Electronic Journal of Probability

Electron. J. Probab. **24** (2019), no. 76, 1–27.
ISSN: 1083-6489 https://doi.org/10.1214/19-EJP334

First passage time of the frog model has a sublinear variance

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Abstract

In this paper, we show that the first passage time in the frog model on $\mathbb{Z}^d$ with $d \geq 2$ has a sublinear variance. This implies that the central limit theorem does not hold at least with the standard diffusive scaling. The proof is based on the method introduced in [4, 11] combined with a control of the maximal weight of paths in a locally dependent site-percolation. We also apply this method to get the linearity of the lengths of optimal paths.

Keywords: frog model; first passage time; sublinear variance.
AMS MSC 2010: 60K35; 60J80; 60J10.
Submitted to EJP on October 1, 2018, final version accepted on June 15, 2019.

1 Introduction

Frog models are simple but well-known models in the study of the spread of infection. In these models, individuals (also called frogs) move on the integer lattice $\mathbb{Z}^d$, which have one of two states infected (active) and healthy (passive). We assume that at the beginning, there is only one infected frog at the origin, and there are healthy frogs at other sites of $\mathbb{Z}^d$. When a healthy frog encounters with an infected one, it becomes infected forever. While the healthy frogs do not move, the frogs perform independent simple random walks once they get infected. We are interested in the long time behavior of the infected individuals.

To the best of our knowledge, the first result on frog models is due to Tecls and Wormald [23], where they proved the recurrence of the model (more precisely, they showed that the origin is visited infinitely often a.s. by infected frogs). Since then, there are numerous results on the behavior of the model under various settings of initial configurations, mechanism of walks, or underlying graphs, see [1, 3, 6, 13, 14, 15, 16, 18].

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In particular, Popov and some authors study the phase transition of the recurrence versus transience for the model with Bernoulli initial configurations and for the model with drift, see [2, 12, 14, 22]. Another interesting feature in the frog model is that it can be described in the first passage percolation context, which is explained below. In fact, Alves, Machado and Popov used this property to prove a shape theorem [1]. Moreover, the large deviation estimate for the first passage time is derived in [8, 19] recently.

The frog model can be defined formally as follows. Let $d \geq 2$ and $\{(\mathrm{S}_j^x)_{j\in\mathbb{N}}, x \in \mathbb{Z}^d\}$ be independent SRWs such that $\mathrm{S}_0^x = x$ for any $x \in \mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, let

$$t(x,y) = \inf\{j \in \mathbb{N}_{\geq 0} : \mathrm{S}_j^x = y\}.$$

The first passage time from x to y is defined by

$$\mathrm{T}(x,y) = \inf\Big\{\sum_{i=1}^{k} t(x_{i-1},x_i) : x = x_0,\dots,x_k = y \text{ for some } k\Big\}.$$

The quantity $\mathrm{T}(x,y)$ can be seen as the first time when the frog at y becomes infected assuming that the frog at x was the only infected one at the beginning. For the simplicity of notation, we write $\mathrm{T}(x)$ instead of $\mathrm{T}(0,x)$. A path $\gamma = (x_i)_{i=0}^{\ell}$ with $x_0 = x$ and $x_\ell = y$ is said to be optimal if $\mathrm{T}(x,y) = \sum_{i=1}^{\ell} t(x_{i-1},x_i)$. For any $x, y \in \mathbb{Z}^d$, such a path certainly exists since $\mathrm{T}(x,y)$ is a finite natural number almost surely by Lemma 2.1.

It has been shown in [1] that the first passage time is subadditive, i.e., for any $x, y, z \in \mathbb{Z}^d$

$$\mathrm{T}(x,z) \leq \mathrm{T}(x,y) + \mathrm{T}(y,z). \tag{1.1}$$

The authors of [1] also show that the sequence $\{\mathrm{T}((k-1)z, kz)\}_{k\geq 1}$ is stationary and ergodic for any $z \in \mathbb{Z}^d$. As a consequence of Kingman's subadditive ergodic theorem (see [17] or [1, Theorem 3.1]), one has

$$\lim_{n\to\infty} \frac{\mathrm{T}(nz)}{n} \to \kappa_z \quad a.s., \tag{1.2}$$

with

$$\kappa_z = \inf_{n\in\mathbb{N}_{\geq 1}} \frac{\mathbb{E}(\mathrm{T}(nz))}{n}.$$

Furthermore, a shape theorem for the set of active frogs has been also proved, see [1, Theorem 1.1]. The convergence (1.2), which can be seen as a law of large numbers, implies that for any $x \in \mathbb{Z}^d$ the first passage time $\mathrm{T}(x)$ grows linearly in $|x|_1$. A natural question is whether the standard central limit theorem holds for $\mathrm{T}(x)$. The first task is to understand the behavior of variance of $\mathrm{T}(x)$. In [19], the author proves some large deviation estimates for $\mathrm{T}(x)$, see in particular Lemma 2.2 below. As a consequence, one can show that $\mathrm{Var}(\mathrm{T}(x)) = \mathcal{O}(|x|_1(1+\log|x|_1)^{2A})$, for some constant A, see Corollary 2.3. However, this result is not enough to answer the question on the standard central limit theorem.

Our main result is to show that the first passage time has sublinear variance and thus the central limit theorem with the standard diffusive scaling[1] is not true.

Theorem 1.1. Let $d \geq 2$. Then there exists a positive constant $C = C(d)$ such that for any $x \in \mathbb{Z}^d$,

$$\mathrm{Var}(\mathrm{T}(x)) \leq \frac{C|x|_1}{\log|x|_1}.$$

[1] Indeed, it follows from Theorem 1.1 and Chebyshev's inequality that $\mathbb{P}\left(\frac{T(x)-\mathbb{E}(T(x))}{\sqrt{\mathbb{E}(T(x))}} \geq t\right) \leq \frac{C}{t^2 \log|x|_1} \to 0$ as $|x|_1 \to \infty$. This rules out the possibility of holding the standard central limit theorem.

The frog model on $\mathbb{Z}$ (i.e., $d = 1$ in our setting) has been carefully investigated by many authors, see e.g., [6, 4, 14]. In particular, Commets, Quastel and Ramírez [10] proved the standard Gaussian fluctuation for the first passage time $\mathrm{T}(x)$. As a consequence, $\mathrm{Var}(\mathrm{T}(x)) \asymp |x|_1$ and the standard central limit theorem for $\mathrm{T}(x)$ holds. We also notice that not only the fluctuation but also the large deviation behavior of $\mathrm{T}(x)$ in one dimension is different from that in higher dimensions. Indeed, in the forthcoming paper [8], we and Kubota prove that $\varphi(x) = -\log \mathbb{P}(\mathrm{T}(x) \geq (1+\varepsilon)\mathbb{E}(\mathrm{T}(x))$ behaves differently when the dimension increases. More precisely, we show that if $d = 1$ then $\varphi(x)$ is of order $\sqrt{|x|_1}$, if $d = 2$ then $\varphi(x)$ is of order $|x|_1/\log |x|_1$ and if $d \geq 3$ then $\varphi(x)$ is of order $|x|_1$ as $|x|_1 \to \infty$.

The sublinearity of variance as in Theorem 1.1, which is also called the superconcentration, was first discovered in the first passage percolation with Bernoulli edge weights by Benjamini, Kalai and Schramm [5]. Hence, this result is sometimes called BKS-inequality. Chatterjee [9] found the connection among properties of superconcentration, chaos and multiple valleys in, for example, the gaussian polymer model and Sherrington-Kirkpatrick model (see Chapter 5 and 10 in [9]). This relation is expected to hold in general models. Therefore, the superconcentration is not only an interesting result itself but also an important property to study the structure of optimal paths and the energy landscape.

The method in [5] has been improved by Benaïm and Rossignol in [4] to show the sublinearity of the variance of $\mathrm{T}(x)$ in the first passage percolation with a wide class of edge weight distributions, which they called "nearly gamma". Finally, Damron, Hanson and Sosoe in [11] generalized the result to all edge weight distributions with $2 + \log$ finite moment. In this paper, we closely follows the method given in [5, 4, 11]. However, there are some other difficulties to prove the sublinear variance in the frog models, which will be explained in a sketch of the proof below.

1.1 Sketch of the proof

First, we define F_m the spatial average of $\mathrm{T}(x)$ as

$$\mathrm{F}_m = \frac{1}{\#\mathrm{B}(m)} \sum_{z \in \mathrm{B}(m)} \mathrm{T}(z, z+x),$$

with $m = [|x|_1^{1/4}]$ and we prove in Proposition 3.1 that $|\mathrm{Var}(\mathrm{T}(x)) - \mathrm{Var}(\mathrm{F}_m)| = \mathcal{O}(|x|_1^{3/4+\varepsilon})$ for any $\varepsilon > 0$. That means we only need to study $\mathrm{Var}(\mathrm{F}_m)$. As in [4, 11] we consider the martingale decomposition of F_m,

$$\mathrm{F}_m - \mathbb{E}(\mathrm{F}_m) = \sum_{k=1}^{\infty} \Delta_k,$$

where

$$\Delta_k = \mathbb{E}(\mathrm{F}_m \mid \mathcal{F}_k) - \mathbb{E}(\mathrm{F}_m \mid \mathcal{F}_{k-1}),$$

with $\mathcal{F}_k$ the sigma-algebra generated by SRWs $\{(\mathrm{S}_j^{x_i})_{j \in \mathbb{N}}, i = 1, \ldots, k\}$ and $\mathcal{F}_0$ the trivial sigma-algebra. Note that here we enumerate $\mathbb{Z}^d$ as $\{x_1, x_2, \ldots\}$. As we will see later, with the help of the weighted logarithmic Sobolev inequality (Lemma 2.8) and the Falik-Smorodnisky inequality (Lemma 3.2), our problem is reduced to prove a series of lemmas 3.4, 3.7 and 3.8. For illustration, we sketch here the proof of Lemma 3.4, where we show that as $L \to \infty$,

$$\mathbb{E}\left(\max_{\gamma = (y_i)_{i=1}^{\ell} \in \mathcal{P}_L} \sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i, y_{i+1}) \right) = \mathcal{O}(L), \tag{1.3}$$

where T_1 is a modified first passage time, and $\mathcal{P}_L$ is the set of paths in the box $[-L, L]^d$ with length less than L (see (3.19) and section 1.3 for precise definitions). Although the passage times $\{\mathrm{T}_1(y_i, y_{i+1})\}_i$ are concentrated around their means, the correlation among them makes the above problem difficult and interesting. Fortunately, the passage times have the local-dependency property. Indeed, we will show in Lemma 3.9 that

(O1) for any $u, v \in \mathbb{Z}^d$, $M \geq 1$, the event $\{\mathrm{T}_1(u, v) = M\}$ depends only on SRWs $\{(S^x_\cdot) : |x - u|_1 \leq M\}$,

(O2) there exist an integer $C_1 \geq 1$ and a constant $\varepsilon_1 > 0$ such that for any $u, v \in \mathbb{Z}^d$,

$$\mathbb{P}(\mathrm{T}_1(u, v) \geq C_1|u - v|_1) \leq \exp(|u - v|_1^{\varepsilon_1}).$$

Starting from these observations, for any path $\gamma = (y_i)_{i=1}^{\ell}$, we consider the following bound

$$\sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i, y_{i+1}) \leq \sum_{M\geq 1} \sum_{k\geq 0} (C_1 M + k) a^{\gamma}_{M,k}, \tag{1.4}$$

where

$$a^{\gamma}_{M,0} = \sum_{y_i \in \gamma} \mathbb{I}(|y_i - y_{i+1}|_1 = M,\ \mathrm{T}_1(y_i, y_{i+1}) \leq C_1 M),$$

and for $k \geq 1$,

$$a^{\gamma}_{M,k} = \sum_{y_i \in \gamma} \mathbb{I}(|y_i - y_{i+1}|_1 = M,\ \mathrm{T}_1(y_i, y_{i+1}) = C_1 M + k).$$

Hence

$$\sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i, y_{i+1}) \leq C_1|\gamma|_1 + \sum_{M\geq 1} \sum_{k\geq 1} k a^{\gamma}_{M,k}, \tag{1.5}$$

with $|\gamma|_1 = \sum_{i=1}^{\ell-1} |y_i - y_{i+1}|$. It is obvious that

$$a^{\gamma}_{M,k} \leq X_{M,k}(\gamma) := \sum_{y\in\gamma} I_y^{M,k}, \tag{1.6}$$

where

$$I_y^{M,k} = \mathbb{I}(\exists z : |z - y|_1 \leq M, \mathrm{T}_1(y, z) = C_1 M + k).$$

Now we arrive at

$$\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L} \sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i, y_{i+1}) \leq C_1 L + \max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L} \sum_{M\geq 1} \sum_{k\geq 1} k X_{M,k}(\gamma). \tag{1.7}$$

Here considering the site-percolation on $\mathbb{Z}^d$ generated by the collection of Bernoulli random variables $\{I_y^{M,k}, y \in \mathbb{Z}^d\}$, $X_{M,k}(\gamma)$ is the total weight of γ on this percolation. Thanks to the observation (O1), the site-percolation is $(C_1 M + k)$-dependent and by the union bound and (O2),

$$q_{M,k} := \sup_{y\in\mathbb{Z}^d} \mathbb{E}(I_y^{M,k}) \leq (2(C_1 M + k) + 1)^d \exp(-(C_1 M + k)^{\varepsilon_1}).$$

In the next section, we prove Lemma 2.6 to control the maximal weight of paths in locally dependent site-percolation by using a known result for independent site-percolation and tessellation arguments. In particular, we can show that, with some constant $C > 0$,

$$\begin{aligned}\mathbb{E}\left(\max_{\gamma\in\mathcal{P}_L} X_{M,k}(\gamma)\right) &\leq CL(C_1 M + k)^d q_{M,k}^{1/d} \\ &\leq CL(C_1 M + k)^{d+1} \exp(-(C_1 M + k)^{\varepsilon_1}/d).\end{aligned}$$

Plugging this estimate into (1.7), we get (1.3).

