

# ANISOTROPIC BOOTSTRAP PERCOLATION IN THREE DIMENSIONS

BY DANIEL BLANQUICETT

*Department of Mathematics, University of California, Davis, drbt@math.ucdavis.edu*

Consider a  $p$ -random subset  $A$  of initially infected vertices in the discrete cube  $[L]^3$ , and assume that the neighborhood of each vertex consists of the  $a_i$  nearest neighbors in the  $\pm e_i$ -directions for each  $i \in \{1, 2, 3\}$ , where  $a_1 \leq a_2 \leq a_3$ . Suppose we infect any healthy vertex  $x \in [L]^3$  already having  $a_3 + 1$  infected neighbors, and that infected sites remain infected forever. In this paper, we determine the critical length for percolation up to a constant factor in the exponent, for all triples  $(a_1, a_2, a_3)$ . To do so, we introduce a new algorithm called the *beams process* and prove an exponential decay property for a family of subcritical two-dimensional bootstrap processes.

**1. Introduction.** The study of bootstrap processes on graphs was initiated in 1979 by Chalupa, Leath and Reich [12], and is motivated by problems arising from statistical physics, such as the Glauber dynamics of the zero-temperature Ising model, and kinetically constrained spin models of the liquid-glass transition (see, e.g., [5, 15, 18, 19], and the recent survey [20]). The  $r$ -neighbor bootstrap process on a locally finite graph  $G$  is a monotone cellular automata on the configuration space  $\{0, 1\}^{V(G)}$  (we call vertices in state 1 “infected”), evolving in discrete time in the following way: 0 becomes 1 when it has at least  $r$  neighbors in state 1, and infected vertices remain infected forever. Throughout this paper,  $A$  denotes the initially infected set, and we write  $\langle A \rangle = G$  if the state of each vertex is eventually 1.

We will focus on *anisotropic* bootstrap models, which are  $d$ -dimensional analogues of a family of (two-dimensional) processes studied by Duminil-Copin, van Enter and Hulshof [13, 14, 23]. In these models, the graph  $G$  has vertex set  $[L]^d$ , and the neighborhood of each vertex consists of the  $a_i$  nearest neighbors in the  $-e_i$  and  $e_i$ -directions for each  $i \in [d]$ , where  $a_1 \leq \dots \leq a_d$  and  $e_i \in \mathbb{Z}^d$  denotes the  $i$ th canonical unit vector. In other words,  $u, v \in [L]^d$  are neighbors if (see Figure 1 for  $d = 3$ )

$$(1) \quad u - v \in N_{a_1, \dots, a_d} := \{\pm e_1, \dots, \pm a_1 e_1\} \cup \dots \cup \{\pm e_d, \dots, \pm a_d e_d\}.$$

We also call this process the  $\mathcal{N}_r^{a_1, \dots, a_d}$ -model. Our initially infected set  $A$  is chosen according to the Bernoulli product measure  $\mathbb{P}_p = \bigotimes_{v \in [L]^d} \text{Ber}(p)$ , and we are interested in the so-called *critical length for percolation*, for small values of  $p$ ,

$$(2) \quad L_c(\mathcal{N}_r^{a_1, \dots, a_d}, p) := \min\{L \in \mathbb{N} : \mathbb{P}_p(\langle A \rangle = [L]^d) \geq 1/2\}.$$

The analysis of these bootstrap processes for  $a_1 = \dots = a_d = 1$  was initiated by Aizenman and Lebowitz [1] in 1988, who determined the magnitude of the critical length up to a constant factor in the exponent for the  $\mathcal{N}_2^{1, \dots, 1}$ -model (in other words, they determined the ‘metastability threshold’ for percolation). In the case  $d = 2$ , Holroyd [17] determined (asymptotically, as  $p \rightarrow 0$ ) the constant in the exponent (this is usually called a sharp metastability threshold), proving that

$$L_c(\mathcal{N}_2^{1,1}, p) = \exp\left(\frac{\pi^2/18 + o(1)}{p}\right).$$

Received September 2019.

*MSC2020 subject classifications.* Primary 60K35; secondary 60C05.

*Key words and phrases.* Anisotropic bootstrap percolation, beams process, exponential decay.

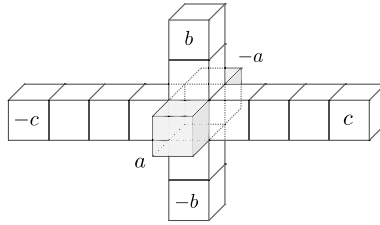


FIG. 1. The neighborhood  $N_{a,b,c}$  with  $a = 1, b = 2$  and  $c = 4$ . The  $e_1$ -axis is towards the reader, the  $e_2$ -axis is vertical and the  $e_3$ -axis is horizontal.

For the general  $\mathcal{N}_r^{1,\dots,1}$ -model with  $2 \leq r \leq d$ , the threshold was determined by Cerf and Cirillo [10] and Cerf and Manzo [11], and the sharp threshold by Balogh, Bollobás and Morris [4] and Balogh, Bollobás, Duminil-Copin and Morris [3]: for all  $d \geq r \geq 2$ , there exists a computable constant  $\lambda(d, r)$  such that, as  $p \rightarrow 0$ ,

$$L_c(\mathcal{N}_r^{1,\dots,1}, p) = \exp_{(r-1)}\left(\frac{\lambda(d, r) + o(1)}{p^{1/(d-r+1)}}\right).$$

In dimension  $d = 2$ , we write  $a_1 = a, a_2 = b$ , and the  $\mathcal{N}_r^{a,b}$ -model is called isotropic when  $a = b$  and anisotropic when  $a < b$ . Hulshof and van Enter [23] determined the threshold for the first interesting anisotropic model given by the family  $\mathcal{N}_3^{1,2}$ , and the corresponding sharp threshold was determined by Duminil-Copin and van Enter [13]: for  $b \geq 2$ , as  $p \rightarrow 0$ ,

$$L_c(\mathcal{N}_{b+1}^{1,b}, p) = \exp\left(\left(\frac{(b-1)^2}{4(b+1)} + o(1)\right)\frac{(\log p)^2}{p}\right).$$

The threshold was also determined in the general case  $r = a + b$  by van Enter and Fey [22] and the proof can be extended to all  $b + 1 \leq r \leq a + b$ : as  $p \rightarrow 0$ ,

$$(3) \quad \log L_c(\mathcal{N}_r^{a,b}, p) = \begin{cases} \Theta(p^{-(r-b)}) & \text{if } b = a, \\ \Theta(p^{-(r-b)}(\log p)^2) & \text{if } b > a. \end{cases}$$

1.1. *Anisotropic bootstrap percolation on  $[L]^3$ .* In this paper, we consider the three-dimensional analogue of the anisotropic bootstrap process studied by Duminil-Copin, van Enter and Hulshof. In dimension  $d = 3$ , we write  $a_1 = a, a_2 = b$  and  $a_3 = c$ . These models were studied by van Enter and Fey [22] for  $r = a + b + c$ ; they determined the following bounds on the critical length, as  $p \rightarrow 0$ :

$$(4) \quad \log \log L_c(\mathcal{N}_{a+b+c}^{a,b,c}, p) = \begin{cases} \Theta(p^{-a}) & \text{if } b = a, \\ \Theta\left(p^{-a}\left(\log \frac{1}{p}\right)^2\right) & \text{if } b > a. \end{cases}$$

Note that by (4) the critical length is doubly exponential in  $p$  when  $r = a + b + c$ . It is not difficult to show that the critical length is polynomial in  $p$  if  $r \leq c$ .

On the other hand, the critical length is singly exponential in the case  $r = c + 1$ ; indeed, we determine the magnitude of the critical length up to a constant factor in the exponent, for all triples  $(a, b, c)$ .

The following is our main result.

THEOREM 1.1. As  $p \rightarrow 0$ :

(i) if  $c = b = a$ , then

$$(5) \quad \log L_c(\mathcal{N}_{c+1}^{a,b,c}, p) = \Theta(p^{-1/2}),$$

(ii) if  $c = b > a$ , then

$$(6) \quad \log L_c(\mathcal{N}_{c+1}^{a,b,c}, p) = \Theta\left(p^{-1/2}\left(\log \frac{1}{p}\right)^{1/2}\right),$$

(iii) if  $c \in \{b + 1, \dots, a + b - 1\}$  with  $a \geq 2$ , then

$$(7) \quad \log L_c(\mathcal{N}_{c+1}^{a,b,c}, p) = \Theta\left(p^{-1/2}\left(\log \frac{1}{p}\right)^{3/2}\right),$$

(iv) if  $c = a + b$ , then

$$(8) \quad \log L_c(\mathcal{N}_{c+1}^{a,b,c}, p) = \Theta(p^{-1}),$$

(v) if  $c > a + b$ , then

$$(9) \quad \log L_c(\mathcal{N}_{c+1}^{a,b,c}, p) = \Theta\left(p^{-1}\left(\log \frac{1}{p}\right)^2\right).$$

We highlight that to deal with the lower bounds in the cases (iv) and (v) (corresponding to  $c \geq a + b$ ), it is necessary to introduce a new algorithm which we call the *beams process*, and to develop new tools in *subcritical* bootstrap percolation (see Theorem 1.2 below).

1.2. *The BSU model.* The model we study here is a special case of the following extremely general class of  $d$ -dimensional monotone cellular automata, which were introduced by Bollobás, Smith and Uzzell [9].

Let  $\mathcal{U} = \{X_1, \dots, X_m\}$  be an arbitrary finite family of finite subsets of  $\mathbb{Z}^d \setminus \{0\}$ . We call  $\mathcal{U}$  the *update family*, each  $X \in \mathcal{U}$  an *update rule* and the process itself  *$\mathcal{U}$ -bootstrap percolation*. Let  $\Lambda$  be either  $\mathbb{Z}^d$  or  $\mathbb{Z}_L^d$  (the  $d$ -dimensional torus of sidelength  $L$ ). Given a set  $A \subset \Lambda$  of initially *infected* sites, set  $A_0 = A$ , and define for each  $t \geq 0$ ,

$$A_{t+1} = A_t \cup \{x \in \Lambda : x + X \subset A_t \text{ for some } X \in \mathcal{U}\}.$$

The set of eventually infected sites is the *closure* of  $A$ , denoted by  $\langle A \rangle_{\mathcal{U}} = \bigcup_{t \geq 0} A_t$ , and we say that there is *percolation* when  $\langle A \rangle_{\mathcal{U}} = \Lambda$ .

Let  $S^{d-1}$  be the unit  $(d - 1)$ -sphere and denote the discrete half-space orthogonal to  $u \in S^{d-1}$  as  $\mathbb{H}_u^d := \{x \in \mathbb{Z}^d : \langle x, u \rangle < 0\}$ . The *stable set*  $\mathcal{S} = \mathcal{S}(\mathcal{U})$  is the set of all  $u \in S^{d-1}$  such that no rule  $X \in \mathcal{U}$  is contained in  $\mathbb{H}_u^d$ . Let  $\mu$  denote the Lebesgue measure on  $S^{d-1}$ . The following classification of families was proposed in [9] for  $d = 2$  and extended to all dimensions in [7]: A family  $\mathcal{U}$  is:

- *subcritical* if for every hemisphere  $\mathcal{H} \subset S^{d-1}$  we have  $\mu(\mathcal{H} \cap \mathcal{S}) > 0$ ;
- *critical* if there exists a hemisphere  $\mathcal{H} \subset S^{d-1}$  such that  $\mu(\mathcal{H} \cap \mathcal{S}) = 0$ , and every open hemisphere in  $S^{d-1}$  has nonempty intersection with  $\mathcal{S}$ ;
- *supercritical* otherwise.

Subcritical families exhibit a behaviour which resembles models in classical site percolation (see, e.g., [2, 16]). For a certain class of subcritical models, we have succeeded in proving an exponential decay property about the cluster size (see Section 4): denote by  $\mathcal{K}$  the connected component containing 0 in  $\langle A \rangle_{\mathcal{U}}$ .

**THEOREM 1.2.** *Assume that  $d = 2$ . Consider  $\mathcal{U}$ -bootstrap percolation with  $\mathcal{S}(\mathcal{U}) = S^1$  and  $A \sim \bigotimes_{v \in \mathbb{Z}^2} \text{Ber}(p)$ . If  $p$  is small enough, then*

$$(10) \quad \mathbb{P}_p(|\mathcal{K}| \geq n) \leq e^{-\Omega(n)},$$

for every  $n \in \mathbb{N}$ .

For dimension  $d = 2$ , Bollobás, Duminil-Copin, Morris and Smith proved a universality result in [7], determining the critical length (with  $A \sim \bigotimes_{v \in \mathbb{Z}_L^2} \text{Ber}(p)$ ),

$$L_c(\mathcal{U}, p) := \min\{L \in \mathbb{N} : \mathbb{P}_p(\langle A \rangle_{\mathcal{U}} = \mathbb{Z}_L^d) \geq 1/2\},$$

up to a constant factor in the exponent for all two-dimensional critical families  $\mathcal{U}$ , which we can briefly state as follows.

