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Exact simulation of two-parameter Poisson-Dirichlet random variables

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Abstract

Consider a random vector $(V_1, \dots, V_n)$ where $\{V_k\}_{k=1,\dots,n}$ are the first n components of a two-parameter Poisson-Dirichlet distribution $PD(\alpha, \theta)$. In this paper, we derive a decomposition for the components of the random vector, and propose an exact simulation algorithm to sample from the random vector. Moreover, a special case arises when θ/α is a positive integer, for which we present a very fast modified simulation algorithm using a compound geometric representation of the decomposition. Numerical examples are provided to illustrate the accuracy and effectiveness of our algorithms.

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1 Introduction

The two-parameter Poisson-Dirichlet distribution is a probability distribution on the set of decreasing positive sequences with sum 1. It can be defined in terms of independent Beta random variables as the following.

Definition 1.1 (Definition 1 of [19]). *For $0 \le \alpha < 1$ and $\theta > -\alpha$, suppose that a probability $\mathbb{P}_{\alpha,\theta}$ governs independent random variables $\tilde{Y}_i$ such that $\tilde{Y}_i$ has Beta$(1-\alpha, \theta + i\alpha)$ distribution. Let*

$$\tilde{V}_1 = \tilde{Y}_1, \quad \tilde{V}_i = (1-\tilde{Y}_1)\dots(1-\tilde{Y}_{i-1})\tilde{Y}_i \quad (i \ge 2)$$

and let $V_1 \ge V_2 \ge \dots$ be the ranked values of the $\tilde{V}_i$. Define the Poisson-Dirichlet distribution with parameters (α, θ), abbreviated $PD(\alpha, \theta)$, to be the $\mathbb{P}_{\alpha,\theta}$ distribution of (V_i).

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Moreover, results of [14], [16] and [17] show that under the $\mathbb{P}_{\alpha,\theta}$ governing, the sequence $\{\tilde{V}_i\}_{i\geq 1}$ is a size-biased permutation of $\{V_i\}_{i\geq 1}$. Based on the size-biased permutation, [6] proposed a residual allocation model for $\{\tilde{V}_i\}_{i\geq 1}$.

The $PD(\alpha,\theta)$ distribution extends the one-parameter family of Poisson-Dirichlet distribution introduced by [11] and denoted by $PD(0,\theta)$, $\theta > 0$. It also generalizes the family of distributions denoted by $PD(\alpha,0)$, which can be interpreted in terms of the ranked lengths of excursion intervals between zeros of a recurrent Bessel process, see [18]. We refer the reader to [19] for the motivation and a collection of existing results of the Poisson-Dirichlet distribution. In particular, [19] includes the distributional properties of $PD(\alpha,0)$ and its connection to random processes, we will use these properties throughout the paper.

The Poisson-Dirichlet distribution arises in many fields, for example, as the asymptotic distribution of the ranked relative cycle lengths in a random permutation, see [21] and [20]; as the limiting proportions of genes in some populations genetics models, see [9] and [24]; as the distribution of the ranked sizes of atoms in the Dirichlet process prior in Bayesian statistics, see [7] and [4]. It also appears in the research fields such as number theory [3], [23] and combinatorics [2], [8]. More recently, the Poisson-Dirichlet distribution is used to approximate the capital distribution curves in equity markets, see [22].

Despite its huge variety of applications, the simulation method for $PD(\alpha,\theta)$ is less attended and we found no exact method in the literature. When $\alpha = 0$, $PD(0,\theta)$ can be approximated by a Dirichlet distribution, see Section 9.3 of [12] and Proposition 5 of [19]. An approximation method for $PD(\alpha,\theta)$ with a general value of α is proposed in [1].

In this paper we develop two exact simulation algorithms for the first n components, $(V_1, V_2, \ldots, V_n)$, of the $PD(\alpha,\theta)$ distribution. The following trivial simulation algorithm is obtained immediately from Definition 1.1. However, this is only an approximation.

Algorithm 1.2 (Trivial algorithm)**.** The approximation algorithm for the random vector $(V_1, V_2, \ldots, V_n)$ is the following.

---

1. Initialize α, θ and n, select a positive integer $m \gg n$ (for example $m = 5n$).
2. For $i = 1, 2, \ldots, m$, generate independent Beta random variables
$$\tilde{Y}_i \sim \text{Beta}(1-\alpha, \theta + i\alpha).$$
3. Set $\tilde{V}_1 = \tilde{Y}_1$, and for each $i = 2, \ldots, m$, set
$$\tilde{V}_i = (1-\tilde{Y}_1)\ldots(1-\tilde{Y}_{i-1})\tilde{Y}_i.$$
4. Sort $\{\tilde{V}_i\}_{i=1,\ldots,m}$ in a descending order and let $V_1 \geq V_2 \geq \cdots \geq V_m$ be the ranked values of $\{\tilde{V}_i\}_{i=1,\ldots,m}$.
5. Truncate the sequence $\{V_i\}_{i=1,\ldots,m}$ at the first n components, then $(V_1, V_2, \ldots, V_n)$ is an approximation of the first n components of the $PD(\alpha,\theta)$ distribution.

Proof. This follows Definition 1.1 directly. □

As $m \to \infty$, Algorithm 1.2 coincides with the definition of the $PD(\alpha,\theta)$ distribution, but in practice m can only take a finite value, so this algorithm is a non-exact approximation for $(V_1, V_2, \ldots, V_n)$.

The rest of the paper is organized as follows. In Section 2 we provide two decompositions in law for the components of the $PD(\alpha,\theta)$ distribution, these decompositions will lead to the exact simulation algorithms directly. In Section 3 we present the main results, namely the subordinator algorithm and the compound geometric representation algorithm to sample from the $PD(\alpha,\theta)$ distribution. Numerical examples and their discussions are given in Section 4.

2 Decompositions of $1/V_k$ under $\mathbb{P}_{\alpha,\theta}$

Denote by $(V_1, V_2, \dots, V_n)$ the first n components of the $PD(\alpha,\theta)$ distribution. In this section, we provide two decompositions for $1/V_k$, $k = 1, \dots, n$ under the probability measure $\mathbb{P}_{\alpha,\theta}$. These decompositions will lead to the exact simulation algorithms. For simplicity, we make the convention throughout the paper that $\prod_{j=n}^{n-1} a_j = 1$. The following lemma provides a preliminary result that will be used in the proof of the main results.

Lemma 2.1 (Existing results under $\mathbb{P}_{\alpha,0}$). *Denote by τ_t a stable subordinator with Lévy measure $C\alpha x^{-\alpha-1}\mathbb{1}_{\{x>0\}}dx$ for $0<\alpha<1$, and Δ_k the ranked jumps of τ_t, such that $\Delta_1 > \Delta_2 > \dots$ and $\tau_t = \sum_{k=1}^{\infty}\Delta_k$. Then for every $C>0$ and $t>0$, the random vector*

$$\left(\frac{\Delta_1}{\tau_t}, \frac{\Delta_2}{\tau_t}, \dots\right) \text{ has } PD(\alpha,0) \text{ distribution.}$$

Moreover, let $V_k := \frac{\Delta_k}{\tau_t}$ be the k-th component of the $PD(\alpha,0)$ distribution, then for $k = 1, \dots, n$, the decomposition

$$\frac{1}{V_k} \overset{law}{=} \left(1 + \left(R_1 + R_1R_2 + \dots + \prod_{j=1}^{n-1} R_j\right) + \left(\prod_{j=1}^{n-1} R_j\right)\Sigma_n\right)\prod_{j=1}^{k-1} R_j^{-1} \tag{2.1}$$

holds under the probability measure $\mathbb{P}_{\alpha,0}$, where $R_j := V_{j+1}/V_j = \Delta_{j+1}/\Delta_j$, and $\Sigma_n|\Delta_1,\dots,\Delta_n$ is a subordinator with truncated Lévy measure $C\alpha x^{-\alpha-1}\mathbb{1}_{\{0<x<1\}}dx$ at time $t\Delta_1^{-\alpha}(\prod_{j=1}^{n-1} R_j^{-\alpha})$.

Proof. For the distribution of the random vector $\left(\frac{\Delta_1}{\tau_t}, \frac{\Delta_2}{\tau_t}, \dots\right)$, see Proposition 6 of [19]. We now proceed to prove the decomposition (2.1) under $\mathbb{P}_{\alpha,0}$. Denote by

$$\Sigma_n := \frac{\tau_t - \Delta_1 - \dots - \Delta_n}{\Delta_n}$$

and $R_j := \Delta_{j+1}/\Delta_j$, it follows that

$$\begin{aligned}\frac{1}{V_1} = \frac{\tau_t}{\Delta_1} &= 1 + \frac{\Delta_2 + \dots + \Delta_n}{\Delta_1} + \frac{\Delta_n}{\Delta_1}\Sigma_n \\ &= 1 + \left(R_1 + R_1R_2 + \dots + \prod_{j=1}^{n-1} R_j\right) + \left(\prod_{j=1}^{n-1} R_j\right)\Sigma_n,\end{aligned}$$

and the decomposition for $1/V_k$, $k = 2, \dots, n$ is given by $V_k = V_1\prod_{j=1}^{k-1} R_j$.