Our approach seems to be robust and useful for other problems. In particular, using a similar method, we also prove the linearity of the length of optimal paths.

1.2 The linearity of the lengths of optimal paths

Given $x, y \in \mathbb{Z}^d$, let us denote by $\mathbb{O}(x,y)$ the set of all optimal paths from x to y. We simply write $\mathbb{O}(x)$ for $\mathbb{O}(0,x)$. For any path $\gamma = (y_i)_{i=1}^{\ell} \subset \mathbb{Z}^d$, we denote the length of γ as $l(\gamma) = \ell$. We will prove that the lengths of optimal paths from 0 to x grow linearly in $|x|_1$ despite of the fact that optimal paths may have jumps with size tending to infinity as $|x|_1 \to \infty$.

Proposition 1.2. Let $d \geq 2$. Then there exist positive constants ε, c and C such that for any $x \in \mathbb{Z}^d$

$$\mathbb{P}\left(c|x|_1 \leq \min_{\gamma \in \mathbb{O}(x)} l(\gamma) \leq \max_{\gamma \in \mathbb{O}(x)} l(\gamma) \leq C|x|_1\right) \geq 1 - e^{-|x|_1^{\varepsilon}}.$$

1.3 Notation

- If $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we denote $|x|_1 = |x_1| + \ldots + |x_d|$.
- For any $n \geq 1$, we denote $\mathrm{B}(n) = [-n, n]^d$.
- For any $\ell \geq 1$, we call a sequence of ℓ distinct vertices $\gamma = (y_i)_{i=1}^{\ell}$ in $\mathbb{Z}^d$ a path of length ℓ, we denote $|\gamma|_1 = |y_2 - y_1|_1 + \ldots + |y_\ell - y_{\ell-1}|_1$.
- Given $y = y_i \in \gamma$, we define $\bar{y} = y_{i+1}$ the next point of y in γ with the convention that $\bar{y}_\ell = y_\ell$.
- We write $y \sim \bar{y} \in \gamma$ if $\bar{y}$ is the next point of y in γ.
- For $L \geq 1$, we write

$$\mathcal{P}_L = \{\gamma = (y_i)_{i=1}^{\ell} \subset \mathrm{B}(L) \,|\, |\gamma|_1 \leq L,\ y_i \neq y_j \ if \ i \neq j\}.$$

- If f and g are two functions, we write $f = \mathcal{O}(g)$ if there exists a positive constant C such that $f(x) \leq Cg(x)$ for any x.
- We use $C > 0$ for a large constant and ε for a small constant. Note that they may change from line to line.
- Given a set $A \subset \mathbb{Z}^d$, we denote by $|A|$ the number of elements of A.

1.4 Organization of this paper

The paper is organized as follows. In Section 2, we present some preliminary results including large deviation estimates on the first passage time and an estimate to control the tail distribution of maximal weight of paths in site-percolation, the introduction and properties of entropy. In Sections 3 and 4, we prove the main theorem 1.1 and Proposition 1.2, respectively.

2 Preliminaries

2.1 Large deviation estimates on first passage times

We present here some useful estimates on the deviation of first passage times.

Lemma 2.1. [19, Proposition 2.4] There exists an integer $C_1 \geq 1$ and a positive constant ε_1 such that for any $x, y \in \mathbb{Z}^d$ and $t \geq C_1|x-y|_1$,

$$\mathbb{P}\left(\mathrm{T}(x,y) \geq t\right) \leq e^{-t^{\varepsilon_1}}.$$

We notice that Lemma 2.1 was first proved in [1, Lemma 4.2] for the case $t = C_1|x-y|_1$. It follows from Lemma 2.1 that there exists $C > 0$ such that for any $x \in \mathbb{Z}^d$,

$$\mathbb{E}\mathrm{T}(x) \le C|x|_1. \tag{2.1}$$

The following concentration inequality is derived in [19].

Lemma 2.2. [19, Theorem 1.4] For any $C > 0$, there exist positive constants a, b and A such that for any $x \in \mathbb{Z}^d$ and $(2 + \log|x|_1)^A \le t \le C\sqrt{|x|_1}$,

$$\mathbb{P}(|\mathrm{T}(x) - \mathbb{E}\mathrm{T}(x)| \ge t\sqrt{|x|_1}) \le e^{-bt^a}.$$

As a direct consequence of Lemmas 2.1 and 2.2, we have the following.

Corollary 2.3. There exists a positive constant A such that

$$\mathrm{Var}(\mathrm{T}(x)) = \mathcal{O}(|x|_1(1 + \log|x|_1)^{2A}).$$

Proof. We take a positive constant C sufficiently large such that Lemma 2.1 and (2.1) hold. By using the fact $\mathbb{E}(X^2) = \int_0^\infty 2t\mathbb{P}(X \ge t)dt$ for any non-negative random variable X, we get

$$\begin{aligned}
\mathrm{Var}(\mathrm{T}(x)) &= \int_0^\infty 2t\mathbb{P}(|\mathrm{T}(x) - \mathbb{E}\mathrm{T}(x)| \ge t)dt \\
&= \int_0^{(2+\log|x|_1)^A\sqrt{|x|_1}} 2t\mathbb{P}(|\mathrm{T}(x) - \mathbb{E}\mathrm{T}(x)| \ge t)dt \\
&\quad + \int_{(2+\log|x|_1)^A\sqrt{|x|_1}}^{2C|x|_1} 2t\mathbb{P}(|\mathrm{T}(x) - \mathbb{E}\mathrm{T}(x)| \ge t)dt \\
&\quad + \int_{2C|x|_1}^\infty 2t\mathbb{P}(|\mathrm{T}(x) - \mathbb{E}\mathrm{T}(x)| \ge t)dt.
\end{aligned} \tag{2.2}$$

The first term of the right hand side (2.2) can be bounded from above by

$$\int_0^{(2+\log|x|_1)^A\sqrt{|x|_1}} 2tdt \le (2 + \log|x|_1)^{2A}|x|_1.$$

By Lemma 2.2, the second term is bounded from above by

$$|x|_1 \int_0^\infty 2te^{-bt^a}dt = \mathcal{O}(|x|_1).$$

Finally, by (2.1) and Lemma 2.1, the third term is bounded from above by

$$\int_{2C|x|_1}^\infty 2t\mathbb{P}(\mathrm{T}(x) \ge t/2)dt \le \int_{2C|x|_1}^\infty 2te^{-(t/2)^{\varepsilon_1}}dt = \mathcal{O}(1).$$

Combining these estimates, we get the conclusion. □

Lemma 2.4. There exists a positive constant ε_2 such that for any $x, y \in \mathbb{Z}^d$ and $M \ge 1$,

$$\mathbb{P}(\mathrm{T}(x,y) = t(x,y) = M) \le e^{-M^{\varepsilon_2}}.$$

Proof. If $|x-y|_1 \le M^{2/3}$, then the result follows from Lemma 2.1. Assume that $|x-y|_1 \ge M^{2/3}$. Then a well-known estimate for the trajectory of random walk (see [20, Proposition 2.1.2]) shows that for some positive constants c and C,

$$\mathbb{P}\left(\max_{0\le j\le k}|\mathrm{S}_j^x - x|_1 \ge r\right) \le Ce^{-cr^2/k}. \tag{2.3}$$

Therefore,

$$\mathbb{P}(t(x,y) = M) \le \mathbb{P}\left(\max_{0\le j\le M}|\mathrm{S}_j^x - x|_1 \ge M^{2/3}\right) \le Ce^{-cM^{1/3}}.$$ □

2.2 The maximal weight of paths in site-percolation

2.2.1 The case of independent percolation

Let $\{I_x\}_{x\in\mathbb{Z}^d}$ be a collection of independent random variables such that $\mathbb{P}(I_x = 1) = 1 - \mathbb{P}(I_x = 0) = p_x \le p$ with a parameter $p \in [0,1]$ for all $x \in \mathbb{Z}^d$. For any $A \subset \mathbb{Z}^d$, we define the weight of A as $X(A) = \sum_{x\in A} I_x$. The maximal weight of paths in $\mathcal{P}_L$ is defined as

$$X_L = \max_{\gamma\in\mathcal{P}_L} X(\gamma).$$

The tail distribution and expectation of X_L can be controlled by the following lemma.

Lemma 2.5. There exist positive constants A_1 and A_2 such that the following statements hold.

(i) If $\min\{sLp^{1/d}, s\} \ge A_1$ then

$$\mathbb{P}\left(X_L \ge sLp^{1/d}\right) \le \exp\left(-sLp^{1/d}/2\right).$$

(ii) For any $p \in (0,1)$ and $L \ge 1$,

$$\mathbb{E}\left(X_L\right) \le A_2 Lp^{1/d}.$$

Proof. We start by recalling a result in [11] on the maximal weight of lattice animals (i.e., connected sets containing 0). Define

$$N_L = \sup\{X(A) : 0 \in A, A \text{ is connected}, |A| \le L+1\}. \tag{2.4}$$

In Lemma 6.8 in [11], the authors show that there exist positive constants A_1' and A_2' such that

(a) if $Lp^{1/d} > 1$ and $s \ge A_1'$, then

$$\mathbb{P}\left(N_L \ge sLp^{1/d}\right) \le \exp\left(-sLp^{1/d}/2\right),$$

(b) for any $p \in (0,1)$ and $L \ge 1$,

$$\mathbb{E}\left(N_L\right) \le A_2' Lp^{1/d}.$$

Let $\gamma = (y_i)_{i=1}^{\ell} \in \mathcal{P}_L$. Then $\gamma \subset \mathrm{B(L)}$ and $\sum_{i=2}^{\ell} |y_i - y_{i-1}|_1 \le L$. Thus $\sum_{i=1}^{\ell} |y_i - y_{i-1}|_1 \le (d+1)L$ with $y_0 = 0$. Considering shortest paths from y_{i-1} to y_i for $1 \le i \le \ell$ in the lattice $\mathbb{Z}^d$, there exists a connected set $A \subset \mathbb{Z}^d$ such that $\gamma \subset A$ and $|A| \le 1 + \sum_{i=1}^{\ell} |y_i - y_{i-1}|_1 \le (d+1)L + 1$. Therefore $X(\gamma) \le X(A) \le N_{(d+1)L}$ for all $\gamma \in \mathcal{P}_L$. Hence

$$X_L \le N_{(d+1)L}. \tag{2.5}$$

Using (2.5) and (b), we obtain (ii). We now prove that (i) holds for $A_1 = (d+1)A_1'$ with A_1' as in (a). Let us denote by $\mathbb{P}_p$ the probability measure of site-percolation with density p. Using (2.5),

$$\mathbb{P}_p\left(X_L > sLp^{1/d}\right) \le \mathbb{P}_p\left(N_{(d+1)L} > sLp^{1/d}\right) = \mathbb{P}_p\left(N_{(d+1)L} > \tfrac{s}{d+1}(d+1)Lp^{1/d}\right). \tag{2.6}$$

Suppose $\min\{sLp^{1/d}, s\} \ge A_1$. If $(d+1)Lp^{1/d} > 1$, then using (a) and $\frac{s}{d+1} \ge A_1'$,

$$\mathbb{P}_p\left(N_{(d+1)L} > \tfrac{s}{d+1}(d+1)Lp^{1/d}\right) \le \exp\left(-sLp^{1/d}/2\right). \tag{2.7}$$

For the case $(d+1)Lp^{1/d} \le 1$, we define $q = L^{-d}$. Then $p < q$ and $(d+1)Lq^{1/d} > 1$. Thus using the monotonicity of $\mathbb{P}_p$ in p and (a),

$$\begin{aligned} \mathbb{P}_p\left(N_{(d+1)L} > sLp^{1/d}\right) &\le \mathbb{P}_q\left(N_{(d+1)L} > sLp^{1/d}\right) \\ &= \mathbb{P}_q\left(N_{(d+1)L} > \tfrac{sLp^{1/d}}{d+1}(d+1)Lq^{1/d}\right) \\ &\le \exp\left(-sLp^{1/d}/2\right), \end{aligned} \tag{2.8}$$

since $\frac{sLp^{1/d}}{d+1} \ge A_1'$. Combining (2.7) and (2.8) we get (i). □

Given $M \ge 1$, let $\{I_x, x \in \mathbb{Z}^d\}$ be a collection of Bernoulli random variables such that

(E1) $\{I_x, x \in \mathbb{Z}^d\}$ is M-dependent, i.e., for all $x \in \mathbb{Z}^d$, the variable I_x is independent of all variables $\{I_y : |y-x|_1 > M\}$,
(E2) $q_M = \sup_{x\in\mathbb{Z}^d} \mathbb{E}(I_x) \le (3M+1)^{-d}$.

