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Corrigendum to "Regularity structures and renormalisation of FitzHugh-Nagumo SPDEs in three space dimensions"*

Nils Berglund[†] Christian Kuehn[‡]

Abstract

Lemma 4.8 in the article [1] contains a mistake, which implies a weaker regularity estimate than the one stated in Proposition 4.11. This does not affect the proof of Theorem 2.1, but Theorems 2.2 and 2.3 only follow from the given proof if either the space dimension d is equal to 2, or the nonlinearity $F(U,V)$ is linear in V. To fix this problem and provide a proof of Theorems 2.2 and 2.3 valid in full generality, we consider an alternative formulation of the fixed-point problem, involving a modified integration operator with nonlocal singularity and a slightly different regularity structure. We provide the multilevel Schauder estimates and renormalisation-group analysis required for the fixed-point argument in this new setting.

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1 Set-up and mistake in the original article [1]

In [1], we considered FitzHugh–Nagumo-type SPDEs on the torus $\mathbb{T}^d$, $d \in \{2,3\}$, of the form

$$\begin{aligned} \partial_t u &= \Delta_x u + F(u,v) + \xi^\varepsilon \;, \\ \partial_t v &= a_1 u + a_2 v \;, \end{aligned} \tag{1.1}$$

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[†]Institut Denis Poisson (IDP), Université d'Orléans, Université de Tours, CNRS – UMR 7013, 45067 Orléans, France.
E-mail: nils.berglund@univ-orleans.fr. https://www.idpoisson.fr/berglund/. ORCID: 0000-0001-7044-7885.
[‡]Technical University of Munich (TUM), Faculty of Mathematics, 85748 Garching bei München, Germany.
E-mail: ckuehn@ma.tum.de. http://www.multiscale.systems/

where $F(u,v)$ is a cubic polynomial, ξ^ε denotes mollified space-time white noise, and $a_1, a_2 \in \mathbb{R}$ are scalar parameters (in the case of vectorial v, a_1 is a vector and a_2 is a square matrix). Duhamel's formula allows us to represent (mild) solutions of (1.1) on a bounded interval $[0,T]$ as

$$\begin{aligned} u_t &= \int_0^t S(t-s)\bigl[\xi^\varepsilon_s + F(u_s,v_s)\bigr]\,\mathrm{d}s + S(t)u_0 \;, \\ v_t &= \int_0^t Q(t-s)u_s\,\mathrm{d}s + \mathrm{e}^{ta_2}\,v_0 \;, \end{aligned} \tag{1.2}$$

where S denotes the heat semigroup and $Q(t) := a_1\,\mathrm{e}^{ta_2}\,\chi(t)$, where $\chi : \mathbb{R}_+ \to [0,1]$ is a smooth cut-off function supported on $[0,2T]$ such that $\chi(t) = 1$ for all $t \in [0,T]$.

In [1], we used a lift of (1.2) to a regularity structure of the form

$$\begin{aligned} U &= (\mathcal{K}_{\bar\gamma} + R_\gamma\mathcal{R})\boldsymbol{R}^+\bigl[\Xi + F(U,V)\bigr] + Gu_0 \;, \\ V &= \mathcal{K}^Q_\gamma \boldsymbol{R}^+ U + \widehat{Q}v_0 \;, \end{aligned} \tag{1.3}$$

where $\mathcal{K}_{\bar\gamma}$ is the standard lift of the heat kernel (cf. [2, Section 5]), and $\mathcal{K}^Q_\gamma$ is a new operator lifting time-convolution with Q.

The problem is that [1, Lemma 4.8] is incorrect (it wrongly assumed translation invariance of the model for space-time white noise). As a consequence, [1, Proposition 4.11] does not prove that $\mathcal{K}^Q_\gamma$ maps $\mathcal{D}^{\gamma,\eta}$ into itself for any $\gamma \in (0,\eta+2)$. Instead, it only shows that $\mathcal{K}^Q_\gamma$ maps $\mathcal{D}^{\gamma,\eta}$ into $\mathcal{D}^{\gamma',\eta}$ for some $\gamma' \leqslant \gamma$ that can at best be slightly less than $1/2$.

If we look for a fixed point of (1.3) with $U \in \mathcal{D}^{\gamma,\eta}$, we have in particular to determine the regularity of $F(U,V)$. Let α be the regularity of the stochastic convolution, that is,

$$\alpha = \begin{cases} -\kappa & \text{if } d = 2 \;, \\ -\frac12 - \kappa & \text{if } d = 3 \;. \end{cases} \tag{1.4}$$

Using [2, Proposition 6.12] and $2\eta + \alpha \geqslant 3\eta \wedge (\eta + 2\alpha)$, we find that $U^3 \in \mathcal{D}^{\gamma+2\alpha, 3\eta\wedge(\eta+2\alpha)}$, while

$$V \in \mathcal{D}^{\gamma',\eta} \;, \qquad V^2 \in \mathcal{D}^{\gamma'+\alpha, 2\eta\wedge(\eta+\alpha)} \;, \qquad V^3 \in \mathcal{D}^{\gamma'+2\alpha, 3\eta\wedge(\eta+2\alpha)} \;. \tag{1.5}$$

This implies that

1. If $d = 2$, then $F(U,V)$ is still in a space of modelled distributions $\mathcal{D}^{\gamma'+2\alpha,3\eta\wedge(\eta+2\alpha)}$ with positive exponent $\gamma' + 2\alpha$. This is sufficient to carry out the fixed-point argument stated in [1, Proposition 6.5], which relies in particular on [2, Theorem 7.1], that requires this exponent to be positive.
2. If $d = 3$ and $F(U,V)$ is linear in V, then $F(U,V) \in \mathcal{D}^{(\gamma+2\alpha)\wedge\gamma',3\eta\wedge(\eta+2\alpha)}$. Since $(\gamma + 2\alpha) \wedge \gamma' > 0$, the fixed-point argument again holds.
3. If $d = 3$ and $F(U,V)$ contains terms in V^2 or V^3, however, we can no longer assume that $F(U,V)$ is in a space of modelled distributions with positive exponent, and we cannot apply [2, Theorem 7.1].

We thus conclude that [1, Theorem 2.1], which concerns the standard FitzHugh–Nagumo case with $F(U,V) = U + V - U^3$, still follows from the given proof. Theorems 2.2 and 2.3, however, are only proved if either $d = 2$ or F does not contain any terms in V^2 or V^3.

2 Corrected results

We now provide a different argument allowing to prove the results in full generality. Consider the system (1.1) on the 3-dimensional torus, for a general cubic nonlinearity of the form

$$F(u,v) = \alpha_1 u + \alpha_2 v + \beta_1 u^2 + \beta_2 uv + \beta_3 v^2 + \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uv^2 + \gamma_4 v^3 \;. \tag{2.1}$$

Its renormalised version is given by

$$\begin{aligned} \partial_t u^\varepsilon &= \Delta_x u^\varepsilon + \bigl[F(u^\varepsilon, v^\varepsilon) + c_0(\varepsilon) + c_1(\varepsilon) u^\varepsilon + c_2(\varepsilon) v^\varepsilon\bigr] + \xi^\varepsilon \;, \\ \partial_t v^\varepsilon &= a_1 u^\varepsilon + a_2 v^\varepsilon \;, \end{aligned} \tag{2.2}$$

where $\xi^\varepsilon = \varrho_\varepsilon * \xi$ is a mollification of space-time white noise, with mollifier $\varrho_\varepsilon(t,x) = \varepsilon^{-5}\varrho(\varepsilon^{-2}t, \varepsilon^{-1}x)$ for a compactly supported function $\varrho : \mathbb{R}^4 \to \mathbb{R}$ of integral 1. Below we provide a proof of the following result, which is in fact a slight generalisation of [1, Theorem 2.2].

Theorem 2.1. *Assume $u_0 \in \mathcal{C}^\eta$ for some $\eta > -\frac{2}{3}$ and $v_0 \in \mathcal{C}^\gamma$ for some $\gamma > 1$. Then there exists a choice of constants $c_0(\varepsilon)$, $c_1(\varepsilon)$ and $c_2(\varepsilon)$ such that the system (2.2) with initial condition (u_0, v_0) admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u,v) as $\varepsilon \to 0$. The limit is independent of the choice of mollifier ϱ.*

This result is more general than [1, Theorem 2.2] because we do not assume that $\gamma_2 = 0$, even though we are in dimension $d = 3$. The renormalisation constants $c_i(\varepsilon)$ are given by

$$\begin{aligned} c_0(\varepsilon) &= -\beta_1\bigl[C_1(\varepsilon) + 3\gamma_1 C_2(\varepsilon)\bigr] \;, \\ c_1(\varepsilon) &= -3\gamma_1\bigl[C_1(\varepsilon) + 3\gamma_1 C_2(\varepsilon)\bigr] \;, \\ c_2(\varepsilon) &= -\gamma_2\bigl[C_1(\varepsilon) + 3\gamma_1 C_2(\varepsilon)\bigr] \;, \end{aligned} \tag{2.3}$$

where

$$C_1(\varepsilon) = \int_{\mathbb{R}^4} G_\varepsilon(z)^2 \,\mathrm{d}z \;, \qquad C_2(\varepsilon) = 2\int_{\mathbb{R}^4} G(z) \biggl(\int_{\mathbb{R}^4} G_\varepsilon(z_1) G_\varepsilon(z_1 - z)\,\mathrm{d}z_1\biggr)^2 \mathrm{d}z \;. \tag{2.4}$$

Here G denotes the heat kernel in dimension $d = 3$, and $G_\varepsilon = G * \varrho_\varepsilon$. It is known that $C_1(\varepsilon)$ diverges as ε^{-1} while $C_2(\varepsilon)$ diverges as $\log(\varepsilon^{-1})$.

An analogous result holds for vectorial variables v, in the same way as in [1, Theorem 2.3], but without the restriction on $F(u,v)$ having no terms in $u^2 v_i$. In that case, γ_2 and $c_2(\varepsilon)$ become row vectors of the same dimension as v. Since all arguments are virtually the same, we do not present here the details for this situation.

The main idea for proving Theorem 2.1 is to replace (1.2) by another fixed-point equation, which always involves convolution in space and time. The price to pay is that this leads to an integral kernel with a singularity that is no longer concentrated at the origin, but “smeared out” along the time axis. Therefore we need to rederive the multilevel Schauder estimates for this type of kernel, which we do in Section 3. The resulting fixed-point argument is then considered in Section 4, and the effect of renormalisation is addressed in Section 5.

3 Alternative integral equation

There is an alternative to using the fixed-point equation (1.3). Indeed, substituting the expression for u_t in (1.2) in the expression of v_t and rearranging, we find that v_t can also be represented as

$$v_t = \int_0^t S^Q(t-s)\bigl[\xi^\varepsilon_s + F(u_s, v_s)\bigr]\,\mathrm{d}s + S^Q(t) u_0 + \mathrm{e}^{ta_2}\, v_0 \;, \tag{3.1}$$

where

$$S^Q(t) = \int_0^t Q(t-s)S(s)\,\mathrm{d}s\;. \tag{3.2}$$

Our aim is thus to lift the operation of convolution with S^Q to the regularity structure, in order to obtain an equivalent fixed-point equation of the form

$$\begin{aligned} U &= (\mathcal{K}_{\bar\gamma} + R_\gamma\mathcal{R})\boldsymbol{R}^+\bigl[\Xi + F(U,V)\bigr] + Gu_0\;, \\ V &= (\mathcal{K}^Q_{\bar\gamma} + R^Q_\gamma\mathcal{R})\boldsymbol{R}^+\bigl[\Xi + F(U,V)\bigr] + G^Q u_0 + \widehat{Q}v_0\;, \end{aligned} \tag{3.3}$$

for some suitable kernels $\mathcal{K}^Q_{\bar\gamma}$ and R^Q_γ. We already know that S is represented by convolution with a kernel $G = K + R$. Hence S^Q corresponds to convolution with a kernel $G^Q = K^Q + R^Q$, where the superscript Q always indicates time-convolution with Q. Thus we have to define the lift $\mathcal{K}^Q_\gamma$ of K^Q to the regularity structure, meaning that it should map $\mathcal{D}^{\gamma,\eta}$ into $\mathcal{D}^{\bar\gamma,\bar\eta}$ for some suitable $\bar\gamma$, $\bar\eta$ and satisfy

$$\mathcal{R}\mathcal{K}^Q_\gamma f = K^Q * \mathcal{R}f\;. \tag{3.4}$$

3.1 Decomposition of the kernel

The difficulty is that since K^Q is obtained by convolution in time of K with Q, its singularity is no longer concentrated at the origin, but is “smeared out” along the time axis. In fact, we have the following decomposition result replacing [2, Assumption 5.1]. Note that here and below, we write $z = (t,x)$ for space-time points.

Proposition 3.1. *Assume Q is supported on $[0,2T]$ for a given $T > 0$, fix a scaling $\mathfrak{s} = (\mathfrak{s}_0, \mathfrak{s}_1, \dots, \mathfrak{s}_d)$, and let K be a regularizing kernel of order β (cf. [2, Assumption 5.1]). The kernel K^Q obtained by convoluting Q and K in time can be decomposed as*

$$K^Q(z) = \sum_{(n,m)\in\mathfrak{N}} K^Q_{nm}(z)\;, \tag{3.5}$$

where $\mathfrak{N} = \{(n,m)\in\mathbb{Z}^2 \colon n \geqslant 0, -1 \leqslant m \leqslant 1 + 2T2^{\mathfrak{s}_0 n}\}$ and the K^Q_{nm} have the following properties.