Moreover, from the proof of Proposition 11 in [19] (see also the calculations in [11] and [15]), we know the Laplace transform of $\Sigma_n \mid \Delta_1, \dots, \Delta_n$ is

$$\begin{aligned}\mathbb{E}\left(e^{-\beta\Sigma_n}|\Delta_1,\dots,\Delta_n\right) &= e^{-t\Delta_n^{-\alpha}\int_0^1(1-e^{-\beta x})C\alpha x^{-\alpha-1}dx} \\ &= e^{-t\Delta_1^{-\alpha}(\prod_{j=1}^{n-1} R_j^{-\alpha})\int_0^1(1-e^{-\beta x})C\alpha x^{-\alpha-1}dx},\end{aligned}$$

this is the Lévy-Khintchine representation (see [13]) of a subordinator with truncated Lévy measure $C\alpha x^{-\alpha-1}\mathbb{1}_{\{0<x<1\}}$ at time $t\Delta_1^{-\alpha}(\prod_{j=1}^{n-1} R_j^{-\alpha})$, then the Lemma is proved. □

Next, we present the decomposition for the first n components of the $PD(\alpha,\theta)$ distribution, this result will permit us to use the subordinator algorithm developed in [5]. Without loss of generality, we set $t=1$ and $C=1$ in the rest of the paper.

Theorem 2.2. *Let $(V_1,V_2,\dots,V_n)$ be the first n components of the $PD(\alpha,\theta)$ distribution, then for every $k=1,\dots,n$, the decomposition*

$$\frac{1}{V_k}\overset{law}{=}\left(1+\left(R_1+R_1R_2+\cdots+\prod_{j=1}^{n-1}R_j\right)+\left(\prod_{j=1}^{n-1}R_j\right)\Sigma_n\right)\prod_{j=1}^{k-1}R_j^{-1}$$

holds under the probability measure $\mathbb{P}_{\alpha,\theta}$, where $(\Delta_1^{-\alpha},R_1,\dots,R_{n-1},\Sigma_n)$ has the joint density

$$g(w,r_1,\dots,r_{n-1},x):=$$
$$\frac{\frac{\Gamma(\theta+1)}{\Gamma(\frac{\theta}{\alpha}+1)}\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}f_{\Sigma_n}\left(x\mid w\prod_{j=1}^{n-1}r_j^{-\alpha}\right)\alpha^{n-1}w^{\frac{\theta}{\alpha}+n-1}e^{-w\prod_{j=1}^{n-1}r_j^{-\alpha}}\left(\prod_{j=1}^{n-1}r_j^{-(n-j)\alpha-1}\right)}{\left(1+\left(r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1}r_j\right)+\left(\prod_{j=1}^{n-1}r_j\right)x\right)^{\theta}},$$

and $f_{\Sigma_n}\left(x\mid w\prod_{j=1}^{n-1}r_j^{-\alpha}\right)$ denotes the density of a subordinator with truncated Lévy measure $\alpha x^{-\alpha-1}\mathbb{1}_{\{0<x<1\}}dx$ at time $w\prod_{j=1}^{n-1}r_j^{-\alpha}$, for $w>0$, $0<r_j<1$, $j=1,\dots,n-1$ and $x>0$.

Proof. This theorem is an analogue of Lemma 2.1 with a changed probability measure. Denote by H the non-negative product measurable function

$$H(x_1,\dots,x_n):=e^{-\beta_1\frac{1}{x_1}}\dots e^{-\beta_n\frac{1}{x_n}},$$

where $\beta_k\geq 0$ and $0<x_k<1$, for $k=1,\dots,n$. From Proposition 14 of [19], we know

$$\mathbb{E}_{\alpha,\theta}(H(V_1,\dots,V_n))=c_{\alpha,\theta}\mathbb{E}_{\alpha,0}\left(\tau_t^{-\theta}H(V_1,\dots,V_n)\right),$$

where

$$c_{\alpha,\theta}=C^{\frac{\theta}{\alpha}}\frac{\Gamma(\theta+1)}{\Gamma(\frac{\theta}{\alpha}+1)}\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}. \tag{2.2}$$

Since $V_1=\Delta_1/\tau_t$ under $\mathbb{P}_{\alpha,0}$, we set $\tau_t^{-\theta}=\Delta_1^{-\theta}V_1^{\theta}$, then

$$\mathbb{E}_{\alpha,\theta}(H(V_1,\dots,V_n))=c_{\alpha,\theta}\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}V_1^{\theta}H(V_1,\dots,V_n)\right). \tag{2.3}$$

From Lemma 24 of [19], we know that under the probability measure $\mathbb{P}_{\alpha,0}$, $\Delta_1^{-\alpha}$ has a standard exponential distribution. Conditioning on $\Delta_1^{-\alpha}$, we have

$$\begin{aligned}\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}V_1^{\theta}H(V_1,\dots,V_n)\right)&=\int_0^{\infty}\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}V_1^{\theta}H(V_1,\dots,V_n)\mid\Delta_1^{-\alpha}\right)e^{-w}dw\\&=\int_0^{\infty}\mathbb{E}_{\alpha,0}\left(V_1^{\theta}H(V_1,\dots,V_n)\mid\Delta_1^{-\alpha}\right)w^{\frac{\theta}{\alpha}}e^{-w}dw.\end{aligned}$$

Moreover, the joint density of $R_1,\dots,R_{n-1}\mid\Delta_1^{-\alpha}=w$ under $\mathbb{P}_{\alpha,0}$ is given by Lemma 3.2 of [10] (see Appendix A),

$$f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1}\mid\Delta_1^{-\alpha}=w):=\alpha^{n-1}w^{n-1}e^{w}e^{-w\prod_{j=1}^{n-1}r_j^{-\alpha}}\prod_{j=1}^{n-1}r_j^{-(n-j)\alpha-1}, \tag{2.4}$$

for $0 < r_j < 1$, $j = 1, \dots, n-1$. Thus, conditioning on $R_1, \dots, R_{n-1}$, we have

$$\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta} V_1^{\theta} H(V_1,\dots,V_n)\right) \\
&= \int_0^{\infty}\int_0^1 \cdots \int_0^1 \mathbb{E}_{\alpha,0}\left(V_1^{\theta} H(V_1,\dots,V_n) \mid \Delta_1^{-\alpha}, R_1,\dots,R_{n-1}\right) \\
&\qquad \times f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1} \mid w) w^{\frac{\theta}{\alpha}} e^{-w} dr_1 \dots dr_{n-1} dw.
\end{aligned}$$

We also denote by $f_{\Sigma_n}(x \mid w \prod_{j=1}^{n-1} r_j^{-\alpha})$ the density of a subordinator with truncated Lévy measure $\alpha x^{-\alpha-1}\mathbb{1}_{\{0<x<1\}}dx$ at time $w \prod_{j=1}^{n-1} r_j^{-\alpha}$. Then, conditioning on Σ_n leads to

$$\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta} V_1^{\theta} H(V_1,\dots,V_n)\right) \\
&= \int_0^{\infty}\int_0^1 \cdots \int_0^1 \int_0^{\infty} \mathbb{E}_{\alpha,0}\left(V_1^{\theta} H(V_1,\dots,V_n) \mid \Delta_1^{-\alpha}, R_1,\dots,R_{n-1}, \Sigma_n\right) \\
&\qquad \times f_{\Sigma_n}\left(x \mid w \prod_{j=1}^{n-1} r_j^{-\alpha}\right) f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1} \mid w) w^{\frac{\theta}{\alpha}} e^{-w} dx dr_1 \dots dr_{n-1} dw.
\end{aligned}$$

From Lemma 2.1, we know $(V_1,\dots,V_n)$ is determined by $(\Delta_1^{-\alpha}, R_1,\dots,R_{n-1},\Sigma_n)$ under the probability measure $\mathbb{P}_{\alpha,0}$. Using the decomposition (2.1), we get

$$\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta} V_1^{\theta} H(V_1,\dots,V_n)\right) \\
&= \int_0^{\infty}\int_0^1 \cdots \int_0^1 \int_0^{\infty} \prod_{k=1}^{n} e^{-\beta_k \left(1+\left(r_1+r_1r_2+\dots+\prod_{j=1}^{n-1} r_j\right)+\left(\prod_{j=1}^{n-1} r_j\right)x\right)\prod_{j=1}^{k-1} r_j^{-1}} \\
&\qquad \times \frac{f_{\Sigma_n}\left(x \mid w \prod_{j=1}^{n-1} r_j^{-\alpha}\right) f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1} \mid w) w^{\frac{\theta}{\alpha}} e^{-w}}{\left(1+\left(r_1+r_1r_2+\dots+\prod_{j=1}^{n-1} r_j\right)+\left(\prod_{j=1}^{n-1} r_j\right)x\right)^{\theta}} dx dr_1 \dots dr_{n-1} dw.
\end{aligned}$$

Taking this into (2.3) and using the expressions of $f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1} \mid \Delta_1^{-\alpha} = w)$ and $c_{\alpha,\theta}$, we obtain the joint Laplace transform of $(\frac{1}{V_1},\dots,\frac{1}{V_n})$ under $\mathbb{P}_{\alpha,\theta}$,

