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Shape theorem and surface fluctuation for Poisson cylinders

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Abstract

We prove a shape theorem for Poisson cylinders, and give a power-law bound on surface fluctuations. In particular, we show that for any $a \in (1/2, 1)$, conditioned on the origin being in the set of cylinders, if a point belongs to this set and has Euclidean norm below R, then this point lies at internal distance less than $R + O(R^a)$ from the origin.

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1 Introduction

We consider the Poisson cylinder model which consists of a random collection of bi-infinite cylinders, sampled through taking a Poisson point process of lines in $\mathbb{R}^d$, $d \geq 3$, and then thickening the obtained lines. This model serves as a natural mathematical model for various random fiber structures and also has many applications in image analysis; we refer the readers to [14] for a detailed survey on this topic. Recently, many geometric properties of this model, especially those related to percolation, have been studied [15, 7, 3] along with other models presenting long-range correlation.

We now describe our main result. Suppose that $d \geq 3$, and let μ be the translation and rotation invariant Haar measure on the space of lines in $\mathbb{R}^d$. Given $u > 0$, we let $\mathbb{P}^u$ be the law of Poisson cylinders in $\mathbb{R}^d$ with intensity measure $u\mu$.[1] By taking the union of cylinders generated by this process, we obtain a random set $\mathcal{C} \subset \mathbb{R}^d$. Let $\rho = \rho(\mathcal{C})$ be the metric on $\mathcal{C}$ defined as follows: if $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$, then $\rho(\boldsymbol{x}, \boldsymbol{y})$ is the length of the shortest path connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathcal{C}$. The metric ρ is referred to as the *internal distance*.

Let B_r and B_r^ρ stand for closed balls of radius r centered at the origin $\mathbf{0}$, with regard to the Euclidean metric and ρ, respectively. Our main result is the following:

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[1] We refer the readers to Section 2 for a precise mathematical construction.

Theorem 1.1 (Shape theorem)**.** *For every $a \in (1/2, 1)$, $u > 0$ and $c > 0$, $\mathbb{P}^u[\cdot|\mathbf{0} \in \mathcal{C}]$-almost surely, there exists a finite $R_0 > 0$, such that*

$$\text{for any } R > R_0, \qquad (\mathcal{C} \cap B_R) \subseteq B^{\rho}_{R+cR^a}. \tag{1.1}$$

We can roughly rephrase this as follows. Given a realization of the set of cylinders such that $\mathbf{0} \in \mathcal{C}$, when R is large, every point in $B_R \cap \mathcal{C}$ can be connected to $\mathbf{0}$ by a path that lies inside $\mathcal{C}$, and whose length is bounded from above by $R\big(1 + O(R^{a-1})\big)$. Note that conditioning on $\mathbf{0} \in \mathcal{C}$ is an arbitrary choice: by the translation invariance of the cylinder model, we could have taken any other point in $\mathbb{R}^d$.

Remark 1.2. Our result is stronger than the usual shape theorems, which in our notation would be stated as follows: there is a convex compact set[2] $D \subset \mathbb{R}^d$, such that for any $\epsilon > 0$, $\mathbb{P}[\cdot|\mathbf{0} \in \mathcal{C}]$-almost surely, there exists a finite $R_0 > 0$, such that

$$B^{\rho}_{(1-\epsilon)R} \subseteq (\mathcal{C} \cap D_R) \subseteq B^{\rho}_{(1+\epsilon)R} \quad \text{for} \quad R \geq R_0, \tag{1.2}$$

where

$$D_R := \{a\boldsymbol{x} \in \mathbb{R}^d;\, a \in [0, R],\, \boldsymbol{x} \in D\}.$$

In other words, we prove the asymptotic equivalence between ρ and the Euclidean metric (which implies the asymptotic shape is the unit ball), by providing a bound on surface fluctuations, in the spirit of results in [9]. For further discussion on shape theorems for models of first passage percolation, see Section 3 of [2]. Note also, that the first inclusion in (1.2) is immediate in our case.

We do not discuss the Poisson cylinder model for $d = 2$. In the planar case, the Poisson process of lines generates a connected set, and thickening this set would not qualitatively change its connectivity properties. Geodesics in models of this type were first investigated by Aldous and Kendall, who proved results that amount to an $O(\log R)$ surface fluctuation in the shape theorem; see, e.g., [1] and [8] for more details. Note that the most natural generalization of the model in those works for higher dimensions is not the Poisson cylinder model, but rather the Poisson flats model (introduced in [11]). The latter is a Poissonian soup of $(d-1)$-dimensional affine spaces, for which the shape fluctuates by $O(\log R)$, as it can be seen using a projection argument.

One can also compare our result with shape theorems obtained for discrete percolation models with long-range correlations, e.g., random interlacements and level sets of the Gaussian free field. In [6], a general scheme is developed for proving shape theorems for these models, which involves checking that the specific model under consideration fulfills a few criteria. Once these criteria are met, one has a shape theorem (in the form of the one in Remark 1.2) for this model, along with lots of other geometric properties. However, it cannot be applied to Poisson cylinders due to the spatial rigidity of cylinders: for (a discretized version of) the Poisson cylinder model, assumption **P3** of [6], which is known as the decoupling inequality in the random interlacement folklore, is not satisfied. Since our result in this work is actually stronger than the statement in Remark 1.2, we do not try to bypass this obstacle and adapt the arguments in [6].

We now explain the proof strategy for (1.1). Precise statements and detailed explanations can be found in Section 3.

• We first reduce the original theorem to a statement regarding the internal distance between $\mathbf{0}$ and a point $\boldsymbol{x} \in \mathcal{C} \cap B_R$. This is summarized in Proposition 3.2.

• Secondly, we consider two "*local networks*" of truncated cylinders with length of order $r = R^a$ near $\mathbf{0}$ and $\boldsymbol{x}$. We show that the μ-measure of said network is, in a sense, comparable to that of a Euclidean ball of radius r; see Proposition 3.6.

[2] Set D is referred to as the asymptotic shape with respect to ρ.

• Finally, we find a "*highway*" (long cylinder) connecting the local networks near $\mathbf{0}$ and $\boldsymbol{x}$. Thanks to the previous step we know that local networks are about as "visible"[3] to each other, as Euclidean balls of the same size. This fact, together with a classical estimate on μ (see, e.g., Lemma 3.1 in [15]), assures the existence of a highway with high probability. This part of the proof corresponds to Lemma 3.5.

One can compare our result with the connectivity results for Poisson cylinders. Imagine a graph where each vertex represents a cylinder in the Poissonian soup, and where edges connect any two intersecting cylinders. In [3] the authors show that for any intensity $u > 0$, this graph is $\mathbb{P}^u$-almost surely connected, and its diameter is equal to $(d-1)$. However, their results do not provide a bound on ρ. On the other hand, our strategy, which also involves connecting cylinders, provides a short path, but it is not designed to optimize the amount of cylinders visited by this path.

It is worth mentioning that the second and third steps above lead to a strong connectivity result in the form of criterion **S1** in [6] for Poisson cylinders. If capacity and hitting probability are interpreted in the context of Poisson cylinders as μ and visibility, respectively, then the analogy with random interlacements becomes apparent. In particular, our results have much in common with Lemma 12 from [12]. It is also worth mentioning that the recursive construction of "local networks" in the second step above bears some similarity in spirit to the construction of the "dense" set in [13] and [10].