- *Let $h_{nm} = (m2^{-\mathfrak{s}_0 n}, 0)$. For all n,m, K^Q_{nm} is supported on the ball*

$$\bigl\{z \in \mathbb{R}^{d+1} \colon \|z - h_{nm}\|_{\mathfrak{s}} \leqslant (1 + 2^{1/\mathfrak{s}_0})2^{-n}\bigr\}\;. \tag{3.6}$$

- *For any multiindex k, there exists a constant C_Q such that*

$$\bigl|\mathrm{D}^k K^Q_{nm}(z)\bigr| \leqslant C_Q 2^{(|\mathfrak{s}|-\mathfrak{s}_0-\beta+|k|_{\mathfrak{s}})n} \tag{3.7}$$

  holds uniformly over all $(n,m)\in\mathfrak{N}$ and all $z\in\mathbb{R}^{d+1}$.
- *For any two multiindices k and ℓ, there exists a constant C_Q such that*

$$\biggl|\int_{\mathbb{R}^{d+1}} z^\ell\,\mathrm{D}^k K^Q_{nm}(z)\,\mathrm{d}z\biggr| \leqslant C_Q 2^{-(\beta+\mathfrak{s}_0)n} \tag{3.8}$$

  holds uniformly over all $(n,m)\in\mathfrak{N}$.

We give the proof in Appendix A. Note the extra $\mathfrak{s}_0$ in the bound (3.7), which compensates the fact that m takes of the order of $2^{\mathfrak{s}_0 n}$ values.

Remark 3.2. We only need these results in the case $\beta = 2$, and for the parabolic scaling $\mathfrak{s} = (2,1,1,1)$. However, since there is no difficulty in dealing with this more general setting, we may as well do so here.

3.2 Extension of the regularity structure

In order to lift convolution with K^Q to the regularity structure, a natural idea is to enlarge the model space of the Allen–Cahn equation (cf. [1, Section 3 and Table 1]) by adding new elements of the form $\mathcal{I}^Q(\tau)$ whenever $|\tau|_\mathfrak{s} \notin \mathbb{Z}$. By convention, $\mathcal{I}^Q(\tau)$ then has homogeneity $|\mathcal{I}^Q(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + \beta$.

In order to extend the model, one can then try to proceed as in [2, Section 5] by first introducing functions

$$\mathcal{J}^Q(z)\tau = \sum_{|k|_\mathfrak{s} < \alpha+\beta} \frac{X^k}{k!} \sum_{(n,m)\in\mathfrak{N}} \big\langle \Pi_z\tau, \mathrm{D}^k K^Q_{nm}(z-\cdot)\big\rangle \tag{3.9}$$

where $\alpha = |\tau|_\mathfrak{s}$. Then the model is formally given by

$$(\Pi_z\mathcal{I}^Q\tau)(\bar z) = \big\langle \Pi_z\tau, K^Q(\bar z - \cdot)\big\rangle - (\Pi_z\mathcal{J}^Q(z)\tau)(\bar z)\;. \tag{3.10}$$

The precise formulation of this relation is that for any test function ψ,

$$\big\langle \Pi_z\mathcal{I}^Q\tau, \psi\big\rangle = \sum_{(n,m)\in\mathfrak{N}} \int_{\mathbb{R}^{d+1}} \big\langle \Pi_z\tau, K^{Q;\alpha}_{nm;z\bar z}\big\rangle \psi(\bar z)\,\mathrm{d}\bar z\;, \tag{3.11}$$

where

$$K^{Q;\alpha}_{nm;z\bar z}(z') = K^Q_{nm}(\bar z - z') - \sum_{|k|_\mathfrak{s}<\alpha+\beta} \frac{(\bar z - z)^k}{k!} \mathrm{D}^k K^Q_{nm}(z-z')\;. \tag{3.12}$$

We still need to verify that all these definitions make sense for the new kernel. We can however exploit the fact that in practice, we will only need to apply this construction to symbols τ whose model does not depend on the reference time in the following sense.

Definition 3.3. *We say that the model $\Pi\tau$ is base-time independent if*

$$(\Pi_{z+h}\tau)(\bar z) = (\Pi_z\tau)(\bar z) \tag{3.13}$$

holds for all $z, \bar z \in \mathbb{R}^{d+1}$ and all time shifts $h = (h_0, 0) \in \mathbb{R}\times\mathbb{R}^d$.

Lemma 3.4. *Assume that $\tau \in T_\alpha$ has a base-time independent model and that $\alpha+\beta \notin \mathbb{N}$. Then the series in* (3.9) *and* (3.11) *are absolutely convergent. Furthermore,*

$$\big|\big\langle \Pi_z\mathcal{I}^Q\tau, \psi^\lambda_z\big\rangle\big| \lesssim \lambda^{\alpha+\beta}\|\Pi\|_{\alpha;\mathfrak{K}_z} \tag{3.14}$$

holds uniformly over $z\in\mathbb{R}^{d+1}$ and $\lambda\in(0,1]$, where $\psi^\lambda_z(\bar z) = \mathcal{S}^\lambda_{\mathfrak{s},z}\psi(\bar z)$ and $\mathfrak{K}_z$ is the ball of radius 2 centred in z. Here $\mathcal{S}^\lambda_{\mathfrak{s},z}\psi(\bar z_0,\dots,\bar z_d) = \lambda^{-|\mathfrak{s}|}\psi(\lambda^{-\mathfrak{s}_0}(\bar z_0 - z_0),\dots,\lambda^{-\mathfrak{s}_d}(\bar z_d - z_d))$.

The proof of this result is very similar to the proof of [2, Lemma 5.19], but there are a few differences due to the nonlocal singularity of K^Q which we explain in Appendix B. The constant in (3.14) does not depend on $\|\Gamma\|$ owing to the fact that Π is base-time independent.

Remark 3.5. In our particular case, the canonical model of the following symbols is base-time independent:

Ξ,  , $\mathbf{1}$. (3.15)

Here $:= \mathcal{I}^Q(\Xi)$ has homogeneity $|\Xi|_\mathfrak{s} + 2$, and we employ the usual notation and additivity rule of homogeneities for products. In fact, the canonical model is completely independent of the base point for these symbols, so that any translation, not only in the time direction, has no effect. Indeed, $(\Pi^\varepsilon_z\Xi)(\bar z) = \xi^\varepsilon(\bar z)$ does not depend on z, and neither does, for instance,

$(\Pi^\varepsilon_z$  $)(\bar z) = \int \xi^\varepsilon(z')K^Q(\bar z - z')\,\mathrm{d}z'$ (3.16) owing to the fact that $|\Xi|_{\mathfrak{s}} + 2$ is strictly negative, so that the sum in (3.9) is empty. A similar argument holds for the other symbols in the list (3.15). In addition, the canonical model is base-time independent for monomials of the form X_i, $i \in \{1,2,3\}$, since $(\Pi^\varepsilon_z X_i)(\bar z) = \bar z_i - z_i$, though it does depend on the spatial part of the base point. By contrast, the model of X_0 is not base-time independent, and neither are symbols such as $\treeIQA = \mathcal{I}^Q(\treeA)$, which have positive regularity (see also Remark 3.7 below).

In order to also extend the structure group, we first extend the coproduct via

$$\Delta(\mathcal{I}^Q\tau) = (\mathcal{I}^Q \otimes \mathrm{Id})\Delta(\tau) + \sum_{|k+\ell|_{\mathfrak{s}} < \alpha+\beta} \frac{X^k}{k!} \otimes \frac{X^\ell}{\ell!} \mathcal{J}^Q_{k+\ell}\tau \tag{3.17}$$

where the $\mathcal{J}^Q_{k+\ell}\tau$ are new symbols satisfying

$$\bigl\langle f_z, \mathcal{J}^Q_\ell \tau \bigr\rangle = -\bigl\langle \Pi_z\tau, \mathrm{D}^\ell K^Q(z-\cdot)\bigr\rangle\;. \tag{3.18}$$

Recall that the f_z are linear forms allowing to define the structure group by setting $\Gamma_{z\bar z} = F_z^{-1}F_{\bar z}$, where

$$F_z\tau = (\mathrm{Id} \otimes f_z)\Delta\tau\;. \tag{3.19}$$

In the particular case $\tau = \Xi$, we obtain that $\mathcal{I}^Q\tau =: \treeIQXi$ satisfies $\Delta(\treeIQXi) = \treeIQXi \otimes \mathbf{1}$ and thus

$$F_z \treeIQXi = \treeIQXi\;, \qquad \Gamma_{z\bar z}\treeIQXi = \treeIQXi\;. \tag{3.20}$$

The model space can then be extended in the usual way to monomials in $\treeIXi$ and $\treeIQXi$ with the usual additivity rule of homogeneities and product rule for the canonical model. Then we can again apply $\mathcal{I}$ and $\mathcal{I}^Q$ to these monomials. In what follows, it will be useful to have explicit expressions for the action of the structure group on such monomials. Such an expression is provided by the next result, proved in Appendix C.

Lemma 3.6. *Assume that $\tau \in T_\alpha$ has a base-time independent model $\Pi_z\tau$ and satisfies $\Delta(\tau) = \tau \otimes \mathbf{1}$. Then the structure group acts via*

$$\Gamma_{z\bar z}\mathcal{I}^Q\tau = \mathcal{I}^Q\tau + \sum_{|k|_{\mathfrak{s}} < \alpha+\beta} \frac{X^k}{k!}\biggl[\chi^k_\tau(z) - \sum_{|\ell|_{\mathfrak{s}} < \alpha+\beta-|k|_{\mathfrak{s}}} \frac{(z-\bar z)^\ell}{\ell!}\chi^{k+\ell}_\tau(\bar z)\biggr] \tag{3.21}$$

where $\chi^k_\tau(z) = \sum_{(n,m)\in\mathfrak{N}} \langle \Pi_z\tau, \mathrm{D}^k K^Q_{nm}(z-\cdot)\rangle$.

Remark 3.7. This result illustrates the fact that [1, Lemma 4.8] is incorrect in general. For instance, in the case $\tau = \treeA$ we obtain

$$\Gamma_{z\bar z}\treeIQA = \treeIQA + \bigl[\chi^0_{\treeA}(z) - \chi^0_{\treeA}(\bar z)\bigr]\mathbf{1}\;. \tag{3.22}$$

Since $\chi^0_{\treeA}(z+h) - \chi^0_{\treeA}(\bar z+h) \neq \chi^0_{\treeA}(z) - \chi^0_{\treeA}(\bar z)$, the operator $\Gamma_{z\bar z}$ is indeed not translation invariant. An even simpler example of the incorrectness of [1, Lemma 4.8] occurs for the canonical model of Ξ, since $(\Pi^\varepsilon_z\Xi)(\bar z) = \xi^\varepsilon(\bar z)$ is constant with respect to z, not with respect to $\bar z$.

3.3 Lifting the convolution operator

Following the strategy in [2, Section 5], it is natural to look for a lift of the operation of convolution with K^Q given for $f \in \mathcal{D}^{\gamma,\eta}$ by

$$(\mathcal{K}^Q_\gamma f)(z) = \mathcal{I}^Q f(z) + \mathcal{J}^Q(z)f(z) + (\mathcal{N}^Q_\gamma f)(z)\;, \tag{3.23}$$

where the nonlocal operator $\mathcal{N}^Q_\gamma$ is defined by

$$(\mathcal{N}^Q_\gamma f)(z) = \sum_{|k|_{\mathfrak{s}} < \gamma+\beta} \frac{X^k}{k!} \sum_{(n,m)\in\mathfrak{N}} \bigl\langle \mathcal{R}f - \Pi_z f(z), \mathrm{D}^k K^Q_{nm}(z-\cdot)\bigr\rangle\;. \tag{3.24}$$

The problem with this definition is that in general, if f is defined on a sector of regularity α, we can only prove a bound of the form

$$\big|\big\langle \mathcal{R}f - \Pi_z f(z), \mathrm{D}^k\, K_{nm}^Q(z-\cdot)\big\rangle\big| \lesssim 2^{(|k|_{\mathfrak{s}} - \mathfrak{s}_0 - \alpha - \beta)n}\;, \tag{3.25}$$

instead of $2^{(|k|_{\mathfrak{s}} - \gamma - \beta)n}$ as in [2, (5.42)]. The reason for this weaker bound is that in general $K_{nm}^Q(z-\cdot)$ is not supported near the origin, so that shifting the model as in the proof of [2, Lemma 5.18] produces an additional factor of order $\|h_{nm}\|_{\mathfrak{s}}^{\gamma-\alpha}$, which can have order 1 instead of order $2^{-(\gamma-\alpha)n}$ as in that Lemma.

The bound (3.25) proves convergence of the sum in (3.24) only for $|k|_{\mathfrak{s}} < \alpha + \beta$. If for instance $f(z) = a(z)\,{}^{\bullet}\!\vee^{\bullet}$, in dimension $d = 3$ the sector has regularity $\alpha = -1 - 2\kappa$, and thus only the term with $k = 0$ is well-defined. Restricting the sum over k to only the term $k = 0$, however, results in $\mathcal{K}_\gamma^Q f$ belonging only to some $\mathcal{D}^{\bar\gamma,\bar\eta}$ with $\bar\gamma < 1$, which is not sufficient to carry out the fixed-point argument for a general cubic $F(U,V)$ for $d = 3$.

A way out of this situation is to work with shift operators. Define, for any $h \in \mathbb{R}^{d+1}$, an operator $T_h : \mathcal{C}_{\mathfrak{s}}^\alpha \to \mathcal{C}_{\mathfrak{s}}^\alpha$ by

$$\langle T_h \upsilon, \psi\rangle = \langle \upsilon, \psi(\cdot - h)\rangle \tag{3.26}$$

for any test function ψ. In case υ is a function, this amounts to setting $T_h\upsilon(z) = \upsilon(z+h)$. We define a shifted model $T_h\Pi =: \Pi^h$ by

$$\Pi_z^h \tau(\bar z) = \Pi_{z+h}\tau(\bar z + h) \qquad \forall z, \bar z \in \mathbb{R}^{d+1} \tag{3.27}$$

(which should be interpreted as $\Pi_z^h\tau = T_h(\Pi_{z+h}\tau)$ if $\Pi_z\tau$ is a distribution). Assume we can define, on some sector of $\mathcal{D}^{\gamma,\eta}(\Pi)$, a map $\mathcal{T}_h$ taking values in $\mathcal{D}^{\gamma,\eta}(\Pi^h)$ and satisfying

$$\mathcal{R}^h \mathcal{T}_h = T_h \mathcal{R}\;, \tag{3.28}$$

where $\mathcal{R}^h$ is the reconstruction operator on $\mathcal{D}^{\gamma,\eta}(\Pi^h)$. If f belongs to a function-like sector, (3.28) is equivalent to

$$(\mathcal{R}^h \mathcal{T}_h f)(z) = (\mathcal{R}f)(z+h) \tag{3.29}$$

where $(\mathcal{R}f)(z+h) = (\Pi_{z+h} f(z+h))(z+h) = (\Pi_z^h f(z+h))(z)$.