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}\left(e^{-\beta_1 \frac{1}{V_1}} \dots e^{-\beta_n \frac{1}{V_n}}\right) \\
&= \int_0^{\infty}\int_0^1 \cdots \int_0^1 \int_0^{\infty} \prod_{k=1}^{n} e^{-\beta_k \left(1+\left(r_1+r_1r_2+\dots+\prod_{j=1}^{n-1} r_j\right)+\left(\prod_{j=1}^{n-1} r_j\right)x\right)\prod_{j=1}^{k-1} r_j^{-1}} \\
&\times \frac{\frac{\Gamma(\theta+1)}{\Gamma(\frac{\theta}{\alpha}+1)}\Gamma(1-\alpha)^{\frac{\theta}{\alpha}} f_{\Sigma_n}\left(x \mid w \prod_{j=1}^{n-1} r_j^{-\alpha}\right) \alpha^{n-1} w^{\frac{\theta}{\alpha}+n-1} e^{-w \prod_{j=1}^{n-1} r_j^{-\alpha}} \left(\prod_{j=1}^{n-1} r_j^{-(n-j)\alpha-1}\right)}{\left(1+\left(r_1+r_1r_2+\dots+\prod_{j=1}^{n-1} r_j\right)+\left(\prod_{j=1}^{n-1} r_j\right)x\right)^{\theta}} \\
&\qquad dx dr_1 \dots dr_{n-1} dw,
\end{aligned}$$

and the theorem is a direct consequence of this result. □

The next theorem gives another decomposition for the components of $PD(\alpha,\theta)$, which will permit us to use a faster simulation algorithm when θ/α is a positive integer.

Theorem 2.3. *Let $(V_1, V_2, \dots, V_n)$ be the first n components of the $PD(\alpha,\theta)$ distribution. If $\theta > 0$ and θ/α is a positive integer, then for every $k = 1,\dots,n$, the decomposition*

$$\frac{1}{V_k} \overset{law}{=} \left(1 + R_1 + R_1R_2 + \dots + \prod_{j=1}^{n-1} R_j\right) \prod_{j=1}^{k-1} R_j^{-1} + \sum_{i=1}^{\frac{\theta}{\alpha}+n} \left(\left(\prod_{j=k}^{n-1} R_j\right)\left(\sum_{j=0}^{N^{(i)}} T_j^{(i)}\right)\right)$$

holds under the probability measure $\mathbb{P}_{\alpha,\theta}$*, where* $(Z, R_1, \ldots, R_{n-1})$ *has the joint density*

$$m(z, r_1, \ldots, r_{n-1}) := \frac{\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}}{\Gamma(\theta)} z^{\theta-1} \left(\prod_{j=1}^{n-1} (j\alpha+\theta) r_j^{j\alpha+\theta-1} \right) \times \frac{e^{-z\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)}}{\left(1+\int_0^1 (1-e^{-z\left(\prod_{j=1}^{n-1} r_j\right)x})\alpha x^{-\alpha-1}dx\right)^{\frac{\theta}{\alpha}+n}},$$

for $z > 0$ *and* $0 < r_j < 1$*,* $j = 1, 2, \ldots, n-1$*. Moreover, let* A *be defined as*

$$A = A(Z, R_1, \ldots, R_{n-1}) := \int_0^1 e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)(v+1)} \frac{v^{-\alpha}-v^{\alpha}}{v+1} dv; \tag{2.5}$$

then for $i = 1, 2, \ldots, \theta/\alpha+n$*,* $N^{(i)} \in \{0, 1, 2, \ldots\}$ *are independent and identical geometric random variables with parameter* q*, where*

$$q = q(Z, R_1, \ldots, R_{n-1}) := 1 - \frac{A}{\pi \csc(\pi\alpha)}. \tag{2.6}$$

Furthermore, for every $i = 1, 2, \ldots, \theta/\alpha+n$*,* $T_0^{(i)} \in (0,1)$ *is a random variable with density*

$$h_0(x|Z, R_1, \ldots, R_{n-1}) = \frac{e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)x} x^{\alpha-1}}{\int_0^1 e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)y} y^{\alpha-1} dy} \quad \textit{for} \quad 0 < x < 1; \tag{2.7}$$

and $T_1^{(i)}, T_2^{(i)}, \ldots$ *are independent and identically distributed random variables with* $T_j^{(i)} \overset{\mathcal{D}}{=} G+1$*,* $j = 1, 2, \ldots, N^{(i)}$*, where* $G \in (0,1)$ *is a random variable with density*

$$h(u \mid Z, R_1, \ldots, R_{n-1}) = \frac{e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)(u+1)} \frac{u^{-\alpha}-u^{\alpha}}{u+1}}{A} \quad \textit{for} \quad 0 < u < 1. \tag{2.8}$$

Proof. From equation (2.3), we know that

$$\begin{aligned}
\mathbb{E}_{\alpha,\theta}\left(\prod_{k=1}^{n} e^{-\beta_k \frac{1}{V_k}}\right) &= c_{\alpha,\theta}\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta} V_1^{\theta} \prod_{k=1}^{n} e^{-\beta_k \frac{1}{V_k}}\right) \\
&= c_{\alpha,\theta}\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta} V_1^{\theta} \prod_{k=1}^{n} e^{-\beta_k \frac{1}{V_1}\left(\prod_{j=1}^{k-1} R_j^{-1}\right)}\right),
\end{aligned}$$

where $R_j := V_{j+1}/V_j$ and $c_{\alpha,\theta}$ is defined as (2.2).

Since $\theta > 0$ and $V_1 > 0$, the Gamma density implies $V_1^{\theta} = \int_0^{\infty} \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz$, then

$$\mathbb{E}_{\alpha,\theta}\left(\prod_{k=1}^{n} e^{-\beta_k \frac{1}{V_k}}\right) = c_{\alpha,\theta}\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}\left(\int_0^{\infty} \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz\right) \prod_{k=1}^{n} e^{-\beta_k \frac{1}{V_1}\left(\prod_{j=1}^{k-1} R_j^{-1}\right)}\right). \tag{2.9}$$

As in the proof of Theorem 2.2, we condition on $(\Delta_1^{-\alpha}, R_1, \dots, R_{n-1})$ under $\mathbb{P}_{\alpha,0}$, then

$$
\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}\left(\int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz\right)\prod_{k=1}^n e^{-\beta_k \frac{1}{V_1}(\prod_{j=1}^{k-1} R_j^{-1})}\right)\\
&= \int_0^\infty \int_0^1 \cdots \int_0^1 \mathbb{E}_{\alpha,0}\left(\int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz \prod_{k=1}^n e^{-\beta_k \frac{1}{V_1}(\prod_{j=1}^{k-1} r_j^{-1})} \mid \Delta_1^{-\alpha}, R_1, \dots, R_{n-1}\right)\\
&\qquad \times f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1} \mid w) w^{\frac{\theta}{\alpha}} e^{-w} dr_1 \dots dr_{n-1} dw\\
&= \int_0^\infty \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} \mathbb{E}_{\alpha,0}\left(e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\frac{1}{V_1}} \mid \Delta_1^{-\alpha}, R_1, \dots, R_{n-1}\right) dz\\
&\qquad \times \alpha^{n-1} w^{\frac{\theta}{\alpha}+n-1} e^{-w\prod_{j=1}^{n-1} r_j^{-\alpha}} \left(\prod_{j=1}^{n-1} r_j^{-(n-j)\alpha-1}\right) dr_1 \dots dr_{n-1} dw,
\end{aligned}
$$

where $f_{R_1,\dots,R_{n-1}}(r_1,\dots,r_{n-1} \mid w)$ is given in (2.4).

Using the decomposition (2.1) for $\frac{1}{V_1}$ under the probability measure $\mathbb{P}_{\alpha,0}$, we get

$$
\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}\left(\int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz\right)\prod_{k=1}^n e^{-\beta_k \frac{1}{V_1}(\prod_{j=1}^{k-1} R_j^{-1})}\right)\\
&= \int_0^\infty \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)}\\
&\quad \times \mathbb{E}_{\alpha,0}\left(e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\left(\prod_{j=1}^{n-1} r_j\right)\Sigma_n} \mid \Delta_1^{-\alpha}, R_1, \dots, R_{n-1}\right) dz\\
&\quad \times \alpha^{n-1} w^{\frac{\theta}{\alpha}+n-1} e^{-w\prod_{j=1}^{n-1} r_j^{-\alpha}} \left(\prod_{j=1}^{n-1} r_j^{-(n-j)\alpha-1}\right) dr_1 \dots dr_{n-1} dw.
\end{aligned}
$$

The distribution of $\Sigma_n \mid \Delta_1^{-\alpha}, R_1, \dots, R_{n-1}$ under $\mathbb{P}_{\alpha,0}$ has been specified in Lemma 2.1, hence we can calculate its Laplace transform using the Lévy-Khintchine representation,

$$
\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}\left(\int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz\right)\prod_{k=1}^n e^{-\beta_k \frac{1}{V_1}(\prod_{j=1}^{k-1} R_j^{-1})}\right)\\
&= \int_0^\infty \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)}\\
&\quad \times e^{-w(\prod_{j=1}^{n-1} r_j^{-\alpha})\int_0^1 (1-e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\left(\prod_{j=1}^{n-1} r_j\right)x})\alpha x^{-\alpha-1} dx} dz\\
&\quad \times \alpha^{n-1} w^{\frac{\theta}{\alpha}+n-1} e^{-w\prod_{j=1}^{n-1} r_j^{-\alpha}} \left(\prod_{j=1}^{n-1} r_j^{-(n-j)\alpha-1}\right) dr_1 \dots dr_{n-1} dw.
\end{aligned}
$$