With our proof strategy, the lower bound for a cannot be improved further. Indeed, if $a \leq 1/2$, then the local networks from the construction above will no longer be visible to each other with high probability. However, we are not able to rule out the possibility of a completely different strategy which could lead to stronger results. For instance, it might be that the shortest paths inside the set of cylinders consist of many short segments, much like as in the models investigated by Aldous and Kendall, as opposed to a highway and a few short segments.

This work allows for various extensions. If $\sqrt{R}$ is indeed the right scaling for surface fluctuations, we are naturally led to the question of whether more can be said on these fluctuations. To mix this problem with classical Bernoulli first passage percolation problems, one can assign random speed on each cylinder, or even between different sections of the same cylinder, and ask a similar question. One can also ask if the same shape theorem holds for the Poisson cylinder set in hyperbolic space, where the connectedness of cylinders undergoes a phase transition as the soup intensity changes, see [4] for more details.[4]

We now explain how this work is organized. In Section 2 we introduce the model and our notations. In Section 3 we state our main result and a few key propositions. Proofs are postponed till Section 4.

2 Model, notation and conventions

In this section, we introduce notations and describe the Poisson cylinder model.

2.1 Notation

Throughout this work, we consider $\mathbb{R}^d$ with $d \geq 3$ and view the integer lattice $\mathbb{Z}^d$ as its subset. Ordered tuples and particularly vectors in $\mathbb{R}^d$ are written in bold. We use $|\cdot|$ for the Euclidean norm on $\mathbb{R}^d$. We denote by $\boldsymbol{x} + B_r$ the closed Euclidean ball of radius $r > 0$ centered at $\boldsymbol{x} \in \mathbb{R}^d$; here the plus sign stands for the sumset operation and $\{\boldsymbol{x}\}$ is

[3] If A is a set of cylinders, and $\boldsymbol{y} \in \mathbb{R}^d$, then *visibility* of A from $\boldsymbol{y}$ is characterized by the μ-measure of lines going near A and $\boldsymbol{y}$.

[4] Even if a similar shape theorem holds in the hyperbolic setting, we expect the proof to be much different.

replaced by $\boldsymbol{x}$ for brevity. Given a metric $\rho : \mathbb{R}^d \times \mathbb{R}^d \mapsto [0, \infty]$, we let

$$B_r^\rho(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^d : \rho(\boldsymbol{x}, \boldsymbol{y}) \leq r\} \subset \mathbb{R}^d,$$

and write $B_r^\rho = B_r^\rho(\mathbf{0})$ for simplicity.

We write $(\cdot_j)_{j \in A}$ for a sequence of elements whose indices take values in an ordered countable set A, and $\{\cdot_j\}_{j \in A}$ for unordered sets with any index set A. To avoid bulky expressions, we will write $\{\cdot\}$ in place of $(\{\cdot\})$. If a set A is finite, we use $|A|$ for its cardinality. For $x \in \mathbb{R}^d$, we write

$$\lfloor x \rfloor := \sup\{y \in \mathbb{Z} : y \leq x\}, \qquad \lceil x \rceil := \inf\{y \in \mathbb{Z} : y \geq x\},$$

and use $[x]$ to denote the set $\{1, 2, \ldots, \lfloor x \rfloor\}$.

We will use a number of positive and finite constants, which will be denoted by c, and whose values might change from line to line. Even when we do not write it explicitly, these constants will always take strictly positive values. When c comes with an integer subscript, its value is kept fixed throughout the paper. Symbolic superscripts are exponents and not indices. For example, c_4^M refers to a fixed constant $c_4 > 0$, raised to the power M.

Given two functions $f, g : (0, \infty) \mapsto (0, \infty)$, we write $f \in O(g)$ and $g \in \Omega(f)$, if there is a c such that $f \leq cg$. If $f \in O(g)$ and $g \in O(f)$, then we write $f \asymp g$.

Finally, unless otherwise specified, $\log$ stands for the natural logarithm.

2.2 Poisson cylinder model

We now turn to the model of Poisson cylinders. Let $\mathbb{L}$ be the set of 1-dimensional affine subspaces of $\mathbb{R}^d$. Fix any line $\hat{l} \in \mathbb{L}$. For every line $l \in \mathbb{L}$ there exist a translation τ by a vector orthogonal to $\hat{l}$, and a rigid rotation θ around the origin, such that $(\theta \circ \tau)(\hat{l}) = l$. Endow $\mathbb{L}$ with the finest topology which makes $\theta \circ \tau$ continuous for all τ and θ. Once the topology is given, we equip $\mathbb{L}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{L})$ and denote by μ the Haar measure on $(\mathbb{L}, \mathcal{B}(\mathbb{L}))$, which is invariant under all isometries of $\mathbb{R}^d$. Also, the measure μ is unique up to a normalizing constant which we decide not to specify, since our results do not depend on the normalization.

For $l \in \mathbb{L}$, define a cylinder of radius 1 around l as $\mathrm{Cyl}(l) = l + B_1$. For open or compact $A \subset \mathbb{R}^d$ let

$$\mathcal{L}(A) := \{l \in \mathbb{L} : \mathrm{Cyl}(l) \cap A \neq \emptyset\},$$

which is $\mathcal{B}(\mathbb{L})$-measurable; see [15], around Eq. (2.11) for a proof.

Another important Borel measurable subset of $\mathbb{L}$ is the set of lines whose angle with respect to a given vector falls within a certain range. For $\boldsymbol{x} \in \mathbb{R}^d$, a unit vector $\boldsymbol{u} \in \mathbb{R}^d$, and $0 \leq \alpha \leq \beta \leq \pi/2$, we let

$$\mathcal{L}_{\boldsymbol{x},\boldsymbol{u}}^{\alpha,\beta} := \{l \in \mathcal{L}\{\boldsymbol{x}\} : |\langle \boldsymbol{u}, \boldsymbol{v}(l) \rangle| \in [\sin\alpha, \sin\beta]\},$$

where $\boldsymbol{v}(l)$ is either of the two unit vectors directing a line $l \in \mathbb{L}$, and $\langle \cdot, \cdot \rangle$ is the scalar product. For example, $\mathcal{L}_{\mathbf{0},\boldsymbol{v}(l_0)}^{0,0}$ is the set of lines hitting B_1 and orthogonal to l_0.

We now consider the Poisson point process on $(\mathbb{L}, \mathcal{B}(\mathbb{L}))$ defined on a probability space $(\mathcal{M}, \mathcal{A}, \mathbb{P}^u)$. Here

$$\mathcal{M} := \Big\{\omega = \sum_{j \geq 0} \delta_{l_j} : l_j \in \mathbb{L} \text{ for all } j, \text{ and } \omega(A) < \infty \text{ for compact } A \in \mathcal{B}(\mathbb{L})\Big\}$$

is the sample space (composed of locally finite point measures). The set of events $\mathcal{A} := \sigma(\{e_A\}_{A \in \mathcal{B}(\mathbb{L})})$ is the σ-algebra generated by the evaluation maps $e_A : \omega \mapsto \omega(A)$.

Finally, $\mathbb{P}^u$ is the probability measure, under which ω is a Poisson point process on $\mathbb{L}$ with intensity measure $u\mu$ for some $u > 0$. Note that $\mathbb{P}^u$ inherits the invariance under translations and rotations from μ. See [15] for a more detailed account of the properties of $\mathbb{P}^u$.