Setting $\mathcal{R}^{nm} := \mathcal{R}^{h_{nm}}$ we have

$$\big\langle \mathcal{R}f, \mathrm{D}^k\, K_{nm}^Q(z-\cdot)\big\rangle = \big\langle \mathcal{R}^{nm} f_{nm}, \mathrm{D}^k\, \hat K_{nm}^Q(z-\cdot)\big\rangle \tag{3.30}$$

where $f_{nm} = \mathcal{T}_{h_{nm}} f$ and

$$\hat K_{nm}^Q(z) = K_{nm}^Q(z + h_{nm}) \tag{3.31}$$

is a shifted kernel, supported in a ball of radius of order 2^{-n} around the origin. Finally, let $\Pi^{nm} = \Pi^{h_{nm}} = T_{h_{nm}}\Pi$ denote the time-shifted models, and assume that for each h_{nm}, we can define an operator $\mathcal{K}_{\gamma,nm}^Q$ from $\mathcal{D}^{\gamma,\eta}(\Pi^{nm})$ to $\mathcal{D}^{\gamma+\beta,\bar\eta}(\Pi)$ satisfying

$$\mathcal{R}\mathcal{K}_{\gamma,nm}^Q = \hat K_{nm}^Q * \mathcal{R}^{nm}\;. \tag{3.32}$$

Then the operator

$$\mathcal{K}_\gamma^Q = \sum_{(n,m)\in\mathfrak{N}} \mathcal{K}_{\gamma,nm}^Q \mathcal{T}_{h_{nm}} \tag{3.33}$$

maps $\mathcal{D}^{\gamma,\eta}(\Pi)$ into $\mathcal{D}^{\gamma+\beta,\bar\eta}(\Pi)$ and satisfies the required identity $\mathcal{R}\mathcal{K}_\gamma^Q = K^Q * \mathcal{R}$.

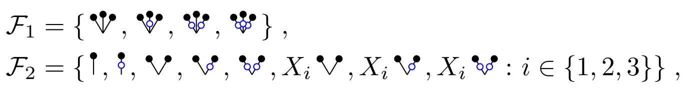
The property (3.32) can be achieved by defining $\mathcal{K}_{\gamma,nm}^Q$ as in (3.23), but replacing the model, kernel and reconstruction operator in (3.24) and (3.9) by their shifted versions. This has the advantage of improving the bound (3.25), since the kernel $\hat K_{nm}^Q$ is now supported near the origin. A drawback is that this forces us to introduce a countable infinity of new symbols $\mathcal{I}^Q_{nm}\tau$, for τ in the sector under consideration. We will now show that in the case of FitzHugh–Nagumo-type SPDEs of the form (1.1), one can indeed construct a shift map realising (3.28) on a specific sector of negative homogeneity. Then we will check that the introduction of infinitely many new symbols does not pose a problem for the renormalisation procedure.

3.4 Multilevel Schauder estimates for FitzHugh–Nagumo-type SPDEs

We now particularise to the FitzHugh–Nagumo-type SPDE (1.1) in dimension $d = 3$. We consider modelled distributions in $\mathcal{D}^{\gamma,\eta}$ of the form

$$\begin{aligned} f(z) &= \sum_{\tau\in\mathcal{F}_1} c_\tau \tau + \sum_{\tau\in\mathcal{F}_2} a_\tau(z)\tau + \varphi(z)\mathbf{1} + \sum_{\tau\in\mathcal{F}_3} a_\tau(z)\tau \\ &=: f_1(z) + f_2(z) + \varphi(z)\mathbf{1} + f_3(z) \;, \end{aligned} \tag{3.34}$$

where



(3.35)

and $\mathcal{F}_3$ is such that any $\tau \in \mathcal{F}_3$ satisfies the diagonal identity

$$\lim_{\lambda\to 0} \langle \Pi_z \tau, \psi_z^\lambda \rangle = 0 \;. \tag{3.36}$$

The reason why we only include polynomial elements X_i in the spatial directions in $\mathcal{F}_2$ is that owing to the polynomial scaling, $|X_0|_\mathfrak{s} = 2$ and thus $|X_0$  $|_\mathfrak{s} > 0$. By linearity, we may define separately the action of $\mathcal{K}^Q_\gamma$ on f_1, f_2, $\varphi\mathbf{1}$, and f_3. In the case of f_1 and $\varphi\mathbf{1}$, we use the standard definition (3.23), which takes here the form

$$\mathcal{K}^Q_\gamma f_1(z) = \sum_{\tau\in\mathcal{F}_1} c_\tau \bigl[\mathcal{I}^Q\tau + \chi^0_\tau(z)\mathbf{1}\bigr] \;, \tag{3.37}$$

$$\mathcal{K}^Q_\gamma \varphi\mathbf{1}(z) = \sum_{|k|_\mathfrak{s}<\gamma+\beta} \frac{X^k}{k!} \langle \varphi, \mathrm{D}^k K^Q(z-\cdot)\rangle \;. \tag{3.38}$$

Here we have set $\mathcal{N}_\gamma f_1 = 0$, since we may choose $\mathcal{R} f_1 = \Pi_z f_1(z) = \sum_{\tau\in\mathcal{F}_1} c_\tau \Pi_z\tau$, owing to the fact that f_1 does not depend on z. Furthermore, we have used the fact that thanks to the vanishing-moments condition, $\mathcal{J}^Q(z)\mathbf{1} = 0$ and $\langle \Pi_z\mathbf{1}, \mathrm{D}^k K^Q(z-\cdot)\rangle = 0$. For f_3 we simply set

$$\mathcal{K}^Q_\gamma f_3(z) = 0 \;, \tag{3.39}$$

which is allowed thanks to the diagonal identity (3.36).

It thus remains to define $\mathcal{K}^Q_\gamma f_2(z)$. Here we use the procedure based on shift operators, as outlined above. Owing to the fact that the only polynomial terms X_i occurring in $\mathcal{F}_2$ are purely spatial, all $\tau \in \mathcal{F}_2$ are base-time independent (cf. Remark 3.5). As a consequence, one can check that the map $\mathcal{T}_{h_{nm}}$ can be realised by

$$\mathcal{T}_{h_{nm}} f_2(z) = \sum_{\tau\in\mathcal{F}_2} a_\tau(z + h_{nm})\tau \;. \tag{3.40}$$

In this way, we obtain

$$\begin{aligned} \mathcal{K}^Q_\gamma f_2(z) = \sum_{(n,m)\in\mathfrak{N}} \biggl\{ &\sum_{\tau\in\mathcal{F}_2} a_\tau(z+h_{nm}) \biggl[\mathcal{I}^Q_{nm}\tau + \sum_{|k|_\mathfrak{s}<|\tau|_\mathfrak{s}+\beta} \frac{X^k}{k!}\hat\chi^k_{\tau,nm}(z)\biggr] \\ &+ \sum_{|k|_\mathfrak{s}<\gamma+\beta} \frac{X^k}{k!} b^k_{nm}(z) \biggr\} \;, \end{aligned} \tag{3.41}$$

where

$$
\begin{aligned}
\hat{\chi}^k_{\tau,nm}(z) &= \bigl\langle \Pi^{nm}_z \tau, \mathrm{D}^k\, \hat{K}^Q_{nm}(z-\cdot) \bigr\rangle\;, \\
b^k_{nm}(z) &= \bigl\langle \mathcal{R}^{nm} f_{nm} - \Pi^{nm}_z f_{nm}(z), \mathrm{D}^k\, \hat{K}^Q_{nm}(z-\cdot) \bigr\rangle\;.
\end{aligned} \tag{3.42}
$$

Furthermore, the $\mathcal{I}^Q_{nm}\tau$ are new symbols with model

$$
(\Pi_z \mathcal{I}^Q_{nm}\tau)(\bar z) = \hat{\chi}^0_{\tau,nm}(\bar z) - \sum_{|k|_{\mathfrak{s}} < |\tau|_{\mathfrak{s}}+\beta} \frac{(\bar z - z)^k}{k!} \hat{\chi}^k_{\tau,nm}(z)\;. \tag{3.43}
$$

Since $T_{|\tau|_{\mathfrak{s}}+\beta}$ is now infinite-dimensional for $\tau \in \mathcal{F}_2$, the choice of norm on these subspaces matters, and we choose it to be the supremum norm. More precisely, writing $\alpha = |\tau|_{\mathfrak{s}} + \beta$, we have

$$
g = \sum_{(n,m)\in\mathfrak{N}} \sum_{\tau\in\mathcal{F}_2} c_{\tau,nm} \mathcal{I}^Q_{nm}\tau \qquad \Rightarrow \qquad \|g\|_\alpha = \sup_{(n,m)\in\mathfrak{N}} \sup_{\tau\in\mathcal{F}_2} \bigl|c_{\tau,nm}\bigr|\;, \tag{3.44}
$$

where the supremum over $\tau \in \mathcal{F}_2$ may be replaced by any other norm on the finite-dimensional span of $\mathcal{F}_2$.

We summarise the construction in the following definition.

Definition 3.8. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be defined by* (3.35) *and let*

$$
\mathcal{F}_3 = \bigl\{\mathcal{I}^Q\tau \colon \tau \in \mathcal{F}_1\bigr\} \cup \bigl\{\mathcal{I}^Q_{nm}\tau \colon \tau \in \mathcal{F}_2, (n,m)\in\mathfrak{N}\bigr\}\;. \tag{3.45}
$$

Take as model space the Banach space

$$
\operatorname{span}(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3) \cup \overline{T}\;, \tag{3.46}
$$

where $\overline{T}$ is the span of all polynomials. If $f \in \mathcal{D}^{\gamma,\eta}$ is of the form (3.34)*, we set*

$$
\mathcal{K}^Q_\gamma f(z) = \mathcal{K}^Q_\gamma f_1(z) + \mathcal{K}^Q_\gamma \varphi\mathbf{1}(z) + \mathcal{K}^Q_\gamma f_2(z)\;, \tag{3.47}
$$

where $\mathcal{K}^Q_\gamma f_1$ and $\mathcal{K}^Q_\gamma\varphi\mathbf{1}$ are defined in (3.37) *and* (3.38) *and $\mathcal{K}^Q_\gamma f_2$ is given in* (3.41)*.*

Remark 3.9. We could also have introduced symbols of the form $\mathcal{I}^Q_{nm}\tau$ for $\tau \in \mathcal{F}_1$, but this is not necessary because $f_1(z)$ does not depend on z.

In this setting, we can now state our central result, which is the following extension of the multilevel Schauder estimates in [2, Theorem 5.12]. Here the notations for $\|\!|f|\!\|_{\gamma,\eta;T}$, $\|\!|f\,;\bar f|\!\|_{\gamma,\eta;T}$ and $\|\!|Z\,;\bar Z|\!\|_{\gamma;O}$ are as in [2, Definition 6.2] and [2, Section 7.1] with P the hyperplane $\{t=0\}$, see also [1, Section 4.3].

Theorem 3.10. *Let $\alpha_0 = |\Psi|_{\mathfrak{s}}$ be the regularity of the sector defining f. Assume $f \in \mathcal{D}^{\gamma,\eta}$ is of the form* (3.34)*, where $\eta < \alpha_0 \wedge \gamma$, and $\gamma+\beta, \eta+\beta \notin \mathbb{N}$. Then $\boldsymbol{R}^+\mathcal{K}^Q_\gamma \boldsymbol{R}^+ f \in \mathcal{D}^{\gamma+\beta,\eta+\beta}$ and*

$$
(\mathcal{R}\mathcal{K}^Q_\gamma \boldsymbol{R}^+ f)(z) = (K^Q * \mathcal{R}\boldsymbol{R}^+ f)(z) \tag{3.48}
$$

holds for every $z=(t,x)$ such that $t>0$. Furthermore, we have

$$
\|\!|\boldsymbol{R}^+\mathcal{K}^Q_\gamma \boldsymbol{R}^+ f|\!\|_{\gamma+\beta,\bar\eta;T} \lesssim T^{\kappa/\mathfrak{s}_0} \|\!|f|\!\|_{\gamma,\eta;T} \tag{3.49}
$$

whenever $\bar\eta = \eta+\beta-\kappa$ with $\kappa>0$. Finally, if $\bar Z = (\bar\Pi,\bar\Gamma)$ is a second model such that $\Pi_z\tau$ is base-time independent for all $\tau \in \mathcal{F}_1 \cup \mathcal{F}_2$, and $\bar f \in \mathcal{D}^{\gamma,\eta}(\bar\Gamma)$ is of the form (3.34)*, then*

$$
\|\!|\boldsymbol{R}^+\mathcal{K}^Q_\gamma \boldsymbol{R}^+ f\,; \boldsymbol{R}^+\bar{\mathcal{K}}^Q_\gamma \boldsymbol{R}^+ \bar f|\!\|_{\gamma+\beta,\bar\eta;T} \lesssim T^{\kappa/\mathfrak{s}_0}\bigl(\|\!|f\,;\bar f|\!\|_{\gamma,\eta;T} + \|\!|Z\,;\bar Z|\!\|_{\gamma;O}\bigr)\;. \tag{3.50}
$$

The proof is given in Appendix D. Note that we have assumed $\eta < \alpha_0$ to simplify the notation (otherwise we need to take $\bar\eta = (\eta\wedge\alpha_0)+\beta-\kappa$). Note also the extra factor $\boldsymbol{R}^+(t,x) = 1_{\{t>0\}}$, which is needed because the translation operators shift singularities along the time axis.