For any path γ in $\mathbb{Z}^d$, we also define

$$X(\gamma) = \sum_{x\in\gamma} I_x, \qquad X_L = \max_{\gamma\in\mathcal{P}_L} X(\gamma). \tag{2.9}$$

Lemma 2.6. Let $M \ge 1$ and $\{I_x, x \in \mathbb{Z}^d\}$ be a collection of random variables satisfying (E1) and (E2). Then there exists a positive constant $C = C(d)$ such that

(i) for any $L \ge 1$,
$$\mathbb{E}(X_L) \le CLM^{d+1}q_M^{1/d}, \tag{2.10}$$

(ii) if $n \ge CM^d \max\{1, MLq_M^{1/d}\}$, then
$$\mathbb{P}(X_L \ge n) \le 2^d \exp(-n/(16M)^d). \tag{2.11}$$

Proof. For each $M \ge 1$, let us consider a standard tessellation of $\mathbb{Z}^d$ constructed as follows. Enumerate $\{0,1\}^d$ as $\{w_i, i = 1, \ldots, 2^d\}$. Then for any $i \in \{1, \ldots, 2^d\}$ and $z \in \mathbb{Z}^d$, we define
$$B_{i,z}^M = 3M(w_i + 2z) + [0, 3M]^d. \tag{2.12}$$
Then $(B_{i,z}^M)_{i,z}$ are boxes of side length $3M$ satisfying

(a) for all $y \in \mathbb{Z}^d$, there exists $\mathrm{B}_{i,z}^M$ containing y,
(b) for any $i = 1, \ldots, 2^d$, the boxes in the i-th group, $(\mathrm{B}_{i,\cdot}^M)$ satisfy that the distance $|\cdot|_1$ between two arbitrary boxes is larger than $3M$.

For $i = 1, \ldots, 2^d$, $z \in \mathbb{Z}^d$ and $\gamma \in \mathcal{P}_L$, define

$$X_{i,z}(\gamma) = X(\gamma \cap \mathrm{B}_{i,z}^M) = \sum_{x\in\gamma\cap\mathrm{B}_{i,z}^M} I_x.$$

Then by (a),

$$X(\gamma) \le \sum_{i=1}^{2^d} \sum_{z\in\mathbb{Z}^d} X_{i,z}(\gamma). \tag{2.13}$$

It is clear that for all $i = 1, \ldots, 2^d$,

$$\sum_{z\in\mathbb{Z}^d} X_{i,z}(\gamma) \le (3M+1)^d \sum_{z\in\eta^{i,M}} Y_z^i, \tag{2.14}$$

where $\eta^{i,M}$ is the projected path of γ defined by

$$\eta^{i,M} = \{z \in \mathbb{Z}^d : \gamma \cap \mathrm{B}^M_{i,z} \neq \varnothing\},$$

and

$$Y^i_z = \mathbb{I}\left(\exists x \in \mathrm{B}^M_{i,z} \text{ such that } I_x = 1\right).$$

Since $\gamma \in \mathcal{P}_L$, we have

$$\eta^{i,M} \in \mathcal{P}_{\lceil L/(3M)\rceil}.$$

Hence,

$$\sum_{z\in\eta^{i,M}} Y^i_z \leq \max_{\eta\in\mathcal{P}_{\lceil L/(3M)\rceil}} \sum_{z\in\eta} Y^i_z =: X^i_{L,M}. \tag{2.15}$$

Combining this inequality with (2.13) and (2.14) yields that

$$X_L = \max_{\gamma\in\mathcal{P}_L} \sum_{i=1}^{2^d} \sum_{z\in\mathbb{Z}^d} X_{i,z}(\gamma) \leq (3M+1)^d \sum_{i=1}^{2^d} X^i_{L,M}. \tag{2.16}$$

By (b) and (E1), $(Y^i_z)_{z\in\mathbb{Z}^d}$ are independent Bernoulli random variables. Moreover, by the union bound and (E2)

$$\begin{aligned} p_M := \sup_{(i,z)} \mathbb{E}(Y^i_z) &= \sup_{(i,z)} \mathbb{P}(\exists x \in \mathrm{B}^M_{i,z} : I_x = 1) \\ &\leq (3M+1)^d q_M \leq 1. \end{aligned} \tag{2.17}$$

Now applying Lemma 2.5 to the set of random variables $(Y^i_z)_{z\in\mathbb{Z}^d}$ and the set of paths $\mathcal{P}_{\lceil L/(3M)\rceil}$, we get

$$\mathbb{E}(X^i_{L,M}) \quad \leq \quad A_2\lceil L/(3M)\rceil p_M^{1/d}, \tag{2.18}$$

with A_2 as in Lemma 2.5 (ii). Combining (2.16), (2.17) and (2.18) gives

$$\begin{aligned} \mathbb{E}(X_L) &\leq A_2 2^d \lceil L/(3M)\rceil (3M+1)^d p_M^{1/d} \\ &\leq CLM^{d+1} q_M^{1/d}, \end{aligned} \tag{2.19}$$

for some $C = C(d)$. This proves (ii). We now turn to prove (i). Observe that by (2.16), for all n

$$\mathbb{P}\left(X_L \geq n\right) \leq \mathbb{P}\left(\sum_{i=1}^{2^d} X^i_{L,M} \geq n/(4M)^d\right) \leq \sum_{i=1}^{2^d} \mathbb{P}\left(X^i_{L,M} \geq n/(8M)^d\right). \tag{2.20}$$

By Lemma 2.5 (i), for all $i = 1, \ldots, 2^i$,

$$\begin{aligned} \mathbb{P}\left(X^i_{L,M} \geq n/(8M)^d\right) &= \mathbb{P}\left(X^i_{L,M} \geq \frac{n}{(8M)^d\lceil L/(3M)\rceil p_M^{1/d}} \lceil L/(3M)\rceil p_M^{1/d}\right) \\ &\leq \exp\left(-n/(2(8M)^d)\right), \end{aligned}$$

provided that

$$\min\left\{\frac{n}{(8M)^d}, \frac{n}{(4M)^d\lceil L/(3M)\rceil p_M^{1/d}}\right\} \geq A_1, \tag{2.21}$$

with A_1 as in Lemma 2.5 (i). Using (2.17), the condition (2.21) follows if

$$n \geq A_1(8M)^d \max\{1, \lceil L/(3M)\rceil (3M+1) q_M^{1/d}\}, \tag{2.22}$$

which is satisfied if

$$n \geq CM^d \max\{1, MLq_M^{1/d}\}, \tag{2.23}$$

for some $C = C(A_1, d)$. In conclusion, if (2.23) holds then

$$\mathbb{P}\left(X_L \geq n\right) \leq 2^d \exp(-n/(16M)^d). \tag{2.24}$$

□

2.3 Entropy

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $X \in L^1(\Omega, \mu)$ an non-negative random variable. Then the entropy of X with respect to μ is defined as

$$\mathrm{Ent}_\mu(X) = \mathbb{E}_\mu(X \log X) - \mathbb{E}_\mu(X) \log \mathbb{E}_\mu(X).$$

Note that by Jensen's inequality, $\mathrm{Ent}_\mu(X) \geq 0$. The following tensorization property of entropy is well-known and we refer the reader to [7] for the proof.

Lemma 2.7. [7, Theorem 4.22] Let X be a non-negative L^2 random variable on a product space

$$\left(\prod_{i=1}^{\infty} \Omega_i, \mathcal{F}, \mu = \prod_{i=1}^{\infty} \mu_i\right),$$

where $\mathcal{F} = \bigvee_{i=1}^{\infty} \mathcal{G}_i$, and each triple $(\Omega_i, \mathcal{G}_i, \mu_i)$ is a probability space. Then

$$\mathrm{Ent}_\mu(X) \leq \sum_{i=1}^{\infty} \mathbb{E}_\mu \mathrm{Ent}_i(X),$$

where $\mathrm{Ent}_i(X)$ is the entropy of $X(\omega) = X((\omega_1, \ldots, \omega_i, \ldots))$ with respect to μ_i, as a function of the i-th coordinate (with all other values fixed).

The following weighted logarithmic Sobolev inequality will be useful for estimating the entropy of martingale difference.

Lemma 2.8. [21, Lemma 2.6] Assume that $k \geq 2$. Let $f : \{1, \ldots, k\} \mapsto \mathbb{R}$ be a function and ν be the uniform distribution on $\{1, \ldots, k\}$. Then

$$\mathrm{Ent}_\nu(f^2) \leq k\mathrm{E}((f(U) - f(\tilde{U}))^2),$$

where E is the expectation with respect to two independent random variables $U, \tilde{U}$, which have the same distribution ν.

3 Proof of Theorem 1.1

3.1 Spatial average of the first passage time

We consider a spatial average of $\mathrm{T}(x)$ defined as

$$\mathrm{F}_m = \frac{1}{\#\mathrm{B}(m)} \sum_{z \in \mathrm{B}(m)} \mathrm{T}(z, z + x), \tag{3.1}$$

where

$$m = [|x|_1^{1/4}].$$

Proposition 3.1. For any $\varepsilon > 0$, it holds that

$$|\mathrm{Var}(\mathrm{T}(x)) - \mathrm{Var}(\mathrm{F}_m)| = \mathcal{O}(|x|_1^{3/4+\varepsilon}).$$

Proof. For any variables X and Y, by writing $\hat{X} = X - \mathbb{E}(X)$ and $||X||_2 = (\mathbb{E}(X^2))^{1/2}$ and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\mathrm{Var}(X) - \mathrm{Var}(Y)| = |E(\hat{X}^2 - \hat{Y}^2)| &\leq ||\hat{X} + \hat{Y}||_2 ||\hat{X} - \hat{Y}||_2 \\ &\leq (||\hat{X}||_2 + ||\hat{Y}||_2) ||\hat{X} - \hat{Y}||_2. \end{aligned} \tag{3.2}$$

We aim to apply (3.2) for $\mathrm{T}(x)$ and F_m. Observe that

$$||\hat{\mathrm{F}}_m||_2 \leq \frac{1}{\#\mathrm{B}(m)} \sum_{z \in B_m} ||\hat{\mathrm{T}}(z, z+x)||_2 = ||\hat{\mathrm{T}}(0,x)||_2, \tag{3.3}$$

by translation invariance. By Corollary 2.3,

$$||\hat{\mathrm{T}}(0,x)||_2 = \sqrt{\mathrm{Var}(\mathrm{T}(x))} = \mathcal{O}(|x|_1^{1/2}(1 + \log|x|_1)^A). \tag{3.4}$$

Using the subadditivity (1.1),

$$\begin{aligned} ||\hat{\mathrm{T}}(0,x) - \hat{\mathrm{F}}_m||_2^2 &= ||\mathrm{T}(x) - \mathrm{F}_m||_2^2 \\ &= \frac{1}{\#\mathrm{B}(m)^2} \left\| \sum_{z \in \mathrm{B}(m)} (\mathrm{T}(x) - \mathrm{T}(z, z+x)) \right\|_2^2 \\ &\leq \frac{1}{\#\mathrm{B}(m)^2} \left\| \sum_{z \in \mathrm{B}(m)} (\mathrm{T}(z) + \mathrm{T}(x, z+x) + \mathrm{T}(z,0) + \mathrm{T}(z+x, x)) \right\|_2^2 . \\ &\leq \frac{4}{\#\mathrm{B}(m)} \sum_{z \in \mathrm{B}(m)} \left[\mathbb{E}\mathrm{T}(z)^2 + \mathbb{E}\mathrm{T}(x, z+x)^2 + \mathbb{E}\mathrm{T}(z,0)^2 + \mathbb{E}\mathrm{T}(z+x, x)^2 \right] \\ &\leq 16 \max_{z \in \mathrm{B}(m)} \mathbb{E}\mathrm{T}(z)^2, \end{aligned} \tag{3.5}$$

where we used the following inequality in the 4-th line,

$$\Big(\sum_{j \in \Lambda} a_j + b_j + c_j + d_j \Big)^2 \leq 4|\Lambda| \sum_{j \in \Lambda} (a_j^2 + b_j^2 + c_j^2 + d_j^2),$$

and we used the translation invariant in the last line.