**THEOREM 1.3 (Universality).** *Let  $\mathcal{U}$  be a critical two-dimensional family. There exists a computable positive integer  $\alpha = \alpha(\mathcal{U})$  such that, as  $p \rightarrow 0$ , either*

$$(11) \quad \log L_c(\mathcal{U}, p) = \Theta(p^{-\alpha}),$$

or

$$(12) \quad \log L_c(\mathcal{U}, p) = \Theta\left(p^{-\alpha} \left(\log \frac{1}{p}\right)^2\right).$$

Proving a universality result of this kind for three (or higher) dimensions is a challenging open problem. However, there is a weaker conjecture concerning all critical families and all  $d \geq 3$ , stated by the authors in [7]; here for simplicity we state only the case  $d = 3$ .

**CONJECTURE 1.4.** *Let  $\mathcal{U}$  be a critical three-dimensional family. As  $p \rightarrow 0$ , either*

$$(13) \quad \log L_c(\mathcal{U}, p) = p^{-\Theta(1)},$$

or

$$(14) \quad \log \log L_c(\mathcal{U}, p) = p^{-\Theta(1)}.$$

Let us say that  $\mathcal{U}$  is *2-critical* if it satisfies condition (13), and is *3-critical* if it satisfies condition (14). Observe that we can also think of our  $\mathcal{N}_r^{a,b,c}$ -model as  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation, where  $\mathcal{N}_r^{a,b,c}$  is the family consisting of all subsets of size  $r$  of the neighborhood  $N_{a,b,c}$  in (1). It is easy to check that the family  $\mathcal{N}_r^{a,b,c}$  is critical if and only if

$$r \in \{c + 1, \dots, a + b + c\}.$$

Moreover, it turns out that the family  $\mathcal{N}_r^{a,b,c}$  is 2-critical for all  $r \in \{c + 1, \dots, c + b\}$  (see Remark A.2). On the other hand, the family  $\mathcal{N}_{a+b+c}^{a,b,c}$  is 3-critical by (4); it is natural to conjecture that this is the case for all  $r \in \{c + b + 1, \dots, c + b + a\}$ .

**1.3. Outline of the proof.** The proofs of all upper bounds are obtained by adapting standard arguments in bootstrap percolation (see Section 2); the same is true for the lower bounds in the cases  $c < a + b$  (see Section 3).

We deal with the lower bounds in the cases  $c \geq a + b$  by introducing an algorithm that we call the *beams process*, which will allow us to control the size of the components that can be created in the intermediate steps of the bootstrap dynamics, the trick will be to cover such components with beams (a *beam* is a finite three-dimensional set of the form  $H \times [w]$ , where  $w$  is taken in the left-right direction,  $H \subset \mathbb{Z}^2$  is connected and  $\langle H \rangle_{\mathcal{N}_{a+b+1}^{a,b}} = H$ ; see Definition 5.1). All initially infected sites are beams, and at every step we merge beams that are within some constant distance, to create a bigger one, then repeat this algorithm and stop it at some finite time; each beam created during the process we call *covered*. When we observe the induced process along the  $e_3$ -direction, it looks like subcritical two-dimensional  $\mathcal{N}_{a+b+1}^{a,b}$ -bootstrap percolation, thus we can couple the original process and apply the exponential

decay property (Theorem 1.2) to bound the probability of a beam been covered; all details are given in Section 5.

Theorem 1.2 provides new machinery in subcritical bootstrap percolation, we prove it in Section 4 and here we summarize the core idea. First, we need to guarantee the existence of *inwards stable droplets*, which are basically, discrete polygons that cannot be infected from outside; it is possible to show the existence of such droplets by considering families  $\mathcal{U}$  such that  $\mathcal{S}(\mathcal{U}) = S^1$ . After that, we combine ideas used by Bollobás–Riordan in classical percolation models to prove that, when the density of initially infected sites is small enough, then the size of the cluster containing the origin decays exponentially fast, in distribution.

**2. Upper bounds.** To prove upper bounds, it is enough to give one possible way of growing from  $A$  step by step until we fill the whole of  $[L]^3$ . The case  $c > a + b$  will be deduced in the Appendix as a particular case of Proposition A.1 (see Remark A.2), and the proof of case  $c = b = a$  is similar to the proof given in [1]; thus we will omit it here. However, we now provide heuristics describing a brief sketch for all 5 cases, in this order: (i), (iii), (ii), (v) and (iv).

*Heuristics.* Roughly speaking, the typical growth patterns in all 5 cases of Theorem 1.1 run as follows: In (i), one has growing cubes starting from  $[c]^3$ , similar to the growing rectangles in  $d = 2$ , and when the 6 boundary squares have size  $C_1/\sqrt{p}$ , where  $C_1$  is a large constant, the cube keeps growing as with high probability one finds an occupied site on all 6 sides. In (iii), we can start with  $[c]^3$  and grow first in the two directions  $e_1$  and  $e_2$  at the same time until we get a block of the form  $[cp^{-\frac{1}{2}}(\log \frac{1}{p})^{\frac{1}{2}}]^2 \times [c]$ , then it becomes easy to grow in the  $e_3$ -direction, and finally in all directions. In (ii), we can start with a tiny block of the form  $[l] \times [l \log \frac{1}{p}]^2$ , where  $l = p^{-\varepsilon}(\log \frac{1}{p})^{-\frac{1}{2}}$ , and alternate between growing along the  $e_1$ -direction and growing in the  $e_2$  and  $e_3$ -directions simultaneously, until we get a block of the form  $[C_2 p^{-1/2}(\log \frac{1}{p})^{-\frac{1}{2}}] \times [p^{-1/2}(\log \frac{1}{p})^{\frac{1}{2}}]^2$ ; the latter will then easily grow.

In (v), we can start with a larger block of the form  $[C_3 p^{-\frac{1}{2}}(\log \frac{1}{p})^{\frac{1}{2}}]^2 \times [c]$ ; the probability of such a block being initially infected is  $\exp(-cC_3 p^{-1}(\log \frac{1}{p})^2)$ . This block easily grows along the  $e_3$ -direction, and finally in all directions. In (iv), start with a tiny block and alternate between growing  $k$  steps along the  $e_1$  and  $e_2$ -directions simultaneously and growing  $k^2$  steps in the  $e_3$ -direction, until we get a block of the form  $[C_4/\sqrt{p}]^2 \times [p^{-1}]$  which will grow with high probability.

**DEFINITION 2.1.** A *rectangular block* is a set of the form  $R = [x] \times [y] \times [w] \subset \mathbb{Z}^3$ . We say that a rectangular block  $R$  is *internally filled* if  $R \subset \langle A \cap R \rangle_{\mathcal{N}_r^{a,b,c}}$ , and denote this event by  $I^\bullet(R)$ .

In this section, we will only focus on the upper bounds for cases (iii), (iv) and (ii). More precisely, we will give a full proof of the case  $c \in \{b + 1, \dots, a + b - 1\}$  in Section 2.1, then we will only sketch the cases  $c = a + b$  and  $c = b > a$  in Sections 2.2 and 2.3, respectively, by pointing out the small differences between these cases.

**2.1. Case  $c \in \{b + 1, \dots, a + b - 1\}$ .** In this section, we consider the families  $\mathcal{N}_{c+1}^{a,b,c}$ , with  $c \in \{b + 1, \dots, a + b - 1\}$  (here  $a > 1$ , otherwise this case does not exist). As usual in bootstrap percolation, we actually prove a stronger proposition.

PROPOSITION 2.2. Fix  $c \in \{b + 1, \dots, a + b - 1\}$  and consider  $\mathcal{N}_{c+1}^{a,b,c}$ -bootstrap percolation. There exists a constant  $\Gamma = \Gamma(c) > 0$  such that, if

$$L = \exp\left(\Gamma p^{-1/2} \left(\log \frac{1}{p}\right)^{3/2}\right),$$

then  $\mathbb{P}_p(I^\bullet([L]^3)) \rightarrow 1$ , as  $p \rightarrow 0$ .

When  $h, w \geq c$ , for simplicity we denote the event

$$I(h, w) := I^\bullet([h]^2 \times [w]).$$

LEMMA 2.3. If  $p$  is small enough, then

$$\mathbb{P}_p(I(h, w + 1) | I(h, w)) \geq 1 - e^{-ph^2},$$

under  $\mathcal{N}_{c+1}^{a,b,c}$ -bootstrap percolation.

PROOF. If  $R_1 := [h]^2 \times [w]$  is completely infected, we just need to infect the right-most face  $Q := [h]^2 \times \{w + 1\}$ , and since we have  $c$  already infected vertices in  $R_1$ , then it is enough to find 1 infected vertex in  $Q$  (see Figure 2(a) below). Thus,

$$\mathbb{P}_p(I^\bullet([h]^2 \times [w + 1]) | I^\bullet(R_1)) \geq 1 - \prod_{v \in Q} (1 - \mathbb{P}_p(v \in A)) \geq 1 - e^{-ph^2}. \quad \square$$

Lemma 2.3 tells us the cost of growing one step along the (easiest)  $e_3$ -direction, and we are also interested in computing the cost of growing along the  $e_1$  and  $e_2$  (harder) directions. To do so, we will consider general values of  $r$ : let us first consider the regime  $r \leq a + b$ ; this implies that given any rectangular block  $R$ , all three induced two-dimensional processes in the faces of  $R$ , namely,  $\mathcal{N}_{r-c}^{a,b}$ ,  $\mathcal{N}_{r-b}^{a,c}$  and  $\mathcal{N}_{r-a}^{b,c}$ , are supercritical.

LEMMA 2.4 (Supercritical faces). If  $r \leq a + b$ , and  $p$  is small enough, then

$$\mathbb{P}_p(I(h + 1, w) | I(h, w)) \geq (1 - e^{-\binom{c}{2}^{-1} p \binom{c}{2} wh})^2,$$

under  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation.

PROOF. For  $s = a, b$ , let  $\Delta_s$  be the discrete right-angled triangle whose legs are  $[r - s] \times \{1\}$  and  $\{1\} \times [r - s]$ . Once  $R_1 = [h]^2 \times [w]$  is completely full, to get  $R_2 = [h + 1]^2 \times [w]$  internally filled it is enough to have one copy of  $\Delta_a$  in  $A \cap (\{h + 1\} \times [h] \times [w])$  (front face), and one copy of  $\Delta_b$  in  $A \cap ([h] \times \{h + 1\} \times [w])$  (top face, see Figure 2(a) below).

Since  $|\Delta_s| = (r - s)(r - s + 1)/2$  and  $a \geq 2$ , then  $|\Delta_b| \leq |\Delta_a| \leq \binom{c}{2}$ . Hence, by independence between the front and top faces,

$$\begin{aligned} \mathbb{P}_p(I^\bullet(R_2) | I^\bullet(R_1)) &\geq (1 - e^{-|\Delta_a|^{-1} p^{|\Delta_a|} wh}) (1 - e^{-|\Delta_b|^{-1} p^{|\Delta_b|} wh}) \\ &\geq (1 - e^{-\binom{c}{2}^{-1} p \binom{c}{2} wh})^2. \end{aligned} \quad \square$$

The next step is to determine the size of a rectangular block (usually called *critical droplet*) such that, once it is internally filled, then it can grow until  $[L]^3$  with high probability.

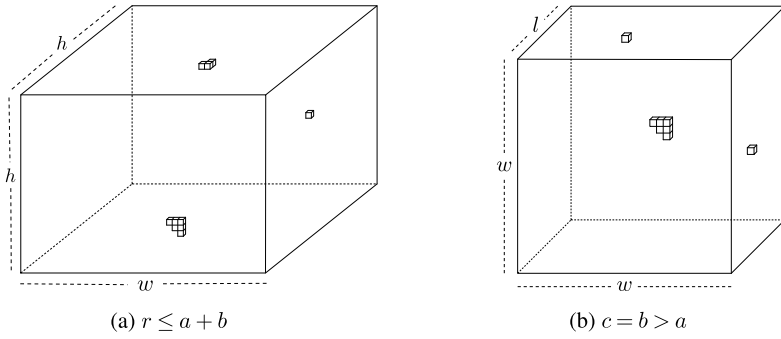


FIG. 2. A single vertex on the right-most side, one copy of  $\Delta_a$  on the front side, and one copy of  $\Delta_b$  on the top side.