4 Fixed point argument

Assume the nonlinearity has the general cubic form (2.1). Note in particular that if $p(z)$ and $q(z)$ are polynomial terms, and $\Phi(z)$ and $\Psi(z)$ are terms of fractional, strictly positive homogeneity, then

$$F(\treeI + p(z) + \Phi(z), \treeIblue + q(z) + \Psi(z)) = f_1(z) + f_2(z) + F(p(z), q(z)) + f_3(z) \;, \tag{4.1}$$

where

$$\begin{aligned} f_1(z) &= \gamma_1 \treeVbbb + \gamma_2 \treeVbbB + \gamma_3 \treeVbBB + \gamma_4 \treeVBBB \;, \\ f_2(z) &= b_1(z) \treeVbb + b_2(z) \treeVbB + b_3(z) \treeVBB + a_1(z) \treeI + a_2(z) \treeIblue \;, \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} b_1(z) &= \beta_1 + 3\gamma_1 p(z) + \gamma_2 q(z) \;, \\ b_2(z) &= \beta_2 + 2\gamma_2 p(z) + 2\gamma_3 q(z) \;, \\ b_3(z) &= \beta_3 + \gamma_3 p(z) + 3\gamma_4 q(z) \;. \\ a_1(z) &= \alpha_1 + 2\beta_1 p(z) + \beta_2 q(z) + 3\gamma_1 p(z)^2 + 2\gamma_2 p(z)q(z) + \gamma_3 q(z)^2 \;, \\ a_2(z) &= \alpha_2 + \beta_2 p(z) + 2\beta_3 q(z) + \gamma_2 p(z)^2 + 2\gamma_3 p(z)q(z) + 3\gamma_4 q(z)^2 \;. \end{aligned} \tag{4.3}$$

Furthermore, all terms of $f_3(z)$ contain at least a factor $\Phi(z)$ or a factor $\Psi(z)$. Thus if the model Π satisfies the two properties

$$|\Pi_z \tau(\bar z)| \lesssim \|z - \bar z\|_{\mathfrak{s}}^{|\tau|_{\mathfrak{s}}} \|\tau\| \;, \qquad \Pi_z(\tau_1\tau_2) = \Pi_z(\tau_1)\Pi_z(\tau_2) \tag{4.4}$$

for all $\tau, \tau_1, \tau_2 \in T$, then f_3 satisfies the diagonal identity (3.36).

Let $\mathcal{D}_*^{\gamma,\eta}(\Pi)$ denote the subspace of modelled distributions in $\mathcal{D}^{\gamma,\eta}(\Pi)$ whose components of negative homogeneity are of the form $c_1 \treeI + c_2 \treeIblue$ for constants $c_1, c_2 \in \mathbb{R}$. Consider the map $\mathcal{M}(U,V) = (\mathcal{M}_1(U,V), \mathcal{M}_2(U,V))$ defined on $\mathcal{D}_*^{\gamma,\eta}(\Pi) \times \mathcal{D}_*^{\gamma,\eta}(\Pi)$ by

$$\begin{aligned} \mathcal{M}_1(U,V) &= \boldsymbol{R}^+(\mathcal{K}_{\bar\gamma} + R_\gamma \mathcal{R})\boldsymbol{R}^+ F(U,V) + W_1 \;, \\ \mathcal{M}_2(U,V) &= \boldsymbol{R}^+(\mathcal{K}^Q_{\bar\gamma} + R^Q_\gamma \mathcal{R})\boldsymbol{R}^+ F(U,V) + W_2 \;, \end{aligned} \tag{4.5}$$

where W_1 and W_2 are placeholders for the stochastic convolution and the initial conditions (we only need the case where $W_1 - \treeI$ and $W_2 - \treeIblue$ take values in the polynomial part of the regularity structure). By iterating the map (4.5), we find that if it admits a fixed point, then it necessarily has the form

$$\begin{aligned} U(z) &= \treeI + \varphi(z)\mathbf{1} + \big[\gamma_1 \treeYbbb + \gamma_2 \treeYbbB + \gamma_3 \treeYbBB + \gamma_4 \treeYBBB\big] + \big[b_1(z) \treeYbb + b_2(z) \treeYbB + b_3(z) \treeYBB\big] + \dots \\ V(z) &= \treeIblue + \psi(z)\mathbf{1} + \big[\gamma_1 \treeYbbbQ + \gamma_2 \treeYbbBQ + \gamma_3 \treeYbBBQ + \gamma_4 \treeYBBBQ\big] \\ &\quad + \sum_{(n,m)\in\mathfrak{N}} \big[b_1(z+h_{nm}) \treeYbbQ_{\,nm} + b_2(z+h_{nm}) \treeYbBQ_{\,nm} + b_3(z+h_{nm}) \treeYBBQ_{\,nm}\big] + \dots \end{aligned} \tag{4.6}$$

where the $b_i(z)$ are as in (4.3) with $p(z) = \varphi(z)$, $q(z) = \psi(z)$, and the dots indicate terms of homogeneity at least 1. As in [1, Proposition 5.2], it is rather straightforward to show that if (U,V) satisfies the fixed-point equation (3.3) with U and V in some $\mathcal{D}^{\gamma,\eta}$ then $(u,v) = (\mathcal{R}U, \mathcal{R}V)$ satisfies (1.2).

[1, Proposition 5.6] is then replaced by the following result, which is all we need for the fixed-point argument to work. Its proof is very similar to the proof of [1, Proposition 5.6], so we omit it here.

Proposition 4.1. *Let* Π *be a model satisfying* (4.4)*, and assume* $-\frac{2}{3} < \eta < \alpha < -\frac{1}{2}$*,* $\eta + 2\alpha > -2$ *and* $\gamma > -2\alpha$*. Then for any* $W_1, W_2 \in \mathcal{D}_*^{\gamma,\eta}(\Pi)$ *of regularity* α*, there exists a time* $T > 0$ *such that* $\mathcal{M}$ *admits a unique fixed point* $(U^*, V^*) \in \mathcal{D}_*^{\gamma,\eta}(\Pi) \times \mathcal{D}_*^{\gamma,\eta}(\Pi)$ *on* $(0, T)$*. Furthermore, the solution map* $\mathcal{S}_T : (W_1, W_2, Z) \mapsto (U^*, V^*)$ *is jointly Lipschitz continuous.*

Note that in the case

$$\begin{aligned} W_1 &= (\mathcal{K}_{\bar\gamma} + R_\gamma \mathcal{R})\boldsymbol{R}^+ \Xi + G u_0 \;, \\ W_2 &= (\mathcal{K}^Q_{\bar\gamma} + R_\gamma \mathcal{R})\boldsymbol{R}^+ \Xi + G^Q u_0 + \widehat{Q} v_0 \;, \end{aligned} \tag{4.7}$$

the fixed point (U^*, V^*) of $\mathcal{M}$ is indeed a fixed point of (3.3). As pointed out in [1, Remark 5.7], the assumptions on u_0 and v_0 guarantee that W_1 and W_2 belong to the right functional space.

5 Renormalisation

It remains to check that the fact that we have modified the regularity structure by adding a countable infinity of symbols does not cause any problems as far as the renormalisation procedure is concerned, and to derive the renormalised equations.

We define a renormalisation transformation, depending on two parameters, given by

$$M_\varepsilon \tau = \exp\Bigl\{-C_1(\varepsilon) L_1 \tau - C_2(\varepsilon) L_2 \tau\Bigr\} \;, \tag{5.1}$$

where the generators L_1 and L_2 are defined by applying the substitution rules (called contractions)

$$L_1 : \vee \mapsto \mathbf{1} \;, \qquad L_2 : \overset{\vee\vee}{\circ} \mapsto \mathbf{1} \tag{5.2}$$

as many times as possible, so that for instance $L_1 \Psi = 3 \,\mathord{\uparrow}$. In particular, we obtain

$$M_\varepsilon \overset{\vee\vee}{\circ}_{nm} = \overset{\vee\vee}{\circ}_{nm} - C_1(\varepsilon) \overset{\vee}{\circ}_{nm} \;. \tag{5.3}$$

Other examples of the action of M_ε are given in [1, (6.12)]. Note that there are no generators acting by contracting symbols that contain at least one edge associated to $\mathcal{I}^Q$, implying that for instance $M_\varepsilon \overset{\vee}{\diamond} = \overset{\vee}{\diamond}$ and $M_\varepsilon \overset{\vee\vee}{\diamond}_{nm} = \overset{\vee\vee}{\diamond}_{nm}$. The fact that these symbols do not require additional renormalisation constants is a consequence of [1, Lemma 6.2] and Lemma 5.1 below.

The renormalisation map M_ε induces a renormalised model $\widehat{\Pi}^\varepsilon = \Pi^{M_\varepsilon}$ which can be computed as described in [2, Section 8.3] and [1, Section 6.1]. In particular, we find

$$\widehat{\Pi}^\varepsilon_z(\overset{\vee\vee}{\circ}_{nm}) = \Pi^\varepsilon_z(\overset{\vee}{\circ}_{nm}) \widehat{\Pi}^\varepsilon_z(\vee) \;. \tag{5.4}$$

Here the canonical model for $\Pi^\varepsilon_z(\overset{\vee}{\circ}_{nm})$ can be computed using (3.43), which yields

$$\begin{aligned} (\Pi^\varepsilon_z \overset{\vee}{\circ}_{nm})(\bar z) &= \hat\chi^0_{\vee,nm}(\bar z) - \hat\chi^0_{\vee,nm}(z) \\ &= \langle \Pi^{\varepsilon,nm}_z \vee, \hat K^Q_{nm}(\bar z - \cdot) - \hat K^Q_{nm}(z - \cdot)\rangle \\ &= \langle \Pi^\varepsilon_z \vee, K^Q_{nm}(\bar z - \cdot) - K^Q_{nm}(z - \cdot)\rangle \\ &= \int \bigl[K^Q_{nm}(\bar z - z_1) - K^Q_{nm}(z - z_1)\bigr] \bigl[(K_\varepsilon * \xi)(z_1)\bigr]^2 \,\mathrm{d}z_1 \;, \end{aligned} \tag{5.5}$$

where $K_\varepsilon = K * \varrho_\varepsilon$, and we have used the expression for the canonical model of $\vee$ in the last line, which is base-time independent, cf. [1, (6.28)]. It follows that

$$\begin{aligned} \widehat{\Pi}^\varepsilon_z(\overset{\vee\vee}{\circ}_{nm})(\bar z) = \int &\bigl[K^Q_{nm}(\bar z - z_1) - K^Q_{nm}(z - z_1)\bigr] \bigl[(K_\varepsilon * \xi)(z_1)\bigr]^2 \,\mathrm{d}z_1 \\ &\times \Bigl(\bigl[(K_\varepsilon * \xi)(\bar z)\bigr]^2 - C_1(\varepsilon)\Bigr) \;. \end{aligned} \tag{5.6}$$

The renormalised models of other symbols are obtained in a similar way, using the expressions given in [1, (6.13)].

We now have to show that the renormalised models converge, for an appropriate choice of the renormalisation constants $C_1(\varepsilon)$ and $C_2(\varepsilon)$, to a well-defined limiting model. This amounts to showing that the Wiener chaos expansions of the renormalised models satisfy the bounds [1, (6.20)]. To a large extent, the computations have already been made in [1, Proposition 6.4], so that we only discuss one representative case involving an infinite collection of symbols. Proceeding as in [1, (6.39)], we find that the contribution to the zeroth Wiener chaos of $\widehat{\Pi}_z^\varepsilon(\text{[symbol]}_{nm})$ is given by

$$(\widehat{\mathcal{W}}_z^{(\varepsilon;0)} \text{[symbol]}_{nm})(\bar z) = 2 \iiint \big[K_{nm}^Q(\bar z - z_1) - K_{nm}^Q(z - z_1)\big] \\ \times K_\varepsilon(z_1 - z_2) K_\varepsilon(z_1 - z_3) K_\varepsilon(\bar z - z_2) K_\varepsilon(\bar z - z_3)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\;. \tag{5.7}$$

As in [1, Proposition 6.4], the crucial term is the one involving $K_{nm}^Q(\bar z - z_1)$, which can be rewritten as $2I_{00;nm}^Q(\varepsilon)$, where

$$I_{00;nm}^Q(\varepsilon) = \int K_{nm}^Q(z_1)\mathcal{Q}_0^\varepsilon(z_1)^2\,\mathrm{d}z_1\;, \qquad \mathcal{Q}_0^\varepsilon(z) = \int K_\varepsilon(z_1)K_\varepsilon(z_1 - z)\,\mathrm{d}z_1\;. \tag{5.8}$$

Note that if K_{nm}^Q is replaced by K, we obtain the renormalisation constant $C_2(\varepsilon)$, which diverges like $\log(\varepsilon^{-1})$, cf. (2.4). The following lemma implies that no renormalisation is needed in the case of [symbol]_{nm}.

Lemma 5.1. *For all $(n,m) \in \mathfrak{N}$, the bound*

$$\big|I_{00;nm}^Q(\varepsilon)\big| \lesssim \frac{2^{-2n}}{m+2} \tag{5.9}$$

holds uniformly in $\varepsilon \in [0,1]$.

The proof is given in Appendix E.1. The important point is that the bound (5.9) is square-summable over all $(n,m) \in \mathfrak{N}$, which is related to the fact that $K^Q(z_1)\mathcal{Q}_0^\varepsilon(z_1)^2$ is integrable uniformly in ε. This is essential in establishing the following convergence result.