Since $\mathbb{E}\mathrm{T}(z)^2 = \mathrm{Var}(\mathrm{T}(z)) + (\mathbb{E}\mathrm{T}(z))^2$, by using (3.5), (2.1) and Corollary 2.3,

$$||\hat{\mathrm{T}}(0,x) - \hat{\mathrm{F}}_m||_2^2 = \mathcal{O}(m^2) = \mathcal{O}(|x|_1^{1/2}). \tag{3.6}$$

Combining (3.2)–(3.6), we get the desired result. □

3.2 Martingale decomposition of F_m and the proof of Theorem 1.1

Enumerate the vertices of $\mathbb{Z}^d$ as $x_1, x_2, \ldots$. We consider the martingale decomposition of F_m as follows

$$\mathrm{F}_m - \mathbb{E}(\mathrm{F}_m) = \sum_{k=1}^{\infty} \Delta_k, \tag{3.7}$$

where

$$\Delta_k = \mathbb{E}(\mathrm{F}_m \mid \mathcal{F}_k) - \mathbb{E}(\mathrm{F}_m \mid \mathcal{F}_{k-1}),$$

with $\mathcal{F}_k$ the sigma-algebra generated by SRWs $\{(\mathrm{S}_j^{x_i})_{j \in \mathbb{N}}, i = 1, \ldots, k\}$ and $\mathcal{F}_0$ the trivial sigma-algebra. In [11], using the Falik-Samorodnitsky lemma, the authors give an upper bound for the variance of F_m in term of $\mathrm{Ent}(\Delta_k^2)$, and $\mathbb{E}(|\Delta_k|)$.

Lemma 3.2. [11, Lemma 3.3] We have

$$\sum_{k\geq 1} \mathrm{Ent}(\Delta_k^2) \geq \mathrm{Var}(\mathrm{F}_m) \log \left[\frac{\mathrm{Var}(\mathrm{F}_m)}{\sum_{k\geq 1}(\mathbb{E}(|\Delta_k|))^2} \right].$$

Now, our main task is to estimate $\mathrm{Ent}(\Delta_k^2)$ and $\mathbb{E}(|\Delta_k|)$.

Proposition 3.3. As $|x|_1$ tends to infinity,

(i)

$$\sum_{k\geq 1} \mathrm{Ent}(\Delta_k^2) = \mathcal{O}(|x|_1).$$

(ii)

$$\sum_{k\geq 1} (\mathbb{E}(|\Delta_k|))^2 = \mathcal{O}\left(|x|_1^{\frac{5-d}{4}} \right).$$

Proof of Theorem 1.1 assuming Proposition 3.3. Since $d \geq 2$, Proposition 3.3 (ii) implies that $\sum_{k\geq 1}(\mathbb{E}(|\Delta_k|))^2 = \mathcal{O}\left(|x|_1^{3/4}\right)$. Therefore, using Propositions 3.1, 3.3 and Lemma 3.2, for any $\varepsilon > 0$, there exists a positive constant C such that

$$\begin{aligned} \mathrm{Var}(\mathrm{T}(x)) &\leq \mathrm{Var}(\mathrm{F}_m) + C|x|_1^{3/4+\varepsilon} \\ &\leq C\left(|x|_1^{3/4+\varepsilon} + |x|_1 \left[\log\left[\frac{\mathrm{Var}(\mathrm{F}_m)}{|x|_1^{3/4}} \right] \right]^{-1} \right). \end{aligned} \tag{3.8}$$

If $\mathrm{Var}(\mathrm{F}_m) \leq |x|_1^{7/8}$, then $\mathrm{Var}(\mathrm{T}(x)) = \mathcal{O}(|x|_1^{7/8})$ and Theorem 1.1 follows. Otherwise, if $\mathrm{Var}(\mathrm{F}_m) \geq |x|_1^{7/8}$, using (3.8) we get that $\mathrm{Var}(\mathrm{T}(x)) = \mathcal{O}(|x|_1/\log|x|_1)$ and Theorem 1.1 also follows. □

3.3 Proof of Proposition 3.3

By the definition of Δ_k, we have

$$\begin{aligned} |\Delta_k| &= \frac{1}{\#\mathrm{B}(m)} \left| \mathbb{E}\left[\sum_{z\in \mathrm{B}(m)} \mathrm{T}(z, z+x) \mid \mathcal{F}_k \right] - \mathbb{E}\left[\sum_{z\in \mathrm{B}(m)} \mathrm{T}(z, z+x) \mid \mathcal{F}_{k-1} \right] \right| \\ &\leq \frac{1}{\#\mathrm{B}(m)} \sum_{z\in \mathrm{B}(m)} \Big| \mathbb{E}\left[\mathrm{T}(z,z+x) \mid \mathcal{F}_k \right] - \mathbb{E}\left[\mathrm{T}(z,z+x) \mid \mathcal{F}_{k-1} \right] \Big|. \end{aligned} \tag{3.9}$$

We precise the dependence of first passage times on trajectories of SRWs by writing

$$\mathrm{T}(u,v) = \mathrm{T}(u,v,(\mathrm{S}_\cdot^{x_i})_{i\in\mathbb{N}}).$$

For any k, let us define

$$X_k(u,v) = \mathbb{E}(\mathrm{T}(u,v) \mid \mathcal{F}_k).$$

Then $X_k(u,v)$ is a function of trajectories of $(\mathrm{S}_\cdot^{x_i})_{i\leq k}$, so we write

$$X_k(u,v) = X_k(u,v)[(\mathrm{S}_\cdot^{x_i})_{i<k}, (\mathrm{S}_\cdot^{x_k})].$$

Let $(\tilde{S}_\cdot^x)_{x\in\mathbb{Z}^d}$ be an independent copy of $(\mathrm{S}_\cdot^x)_{x\in\mathbb{Z}^d}$. We observe that

$$\mathbb{E}(|X_k(u,v) - \mathbb{E}^k(X_k(u,v))|) \leq \mathbb{E}^{<k}\mathbb{E}^k\tilde{\mathbb{E}}^k(|X_k(u,v) - \tilde{X}_k(u,v)|), \tag{3.10}$$

where

$$\tilde{X}_k(u,v) = X_k(u,v)[(\mathrm{S}_\cdot^{x_i})_{i<k}, (\tilde{S}_\cdot^{x_k})],$$

and $\mathbb{E}^{<k}, \mathbb{E}^k$, and $\tilde{\mathbb{E}}^k$ denote the expectations with respect to SRWs $(\mathrm{S}^{x_i}_\cdot)_{i<k}$, $(\mathrm{S}^{x_k}_\cdot)$ and $(\tilde{S}^{x_k}_\cdot)$ respectively. Then the inequality (3.10) becomes

$$\mathbb{E}\Big|\mathbb{E}\left[\mathrm{T}(z,z+x)\mid\mathcal{F}_k\right]-\mathbb{E}\left[\mathrm{T}(z,z+x)\mid\mathcal{F}_{k-1}\right]\Big|\leq\mathbb{E}\tilde{\mathbb{E}}^k\Big|\mathrm{T}(z,z+x)-\tilde{\mathrm{T}}_{x_k}(z,z+x)\Big|,\quad(3.11)$$

where for $u,v\in\mathbb{Z}^d$ and $k\geq 1$

$$\tilde{\mathrm{T}}_{x_k}(u,v)=\mathrm{T}(u,v)[(\mathrm{S}^{x_i}_\cdot)_{i<k},(\tilde{S}^{x_k}_\cdot),(\mathrm{S}^{x_i}_\cdot)_{i>k}].$$

By the symmetry $\mathrm{T}(z,z+x)-\tilde{\mathrm{T}}_{x_k}(z,z+x)$,

$$\begin{aligned}&\mathbb{E}\tilde{\mathbb{E}}^k\Big|\mathrm{T}(z,z+x)-\tilde{\mathrm{T}}_{x_k}(z,z+x)\Big|\\ =\;&2\mathbb{E}\tilde{\mathbb{E}}^k\left((\tilde{\mathrm{T}}_{x_k}(z,z+x)-\mathrm{T}(z,z+x))\mathbb{I}(\tilde{\mathrm{T}}_{x_k}(z,z+x)\geq\mathrm{T}(z,z+x))\right).\end{aligned}\quad(3.12)$$

For any $u,v\in\mathbb{Z}^d$, we choose an optimal path for $\mathrm{T}(u,v)$ with a deterministic rule breaking ties and denote it by $\gamma_{u,v}$. We observe that if $x_k\notin\gamma_{u,v}$ then $\tilde{\mathrm{T}}_{x_k}(u,v)\leq\mathrm{T}(u,v)$. Otherwise, if $x_k\in\gamma_{u,v}$, then

$$\mathrm{T}(u,v)=\mathrm{T}(u,x_k)+\mathrm{T}(x_k,\bar{x}_k)+\mathrm{T}(\bar{x}_k,v),\quad(3.13)$$

where $\bar{x}_k$ is the next point of x_k in $\gamma_{u,v}$ (recall also that we denote by $y\sim\bar{y}\in\gamma$ if $\bar{y}$ is the next point of y in γ). Due to the subadditivity,

$$\tilde{\mathrm{T}}_{x_k}(u,v)\leq\tilde{\mathrm{T}}_{x_k}(u,x_k)+\tilde{\mathrm{T}}_{x_k}(x_k,\bar{x}_k)+\tilde{\mathrm{T}}_{x_k}(\bar{x}_k,v).\quad(3.14)$$

It is clear that any optimal path for $\mathrm{T}(u,x_k)$ does not use the simple random walk $(\mathrm{S}^{x_k}_\cdot)$. Hence,

$$\tilde{\mathrm{T}}_{x_k}(u,x_k)\leq\mathrm{T}(u,x_k).\quad(3.15)$$

In addition, since $\bar{x}_k$ is the next point of x_k in $\gamma_{u,v}$, the optimal path for $\mathrm{T}(\bar{x}_k,v)$ does not use the simple random walk $(\mathrm{S}^{x_k}_\cdot)$. Thus

$$\tilde{\mathrm{T}}_{x_k}(\bar{x}_k,v)\leq\mathrm{T}(\bar{x}_k,v).\quad(3.16)$$

It follows from (3.13)–(3.16) that

$$\tilde{\mathrm{T}}_{x_k}(u,v)-\mathrm{T}(u,v)\leq\tilde{\mathrm{T}}_{x_k}(x_k,\bar{x}_k).$$

Therefore, we have

$$\begin{aligned}&(\tilde{\mathrm{T}}_{x_k}(z,z+x)-\mathrm{T}(z,z+x))\mathbb{I}(\tilde{\mathrm{T}}_{x_k}(z,z+x)\geq\mathrm{T}(z,z+x))\\ \leq\;&\tilde{\mathrm{T}}_{x_k}(x_k,\bar{x}_k)\mathbb{I}(x_k\in\gamma_{z,z+x}).\end{aligned}\quad(3.17)$$