LEMMA 2.5. Let  $\Gamma > 0$  be a large constant and set  $h = cp^{-\frac{1}{2}}(\log \frac{1}{p})^{\frac{1}{2}}$ ,  $R_1 := [h]^2 \times [c]$  and

$$L = \exp\left(\Gamma p^{-\frac{1}{2}}\left(\log \frac{1}{p}\right)^{\frac{3}{2}}\right).$$

Conditionally on  $I^\bullet(R_1)$ , the probability of  $I^\bullet([L]^3)$  goes to 1, as  $p \rightarrow 0$ .

PROOF. Consider the rectangular blocks  $R_2 \subset R_3 \subset R_4 \subset R_5 := [L]^3$  containing  $R_1$ , defined by

$$\begin{aligned} R_2 &:= [h]^2 \times \left[ c^2 p^{-\binom{c}{2} + \frac{1}{2}} \left(\log \frac{1}{p}\right)^{\frac{1}{2}} \right], \\ R_3 &:= [h^2]^2 \times \left[ c^2 p^{-\binom{c}{2} + \frac{1}{2}} \left(\log \frac{1}{p}\right)^{\frac{1}{2}} \right], \\ R_4 &:= [h^2]^2 \times [L]. \end{aligned}$$

Note that  $\mathbb{P}_p(I^\bullet([L]^3) | I^\bullet(R_1)) \geq \prod_{k=1}^4 \mathbb{P}_p(I^\bullet(R_{k+1}) | I^\bullet(R_k))$ . We apply Lemma 2.3 to deduce

$$\mathbb{P}_p(I^\bullet(R_2) | I^\bullet(R_1)) \geq (1 - e^{-ph^2}) c^2 p^{-\binom{c}{2} + \frac{1}{2}} (\log \frac{1}{p})^{\frac{1}{2}} \geq e^{-2p^{\frac{c^2}{2} - \binom{c}{2}}} \rightarrow 1,$$

and by Lemma 2.4,

$$\mathbb{P}_p(I^\bullet(R_3) | I^\bullet(R_2)) \geq (1 - e^{-\Omega(p) p^{-\binom{c}{2} + \frac{1}{2}} (\log \frac{1}{p})^{\frac{1}{2}} \cdot h}) 2h^2 \geq \exp(-4h^2 p^{2c}) \rightarrow 1.$$

We apply these lemmas again and the fact that  $ph^4 \geq p^{-1} \gg \Gamma p^{-\frac{1}{2}} (\log \frac{1}{p})^{\frac{3}{2}}$  to get  $\mathbb{P}_p(I^\bullet(R_4) | I^\bullet(R_3)) \rightarrow 1$ , and also  $\mathbb{P}_p(I^\bullet(R_5) | I^\bullet(R_4)) \rightarrow 1$ . So,  $\mathbb{P}_p(I^\bullet([L]^3) | I^\bullet(R_1)) \rightarrow 1$ , as  $p \rightarrow 0$ .  $\square$

Now, we are ready to show the upper bound.

PROOF OF PROPOSITION 2.2. Set  $L = \exp(\Gamma p^{-\frac{1}{2}} (\log \frac{1}{p})^{\frac{3}{2}})$ , where  $\Gamma > 0$  is a large constant to be chosen. Consider the rectangular block

$$R := \left[ cp^{-\frac{1}{2}} \left(\log \frac{1}{p}\right)^{\frac{1}{2}} \right]^2 \times [c] \subset [L]^3,$$

and the events  $F_L := \{\exists \text{ an internally filled copy of } R \text{ in } [L]^3\}$ , and  $G_L := \{\langle A \cup R \rangle = [L]^3\}$ . It follows that  $\mathbb{P}_p(I^\bullet([L]^3)) \geq \mathbb{P}_p(F_L)\mathbb{P}_p(G_L|I^\bullet(R))$ , and  $\mathbb{P}_p(G_L|I^\bullet(R)) \rightarrow 1$ , as  $p \rightarrow 0$ , by the previous lemma. Therefore, it remains to show that  $\mathbb{P}_p(F_L) \rightarrow 1$ , too.

Indeed, we claim that there exists a constant  $C' > 0$  such that

$$(15) \quad \mathbb{P}_p(I^\bullet(R)) \geq \exp\left(-C' p^{-\frac{1}{2}} \left(\log \frac{1}{p}\right)^{\frac{3}{2}}\right),$$

so using the fact that there are roughly  $L^3/|R|$  disjoint (therefore independent) copies of  $R$  (which we label  $Q_1, \dots, Q_{L^3/|R|}$ ), and  $|R| \leq p^{-3}$ , (15) immediately gives

$$\begin{aligned} \mathbb{P}_p(F_L^c) &\leq \mathbb{P}_p\left(\bigcap_i I^\bullet(Q_i)^c\right) \leq [1 - \mathbb{P}_p(I^\bullet(R))]^{L^3/|R|} \\ &\leq \exp(-e^{3 \log L - 3 \log(1/p) - C' p^{-\frac{1}{2}} (\log \frac{1}{p})^{\frac{3}{2}}}). \end{aligned}$$

Since  $\log L = \Gamma p^{-\frac{1}{2}} (\log \frac{1}{p})^{\frac{3}{2}}$ , by taking  $\Gamma > C'/3$  we conclude  $\mathbb{P}_p(F_L) \rightarrow 1$ , as  $p \rightarrow 0$ . To finish, it is only left to prove inequality (15).

In fact, note that a way to make  $R$  be internally filled is the following: start with  $[c]^3 \subset A$ , and then grow from  $R_k = [k]^2 \times [c]$  to  $R_{k+1}$ , for  $k = c, \dots, m := cp^{-\frac{1}{2}} (\log \frac{1}{p})^{\frac{1}{2}}$ . This gives us

$$\begin{aligned} \mathbb{P}_p(I^\bullet(R)) &\geq \mathbb{P}_p([c]^3 \subset A) \prod_{k=c}^m \mathbb{P}_p(I^\bullet(R_{k+1})|I^\bullet(R_k)) \geq p^{c^3} \prod_{k=c}^m (1 - e^{-\binom{c}{2}^{-1} p^{\binom{c}{2} ck}})^2 \\ &\geq p^{c^3+c^2m} \geq \exp\left(-C' p^{-\frac{1}{2}} \left(\log \frac{1}{p}\right)^{\frac{3}{2}}\right), \end{aligned}$$

for  $C' > c^3$ , as we claimed.  $\square$

2.2. *Case  $c = a + b$ .* In this section, we consider the families  $\mathcal{N}_{a+b+1}^{a,b,a+b}$ , corresponding to the case  $r = a + b + 1$ . To do so, we first compute the cost of growing for all cases  $a + b < r \leq a + c$ , where, the induced  $\mathcal{N}_{r-c}^{a,b}$  process is still supercritical, but the induced processes  $\mathcal{N}_{r-b}^{a,c}$  and  $\mathcal{N}_{r-a}^{b,c}$  are critical.

LEMMA 2.6 (Critical faces). *If  $r \in \{a + b + 1, \dots, a + c\}$  and  $p$  is small, then*

$$\mathbb{P}_p(I(h + 1, w)|I(h, w)) \geq (1 - e^{-\frac{1}{r-a} p^{r-a} w})^r (1 - e^{-\frac{1}{m} p^m w})^{2h},$$

under  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation, with  $m := r - (a + b)$ .

PROOF. Once  $[h]^2 \times [w]$  is completely full, to fill  $[h + 1]^2 \times [w]$  it is enough to have the occurrence of the events  $F_h^{e_1}$  and  $F_h^{e_2}$  defined as follows:  $F_h^{e_1}$  as (growing along the  $e_1$ -direction) there exist  $r - a$  adjacent vertices in  $A \cap (\{h + 1\} \times \{1\} \times [w])$ ,  $r - (a + 1)$  adjacent vertices in  $A \cap (\{h + 1\} \times \{2\} \times [w])$ ,  $\dots$ ,  $r - (a + b - 1)$  adjacent vertices in  $A \cap (\{h + 1\} \times \{b\} \times [w])$ , and for each  $i = b + 1, \dots, h$ , there exist  $m = r - (a + b)$  adjacent vertices in  $A \cap (\{h + 1\} \times \{i\} \times [w])$ :

$$\begin{aligned} \mathbb{P}_p(F_h^{e_1}) &\geq \prod_{k=m+1}^{r-a} (1 - (1 - p^k)^{\frac{w}{k}}) \prod_{i=1}^h (1 - (1 - p^m)^{\frac{w}{m}}) \\ &\geq (1 - e^{-\frac{1}{r-a} p^{r-a} w})^b (1 - e^{-\frac{1}{m} p^m w})^h. \end{aligned}$$



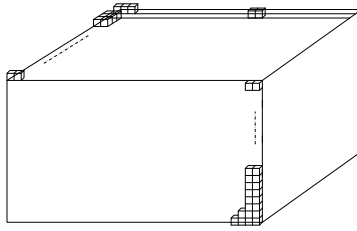


FIG. 3.  $m$  vertices in each of the lines along the  $e_3$ -direction ( $m = 2$ ).

$F'_h$  is defined analogously, this time growing along the  $e_2$ -direction (see Figure 3), thus

$$\mathbb{P}_p(F_h^{e_2}) \geq (1 - e^{-\frac{1}{r-b}p^{r-b}w})^a (1 - e^{-\frac{1}{m}p^m w})^h.$$

Finally,  $p_r(h+, w) \geq \mathbb{P}_p(F_h^{e_1})\mathbb{P}_p(F_h^{e_2})$ .  $\square$

REMARK 2.7. In the regime  $a + c < r \leq a + b + c$ , all three the induced two-dimensional processes  $\mathcal{N}_{r-c}^{a,b}$ ,  $\mathcal{N}_{r-b}^{a,c}$  and  $\mathcal{N}_{r-a}^{b,c}$  are critical.

As before, we need to set the size of a critical droplet.

LEMMA 2.8. Fix  $\varepsilon > 0$  and let  $\Gamma$  be a large constant. Set  $h = p^{-1/2-\varepsilon}$ ,  $R_1 := [h]^2 \times [h^2]$  and

$$L = \exp(\Gamma p^{-1}).$$

Conditionally on  $I^\bullet(R_1)$ , the probability of  $I^\bullet([L]^3)$  goes to 1, as  $p \rightarrow 0$ .

The proof of this lemma is very similar to that of Lemma 2.5, thus we omit it. Finally, to deduce the upper bound, we proceed in the same way that we used to prove Proposition 2.2, this time by showing that the critical droplet  $R_1$  satisfies

$$(16) \quad \mathbb{P}_p(I^\bullet(R_1)) \geq \exp(-C' p^{-1}),$$

for some constant  $C' > 0$ , depending on the integral of the function  $f_1 : (0, \infty) \rightarrow (0, \infty)$ , defined by  $f_1(z) = -\log(1 - e^{-z})$  (see, e.g., [1] and [17]).

2.3. Case  $c = b > a$ . In this section, we sketch the proof of the last case. Consider the families

$$\mathcal{N}_{c+1}^{a,c,c}.$$

We follow the same steps, taking into account that the way to grow is slightly different: in this case, to grow along the  $e_2$ -direction is as easy as grow along the  $e_3$ -direction, so that it is enough to find a single infected vertex on the right-most and top sides, while to grow along the  $e_1$ -direction we still need to find one copy of  $\Delta_a$  on the front side (see Figure 2(b) above).

LEMMA 2.9. Fix  $l, w \geq c$  and let  $I = I^\bullet([l] \times [w]^2)$ . If  $p$  is small enough, then:

- (i)  $\mathbb{P}_p(I^\bullet([l] \times [w + 1]^2) | I) \geq (1 - e^{-plw})^2$ .
- (ii)  $\mathbb{P}_p(I^\bullet([l + 1] \times [w]^2) | I) \geq 1 - e^{-\Omega(p^{c(c+1)/2} w^2)}$ .

PROOF. Similar to the proof of Lemmas 2.3 and 2.4.  $\square$

The size of the critical droplet is given by the following lemma, again, we omit the proof.

LEMMA 2.10. *Let  $\Gamma$  be a large constant. Set  $R_1 := [p^{-1/2}(\log \frac{1}{p})^{-\frac{1}{2}}] \times [2\Gamma p^{-1} \log \frac{1}{p}]^2$  and*

$$L = \exp\left(\Gamma p^{-1/2} \sqrt{\log \frac{1}{p}}\right).$$

*Conditionally on  $I^\bullet(R_1)$ , the probability of  $I^\bullet([L]^3)$  goes to 1, as  $p \rightarrow 0$ .*

Finally, to deduce the upper bound, we proceed as before, this time by showing that

$$(17) \quad \mathbb{P}_p(I^\bullet(R_1)) \geq \exp\left(-C' p^{-\frac{1}{2}} \sqrt{\log \frac{1}{p}}\right),$$

for some constant  $C' > 0$ , depending on  $c$  and the function  $f_2(z) = -\log(1 - e^{-z^2})$ .