Next, we carry out the integral with respect to w using a Gamma density; it follows that

$$
\begin{aligned}
&\mathbb{E}_{\alpha,0}\left(\Delta_1^{-\theta}\left(\int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\frac{z}{V_1}} dz\right)\prod_{k=1}^n e^{-\beta_k \frac{1}{V_1}(\prod_{j=1}^{k-1} R_j^{-1})}\right)\\
&= \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)}\\
&\quad \times \frac{\Gamma(\frac{\theta}{\alpha}+n)\alpha^{n-1}\left(\prod_{j=1}^{n-1} r_j^{j\alpha+\theta-1}\right)}{\left(1+\int_0^1 (1-e^{-\left(z+\sum_{k=1}^n \beta_k(\prod_{j=1}^{k-1} r_j^{-1})\right)\left(\prod_{j=1}^{n-1} r_j\right)x})\alpha x^{-\alpha-1} dx\right)^{\frac{\theta}{\alpha}+n}} dz dr_1 \dots dr_{n-1}.
\end{aligned}
\tag{2.10}
$$

We focus on the fraction in the integrand, denoted by

$$I := \frac{1}{1+\int_0^1 (1-e^{-\left(z+\sum_{k=1}^n \beta_k (\prod_{j=1}^{k-1} r_j^{-1})\right)(\prod_{j=1}^{n-1} r_j)x})\alpha x^{-\alpha-1}dx}.$$

For the denominator of I, we integrate by parts; then we multiply both the numerator and denominator of I by $\int_0^1 e^{-(z+\sum_{k=1}^n \beta_k (\prod_{j=1}^{k-1} r_j^{-1}))(\prod_{j=1}^{n-1} r_j)x} x^{\alpha-1}dx$. We also divide both the numerator and denominator of I by $\pi\csc(\pi\alpha)$, it follows that

$$\begin{aligned}
I &= \frac{\int_0^1 e^{-\left(z+\sum_{k=1}^n \beta_k (\prod_{j=1}^{k-1} r_j^{-1})\right)(\prod_{j=1}^{n-1} r_j)x} x^{\alpha-1}dx}{\pi\csc(\pi\alpha) - \int_0^1 e^{-\left(z+\sum_{k=1}^n \beta_k (\prod_{j=1}^{k-1} r_j^{-1})\right)(\prod_{j=1}^{n-1} r_j)(u+1)} \frac{u^{-\alpha}-u^{\alpha}}{u+1}du} \\
&= \frac{\frac{B}{\pi\csc(\pi\alpha)} \frac{1}{(1-\frac{A}{\pi\csc(\pi\alpha)})}(1-\frac{A}{\pi\csc(\pi\alpha)}) \int_0^1 e^{-\sum_{k=1}^n \beta_k \left(\prod_{j=k}^{n-1} r_j\right)x} \frac{e^{-z\left(\prod_{j=1}^{n-1} r_j\right)x} x^{\alpha-1}}{B}dx}{1-\frac{A}{\pi\csc(\pi\alpha)}\int_0^1 e^{-\sum_{k=1}^n \beta_k \left(\prod_{j=k}^{n-1} r_j\right)(u+1)} \frac{e^{-z\left(\prod_{j=1}^{n-1} r_j\right)(u+1)}\frac{u^{-\alpha}-u^{\alpha}}{u+1}}{A}du},
\end{aligned}$$

where we have defined

$$\begin{aligned}
A = A(z, r_1, \dots, r_{n-1}) &:= \int_0^1 e^{-z\left(\prod_{j=1}^{n-1} r_j\right)(v+1)} \frac{v^{-\alpha}-v^{\alpha}}{v+1}dv, \\
B = B(z, r_1, \dots, r_{n-1}) &:= \int_0^1 e^{-z\left(\prod_{j=1}^{n-1} r_j\right)y} y^{\alpha-1}dy.
\end{aligned}$$

Also, let q, $h_0(x \mid z, r_1, \dots, r_{n-1})$ and $h(u \mid z, r_1, \dots, r_{n-1})$ be defined as (2.6), (2.7) and (2.8), then I can be written as

$$\begin{aligned}
I &= \frac{B}{\pi\csc(\pi\alpha)-A} \times \frac{q\int_0^1 e^{-\sum_{k=1}^n \beta_k \left(\prod_{j=k}^{n-1} r_j\right)x} h_0(x \mid z, r_1, \dots, r_{n-1})dx}{1-(1-q)\int_0^1 e^{-\sum_{k=1}^n \beta_k \left(\prod_{j=k}^{n-1} r_j\right)(u+1)} h(u \mid z, r_1, \dots, r_{n-1})du} \\
&= \frac{1}{1+\int_0^1 (1-e^{-z\left(\prod_{j=1}^{n-1} r_j\right)x})\alpha x^{-\alpha-1}dx} \times \mathbb{E}\left(e^{-\sum_{k=1}^n \beta_k (\prod_{j=k}^{n-1} r_j)(\sum_{j=0}^N T_j)}\right),
\end{aligned}$$

where $N \in \{0, 1, 2, \dots\}$ is a geometric random variable with parameter q. Taking this back to equation (2.10) and using equation (2.9), we get the joint Laplace transform of $(\frac{1}{V_1}, \dots, \frac{1}{V_n})$ under $\mathbb{P}_{\alpha,\theta}$,

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}\left(e^{-\beta_1 \frac{1}{V_1}} \dots e^{-\beta_n \frac{1}{V_n}}\right) \\
&= \int_0^1 \cdots \int_0^1 \int_0^\infty e^{-\sum_{k=1}^n \beta_k (\prod_{j=1}^{k-1} r_j^{-1})\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)} \\
&\quad \times \left(\mathbb{E}\left(e^{-\sum_{k=1}^n \beta_k (\prod_{j=k}^{n-1} r_j)(\sum_{j=0}^N T_j)}\right)\right)^{\frac{\theta}{\alpha}+n} \frac{\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}}{\Gamma(\theta)} z^{\theta-1} \\
&\quad \times \frac{\left(\prod_{j=1}^{n-1}(j\alpha+\theta) r_j^{j\alpha+\theta-1}\right) e^{-z\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)}}{\left(1+\int_0^1 (1-e^{-z\left(\prod_{j=1}^{n-1} r_j\right)x})\alpha x^{-\alpha-1}dx\right)^{\frac{\theta}{\alpha}+n}} dz dr_1 \dots dr_{n-1},
\end{aligned}$$

and the theorem is a direct consequence of this result. □

3 Exact simulation algorithms

We start this section with introducing the subordinator algorithm for the random vector $(V_1, \dots, V_n)$. We call it the 'subordinator algorithm' because it is based on the exact simulation algorithm of truncated subordinator. It is Algorithm 4.3 of [5], which we refer to as Algorithm(α, t) and attach the full steps in Appendix B.

Algorithm 3.1 (Subordinator algorithm)**.** For $\alpha \in (0,1)$ and $\theta \geq 0$, the exact simulation algorithm for $(V_1, V_2, \ldots, V_n)$ is the following.

---

1. Initialize $\alpha \in (0,1)$, $\theta \geq 0$ and $n \geq 2$.
2. Sample from the random vector $(R_1, \ldots, R_{n-1}, Y, \Sigma_n)$ via the following steps.

   (a) Generate a Gamma random variable Y by setting

   $$Y \sim \text{Gamma}\left(\frac{\theta}{\alpha} + n, 1\right).$$

   (b) For $j = 1, \ldots, n-1$, generate a Beta random variable R_j by setting

   $$R_j \sim \text{Beta}(j\alpha + \theta, 1).$$

   (c) Generate a truncated subordinator Σ_n by setting

   $$\Sigma_n = \text{Algorithm}(\alpha, \Gamma(1-\alpha)Y).$$

   (d) Set $V \sim U[0,1]$, if

   $$V \leq \frac{1}{\left(1 + R_1 + R_1 R_2 + \cdots + \prod_{j=1}^{n-1} R_j + \left(\prod_{j=1}^{n-1} R_j\right) \Sigma_n\right)^{\theta}},$$

   accept these candidates and go to Step 3; Otherwise go back to Step 2(a).
3. For $k = 1, \ldots, n$ output

$$V_k = \frac{1}{\left(1 + R_1 + R_1 R_2 + \cdots + \prod_{j=1}^{n-1} R_j + \left(\prod_{j=1}^{n-1} R_j\right) \Sigma_n\right)} \prod_{j=1}^{k-1} R_j$$

Proof. We apply the acceptance rejection method to sample from the random vector $(\Delta_1^{-\alpha}, R_1, \ldots, R_{n-1}, \Sigma_n)$ given in Theorem 2.2 with the envelope

$$g^*(r_1, \ldots, r_{n-1}, y, x) := \\ \left(\prod_{j=1}^{n-1} (j\alpha + \theta) r_j^{j\alpha+\theta-1}\right) \frac{1}{\Gamma(\frac{\theta}{\alpha} + n)} y^{\frac{\theta}{\alpha}+n-1} e^{-y} f_{\Sigma_n}(x \mid y) dx dy dr_1 \ldots dr_{n-1},$$

where $f_{\Sigma_n}(x \mid y)$ denotes the density of a subordinator with truncated Lévy measure $\alpha x^{-\alpha-1} \mathbb{1}_{\{0<x<1\}} dx$ at time y, for $x > 0$, $y > 0$ and $0 < r_j < 1$, $j = 1, \ldots, n-1$. To sample from the envelope, we generate independent $\text{Gamma}(\frac{\theta}{\alpha} + n, 1)$ and $\text{Beta}(j\alpha + \theta, 1)$ random variables in Step 2(a, b), then simulate the subordinator via Step 2(c).