We notice here that the complete notation of $\bar{x}_k$ should be $\bar{x}_k(\gamma_{z,z+x})$ to highlight the dependence of $\bar{x}_k$ on the path $\gamma_{z,z+x}$. However, for the simplicity of notation, we shortly write it by $\bar{x}_k$ when the fact $x_k\in\gamma_{z,z+x}$ is precise. Combining (3.9), (3.11), (3.12) and (3.17), we get

$$\begin{aligned}\mathbb{E}(|\Delta_k|)&\leq\frac{2}{\#\mathrm{B}(m)}\mathbb{E}^{\otimes 2}\left(\sum_{z\in\mathrm{B}(m)}\tilde{\mathrm{T}}_{x_k}(x_k,\bar{x}_k)\mathbb{I}(x_k\in\gamma_{z,z+x})\right)\\ &=\frac{2}{\#\mathrm{B}(m)}\mathbb{E}^{\otimes 2}\left(\sum_{z\in\mathrm{B}(m)}\tilde{\mathrm{T}}_{x_k-z}(x_k-z,\overline{x_k-z})\mathbb{I}(x_k-z\in\gamma_{0,x})\right)\\ &=\frac{2}{\#\mathrm{B}(m)}\mathbb{E}^{\otimes 2}\left(\sum_{y\in x_k-\mathrm{B}(m)}\tilde{\mathrm{T}}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\right)\\ &=\frac{2}{\#\mathrm{B}(m)}\sum_{L\geq 0}\mathbb{E}^{\otimes 2}\left(\sum_{y\in x_k-\mathrm{B}(m)}\tilde{\mathrm{T}}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L})\right),\end{aligned}\quad(3.18)$$

where $\mathbb{E}^{\otimes 2}$ is the expectation with respect to two independent collections of SRWs $(\mathrm{S}^{x_i}_\cdot)_{i\in\mathbb{N}}$ and $(\tilde{S}^{x_i}_\cdot)_{i\in\mathbb{N}}$, and we define

$$\mathcal{E}_{k,L}=\left\{\sum_{y\in\gamma_{0,x}\cap(x_k-\mathrm{B}(m))}|y-\bar{y}|_1=L\right\}.$$

Notice that for the second equation, we have used the invariant translation. Let us define

$$\mathrm{T}^{[z]}(u,v)=\inf\Big\{\sum_{l=1}^{k}t(y_{l-1},y_l):u=y_0,\dots,y_k=v,y_l\neq z\,\forall\, l\geq 1,\text{ for some } k\Big\},$$

as the first passage time from u to v not using the frog at z, and set

$$\mathrm{T}_1(u,v)=\max_{z:\,|z-u|_1=1}\mathrm{T}^{[u]}(z,v)+1. \tag{3.19}$$

Then, it holds that

$$\tilde{\mathrm{T}}_u(u,v)\leq\mathrm{T}_1(u,v). \tag{3.20}$$

Using (3.20), we obtain

$$\begin{aligned}\sum_{y\in(x_k-\mathrm{B}(m))}\tilde{\mathrm{T}}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L}) &\leq \max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset(x_k-\mathrm{B}(m+L))\\|\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}\tilde{\mathrm{T}}_{y_i}(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L})\\ &\leq \max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset(x_k-\mathrm{B}(m+L))\\|\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L}).\end{aligned}$$

Therefore, with $C_1\geq 1$ as in Lemma 2.1,

$$\begin{aligned}&\sum_{L=0}^{4dC_1m}\mathbb{E}^{\otimes 2}\left(\sum_{y\in(x_k-\mathrm{B}(m+L))}\tilde{\mathrm{T}}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\mathbb{I}(\mathcal{E}_{k,L})\right)\\ \leq\;& \mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset(x_k-\mathrm{B}(8dC_1m))\\|\gamma|_1\leq 4dC_1m}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)\\ =\;& \mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset\mathrm{B}(8dC_1m)\\|\gamma|_1\leq 4dC_1m}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)\\ \leq\;& \mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{8dC_1m}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right),\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}&\sum_{L\geq 4dC_1m+1}\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset(x_k-\mathrm{B}(m+L))\\|\gamma|_1\leq L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L})\\ \leq\;& \sum_{L\geq 4dC_1m+1}\mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset(x_k-\mathrm{B}(2L))\\|\gamma|_1\leq 2L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\mathbb{I}(\mathcal{E}_{k,L})\right)\\ \leq\;& \sum_{L\geq 4dC_1m+1}\left[\mathbb{E}\left(\max_{\substack{\gamma=(y_i)_{i=1}^{\ell}\subset(x_k-\mathrm{B}(2L))\\|\gamma|_1\leq 2L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{k,L})^{1/2}\end{aligned}$$

$$\leq \sum_{L\geq 4dC_1m+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{2L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{k,L})^{1/2}, \tag{3.22}$$

where we have used the Cauchy-Schwarz inequality in the second inequality.

These yield that

$$\begin{aligned}\mathbb{E}|\Delta_k| \leq& \frac{2}{\#\mathrm{B}(m)}\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{8dC_1m}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)\\ &+\frac{2}{\#\mathrm{B}(m)}\sum_{L\geq 4dC_1m+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{2L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{k,L})^{1/2}.\end{aligned} \tag{3.23}$$

Using similar arguments for (3.18), (3.21) and (3.22), we can show that

$$\sum_{k=1}^{\infty}\mathbb{E}(|\Delta_k|) \tag{3.24}$$

$$\begin{aligned}\leq&\ \frac{2}{\#\mathrm{B}(m)}\sum_{k=1}^{\infty}\sum_{z\in\mathrm{B}(m)}\mathbb{E}^{\otimes 2}\tilde{\mathrm{T}}_{x_k}(x_k,\bar{x}_k)\mathbb{I}(x_k\in\gamma_{z,z+x})\\ =&\ 2\mathbb{E}^{\otimes 2}\left(\sum_{y\in\mathbb{Z}^d}\tilde{\mathrm{T}}_y(y,\bar{y})\mathbb{I}(y\in\gamma_{0,x})\right)\\ \leq&\ 2\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{C_1|x|_1}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)\\ &+2\sum_{L\geq C_1|x|_1+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_{L})^{1/2},\end{aligned} \tag{3.25}$$

where we define

$$\mathcal{E}_L=\{|\gamma_{0,x}|_1=L\}.$$

Lemma 3.4. There exists a positive constant C such that for all $L\geq 1$,

(i)

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)\leq CL.$$

(ii)

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{L}}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)^2\leq CL^4.$$

We postpone the proof of this lemma to Section 3.4.

Lemma 3.5. Given a path $\gamma=(y_i)_{i=1}^{\ell}\subset\mathbb{Z}^d$, we define the maximal jump

$$\mathcal{M}(\gamma)=\max_{1\leq i\leq \ell-1}|y_i-y_{i+1}|_1.$$

Then, there exists $\varepsilon>0$ independent of x such that for any $L\geq m=|x|_1^{1/4}$,

$$\mathbb{P}(\mathcal{M}(\gamma_{0,x})\geq L)\leq e^{-L^{\varepsilon}}.$$

Proof. We write $\gamma_{0,x} = (y_i)_{i=1}^{\ell}$. If $|y_i - y_{i+1}|_1 \geq L$, then $\mathrm{T}(y_i, y_{i+1}) = t(y_i, y_{i+1}) \geq L$. By the union bound, Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned}
&\mathbb{P}(\mathcal{M}(\gamma_{0,x}) \geq L) \\
&\leq \mathbb{P}(\exists u, v \in \mathrm{B}(C_1|x|_1 + L) \text{ s.t. } \mathrm{T}(u,v) = t(u,v) \geq L) + \mathbb{P}(|\gamma_{0,x}|_1 \geq C_1|x_1| + L) \\
&\leq [\#\mathrm{B}(C_1|x|_1 + L)]^2 \max_{u,v \in \mathrm{B}(C_1|x|_1+L)} \mathbb{P}(\mathrm{T}(u,v) = t(u,v) \geq L) + \mathbb{P}(\mathrm{T}(0,x) \geq C_1|x_1| + L) \\
&\leq e^{-L^{\varepsilon}},
\end{aligned} \tag{3.26}$$

for some constant $\varepsilon > 0$. □

3.3.1 Proof of Proposition 3.3 (ii)

Fix $k \geq 1$. We first estimate $\mathbb{P}(\mathcal{E}_{k,L})$. Assume that $\mathcal{E}_{k,L}$ occurs and $\gamma_{0,x} \cap (x_k - \mathrm{B}(m)) = (y_i)_{i=1}^{\ell}$. Then

$$L = \sum_{y \in \gamma_{0,x} \cap x_k - \mathrm{B}(m)} |y - \bar{y}|_1 \leq \sum_{i=1}^{\ell-1} t(y_i, y_{i+1}) + t(y_\ell, \bar{y}_\ell) = \mathrm{T}(y_1, \bar{y}_\ell).$$

Moreover, $\bar{y}_\ell \in x_k - \mathrm{B}(m + \mathcal{M}(\gamma_{0,x}))$, since $|y_\ell - \bar{y}_\ell|_1 \leq \mathcal{M}(\gamma_{0,x})$ and $y_\ell \in x_k - \mathrm{B}(m)$. Therefore, using the union bound, Lemma 2.1 and Lemma 3.5, for $L \geq 4dC_1 m + 1$,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_{k,L}) &\leq \mathbb{P}\left(\exists u, v \in x_k - \mathrm{B}(m + \mathcal{M}(\gamma_{0,x})) \text{ such that } \mathrm{T}(u,v) \geq L\right) \\
&\leq \mathbb{P}\left(\exists u, v \in \mathrm{B}(m + (L/4dC_1)) \text{ such that } \mathrm{T}(u,v) \geq L\right) + \mathbb{P}(M(\gamma_{0,x}) \geq L/4dC_1) \\
&\leq (2(m+L))^{2d} e^{-L^{\varepsilon}} + e^{-L^{\varepsilon}} \leq (4(m+L))^{2d} e^{-L^{\varepsilon}}.
\end{aligned}$$

Combining this inequality with (3.23) and Lemma 3.4, we obtain that there exists $C > 0$ such that for any $k \geq 1$

$$\begin{aligned}
\mathbb{E}(|\Delta_k|) &\leq \frac{C}{\#\mathrm{B}(m)} \left(m + \sum_{L \geq 4dC_1 m} L^2 (4(m+L))^d e^{-L^{\varepsilon}/2} \right) \\
&= \mathcal{O}(m^{1-d}) = \mathcal{O}(|x|_1^{(1-d)/4}).
\end{aligned} \tag{3.27}$$

Since $\mathrm{T}(x) \geq |\gamma_{0,x}|_1$, by using Lemma 2.1, for any $L \geq C_1|x|_1$

$$\mathbb{P}(\mathcal{E}_L) \leq \mathbb{P}(\mathrm{T}(x) \geq L) \leq e^{-L^{\varepsilon_1}}. \tag{3.28}$$

Using this inequality, (3.24) and Lemma 3.4, we get

$$\begin{aligned}
\sum_{k \geq 1} \mathbb{E}(|\Delta_k|) &\leq C \left(|x|_1 + \sum_{L \geq C_1|x|_1} L^2 e^{-L^{\varepsilon}/2} \right) \\
&= \mathcal{O}(|x|_1).
\end{aligned} \tag{3.29}$$

Now, Proposition 3.3 (ii) follows from (3.27) and (3.29), since

$$\sum_{k \geq 1} (\mathbb{E}(|\Delta_k|))^2 \leq \left(\max_{k \geq 1} \mathbb{E}|\Delta_k| \right) \left(\sum_{k \geq 1} \mathbb{E}|\Delta_k| \right).$$

3.3.2 Proof of Proposition 3.3 (i)

To estimate $\mathrm{Ent}(\Delta_k)$, we decompose a simple random walk $(\mathrm{S}_\cdot^{x_i})$ into the sum of i.i.d. random variables. More precisely, for any $x_i \in \mathbb{Z}^d$ and $j \geq 1$, we write

$$\mathrm{S}_j^{x_i} = x_i + \sum_{r=1}^{j} \omega_{i,r},$$

where $(\omega_{i,r})_{i,r\geq 1}$ is an array of i.i.d. uniform random variables taking value in the set of canonical coordinates in $\mathbb{Z}^d$, denoted by

$$\mathcal{B}_d = \{e_1, \dots, e_{2d}\}.$$

Therefore, we can view $\mathrm{T}(u,v)$ and F_m as a function of $(\omega_{i,r})$, and hence we sometimes write $\mathrm{T}(u,v) = \mathrm{T}(u,v,\omega)$ to make the dependence of $\mathrm{T}(u,v)$ on ω precise. We define

$$\Omega = \prod_{i,j\in\mathbb{N}} \Omega_{i,j},$$

where $\Omega_{i,j}$ is a copy of $\mathcal{B}_d$. The measure on Ω is $\pi = \prod_{i,j\in\mathbb{N}} \pi_{i,j}$, where $\pi_{i,j}$ is the uniform measure on $\Omega_{i,j}$. Then we can consider F_m as a random variable on the probability space (Ω, π). Given $\omega \in \Omega, e \in \mathcal{B}_d$ and $i, j \in \mathbb{N}$, we define a new configuration $\omega^{i,j,e}$ as

$$\omega_{k,r}^{i,j,e} = \begin{cases} \omega_{k,r} & \text{if } (k,r) \neq (i,j) \\ e & \text{if } (k,r) = (i,j). \end{cases}$$

We define

$$\Delta_{i,j} f = \Big[\mathrm{E}\Big(|f(\omega^{i,j,U}) - f(\omega^{i,j,\tilde{U}})|^2\Big)\Big]^{1/2}, \tag{3.30}$$

where the expectation runs over two independent random variables U and $\tilde{U}$, with the same law as the uniform distribution on $\mathcal{B}_d$.

