTY, A. and CHAUDHURI, P. (2014). The spatial distribution in infinite dimensional spaces and related quantiles and depths. *Ann. Statist.* **42** 1203–1231. [MR3224286](https://doi.org/10.1214/14-AOS1226) <https://doi.org/10.1214/14-AOS1226>
- CHEBANA, F. and OUARDA, T. B. M. J. (2008). Depth and homogeneity in regional flood frequency analysis. *Water Resour. Res.* **44** W11422. <https://doi.org/10.1029/2007WR006771>
- CHEBANA, F. and OUARDA, T. B. M. J. (2011). Depth-based multivariate descriptive statistics with hydrological applications. *J. Geophys. Res.* **116** D10120. <https://doi.org/10.1029/2010JD015338>
- CHEN, M., GAO, C. and REN, Z. (2018). Robust covariance and scatter matrix estimation under Huber's contamination model. *Ann. Statist.* **46** 1932–1960. [MR3845006](https://doi.org/10.1214/17-AOS1607) <https://doi.org/10.1214/17-AOS1607>
- CHERNOZHUKOV, V., GALICHON, A., HALLIN, M. and HENRY, M. (2017). Monge–Kantorovich depth, quantiles, ranks and signs. *Ann. Statist.* **45** 223–256. [MR3611491](https://doi.org/10.1214/16-AOS1450) <https://doi.org/10.1214/16-AOS1450>
- CLAESKENS, G., HUBERT, M., SLAETS, L. and VAKILI, K. (2014). Multivariate functional halfspace depth. *J. Amer. Statist. Assoc.* **109** 411–423. [MR3180573](https://doi.org/10.1080/01621459.2013.856795) <https://doi.org/10.1080/01621459.2013.856795>
- CUI, X., LIN, L. and YANG, G. (2008). An extended projection data depth and its applications to discrimination. *Comm. Statist. Theory Methods* **37** 2276–2290. [MR2526679](https://doi.org/10.1080/03610920701858396) <https://doi.org/10.1080/03610920701858396>
- DANG, X. and SERFLING, R. (2010). Nonparametric depth-based multivariate outlier identifiers, and masking robustness properties. *J. Statist. Plann. Inference* **140** 198–213. [MR2568133](https://doi.org/10.1016/j.jspi.2009.07.004) <https://doi.org/10.1016/j.jspi.2009.07.004>
- DONOHO, D. L. (1982). Breakdown properties of multivariate location estimators. Ph.D. qualifying paper, Harvard Univ.
- DONOHO, D. L. and GASKO, M. (1992). Breakdown properties of location estimates based on halfspace depth and projected outlyingness. *Ann. Statist.* **20** 1803–1827. [MR1193313](https://doi.org/10.1214/aos/1176348890) <https://doi.org/10.1214/aos/1176348890>
- DYCKERHOFF, R. (2004). Data depths satisfying the projection property. *Allg. Stat. Arch.* **88** 163–190. [MR2074729](https://doi.org/10.1007/s101820400167) <https://doi.org/10.1007/s101820400167>
- EISENHAUER, J. G. (2003). Regression through the origin. *Teach. Stat.* **25** 76–80.

- FEBRERO, M., GALEANO, P. and GONZÁLEZ-MANTEIGA, W. (2008). Outlier detection in functional data by depth measures, with application to identify abnormal NO_x levels. *Environmetrics* **19** 331–345. MR2440036 <https://doi.org/10.1002/env.878>
- GHOSH, A. K. and CHAUDHURI, P. (2005). On data depth and distribution-free discriminant analysis using separating surfaces. *Bernoulli* **11** 1–27. MR2121452 <https://doi.org/10.3150/bj/1110228239>
- GIJBELS, I. and NAGY, S. (2017). On a general definition of depth for functional data. *Statist. Sci.* **32** 630–639. MR3730526 <https://doi.org/10.1214/17-STS625>
- HALLIN, M., PAINDAVEINE, D. and ŠIMAN, M. (2010). Multivariate quantiles and multiple-output regression quantiles: From L_1 optimization to halfspace depth. *Ann. Statist.* **38** 635–669. MR2604670 <https://doi.org/10.1214/09-AOS723>
- HAMPEL, F. R. (1974). The influence curve and its role in robust estimation. *J. Amer. Statist. Assoc.* **69** 383–393. MR0362657
- HOBERG, R. (2000). Cluster analysis based on data depth. In *Data Analysis, Classification and Related Methods* (H. Kiers, J. P. Rason, P. Groenen and M. Schader, eds.) 17–22. Springer, Berlin.