To justify the acceptance rejection algorithm, we re-parametrize the envelope with a new variable $w := y \prod_{j=1}^{n-1} r_j^{\alpha}$, $w > 0$, then we have

$$g^*(w, r_1, \ldots, r_{n-1}, x) = f_{\Sigma_n}\left(x \mid w \prod_{j=1}^{n-1} r_j^{-\alpha}\right) \frac{\alpha^{n-1}}{\Gamma(\frac{\theta}{\alpha} + 1)} w^{\frac{\theta}{\alpha}+n-1} e^{-w(\prod_{j=1}^{n-1} r_j^{-\alpha})} \\ \times \left(\prod_{j=1}^{n-1} r_j^{-(n-j)\alpha-1}\right) dx dw dr_1 \ldots dr_{n-1}.$$

Since $\theta \geq 0$, we know

$$\begin{aligned}
&\max_{w>0,0<r_1<1,\ldots,0<r_{n-1}<1,x>0} \frac{g(w,r_1,\ldots,r_{n-1},x)}{g^*(w,r_1,\ldots,r_{n-1},x)} \\
=&\max_{w>0,0<r_1<1,\ldots,0<r_{n-1}<1,x>0} \frac{\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}}{\left(1+\left(r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)+\left(\prod_{j=1}^{n-1} r_j\right)x\right)^{\theta}} \\
=&\,\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}},
\end{aligned}$$

then we accept the candidates via Step 2(d). □

Next, we consider a special case when $\theta > 0$ and θ/α is a positive integer, and develop the compound geometric representation algorithm for the random vector $(V_1,\ldots,V_n)$.

Algorithm 3.2 (Compound geometric representation algorithm)**.** For $\alpha \in (0,1)$ and $\theta > 0$, if θ/α is a positive integer, the exact simulation algorithm for $(V_1,V_2,\ldots,V_n)$ is the following.

---

1. Initialize $\alpha \in (0,1)$, $\theta > 0$ and $n \geq 2$.
2. Sample from the random vector $(Z,R_1,\ldots,R_{n-1})$ via the following steps.
   (a) Generate a Gamma random variable Z by setting
   $$Z \sim \text{Gamma}(\theta,1).$$
   (b) For $j=1,2,\ldots,n-1$, generate a Beta random variable R_j by setting
   $$R_j \sim \text{Beta}(j\alpha+\theta,1).$$
   (c) Set $V \sim U[0,1]$, if
   $$V \leq \frac{e^{-Z\left(R_1+R_1R_2+\cdots+\prod_{j=1}^{n-1} R_j\right)}}{\left(1+\int_0^1 (1-e^{-Z(\prod_{j=1}^{n-1} R_j)x})\alpha x^{-\alpha-1}dx\right)^{\frac{\theta}{\alpha}+n}},$$
   accept these candidates; otherwise go back to Step 2(a).

   With the accepted candidates, calculate A and q numerically by setting
   $$A=\int_0^1 e^{-Z(\prod_{j=1}^{n-1} R_j)(v+1)}\frac{v^{-\alpha}-v^{\alpha}}{v+1}dv \quad \text{and} \quad q=1-\frac{A}{\pi\csc(\pi\alpha)},$$
   then go to Step 3.
3. For every $i=1,2,\ldots,\theta/\alpha+n$, execute the following Step (a), (b) and (c):
   (a) Generate a geometric random variable $N^{(i)}$ by setting
   $$N^{(i)} \sim \text{Geometric}(q).$$
   (b) Generate a random variable $T_0^{(i)}$ via the following Step i. and ii.:
      i. Generate a Beta random variable T_0^* by setting
      $$T_0^* \sim \text{Beta}(\alpha,1).$$
      ii. Set $V \sim U[0,1]$, if
      $$V \leq e^{-Z(\prod_{j=1}^{n-1} R_j)T_0^*},$$
      accept this candidate and set $T_0^{(i)} = T_0^*$; otherwise go back to 3.b.(i).

(c) If $N^{(i)} > 0$, generate $T_j^{(i)}$ by executing the following Step i. and ii. for every $j = 1, 2, \ldots, N^{(i)}$; otherwise skip this step:

i. Generate a random variable G^* whose density is

$$h^*(u) = \frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}} \frac{u^{-\alpha} - u^{\alpha}}{u+1} \quad \text{for} \quad 0 < u < 1,$$

using the algorithm provided in Appendix C.

ii. Set $V \sim U[0,1]$, if

$$V \leq e^{-Z(\prod_{j=1}^{n-1} R_j)G^*},$$

accept this candidate and set $G = G^*$, then set $T_j^{(i)} = G + 1$; otherwise go back to 3.c.(i).

4. For $k = 1, 2, \ldots, n$, output

$$V_k = \frac{1}{\left(1 + R_1 + \cdots + \prod_{j=1}^{n-1} R_j\right) \prod_{j=1}^{k-1} R_j^{-1} + \sum_{i=1}^{\frac{\theta}{\alpha}+n} \left(\left(\prod_{j=k}^{n-1} R_j\right) \left(\sum_{j=0}^{N^{(i)}} T_j^{(i)}\right)\right)}.$$

Proof. We apply the acceptance rejection method to sample from the random vector $(Z, R_1, \ldots, R_{n-1})$ given in Theorem 2.3 with the envelope

$$m^*(z, r_1, \ldots, r_{n-1}) := \left(\prod_{j=1}^{n-1} (j\alpha + \theta) r_j^{j\alpha+\theta-1}\right) \frac{1}{\Gamma(\theta)} z^{\theta-1} e^{-z},$$

for $z > 0$ and $0 < r_j < 1$, $j = 1, \ldots, n-1$. To sample from the envelope, we generate independent Gamma$(\theta, 1)$ and Beta$(j\alpha + \theta, 1)$ random variables via Step 2(a-b). Since $\frac{\theta}{\alpha} + n > 0$, we know

$$\begin{aligned}
&\max_{z>0, 0<r_1<1, \ldots, 0<r_{n-1}<1} \frac{m(z, r_1, \ldots, r_{n-1})}{m^*(z, r_1, \ldots, r_{n-1})} \\
&= \max_{z>0, 0<r_1<1, \ldots, 0<r_{n-1}<1} \frac{\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}} e^{-z\left(r_1 + r_1 r_2 + \cdots + \prod_{j=1}^{n-1} r_j\right)}}{\left(1 + \int_0^1 (1 - e^{-z\left(\prod_{j=1}^{n-1} r_j\right)x}) \alpha x^{-\alpha-1} dx\right)^{\frac{\theta}{\alpha}+n}} \\
&= \Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}},
\end{aligned}$$

then we accept the candidates via Step 2(c).

Next, we use the acceptance rejection method to sample from $T_0^{(i)}$ with the envelope $h_0^*(x) := \alpha x^{\alpha-1}$, for $0 < x < 1$. Since

$$\begin{aligned}
\max_{0<x<1} \frac{h_0(x \mid Z, R_1, \ldots, R_{n-1})}{h_0^*(x)} &= \max_{0<x<1} \frac{e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)x}}{\alpha \int_0^1 e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)y} y^{\alpha-1} dy} \\
&= \frac{1}{\alpha \int_0^1 e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)y} y^{\alpha-1} dy}
\end{aligned}$$

we accept the candidate via Step 3.b.(ii).

We also use the acceptance rejection method to sample from G with the envelope

$$h^*(u) = \frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}} \frac{u^{-\alpha} - u^{\alpha}}{u+1} \quad \text{for} \quad 0 < u < 1,$$

note that we can sample from $h^*(u)$ using the algorithm given in Appendix C. Since

$$\max_{0<u<1} \frac{h(u \mid Z, R_1, \dots, R_{n-1})}{h^*(u)} = \max_{0<u<1} \frac{\frac{1}{A} e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)(u+1)}}{\frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}}}$$
$$= \frac{1}{A}\left(\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}\right) e^{-Z\left(\prod_{j=1}^{n-1} R_j\right)},$$

we accept the candidate via Step 3.c.(ii). □

4 Numerical results

In this section we present some numerical results for Algorithm 3.1 and 3.2. We will use the expectation and covariance of $(V_1, \dots, V_n)$ as benchmarks to illustrate the accuracy of these algorithms. The complexity of the algorithms are also considered. The following theorem gives an expression for the moments of V_n.