Lemma 3.6. We have

$$\sum_{k=1}^{\infty} \mathrm{Ent}(\Delta_k^2) \leq 2d \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\mathrm{F}_m)^2].$$

Proof. We recall that $\Delta_k = \mathbb{E}(\mathrm{F}_m \mid \mathcal{F}_k) - \mathbb{E}(\mathrm{F}_m \mid \mathcal{F}_{k-1})$, where

$$\mathcal{F}_k = \sigma((\mathrm{S}_j^{x_i}), i \leq k, j \geq 1) = \sigma(\omega_{i,j}, i \leq k, j \geq 1).$$

Notice that $\Delta_k^2 \in \mathrm{L}^2$, since $\mathrm{T}(x) \in \mathrm{L}^4$ by Lemma 2.1. Hence, using the tensorization of entropy (Lemma 2.7), we have for $k \geq 1$,

$$\mathrm{Ent}(\Delta_k^2) = \mathrm{Ent}_\pi(\Delta_k^2) \leq \mathbb{E}_\pi \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{Ent}_{\pi_{i,j}} \Delta_k^2.$$

By Lemma 2.8,

$$\mathrm{Ent}_{\pi_{i,j}} \Delta_k^2 \leq 2d(\Delta_{i,j}\Delta_k)^2.$$

Thus

$$\sum_{k=1}^{\infty} \mathrm{Ent}(\Delta_k^2) \leq 2d \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\Delta_k)^2]. \tag{3.31}$$

We fix i, j. We define the filtration $\tilde{\mathcal{F}}_k$ as $\tilde{\mathcal{F}}_k = \mathcal{F}_k$ if $k < i$, and $\tilde{\mathcal{F}}_k = \mathcal{F}_k \vee \sigma(U, \tilde{U})$ if $k \geq i$. For simplicity of notation, we denote $\mathbb{E} = \mathbb{E}_\pi \mathrm{E}$. Since

$$\begin{aligned}
&\mathbb{E}_\pi[(\Delta_{i,j}\Delta_k)^2] \\
&= \mathbb{E}[(\mathbb{E}[\mathrm{F}_m(\omega^{i,j,U}) - \mathrm{F}_m(\omega^{i,j,\tilde{U}}) |\ \tilde{\mathcal{F}}_k] - \mathbb{E}[\mathrm{F}_m(\omega^{i,j,U}) - \mathrm{F}_m(\omega^{i,j,\tilde{U}}) |\ \tilde{\mathcal{F}}_{k-1}])^2] \\
&= \mathbb{E}[(\mathbb{E}[\mathrm{F}_m(\omega^{i,j,U}) - \mathrm{F}_m(\omega^{i,j,\tilde{U}}) |\ \tilde{\mathcal{F}}_k])^2] - \mathbb{E}[(\mathbb{E}[\mathrm{F}_m(\omega^{i,j,U}) - \mathrm{F}_m(\omega^{i,j,\tilde{U}}) |\ \tilde{\mathcal{F}}_{k-1}])^2],
\end{aligned}$$

$$\mathbb{E}[(\mathbb{E}[\mathrm{F}_m(\omega^{i,j,U}) - \mathrm{F}_m(\omega^{i,j,\tilde{U}}) |\ \tilde{\mathcal{F}}_0])^2] = 0,$$

and

$$\lim_{k\to\infty} \mathbb{E}[(\mathbb{E}[\mathrm{F}_m(\omega^{i,j,U}) - \mathrm{F}_m(\omega^{i,j,\tilde{U}}) |\ \tilde{\mathcal{F}}_k])^2] = \mathbb{E}_\pi[(\Delta_{i,j}\mathrm{F}_m)^2],$$

we get

$$\sum_{k=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\Delta_k)^2] = \mathbb{E}_\pi[(\Delta_{i,j}\mathrm{F}_m)^2],$$

for any i, j. Combining this equation with (3.31), we get the desired result. □

Proof of Proposition 3.3 (i). Using Lemma 3.6 and Jensen's inequality, we get

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathrm{Ent}(\Delta_k^2) &\leq 2d \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\mathrm{F}_m)^2] \\
&\leq \frac{2d}{\#\mathrm{B}(m)} \sum_{z\in\mathrm{B}(m)} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\mathrm{T}(z, z+x))^2]. \qquad (3.32)
\end{aligned}$$

By the translation invariance of the passage times, we reach

$$\sum_{k=1}^{\infty} \mathrm{Ent}(\Delta_k^2) \leq 2d \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi[(\Delta_{i,j}\mathrm{T}(x))^2]. \qquad (3.33)$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}_\pi[(\Delta_{i,j}\mathrm{T}(x))^2] = \mathbb{E}_\pi[(\mathrm{E}|\mathrm{T}(x, \omega^{i,j,U}) - \mathrm{T}(x, \omega^{i,j,\tilde{U}})|)^2] \\
\leq\ & \mathbb{E}_\pi \mathrm{E}[|\mathrm{T}(x, \omega^{i,j,U}) - \mathrm{T}(x, \omega^{i,j,\tilde{U}})|^2] \\
=\ & \mathbb{E}_\pi \mathrm{E}[|\mathrm{T}(x, \omega^{i,j,U}) - \mathrm{T}(x)|^2] \\
=\ & 2\mathbb{E}_\pi \mathrm{E}[(\mathrm{T}(x, \omega^{i,j,U}) - \mathrm{T}(x))^2 \mathbb{I}(\mathrm{T}(x, \omega^{i,j,U}) \geq \mathrm{T}(x))].
\end{aligned}$$

We observe that if $x_i \notin \gamma_{0,x}$, or $x_i \in \gamma_{0,x}$ but $\mathrm{T}(x_i, \bar{x}_i) < j$, then

$$\mathrm{T}(x, \omega^{i,j,U}) \leq \mathrm{T}(x).$$

Otherwise, assume that $x_i \in \gamma_{0,x}$ and $\mathrm{T}(x_i, \bar{x}_i) \geq j$. Then for any $e \in \mathcal{B}_d$,

$$\mathrm{T}(x_i, \bar{x}_i) \geq \mathrm{T}(x_i, \bar{x}_i + e - \omega_{i,j}, \omega^{i,j,e}),$$

since if we only replace $\omega_{i,j}$ by e, by $t(x_i, \bar{x}_i)$ (also equals $\mathrm{T}(x_i, \bar{x}_i)$, as $x_i \sim \bar{x}_i \in \gamma_{0,x}$) steps, the simple random walk $(\mathrm{S}^{x_i}_{\cdot})$ arrives at $\bar{x}_i + e - \omega_{i,j}$. Moreover,

$$\mathrm{T}(x_i, \omega^{i,j,e}) = \mathrm{T}(x_i), \quad \mathrm{T}(\bar{x}_i, x, \omega^{i,j,e}) \leq \mathrm{T}(\bar{x}_i, x),$$

and

$$\begin{aligned}
\mathrm{T}(x) &= \mathrm{T}(x_i) + \mathrm{T}(x_i, \bar{x}_i) + \mathrm{T}(\bar{x}_i, x), \\
\mathrm{T}(x, \omega^{i,j,e}) &\leq \mathrm{T}(x_i, \omega^{i,j,e}) + \mathrm{T}(x_i, \bar{x}_i - e + \omega_{i,j}, \omega^{i,j,e}) \\
&\quad + \mathrm{T}(\bar{x}_i - e + \omega_{i,j}, \bar{x}_i, \omega^{i,j,e}) + \mathrm{T}(\bar{x}_i, x, \omega^{i,j,e}).
\end{aligned}$$

Therefore, we reach

$$\mathrm{T}(x,\omega^{i,j,U})-\mathrm{T}(x)\le \mathrm{T}(\bar{x}_i-U+\omega_{i,j},\bar{x}_i,\omega^{i,j,U})\le \max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}(y,\bar{x}_i,\omega^{i,j,U}).$$

Furthermore, since ω differs from $\omega^{i,j,U}$ only in the trajectory of $(\mathrm{S}^{x_i}_{\cdot})$, for any $u,v\in\mathbb{Z}^d$,

$$\mathrm{T}(u,v,\omega^{i,j,U})\le \mathrm{T}^{[x_i]}(u,v)\le \mathrm{T}_2(u,v), \tag{3.34}$$

where we define

$$\mathrm{T}_2(u,v)=\sup_{z\in\mathbb{Z}^d}\mathrm{T}^{[z]}(u,v). \tag{3.35}$$

Therefore, we have

$$\mathbb{E}_\pi[(\Delta_{i,j}\mathrm{T}(x))^2]\le 2\mathbb{E}\Big[\max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}_2(y,\bar{x}_i)^2\mathbb{I}(x_i\sim\bar{x}_i\in\gamma_{0,x},\mathrm{T}(x_i,\bar{x}_i)\ge j)\Big],$$

and thus

$$\begin{aligned}
&\sum_{j=1}^{\infty}\mathbb{E}_\pi(\Delta_{i,j}\mathrm{T}(x))^2\\
\le\ & 2\mathbb{E}\Big[\sum_{j=1}^{\infty}\max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}_2(y,\bar{x}_i)^2\mathbb{I}(x_i\sim\bar{x}_i\in\gamma_{0,x},\mathrm{T}(x_i,\bar{x}_i)\ge j)\Big]\\
=\ & 2\mathbb{E}\Big[\mathrm{T}(x_i,\bar{x}_i)\max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}_2(y,\bar{x}_i)^2\mathbb{I}(x_i\sim\bar{x}_i\in\gamma_{0,x})\Big]\\
\le\ & \mathbb{E}\Big[(\mathrm{T}(x_i,\bar{x}_i)^2+\max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}_2(y,\bar{x}_i)^4)\mathbb{I}(x_i\sim\bar{x}_i\in\gamma_{0,x})\Big].
\end{aligned}$$

This yields that

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}_\pi(\Delta_{i,j}\mathrm{T}(x))^2\\
\le\ & \mathbb{E}\Big[\sum_{i=1}^{\infty}(\mathrm{T}(x_i,\bar{x}_i)^2+\max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}_2(y,\bar{x}_i)^4)\mathbb{I}(x_i\sim\bar{x}_i\in\gamma_{0,x})\Big]\\
=\ & \mathbb{E}\Big[\sum_{i=1}^{\infty}(\mathrm{T}(x_i,\bar{x}_i)^2+\max_{y:|y-\bar{x}_i|_1\le 2}\mathrm{T}_2(y,\bar{x}_i)^4)\mathbb{I}(x_i\sim\bar{x}_i\in\gamma_{0,x})\Big]\\
\le\ & \mathbb{E}\left(\sum_{y\in\gamma_{0,x}}\mathrm{T}(y,\bar{y})^2\right)+\mathbb{E}\left(\sum_{y\in\gamma_{0,x}}\max_{u:|u-y|_1\le 2}\mathrm{T}_2(u,y)^4\right).
\end{aligned} \tag{3.36}$$

Now using the same arguments for (3.22) and (3.24), we get

$$\begin{aligned}
&\mathbb{E}\left(\sum_{y\in\gamma_{0,x}}\max_{|u-y|_1\le 2}\mathrm{T}_2(u,y)^4\right) \qquad (3.37)\\
\le\ & \mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_{C_1|x|_1}}\sum_{i=1}^{\ell}\max_{|u-y_i|_1\le 2}\mathrm{T}_2(u,y_i)^4\right)\\
&+\sum_{L\ge C_1|x|_1+1}\left[\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell}\max_{|u-y_i|_1\le 2}\mathrm{T}_2(u,y_i)^4\right)^2\right]^{1/2}\mathbb{P}(\mathcal{E}_L)^{1/2}.
\end{aligned}$$

Lemma 3.7. As $|x|_1$ tends to infinity,

$$\mathbb{E}\left(\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2\right) = \mathcal{O}(|x|_1).$$

Lemma 3.8. There exists a positive constant C such that for any $L \geq 1$,

(i)
$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L} \sum_{i=1}^{\ell} \max_{u:|u-y_i|_1\leq 2} \mathrm{T}_2(u,y_i)^4\right) \leq CL.$$

(ii)
$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L} \sum_{i=1}^{\ell-1} \max_{u:|u-y_i|_1\leq 2} \mathrm{T}_2(u,y_i)^4\right)^2 \leq CL^{10}.$$

We postpone the proofs of the above two lemmas to Section 3.4 and complete the proof of Proposition 3.3. Combining (3.28), (3.36), (3.37) and Lemmas 3.7 and 3.8, we get

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}_\pi(\Delta_{i,j}\mathrm{T}(x))^2 &\leq C\left(|x|_1 + \sum_{L\geq C_1|x|_1} L^2 e^{-L^{\varepsilon_1}/2}\right)\\ &= \mathcal{O}(|x|_1).\end{aligned}$$

Thus, we can conclude the proof of Proposition 3.3 by (3.33). □

3.4 Proof of Lemmas 3.4, 3.7 and 3.8

Before presenting the proof of these lemmas, we first show the large deviation estimates as in Lemma 2.1 for T_1 and T_2.