Proposition 5.2. *Let $C_1(\varepsilon)$ and $C_2(\varepsilon)$ be the constants defined in (2.4). Then there exists a random model $\widehat Z = (\widehat\Pi, \widehat\Gamma)$, independent of the choice of mollifier ϱ, such that for any $\theta < -\frac{5}{2} - \alpha_0 = \kappa$ and any compact set $\mathfrak{K}$, one has*

$$\mathbb{E}\|\widehat Z^\varepsilon ; \widehat Z\|_{\gamma;\mathfrak{K}} \lesssim \varepsilon^\theta\;, \tag{5.10}$$

provided $\gamma < \zeta$, where ζ is such that all moments of K up to parabolic degree ζ vanish.

Proof. The proof follows along the lines of [1, Proposition 6.4], which is closely based on [2, Theorem 10.22]. The crucial point to note here is that [2, Theorem 10.7] can still be applied in this case, even though there is a countable infinity of symbols such as [symbol]_{nm} that need to be renormalised. Indeed, the bound [2, (10.4)] involves a sum, over all basis vectors τ of a given sector, of the p-th power of the second moment of $\langle\widehat\Pi_0\tau, \psi_0^\lambda\rangle$. To be applicable, the bounds

$$\langle\widehat\Pi_z \text{[symbol]}_{nm}, \psi_z^\lambda\rangle^2 \leqslant C_{nm}^2 \lambda^{2|\tau|_\mathfrak{s}+\kappa}\;,$$
$$\mathbb{E}\big|\langle\widehat\Pi_z \text{[symbol]}_{nm} - \widehat\Pi_z^\varepsilon \text{[symbol]}_{nm}, \psi_z^\lambda\rangle\big|^2 \leqslant C_{nm}^2 \varepsilon^{2\theta}\lambda^{2|\tau|_\mathfrak{s}+\kappa} \tag{5.11}$$

should hold for some $\kappa, \theta > 0$, with proportionality constants C_{nm}^2 that are summable over all $(n, m) \in \mathfrak{N}$, cf. [2, (10.2), (10.3)]. By [2, Proposition 10.11], this is the case if the Wiener chaos expansion of these τ satisfies the bounds

$$\begin{aligned}\Big|\big\langle(\widehat{\mathcal{W}}^{(k)}\,\text{(tree)}_{nm})(z), (\widehat{\mathcal{W}}^{(k)}\,\text{(tree)}_{nm})(\bar z)\big\rangle\Big| &\leqslant C_{nm}^2 \sum_{\varsigma>0} \big(\|z\|_{\mathfrak{s}} + \|\bar z\|_{\mathfrak{s}}\big)^{\varsigma} \|z - \bar z\|_{\mathfrak{s}}^{\bar\kappa + 2\alpha - \varsigma} ,\\ \Big|\big\langle(\delta\widehat{\mathcal{W}}^{(\varepsilon;k)}\,\text{(tree)}_{nm})(z), (\delta\widehat{\mathcal{W}}^{(\varepsilon;k)}\,\text{(tree)}_{nm})(\bar z)\big\rangle\Big| &\leqslant C_{nm}^2 \varepsilon^{2\theta} \sum_{\varsigma>0} \big(\|z\|_{\mathfrak{s}} + \|\bar z\|_{\mathfrak{s}}\big)^{\varsigma} \|z - \bar z\|_{\mathfrak{s}}^{\bar\kappa + 2\alpha - \varsigma}\end{aligned} \tag{5.12}$$

for some $\bar\kappa, \theta > 0$, where $\alpha = |\text{(tree)}_{nm}|_{\mathfrak{s}}$ and the sums run over finitely many positive ς. This in turns follows from the square-summability of integrals such as (5.9). □

The final step is to compute the renormalised equations corresponding to the renormalisation map M_ε. It is straightforward to check that Lemma 6.5 and Proposition 6.7 in [1] still hold in the present situation. It is thus sufficient to compute the non-positive-homogeneous part $\widehat F(U, V)$ of $M_\varepsilon F(U, V)$, for a cubic nonlinearity F as in (2.1). This yields the following result, which is proved in Appendix E.2.

Proposition 5.3. *In the situation just described, we have*

$$\widehat F(u, v) = F(u, v) + c_0(\varepsilon) + c_1(\varepsilon)u + c_2(\varepsilon)v , \tag{5.13}$$

where the $c_i(\varepsilon)$ are defined in (2.3)*.*

The proof of Theorem 2.1 now follows in the same way as in [1, Section 7].

A Proof of Proposition 3.1

Let $\varphi : \mathbb{R} \to [0, 1]$ be a partition of unity, i.e., a function satisfying

- φ is of class $\mathcal{C}^\infty$ and of compact support, say $[-1, 1]$;
- for any $t \in \mathbb{R}$ one has

$$\sum_{m\in\mathbb{Z}} \varphi(m + t) = 1 . \tag{A.1}$$

Remark A.1. An example of such a function would be a smooth even function, satisfying $\varphi(\theta) = 1 - \varphi(1 - \theta)$ for any $\theta \in [0, 1]$. For instance one can take

$$\varphi(\theta) = \frac{1}{2} + \frac{1}{2}\tanh\Big(\frac{1}{\theta} - \frac{1}{1-\theta}\Big) \qquad \forall\theta \in (0, 1) \tag{A.2}$$

and set $\varphi(0) = 1$, $\varphi(\theta) = 0$ for $\theta \geqslant 1$ and $\varphi(-\theta) = \varphi(\theta)$ for all θ.

Given such a function φ, we set

$$\varphi_n(\theta) = \varphi(2^{\mathfrak{s}_0 n}\theta) \tag{A.3}$$

for all $n \in \mathbb{N}_0$. Then φ_n is supported on $[-2^{-\mathfrak{s}_0 n}, 2^{-\mathfrak{s}_0 n}]$. Observe that for any $n \in \mathbb{N}_0$ and any $t \in \mathbb{R}$, we have

$$\sum_{m\in\mathbb{Z}} \varphi_n(2^{-\mathfrak{s}_0 n}m + t) = 1 , \tag{A.4}$$

and thus

$$\sum_{m\in\mathbb{Z}} Q_{nm}(t) = Q(t) \qquad \text{where} \quad Q_{nm}(t) = Q(t)\varphi_n(2^{-\mathfrak{s}_0 n}m + t) . \tag{A.5}$$

Since Q is compactly supported, the above sum only contains a finite number of terms, of order $2^{\mathfrak{s}_0 n}$. In fact, for any given t, there are at most two nonzero terms in the sum, and Q_{nm} is supported on the interval

$$\left[(m-1)2^{-\mathfrak{s}_0 n}, (m+1)2^{-\mathfrak{s}_0 n}\right] . \tag{A.6}$$

Recall that the kernel K being regularizing of order β means that K and its derivatives satisfy the bounds given in [2, Assumption 5.1]. Thus if we define for $n \in \mathbb{N}_0$, $m \in \mathbb{Z}$

$$K^Q_{nm}(t,x) = \int_{t-2T}^{t} Q_{nm}(t-s) K_n(s,x)\,\mathrm{d}s = \int_0^{2T} Q_{nm}(u) K_n(t-u,x)\,\mathrm{d}u ,$$

then we obtain a decomposition

$$K^Q(z) = \sum_{n\geqslant 0} \sum_{m=-1}^{1+2T2^{\mathfrak{s}_0 n}} K^Q_{nm}(z) , \tag{A.7}$$

where the range of m is due to (A.6) and the fact that Q is supported on $[0,2T]$.

First note that since Q_{nm} is supported on the interval given in (A.6), $K^Q_{nm}(t,x)$ can be nonzero only if

$$(m-2)2^{-\mathfrak{s}_0 n} \leqslant t \leqslant (m+2)2^{-\mathfrak{s}_0 n} \tag{A.8}$$

and x is in a ball of radius 2^{-n}. The condition on t is equivalent to $|t - m2^{-\mathfrak{s}_0 n}| \leqslant 2^{1-\mathfrak{s}_0 n}$, from which (3.6) follows.

Since φ takes values in $[0,1]$, we have

$$|Q_{nm}(t)| \leqslant |Q(t)| \leqslant \|Q\|_\infty := \sup_{t\in[0,2T]} |Q(t)| . \tag{A.9}$$

Therefore, by Condition (5.4) in [2, Assumption 5.1], we have

$$\begin{aligned}
|\mathrm{D}^k K^Q_{nm}(z)| &\leqslant \int_{(m-1)2^{-\mathfrak{s}_0 n}}^{(m+1)2^{-\mathfrak{s}_0 n}} |Q_{nm}(u)||\mathrm{D}^k K_n(t-u,x)|\,\mathrm{d}u \\
&\leqslant C2^{(|\mathfrak{s}|-\beta+|k|_\mathfrak{s})n} \int_{(m-1)2^{-\mathfrak{s}_0 n}}^{(m+1)2^{-\mathfrak{s}_0 n}} |Q_{nm}(u)|\,\mathrm{d}u \\
&\leqslant 2C2^{(|\mathfrak{s}|-\mathfrak{s}_0-\beta+|k|_\mathfrak{s})n} \|Q_{nm}\|_\infty ,
\end{aligned} \tag{A.10}$$

which implies (3.7) with $C_Q = 2C\|Q\|_\infty$.

Finally, we have

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}} z^\ell \,\mathrm{D}^k K^Q_{nm}(z)\,\mathrm{d}z &= \int_{(m-1)2^{-\mathfrak{s}_0 n}}^{(m+1)2^{-\mathfrak{s}_0 n}} Q_{nm}(u) \int_{\mathbb{R}^{d+1}} z^\ell \,\mathrm{D}^k K_n(t-u,x)\,\mathrm{d}z\,\mathrm{d}u \\
&= \int_{(m-1)2^{-\mathfrak{s}_0 n}}^{(m+1)2^{-\mathfrak{s}_0 n}} Q_{nm}(u) \int_{\mathbb{R}^{d+1}} (\bar{z}+(u,0))^\ell \,\mathrm{D}^k K_n(\bar{z})\,\mathrm{d}\bar{z}\,\mathrm{d}u .
\end{aligned} \tag{A.11}$$

It follows from Condition (5.5) in [2, Assumption 5.1], applied to all ℓ' of degree less or equal ℓ, that there exists a constant C', depending only on k and ℓ, such that the absolute value of the integral over $\mathbb{R}^{d+1}$ is bounded by $C'2^{-\beta n}$ uniformly in n. Therefore, (3.8) follows with $C_Q = 2C'$. □

B Proof of Lemma 3.4

As in [2, Lemma 5.19], the cases $2^{-n} > \lambda$ and $2^{-n} \leqslant \lambda$ are treated differently. We start by dealing with the case $2^{-n} > \lambda$. The base-time independence assumption $\Pi_{z+h_{nm}}\tau = \Pi_z\tau$ implies

$$\big\langle \Pi_z\tau, \mathrm{D}^k K^Q_{nm}(z-\cdot)\big\rangle = \big\langle \Pi_{z+h_{nm}}\tau, \mathrm{D}^k K^Q_{nm}(z-\cdot)\big\rangle \;. \tag{B.1}$$

Since the singularity of $K^Q_{nm}(z-\cdot)$ is located at $z+h_{nm}$, we can apply [2, Remark 2.21], which together with the bound (3.7) on $|\mathrm{D}^k K^Q_{nm}|$ yields

$$\big|\big\langle \Pi_z\tau, \mathrm{D}^k K^Q_{nm}(z-\cdot)\big\rangle\big| \lesssim \|\Pi\|_{\alpha;\mathfrak{K}_z} 2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta)n} \;. \tag{B.2}$$

Note that owing to base-time independence of the model, we have avoided making use of Γ as in [2, Lemma 5.18]. We now use the Taylor expansion representation of [2, Appendix A] to get

$$K^{Q;\alpha}_{nm;z\bar z}(z') = \sum_{\ell\in\partial A}\int_{\mathbb{R}^{d+1}} \mathrm{D}^\ell K^Q_{nm}(\bar z + h - z')\mathcal{Q}^\ell(z-\bar z, \mathrm{d}h) \tag{B.3}$$

where $A = \{\ell\colon |\ell|_\mathfrak{s} < \alpha+\beta\}$ and $\mathcal{Q}^\ell$ is a measure with total mass $\|z-\bar z\|_\mathfrak{s}^{|\ell|_\mathfrak{s}}$. It follows from (B.2) that

$$\big|\big\langle \Pi_z\tau, K^{Q;\alpha}_{nm;z\bar z}\big\rangle\big| \lesssim \|\Pi\|_{\alpha;\mathfrak{K}_z} \sum_{\ell\in\partial A} \|z-\bar z\|_\mathfrak{s}^{|\ell|_\mathfrak{s}} 2^{(|\ell|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta)n} \;. \tag{B.4}$$

Together with the fact that

$$\int_{\mathbb{R}^{d+1}} \|z-\bar z\|_\mathfrak{s}^{|\ell|_\mathfrak{s}} \psi^\lambda_z(\bar z)\,\mathrm{d}\bar z \lesssim \lambda^{|\ell|_\mathfrak{s}} \tag{B.5}$$

this yields

$$\begin{aligned}\sum_{\substack{(n,m)\in\mathfrak{N}\\ 2^{-n}>\lambda}} \bigg|\int_{\mathbb{R}^{d+1}} \big\langle \Pi_z\tau, K^{Q;\alpha}_{nm;z\bar z}\big\rangle \psi^\lambda_z(\bar z)\,\mathrm{d}\bar z\bigg| &\lesssim \sum_{\ell\in\partial A}\sum_{\substack{(n,m)\in\mathfrak{N}\\ 2^{-n}>\lambda}} 2^{(|\ell|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta)n}\lambda^{|\ell|_\mathfrak{s}}\|\Pi\|_{\alpha;\mathfrak{K}_z} \\ &\lesssim \lambda^{\alpha+\beta}\|\Pi\|_{\alpha;\mathfrak{K}_z} \;.\end{aligned} \tag{B.6}$$