- HUBER, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Stat.* **35** 73–101. MR0161415 <https://doi.org/10.1214/aoms/1177703732>
- HUBER, P. J. (1973). Robust regression: Asymptotics, conjectures and Monte Carlo. *Ann. Statist.* **1** 799–821. MR0356373
- HUBERT, M., ROUSSEEUW, P. J. and SEGAERT, P. (2015). Multivariate functional outlier detection. *Stat. Methods Appl.* **24** 177–202. MR3376852 <https://doi.org/10.1007/s10260-015-0297-8>
- HUBERT, M., ROUSSEEUW, P. J. and VAN AELST, S. (2001). Similarities between location depth and regression depth. In *Statistics in Genetics and in the Environmental Sciences* (Ascona, 1999) (L. Fernholz, S. Morgenthaler and W. Stahel, eds.). Trends Math. 159–172. Birkhäuser, Basel. MR1843176
- JAECKEL, L. A. (1972). Estimating regression coefficients by minimizing the dispersion of the residuals. *Ann. Math. Stat.* **43** 1449–1458. MR0348930 <https://doi.org/10.1214/aoms/1177692377>
- JÖRNSTEN, R. (2004). Clustering and classification based on the L_1 data depth. *J. Multivariate Anal.* **90** 67–89. MR2064937 <https://doi.org/10.1016/j.jmva.2004.02.013>
- JUREČKOVÁ, J. (1971). Nonparametric estimate of regression coefficients. *Ann. Math. Stat.* **42** 1328–1338. MR0295487 <https://doi.org/10.1214/aoms/1177693245>
- KOENKER, R. and BASSETT, G. JR. (1978). Regression quantiles. *Econometrica* **46** 33–50. MR0474644 <https://doi.org/10.2307/1913643>
- KOSHEVOY, G. and MOSLER, K. (1997). Zonoid trimming for multivariate distributions. *Ann. Statist.* **25** 1998–2017. MR1474078 <https://doi.org/10.1214/aos/1069362382>
- KOUL, H. L. (1970). Some convergence theorems for ranks and weighted empirical cumulatives. *Ann. Math. Stat.* **41** 1768–1773. MR0267631 <https://doi.org/10.1214/aoms/1177696824>
- KOUL, H. L. (1971). Asymptotic behavior of a class of confidence regions based on ranks in regression. *Ann. Math. Stat.* **42** 466–476. MR0288896 <https://doi.org/10.1214/aoms/1177693398>
- LANGE, T., MOSLER, K. and MOZHAROVSKYI, P. (2014). Fast nonparametric classification based on data depth. *Statist. Papers* **55** 49–69. MR3152767 <https://doi.org/10.1007/s00362-012-0488-4>
- LI, J., CUESTA-ALBERTOS, J. A. and LIU, R. Y. (2012). DD-classifier: Nonparametric classification procedure based on DD-plot. *J. Amer. Statist. Assoc.* **107** 737–753. MR2980081 <https://doi.org/10.1080/01621459.2012.688462>
- LI, J. and LIU, R. Y. (2004). New nonparametric tests of multivariate locations and scales using data depth. *Statist. Sci.* **19** 686–696. MR2185590 <https://doi.org/10.1214/088342304000000594>
- LIU, R. Y. (1990). On a notion of data depth based on random simplices. *Ann. Statist.* **18** 405–414. MR1041400 <https://doi.org/10.1214/aos/1176347507>
- LIU, R. Y. (1992). Data depth and multivariate rank tests. In *L_1 -Statistical Analysis and Related Methods* (Neuchâtel, 1992) (Y. Dodge, ed.) 279–294. North-Holland, Amsterdam. MR1214839
- LIU, R. Y. (1995). Control charts for multivariate processes. *J. Amer. Statist. Assoc.* **90** 1380–1387. MR1379481
- LIU, R. Y., PARELIUS, J. M. and SINGH, K. (1999). Multivariate analysis by data depth: Descriptive statistics, graphics and inference. *Ann. Statist.* **27** 783–858. MR1724033 <https://doi.org/10.1214/aos/1018031260>
- LIU, R. Y. and SINGH, K. (1993). A quality index based on data depth and multivariate rank tests. *J. Amer. Statist. Assoc.* **88** 252–260. MR1212489
- LÓPEZ-PINTADO, S. and ROMO, J. (2009). On the concept of depth for functional data. *J. Amer. Statist. Assoc.* **104** 718–734. MR2541590 <https://doi.org/10.1198/jasa.2009.0108>
- MAJUMDAR, S. and CHATTERJEE, S. (2018). Non-convex penalized multitask regression using data depth-based penalties. *Stat* **7** e174. MR3769054 <https://doi.org/10.1002/sta4.174>