Lemma 3.9. The following statements hold.

(i) For any $u, v \in \mathbb{Z}^d$ and $n \geq 1$, the events $\{\mathrm{T}(u,v) \leq n\}$, $\{\mathrm{T}_1(u,v) \leq n\}$ and $\{\mathrm{T}_2(u,v) \leq n\}$ depend only on SRWs $\{(S^x_\cdot) : |x-u|_1 \leq n\}$.

(ii) There exist an integer $C_1 \geq 1$ and a positive constant ε_1 such that for $k \geq C_1|y|_1$,

$$\max\{\mathbb{P}(\mathrm{T}(0,y)\geq k), \mathbb{P}(\mathrm{T}_1(0,y)\geq k), \mathbb{P}(\mathrm{T}_2(0,y)\geq k)\} \leq e^{-k^{\varepsilon_1}}.$$

Proof. For any $u \in \mathbb{Z}^d$ and $n \geq 1$, an event $\mathcal{A}$ is called $\mathcal{F}^u_n$-measurable if $\mathcal{A}$ depends only on the SRWs $\{(S^x_\cdot) : |x-u|_1 \leq n\}$. It directly follows from definition of T that the event $\{\mathrm{T}(u,v) \leq n\}$ is $\mathcal{F}^u_n$-measurable. By definition of T_1 as in (3.19),

$$\{\mathrm{T}_1(u,v) \leq n\} = \bigcap_{z:|z-u|\leq 1} \{\mathrm{T}(z,v) \leq n-1\}. \tag{3.38}$$

In addition the event $\{\mathrm{T}(z,v) \leq n-1\}$ is $\mathcal{F}^z_{n-1}$-measurable and $\mathcal{F}^{u_1}_{n-1} \subset \mathcal{F}^u_n$ if $|z-u|_1 \leq 1$, so the event $\{\mathrm{T}_1(u,v) \leq n\}$ is $\mathcal{F}^u_n$-measurable. Moreover, since $\{\mathrm{T}^{[z]}(u,v) \leq n\}$ is $\mathcal{F}^u_n$-measurable for any $z \in \mathbb{Z}^d$, the event $\{\mathrm{T}_2(u,v) \leq n\} = \cap_z\{\mathrm{T}^{[z]}(u,v) \leq n\}$ is $\mathcal{F}^u_n$-measurable. We now prove (ii).

By repeating the arguments of the proof of Lemma 2.1 (see [19, Proposition 2.4] or [1, Lemma 4.2]), we can show that there exist positive constants C and ε such that for any $y, z \in \mathbb{Z}^d$, and $t \geq C|y|_1$,

$$\mathbb{P}\left(\mathrm{T}^{[z]}(0,y) \geq t\right) \leq e^{-t^\varepsilon}. \tag{3.39}$$

By the union bound, for $t \geq C_2|y|_1$ with $C_2 = 2C$, we have

$$\begin{aligned} \mathbb{P}(\mathrm{T}_1(0,y) \geq t) &\leq \sum_{z\in\mathbb{Z}^d:|z|_1=1} \mathbb{P}(\mathrm{T}^{[0]}(z,y) \geq t-1) \\ &\leq 2de^{-(t-1)^{\varepsilon}} \leq e^{-t^{\varepsilon_2}}, \end{aligned} \tag{3.40}$$

with some $\varepsilon_2 > 0$, where we have used (3.39) for $t - 1 \geq 2C|y|_1 - 1 \geq C|z-y|_1$.

We observe also that if $\mathrm{T}(y) \leq k$ then $\mathrm{T}^{[z]}(0,y) = \mathrm{T}(y)$ for $z \notin \mathrm{B}(k)$. Therefore, for $k \geq C_3|y|_1$ with $C_3 = \max\{C_1, C_2\}$,

$$\begin{aligned} \mathbb{P}\left(\mathrm{T}_2(0,y) \geq k\right) &\leq \mathbb{P}(\mathrm{T}(y) \geq k) + \mathbb{P}(\mathrm{T}(y) < k, \mathrm{T}_2(0,y) \geq k) \\ &\leq \mathbb{P}(\mathrm{T}(y) \geq k) + \sum_{z\in\mathrm{B}(k)} \mathbb{P}\left(\mathrm{T}^{[z]}(0,y) \geq k\right) \\ &\leq e^{-k^{\varepsilon_1}} + (2k+1)^d e^{-k^{\varepsilon_2}} \leq e^{-k^{\varepsilon_3}}, \end{aligned} \tag{3.41}$$

with some $\varepsilon_3 > 0$. Combining (3.40) and (3.41) with Lemma 2.1, we get (ii). □

3.4.1 Proof of Lemma 3.7

We decompose

$$\begin{aligned} &\mathbb{E}\left(\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2\right) \\ = \quad &\mathbb{E}\left[\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2;\ \mathrm{T}(x) \leq C|x|_1\right] + \mathbb{E}\left[\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2;\ \mathrm{T}(x) > C|x|_1\right]. \end{aligned}$$

By a similar argument as in Lemma 2.2, the second term can be bounded from above by

$$\begin{aligned} &\left(\mathbb{E}\left[\left(\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2\right)^2\right]\right)^{1/2} \mathbb{P}(|\gamma_{0,x}|_1 > C_1|x|_1|)^{1/2} \\ &\leq \left(\mathbb{E}\left[\mathrm{T}(x)^4\right]\right)^{1/2} \mathbb{P}(\mathrm{T}(x) > C_1|x|_1|)^{1/2} \\ &\leq C|x|_1^2 e^{-|x|_1^{\varepsilon}/2}, \end{aligned} \tag{3.42}$$

and thus for all $|x|_1$ large enough,

$$\mathbb{E}\left(\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2\right) \leq \mathbb{E}\left[\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2;\ \mathrm{T}(x) \leq C|x|_1\right] + 1. \tag{3.43}$$

For any $\gamma = (y_i)_{i=1}^{\ell}$, we define

$$\mathrm{A}_M^{\gamma} \quad = \quad \{y_i \in \gamma : \mathrm{T}(y_i, y_{i+1}) = M\}.$$

Then, we can express

$$\sum_{i=1}^{\ell-1} \mathrm{T}(y_i, y_{i+1})^2 = \sum_{M\geq 1} M^2 \# \mathrm{A}_M^{\gamma}. \tag{3.44}$$

By definition of $\mathrm{A}_M^{\gamma_{0,x}}$,

$$\begin{aligned}
\#\mathrm{A}_M^{\gamma_{0,x}}\mathbb{I}(\mathrm{T}(x) \le C_1|x|_1) &\le \mathbb{I}(\gamma_{0,x} \in \mathcal{P}_{C_1|x|_1}) \sum_{y\in\gamma_{0,x}} \mathbb{I}(T(y,\bar{y}) = M) \\
&= \mathbb{I}(\gamma_{0,x} \in \mathcal{P}_{C_1|x|_1}) \sum_{y\in\gamma_{0,x}} \mathbb{I}(T(y,\bar{y}) = t(y,\bar{y}) = M) \\
&\le \mathbb{I}(\gamma_{0,x} \in \mathcal{P}_{C_1|x|_1}) \sum_{y\in\gamma_{0,x}} I_y, \qquad (3.45)
\end{aligned}$$

where

$$I_y = \{\exists\, z \in \mathbb{Z}^d : |z-y|_1 \le M, \mathrm{T}(y,z) = t(y,z) = M\}. \qquad (3.46)$$

By Lemma 3.9 (i), $\{I_y, y \in \mathbb{Z}^d\}$ is a collection of M-dependent Bernoulli random variables, and thus the condition (E1) in Lemma 2.6 holds. In addition, it follows from the union bound and Lemma 2.4 that

$$\begin{aligned}
q_M &= \sup_{y\in\mathbb{Z}^d} \mathbb{P}(\exists\, z \in \mathbb{Z}^d : |z-y|_1 \le M, T(y,z) = t(y,z) = M) \\
&\le (2M+1)^d e^{-M^\varepsilon},
\end{aligned} \qquad (3.47)$$

with $\varepsilon > 0$ as in Lemma 2.4. Therefore, the condition (E2) that $q_M \le (3M+1)^{-d}$ follows if $\exp(M^\varepsilon) \ge ((2M+1)(3M+1))^d$, which holds for all $M \ge M_0$, with $M_0 = M_0(d,\varepsilon)$ a large constant. Now using (3.45) and Lemma 2.6, we obtain that for $M \ge M_0$,

$$\begin{aligned}
\mathbb{E}(\#\mathrm{A}_M^{\gamma_{0,x}}\mathbb{I}(\mathrm{T}(x) \le C_1|x|_1)) \le \mathbb{E}\left(\max_{\gamma\in\mathcal{P}_{C_1|x|_1}} \sum_{y\in\gamma} I_y\right) &\le C|x|_1 M^{d+1} q_M^{1/d} \\
&\le C'|x|_1 M^{d+2} e^{-M^\varepsilon/d}. \qquad (3.48)
\end{aligned}$$

For $M \le M_0$, it is obvious that

$$\#\mathrm{A}_M^{\gamma_{0,x}}\mathbb{I}(\mathrm{T}(x) \le C_1|x|_1) \le \frac{C_1|x|_1}{M}. \qquad (3.49)$$

Combining the last two estimates with (3.44), we arrive at

$$\begin{aligned}
&\mathbb{E}\left[\sum_{y\in\gamma_{0,x}} \mathrm{T}(y,\bar{y})^2;\ \mathrm{T}(x) \le C|x|_1\right] \\
&= \mathbb{E}\left[\sum_{M\ge 1} M^2 \#\mathrm{A}_M^{\gamma_{0,x}};\ \mathrm{T}(x) \le C|x|_1\right] \\
&\le C|x|_1 \left[\sum_{M=1}^{M_0-1} M + \sum_{M\ge M_0} M^{d+4}\exp(-M^\varepsilon/d)\right] = \mathcal{O}(|x|_1).
\end{aligned}$$

Combining this estimate with (3.43), we get the desired result.