In the case $2^{-n} \leqslant \lambda$, we use the representation

$$\int_{\mathbb{R}^{d+1}} \big\langle \Pi_z\tau, K^{Q;\alpha}_{nm;z\bar z}\big\rangle \psi^\lambda_z(\bar z)\,\mathrm{d}\bar z = \big\langle \Pi_z\tau, Y^\lambda_{nm}\big\rangle - \sum_{|\ell|_\mathfrak{s}<\alpha+\beta} \big\langle \Pi_z\tau, Z^\lambda_{nm;\ell}\big\rangle \tag{B.7}$$

where

$$\begin{aligned} Y^\lambda_{nm}(z') &= \int_{\mathbb{R}^{d+1}} K^Q_{nm}(\bar z - z')\psi^\lambda_z(\bar z)\,\mathrm{d}\bar z \;, \\ Z^\lambda_{nm;\ell}(z') &= \mathrm{D}^\ell K^Q_{nm}(\bar z - z') \int_{\mathbb{R}^{d+1}} \frac{(\bar z - z)^\ell}{\ell!}\psi^\lambda_z(\bar z)\,\mathrm{d}\bar z \;.\end{aligned} \tag{B.8}$$

Here one readily checks that the arguments used in the proof of [2, Lemma 5.19] to bound $\langle \Pi_z\tau, Y^\lambda_{nm}\rangle$ and $\langle \Pi_z\tau, Z^\lambda_{nm;\ell}\rangle$ are not affected by the location of the support of K^Q_{nm}, so that as a result we obtain in the same way as there the bounds

$$\begin{aligned} \big|\big\langle \Pi_z\tau, Y^\lambda_{nm}\big\rangle\big| &\lesssim \lambda^\alpha 2^{-(\mathfrak{s}_0+\beta)n} \;, \\ \big|\big\langle \Pi_z\tau, Z^\lambda_{nm;\ell}\big\rangle\big| &\lesssim \lambda^{|\ell|_\mathfrak{s}} 2^{(|\ell|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta)n} \;.\end{aligned} \tag{B.9}$$

This yields

$$\bigg|\int_{\mathbb{R}^{d+1}} \big\langle \Pi_z\tau, K^{Q;\alpha}_{nm;z\bar z}\big\rangle \psi^\lambda_z(\bar z)\,\mathrm{d}\bar z\bigg| \lesssim 2^{-\mathfrak{s}_0 n}\lambda^{\alpha+\beta} \sum_{|\ell|_\mathfrak{s}<\alpha+\beta} (\lambda 2^n)^{|\ell|_\mathfrak{s}-\alpha-\beta} \;, \tag{B.10}$$

and summing over (n,m) with $2^{-n} \leqslant \lambda$ gives the result. □

C Proof of Lemma 3.6

Using the fact that $\langle f_z, X^\ell \rangle = (-z)^\ell$ and multiplicativity of $\langle f_z, \cdot \rangle$, we obtain

$$\bigl\langle f_z, X^\ell \mathcal{J}^Q_{k+\ell}\tau \bigr\rangle = -(-z)^\ell \chi^{k+\ell}_\tau(z) \; . \tag{C.1}$$

From the expression (3.17) of $\Delta(\mathcal{I}^Q\tau)$ we thus deduce

$$F_z\mathcal{I}^Q\tau = (\mathrm{Id}\otimes f_z)\Delta(\mathcal{I}^Q\tau) = \mathcal{I}^Q\tau - \sum_{|k+\ell|_{\mathfrak{s}}<\alpha+\beta} \frac{X^k}{k!}\frac{(-z)^\ell}{\ell!}\chi^{k+\ell}_\tau(z) \; . \tag{C.2}$$

In the basis $(\{X^k\}_{|k|_{\mathfrak{s}}<\alpha+\beta}, \mathcal{I}^Q\tau)$ we can thus identify F_z and its inverse with matrices

$$F_z = \begin{pmatrix} T(z) & T_*(z) \\ 0 & 1 \end{pmatrix} \; , \qquad F_z^{-1} = \begin{pmatrix} T(z)^{-1} & -T(z)^{-1}T_*(z) \\ 0 & 1 \end{pmatrix} \; . \tag{C.3}$$

Here $T(z)$, which represents the action of F_z on monomials X^k, is an upper triangular matrix with elements

$$\bigl[T(z)\bigr]_{kj} = \frac{j!}{k!(j-k)!}(-z)^{j-k} \; , \tag{C.4}$$

while $T_*(z)$ is a column vector given by the coefficients of X^k in the sum on the right-hand side of (C.2). It follows that $\Gamma_{z\bar z}$ is represented by the matrix

$$F_z^{-1}F_{\bar z} = \begin{pmatrix} T(z)^{-1}T(\bar z) & T(z)^{-1}[T_*(\bar z) - T_*(z)] \\ 0 & 1 \end{pmatrix} \; . \tag{C.5}$$

Since $F_z^{-1} = F_{-z}$ for elements of the polynomial part of the regularity structure, one has $T(z)^{-1} = T(-z)$. A direct computation of the upper right matrix element then yields the result, making use of the binomial identity. $\square$

Remark C.1. Another way of deriving the result is by using the identity

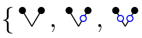
$$F_z^{-1}\mathcal{I}^Q\tau = \mathcal{I}^Q\tau + \sum_{|k|_{\mathfrak{s}}<\alpha+\beta} \frac{X^k}{k!}\chi^k_\tau(z) \; , \tag{C.6}$$

which can be readily checked by showing that $F_z^{-1}F_z = \mathrm{Id}$.

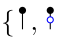
D Proof of Theorem 3.10

It follows from [2, Proposition 6.16 and Thm. 7.1] that $\mathcal{K}^Q_\gamma f_1$ and $\mathcal{K}^Q_\gamma \varphi\mathbf{1}$ satisfy the theorem. It thus remains to prove the statement for $\mathcal{K}^Q_\gamma f_2$ and check the convolution identity for f_3. By linearity, it is sufficient to consider the case

$$f_2(z) = a(z)\tau + \sum_{i=1}^{3} a_i(z)X_i\tau \; , \qquad \tau \in \{\ \ldots\ \} \; , \tag{D.1}$$



the cases $f_2(z) = a(z)\tau$ with $\tau \in \{\ \ldots\ \}$ being similar but simpler. Therefore we fix $\alpha = |\tau|_{\mathfrak{s}} = -1-2\kappa$. To further simplify the notations, we will drop the notation $\sum_{i=1}^3$, and not indicate the dependence of proportionality constants on $\|\Pi\|$ and $\|\Gamma\|$.



It is crucial to keep track of the sign of the first component t of z. We write $\boldsymbol{R}^+ f_2(z) = a^+(z)\tau + a_i^+(z)X_i\tau$, where $a^+(z) = a(z)1_{\{t>0\}}$ and similarly for $a_i^+(z)$, and $t_+ = t1_{\{t>0\}}$. We also use the notations $f^+_{nm}(z) = f_2(z+h_{nm})1_{\{t>0\}}$ and

$$b^{k;+}_{nm}(z) = \langle \mathcal{R}^{nm} f^+_{nm} - \Pi^{nm}_z f^+_{nm}(z), \mathrm{D}^k\,\hat K^Q_{nm}(z-\cdot)\rangle \; . \tag{D.2}$$

For all $z,\bar z\in\mathbb{R}^{d+1}$, $(n,m)\in\mathfrak{N}$ and τ as in (D.1), we have the relations

$$\Pi_z^{nm}\tau=\Pi_{\bar z}^{nm}\tau\;,\qquad \Pi_z^{nm}X_i\tau=\Pi_{\bar z}^{nm}X_i\tau+(\bar z_i-z_i)\Pi_z^{nm}\tau\;, \tag{D.3}$$

and the estimates

$$\big|\hat\chi^k_{\tau,nm}(z)\big|\lesssim 2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta)n}\;, \tag{D.4}$$

$$\big|\hat\chi^k_{X_i\tau,nm}(z)\big|\lesssim 2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta-1)n}\;, \tag{D.5}$$

$$\big|\boldsymbol{R}^+ b^{k,+}_{nm}(z)\big|\leqslant (1\wedge t_+)^{(\eta-\gamma)/\mathfrak{s}_0}1_{\{t>0\}}2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\gamma-\beta)n}|\!|\!|\boldsymbol{R}^+f_2|\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}}\;. \tag{D.6}$$

Indeed, the first two bounds follows in the same way as (B.2) (using the fact that $\bar z_i-z_i=0$ for time-translations), while the last one is a consequence of the improved reconstruction theorem [2, Lemma 6.7], (3.7) and the fact that $f^+_{nm}\in\mathcal{D}^{\gamma,\eta}(\Pi^{nm})$. This shows that the infinite series in the definition (3.41) of $\mathcal{K}^Q_\gamma$ are indeed convergent. Note that this is exactly the point where the introduction of shifted models is necessary, since otherwise b^k_{nm} would only satisfy a weaker bound of the form (3.25), which guarantees summability only for $|k|_\mathfrak{s}<\alpha+\beta$.

D.1 Bounds on $|\!|\!|\boldsymbol{R}^+\mathcal{K}^Q_\gamma\boldsymbol{R}^+f|\!|\!|_{\gamma+\beta,\bar\eta;T}$

Since $\Gamma_{z\bar z}\tau=\tau$ and $\Gamma_{z\bar z}X_i\tau=X_i\tau+(z_i-\bar z_i)\tau$, the fact that $f_2\in\mathcal{D}^{\gamma,\eta}$ implies that

$$\begin{aligned}|a^+(z)|&\lesssim(1\wedge t_+)^{(\eta-\alpha)/\mathfrak{s}_0}1_{\{t>0\}}|\!|\!|\boldsymbol{R}^+f_2|\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}}\;,\\ |a_i^+(z)|&\lesssim(1\wedge t_+)^{(\eta-\alpha-1)/\mathfrak{s}_0}1_{\{t>0\}}|\!|\!|\boldsymbol{R}^+f_2|\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}}\;,\\ |a^+(z)-a^+(\bar z)-(z_i-\bar z_i)a_i^+(\bar z)|&\lesssim\|z-\bar z\|_\mathfrak{s}^{\gamma-\alpha}(1\wedge t_+\wedge\bar t_+)^{(\eta-\gamma)/\mathfrak{s}_0}1_{\{t,\bar t>0\}}|\!|\!|\boldsymbol{R}^+f_2|\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}}\;,\\ |a_i^+(z)-a_i^+(\bar z)|&\lesssim\|z-\bar z\|_\mathfrak{s}^{\gamma-\alpha-1}(1\wedge t_+\wedge\bar t_+)^{(\eta-\gamma)/\mathfrak{s}_0}1_{\{t,\bar t>0\}}|\!|\!|\boldsymbol{R}^+f_2|\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}}\end{aligned} \tag{D.7}$$

holds for all $z,\bar z\in\bar{\mathfrak{K}}$ (the 1-fattening of $\mathfrak{K}$). We start by estimating

$$\|\boldsymbol{R}^+\mathcal{K}^Q_\gamma\boldsymbol{R}^+f_2\|_{\gamma+\beta,\bar\eta;\mathfrak{K}}=\sup_{z\in\mathfrak{K}}\max_{\delta\in\{\alpha+\beta,\alpha+\beta+1,0,\dots,\lfloor\gamma+\beta\rfloor\}}\frac{\|\boldsymbol{R}^+\mathcal{K}^Q_\gamma\boldsymbol{R}^+f_2\|_\delta}{1\wedge t_+^{(\bar\eta-\delta)\wedge0}}\;. \tag{D.8}$$

In the case $\delta=\alpha+\beta$, we have

$$\begin{aligned}\|\boldsymbol{R}^+\mathcal{K}^Q_\gamma\boldsymbol{R}^+f_2(z)\|_{\alpha+\beta}&=\sup_{(n,m)\in\mathfrak{N}}|a^+(z+h_{nm})|1_{\{t>0\}}\\ &\leqslant(1\wedge t_+)^{(\eta-\alpha)/\mathfrak{s}_0}1_{\{t>0\}}|\!|\!|\boldsymbol{R}^+f_2|\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}}\;.\end{aligned} \tag{D.9}$$

For $\bar\eta=\eta+\beta$, this provides the first bound required to obtain $\boldsymbol{R}^+\mathcal{K}^Q_\gamma\boldsymbol{R}^+f_2\in\mathcal{D}^{\gamma+\beta,\eta+\beta}$. Note that the factor $1_{\{t>0\}}$, which is due to the first $\boldsymbol{R}^+$, is required to kill the singularities of $a^+(z+h_{nm})$ for negative time. In the particular case $\mathfrak{K}=O_T=(-\infty,T]\times\mathbb{R}^d$, we can further bound the factor $(1\wedge t_+)^{(\eta-\alpha)/\mathfrak{s}_0}1_{\{t>0\}}$ by $T^{\kappa/\mathfrak{s}_0}(1\wedge t_+)^{(\eta-\alpha-\kappa)/\mathfrak{s}_0}1_{\{t>0\}}$, with O_T instead of $\bar{\mathfrak{K}}$ since the kernel is non-anticipative. This yields one of the bounds required to prove (3.49). The case $\delta=\alpha+\beta+1$ is treated similarly.