As mentioned before, the statements in this paper hold for any intensity parameter $u > 0$ and dimension $d \geq 3$, so we will often not be explicit about the dependence of constants and probability measures on them. For example, we write $\mathbb{P}(\text{event}) < c$, if for any $u > 0$ there is a constant $c = c(u, d) > 0$, such that $\mathbb{P}^u(\text{event}) < c$. The same convention applies to the asymptotic notations that we have introduced previously.

Having constructed the Poisson point process of lines, we denote the set of cylinders by

$$\mathcal{C} = \mathcal{C}(\omega) := \bigcup_{l \in \omega} \mathrm{Cyl}(l).$$

Here and in what follows we write $l \in \omega$ and $\mathrm{Cyl}(l) \in \omega$ instead of $l \in \mathrm{supp}(\omega)$ for brevity. Note that the law of $\mathcal{C}$ is invariant under isometries of $\mathbb{R}^d$.

Given two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, we denote by $[\boldsymbol{x}, \boldsymbol{y}]$ the line segment connecting $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$[\boldsymbol{x}, \boldsymbol{y}] := \big\{ (1-t)\, \boldsymbol{x} + t\, \boldsymbol{y} : t \in [0,1] \big\}.$$

For any $A \subseteq \mathbb{R}^d$, we define the set $\mathcal{P}_A(\boldsymbol{x}, \boldsymbol{y})$ of *polygonal paths* from $\boldsymbol{x}$ to $\boldsymbol{y}$ in A, as a set of finite sequences of vertices, for which the line segments between consecutive elements are within A:

$$\mathcal{P}_A(\boldsymbol{x}, \boldsymbol{y}) := \big\{ (\boldsymbol{z}_j)_{j=0}^n : n \geq 1,\ \boldsymbol{z}_0 = \boldsymbol{x},\ \boldsymbol{z}_n = \boldsymbol{y} \ \text{ and } \ [\boldsymbol{z}_{k-1}, \boldsymbol{z}_k] \subseteq A \quad \forall k \in [n] \big\}.$$

We define the *internal distance* $\rho = \rho(\omega)$ as follows:

$$\rho(\boldsymbol{x}, \boldsymbol{y}) := \inf \left\{ \sum_{j=1}^n |\boldsymbol{z}_j - \boldsymbol{z}_{j-1}| : (\boldsymbol{z}_j)_{j=0}^n \subset \mathcal{P}_{\mathcal{C}(\omega)}(\boldsymbol{x}, \boldsymbol{y}) \right\}, \qquad \text{for all} \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d.$$

We follow the convention that $\inf\{\emptyset\} = +\infty$. It is proved in Theorem 6.1 from [3], that $\mathcal{C}(\omega)$ is a connected set for $\mathbb{P}$-almost all ω. In other words, $\rho(\boldsymbol{x}, \boldsymbol{y}) < \infty \iff \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.

3 Proof strategy and intermediate results

In this section, we outline the proof of the shape theorem and state intermediate results. First, in Section 3.1 we reduce our theorem to the study of the internal distance between a pair of points. Then we divide the problem into finding a "highway" and "local networks" in Section 3.2. Section 3.3 is dedicated to the study of local networks.

We start by restating (1.1) in a form which is easier to analyze.

Theorem 3.1 (Shape theorem restated)**.** *For every $a \in (1/2, 1)$ and $c > 0$,*

$\mathbb{P}[\cdot | \mathbf{0} \in \mathcal{C}]$*-almost surely,* $\exists\, R_0 > 0$ *such that* $\rho(\mathbf{0}, \boldsymbol{x}) \leq R + cR^a, \quad \forall\, \boldsymbol{x} \in \mathcal{C} \cap B_R,\ \forall\, R > R_0.$

3.1 A bound on the internal distance

To prove (1.1), we first show that with high probability the internal distance between any pair of points $\mathcal{C}$ cannot be much larger than the Euclidean distance between them.

Proposition 3.2. *Given $\delta \in (0, 1)$ such that $\delta < (2a - 1)(d - 1)$, there exists c_1 such that,*

$$\mathbb{P}\big[\rho(\mathbf{0}, \boldsymbol{x}) > |\boldsymbol{x}| + c_1 |\boldsymbol{x}|^a \big| \mathbf{0}, \boldsymbol{x} \in \mathcal{C}\big] \in O\big(\exp\{-|\boldsymbol{x}|^\delta\}\big) \tag{3.1}$$

for all

$$\boldsymbol{x} \in \frac{1}{2\sqrt{d}} (\mathbb{R}^d \backslash B_1). \tag{3.2}$$

Remark 3.3. Proposition 3.2 is stronger than what we really need. It suffices to provide a bound like $O(|\boldsymbol{x}|^{-d-1-c})$ on the right-hand side of (3.1) with some $c > 0$.

We pick an $\boldsymbol{x}$ that satisfies (3.2), and let $R := |\boldsymbol{x}|$ and $r := R^a$. To avoid conditioning on the event $\{\mathbf{0}, \boldsymbol{x} \in \mathcal{C}\}$, we prove an unconditional version of Proposition 3.2, which gives a uniform bound over all possible pairs of $l_0 \in \mathcal{L}\{\mathbf{0}\}$ and $l_x \in \mathcal{L}\{\boldsymbol{x}\}$. To state it precisely, we let

$$\omega^- := \omega - \omega\, \mathbb{1}_{\mathcal{L}\{\mathbf{0},\boldsymbol{x}\}}, \qquad \mathcal{C}_{l_0,l_x}(\omega) := \mathcal{C}(\omega^-) \cup \mathrm{Cyl}(l_0) \cup \mathrm{Cyl}(l_x),$$

and define the event

$$E_{l_0,l_x}(c) := \Big\{ \forall\, (\boldsymbol{z}_j)_{j=0}^n \subset \mathcal{P}_{\mathcal{C}_{l_0,l_x}(\omega)}(\mathbf{0}, \boldsymbol{x}),\ \sum_{j=1}^n |\boldsymbol{z}_j - \boldsymbol{z}_{j-1}| > R + cr \Big\}.$$

Proposition 3.4. *For any δ as in Proposition 3.2 there is $c_1 > 0$ such that*

$$-\log \sup_{l_0 \in \mathcal{L}\{\mathbf{0}\},\, l_x \in \mathcal{L}\{\boldsymbol{x}\}} \mathbb{P}\big(E_{l_0,l_x}(c_1)\big) \in O(R^\delta).$$

3.2 Highway and local connections

Although Propositions 3.2 and 3.4 do not specify how to construct a short path, the bound on ρ strongly suggests the following strategy:

- take B_r and $B_r + \boldsymbol{x}$;
- try to find a cylinder which intersects both balls (we call such a cylinder a *highway*);
- if a highway exists, and we manage to connect $\mathbf{0}$ and $\boldsymbol{x}$ to it by finitely many truncated cylinders with heights of order r, then $\rho(\mathbf{0}, \boldsymbol{x})$ is bounded from above by $R + c_1 r$ for some $c_1 > 0$.

It is not difficult to see that a highway exists with high probability. Thanks to Lemma 3.1 in [15], for some absolute c,

$$\mu\big(\mathcal{L}(B_r) \cap \mathcal{L}(B_r + \boldsymbol{x})\big) > c(r^2/R)^{d-1}.$$

Since $a > 1/2$, the probability of having no highway between B_r and $B_r + \boldsymbol{x}$ decays rapidly as we increase $R = |\boldsymbol{x}|$:

$$-\log \mathbb{P}\big\{\omega\big(\mathcal{L}(B_r) \cap \mathcal{L}(B_r + \boldsymbol{x})\big) = 0\big\} > cuR^{(2a-1)(d-1)}.$$

The purpose of this subsection is to set up a structure in space (referred to in the text as a *crossing*) which mediates connections between local networks and the highway. We will often "discretize" various geometric objects in $\mathbb{R}^d$, such as disks and cylinders, by replacing them with finite sets of points. Doing so allows us to use finite sequences of random variables to provide the concentration bounds from the previous subsection.