3.4.2 Proof of Lemma 3.4

We begin with part (ii), which is easier than (i). Observe that

$$\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L} \sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i,y_{i+1}) \le L \max_{u,v\in\mathrm{B}(L)} \mathrm{T}_1(u,v)$$

Using the union bound and Lemma 3.9 (ii), for any $k \geq 4dC_1L$,

$$\mathbb{P}\left(\max_{u,v\in \mathrm{B}(L)} \mathrm{T}_1(u,v) \geq k\right) \leq (2L+1)^{2d} e^{-k^{\varepsilon_1}}.$$

The last two inequalities yield that

$$\begin{aligned}
\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L} \sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i,y_{i+1})\right)^2 \quad &\leq \quad CL^4\left(1+(2L+1)^{2d}\sum_{k\geq 4dC_1L} k^2 e^{-k^{\varepsilon_1}}\right)\\
&= \quad \mathcal{O}(L^4).
\end{aligned}$$

We now prove (i). For any $\gamma = (y_i)_{i=1}^{\ell} \in \mathcal{P}_L$, we define

$$\begin{aligned}
\bar{A}_M^{\gamma} &= \{y_i \in \gamma : |y_i - y_{i+1}|_1 = M\},\\
\bar{A}_{M,0}^{\gamma} &= \{y_i \in \bar{A}_M^{\gamma} : \mathrm{T}_1(y_i,y_{i+1}) \leq C_1M\},\\
\bar{A}_{M,k}^{\gamma} &= \{y_i \in \bar{A}_M^{\gamma} : \mathrm{T}_1(y_i,y_{i+1}) = C_1M+k\},
\end{aligned}$$

with C_1 as in Lemma 3.9 (ii). Then

$$\#\bar{A}_M^{\gamma} = \sum_{k\geq 0} \#\bar{A}_{M,k}^{\gamma}, \quad \sum_{M\geq 1} M\#\bar{A}_M^{\gamma} = |\gamma|_1 \leq L. \tag{3.50}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{\ell-1} \mathrm{T}_1(y_i,y_{i+1}) \quad &\leq \quad \sum_{M\geq 1}\left(C_1M\#\bar{A}_{M,0}^{\gamma} + \sum_{k\geq 1}(C_1M+k)\#\bar{A}_{M,k}^{\gamma}\right)\\
&\leq \quad C_1L + \sum_{M\geq 1}\sum_{k\geq 1} k\#\bar{A}_{M,k}^{\gamma}.
\end{aligned} \tag{3.51}$$

We shall apply the same arguments as in the proof of Lemma 3.7 to deal with the sum above. Similarly to (3.45),

$$\#\bar{A}_{M,k}^{\gamma} \leq \sum_{y\in\gamma} \bar{I}_y, \tag{3.52}$$

where

$$\bar{I}_y = \mathbb{I}\left(\exists z \in \mathbb{Z}^d : |z-y|_1 \leq C_1M+k, \mathrm{T}_1(y,z) = C_1M+k\right). \tag{3.53}$$

By Lemma 3.9 (i), $\{\bar{I}_y, y \in \mathbb{Z}^d\}$ is a collection of (C_1M+k)-dependent Bernoulli random variables. Hence, using the same arguments for (3.48), we can prove that for $C_1M+k \geq M_0$, with $M_0 = M_0(d)$ some large constant,

$$\mathbb{E}\left(\max_{\gamma\in\mathcal{P}_L} \#\bar{A}_{M,k}^{\gamma}\right) \leq CL(C_1M+k)^{d+1} q_{M,k}^{1/d}, \tag{3.54}$$

where

$$\begin{aligned}
q_{M,k} \quad &= \quad \sup_{y\in\mathbb{Z}^d} \mathbb{P}\left(\exists z \in \mathbb{Z}^d : |y-z|_1 \leq C_1M+k, \mathrm{T}_1(u,v) = C_1M+k\right)\\
&\leq \quad (2(C_1M+k)+1)^d e^{-(C_1M+k)^{\varepsilon_1}},
\end{aligned}$$

by using the union bound and Lemma 3.9 (ii). It is obvious that $\#\bar{A}_{M,k}^{\gamma} \leq |\gamma|_1/(C_1M+k)$ for all M,k. Hence,

$$\sum_{M,k:C_1M+k\leq M_0} k\#\bar{A}_{M,k}^{\gamma} \leq M_0^2|\gamma|_1. \tag{3.55}$$

Combining (3.51) and (3.54), we have

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}\mathrm{T}_1(y_i,y_{i+1})\right)\leq CL\left(1+\sum_{M,k:C_1M+k\geq M_0}(C_1M+k)^{d+3}e^{-(C_1M+k)^{\varepsilon_1}/d}\right)$$
$$=\mathcal{O}(L),$$

for some $C=C(d,M_0)$, which proves (i).

3.4.3 Proof of Lemma 3.8

To show (ii), we notice that

$$\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell-1}\max_{u:|u-y_i|_1\leq 2}\mathrm{T}_2(u,y_i)^4\leq L\max_{u,v\in\mathrm{B}(L+2)}\mathrm{T}_2(u,v)^4. \tag{3.56}$$

Now part (ii) follows from (3.9) and (3.56) by using the same arguments as in Lemma 3.4 (ii).

The proof of (i) is similar to that of Lemma 3.7. As in Lemma 3.7, we define for $\gamma\in\mathcal{P}_L$, and $M\geq 1$,

$$A'^{\gamma}_M \quad = \quad \#\{y\in\gamma:\max_{u:|u-y|_1\leq 2}\mathrm{T}_2(u,y)^4=M\}=\sum_{y\in\gamma}I'_y,$$

where

$$I'_y=\mathbb{I}\left(\max_{u:|u-y|_1\leq 2}\mathrm{T}_2(u,y)^4=M\right). \tag{3.57}$$

By Lemma 3.9 (i), for $M\geq 16$, $\{I'_y, y\in\mathbb{Z}^d\}$ is a collection of M-dependent Bernoulli random variables. By Lemma 3.9 (ii) and the union bound,

$$q'_M=\sup_{y\in\mathbb{Z}^d}\mathbb{P}\left(\max_{u:|u-y|_1\leq 2}\mathrm{T}_2(u,y)^4=M\right)\quad\leq\quad e^{-M^{\varepsilon}}, \tag{3.58}$$

for some $\varepsilon>0$ small. Repeating the arguments as in the proof of Lemma 3.7 with A'^{γ}_M, q'_M instead of $\mathrm{A}^{\gamma}_M, q_M$, we can show that

$$\mathbb{E}\left(\max_{\gamma=(y_i)_{i=1}^{\ell}\in\mathcal{P}_L}\sum_{i=1}^{\ell}\max_{u:|u-y_i|_1\leq 2}\mathrm{T}_2(u,y_i)^4\right)=\mathcal{O}(L)\left(M_0^2+\sum_{M\geq M_0}M^{d+2}e^{-M^{\varepsilon}/d}\right)=\mathcal{O}(L),$$

with $M_0=M_0(d)$ a large constant, which proves (i).

4 Proof of Proposition 1.2

Proof. The upper bound on the length of optimal paths is a consequence of Lemma 2.1. Indeed, if $\gamma\in\mathbb{O}(x)$, then $l(\gamma)\leq\mathrm{T}(x)$. Hence, by Lemma 2.1,

$$\mathbb{P}\left(\max_{\gamma\in\mathbb{O}(x)}l(\gamma)>C_1|x|_1\right)\quad\leq\quad\mathbb{P}(\mathrm{T}(x)>C_1|x|_1) \tag{4.1}$$
$$\leq\quad e^{-|x|_1^{\varepsilon_1}}, \tag{4.2}$$

with ε_1 and C_1 positive constants as in Lemma 2.1. We start the proof of the lower bound by recalling a definition in the proof of Lemma 3.7. Given a path $\gamma=(y_i)_{i=0}^{\ell}$, define

$$\mathrm{A}^{\gamma}_M=\{0\leq i\leq\ell-1:\mathrm{T}(y_i,y_{i+1})=t(y_i,y_{i+1})=M\}.$$

Note that $l(\gamma) \geq \sum_{M\geq 1} \#\mathrm{A}_M^\gamma$ for any γ. Thus, for any $\gamma \in \mathbb{O}(x)$ and $K \geq 1$

$$\begin{aligned} |x|_1 \leq \mathrm{T}(x) = \sum_{M\geq 1} M\#\mathrm{A}_M^\gamma &\leq K\sum_{M=1}^{K}\#\mathrm{A}_M^\gamma + \sum_{M\geq K} M\#\mathrm{A}_M^\gamma \\ &\leq Kl(\gamma) + \sum_{M\geq K} M\#\mathrm{A}_M^\gamma. \end{aligned} \tag{4.3}$$

Rearranging it, we obtain that for any $K \geq 1$,

$$\begin{aligned} \min_{\gamma\in\mathbb{O}(x)} l(\gamma) &\geq \frac{1}{K}\left(|x|_1 - \max_{\gamma\in\mathbb{O}(x)}\sum_{M\geq K} M\#\mathrm{A}_M^\gamma\right) \\ &\geq \frac{1}{K}\left(|x|_1 - \sum_{M\geq K} M\max_{\gamma\in\mathbb{O}(x)}\#\mathrm{A}_M^\gamma\right). \end{aligned} \tag{4.4}$$

Note that if $\mathrm{T}(x) \leq C_1|x|_1$, then $\gamma \in \mathcal{P}_{C_1|x|_1}$ for any $\gamma \in \mathbb{O}(x)$, and thus

$$\sum_{M\geq K} M\max_{\gamma\in\mathbb{O}(x)}\#\mathrm{A}_M^\gamma \leq \sum_{M\geq K} M\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#\mathrm{A}_M^\gamma. \tag{4.5}$$

We define

$$M_x = [|x|_1^{1/2(d+3)}],$$

and

$$\mathcal{E} = \{\forall\, M \geq M_x,\, \forall\gamma \in \mathcal{P}_{C_1|x|_1},\ \#\mathrm{A}_M^\gamma = 0\}.$$

Then, by using the union bound and Lemma 2.4, we get

$$\begin{aligned} \mathbb{P}(\mathcal{E}^c) &\leq \mathbb{P}(\exists u, v \in \mathrm{B}(C_1|x|_1) \text{ such that } \mathrm{T}(u,v) = t(u,v) \geq M_x) \\ &\leq (2C_1|x|_1+1)^{2d}\sum_{M\geq M_x} e^{-M^{\varepsilon_1}} \leq Ce^{-|x|_1^{\varepsilon}}, \end{aligned} \tag{4.6}$$

for some positive constants C and ε. We recall from the proof of Lemma 3.7 that

$$\#\mathrm{A}_M^\gamma \leq \sum_{y\in\gamma} I_y, \tag{4.7}$$

where $\{I_y, y \in \mathbb{Z}^d\}$ is a collection of M-dependent Bernoulli random variables

$$I_y = \mathbb{I}(\exists\, z \in \mathbb{Z}^d : |z-y|_1 \leq M, \mathrm{T}(y,z) = t(y,z) = M),$$

and

$$q_M = \sup_{y\in\mathbb{Z}^d} \mathbb{E}(I_y) \leq (2M+1)^d e^{-M^{\varepsilon}}. \tag{4.8}$$

Then, the conditions (E1) and (E2) of Lemma 2.6 are satisfied. Using Lemma 2.6 (i), we obtain that

$$\mathbb{P}\left(\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#\mathrm{A}_M^\gamma \geq |x|_1M^{-3}\right) \leq 2^d\exp\left(-|x|_1/((16M)^{d+3})\right), \tag{4.9}$$

provided that $|x|_1M^{-3} \geq CM^d\max\{1, |x|_1Mq_M^{1/d}\}$, which holds for $|x|_1 \geq 2CM^{d+5}$ and $M \geq K$ with K a large constant. By (4.9) and the fact that $M_x = [|x|_1^{1/2(d+3)}] = o(|x|_1^{1/(d+5)})$,

$$\begin{aligned} \mathbb{P}\left(\sum_{M=K}^{M_x} M\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#\mathrm{A}_M^\gamma \geq |x|_1\sum_{M=K}^{M_x}M^{-2}\right) &\leq \sum_{M=K}^{M_x}\mathbb{P}\left(M\max_{\gamma\in\mathcal{P}_{C_1|x|_1}}\#\mathrm{A}_M^\gamma \geq |x|_1M^{-2}\right) \\ &\leq 2^d\sum_{M=K}^{M_x}\exp\left(-\tfrac{|x|_1}{(16M)^{d+3}}\right). \end{aligned}$$

Therefore,

$$\mathbb{P}\left(\sum_{M=K}^{M_x} M \max_{\gamma\in\mathcal{P}_{C_1|x|_1}} \#\mathrm{A}_M^\gamma > \frac{|x|_1}{2}\right) \le e^{-|x|_1^\varepsilon}, \tag{4.10}$$

for some $\varepsilon > 0$. Combining (4.4), (4.5), (4.6) and (4.10) yields that

$$\begin{aligned}\mathbb{P}\left(\min_{\gamma\in\mathcal{O}(x)} l(\gamma) < \frac{|x|_1}{2K}\right) &\le \mathbb{P}(\mathrm{T}(x) > C_1|x|_1) + \mathbb{P}(\mathcal{E}^c) + \mathbb{P}\left(\sum_{M=K}^{M_x} M \max_{\gamma\in\mathcal{P}_{C_1|x|_1}} \#\mathrm{A}_M^\gamma > \frac{|x|_1}{2}\right)\\ &\le Ce^{-|x|_1^\varepsilon},\end{aligned}$$

which completes the proof of Proposition 1.2. □

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Acknowledgments. We would like to thank the anonymous referees for carefully reading the manuscript and many valuable comments. The work of V. H. Can is supported by the fellowship of the Japan Society for the Promotion of Science and the Grant-in-Aid for JSPS fellows Number 17F17319, and by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant 101.03-2017.07.