- MARONNA, R. A., MARTIN, R. D. and YOHAI, V. J. (2006). *Robust Statistics: Theory and Methods*. Wiley Series in Probability and Statistics. Wiley, Chichester. MR2238141 <https://doi.org/10.1002/0470010940>
- MARONNA, R. A. and YOHAI, V. J. (1993). Bias-robust estimates of regression based on projections. *Ann. Statist.* **21** 965–990. MR1232528 <https://doi.org/10.1214/aos/1176349160>
- MIZERA, I. (2002). On depth and deep points: A calculus. *Ann. Statist.* **30** 1681–1736. MR1969447 <https://doi.org/10.1214/aos/1043351254>
- MIZERA, I. and MÜLLER, C. H. (2004). Location-scale depth. *J. Amer. Statist. Assoc.* **99** 949–989. MR2109488 <https://doi.org/10.1198/016214504000001312>
- MOSLER, K. (2002). *Multivariate Dispersion, Central Regions and Depth: The Lift Zonoid Approach*. Lecture Notes in Statistics **165**. Springer, Berlin. MR1913862 <https://doi.org/10.1007/978-1-4613-0045-8>
- MOSLER, K. and BAZOVKIN, P. (2014). Stochastic linear programming with a distortion risk constraint. *OR Spectrum* **36** 949–969. MR3258501 <https://doi.org/10.1007/s00291-014-0372-9>
- MOSLER, K. and HOBERG, R. (2006). Data analysis and classification with the zonoid depth. In *Data Depth: Robust Multivariate Analysis, Computational Geometry and Applications* (R. Liu, R. Serfling and D. Souvaine, eds.). DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **72** 49–59. Amer. Math. Soc., Providence, RI. MR2343112
- NIETO-REYES, A. and BATTEY, H. (2016). A topologically valid definition of depth for functional data. *Statist. Sci.* **31** 61–79. MR3458593 <https://doi.org/10.1214/15-STS532>
- PAINDAVEINE, D. and VAN BEVER, G. (2013). From depth to local depth: A focus on centrality. *J. Amer. Statist. Assoc.* **108** 1105–1119. MR3174687 <https://doi.org/10.1080/01621459.2013.813390>
- PAINDAVEINE, D. and VAN BEVER, G. (2015). Nonparametrically consistent depth-based classifiers. *Bernoulli* **21** 62–82. MR3322313 <https://doi.org/10.3150/13-BEJ561>
- PAINDAVEINE, D. and VAN BEVER, G. (2018). Halfspace depths for scatter, concentration and shape matrices. *Ann. Statist.* **46** 3276–3307. MR3852652 <https://doi.org/10.1214/17-AOS1658>
- POLLARD, D. (1984). *Convergence of Stochastic Processes*. Springer Series in Statistics. Springer, New York. MR0762984 <https://doi.org/10.1007/978-1-4612-5254-2>
- PORTNOY, S. (2003). Censored regression quantiles. *J. Amer. Statist. Assoc.* **98** 1001–1012. MR2041488 <https://doi.org/10.1198/016214503000000954>

- PORTNOY, S. (2012). Nearly root- n approximation for regression quantile processes. *Ann. Statist.* **40** 1714–1736. MR3015041 <https://doi.org/10.1214/12-AOS1021>
- ROUSSEEUW, P. J. (1984). Least median of squares regression. *J. Amer. Statist. Assoc.* **79** 871–880. MR0770281
- ROUSSEEUW, P. J. and HUBERT, M. (1999). Regression depth. *J. Amer. Statist. Assoc.* **94** 388–433. MR1702314 <https://doi.org/10.2307/2670155>
- ROUSSEEUW, P. J. and LEROY, A. M. (1987). *Robust Regression and Outlier Detection*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. Wiley, New York. MR0914792 <https://doi.org/10.1002/0471725382>
- ROUSSEEUW, P. J. and STRUYF, A. (2004). Characterizing angular symmetry and regression symmetry. *J. Statist. Plann. Inference* **122** 161–173. MR2057920 <https://doi.org/10.1016/j.jspi.2003.06.015>
- RUPPERT, D. and CARROLL, R. J. (1980). Trimmed least squares estimation in the linear model. *J. Amer. Statist. Assoc.* **75** 828–838. MR0600964
- SERFLING, R. (2006). Depth functions in nonparametric multivariate inference. In *Data Depth: Robust Multivariate Analysis, Computational Geometry and Applications*. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **72** 1–16. Amer. Math. Soc., Providence, RI. MR2343109
- SERFLING, R. (2019). Perspectives on depth functions on general data spaces, with consideration of the Tukey, projection, spatial, ‘density’, ‘local’, and ‘contour’ depths. Preprint.