For some big $c_2 > 0$, which will be picked in Lemma 3.9, we put two identical $(d-1)$-dimensional disks of radius r between $\mathbf{0}$ and $\boldsymbol{x}$, at a distance $c_2 r$ from these two points. On each one of those disks, we place a square grid of points with mesh $c_0 := 10d$ which will be called *crossings*. More precisely, we take an orthogonal coordinate system so that $\boldsymbol{x}$ points in the direction of the last basis vector, and define the crossings as follows:

$$\mathfrak{C}_0 := \Big\{ \big(c_0\, \mathbb{Z}^{d-1} \cap B_r\big) \times \{0\} + c_2 r\, \frac{\boldsymbol{x}}{R} \Big\}, \quad \mathfrak{C}_x := \Big\{ \big(c_0\, \mathbb{Z}^{d-1} \cap B_r\big) \times \{0\} + \Big(\boldsymbol{x} - c_2 r\, \frac{\boldsymbol{x}}{R}\Big) \Big\};$$

see Fig. 1 for a sketch.

Note that $|\mathfrak{C}_j| \asymp r^{d-1}$ for $j \in \{0, x\}$. The next statement implies that under certain conditions a highway exists with high probability.

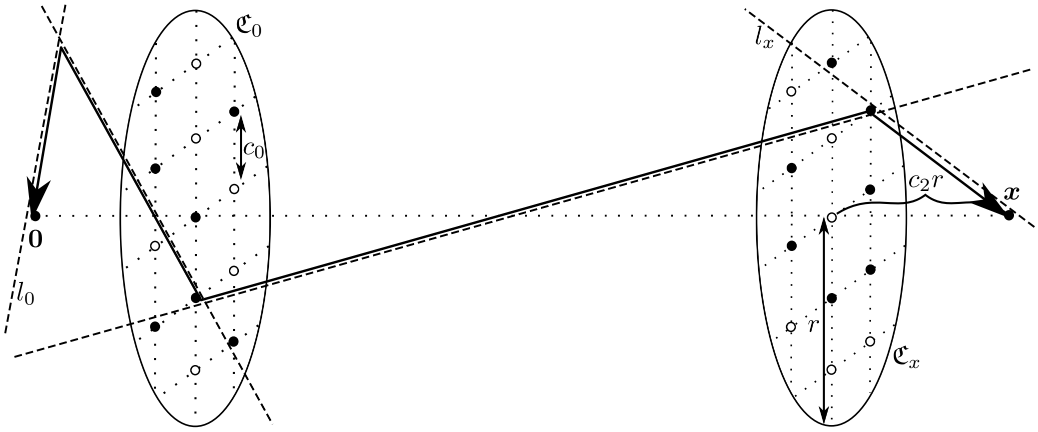

Figure 1: Dashed lines represent lines in the soup ω (more specifically, here we represent the two "given" lines l_0 and l_x, one line from the local network near $\mathbf{0}$ and the highway). The solid broken line depicts a polygonal path from $\mathcal{P}_{\mathcal{C}}(\mathbf{0}, \boldsymbol{x})$. The unions of filled and empty dots on disks are the crossings $\mathfrak{C}_0$ and $\mathfrak{C}_x$. The filled dots correspond to $\mathfrak{C}'_0$ and $\mathfrak{C}'_x$, and they are used to lay down a highway.

Lemma 3.5. *Suppose that* $\mathfrak{C}'_j \subseteq \mathfrak{C}_j$, $j \in \{0, x\}$ *satisfy* $|\mathfrak{C}'_j| \in \Omega(r^\chi)$ *with* $\chi(\epsilon) := (d-1)(1-\epsilon)$ *for some* $\epsilon > 0$ *such that* $2a(1-\epsilon) > 1$*. Then*

$$-\log \mathbb{P}\big\{\omega^-\big(\mathcal{L}(\mathfrak{C}'_0) \cap \mathcal{L}(\mathfrak{C}'_x)\big) = 0\big\} \in \Omega(r^{2\chi}/R^{d-1}).$$

Note that for $d \geq 3$ our restriction on the values of a and ϵ implies that $\chi(\epsilon) > 1$. It now suffices to show, that l_0 can be connected to at least r^χ points on $\mathfrak{C}_0$ via finitely many truncated cylinders of length $O(r)$ with high probability. A similar statement about a local network near $\boldsymbol{x}$ will then follow thanks to the translation and rotation invariance of the model.

Proposition 3.6. *For any* ϵ *and* $\chi(\epsilon)$ *be given as in Lemma 3.5, there exists* $c_3 = c_3(\epsilon)$ *such that for any* $l_0 \in \mathcal{L}\{\mathbf{0}\}$,

$$-\log \mathbb{P}\left\{\left|\mathfrak{C}_0 \cap B_{c_3 r}^{\rho(\omega^- + \delta_{l_0})}\right| < r^\chi\right\} \in \Omega(r). \tag{3.3}$$

Proposition 3.6 ensures, that with high probability we have sufficiently many points to hook onto in Lemma 3.5, and those points are not too far from $\mathbf{0}$ in $\mathcal{C}$. It implies, that the μ-measure of a local network of size r is in $\Omega(r^\chi)$, where χ can be made arbitrarily close to $d-1$. Therefore, the "visibility" of a local network is asymptotically close to that of a Euclidean ball. The purpose of the next subsection is to outline the proof of this.

3.3 Local networks

Here we describe the strategy used to prove (3.3). We first fix some positive integer M. In order to exploit some independence, we split the original cylinder process into a union of $(M+1)$ independent cylinder processes with intensity $w := u/(M+1)$. Next we draw concentric spheres having radius of order r. We then sample cylinders from the first portion of the process to connect l_0 to the first sphere and then sample cylinders from the second portion to connect the first sphere to the second one. Here, *connecting* a set A to a set B means that there is at least one cylinder $C \in \omega^-$ such that $A \cap C \neq \emptyset$ and there exists a point in $B \cap C$, which is referred to as a *connection*. We proceed iteratively, until the M^{th} sphere is reached. We guarantee that the number of connections grows rapidly between each step with high probability. In particular, after the M^{th} step we have many points near the last sphere, which are connected to $\mathrm{Cyl}(l_0)$ by a path of length in $O(r)$. This ensures that the last portion of the process contains enough cylinders to connect those points to at least r^χ points in $\mathfrak{C}_0$.

To describe the strategy more precisely, we fix $\epsilon > 0$ such that $2a(1-\epsilon) > 1$ (as in Lemma 3.5), and then let

$$M := \lceil \log_{\frac{d-2}{d-1}} \epsilon \rceil + 1, \quad a_j := \Big(\frac{d-2}{d-1}\Big)^j, \quad b_j := (d-1)(1-a_j), \qquad \forall j \in [M].$$

Here M corresponds to the number of steps in the scheme we described previously. To form the connection on the j^{th} step of this scheme, we use truncated cylinders, which are close to the j^{th} sphere, and have length of order r^{a_j}. The number of connections to $\mathrm{Cyl}(l_0)$ provided at step j is of order r^{b_j}. Note that M is chosen so that $a_M < \epsilon$ and $b_M > \chi$. Also, $b_{j+1} = a_j + b_j$ for all $j \in [M-1]$.