**3. Lower bounds via components process.** In this section, we only prove the lower bounds corresponding to the cases  $c < a + b$ , since the proof is an application of the *components process* (see Definition 3.8 below), a variant of an algorithm introduced Bollobás, Duminil-Copin, Morris, and Smith [7]. The lower bound in the case  $a = b = c = 1$  was proved in [1], and the general case  $a = b = c$  follows by using the same arguments. Thus, we will omit this case, and prove the following.

PROPOSITION 3.1. *If  $c > a$ , there is a constant  $\gamma = \gamma(c) > 0$  such that, for*

$$L < \exp\left(\gamma p^{-1/2} \left(\log \frac{1}{p}\right)^{1/2}\right),$$

$\mathbb{P}_p(I^\bullet([L]^3)) \rightarrow 0$ , as  $p \rightarrow 0$ , under  $\mathcal{N}_{c+1}^{a,c,c}$ -bootstrap percolation.

PROPOSITION 3.2. *If  $c \in \{b + 1, \dots, a + b - 1\}$ , there exists  $\gamma = \gamma(c) > 0$  such that, for*

$$L < \exp\left(\gamma p^{-1/2} \left(\log \frac{1}{p}\right)^{3/2}\right),$$

$\mathbb{P}_p(I^\bullet([L]^3)) \rightarrow 0$ , as  $p \rightarrow 0$ , under  $\mathcal{N}_{c+1}^{a,b,c}$ -bootstrap percolation.

NOTATION 3.3. Throughout this paper, when  $\mathcal{U} = \mathcal{N}_r^{a,b,c}$  we will omit the subscript in the closure and simply write  $\langle \cdot \rangle$  instead of  $\langle \cdot \rangle_{\mathcal{N}_r^{a,b,c}}$ .

Aizenman and Lebowitz [1] obtained the matching lower bound for the family  $\mathcal{N}_2^{1,1,1}$  by using the so-called *rectangles process*, and they exploited the fact that for this model, the closure  $\langle A \rangle$  is a union of rectangular blocks which are separated by distance at least 2.

In our case, the closure  $\langle A \rangle$  is more complicated. Thus, we need to introduce a notion about rectangular blocks which is an approximation to being internally filled, and this notion requires a strong concept of connectedness; we define both concepts in the following. Consider the superset of  $N_{a,b,c}$  (see (1)) given by

$$\bar{N}_{a,b,c} := \{(u_1, u_2, u_3) \in \mathbb{Z}^3 : |u_1| \leq a, |u_2| \leq b, |u_3| \leq c \text{ and } u_1 u_2 u_3 = 0\}.$$

DEFINITION 3.4. Let  $G = (V, E)$  be the graph with vertex set  $[L]^3$  and edge set given by  $E = \{(u, v) : u - v \in \bar{N}_{a,b,c}\}$ . We say that a set  $S \subset [L]^3$  is *strongly connected* if it is connected in the graph  $G$ .

DEFINITION 3.5. We say that the rectangular block  $R \subset [L]^3$  is *internally spanned* by  $A$ , if there exists a strongly connected set  $S \subset \langle A \cap R \rangle$  such that  $R$  is the smallest rectangular block containing  $S$ . We denote this event by  $I^\times(R)$ .

Note that when a rectangular block is internally filled then it is also internally spanned, therefore, Propositions 3.1 and 3.2 are consequences of the following results.

PROPOSITION 3.6. If  $c > a$ , there is a constant  $\gamma = \gamma(c) > 0$  such that, for

$$L < \exp\left(\gamma p^{-1/2} \left(\log \frac{1}{p}\right)^{1/2}\right),$$

$\mathbb{P}_p(I^\times([L]^3)) \rightarrow 0$ , as  $p \rightarrow 0$ , under  $\mathcal{N}_{c+1}^{a,c,c}$ -bootstrap percolation.

PROPOSITION 3.7. If  $c \in \{b + 1, \dots, a + b - 1\}$ , there exists  $\gamma = \gamma(c) > 0$  such that, for

$$L < \exp\left(\gamma p^{-1/2} \left(\log \frac{1}{p}\right)^{3/2}\right),$$

$\mathbb{P}_p(I^\times([L]^3)) \rightarrow 0$ , as  $p \rightarrow 0$ , under  $\mathcal{N}_{c+1}^{a,b,c}$ -bootstrap percolation.

We will prove them in Sections 3.2 and 3.3, respectively.

3.1. *The components process.* The following is an adaptation of the spanning algorithm in [7], Section 6.2. We will use it to show an Aizenman–Lebowitz-type lemma, which says that when a rectangular block is internally spanned, then it contains internally spanned rectangular blocks of all intermediate sizes (see Lemmas 3.12 and 3.13 below).

DEFINITION 3.8 (The components process). Let  $A = \{v_1, \dots, v_{|A|}\} \subset [L]^3$  and fix  $r \geq c + 1$ . Set  $\mathcal{R} := \{S_1, \dots, S_{|A|}\}$ , where  $S_i = \{v_i\}$  for each  $i = 1, \dots, |A|$ . Then repeat the following steps until STOP:

1. If there exist distinct sets  $S_1, S_2 \in \mathcal{R}$  such that

$$S_1 \cup S_2$$

is strongly connected, then remove them from  $\mathcal{R}$ , and replace by  $\langle S_1 \cup S_2 \rangle$ .

2. If there do not exist such sets in  $\mathcal{R}$ , then STOP.

REMARK 3.9. We highlight two properties that are due to the way the algorithm evolves:

- At any stage of the component process, any set  $S = \langle S_1 \cup S_2 \rangle$  added to the collection  $\mathcal{R}$  satisfies  $S = \langle A \cap S \rangle = \langle S \rangle \subset [L]^3$  (since  $r \geq c + 1$ ). In particular, the smallest rectangular block containing  $S$  is internally spanned.
- Since  $G$  is finite, the process stops in finite time; so that we can consider the final collection  $\mathcal{R}'$  and set  $V(\mathcal{R}') = \bigcup_{S \in \mathcal{R}'} S$ .

LEMMA 3.10.  $V(\mathcal{R}') = \langle A \rangle$ .

PROOF. Clearly,  $A \subset V(\mathcal{R}') \subset \langle A \rangle$ , and to prove that  $\langle A \rangle \subset V(\mathcal{R}')$  we argue by contradiction. Suppose this is not the case, since  $A \subset V(\mathcal{R}')$ , there would exist vertices  $v \in \langle A \rangle \setminus V(\mathcal{R}')$  and  $v_1, \dots, v_r \in V(\mathcal{R}')$  such that  $v - v_i \in N_{a,b,c}$ , for  $i = 1, \dots, r$ . Let us say that  $v_i \in S'_i$  for some sets  $S'_i \in \mathcal{R}'$ .

Since  $S'_1 = \langle S'_1 \rangle$ ,  $v_k \notin S'_1$  for some  $k \neq 1$ , so that  $S'_k \neq S'_1$ . In particular,  $S'_1 \cup S'_k$  is strongly connected via  $v_1, v, v_k$  and  $\langle S'_1 \cup S'_k \rangle \notin \mathcal{R}'$ ; this contradicts the definition of  $\mathcal{R}'$ .  $\square$

NOTATION 3.11. From now on, we allow some abuse of notation by denoting as  $[x] \times [y] \times [z]$  any translate of the rectangular block  $R = [x] \times [y] \times [z]$  located at the origin.

3.2. *Case  $c = b > a$ .* The following is a variant of the Aizenman–Lebowitz lemma in [1].

LEMMA 3.12. *Consider  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation with  $r \geq c + 1$ . If  $[L]^3$  is internally spanned then, for every  $h, k \leq L$  there exists an internally spanned block  $[x] \times [y] \times [z]$  inside  $[L]^3$  satisfying  $(y + z)/2 \leq 2ck$ , and either:*

- (a)  $x \geq h$ , or
- (b)  $x < h$  and  $(y + z)/2 \geq k$ .

PROOF. Let  $S$  be the first set that appears in the components process such that, the smallest block  $Q := [x] \times [y] \times [z]$  containing  $S$  satisfies either  $x \geq h$  or  $(y + z)/2 \geq k$  (such a set exists since  $V(\mathcal{R}') = \langle A \rangle$  and  $[L]^3$  is internally spanned). Since  $Q$  is internally spanned, it only remains to show that the semiperimeter  $(y + z)/2$  is at most  $2ck$ .

In fact, we know that  $S = \langle S_1 \cup S_2 \rangle$  for some sets  $S_t$  such that, for each  $t = 1, 2$ , the smallest block  $[x_t] \times [y_t] \times [z_t]$  containing  $S_t$  satisfies  $(y_t + z_t)/2 \leq k - 1/2$ . Since  $S$  is strongly connected, the new semiperimeter is

$$\frac{y + z}{2} \leq 2 \max_{t=1,2} \left\{ \frac{y_t + z_t}{2} \right\} + \frac{b + c}{2} \leq 2c \left( k - \frac{1}{2} \right) + c = 2ck. \quad \square$$

PROOF OF PROPOSITION 3.6. Fix a small  $\delta > 0$  and take  $L < \exp(\gamma p^{-1/2} (\log \frac{1}{p})^{1/2})$ , where  $\gamma = \gamma(\delta) > 0$  is another small constant to be chosen. Let us show that  $\mathbb{P}_p(I^\times([L]^3))$  goes to 0, as  $p \rightarrow 0$ . Set

$$h = \delta p^{-\frac{1}{2}} \left( \log \frac{1}{p} \right)^{-\frac{1}{2}}, \quad k = p^{-\frac{1}{2}} \sqrt{\log \frac{1}{p}}.$$

If  $[L]^3$  is internally spanned, by Lemma 3.12, the following event occurs: there exists an internally spanned rectangular block  $Q = [x] \times [y] \times [z] \subset [L]^3$  satisfying  $(y + z)/2 \leq 2ck$ , and either  $x \geq h$ , or  $x < h$  and  $(y + z)/2 \geq k$ .

Suppose first that  $x < h$  and  $(y + z)/2 \geq k$ , thus, either  $y$  or  $z$  is at least  $k$ , by symmetry ( $b = c$ ), we can assume  $z \geq k$ . Since  $Q$  is internally spanned, every copy of the slab  $[x] \times [y] \times [2c]$  must contain at least 1 element of  $A$ . Consider only the  $z/2c$  disjoint slabs that partition  $Q$ ; since  $xy = O(hk)$ , if  $\delta$  is small, the probability of this event is at most

$$(O(pxy))^{z/2c} \leq (O(phk))^{k/2c} = (O(\delta))^{k/2c} \leq e^{-k}.$$

On the other hand, if  $x \geq h$  we use the fact that  $a \leq (c + 1) - 2$ , thus, since  $Q$  is internally spanned, every copy of the slab  $[3a] \times [y] \times [z]$  must contain at least 2 elements  $u, v \in A$  such that  $u - v \in \bar{N}_{a,c,c}$ . Since  $x \geq h$ , the probability of this event is at most

$$(O(p^2yz))^{x/3a} \leq (O(p^2k^2))^{h/3a} \leq \left( O \left( p^2 p^{-1} \log \frac{1}{p} \right) \right)^{h/3a} \leq e^{-\Omega(\delta k)}.$$

Therefore, the probability that  $Q$  is internally spanned is at most  $e^{-c(\delta)k}$  for some small constant  $c(\delta) > 0$ . Finally, denoting by  $\mathcal{R}_k$  the collection of blocks  $[x] \times [y] \times [z] \subset [L]^3$  satisfying  $y + z \leq 4ck$ , it follows by union bound that

$$\mathbb{P}_p(I^\times([L]^3)) \leq \sum_{Q \in \mathcal{R}_k} \mathbb{P}_p(I^\times(Q)) \leq |\mathcal{R}_k| e^{-c(\delta)k} \leq L^7 \exp \left( -c(\delta) p^{-1/2} \left( \log \frac{1}{p} \right)^{1/2} \right) \rightarrow 0,$$

as  $p \rightarrow 0$ , for  $7\gamma < c(\delta)$ , and we are finished.  $\square$

3.3. *Case  $c \in \{b + 1, \dots, a + b - 1\}$ .* In this case, the corresponding analogue of the Aizenman–Lebowitz lemma is as follows.

LEMMA 3.13. *Consider  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation with  $r \geq c + 1$ . If  $[L]^3$  is internally spanned then, for every  $h, k \leq L$  there exists an internally spanned block  $[x] \times [y] \times [z] \subset [L]^3$  satisfying  $(x + y)/2 \leq rh$ , and either:*

- (a)  $z \geq k$ , or
- (b)  $z < k$  and  $(x + y)/2 \geq h$ .

The proof of this lemma is identical to that of Lemma 3.12; we therefore omit it and proceed to the proof of the lower bound.