Theorem 4.1 (Proposition 17 of [19]). *Let V_n be the n-th component of the $PD(\alpha, \theta)$ distribution. For $p > 0$,*

$$\mathbb{E}_{\alpha,\theta}(V_n^p) = \frac{\Gamma(1-\alpha)^{\frac{\theta}{\alpha}} \Gamma(\theta+1) \Gamma(\frac{\theta}{\alpha}+n)}{\Gamma(n)\Gamma(\theta+p)\Gamma(\frac{\theta}{\alpha}+1)} \int_0^\infty t^{p+\theta-1} e^{-t} \frac{\phi_\alpha(t)^{n-1}}{\psi_\alpha(t)^{\frac{\theta}{\alpha}+n}} dt,$$

where

$$\phi_\alpha(\lambda) := \alpha \int_1^\infty e^{-\lambda x} x^{-\alpha-1} dx \quad \text{and} \quad \psi_\alpha(\lambda) := \Gamma(1-\alpha)\lambda^\alpha + \phi_\alpha(\lambda).$$

Proof. See Proposition 17 of [19]. □

Next, we derive an expression for $\mathbb{E}_{\alpha,\theta}(V_m V_n)$.

Theorem 4.2 (Covariance). *For positive integers m and n, such that $1 \le m \le n$, let V_m and V_n be the m-th and n-th components of the $PD(\alpha, \theta)$ distribution, then*

$$\mathbb{E}_{\alpha,\theta}(V_m V_n)$$
$$= \frac{\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}\Gamma(\frac{\theta}{\alpha}+n)\alpha^{n-1}}{\Gamma(\frac{\theta}{\alpha}+1)} \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{y^{\theta+1} e^{-y}}{\Gamma(\theta+2)} \left(\prod_{j=1}^{m-1} r_j^{\theta+j\alpha+1}\right) \left(\prod_{j=m}^{n-1} r_j^{\theta+j\alpha}\right)$$
$$\times \frac{e^{-y\left(r_1 + r_1 r_2 + \cdots + \prod_{j=1}^{n-1} r_j\right)}}{\left(1 + \int_0^1 (1 - e^{-y(\prod_{j=1}^{n-1} r_j)x}) \alpha x^{-\alpha-1} dx\right)^{\frac{\theta}{\alpha}+n}} dy dr_1 \dots dr_{n-1}.$$

Proof. Since $(V_m V_n)^{-1} > 0$, we know $\int_0^\infty (V_m V_n)^{-1} e^{-(V_m V_n)^{-1}\beta} d\beta = 1$; it follows that

$$\mathbb{E}_{\alpha,\theta}(V_m V_n) = \mathbb{E}_{\alpha,\theta}\left(\int_0^\infty e^{-\frac{\beta}{V_m V_n}} d\beta\right) = \int_0^\infty \mathbb{E}_{\alpha,\theta}\left(e^{-\frac{\beta}{V_m V_n}}\right) d\beta. \tag{4.1}$$

We concentrate on the integrand first. As in the proof of Theorem 2.2, we change the probability measure to $\mathbb{P}_{\alpha,0}$ using (2.3) and condition on $(\Delta_1^{-\alpha}, R_1, \dots, R_{n-1})$, then

$$\mathbb{E}_{\alpha,\theta}\left(e^{-\frac{\beta}{V_m V_n}}\right) = c_{\alpha,\theta} \int_0^\infty \int_0^1 \cdots \int_0^1 \mathbb{E}_{\alpha,0}\left(V_1^\theta e^{-\frac{\beta}{V_m V_n}} \mid \Delta_1^{-\alpha}, R_1, \dots, R_{n-1}\right)$$
$$\times \alpha^{n-1} w^{\frac{\theta}{\alpha}+n-1} e^{-w \prod_{j=1}^{n-1} r_j^{-\alpha}} \left(\prod_{j=1}^{n-1} r_j^{-(n-j)\alpha-1}\right) dr_1 \dots dr_{n-1} dw.$$

Using the decomposition (2.1) for V_1, V_m and V_n under $\mathbb{P}_{\alpha,0}$, we get

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}\left(e^{-\frac{\beta}{V_m V_n}}\right)\\
&= c_{\alpha,\theta}\int_0^\infty\int_0^1\cdots\int_0^1\\
&\mathbb{E}_{\alpha,0}\left(\frac{e^{-\beta\left(1+\left(r_1+\cdots+\prod_{j=1}^{n-1}r_j\right)+\left(\prod_{j=1}^{n-1}r_j\right)\Sigma_n\right)^2\left(\prod_{j=1}^{m-1}r_j^{-1}\right)\left(\prod_{j=1}^{n-1}r_j^{-1}\right)}}{\left(1+\left(r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1}r_j\right)+\left(\prod_{j=1}^{n-1}r_j\right)\Sigma_n\right)^{\theta}}\mid \Delta_1^{-\alpha},R_1,\ldots,R_{n-1}\right)\\
&\times\alpha^{n-1}w^{\frac{\theta}{\alpha}+n-1}e^{-w\prod_{j=1}^{n-1}r_j^{-\alpha}}\left(\prod_{j=1}^{n-1}r_j^{-(n-j)\alpha-1}\right)dr_1\ldots dr_{n-1}dw.
\end{aligned}$$

Taking this into (4.1) and calculating the integration with respect to β, we have

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}(V_mV_n)\\
&= c_{\alpha,\theta}\int_0^\infty\int_0^1\cdots\int_0^1\\
&\mathbb{E}_{\alpha,0}\left(\frac{\left(\prod_{j=1}^{m-1}r_j\right)\left(\prod_{j=1}^{n-1}r_j\right)}{\left(1+\left(r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1}r_j\right)+\left(\prod_{j=1}^{n-1}r_j\right)\Sigma_n\right)^{\theta+2}}\mid \Delta_1^{-\alpha},R_1,\ldots,R_{n-1}\right)\\
&\times\alpha^{n-1}w^{\frac{\theta}{\alpha}+n-1}e^{-w\prod_{j=1}^{n-1}r_j^{-\alpha}}\left(\prod_{j=1}^{n-1}r_j^{-(n-j)\alpha-1}\right)dr_1\ldots dr_{n-1}dw.
\end{aligned}$$

Since $\theta+2>0$, we use a Gamma density to rewrite the denominator of the expression inside the conditional expectation, then rearrange the terms; it follows that

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}(V_mV_n)\\
&= c_{\alpha,\theta}\int_0^\infty\int_0^1\cdots\int_0^1\left(\prod_{j=1}^{m-1}r_j\right)\left(\prod_{j=1}^{n-1}r_j\right)\int_0^\infty\frac{1}{\Gamma(\theta+2)}y^{\theta+1}\\
&\quad\times e^{-y\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1}r_j\right)}\mathbb{E}_{\alpha,0}\left(e^{-y\left(\prod_{j=1}^{n-1}r_j\right)\Sigma_n}\mid\Delta_1^{-\alpha},R_1,\ldots,R_{n-1}\right)dy\\
&\quad\times\alpha^{n-1}w^{\frac{\theta}{\alpha}+n-1}e^{-w\prod_{j=1}^{n-1}r_j^{-\alpha}}\left(\prod_{j=1}^{n-1}r_j^{-(n-j)\alpha-1}\right)dr_1\ldots dr_{n-1}dw.
\end{aligned}$$

Then we calculate the Laplace transform of $\Sigma_n \mid \Delta_1^{-\alpha},R_1,\ldots,R_{n-1}$ using the Lévy-Khintchine representation given in Lemma 2.1,

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}(V_mV_n)\\
&= c_{\alpha,\theta}\int_0^\infty\int_0^1\cdots\int_0^1\left(\prod_{j=1}^{m-1}r_j\right)\left(\prod_{j=1}^{n-1}r_j\right)\int_0^\infty\frac{1}{\Gamma(\theta+2)}y^{\theta+1}\\
&\quad\times e^{-y\left(1+r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1}r_j\right)}e^{-w\left(\prod_{j=1}^{n-1}r_j^{-\alpha}\right)\int_0^1\left(1-e^{-y\left(\prod_{j=1}^{n-1}r_j\right)x}\right)\alpha x^{-\alpha-1}dx}dy\\
&\quad\times\alpha^{n-1}w^{\frac{\theta}{\alpha}+n-1}e^{-w\prod_{j=1}^{n-1}r_j^{-\alpha}}\left(\prod_{j=1}^{n-1}r_j^{-(n-j)\alpha-1}\right)dr_1\ldots dr_{n-1}dw.
\end{aligned}$$

Finally, we carry out the integration with respect to w using a Gamma density and rearrange the terms; it follows that

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta}(V_m V_n) \\
&= \frac{\Gamma(\theta+1)\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}\Gamma(\frac{\theta}{\alpha}+n)\alpha^{n-1}}{\Gamma(\frac{\theta}{\alpha}+1)} \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{y^{\theta+1}e^{-y}}{\Gamma(\theta+2)} \left(\prod_{j=1}^{m-1} r_j^{\theta+j\alpha+1}\right) \left(\prod_{j=m}^{n-1} r_j^{\theta+j\alpha}\right) \\
&\quad \times \frac{e^{-y\left(r_1+r_1r_2+\cdots+\prod_{j=1}^{n-1} r_j\right)}}{\left(1+\int_0^1 (1-e^{-y(\prod_{j=1}^{n-1} r_j)x})\alpha x^{-\alpha-1}dx\right)^{\frac{\theta}{\alpha}+n}} dy dr_1 \ldots dr_{n-1},
\end{aligned}$$

and the theorem is proved. □

Next, we present numerical results of the algorithms.