We fix a constant c_4, whose actual value will be given by Lemma 3.8 below. We define annuli $A_j := B_{c_4^j r} \backslash B_{c_4^j r - r^{a_j}}$ for $j \in [M]$. Note that if $c_4 > 10$, then for any $l \in \mathcal{L}(B_r)$ the length of both segments of $l \cap B_{c_4 r} \backslash B_{(c_4-1)r}$ is bounded above by $2r$ for every r large enough; we assume this in the sequel.

A sequence of points on a line segment $[\boldsymbol{y}, \boldsymbol{z}] \subset \mathbb{R}^d$ is called a c_0-*grid*, if the spacing between consecutive points along the segment is equal to c_0, and no more points can be added to the sequence without violating this spacing constraint. We take a c_0-grid on $l_0 \cap B_r$ and exclude $\mathbf{0}$ from it:

$$(\boldsymbol{l}_{0,1}^m)_{m=1}^{N_{0,1}} := c_0\big(\mathbb{Z} \cap [-r/c_0, r/c_0] \backslash \{\mathbf{0}\}\big);$$

note that we write $N_{0,1}$ for the amount of points in the sequence. We let $\boldsymbol{L}_0 := \big((\boldsymbol{l}_{0,1}^m)_{m=1}^{N_{0,1}}, l_{0,1}\big)$ and call it the *zeroth layer*.

Next we stack a collection of *layers* (essentially, sets of points) onto the zeroth layer recursively. Layers consist of “beaded” *threads*, each thread being a pair formed by a c_0-grid and the line on which the grid lies. Given $j \in [M]$, the j^{th} layer corresponds to a collection of truncated cylinders, that are connected to the origin in $(c_4^j B_r) \cap \mathcal{C}$ by at most $(j-1)$ intermediate cylinders from the previous layers.

More specifically, we define the set $\mathfrak{L}_j$ of j^{th} layers for $j \in [M]$ recursively as follows:

$$\mathfrak{L}_j(\boldsymbol{L}_{j-1}) := \Big\{ \boldsymbol{L}_j := (T_{j,k})_{k=1}^{|\boldsymbol{L}_j|} \;\Big|\; \text{each } \textit{thread } T_{j,k} = \big((\boldsymbol{l}_{j,k}^m)_{m=1}^{N_{j,k}}, l_{j,k}\big) \text{ satisfies 1.-3. below} \Big\}$$

1. The k^{th} thread $T_{j,k}$ is formed by a line $l_{j,k} \in \mathbb{L}$, and a c_0-grid of $N_{j,k} \geq 1$ points $(\boldsymbol{l}_{j,k}^m)_{m=1}^{N_{j,k}}$ lying on either segment of $l_{j,k} \cap A_j$.
2. There exists $q \in \big[\boldsymbol{L}_{j-1}\big]$, such that $\boldsymbol{l}_{j-1,q}^s \in \mathrm{Cyl}(l_{j,k}) \cap l_{j-1,q}$ for some $s \in [N_{j-1,q}]$. Also, $l_{j,k} \in \mathcal{G}(\boldsymbol{l}_{j-1,q}^s; l_{j-1,q})$, where $\mathcal{G}(\boldsymbol{y}; l) := \mathcal{L}_{\boldsymbol{y},\boldsymbol{v}(l)}^{\alpha,\beta} \backslash \mathcal{L}\{\mathbf{0}, \boldsymbol{x}\}$ for $\boldsymbol{y} \in \mathbb{R}^d$ and $l \in \mathbb{L}$, $\alpha := 2\arctan c_0^{-1}$ and $\beta := \arccos c_0^{-1}$.
3. For every $q \in \big[|\boldsymbol{L}_j|\big] \backslash \{k\}$ and $s \in [N_{j,q}]$, we have $|\boldsymbol{l}_{j,k}^m - \boldsymbol{l}_{j,q}^s| > c_0 r^{a_j}$.

See Fig. 2 for an illustration.

In order to prove (3.3), we start by building the first layer. For the j^{th} layer $\boldsymbol{L}_j$, we say that it exists if all directing lines $\{l_{j,k}\}_{k=1}^{|\boldsymbol{L}_j|}$ in its threads belong to ω^-. We can show that there is a procedure allowing to construct the first layer $\boldsymbol{L}_1$ with $\Omega(r)$ threads.

Lemma 3.7. *There exist $f_1 > 0$ and $c > 0$, such that for any $l_0 \in \mathcal{L}\{\mathbf{0}\}$ and $l_x \in \mathcal{L}\{\boldsymbol{x}\}$,*

$$-\log \mathbb{P}\{\nexists\, \boldsymbol{L}_1 \in \mathfrak{L}_1(\boldsymbol{L}_0) : |\boldsymbol{L}_1| > f_1 r\} \in \Omega(r).$$

Similarly, for each $j \in [M-1]$, layer $\boldsymbol{L}_{j+1}$ can be constructed with high probability.

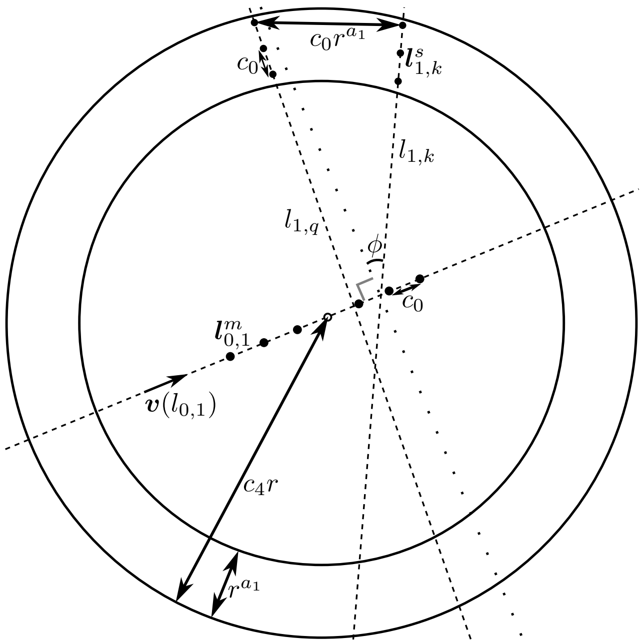

Figure 2: Construction of the first layer. Dashed lines represent the “given” l_0 in $\boldsymbol{L}_0$, and two lines $l_{1,k}$ and $l_{1,q}$ from threads k and $q \neq k$ in $\boldsymbol{L}_1$, respectively. The image is a projection onto a flat surface which is parallel to $l_{1,k}$ and l_0. We demand that $\phi = \arcsin\langle \boldsymbol{v}(l_0), \boldsymbol{v}(l_{1,k})\rangle \in [\alpha, \beta]$. Filled dots on the picture are “beads” lying on $l_0, l_{1,k}$ and $l_{1,q}$.

Lemma 3.8. *There is c_4 such that given any $f_j > 0$, there exists $f_{j+1} > 0$, so that for any j^{th} layer $\boldsymbol{L}_j$ satisfying $|\boldsymbol{L}_j| \geq f_j r^{b_j}$,*

$$-\log \mathbb{P}\big\{ \nexists\, \boldsymbol{L}_{j+1} \in \mathfrak{L}_{j+1}(\boldsymbol{L}_j) : |\boldsymbol{L}_{j+1}| > f_{j+1} r^{b_{j+1}} \big\} \in \Omega(r^{b_{j+1}}) \tag{3.4}$$

for any $j \in [M-1]$.