PROOF OF PROPOSITION 3.7. Take  $L < \exp(\gamma p^{-\frac{1}{2}}(\log \frac{1}{p})^{\frac{3}{2}})$ , where  $\gamma > 0$  is some small constant. Fix  $\delta > 0$  and set

$$h = \delta p^{-\frac{1}{2}} \left( \log \frac{1}{p} \right)^{\frac{1}{2}}, \quad k = p^{-1}.$$

If  $[L]^3$  is internally spanned, by Lemma 3.13, there is an internally spanned rectangular block  $Q = [x] \times [y] \times [z]$  satisfying  $(x + y)/2 \leq rh$ , and either  $z \geq k$ , or  $z < k$  and  $(x + y)/2 \geq h$ .

In the case that  $z \geq k$ , we also know that  $xy = O(h^2)$ . As before, every copy of the slab  $S := [x] \times [y] \times [r]$  intersects  $A$ . Thus, by considering the  $z/r$  disjoint slabs; if  $\delta$  is small, the probability of this event is at most

$$\mathbb{P}_p(S \cap A \neq \emptyset)^{z/r} \leq (1 - e^{-\Omega(ph^2)})^{k/r} = (1 - p^{\Omega(\delta^2)})^{p^{-1}/r} \leq e^{-p^{-3/4}}.$$

In the case that  $z < k$  and  $(x + y)/2 \geq h$ , we can assume w.l.o.g. that  $y \geq h$  and use the fact that  $b \leq c - 1 = r - 2$ . This time there is no gap along the  $e_2$ -direction, so, every copy of the slab  $[x] \times [2r] \times [z]$  must contain at least 2 elements of  $A$  within constant distance. The probability of this event is at most

$$(O(p^2xz))^{y/2r} \leq (O(p^2hk))^{h/2r} \leq e^{-\Omega(h \log \frac{1}{p})}.$$

Therefore, the probability that  $Q$  is internally spanned is at most  $e^{-c(\delta)p^{-\frac{1}{2}}(\log \frac{1}{p})^{\frac{3}{2}}}$ , for some small constant  $c(\delta) > 0$ . Denote by  $\mathcal{R}'_h$  the collection of blocks  $[x] \times [y] \times [z] \subset [L]^3$  satisfying  $x + y \leq 2rh$ , it follows by union bound that

$$\mathbb{P}_p(I^\times([L]^3)) \leq \sum_{Q \in \mathcal{R}'_h} \mathbb{P}_p(I^\times(Q)) \leq |\mathcal{R}'_h| e^{-c(\delta)p^{-\frac{1}{2}}(\log \frac{1}{p})^{\frac{3}{2}}} \rightarrow 0,$$

as  $p \rightarrow 0$ , if  $\gamma > 0$  is small.  $\square$

**4. Exponential decay for subcritical families.** In this section, we develop new machinery for  $\mathcal{U}$ -bootstrap percolation in  $\mathbb{Z}^2$  with  $\mathcal{U}$  subcritical. The first paper studying these families in such generality is [2], it turns out that these families exhibit a behavior which resembles models in classical site percolation, for instance, in [2] it is proved that  $p_c(\mathbb{Z}^2, \mathcal{U}) > 0$ , for every subcritical family  $\mathcal{U}$ , where

$$p_c(\mathbb{Z}^2, \mathcal{U}) := \inf\{p : \mathbb{P}_p(\langle A \rangle_{\mathcal{U}} = \mathbb{Z}^2) = 1\}.$$

We will only deal with subcritical families  $\mathcal{U}$  satisfying  $p_c(\mathbb{Z}^2, \mathcal{U}) = 1$ ; the authors of [2] proved that this condition is equivalent to  $\mathcal{S}(\mathcal{U}) = S^1$ . Our aim is to show that for such families, if we choose the initial infected set  $A$  to be  $\varepsilon$ -random with  $\varepsilon$  small enough, then the size of the cluster in  $\langle A \rangle_{\mathcal{U}}$  containing the origin decays exponentially fast. More precisely, we have the following.

DEFINITION 4.1. We define the *component* (or *cluster*) of  $0 \in \mathbb{Z}^2$  as the connected component containing 0 in the graph induced by  $\langle A \rangle_{\mathcal{U}}$ , and we denote it by  $\mathcal{K} = \mathcal{K}(\mathcal{U}, A)$ . If  $0 \notin \langle A \rangle_{\mathcal{U}}$ , then we set  $\mathcal{K} = \emptyset$ .

The following is the main result in this section. It will be essential to prove the remaining lower bounds (cases  $c \geq a + b$ ) in Section 5.

THEOREM 4.2. Consider  $\mathcal{U}$ -bootstrap percolation with  $\mathcal{S}(\mathcal{U}) = S^1$ . If  $p$  is small enough, then

$$\mathbb{P}_p(|\mathcal{K}| \geq n) \leq e^{-\Omega(n)},$$

for every  $n \in \mathbb{N}$ .

In order to prove this theorem, first we need to guarantee the existence of *inwards stable droplets*, which are basically discrete polygons that can not be infected from outside, it is possible to do so by using the condition  $\mathcal{S}(\mathcal{U}) = S^1$ . After that, we introduce the *dilation radius*, which is a constant depending on  $\mathcal{U}$ , used to obtain an extremal lemma that gives us a quantitative measure of the ratio  $|\langle A \rangle_{\mathcal{U}}|/|A|$ . Finally, we combine ideas used by Bollobás and Riordan in classical percolation models to conclude.

4.1. *Inwards stable droplets and the dilation radius.* Given  $x, y \in \mathbb{R}^2$  we denote the usual euclidean distance between  $x$  and  $y$  by  $\|x - y\|$ , and  $B_\rho(x)$  is the ball of radius  $\rho > 0$  centered at  $x$ :

$$(18) \quad B_\rho(x) := \{y \in \mathbb{R}^2 : \|x - y\| \leq \rho\}.$$

For simplicity, we denote  $B_\rho := B_\rho(0)$ . Imagine for a moment that we have a convex set  $D$  in the plane and suppose it is inscribed in  $B_\rho$ , then we know that any other ball with radius  $\rho$  and center outside  $B_{3\rho}$  is disjoint from  $D$ . This simple remark will be important to prove Theorem 4.2 (see Lemma 4.9).

DEFINITION 4.3. Let us define a *rounded droplet*  $D$  as the intersection of  $\mathbb{Z}^2$  with a bounded convex set in the plane. We say that  $D \subset \mathbb{Z}^2$  is *inwards stable* for  $\mathcal{U}$  if

$$(19) \quad \langle \mathbb{Z}^2 \setminus D \rangle_{\mathcal{U}} = \mathbb{Z}^2 \setminus D.$$

We need to guarantee the existence of inwards stable (rounded) droplets, note that they are finite; this is the only point where we use the hypothesis  $\mathcal{S}(\mathcal{U}) = S^1$ .

LEMMA 4.4 (Existence, [2]). *If  $\mathcal{S}(\mathcal{U}) = S^1$  then, there exist an inwards stable droplet  $D$  such that  $0 \in D$ .*

The origin  $0 \in \mathbb{Z}^2$  has no special role here, it is just a reference point to locate the droplet  $D$ . Any translate of  $D$  is inwards stable as well.

There are several choices for the shape of inwards stable droplets. The following proof is included in [2], and shows that  $D$  could be a polygon or not; this fact justifies the *rounded* term in the definition.

SKETCH OF PROOF OF LEMMA 4.4. Suppose that  $B_\rho$  is initially healthy. If  $\rho$  is large enough, then every rule  $X \in \mathcal{U}$  can only infect sites in disjoint circular segments ‘cut off’ from  $B_\rho$  using chords of length at most

$$\nabla(\mathcal{U}) := \max_{X \in \mathcal{U}} \max_{x, y \in X} \|x - y\|,$$

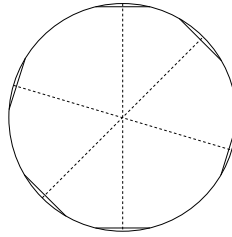


FIG. 4. Set of disjoint circular segments cut off from  $B_\rho$  using chords perpendicular to directions  $\pi/4, \pi/2$  and  $7\pi/8$ .

and parallel to the sides of  $\text{Hull}(X)$  (see Figure 4), and these segments are all either disjoint or contained in each other for different rules, since  $\rho$  is large. No additional infection takes place in  $B_\rho$ , therefore  $D = B_\rho \setminus \langle \mathbb{Z}^2 \setminus B_\rho \rangle_{\mathcal{U}}$  is inwards stable.  $\square$

Now, given  $\rho > 0$  we denote the discrete ball as

$$B'_\rho := \mathbb{Z}^2 \cap B_\rho.$$

An immediate consequence of the above lemma is the fact that every vertex which is eventually infected should be within some constant distance from an initially infected vertex.

COROLLARY 4.5. *If  $\mathcal{S}(\mathcal{U}) = S^1$ , there exists  $\hat{\rho} > 0$  such that, for every  $x \in \langle A \rangle_{\mathcal{U}}$ ,*

$$(20) \quad A \cap B'_{\hat{\rho}}(x) \neq \emptyset.$$

PROOF. Let  $D$  be an inwards stable droplet with  $0 \in D$ , and  $\hat{\rho} > 0$  such that  $D \subset B_{\hat{\rho}}$ . Given  $x \in \langle A \rangle_{\mathcal{U}}$ , the translation  $x + D$  is also inwards stable and  $x \in \langle A \rangle_{\mathcal{U}} \cap (x + D)$ . Thus

$$A \cap B_{\hat{\rho}}(x) \supset A \cap (x + D) \neq \emptyset. \quad \square$$

DEFINITION 4.6 (Dilation radius). We define the *dilation radius*  $\beta := \beta(\mathcal{U})$  to be the smallest radius  $\hat{\rho} \geq 1$  satisfying the conclusion in Corollary 4.5.

Note that

$$(21) \quad |B'_{3\beta}| \leq 30\beta^2.$$

4.2. *Exponential decay.* We will use a specific collection of finite subtrees of  $\mathbb{Z}^2$ .

DEFINITION 4.7. For  $n \geq 0$ , we let  $\mathcal{T}_{0,n}$  to be the collection of all trees  $T \subset \mathbb{Z}^2$  containing the origin  $0 \in \mathbb{Z}^2$  and other  $n$  vertices (so that  $|T| = n + 1$ ). We also define the collection of all trees containing 0 and having at most  $n$  vertices ( $|T| \leq n$ ) by

$$(22) \quad \mathcal{T}_{\leq n} := \bigcup_{k=1}^n \mathcal{T}_{0,k-1}.$$

A key ingredient to prove the exponential decay theorem is an upper bound for  $|\mathcal{T}_{\leq n}|$ . The following proposition is a particular case of a beautiful problem in the book *The art of mathematics: Coffee time in Memphis* (see Problem 45 in [6]).

PROPOSITION 4.8. *For every  $n \geq 1$ , we have  $|\mathcal{T}_{0,n}| \leq (3e)^n$ . As a consequence, we also have  $|\mathcal{T}_{\leq n}| \leq (3e)^n$ .*

Consider  $\mathcal{U}$ -bootstrap percolation with initially infected set  $A \subset \mathbb{Z}^2$ , where  $\mathcal{S}(\mathcal{U}) = S^1$  and let  $\beta$  be the dilation radius.

LEMMA 4.9 (Extremal lemma for  $\mathcal{K}$ ). *If  $|\mathcal{K}| \geq n$  then, there exists a tree  $T \in \mathcal{T}_{\leq n}$  such that*

$$(23) \quad |A \cap T| \geq (30\beta^2)^{-1}n.$$

PROOF. In fact, let us suppose that  $|\mathcal{K}| \geq 30\beta^2n$ , and recursively find  $n$  distinct vertices  $x'_1, \dots, x'_n \in A \cap T$ , for some tree  $T \in \mathcal{T}_{\leq 30\beta^2n}$ .

By definition of  $\beta$ , for  $x_1 = 0 \in \langle A \rangle_{\mathcal{U}}$  there exists  $x'_1 \in A \cap B'_\beta(x_1)$ . Then set  $K_1 = B'_\beta(x_1)$ , and since  $|K_1| \leq 30\beta^2$  we can find a vertex  $x_2 \in \mathcal{K} \setminus K_1$ , which is at distance 1 from  $K_1$ ; now we apply Corollary 4.5 to  $x_2 \in \langle A \rangle_{\mathcal{U}}$  and find a new vertex  $x'_2 \in A \cap B'_\beta(x_2)$ . Proceed in this way, for  $i \leq n$ , assume we have found vertex  $x'_i \in A \cap B'_\beta(x_{i-1})$ , then set

$$K_i = B'_\beta(x_i) \cup K_{i-1}.$$

Since  $|K_i| \leq 30\beta^2i$ , for  $i = 1, \dots, n - 1$ , we have

$$|\mathcal{K} \setminus K_i| \geq 30\beta^2n - 30\beta^2i \geq 1,$$

so we can find a vertex  $x_{i+1} \in \mathcal{K} \setminus K_i$ , which is at distance 1 from  $K_i$ . Observe that at step  $n - 1$  we still have  $|\mathcal{K} \setminus K_{n-1}| \geq 30\beta^2 \geq 1$ , so for  $x_n \in \mathcal{K} \setminus K_{n-1}$  we can apply the corollary one more time to get our last vertex  $x'_n \in A$ . For  $i = 1, \dots, n$ , the vertices  $x'_i$  are all distinct because all balls  $B'_\beta(x_i)$  are pairwise disjoint by construction.