- SERFLING, R. and WANG, S. (2014). General foundations for studying masking and swamping robustness of outlier identifiers. *Stat. Methodol.* **20** 79–90. MR3205723 <https://doi.org/10.1016/j.stamet.2013.08.004>
- STAHEL, W. A. (1981). Robuste Schätzungen: Infinitesimale Optimalität und Schätzungen Von Kovarianzmatrizen. Ph.D. dissertation, ETH, Zurich.
- TUKEY, J. W. (1975). Mathematics and the picturing of data. In *Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974)*, Vol. 2 (R. D. James, ed.) 523–531. MR0426989
- VAN AELST, S. and ROUSSEEUW, P. J. (2000). Robustness of deepest regression. *J. Multivariate Anal.* **73** 82–106. MR1766122 <https://doi.org/10.1006/jmva.1999.1870>
- VARDI, Y. and ZHANG, C.-H. (2000). The multivariate L_1 -median and associated data depth. *Proc. Natl. Acad. Sci. USA* **97** 1423–1426. MR1740461 <https://doi.org/10.1073/pnas.97.4.1423>
- VELASCO-FORERO, S. and ANGULO, J. (2011). Mathematical morphology for vector images using statistical depth. In *Mathematical Morphology and Its Applications to Image and Signal Processing* (P. Soille, M. Pesaresi and G. K. Ouzounis, eds.) 355–366. Springer, Berlin.
- VELASCO-FORERO, S. and ANGULO, J. (2012). Random projection depth for multivariate mathematical morphology. *IEEE J. Sel. Top. Signal Process.* **6** 753–763.
- WANG, S. and SERFLING, R. (2015). On masking and swamping robustness of leading nonparametric outlier identifiers for univariate data. *J. Statist. Plann. Inference* **162** 62–74. MR3323104 <https://doi.org/10.1016/j.jspi.2015.02.002>
- WANG, S. and SERFLING, R. (2018). On masking and swamping robustness of leading nonparametric outlier identifiers for multivariate data. *J. Multivariate Anal.* **166** 32–49. MR3799633 <https://doi.org/10.1016/j.jmva.2018.02.003>
- WU, M. and ZUO, Y. (2008). Trimmed and Winsorized standard deviations based on a scaled deviation. *J. Nonparametr. Stat.* **20** 319–335. MR2436381 <https://doi.org/10.1080/10485250802036909>
- WU, M. and ZUO, Y. (2009). Trimmed and Winsorized means based on a scaled deviation. *J. Statist. Plann. Inference* **139** 350–365. MR2474011 <https://doi.org/10.1016/j.jspi.2008.03.039>
- YEH, A. B. and SINGH, K. (1997). Balanced confidence regions based on Tukey’s depth and the bootstrap. *J. Roy. Statist. Soc. Ser. B* **59** 639–652. MR1452031 <https://doi.org/10.1111/1467-9868.00088>
- ZUO, Y. (1998). *Contributions to the Theory and Applications of Statistical Depth Functions*. ProQuest LLC, Ann Arbor, MI. Ph.D. thesis, Univ. Texas at Dallas. MR2697586
- ZUO, Y. (2003). Projection-based depth functions and associated medians. *Ann. Statist.* **31** 1460–1490. MR2012822 <https://doi.org/10.1214/aos/1065705115>
- ZUO, Y. (2004). Robustness of weighted L^p -depth and L^p -median. *Allg. Stat. Arch.* **88** 215–234. MR2074731 <https://doi.org/10.1007/s101820400169>
- ZUO, Y. (2009). Data depth trimming procedure outperforms the classical t (or T^2) one. *J. Probab. Stat.* <https://doi.org/10.1155/2009/373572>
- ZUO, Y. (2010). Is the t confidence interval $\bar{X} \pm t_\alpha(n-1)s/\sqrt{n}$ optimal? *Amer. Statist.* **64** 170–173. MR2757009 <https://doi.org/10.1198/tast.2010.09021>
- ZUO, Y. (2018). Robustness of deepest projection regression depth functional. *Statist. Papers.* <https://doi.org/10.1007/s00362-019-01129-4>
- ZUO, Y. (2019a). Asymptotics for the maximum regression depth estimator. Available at arXiv:1809.09896.
- ZUO, Y. (2019b). Computation of projection regression depth and its induced median. Available at arXiv:1905.11846.
- ZUO, Y. and SERFLING, R. (2000). General notions of statistical depth function. *Ann. Statist.* **28** 461–482. MR1790005 <https://doi.org/10.1214/aos/1016218226>