The statements in Lemmas 3.7 and 3.8 are very similar, but proving the latter is slightly more challenging, because unlike $\boldsymbol{L}_0$, the layers $(\boldsymbol{L}_j)_{j=1}^{M-1}$ contain more than one thread for large r. This is the reason why our construction of layers above Lemma 3.7 is so overly specific.

We will need to consider balls of the metric $\rho(\omega^-)$ around the beads of a layer $\boldsymbol{L}_j$. That is, we will need to consider sets of the type

$$B_c^{\rho(\omega^-)}(\boldsymbol{L}_j) := \bigcup_{k \in [|\boldsymbol{L}_j|]} \bigcup_{m \in [N_{j,k}]} B_c^{\rho(\omega^-)}(\boldsymbol{l}^m_{j,k}).$$

Once we show that $\mathrm{Cyl}(l_0)$ is connected to sufficiently many points in A_M, we bridge the last layer and $\mathfrak{C}_0$.

Lemma 3.9. *Pick ϵ and χ as in Lemma 3.5. There exist $f_M > 0$ and c_2, such that given any M^{th} layer $\boldsymbol{L}_M$ which satisfies $|\boldsymbol{L}_M| \geq f_M r^{b_M}$, we have*

$$-\log \mathbb{P}\Big\{ \big| \mathfrak{C}_0 \cap B^{\rho(\omega^-)}_{(c_2 + c_4^M + 1)r}(\boldsymbol{L}_M) \big| < r^{\chi} \Big\} \in \Omega(r^{\chi}).$$

4 Proofs

In what follows, we will use a result from [3], which is contained in the remark following Lemma 4.1 of that work. We use it in the following form:

$$c \le \mu\big(\mathcal{L}\{\mathbf{0}\} \cap \mathcal{L}\{\boldsymbol{x}\}\big)\, |\boldsymbol{x}|^{d-1} \le c + c'|\boldsymbol{x}|^{-2}, \qquad \forall \boldsymbol{x} \in \mathbb{R}^d \backslash B_4. \tag{4.1}$$

This fact has an immediate corollary which will be useful later. It can be deduced by covering $\boldsymbol{x} + \partial B_{cR^a}$ with $cR^{a(d-1)}$ unit balls and then using (4.1).

Corollary 4.1. *For any $a \in (0,1)$, $c > 0$ and $\boldsymbol{x} \in \mathbb{R}^d$,*

$$\mu\big(\mathcal{L}\{\mathbf{0}\} \cap \mathcal{L}(\boldsymbol{x} + B_{c|\boldsymbol{x}|^a})\big) \in O\Big(\min(1, |\boldsymbol{x}|^{(a-1)(d-1)})\Big). \tag{4.2}$$

We will also use the following corollary of the Azuma-Hoeffding inequality to provide concentration bounds.

Lemma 4.2. *Suppose that a filtration $\mathbb{F} := (\mathcal{F}_n)_{n=1}^N$ supports a sequence of random variables $(X_n)_{n=1}^N$ such that $X_n|\mathcal{F}_{n-1} \sim$ Poisson(μ_n), where $(\mu_n)_{n=1}^N \subset \mathbb{R}^d$ is such that $\mu_n \ge \mu_0 > 0$ for some $\mu_0 > 0$ and any $n \in [N]$. Let $I_n := \mathbb{1}\{X_n > 0\}$. Then,*

$$-\log \mathbb{P}\left\{\sum_{n=1}^N I_n \le \frac{p}{2}N\right\} > \frac{p^2}{8}N, \qquad \textit{where } \; p := 1 - \exp\{-u\mu_0\} > 0.$$

Proof. Note that

$$\mathbb{E}[I_n] = 1 - \exp\{-u\mu_n\} \ge p, \qquad \forall n \in [N].$$

For every $n \in [N-1]$ we have $|I_{n+1} - I_n| \le 1$. Let $Y_n = \sum_{m=1}^n (I_m - p)$ so that $(Y_n)_{n=1}^N$ is a sub-martingale with respect to $\mathbb{F}$ with increments bounded by 1 in absolute value. We can apply the Azuma-Hoeffding inequality in order to obtain:

$$-\log \mathbb{P}\left\{\sum_{n=1}^N I_n \le Np/2\right\} = -\log \mathbb{P}\big\{Y_N - Y_0 \le -Np/2\big\} \ge \frac{(Np/2)^2}{2N} = \frac{p^2}{8}N.$$

This finishes the proof. □

In the remainder of this article we prove the results stated in the Section 3. We start with a proof which covers Lemmas 3.7 and 3.8 followed by a proof for Lemma 3.9. Then we prove the remaining results in reverse order, finishing with the proof of the Shape Theorem.

*Proof of **Lemmas 3.7 and 3.8**.* Order the indices $\big\{\{(m,k)\}_{m=1}^{N_{j,k}}\big\}_{k=1}^{|\boldsymbol{L}_j|}$ in such a way that indices corresponding to earlier beads within earlier threads appear first. Specifically, define an injection $q : (m,k) \mapsto \mathbb{N}$ for each $m \in [N_{j,k}]$ and $k \in \big[|\boldsymbol{L}_j|\big]$ so that $q(m,k) < q(m',k')$ whenever $k' > k$ and $q(m,k) < q(m',k)$ if $1 \le m < m' \le N_{j,k}$. For simplicity, suppose that q is a bijection to $[N]$ for some integer N. We have supposed that $|\boldsymbol{L}_j| \ge f_j r^{b_j}$, so we assume that $N \asymp r^{a_j+b_j} = r^{b_{j+1}}$ which is possible by ignoring every thread after the one with index $\lfloor f_j r^{b_j} \rfloor$. Associate a filtration $\mathbb{F} := (\mathcal{F}_n)_{n=1}^N$ to the threads of $\boldsymbol{L}_j$ as follows:

$$\mathcal{F}_n := \sigma\Big(\big\{e_L : \omega \to \omega(L) \big| L \in \mathbb{L}_n\big\}\Big), \quad \mathbb{L}_n := \mathcal{L}\{\boldsymbol{l}_{j,k}^m : q(m,k) \in [n]\}, \quad n \in [N].$$

Next we define recursively a sequence of tuples:

$$\Big(\mathcal{L}_n, \;\; X_n, \;\; I_n, \;\; T_n = \big((\boldsymbol{l}_{j+1,n}^m)_{m=1}^{N_{j+1,n}}, \;\; l_{j+1,n}\big), \;\; \boldsymbol{y}_n\Big)_{n=1}^N,$$

where, for each $n \in [N]$ we write $(m,k) = q^{-1}(n)$ and define

$$\mathcal{L}_n := \mathcal{G}(\boldsymbol{l}^m_{j,k}; l_{j,k}) \backslash \left(\bigcup_{q=1}^{n-1} \mathcal{L}\big(B_{c_0 r^{a_{j+1}}} + \{\boldsymbol{l}^m_{j+1,q}\}_{m=1}^{N_{j+1,q}}\big) \cup \mathcal{L}\{\boldsymbol{l}^{m'}_{j,k'} : q(m',k') > n\} \right), \qquad (4.3)$$

$X_n := \omega(\mathcal{L}_n)$ and $I_n := \mathbb{1}\{X_n > 0\}$ (we clarify (4.3) in the Remark 4.3 presented right after this proof).