Finally, consider a spanning tree  $T$  of  $K_n$ , and note that  $x_i, x'_i \in T$  for all  $i = 1, \dots, n$ . In particular,  $|A \cap T| \geq n$ , and the fact that  $T \in \mathcal{T}_{\leq 30\beta^2n}$  follows from  $0 = x_1 \in T$  and  $|T| \leq |K_n| \leq 30\beta^2n$ .  $\square$

The same proof allows us to deduce another similar extremal lemma.

LEMMA 4.10. *There exists a constant  $\lambda \in (0, 30\beta^2]$  such that, if  $\langle A \rangle_{\mathcal{U}}$  is connected then,*

$$(24) \quad |\langle A \rangle_{\mathcal{U}}| \leq \lambda|A|.$$

PROOF. If  $A$  is infinite, we have nothing to show. Assume  $A$  is finite, then it is contained in a big rectangle  $R \subset \mathbb{Z}^2$ , since  $\pm e_1, \pm e_2 \in \mathcal{S}$ , so  $\langle A \rangle_{\mathcal{U}} \subset R$  is also finite. Since  $\langle A \rangle_{\mathcal{U}}$  is connected, the above proof shows that  $|\langle A \rangle_{\mathcal{U}}| > 30\beta^2n$  implies  $|A| > n$ . In other words,  $|A| = n$  implies  $|\langle A \rangle_{\mathcal{U}}| \leq 30\beta^2n = 30\beta^2|A|$ .  $\square$

The following is a quantitative reformulation of Theorem 4.2, whose proof is inspired by lines through the book *Percolation* of Bollobás and Riordan (see page 70 in [8]).

THEOREM 4.11 (Exponential decay for the cluster size). *Consider subcritical  $\mathcal{U}$ -bootstrap percolation on  $\mathbb{Z}^2$  with  $\mathcal{S}(\mathcal{U}) = S^1$  and let  $\beta \geq 1$  be the dilation radius. If  $0 < \varepsilon < e^{-150\beta^2}$  and  $C = C(\varepsilon) := -\frac{1}{60\beta^2} \log(\varepsilon)$ , then*

$$(25) \quad \mathbb{P}_\varepsilon(|\mathcal{K}| \geq n) \leq \varepsilon^{\frac{1}{60\beta^2}n} = e^{-Cn},$$

for every  $n \in \mathbb{N}$ .



PROOF. By Lemma 4.9 and Proposition 4.8, with  $\delta = (30\beta^2)^{-1}$ , we obtain

$$\begin{aligned} \mathbb{P}_\varepsilon(|\mathcal{K}| \geq n) &\leq \mathbb{P}_\varepsilon\left(\bigcup_{T \in \mathcal{T}_{\leq n}} \{|A \cap T| \geq \delta n\}\right) \leq \sum_{T \in \mathcal{T}_{\leq n}} \mathbb{P}_\varepsilon(|A \cap T| \geq \delta n) \leq \sum_{T \in \mathcal{T}_{\leq n}} \binom{n}{\delta n} \varepsilon^{\delta n} \\ &\leq \sum_{T \in \mathcal{T}_{\leq n}} (e\delta^{-1}\varepsilon)^{\delta n} \leq ([3e][e\delta^{-1}\varepsilon]^\delta)^n \leq e^{-Cn}, \end{aligned}$$

and we are done.  $\square$

**5. Lower bounds via beams process.** To deal with the cases  $c \geq a + b$ , we introduce a new tool which we call the *beams process*. This time, instead of covering the infected vertices step by step with components, we cover them with beams, so that when we observe this induced process along the  $e_3$ -direction it looks like subcritical two-dimensional bootstrap percolation.

Consider the family  $\mathcal{N}_m^{a,b}$  given by the collection of all subsets of size  $m$  of

$$(26) \quad N_{a,b} = \{a'e_1 : \pm a' \in [a]\} \cup \{b'e_2 : \pm b' \in [b]\}.$$

Observe that  $\mathcal{S}(\mathcal{N}_m^{a,b}) = S^1$  if and only if  $m \geq a + b + 1$ , in particular, our exponential decay result (Theorem 4.11) holds for these families. From now on, we set

$$(27) \quad m := a + b + 1.$$

DEFINITION 5.1. A *beam* is a finite subset of  $\mathbb{Z}^3$  of the form  $H \times [w]$ , where  $H \subset \mathbb{Z}^2$  is connected and  $\langle H \rangle_{\mathcal{N}_m^{a,b}} = H$ .

It will be important for us to have an upper bound on the number of beams of a given size, which are contained in  $[L]^3$ . The following lemma is another consequence of Proposition 4.8.

LEMMA 5.2 (Counting beams). Let  $\mathcal{B}_{n_1,n_2}$  be the collection of all copies of the beam  $H \times [w]$  contained in  $[L]^3$  satisfying  $w \leq n_1$  and  $|H| \leq n_2$ . Then

$$|\mathcal{B}_{n_1,n_2}| \leq n_1 L^3 (3e)^{n_2}.$$

PROOF. The number of segments inside  $[L]$  with at most  $n_1$  vertices, is at most  $n_1 L$ .

Now we give an upper bound for the number of  $H$ 's. Let  $\mathcal{H}_h$  denote the collection of all connected sets  $H \subset [L]^2$  such that  $|H| = h$ , so we can write

$$h|\mathcal{H}_h| = \sum_{H \in \mathcal{H}_h} |H| = \sum_{x \in [L]^2} \sum_{H \in \mathcal{H}_h} \mathbf{1}_{\{x \in H\}} = \sum_{x \in [L]^2} \text{cs}(x),$$

where  $\text{cs}(x)$  is the number of connected subsets of  $[L]^2$  with size  $h + 1$ , containing a fixed point  $x$ . To each of such sets, we can associate an spanning tree in an injective fashion, so by Proposition 4.8,  $|\mathcal{H}_h| \leq L^2 (3e)^{h-1}$ . It follows that the number of  $H$ 's is at most

$$\sum_{h=1}^{n_2} |\mathcal{H}_h| \leq L^2 \sum_{h=1}^{n_2} (3e)^{h-1} \leq L^2 (3e)^{n_2}.$$

$\square$

5.1. *The beams process.*

DEFINITION 5.3. Given finite connected sets  $S_1, S_2 \subset \mathbb{Z}^3$ , we say that a beam  $H \times [w]$  is *generated by*  $(S_1, S_2)$  if it can be constructed in the following way: by translating  $S_1 \cup S_2$  if necessary, we can assume that the smallest rectangular block containing it is  $R \times [w]$ , then consider the connected sets  $H_1, H_2 \subset \mathbb{Z}^2$  given by

$$H_t := \{x \in R : (\{x\} \times [w]) \cap S_t \neq \emptyset\}, \quad t = 1, 2.$$

If  $\langle H_1 \cup H_2 \rangle_{\mathcal{N}_m^{a,b}}$  is connected, then we take  $H := \langle H_1 \cup H_2 \rangle_{\mathcal{N}_m^{a,b}}$ . Otherwise, we let  $P \subset R$  be any path with minimal size connecting  $H_1$  to  $H_2$  and then set  $H := \langle H_1 \cup H_2 \cup P \rangle_{\mathcal{N}_m^{a,b}}$ .

In this definition  $\langle S_1 \cup S_2 \rangle \subset H \times [w]$  for each  $r \geq m$ , and generated beams could depend on the choice of the path  $P$ . However, such minimal paths are not relevant for our purposes.

NOTATION 5.4. We will denote any fixed beam generated by  $(S_1, S_2)$  as  $B(S_1 \cup S_2)$ , regardless the choice of  $P$ .

We want to track the process of infection by covering all possible infected sites with beams, we do that step by step in order to get some control over the sizes. The following algorithm is a variation of the components process. We will use it to show an Aizenman–Lebowitz-type lemma which says that when  $[L]^3$  is internally filled, then it contains *covered* beams of all intermediate sizes (see Lemma 5.7 below).

DEFINITION 5.5 (The beams process). Let  $A = \{x_1, \dots, x_{|A|}\} \subset [L]^3$  and fix  $r \geq c + 1$ . Set  $\mathcal{B} := \{S_1, \dots, S_{|A|}\}$ , where  $S_i = \{x_i\}$  for each  $i = 1, \dots, |A|$ , and repeat until STOP:

1. If there exist distinct beams  $S_1, S_2 \in \mathcal{B}$  such that

$$S_1 \cup S_2$$

is strongly connected, then remove it from  $\mathcal{B}$ , and replace by  $B(S_1 \cup S_2)$ .

2. If there do not exist such a family of sets in  $\mathcal{B}$ , then STOP.

We call any beam  $S = B(S_1 \cup S_2) \subset [L]^3$  added to the collection  $\mathcal{B}$  a *covered beam*, and denote the event that  $S$  is covered by  $I^\#(S)$ .

Again, there are two properties that are due to the way the algorithm evolves:

- Any covered beam  $S$  satisfies  $\langle A \cap S \rangle \subset \langle S \rangle = S$ .
- The process stops in finite time, thus, we can consider the final collection  $\mathcal{B}'$  and set  $V(\mathcal{B}') := \bigcup_{S \in \mathcal{B}'} S$ . By using the same arguments in the proof of Lemma 3.10, it follows that  $\langle A \rangle \subset V(\mathcal{B}')$ .

5.2. *Case  $c = a + b$ .* In this section, we prove the following.

PROPOSITION 5.6. Under  $\mathcal{N}_m^{a,b,a+b}$ -bootstrap percolation, there is a constant  $\gamma = \gamma(a, b) > 0$  such that, if

$$L < \exp(\gamma p^{-1}),$$

then  $\mathbb{P}_p[I^\bullet([L]^3)] \rightarrow 0, asp \rightarrow 0$ .

The beams process and Lemma 4.10 allow us to prove a beams version of the Aizenman–Lebowitz lemma for this case. Let  $\lambda > 0$  be the constant in Lemma 4.10 associated to the subcritical two-dimensional family  $\mathcal{N}_m^{a,b}$ .

LEMMA 5.7. Consider  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation with  $r \geq c + 1$ . If  $[L]^3$  is internally filled, then there is a constant  $\kappa \in \mathbb{N}$  such that for every  $k = \kappa, \dots, L$ , there exists a covered beam  $H \times [w]$  satisfying  $w, |H| \leq 3\lambda k$ , and either  $w \geq k$  or  $|H| \geq k$ .

PROOF. Let  $S = H \times [w]$  be the first beam that appears in the beam process satisfying either  $w \geq k$  or  $|H| \geq k$  (such a set exists since  $V(\mathcal{B}^r) = [L]^3$ ). Then it is enough to show that  $w \leq rk$  and  $|H| \leq 3\lambda k$ .

We know that  $S = B(S_1 \cup S_2)$  for some beams  $S_t = H_t \times [w_t]$  such that  $S_1 \cup S_2$  is strongly connected. Moreover, by definition of  $S$ ,  $w_t \leq k - 1$  for  $t = 1, 2$ , so

$$(28) \quad w \leq 2 \max_{t=1,2} \{w_t\} + r \leq r(k - 1) + r \leq rk.$$

Analogously,  $|H_t| \leq k - 1$ , and we know that  $H = \langle H_1 \cup H_2 \cup P \rangle_{\mathcal{N}_m^{a,b}}$  for some path  $P$  with bounded (or zero) size,  $H$  is connected, so by Lemma 4.10,

$$(29) \quad |H| \leq \lambda \cdot 2 \max_{t=1,2} \{|H_t|\} + \lambda|P| \leq 2\lambda(k - 1) + O(1) \leq 3\lambda k. \quad \square$$

Now, let us prove the lower bound in the case  $c = a + b$ .

PROOF OF PROPOSITION 5.6. Take  $L < \exp(\gamma p^{-1})$ , where  $\gamma > 0$  is some small constant. Let us show that  $\mathbb{P}_p(I^\bullet([L]^3))$  goes to 0, as  $p \rightarrow 0$ . Fix  $\varepsilon > 0$ .