4.1 Sample average

We illustrate the accuracy of our algorithms by comparing the expectation to the sample average. Consider the first 10 components, $(V_1, \ldots, V_{10})$, of the $PD(\alpha,\theta)$ distribution. We use Theorem 4.1 to calculate $\mathbb{E}_{\alpha,\theta}(V_k)$, $k = 1, \ldots, 10$ numerically. Then we generate samples from the random vector using Algorithm 1.2, 3.1 and 3.2, and calculate the sample average of V_k. The results are recorded in Table 1, 2 and 3, we see from the tables that the algorithms can generate exact samples of the random vector.

4.2 Covariance

We also present numerical results for the covariance between different components of the random vector $(V_1, \ldots, V_5)$, for simplicity we focus on $\mathbb{E}_{\alpha,\theta}(V_m V_n)$ only. We use Theorem 4.2 to calculate $\mathbb{E}_{\alpha,\theta}(V_m V_n)$ for $1 \le m \le n \le 5$ numerically, then generate samples from $V_m V_n$ using Algorithm 1.2, 3.1 and 3.2 and calculate their averages. The results are recorded in Table 4, 5 and 6, the tables show that our algorithms are accurate in estimating the covariance.

4.3 Complexity

We are also interested in the complexity of the algorithms, which indicates how many resources the algorithms will costume. Instead of CPU times, we first consider the total number of random variables generated by the algorithms, because it is consistent and does not depend on the performance of the computer.

From the definition we know the complexity of Algorithm 1.2 is m, that is, the algorithm will generate m number of Beta random variables in total. In the previous subsection we have taken $n = 10$ and $m = 50$.

We record the average number of random variables generated by Algorithm 3.1 and 3.2 for $(V_1, \ldots, V_{10})$ in Table 7 and 8. From the tables we see that when θ/α is an integer and relatively large, Algorithm 3.2 has a lower complexity than Algorithm 3.1, this is because the truncated subordinator is not involved in Algorithm 3.2.

4.4 CPU time

We record the CPU times of Algorithm 1.2, 3.1 and 3.2 for 10^4 samples of $(V_1, \ldots, V_{10})$ in Table 9, 10 and 11. The experiments are implemented on an AMD Ryzen 7 4800U CPU@1.80GHz processor, 16.00GB RAM, Windows 10, 64-bit Operating System and performed in Matlab R2019b. The tables show that when applicable, the compound geometric representation algorithm is preferable in general.

Table 1: Expectation and sample average of V_k, $k = 1, \dots, 10$ for $\alpha = \frac{1}{3}$ and $\theta = \frac{1}{3}$, the sample size is 10^5. For the trivial algorithm $n = 10$ and $m = 50$.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}
expectation	0.6273	0.1695	0.0729	0.0386	0.0230	0.0149	0.0102	0.0073	0.0054	0.0041
trivial algorithm	0.6265	0.1700	0.0737	0.0393	0.0235	0.0154	0.0106	0.0076	0.0057	0.0043
subordinator algorithm	0.6273	0.1696	0.0734	0.0391	0.0235	0.0153	0.0105	0.0076	0.0056	0.0043
compound algorithm	0.6282	0.1695	0.0732	0.0390	0.0234	0.0153	0.0105	0.0076	0.0056	0.0043

Table 2: Expectation and sample average of V_k, $k = 1, \dots, 10$ for $\alpha = \frac{1}{3}$ and $\theta = \frac{1}{5}$, the sample size is 10^5. For the trivial algorithm $n = 10$ and $m = 50$. The compound geometric representation algorithm is not applicable because θ/α is not an integer.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}
expectation	0.6727	0.1598	0.0648	0.0332	0.0195	0.0125	0.0085	0.0061	0.0046	0.0035
trivial algorithm	0.6715	0.1592	0.0648	0.0331	0.0194	0.0124	0.0084	0.0060	0.0044	0.0033
subordinator algorithm	0.6725	0.1589	0.0647	0.0330	0.0193	0.0123	0.0084	0.0060	0.0044	0.0033
Algorithm compound	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A

Table 3: Expectation and sample average of V_k, $k = 1, \dots, 10$ for $\alpha = \frac{2}{3}$ and $\theta = \frac{4}{3}$, the sample size is 10^5. For the trivial algorithm $n = 10$ and $m = 50$.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}
expectation	0.2873	0.1204	0.0718	0.0492	0.0364	0.0284	0.0231	0.0195	0.0169	0.0150
trivial algorithm	0.2879	0.1204	0.0721	0.0498	0.0372	0.0292	0.0237	0.0197	0.0166	0.0142
subordinator algorithm	0.2879	0.1205	0.0724	0.0500	0.0374	0.0294	0.0240	0.0200	0.0171	0.0148
Algorithm compound	0.2875	0.1205	0.0723	0.0500	0.0374	0.0295	0.0240	0.0200	0.0171	0.0148

Table 4: Expectation and sample average of $V_m V_n$, $1 \le m \le n \le 5$ for $\alpha = \frac{1}{2}$ and $\theta = \frac{1}{2}$ with the trivial algorithm, the sample size is 10^5. The data are in the format (a, b) where a represents the expectation and b represents the sample average.

	V_1	V_2	V_3	V_4	V_5
V_1	0.2830, 0.2841	0.0686, 0.0687	0.0324, 0.0322	0.0192, 0.0191	0.0129, 0.0127
V_2		0.0329, 0.0330	0.0149, 0.0147	0.0086, 0.0085	0.0057, 0.0056
V_3			0.0090, 0.0090	0.0052, 0.0052	0.0034, 0.0034
V_4				0.0036, 0.0036	0.0023, 0.0023
V_5					0.0017, 0.0017

Table 5: Expectation and sample average of $V_m V_n$, $1 \le m \le n \le 5$ for $\alpha = \frac{1}{2}$ and $\theta = \frac{1}{2}$ with the subordinator algorithm, the sample size is 10^5. The data are in the format (a, b) where a represents the expectation and b represents the sample average.

	V_1	V_2	V_3	V_4	V_5
V_1	0.2830, 0.2837	0.0686, 0.0688	0.0324, 0.0321	0.0192, 0.0191	0.0129, 0.0127
V_2		0.0329, 0.0331	0.0149, 0.0147	0.0086, 0.0085	0.0057, 0.0057
V_3			0.0090, 0.0091	0.0052, 0.0052	0.0034, 0.0034
V_4				0.0036, 0.0036	0.0023, 0.0023
V_5					0.0017, 0.0017

Table 6: Expectation and sample average of V_mV_n, $1 \le m \le n \le 5$ for $\alpha = \frac{1}{2}$ and $\theta = \frac{1}{2}$ with the compound geometric representation algorithm, the sample size is 10^5. The data are in the format (a, b) where a represents the expectation and b represents the sample average.

	V_1	V_2	V_3	V_4	V_5
V_1	0.2830, 0.2830	0.0686, 0.0685	0.0324, 0.0323	0.0192, 0.0191	0.0129, 0.0127
V_2		0.0329, 0.0330	0.0149, 0.0148	0.0086, 0.0086	0.0057, 0.0057
V_3			0.0090, 0.0091	0.0052, 0.0052	0.0034, 0.0034
V_4				0.0036, 0.0036	0.0023, 0.0023
V_5					0.0017, 0.0017

Table 7: Average number of random numbers (rounding to the nearest integer) generated by the subordinator algorithm for $(V_1, \ldots, V_{10})$, the sample size is 10^4. The data are in the format $a + b + c$ where a, b, c represent the number of uniform, Gamma and Beta random variables respectively.

	$\theta = 0.3$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 1.5$	$\theta = 1.6$
$\alpha = 0.3$	175+56+10	216+69+12	421+134+21	952+304+44	1144+365+52
$\alpha = 0.4$	177+55+11	224+70+13	458+144+24	1076+339+53	1302+410+63
$\alpha = 0.5$	192+59+11	251+77+14	537+166+28	1355+418+67	1664+514+80
$\alpha = 0.8$	435+123+15	647+183+21	1993+565+62	7336+2081+217	9650+2737+283

Table 8: Average number of random numbers (rounding to the nearest integer) generated by the compound geometric representation algorithm for $(V_1, \ldots, V_{10})$, the sample size is 10^4. The data are in the format $a + b + c + d$ where a, b, c, d represent the number of uniform, Gamma, Beta and geometric random variables respectively.

	$\theta = 0.3$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 1.5$	$\theta = 1.6$
$\alpha = 0.3$	16+1+24+11	N/A	N/A	25+5+62+15	N/A
$\alpha = 0.4$	N/A	N/A	N/A	N/A	30+7+82+14
$\alpha = 0.5$	N/A	26+2+32+11	29+3+47+12	36+7+87+13	N/A
$\alpha = 0.8$	N/A	N/A	N/A	N/A	115+30+318+12

Table 9: CPU time (in seconds) of the trivial algorithm for $(V_1, \ldots, V_{10})$, the sample size is 10^4.