• If $I_n = 1$, we select an arbitrary line $l_{j+1,n} \in \mathcal{L}_n$ from ω. Choose either one of the two line segments in $l_{j+1,n} \cap A_{j+1}$ and let $\boldsymbol{y}_n$ be its center. Put a c_0-grid $(\boldsymbol{l}^m_{j+1,n})_{m=1}^{N_{j+1,n}}$ on this segment and pair it with $l_{j+1,n}$ to form a thread T_n.

• If $I_n = 0$, we let T_n and $\boldsymbol{y}_n$ be empty sets.

Choose c_4 big enough so that for any $j \in [M-1]$ and $l \in \mathcal{G}(\boldsymbol{l}^m_{j,k}; l_{j,k})$, the only threads in $\boldsymbol{L}_j$ that can be intersected by Cyl(l), except for the k^{th} thread, are at a distance larger than cr from $\boldsymbol{l}^m_{j,k}$ for some c. See Fig. 3 for clarification.

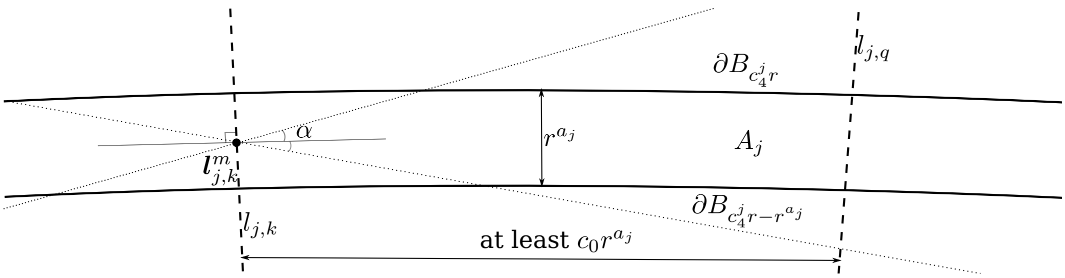

Figure 3: For large c_4, threads become nearly orthogonal to the boundaries of A_j. If $l \in \mathcal{G}(\boldsymbol{l}^m_{j,k}; l_{j,k})$, then it must form an angle larger than α with any plane orthogonal to $l_{j,k}$. In particular, for c_4 large enough $l \cap A_j$ consists of two segments separated by distance $\Omega(r)$ from each other. The length of each segment is less then $c_0 r^{a_j}$, and Cyl(l) can pass through at most one thread in $\boldsymbol{L}_j \backslash T_k$ that is at least $\Omega(r)$ apart from T_k.

We can now show that $\min_{n \in [\eta N]} \mu(\mathcal{L}_n) \in \Omega(1)$ for some $\eta \in (0,1)$. In view of (4.3), $\mu(\mathcal{L}_n)$ is bounded from below by $M_{\text{main}} - M_{\text{prev}} - M_{\text{cur}}$, where

$$\begin{aligned} M_{\text{main}} &:= \mu(\mathcal{L}^{\alpha,\beta}_{\boldsymbol{l}^m_{j,k}, \boldsymbol{v}(l_{j,k})} \backslash \mathcal{L}\{\boldsymbol{0}, \boldsymbol{x}\}), \\ M_{\text{prev}} &:= \sum_{q=1}^{n-1} \mu\Big(\mathcal{L}^{\alpha,\beta}_{\boldsymbol{l}^m_{j,k}, \boldsymbol{v}(l_{j,k})} \cap \mathcal{L}\big(B_{c_0 r^{a_{j+1}}} + \{\boldsymbol{l}^m_{j+1,q}\}_{m=1}^{N_{j+1,q}}\big)\Big), \\ M_{\text{cur}} &:= \mu\big(\mathcal{L}^{\alpha,\beta}_{\boldsymbol{l}^m_{j,k}, \boldsymbol{v}(l_{j,k})} \cap \mathcal{L}\{\boldsymbol{l}^{m'}_{j,k'} : q(m',k') > n\}\big). \end{aligned}$$

Note, that $M_{\text{main}} = \Omega(1)$. Indeed,

$$M_{\text{main}} \ge \mu(\mathcal{L}^{\alpha,\beta}_{\boldsymbol{l}^m_{j,k}, \boldsymbol{v}(l_{j,k})}) - \mu(\mathcal{L}\{\boldsymbol{l}^m_{j,k}\} \cap \mathcal{L}\{\boldsymbol{x}\}) - \mu(\mathcal{L}\{\boldsymbol{l}^m_{j,k}\} \cap \mathcal{L}\{\boldsymbol{0}\}),$$

where the first term is in $\Omega(1)$, and the following two terms are in $O(R^{1-d})$ and $O(r^{1-d})$, respectively, thanks to (4.1). Next, we bound M_{prev} from above by

$$\sum_{q \in [n-1]: \boldsymbol{y}_q \neq \emptyset} \mu\big(\mathcal{L}^{\alpha,\beta}_{\boldsymbol{l}^m_{j,k}, \boldsymbol{v}(l_{j,k})} \cap \mathcal{L}(B_{c_0 r^{a_{j+1}}} + \boldsymbol{y}_q)\big).$$

This sum involves up to $O(r^{b_{j+1}})$ terms, and each term is in $O(r^{(d-1)(a_{j+1}-1)}) = O(r^{-b_{j+1}})$, according to (4.2). Finally, we bound M_{cur} from above by

$$\sum_{(m',k'): q(m',k') \in (n,N]} \mu\big(\mathcal{L}^{\alpha,\beta}_{\boldsymbol{l}^m_{j,k}, \boldsymbol{v}(l_{j,k})} \cap \mathcal{L}\{\boldsymbol{l}^{m'}_{j,k'}\}\big).$$

This expression is either zero (for $j = 0$) or involves at most $N \in O(r^{b_{j+1}})$ terms, all bounded above by cr^{1-d} for some $c > 0$, therefore the whole sum is in $O(r^{-(d-1)a_{j+1}})$. We have thus showed, that there must exist an $\eta \in (0,1)$ such that $\min_{n\in[\eta N]} \mu(\mathcal{L}_n) \in \Omega(1)$.

We can now apply Lemma 4.2 to see that

$$-\log \mathbb{P}\left\{\sum_{n=1}^{\lfloor \eta N\rfloor} I_n \le f_{j+1} r^{b_{j+1}}\right\} \in \Omega(r^{b_{j+1}})$$

for some $f_{j+1} > 0$. We have thus constructed the $(j+1)^{\text{th}}$ layer

$$\boldsymbol{L}_{j+1} := \{T_n : n \in [\eta N],\ I_n = 1\},$$

with required properties. □

Remark 4.3. Roughly speaking, if $q(m,k) = n$, then $\mathcal{L}_n$ is a set of lines that

- do not hit B_1 or $\boldsymbol{x} + B_1$, but go through $\boldsymbol{l}^m_{j,k} + B_1$, and form angle $\phi \in [\alpha, \beta]$ with a flat surface orthogonal to $l_{j,k}$;
- are separated by at least $c_0 r^{a_{j+1}}$ from threads formed in $\boldsymbol{L}_{j+1}$ before step n;
- stay away from the beads in $\boldsymbol{L}_j$ that follow after step n.