If  $[L]^3$  is internally filled, by Lemma 5.7 there exists a covered beam  $S = H \times [w] \subset [L]^3$  satisfying  $w, |H| \leq \varepsilon/p$ , and moreover, either  $w \geq \varepsilon/3\lambda p$  or  $|H| \geq \varepsilon/3\lambda p$ , hence, by union bound,  $\mathbb{P}_p[I^\bullet([L]^3)]$  is at most

$$\sum_{S \in \mathcal{B}_{\frac{\varepsilon}{p}, \frac{\varepsilon}{p}}} (\mathbb{P}_p[I^\#(S) \cap \{w \geq \varepsilon/3\lambda p\}] + \mathbb{P}_p[I^\#(S) \cap \{|H| \geq \varepsilon/3\lambda p\}]).$$

To bound the first term, we use the fact that  $H \times [w]$  is covered; this implies that there is no gap of size  $r$  along the  $e_3$ -direction. Therefore, by considering the  $w/r$  disjoint slabs, if  $\varepsilon$  is small, then there exists some  $c_1 = c_1(\varepsilon, r) > 0$  such that

$$\mathbb{P}_p[I^\#(H \times [w]) \cap \{w \geq \varepsilon/3\lambda p\}] \leq (1 - e^{-\Omega(p|H|)})^{w/r} = (1 - e^{-\Omega(\varepsilon)})^{\varepsilon/2r\lambda p} \leq e^{-c_1/p}.$$

To bound the second term, for each  $S \in \mathcal{B}_{\frac{\varepsilon}{p}, \frac{\varepsilon}{p}}$  consider the set

$$A' := \{x \in [L]^2 : (\{x\} \times [w]) \cap \langle A \cap S \rangle \neq \emptyset\}.$$

In other words,  $x \in A'$  if and only if there exists  $v \in \{x\} \times [w]$  such that either  $v \in A$ , or  $v \in S$  got infected by using at least  $m$  infected neighbors in  $v + N_{a,b} \times \{0\}$ , where  $N_{a,b}$  is given by (26). Now, by applying Markov's inequality,

$$\mathbb{P}_p(A \cap (\{x\} \times [w]) \neq \emptyset) = O(wp) \leq \varepsilon.$$

Therefore, by monotonicity we can couple the process on  $[L]^2 \times [w]$  having initial infected set  $A$ , with  $\mathcal{N}_m^{a,b}$ -bootstrap percolation on  $[L]^2 \times \{0\} \subset \mathbb{Z}^2$  where the initial infected set is chosen to be  $\varepsilon$ -random.

In particular, under  $\mathcal{N}_m^{a,b}$ -bootstrap percolation there should exist a connected component of size at least  $|H| \geq \varepsilon/3\lambda p$  inside  $[L]^2$ . On the other hand, there are at most  $L^2$  possible ways to place the origin in  $H$ , so if  $\mathcal{K}$  denotes the cluster of 0, Theorem 4.11 implies

$$\begin{aligned} \mathbb{P}_p[I^\#(S) \cap \{|H| \geq \varepsilon/3\lambda p\}] &\leq \sum_{x \in [L]^2} \mathbb{P}_\varepsilon(\{|\mathcal{K}| > \varepsilon/3\lambda p\} \cap \{x = 0\}) \leq L^2 \mathbb{P}_\varepsilon(|\mathcal{K}| \geq \varepsilon/3\lambda p) \\ &\leq e^{2\gamma/p} e^{-C\varepsilon/3\lambda p} = e^{-(C\varepsilon/3\lambda - 2\gamma)/p}, \end{aligned}$$

where  $C = -\frac{1}{60\beta^2} \log \varepsilon$  and we choose  $\varepsilon > 0$  such that  $C\varepsilon > 0$  and  $\gamma < C\varepsilon/6\lambda$  at first. By Lemma 5.2, we conclude that

$$\begin{aligned} \mathbb{P}_p[I^\bullet([L]^3)] &\leq \sum_{S \in \mathcal{B}_{\frac{\varepsilon}{p}, \frac{\varepsilon}{p}}} (e^{-c_1/p} + e^{-(C\varepsilon/3\lambda - 2\gamma)/p}) \leq \frac{\varepsilon}{p} L^3 (3e)^{\varepsilon/p} e^{-c_2/p} \\ &\leq e^{4\gamma/p} e^{\varepsilon \log(3e)/p} e^{-c_2/p} \rightarrow 0, \end{aligned}$$

for  $c_2, \gamma > 0$  small enough.  $\square$

5.3. *The coarse beams process.* In this section, we study the last case  $c \geq a + b + 1$ . The lower bound will be proved by using a coupling with subcritical two-dimensional bootstrap percolation again, as we did in the previous section, however, this time we infect squares instead of single vertices. The trick now is to consider the following coarser process.

DEFINITION 5.8 (Coarse bootstrap percolation). Assume that  $b + 1$  divides  $L$  and we partition  $[L]^2$  as  $L^2/(b + 1)^2$  copies of  $\boxplus := [b + 1]^2$  in the obvious way, and think of  $\boxplus$  as a single vertex in the new scaled grid  $[L/(b + 1)]^2$ . Given a two-dimensional family  $\mathcal{U}$ , suppose we have some fully infected copies of  $\boxplus \in [L/(b + 1)]^2$  and denote this initially infected set by  $A$ , then we define *coarse  $\mathcal{U}$ -bootstrap percolation* to be the result of applying  $\mathcal{U}$ -bootstrap percolation to the new rescaled vertices. We denote the closure of this process by  $\langle A \rangle_b$ .

To avoid trivialities, we assume that  $b + 1$  divides  $L$ . Set

$$m := a + b + 1 < c + 1 = r.$$

DEFINITION 5.9. A *coarse beam* is a finite set of the form  $H \times [w]$ , where  $H \subset \mathbb{Z}^2$  is connected and  $\langle H \rangle_b = H$  under coarse  $\mathcal{N}_m^{a,b}$ -bootstrap percolation.

NOTATION 5.10. Given finite connected sets  $S_1, S_2 \subset [L]^2 \times [L]$ , we partition  $[L]^2$  as in Definition 5.8 and denote by  $B_b(S_1 \cup S_2)$  the coarse beam generated by  $(S_1, S_2)$  which is constructed in the (coarse) analogous way, as we did in Definition 5.3, using coarse paths when needed. Note that every coarse beam is a beam in the sense of the previous section.

The following algorithm is a refinement of that one given in Definition 5.5.

DEFINITION 5.11 (The coarse beams process). Let  $A = \{x_1, \dots, x_{|A|}\} \subset [L]^3$  and fix  $r \geq c + 1$ . Set  $\mathcal{B} := \{S_1, \dots, S_{|A|}\}$ , where  $S_i = \{x_i\}$  for each  $i = 1, \dots, |A|$ , and repeat until STOP:

1. If there exist distinct beams  $S_1, S_2 \in \mathcal{B}$  such that

$$S_1 \cup S_2$$

is strongly connected, and  $\langle S_1 \cup S_2 \rangle \neq S_1 \cup S_2$ , then choose a minimal such family, remove it from  $\mathcal{B}$ , and replace by a coarse beam  $B_b(S_1 \cup S_2)$ .

2. If there do not exist such a family of sets in  $\mathcal{B}$ , then STOP.

We call any beam  $S = B_b(S_1 \cup S_2)$  added to the collection  $\mathcal{B}$  a *covered beam*, and denote the event that  $S$  is covered by  $I_b^\#(S)$ .

The two highlighted usual properties are preserved for this algorithm, too:

- Any covered beam  $S$  satisfies  $\langle A \cap S \rangle \subset \langle S \rangle = S$ .
- There is a final collection  $\mathcal{B}'$  and we can set  $V(\mathcal{B}') := \bigcup_{S \in \mathcal{B}'} S$ . Then we also have  $\langle A \rangle \subset V(\mathcal{B}')$ .

5.4. *Case  $c > a + b$ .* In this section, we prove the lower bound corresponding to the last case.

PROPOSITION 5.12. *Under  $\mathcal{N}_{c+1}^{a,b,c}$ -bootstrap percolation with  $c > a + b$ , there exists a constant  $\gamma = \gamma(c) > 0$  such that, if*

$$L < \exp(\gamma p^{-1}(\log p)^2),$$

then  $\mathbb{P}_p[I^\bullet([L]^3)] \rightarrow 0$ , as  $p \rightarrow 0$ .

We state an analogue of Lemma 5.7 for the coarse beams setting without proof because the arguments are exactly the same. However, we obtain slightly different constants since the number of vertices of the form  $\boxplus$  in a coarse beam  $H$  equals  $|H|/(b + 1)^2$ .

Consider  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation with  $r \geq c + 1$ , and let  $\kappa_0$  be a large positive integer depending on  $a, b, c$  and  $r$ .

LEMMA 5.13. *If  $[L]^3$  is internally filled then for every  $h, k = \kappa_0, \dots, L$ , there exists a covered (coarse) beam  $H \times [w] \subset [L]^3$  satisfying  $w \leq rk, |H| \leq 2(b + 1)^2\lambda h$ , and either  $w \geq k$  or  $|H| \geq h$ .*

Finally, we prove the lower bound for the remaining case.

PROOF OF PROPOSITION 5.12. Take  $L < \exp(\gamma p^{-1}(\log p)^2)$ , where  $\gamma > 0$  is some small constant. Let us show that  $\mathbb{P}_p(I^\bullet([L]^3))$  goes to 0, as  $p \rightarrow 0$ . Fix  $\delta > 0$  and set

$$h = \delta p^{-1} \log \frac{1}{p}, \quad k = p^{-\frac{3}{2}}.$$

If  $[L]^3$  is internally filled, by Lemma 5.13 there exists a covered beam  $S = H \times [w] \subset [L]^3$  satisfying  $w \leq k, |H| \leq (b + 1)^2h$ , and either  $w \geq k/2\lambda$  or  $|H| \geq h/2\lambda$  (as we said, the cardinality of  $H$  viewing  $S$  as a beam equal  $(b + 1)^2|H|$  viewing  $S$  as a coarse beam), hence  $\mathbb{P}_p[I^\bullet([L]^3)]$  is at most

$$\sum_{S \in \mathcal{B}_{k,(b+1)^2h}} (\mathbb{P}_p[I_b^\#(S) \cap \{w \geq k/2\lambda\}] + \mathbb{P}_p[I_b^\#(S) \cap \{|H| \geq h/2\lambda\}]).$$

To bound the first term, we use the fact that  $A \cap (H \times \{rk + 1, \dots, rk + r\}) \neq \emptyset$  for all  $k = 0, \dots, w/r - 1$ , since  $H \times [w]$  is covered. Therefore, for some  $c_1 > 0$ ,

$$\begin{aligned} \mathbb{P}_p[I_b^\#(H \times [w]) \cap \{w \geq k/2\lambda\}] &\leq (1 - (1 - p)^{rh})^{w/r} \leq (1 - e^{-2r\epsilon \log \frac{1}{p}})^{k/2r\lambda} \\ &\leq (1 - p^{2r\epsilon})^{k/2r\lambda} \leq e^{-p^{2r\epsilon-3/2}/2r\lambda} = e^{-c_1 p^{-1}(\log \frac{1}{p})^2}. \end{aligned}$$

To bound the second term, we use the fact that  $r = c + 1 \geq a + b + 2$ . More precisely, if  $[L]^3$  is internally filled, then every copy of  $[b + 1]^2 \times [L]$  should contain at least 2 vertices of  $A$  within some constant distance, otherwise, there is no way to infect such a copy.

Then, given  $S = H \times [w] \in \mathcal{B}_{k,(b+1)^2h}$  consider the set  $A'$  consisting of all copies of  $\boxplus \subset [L]^2$  (as in Definition 5.8) such that the rectangular block  $\boxplus \times [w] \subset S$  contains at least 2 vertices of  $A$  within distance  $r$ . By union bound, the probability of finding such vertices is at most

$$\sum_{x \in \boxplus \times [w]} \sum_{0 < \|y-x\| \leq r} \mathbb{P}_p(x, y \in A) \leq \tilde{C} w p^2 \leq p^{\frac{1}{3}}.$$

Therefore, by monotonicity we can couple the process in  $[L]^2 \times [w]$  having initial infected set  $A$ , with coarse  $\mathcal{N}_m^{a,b}$ -bootstrap percolation on  $[L/(b+1)]^2$  and initial infected set  $\varepsilon$ -random with  $\varepsilon = \varepsilon(p) := p^{1/3}$ .