For polynomial components of exponent $\ell\in\mathbb{N}$, (3.41) implies

$$\begin{aligned}\|\boldsymbol{R}^+\mathcal{K}^Q_\gamma\boldsymbol{R}^+f_2(z)\|_\ell\lesssim\sum_{|k|_\mathfrak{s}=\ell}\frac{1}{k!}\bigg|\sum_{(n,m)\in\mathfrak{N}}\Big[&b^{k,+}_{nm}(z)1_{\{\ell<\gamma+\beta\}}+a^+(z+h_{nm})\hat\chi^k_{\tau,nm}(z)1_{\{\ell<\alpha+\beta\}}\\ &+a_i^+(z+h_{nm})\hat\chi^k_{X_i\tau,nm}(z)1_{\{\ell<\alpha+\beta+1\}}\Big]\bigg|1_{\{t>0\}}\;.\end{aligned} \tag{D.10}$$

Here we treat separately the cases $1 \wedge t_+^{1/\mathfrak{s}_0} \geqslant 2^{-n}$ and $1 \wedge t_+^{1/\mathfrak{s}_0} < 2^{-n}$. In the first case, we use the bounds

$$\begin{aligned}\bigl|a^+(z+h_{nm})\hat\chi^k_{\tau,nm}(z)\bigr| &\lesssim (1\wedge t_+)^{(\eta-\alpha)/\mathfrak{s}_0} 2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta)n} \|\!|\boldsymbol{R}^+ f_2|\!\|_{\gamma,\eta;\bar{\mathfrak{K}}}\;,\\ \bigl|a_i^+(z+h_{nm})\hat\chi^k_{X_i\tau,nm}(z)\bigr| &\lesssim (1\wedge t_+)^{(\eta-\alpha-1)/\mathfrak{s}_0} 2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\alpha-\beta-1)n} \|\!|\boldsymbol{R}^+ f_2|\!\|_{\gamma,\eta;\bar{\mathfrak{K}}}\;, \end{aligned}\tag{D.11}$$

which follow directly from (D.4), (D.5) and (D.7). Using (D.6) and summing over the relevant (n,m) yields indeed a bound of order $(1\wedge t_+)^{[(\eta+\beta-|k|_\mathfrak{s})\wedge 0]/\mathfrak{s}_0}$ for (D.10), and in the case $\mathfrak{K} = O_T$ we can again extract a factor $T^{\kappa/\mathfrak{s}_0}$ by decreasing $\bar\eta$.

In the case $1 \wedge t_+^{1/\mathfrak{s}_0} < 2^{-n}$, if $\ell < \alpha+\beta$ we use (3.42) to get

$$b^{k,+}_{nm}(z) + a^+(z+h_{nm})\hat\chi^k_{\tau,nm}(z) + a_i^+(z+h_{nm})\hat\chi^k_{X_i\tau,nm}(z) = \bigl\langle \mathcal{R}^{nm} f^+_{nm}, \mathrm{D}^k\,\hat K^Q_{nm}(z-\cdot)\bigr\rangle\;. \tag{D.12}$$

Here we apply [2, Proposition 6.9], which states that $\mathcal{R}^{nm} f^+_{nm} \in \mathcal{C}^\eta_\mathfrak{s}$, showing that the above quantity has order $2^{(|k|_\mathfrak{s}-\mathfrak{s}_0-\beta-\eta)n}$. Since by assumption $|k|_\mathfrak{s}-\beta-\eta \neq 0$, regardless of its sign, summing over (n,m) yields a bound of order $(1\wedge t_+)^{[(\eta+\beta-|k|_\mathfrak{s})\wedge 0]/\mathfrak{s}_0}$. If $\alpha+\beta < \ell < \alpha+\beta+1$, we use

$$b^{k,+}_{nm}(z) + a_i^+(z+h_{nm})\hat\chi^k_{X_i\tau,nm}(z) = \bigl\langle \mathcal{R}^{nm} f^+_{nm}, \mathrm{D}^k\,\hat K^Q_{nm}(z-\cdot)\bigr\rangle - a^+(z+h_{nm})\hat\chi^k_{\tau,nm}(z)\;, \tag{D.13}$$

so that a bound of the same order follows by combining the two previous estimates. The case $\ell > \alpha+\beta+1$ is treated similarly.

It remains to obtain estimates involving two different points $z,\bar z$. Here we first note that the definition of $\mathfrak{K}_P$ entering the definition of $\|\!|f|\!\|$ implies that if $(z,\bar z)\in\mathfrak{K}_P$ then t and $\bar t$ necessarily have the same sign and are comparable. Thus we only need to consider the case where both t and $\bar t$ are strictly positive, and we may drop one of the factors $\boldsymbol{R}^+$.

Lemma 3.6 extends naturally to $\mathcal{I}^Q_{nm}\tau$ and $\hat\chi^k_{nm}(z)$. Proceeding similarly as in Appendix C, but in the basis $(\{X^k\}_{|k|_\mathfrak{s}<\alpha+\beta}, \mathcal{I}^Q_{nm}\tau, \mathcal{I}^Q_{nm}X_i\tau)$, we also obtain

$$\begin{aligned}\Gamma_{z\bar z}\mathcal{I}^Q_{nm}(X_i\tau) &= \mathcal{I}^Q_{nm}(X_i\tau) + (z_i-\bar z_i)\mathcal{I}^Q_{nm}\tau\\ &\quad + \sum_{|k|_\mathfrak{s}<\alpha+\beta+1} \frac{X^k}{k!}\biggl[\hat\chi^k_{X_i\tau,nm}(z) - \sum_{|\ell|_\mathfrak{s}<\alpha+\beta+1-|k|_\mathfrak{s}} \frac{(z-\bar z)^\ell}{\ell!}\hat\chi^{k+\ell}_{X_i\tau,nm}(\bar z)\\ &\qquad + (z_i-\bar z_i)\hat\chi^k_{\tau,nm}(z)1_{\{|k|_\mathfrak{s}<\alpha+\beta\}}\biggr]\;. \end{aligned}\tag{D.14}$$

This yields the expression

$$\begin{aligned}\mathcal{K}^Q_\gamma \boldsymbol{R}^+ f_2(z) - \Gamma_{z\bar z}\mathcal{K}^Q_\gamma\boldsymbol{R}^+ f_2(\bar z) = \sum_{(n,m)\in\mathfrak{N}} \biggl\{ & Q^{\alpha+\beta}_{nm}(z,\bar z)\mathcal{I}^Q_{nm}\tau + Q^{\alpha+\beta+1}_{nm}(z,\bar z)\mathcal{I}^Q_{nm}(X_i\tau)\\ &+ \sum_{|k|_\mathfrak{s}<\gamma+\beta}\frac{X^k}{k!}Q^k_{nm}(z,\bar z)\biggr\}\;, \end{aligned}\tag{D.15}$$

where

$$\begin{aligned} Q^{\alpha+\beta}_{nm}(z,\bar z) &= a^+(z+h_{nm}) - a^+(\bar z+h_{nm}) - (z_i-\bar z_i)a_i^+(\bar z+h_{nm})\;,\\ Q^{\alpha+\beta+1}_{nm}(z,\bar z) &= a_i^+(z+h_{nm}) - a_i^+(\bar z+h_{nm})\;,\\ Q^k_{nm}(z,\bar z) &= b^{k,+}_{nm}(z) - \sum_{|\ell|_\mathfrak{s}<\beta+\gamma-|k|_\mathfrak{s}} \frac{(z-\bar z)^\ell}{\ell!} b^{k+\ell,+}_{nm}(\bar z)\\ &\quad + Q^{\alpha+\beta}_{nm}(z,\bar z)\hat\chi^k_{\tau,nm}(z)1_{\{|k|_\mathfrak{s}<\alpha+\beta\}}\\ &\quad + Q^{\alpha+\beta+1}_{nm}(z,\bar z)\hat\chi^k_{X_i\tau,nm}(z)1_{\{|k|_\mathfrak{s}<\alpha+\beta+1\}} \end{aligned}\tag{D.16}$$

(note that the terms in $a^+(z+h_{nm})\hat{\chi}^{k+\ell}_{\tau,nm}(\bar z)$ stemming from $\Gamma_{z\bar z}\mathcal{I}^Q_{nm}$ and $\Gamma_{z\bar z}X^k$ in the first sum over k in (3.41) cancel). For the components of non-integer regularity, we obtain using (D.7) the required bounds

$$\begin{aligned}\big|Q^{\alpha+\beta}_{nm}(z,\bar z)\big| &\lesssim \|z-\bar z\|_{\mathfrak{s}}^{\gamma-\alpha}(1\wedge t_+\wedge \bar t_+)^{(\eta-\gamma)/\mathfrak{s}_0}\|\!|\boldsymbol{R}^+ f_2|\!\|_{\gamma,\eta;\bar{\mathfrak{K}}}\;,\\ \big|Q^{\alpha+\beta+1}_{nm}(z,\bar z)\big| &\lesssim \|z-\bar z\|_{\mathfrak{s}}^{\gamma-\alpha-1}(1\wedge t_+\wedge \bar t_+)^{(\eta-\gamma)/\mathfrak{s}_0}\|\!|\boldsymbol{R}^+ f_2|\!\|_{\gamma,\eta;\bar{\mathfrak{K}}}\;,\end{aligned}\tag{D.17}$$

from which a factor $T^{\kappa/\mathfrak{s}_0}$ can be extracted as before.

Finally, in the case of polynomial terms, we now consider three different regimes, depending on the value of 2^{-n} compared to $\|z-\bar z\|_{\mathfrak{s}}$ and $\frac12(1\wedge t_+\wedge\bar t_+)^{1/\mathfrak{s}_0})$ (recall that for $(z,\bar z)\in\mathfrak{K}_P$ we always have $\|z-\bar z\|_{\mathfrak{s}}\leqslant(1\wedge t_+\wedge\bar t_+)^{1/\mathfrak{s}_0})$. In the case $2^{-n}\leqslant\|z-\bar z\|_{\mathfrak{s}}$, we again estimate separately the summands in $Q^k_{nm}(z,\bar z)$, yielding the bound

$$\sum_{\substack{(n,m)\in\mathfrak{N}\\ 2^{-n}\leqslant\|z-\bar z\|_{\mathfrak{s}}}}\big|Q^k_{nm}(z,\bar z)\big|\lesssim\|z-\bar z\|_{\mathfrak{s}}^{\beta+\gamma-|k|_{\mathfrak{s}}}(1\wedge t_+\wedge\bar t_+)^{(\eta-\gamma)/\mathfrak{s}_0}\|\!|\boldsymbol{R}^+ f_2|\!\|_{\gamma,\eta;\bar{\mathfrak{K}}}\;.\tag{D.18}$$

For $\|z-\bar z\|_{\mathfrak{s}}<2^{-n}\leqslant\frac12(1\wedge t_+\wedge\bar t_+)^{1/\mathfrak{s}_0}$ and $|k|_{\mathfrak{s}}<\alpha+\beta$, we use the fact that

$$Q^k_{nm}(z,\bar z)=\big\langle\mathcal{R}^{nm}f^+_{nm}-\Pi^{nm}_{\bar z}f^+_{nm}(\bar z),\hat K^{Q,k;\gamma}_{nm;\bar z z}\big\rangle\;,\tag{D.19}$$

where $K^{Q,k;\gamma}_{nm;\bar z z}$ is defined as in (3.12), but with $\mathrm{D}^k\,\hat K^Q_{nm}$ instead of K^Q_{nm}. It thus admits the integral representation

$$K^{Q,k;\gamma}_{nm;\bar z z}(z')=\sum_{\ell\in\partial A}\int_{\mathbb{R}^{d+1}}\mathrm{D}^{k+\ell}\,\hat K^Q_{nm}(z+h-z')\mathcal{Q}^\ell(\bar z-z,\mathrm{d}h)\tag{D.20}$$

where $A=\{\ell\colon|k+\ell|_{\mathfrak{s}}<\gamma+\beta\}$. Here we use an argument similar to the one used in the proof of Lemma 3.4. Owing to lack of translation invariance, however, we have to decompose, writing $\tilde z=z+h$,

$$\begin{aligned}\big\langle\mathcal{R}^{nm}f^+_{nm}&-\Pi^{nm}_{\bar z}f^+_{nm}(\bar z),\mathrm{D}^{k+\ell}\,\hat K^Q_{nm}(\tilde z-\cdot)\big\rangle\\ &=\big\langle\mathcal{R}^{nm}f^+_{nm}-\Pi^{nm}_{\tilde z}f^+_{nm}(\tilde z),\mathrm{D}^{k+\ell}\,\hat K^Q_{nm}(\tilde z-\cdot)\big\rangle\\ &\quad+\big\langle\Pi^{nm}_{\tilde z}[f^+_{nm}(\tilde z)-\Gamma_{\tilde z\bar z}f^+_{nm}(\bar z)],\mathrm{D}^{k+\ell}\,\hat K^Q_{nm}(\tilde z-\cdot)\big\rangle\;.\end{aligned}\tag{D.21}$$

For the first term on the right-hand side, we apply again the improved reconstruction theorem [2, Lemma 6.7] to obtain a bound of order $(1\wedge\tilde t_+)^{(\eta-\gamma)/\mathfrak{s}_0}2^{(|k+\ell|_{\mathfrak{s}}-\mathfrak{s}_0-\gamma-\beta)n}$. Since $\mathcal{Q}^\ell(\bar z-z,\cdot)$ is supported on values of h such that $\|h\|_{\mathfrak{s}}\leqslant\|z-\bar z\|_{\mathfrak{s}}\leqslant\frac12(1\wedge t_+\wedge\bar t_+)^{1/\mathfrak{s}_0}$, we can replace $\tilde t_+$ by $t_+\wedge\bar t_+$ in this expression. For the second term, we use the fact that

$$\Gamma_{\tilde z\bar z}f^+_{nm}(\bar z)=\big[a^+(\tilde z+h_{nm})+(\tilde z_i-\bar z_i)a^+(\bar z+h_{nm})\big]\tau+a^+(\bar z+h_{nm})X_i\tau\;,\tag{D.22}$$

as well as (D.7), (D.4) and (D.5) to get the bound

$$\|\tilde z-z\|_{\mathfrak{s}}^{\gamma-\alpha}(1\wedge t_+\wedge\tilde t_+)^{(\eta-\gamma)/\mathfrak{s}_0}2^{(|k+\ell|_{\mathfrak{s}}-\mathfrak{s}_0-\alpha-\beta)n}\;.\tag{D.23}$$

Again, we can replace $\tilde t_+$ by $\bar t_+$, and also bound $\|\tilde z-z\|_{\mathfrak{s}}^{\gamma-\alpha}$ by $2^{-(\gamma-\alpha)n}$. Adding the last two bounds and using the fact that $\mathcal{Q}^\ell(z-\bar z,\cdot)$ has total mass $\|z-\bar z\|_{\mathfrak{s}}^{|\ell|_{\mathfrak{s}}}$, we obtain

$$\big|Q^k_{nm}(z,\bar z)\big|\lesssim(1\wedge t_+\wedge\bar t_+)^{(\eta-\gamma)/\mathfrak{s}_0}\sum_{\ell\in\partial A}\|z-\bar z\|_{\mathfrak{s}}^{|\ell|_{\mathfrak{s}}}2^{(|k+\ell|_{\mathfrak{s}}-\mathfrak{s}_0-\gamma-\beta)n}\|\!|\boldsymbol{R}^+ f_2|\!\|_{\gamma,\eta;\bar{\mathfrak{K}}}\;.\tag{D.24}$$

Summing over the relevant values of (n,m), we again obtain a bound as in the right-hand side of (D.18). The same bound holds also in the case $|k|_{\mathfrak{s}}>\alpha+\beta$, by combining the previous arguments.