	$\theta = 0.3$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 1.5$	$\theta = 1.6$
$\alpha = 0.3$	0.628853	0.698557	0.656535	0.667601	0.583066
$\alpha = 0.4$	0.572390	0.583742	0.570028	0.603642	0.553737
$\alpha = 0.5$	0.651893	0.557833	0.545066	0.561283	0.587276
$\alpha = 0.8$	0.545630	0.560253	0.558486	0.579318	0.610685

Table 10: CPU time (in seconds) of the subordinator algorithm for $(V_1, \ldots, V_{10})$, the sample size is 10^4.

	$\theta = 0.3$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 1.5$	$\theta = 1.6$
$\alpha = 0.3$	2.919347	3.407936	5.905766	12.709555	16.287444
$\alpha = 0.4$	4.129321	6.928111	11.446559	23.492180	25.992863
$\alpha = 0.5$	8.535146	10.143674	15.590900	31.600125	38.378702
$\alpha = 0.8$	18.097517	26.450564	85.147071	283.616955	372.585268

Table 11: CPU time (in seconds) of the compound geometric representation algorithm for $(V_1, \ldots, V_{10})$, the sample size is 10^4.

	$\theta = 0.3$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 1.5$	$\theta = 1.6$
$\alpha = 0.3$	7.394960	N/A	N/A	10.042571	N/A
$\alpha = 0.4$	N/A	N/A	N/A	N/A	12.311978
$\alpha = 0.5$	N/A	10.326533	11.250926	13.745608	N/A
$\alpha = 0.8$	N/A	N/A	N/A	N/A	42.778541

A Joint density of the random vector $(R_1, \ldots, R_{n-1} \mid \Delta_1^{-\alpha} = w)$.

From Proposition 10 of [19], we know that under the probability measure $\mathbb{P}_{\alpha,0}$,

$$R_k := \frac{\Delta_{k+1}}{\Delta_k} \stackrel{law}{=} \frac{(\sum_{i=1}^{k+1} \mathbf{e}_i)^{-1/\alpha}}{(\sum_{i=1}^{k} \mathbf{e}_i)^{-1/\alpha}} \quad \text{for} \quad k = 1, 2, \ldots,$$

where $\mathbf{e}_i$ are independent standard exponential random variables. In particular, it is known that $\Delta_1^{-\alpha} \stackrel{\mathcal{D}}{=} \mathbf{e}_1$, see Lemma 24 of [19].

On the other hand, define

$$R_k(\lambda) := \left(\frac{\sum_{i=1}^{k-1} \mathbf{e}_i + \lambda^\alpha}{\sum_{i=1}^{k} \mathbf{e}_i + \lambda^\alpha} \right)^{\frac{1}{\alpha}} \quad \text{for} \quad k = 1, 2, \ldots,$$

then Lemma 3.2 of [10] implies that $(R_1(\lambda), \ldots, R_{n-1}(\lambda))$ has the joint density

$$f_{R_1,\ldots,R_{n-1}}(r_1, \ldots, r_{n-1}) = \alpha^{n-1} \lambda^{(n-1)\alpha} e^{\lambda^\alpha} e^{-\lambda^\alpha \prod_{j=1}^{n-1} r_j^{-\alpha}} \prod_{j=1}^{n-1} r_j^{-(n-j)\alpha - 1}. \tag{A.1}$$

We set $\lambda = \Delta_1^{-1}$, then $\lambda^\alpha \stackrel{\mathcal{D}}{=} \mathbf{e}_1$ and

$$R_k(\lambda) \stackrel{law}{=} \left(\frac{\sum_{i=1}^{k} \mathbf{e}_i}{\sum_{i=1}^{k+1} \mathbf{e}_i} \right)^{\frac{1}{\alpha}} = \frac{(\sum_{i=1}^{k+1} \mathbf{e}_i)^{-1/\alpha}}{(\sum_{i=1}^{k} \mathbf{e}_i)^{-1/\alpha}},$$

hence the random vector $(R_1(\lambda), \ldots, R_{n-1}(\lambda) \mid \lambda = \Delta_1^{-1})$ has the identical distribution as $(R_1, \ldots, R_{n-1})$, and we obtain the joint density (2.4) by setting $\lambda = w^{\frac{1}{\alpha}}$ in (A.1).

B Exact simulation of truncated subordinator.

In this appendix we attach the Algorithm 4.3 of [5], the algorithm exactly generates samples from a truncated subordinator σ_t with Laplace transform

$$\mathbb{E}(e^{-\beta\sigma_t}) = \exp\left(-\frac{t}{\Gamma(1-\alpha)} \int_0^1 (1 - e^{-\beta z}) \alpha z^{-\alpha-1} dz \right).$$

We first present Algorithm 4.1 and 4.2 of [5], there will be used in the Algorithm 4.3.

Algorithm B.1 (Algorithm 4.1 of [5])**.** Exact simulation of (T, W).

1. Set $\xi = \Gamma(1-\alpha)^{-1}$; $A_0 = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$.
2. minimise $C(\lambda) = A_0 e^{\xi^{\frac{1}{\alpha}} \lambda^{1-\frac{1}{\alpha}} \alpha(1-\alpha)^{\frac{1}{\alpha}-1}} (A_0 - \lambda)^{\alpha-2}$; record critical value λ^*.
3. set $C = C(\lambda^*)$.
4. repeat {
5. sample $U \sim U[0, \pi]$; $U_1 \sim U[0, 1]$,
6. set $Y = 1 - U_1^{\frac{1}{1-\alpha}}$; $A_U = [\sin^\alpha(\alpha U) \sin^{1-\alpha}((1-\alpha)U)/\sin(U)]^{\frac{1}{1-\alpha}}$,
7. $R \sim \Gamma(2-\alpha, A_u - \lambda)$; $V \sim U[0, 1]$
8. if $(V \leq A_U e^{\xi R^{1-\alpha} Y^\alpha} e^{-\lambda^* R} (A_U - \lambda^*)^{\alpha-2} Y^{\alpha-1} (1 - (1-Y)^\alpha)/C)$, break
9. }
10. sample $U_2 \sim U[0, 1]$,
11. set $T = R^{1-\alpha} Y^\alpha$; $W = Y - 1 + [(1-Y)^{-\alpha} - U_2((1-Y)^{-\alpha} - 1)]^{-\frac{1}{\alpha}}$
12. return (T, W).

Algorithm B.2 (Algorithm 4.2 of [5])**.** Exact simulation of $\{Z_t|T>t\}$.

1. sample $U_1 \sim U[0,\pi]$
2. set $A_{U_1} = [\sin^{\alpha}(\alpha U_1)\sin^{1-\alpha}((1-\alpha)U_1)/\sin(U_1)]^{\frac{1}{1-\alpha}}$
3. repeat {
4. sample $U_2 \sim U[0,1]$
5. set
$$Z = \left[-\frac{\log(U_2)}{A_{U_1} t^{\frac{1}{1-\alpha}}}\right]^{-\frac{1-\alpha}{\alpha}}$$
6. if $(Z<1)$ break
7. }
8. return Z

Next we provide the Algorithm 4.3 of [5].

Algorithm B.3 (Algorithm 4.3 of [5])**.** Exact simulation of a truncated stable $Z_t \sim TS(\alpha, t)$.

1. set $Z=0$; $S=0$
2. repeat {
3. sample (T,W) via Algorithm 4.1
4. set $S=S+T$, $Z=Z+1+W$
5. if $(S>t)$ break
6. }
7. set $Z_{S-T} = Z-1-W$
8. sample $Z_{t-(S-T)}$ via Algorithm 4.2
9. return $Z_{S-T} + Z_{t-(S-T)}$

Proof. For the proof as well as the motivation of these algorithms, see [5]. □

C Simulation of G^*.

We give the algorithm for sampling from the density $h^*(u)$.

Algorithm C.1. Let G^* be a random variable with the probability density function
$$h^*(u) = \frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}} \frac{u^{-\alpha} - u^{\alpha}}{u+1} \quad \text{for} \quad 0<u<1,$$
then G^* can be generated via the following steps.

1. Numerically maximising
$$C(u) = \frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}} \frac{u^{-\alpha} - u^{\alpha}}{u+1} \frac{B(\theta,2)}{u^{\theta-1}(1-u)},$$
where $\theta = 0.59 - 0.01\alpha - 0.60\alpha^2$ and $B(.,.)$ is the standard Beta function, record the optimal u^* and set $C = C(u^*,\theta)$.
2. Generate a Beta random variable G' by setting
$$G' \sim \text{Beta}(\theta, 2).$$
3. Set $V \sim U[0,1]$, if
$$V \le \frac{1}{C} \frac{1}{\frac{\pi}{\sin(\alpha\pi)} - \frac{1}{\alpha}} \frac{(G')^{-\alpha} - (G')^{\alpha}}{G'+1} \frac{B(\theta,2)}{(G')^{\theta-1}(1-G')},$$
accept this candidate and return $G^* = G'$, otherwise go back to Step 2.

Proof. This is a direct consequence of the accept rejection method, see [5] for details. □

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