In particular, under  $\mathcal{N}_m^{a,b}$  (coarse) there should exist a connected component of size at least  $|H| \geq h/2\lambda$  inside  $[L]^2$ . Once more, there are at most  $L^2$  possible ways to place the origin in  $H$ , so if  $\mathcal{K}$  denotes the (coarse) cluster of 0, Theorem 4.11 implies

$$\begin{aligned} \mathbb{P}_p[I_b^\#(S) \cap \{|H| \geq h/2\lambda\}] &\leq \sum_{\boxplus \subset [L]^2} \mathbb{P}_\varepsilon(\{|\mathcal{K}| > h/2\lambda\} \cap \{\boxplus = 0\}) \leq L^2 \mathbb{P}_\varepsilon(|\mathcal{K}| \geq h/2\lambda) \\ &\leq e^{2\gamma p^{-1}(\log \frac{1}{p})^2} e^{-Ch/2\lambda} = e^{-(c'-2\gamma)p^{-1}(\log \frac{1}{p})^2}, \end{aligned}$$

for some constant  $c' = c'(\beta, \lambda) > 0$  (recall that  $C \sim -\log p$  asymptotically, by Theorem 4.11). Take  $\gamma < c'/2$  at first; by Lemma 5.2 we conclude that

$$\begin{aligned} \mathbb{P}_p[I^\bullet([L]^3)] &\leq \sum_{S \in \mathcal{B}_{k,(b+1)^2h}} (e^{-c_1 p^{-1}(\log \frac{1}{p})^2} + e^{-(c'-2\gamma)p^{-1}(\log \frac{1}{p})^2}) \\ &\leq kL^3 (3e)^{(b+1)^2h} e^{-c_3 p^{-1}(\log \frac{1}{p})^2} \leq e^{4\gamma p^{-1}(\log \frac{1}{p})^2} e^{-c_3 p^{-1}(\log \frac{1}{p})^2} \rightarrow 0, \end{aligned}$$

as  $p \rightarrow 0$ , for  $c_3, \gamma > 0$  small enough, and we are finished.  $\square$

**6. Future work.** All proofs in this paper extend to the case  $r = c + 2$ , and can be used to determine  $\log L_c(\mathcal{N}_{c+2}^{a,b,c}, p)$  up to a constant factor, for all triples  $(a, b, c)$ , except for  $c = a + b - 1$  which is a new interesting case to be studied separately.

In general, a problem which remains open is the determination of the threshold for other values of  $r$ . We believe that the techniques used to prove Theorem 1.1 can be adapted to cover all  $c + 1 < r \leq b + c$  (though significant technical obstacles remain); in this case, by Proposition A.1 below, the critical length is singly exponential. However, to deal with the cases  $b + c < r < a + b + c$ , the techniques required are likely to be more similar to those of [10] and [22], and the critical length should be doubly exponential.

Finally, Theorem 4.11 can be generalized to all dimensions  $d \geq 3$  and all families  $\mathcal{U}$  such that  $\mathcal{S}(\mathcal{U}) = S^{d-1}$ . However, we do not know if this property holds for subcritical families  $\mathcal{U}$  satisfying  $\mathcal{S}(\mathcal{U}) \neq S^{d-1}$ . In order to determine the critical lengths for general critical models, it could be useful to extend this result to a wider class of subcritical families.

**PROBLEM 6.1.** Characterize the subcritical  $d$ -dimensional update families  $\mathcal{U}$  such that  $\mathcal{K}$  has the exponential decay property.

### APPENDIX: A GENERAL UPPER BOUND

In this Appendix, we assume that  $r \leq b + c$  and show that the critical length is at most singly exponential in this case, as we claimed above. Consider  $\mathcal{N}_r^{a,b,c}$ -bootstrap percolation.

**PROPOSITION A.1.** *Given  $r \in \{c + 1, \dots, c + b\}$ , there exists  $\Gamma = \Gamma(c) > 0$  such that, if  $L > L_c(\mathcal{N}_r^{b,c}, p)^\Gamma$ , then  $\mathbb{P}_p(\langle A \rangle_{\mathcal{N}_r^{a,b,c}} = [L]^3) \rightarrow 1$ , as  $p \rightarrow 0$ . Thus,*

$$\log L_c(\mathcal{N}_r^{a,b,c}, p) = O(\log L_c(\mathcal{N}_r^{b,c}, p)) = O(p^{-(r-c)}(\log p)^2).$$

**REMARK A.2.** This proposition, in particular, already gives us the upper bound in the case  $c > a + b$  of our main Theorem 1.1. It also shows that  $\mathcal{N}_r^{a,b,c}$  is 2-critical for all  $r \in \{c + 1, \dots, c + b\}$ ; in fact, since  $L_c(\mathcal{N}_r^{a,b,c}, p)$  is increasing in  $r$ , by Proposition A.1,

$$\log L_c(\mathcal{N}_r^{a,b,c}, p) \leq \log L_c(\mathcal{N}_{c+b}^{a,b,c}, p) \leq O\left(p^{-b} \left(\log \frac{1}{p}\right)^2\right).$$

Moreover, by Theorem 1.1 we also have

$$\log L_c(\mathcal{N}_r^{a,b,c}, p) \geq \log L_c(\mathcal{N}_{c+1}^{a,b,c}, p) \geq \Omega(p^{-1/2}).$$

To prove this proposition, we will use dimensional reduction by means of a renormalization argument, and show that filling the whole of  $[L]^3$  is at most as hard as filling  $L$  disjoint copies of  $[L]^2$  which are orthogonal to the  $e_1$ -direction.

To do so in this regime, we will compare the family  $\mathcal{N}_r^{a,b,c}$  with the two-dimensional family  $\mathcal{N}_r^{b,c}$  consisting of all subsets of size  $r$  of the set  $N_{b,c}$  given by (26). It turns out that  $\mathcal{N}_r^{b,c}$  is critical if and only if  $r$  belongs to this regime, and in this case  $\mathcal{S}(\mathcal{N}_r^{b,c}) = \{\pm e_1, \pm e_2\}$ . The key step is to refine the upper bounds in (3), by using standard renormalization techniques.

**LEMMA A.3 (Renormalization).** *Fix  $r \in \{c+1, \dots, c+b\}$ . Under  $\mathcal{N}_r^{b,c}$ -bootstrap percolation, there exists a constant  $\Gamma' > 0$  depending on  $c$  such that*

$$(30) \quad \mathbb{P}_p(\langle A \rangle_{\mathcal{N}_r^{b,c}} = [L]^2) \geq 1 - \exp(-L^{1/2}),$$

for all  $p$  small enough and  $L > L_c(\mathcal{N}_r^{b,c}, p)^{\Gamma'}$ .

**PROOF.** See, for example, [21].  $\square$

Now, we prove the general upper bound.

**PROOF OF PROPOSITION A.1.** Decompose  $[L]^3$  as  $L$  consecutive copies of  $[L]^2$  all of them orthogonal to the  $e_1$ -direction, and call those copies  $R_i := \{i\} \times [L]^2$ .

Now, we couple the original process with the reduced two-dimensional processes; if for each  $i \in \{1, \dots, L\}$ ,  $\langle A \cap R_i \rangle_{\mathcal{N}_r^{b,c}} = R_i$  in the  $\mathcal{N}_r^{b,c}$ -bootstrap process, then  $[L]^3$  is internally filled. Therefore, by Lemma A.3 we have

$$\begin{aligned} \mathbb{P}_p(\langle A \rangle_{\mathcal{N}_r^{a,b,c}} = [L]^3) &\geq \mathbb{P}_p\left(\bigcap_{i=1}^L \{\langle A \cap R_i \rangle_{\mathcal{N}_r^{b,c}} = R_i\}\right) = \prod_{i=1}^L \mathbb{P}_p(\langle A \cap R_i \rangle_{\mathcal{N}_r^{b,c}} = R_i) \\ &\geq [1 - \exp(-L^{1/2})]^L \xrightarrow{p \rightarrow 0} 1, \end{aligned}$$

if  $L > \exp(\Gamma' p^{-(r-c)} (\log p)^{2 \cdot 1_{\{c>b\}}})$ .  $\square$

**Acknowledgements.** The author would like to thank Rob Morris for introducing him to bootstrap percolation, and for his many invaluable suggestions. The author is very grateful to the Instituto Nacional de Matemática Pura e Aplicada (IMPA) for the time and space to create, research and write in this strong academic environment, and to CAPES in Brasil for partially supporting this project. The author also wants to thank the referees for their very helpful comments.

## REFERENCES

- [1] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. [MR0968311](#)
- [2] BALISTER, P., BOLLOBÁS, B., PRZYKUCKI, M. and SMITH, P. (2016). Subcritical  $\mathcal{U}$ -bootstrap percolation models have non-trivial phase transitions. *Trans. Amer. Math. Soc.* **368** 7385–7411. [MR3471095](#) <https://doi.org/10.1090/tran/6586>
- [3] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701. [MR2888224](#) <https://doi.org/10.1090/S0002-9947-2011-05552-2>

- [4] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2009). Bootstrap percolation in three dimensions. *Ann. Probab.* **37** 1329–1380. MR2546747 <https://doi.org/10.1214/08-AOP433>
- [5] BLANQUICETT, D. Fixation for two-dimensional  $\mathcal{U}$ -Ising and  $\mathcal{U}$ -voter dynamics. Submitted. Available at [arXiv:2003.02420](https://arxiv.org/abs/2003.02420).
- [6] BOLLOBÁS, B. (2006). *The Art of Mathematics: Coffee Time in Memphis*. Cambridge Univ. Press, New York. MR2285090 <https://doi.org/10.1017/CBO9780511816574>
- [7] BOLLOBÁS, B., DUMINIL-COPIN, H., MORRIS, R. and SMITH, P. Universality of two-dimensional critical cellular automata. *Proc. Lond. Math. Soc.* To appear. Available at [arXiv:1406.6680v3](https://arxiv.org/abs/1406.6680v3).
- [8] BOLLOBÁS, B. and RIORDAN, O. (2006). *Percolation*. Cambridge Univ. Press, New York. MR2283880 <https://doi.org/10.1017/CBO9781139167383>
- [9] BOLLOBÁS, B., SMITH, P. and UZZELL, A. (2015). Monotone cellular automata in a random environment. *Combin. Probab. Comput.* **24** 687–722. MR3350030 <https://doi.org/10.1017/S0963548315000012>
- [10] CERF, R. and CIRILLO, E. N. M. (1999). Finite size scaling in three-dimensional bootstrap percolation. *Ann. Probab.* **27** 1837–1850. MR1742890 <https://doi.org/10.1214/aop/1022677550>
- [11] CERF, R. and MANZO, F. (2002). The threshold regime of finite volume bootstrap percolation. *Stochastic Process. Appl.* **101** 69–82. MR1921442 [https://doi.org/10.1016/S0304-4149\(02\)00124-2](https://doi.org/10.1016/S0304-4149(02)00124-2)
- [12] CHALUPA, J., LEATH, P. L. and REICH, G. R. (1979). Bootstrap percolation on a Bethe lattice. *J. Phys. C* **12** L31–L35.
- [13] DUMINIL-COPIN, H. and VAN ENTER, A. C. D. (2013). Sharp metastability threshold for an anisotropic bootstrap percolation model. *Ann. Probab.* **41** 1218–1242. MR3098677 <https://doi.org/10.1214/11-AOP722>
- [14] DUMINIL-COPIN, H., VAN ENTER, A. C. D. and HULSHOF, T. (2018). Higher order corrections for anisotropic bootstrap percolation. *Probab. Theory Related Fields* **172** 191–243. MR3851832 <https://doi.org/10.1007/s00440-017-0808-7>
- [15] FONTES, L. R., SCHONMANN, R. H. and SIDORAVICIUS, V. (2002). Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.* **228** 495–518. MR1918786 <https://doi.org/10.1007/s002200200658>
- [16] HARTARSKY, I.  $\mathcal{U}$ -Bootstrap percolation: Critical probability, exponential decay and applications. Submitted. Available at [arXiv:1806.11405v3](https://arxiv.org/abs/1806.11405v3).
- [17] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342 <https://doi.org/10.1007/s00440-002-0239-x>
- [18] MARTINELLI, F., MORRIS, R. and TONINELLI, C. (2019). Universality results for kinetically constrained spin models in two dimensions. *Comm. Math. Phys.* **369** 761–809. MR3962008 <https://doi.org/10.1007/s00220-018-3280-z>
- [19] MORRIS, R. (2011). Zero-temperature Glauber dynamics on  $\mathbb{Z}^d$ . *Probab. Theory Related Fields* **149** 417–434. MR2776621 <https://doi.org/10.1007/s00440-009-0259-x>
- [20] MORRIS, R. (2017). Bootstrap percolation, and other automata. *European J. Combin.* **66** 250–263. MR3692148 <https://doi.org/10.1016/j.ejc.2017.06.024>
- [21] SCHONMANN, R. H. (1992). On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.* **20** 174–193. MR1143417
- [22] VAN ENTER, A. C. D. and FEY, A. (2012). Metastability thresholds for anisotropic bootstrap percolation in three dimensions. *J. Stat. Phys.* **147** 97–112. MR2922761 <https://doi.org/10.1007/s10955-012-0455-4>
- [23] VAN ENTER, A. C. D. and HULSHOF, T. (2007). Finite-size effects for anisotropic bootstrap percolation: Logarithmic corrections. *J. Stat. Phys.* **128** 1383–1389. MR2352015 <https://doi.org/10.1007/s10955-007-9377-y>