In the last case $2^{-n} > \frac{1}{2}(1 \wedge t_+ \wedge \bar{t}_+)^{1/\mathfrak{s}_0}$, we have the bound

$$\big|\big\langle \mathcal{R}^{nm} f^+_{nm} - \Pi^{nm}_{\bar{z}} f^+_{nm}(\bar{z}), \mathrm{D}^{k+\ell}\, \hat{K}^Q_{nm}(z-\cdot)\big\rangle\big| \lesssim 2^{(|k+\ell|_{\mathfrak{s}} - \mathfrak{s}_0 - \eta - \beta)n} |\!|\!| \boldsymbol{R}^+ f_2 |\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}} \,. \tag{D.25}$$

Indeed, such a bound holds separately for $|\langle \mathcal{R}^{nm} f^+_{nm}, \mathrm{D}^{k+\ell}\, \hat{K}^Q_{nm}(z-\cdot)\rangle|$ since $\mathcal{R}^{nm} f^+_{nm} \in \mathcal{C}^\eta_{\mathfrak{s}}$, and for $|\langle \Pi^{nm}_{\bar{z}} f^+_{nm}(\bar{z}), \mathrm{D}^{k+\ell}\, \hat{K}^Q_{nm}(z-\cdot)\rangle|$, as a consequence of (D.11) and the condition on 2^{-n}. Substituting in the integral expression (D.20) shows that

$$\begin{aligned} \big|Q^k_{nm}(z,\bar{z})\big| & \\ &\lesssim \sum_{\ell \in \partial A} \|z-\bar{z}\|_{\mathfrak{s}}^{|\ell|_{\mathfrak{s}}} 2^{(|k+\ell|_{\mathfrak{s}} - \mathfrak{s}_0 - \eta - \beta)n} |\!|\!| \boldsymbol{R}^+ f_2 |\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}} \\ &\lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{\beta+\gamma-|k|_{\mathfrak{s}}} \sum_{\ell \in \partial A} (1 \wedge t_+ \wedge \bar{t}_+)^{(|k+\ell|_{\mathfrak{s}} - \beta - \gamma)/\mathfrak{s}_0} 2^{(|k+\ell|_{\mathfrak{s}} - \mathfrak{s}_0 - \eta - \beta)n} |\!|\!| \boldsymbol{R}^+ f_2 |\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}} \,. \end{aligned} \tag{D.26}$$

Summing over the relevant values of (n,m) yields again a bound of the form (D.18). This completes the proof of the fact that $\boldsymbol{R}^+ \mathcal{K}^Q_\gamma \boldsymbol{R}^+ f_2 \in \mathcal{D}^{\gamma,\eta}$. As before, a factor $T^{\kappa/\mathfrak{s}_0}$ can be extracted when $\mathfrak{K} = O_T$, which also completes the proof of (3.49).

The proof of (3.50) is very similar to the one just given, using the estimate [2, (3.4)] of the reconstruction theorem in place of [2, (3.3)].

D.2 Convolution identity

It remains to prove that the convolution identity (3.48) holds. This will follow from the next result, combined with the reconstruction theorem.

Lemma D.1. *For every $z = (t,x) \in \mathbb{R}^{d+1}$ such that $t > 0$, the bound*

$$\big|\big\langle \Pi_z \mathcal{K}^Q_\gamma \boldsymbol{R}^+ f_2(z) - K^Q * \mathcal{R}\boldsymbol{R}^+ f_2, \psi^\lambda_z\big\rangle\big| \lesssim \lambda^{\gamma+\beta} (1 \wedge t)^{(\eta-\gamma)/\mathfrak{s}_0} \tag{D.27}$$

holds uniformly in $\lambda \in (0, 1 \wedge t^{1/\mathfrak{s}_0}]$.

Proof. Using the representation

$$\big\langle K^Q * \mathcal{R}\boldsymbol{R}^+ f_2, \psi^\lambda_z\big\rangle = \sum_{(n,m)\in\mathfrak{N}} \int_{\mathbb{R}^{d+1}} \big\langle \mathcal{R}^{nm} f^+_{nm}, \hat{K}^Q_{nm}(\bar{z}-\cdot)\big\rangle \psi^\lambda_z(\bar{z}) \,\mathrm{d}z \tag{D.28}$$

and the definition (3.43) of the model, we obtain

$$\big\langle \Pi_z \mathcal{K}^Q_\gamma \boldsymbol{R}^+ f_2(z) - K^Q * \mathcal{R}\boldsymbol{R}^+ f_2, \psi^\lambda_z\big\rangle = -\sum_{(n,m)\in\mathfrak{N}} \int_{\mathbb{R}^{d+1}} Q^0_{nm}(\bar{z},z) \psi^\lambda_z(\bar{z}) \,\mathrm{d}\bar{z} \,, \tag{D.29}$$

with Q^0_{nm} as in (D.19). Since ψ^λ_z is supported in the set $\{\bar{z}\colon \|\bar{z}-z\|_{\mathfrak{s}} \leqslant \lambda\}$, whenever $\lambda \leqslant 2^{-n} \leqslant \frac{1}{2}(1 \wedge t \wedge \bar{t})^{1/\mathfrak{s}_0}$, (D.24) provides the bound

$$\big|Q^0_{nm}(\bar{z},z)\big| \lesssim (1 \wedge t \wedge \bar{t})^{(\eta-\gamma)/\mathfrak{s}_0} \sum_{\ell \in \partial A} \|z-\bar{z}\|_{\mathfrak{s}}^{|\ell|_{\mathfrak{s}}} 2^{(|\ell|_{\mathfrak{s}} - \mathfrak{s}_0 - \gamma - \beta)n} |\!|\!| \boldsymbol{R}^+ f_2 |\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}} \tag{D.30}$$

where $A = \{\ell\colon |\ell|_{\mathfrak{s}} < \gamma+\beta\}$. For $2^{-n} > \frac{1}{2}(1 \wedge t \wedge \bar{t})^{1/\mathfrak{s}_0}$, (D.26) yields

$$\big|Q^0_{nm}(\bar{z},z)\big| \lesssim \|z-\bar{z}\|_{\mathfrak{s}}^{\beta+\gamma} \sum_{\ell \in \partial A} (1 \wedge t \wedge \bar{t})^{(|\ell|_{\mathfrak{s}} - \gamma - \beta)/\mathfrak{s}_0} 2^{(|\ell|_{\mathfrak{s}} - \mathfrak{s}_0 - \eta - \beta)n} |\!|\!| \boldsymbol{R}^+ f_2 |\!|\!|_{\gamma,\eta;\bar{\mathfrak{K}}} \,. \tag{D.31}$$

Combining this with (B.5), we obtain

$$\sum_{\substack{(n,m)\in\mathfrak{N} \\ 2^{-n} \geqslant \lambda}} \bigg| \int_{\mathbb{R}^{d+1}} Q^0_{nm}(\bar{z},z) \psi^\lambda_z(\bar{z}) \,\mathrm{d}\bar{z} \bigg| \lesssim \lambda^{\gamma+\beta} (1 \wedge t)^{(\eta-\gamma)/\mathfrak{s}_0} \,. \tag{D.32}$$

Finally, in the case $2^{-n} < \lambda$, we use the same argument as in (B.7), yielding

$$\int_{\mathbb{R}^{d+1}} Q^0_{nm}(\bar z, z)\psi^\lambda_z(\bar z)\,\mathrm{d}\bar z = \Big\langle \mathcal{R}^{nm} f^+_{nm} - \Pi^{nm}_z f^+_{nm}(z), Y^\lambda_{nm} - \sum_{|\ell|_{\mathfrak{s}}<\gamma+\beta} Z^\lambda_{nm;\ell}\Big\rangle \;. \tag{D.33}$$

Here we obtain bounds similar to (B.9), but with γ instead of α and an extra factor $(1\wedge t)^{(\eta-\gamma)/\mathfrak{s}_0}$. Summing over m and n yields again a bound as on the right-hand side of (D.32). □

Combining (D.27) with [2, Lemma 6.7], we obtain

$$\big|\big\langle \mathcal{R}\mathcal{K}^Q_\gamma \boldsymbol{R}^+ f_2(z) - K^Q * \mathcal{R}\boldsymbol{R}^+ f_2, \psi^\lambda_z\big\rangle\big| \lesssim \lambda^{\gamma+\beta}(1\wedge t)^{(\eta-\gamma)/\mathfrak{s}_0} \;, \tag{D.34}$$

which proves the convolution identity by the uniqueness part of the reconstruction theorem.

To complete the proof of Theorem 3.10, we have to show that $(K^Q * \mathcal{R}\boldsymbol{R}^+ f_3)(z) = 0$ for all $z=(t,x)$ such that $t>0$. Here we use the fact that

$$\big\langle K^Q * \mathcal{R}\boldsymbol{R}^+ f_3, \psi^\lambda_z\big\rangle = \sum_{(n,m)\in\mathfrak{N}} \int_{\mathbb{R}^{d+1}} \big\langle \mathcal{R}\boldsymbol{R}^+ f_3, \psi^\lambda_z\big\rangle K^Q_{nm}(z-\bar z)\,\mathrm{d}\bar z \;. \tag{D.35}$$

Since by the diagonal identity (3.36)

$$\lim_{\lambda\to0} \big\langle \Pi_z \boldsymbol{R}^+ f_3(z), \psi^\lambda_z\big\rangle = \lim_{\lambda\to0} \sum_{\tau\in\mathcal{F}_3} a^+_\tau(z)\big\langle \Pi_z\tau, \psi^\lambda_z\big\rangle = 0 \;, \tag{D.36}$$

the reconstruction theorem implies that $\langle \mathcal{R}\boldsymbol{R}^+ f_3, \psi^\lambda_z\rangle$ converges to 0 as well, and the desired conclusion follows. □

E Proofs for Section 5

E.1 Proof of Lemma 5.1

Applying Proposition 3.1 with $|\mathfrak{s}| = 5$, $\mathfrak{s}_0 = 2$ and $\beta = 2$, we find that $K^Q_{nm}(t,x)$ is supported in the ball $\|(t,x) - (m2^{-2n}, 0)\|_{\mathfrak{s}} \leqslant (1+\sqrt{2})2^{-n}$, and is of order 2^n. Using the bound on $\mathcal{Q}^\varepsilon_0(z)$ given in [1, Lemma 6.2], it follows that

$$\big|I^Q_{00;mn}(\varepsilon)\big| \lesssim \int_{(m-1)2^{-2n}}^{(m+1)2^{-2n}} \int_{\|x\|\lesssim 2^{-n}} 2^n \frac{1}{|t| + \|x\|^2 + \varepsilon^2}\,\mathrm{d}x\,\mathrm{d}t \tag{E.1}$$

$$\lesssim 2^n \int_{(m-1)2^{-2n}}^{(m+1)2^{-2n}} \int_0^{2^{-n}} \frac{r^2\,\mathrm{d}r}{|t| + r^2 + \varepsilon^2}\,\mathrm{d}t \;, \tag{E.2}$$

where we have used polar coordinates, and the equivalence of the ℓ^1 norm $|x|$ and the Euclidean norm $\|x\|$. For $m<2$, we obtain a bound of order 2^{-2n} by bounding $|t| + r^2 + \varepsilon^2$ below by r^2. For $m \geqslant 2$, bounding $|t| + r^2 + \varepsilon^2$ below by $r^2 + (m-1)2^{-2n}$ yields a bound of order $2^{-2n}m^{-1} \leqslant 2^{1-2n}(m+2)^{-1}$. □

E.2 Proof of Proposition 5.3

It suffices to apply the renormalisation map M_ε to all monomials in U and V of degree 2 and 3, when U and V are given by (4.6). Using the expressions [1, (6.12)] for the action of M_ε, one obtains

$$\begin{aligned} M_\varepsilon U^2 &= U^2 - C_1(\varepsilon)\mathbf{1} \;, \\ M_\varepsilon U^3 &= U^3 - 3\big[\varphi C_1(\varepsilon) + b_1 C_2(\varepsilon)\big]\mathbf{1} - 3\big[C_1(\varepsilon) + 3\gamma_1 C_2(\varepsilon)\big]\stackrel{\bullet}{|} - 9\gamma_2 C_2(\varepsilon)\stackrel{\bullet}{\circ} + \varrho_{U^3}(U,V) \;, \\ M_\varepsilon U^2 V &= U^2 V - \psi C_1(\varepsilon)\mathbf{1} - C_1(\varepsilon)\stackrel{\bullet}{\circ} + \varrho_{U^2V}(U,V) \;, \end{aligned} \tag{E.3}$$

where $\varrho_{U^3}(U,V)$ and $\varrho_{U^2V}(U,V)$ are remainder terms of strictly positive homogeneity. All other monomials are invariant under M_ε up to remainders of strictly positive homogeneity. The result follows, using the expression (4.3) for b_1 with $p = \varphi$ and $q = \psi$, and the expansion (4.6) in order to express ⫯ and ⫰ in terms of U and V. □

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