When $j = 0$, the third item is redundant, because $|\boldsymbol{L}_j| = 1$ and $\phi \le \beta$.

Proof of **Lemma 3.9**. Let $N := |\boldsymbol{L}_M| \ge f_M r^{b_M}$. Since $\boldsymbol{L}_M$ is a layer, its threads must be separated from each other by $c_0 r^{a_M}$, thus $N \in O(r^{d-1}) = O\big(|\mathfrak{C}_0|\big)$. For each thread $k \in [N]$ select only the point $\boldsymbol{l}_k$ closest to the outer boundary of A_M, that is, $\partial B_{c_4^M r}$. Let $P := (\boldsymbol{l}_k)_{k=1}^N$, $A := B_{c_4^M r} \backslash B_{c_4^M r - c_0}$ and note that $P \subset A$. We demand that c_2 is large enough, so that for any $l \in \mathcal{L}(A)$ there is at most one point in $\mathrm{Cyl}(l) \cap \mathfrak{C}_0$; for example, $c_2 > 10 c_4^M d$ would suffice. Associate to the points in P a filtration $\mathbb{F} := (\mathcal{F}_n)_{n=1}^N$:

$$\mathcal{F}_n := \sigma\Big(\big\{e_L : \omega \to \omega(L) \big| L \in \mathbb{L}_n\big\}\Big), \qquad \mathbb{L}_n := \mathcal{L}\big(\{\boldsymbol{l}_k\}_{k=1}^n\big), \qquad n \in [N].$$

We build a sequence $(S_n, \mathcal{L}_n, X_n, I_n, \boldsymbol{p}_n)_{n=1}^N$ recursively. Let

$$\begin{aligned} &\mathcal{L}_n := \mathcal{L}^{\gamma,\pi/2}_{\boldsymbol{l}_n,\boldsymbol{u}_n} \cap \mathcal{L}(\mathfrak{C}_0) \backslash \mathcal{L}\Big(\{\boldsymbol{0}, \boldsymbol{x}\} \cup S_n \cup P \backslash \{\boldsymbol{l}_n\}\Big), \qquad && \boldsymbol{u}_n := \boldsymbol{l}_n / |\boldsymbol{l}_n|, \\ &S_n := \{\boldsymbol{p}_q : q \in [n-1],\ I_q = 1\}, \qquad X_n = \omega(\mathcal{L}_n), && I_n := \mathbb{1}\{X_n > 0\}, \end{aligned}$$

where $\gamma > 0$ is some small angle, such that any $l \in \mathcal{L}^{\gamma,\pi/2}_{\boldsymbol{l}_n,\boldsymbol{u}_n}$ can go through at least $|\mathfrak{C}_0|/2$ points on $\mathfrak{C}_0$ for any $n \in N$.

- If $I_n = 1$, we take any $l \in \omega$ from $\mathcal{L}_n$ and let $\boldsymbol{p}_n := \mathrm{Cyl}(l) \cap \mathfrak{C}_0$.
- If $I_n = 0$, we let $\boldsymbol{p}_n := \emptyset$.

Since $\boldsymbol{L}_M$ is the M^{th} layer, and since $\{\boldsymbol{p}_n\}_{n=1}^N$ is disjoint and contained in $\mathfrak{C}_0$, we have

$$\sum_{n=1}^N I_n \le \Big|\mathfrak{C}_0 \cap B^{\rho(\omega^-)}_{(c_2 + c_4^M + 1)r}(\boldsymbol{L}_M)\Big|.$$

We can show that $\mu(\mathcal{L}_n)$ is bounded away from 0:

$$\begin{aligned} \mu(\mathcal{L}_n) \ge \sum_{\boldsymbol{p} \in \mathfrak{C}_0 \backslash S_n} &\mu\big(\mathcal{L}^{\gamma,\pi/2}_{\boldsymbol{l}_n,\boldsymbol{u}_n} \cap \mathcal{L}\{\boldsymbol{p}\}\big) - \mu(\mathcal{L}\{\boldsymbol{l}_n\} \cap \mathcal{L}\{\boldsymbol{0}\}) \\ &- \mu(\mathcal{L}\{\boldsymbol{l}_n\} \cap \mathcal{L}\{\boldsymbol{x}\}) - \mu\big(\mathcal{L}^{\gamma,\pi/2}_{\boldsymbol{l}_n,\boldsymbol{u}_n} \cap \mathcal{L}(\mathfrak{C}_0) \cap \mathcal{L}(P \backslash \{\boldsymbol{l}_n\})\big). \end{aligned}$$

Since $N \in O(r^{d-1})$, we have $|\mathfrak{C}_0 \backslash S_n| \in \Omega(r^{d-1})$ and the first sum on the right-hand side is in $\Omega\big(|\mathfrak{C}_0|/r^{d-1}\big) = \Omega(1)$ due to (4.1). Inequality (4.1) also implies that the second term is in $O(r^{1-d})$ and the third is in $O(R^{1-d})$.

We now show that the fourth term decays as $r \uparrow \infty$. Define a "cone" $L := \mathcal{L}^{\gamma,\pi/2}_{\boldsymbol{l}_n,\boldsymbol{u}_n} \cap \mathcal{L}(\mathfrak{C}_0)$. Since $\gamma > 0$, any $l \in L$ intersects A twice, the lengths of both segments are in $O(1)$ and the distance between them is in $\Omega(r)$. Points in P are separated by a distance of at least r^{a_M} with $a_M > 0$, so $l \in L$ implies that l can intersect at most one point in $P \backslash \{\boldsymbol{l}_n\}$. On the other hand, the area of $\{\boldsymbol{y} \in l \cap \partial B_{c_4^M r} : l \in L\}$ is in $O(|\mathfrak{C}_0|^{d-1}) = O(r^{d-1})$, therefore it contains at most $O(r^{d-1-a_M})$ many points from P. The distance between $\boldsymbol{l}_n$ and other points in P that could lie in Cyl(l) for $l \in L$ is in $\Omega(r)$, therefore due to (4.1), we have $\mu\big(L \cap \mathcal{L}(P \backslash \{\boldsymbol{l}_n\})\big) \in O(r^{-b_M})$.

We have thus proved that $\min_{n \in [N]} \mu(\mathcal{L}_n) \in \Omega(1)$, and Lemma 4.2 now implies that

$$-\log \mathbb{P}\left\{\sum_{n=1}^{N} I_n \le cN\right\} \in \Omega(N)$$

for some c. In particular, since $N \in \Omega(r^{b_M})$ and $b_M > \chi$,

$$-\log \mathbb{P}\left\{\left|\mathfrak{C}_0 \cap B^{\rho(\omega^-)}_{(c_2+c_4^M+1)r}(\boldsymbol{L}_M)\right| < r^\chi\right\} \ge -\log \mathbb{P}\Big\{\sum_{n=1}^{N} I_n < r^\chi\Big\} \in \Omega(r^\chi).$$

This finishes the proof. □

Proof of **Proposition 3.6**. Split $\omega \sim \mathbb{P}^u$ into $M+1$ independent Poisson cylinder processes of intensity $w = u/(M+1)$, that is, we couple ω with $(\omega_j)_{j=1}^{M+1} \overset{\text{iid}}{\sim} \mathbb{P}^w$ such that $\sum_{j=1}^{M+1} \omega_j = \omega$. Pick c_2, $(f_j)_{j=1}^M$ and build a system of layers $(\boldsymbol{L}_j)_{j=1}^M$ recursively as in Lemmas 3.7, 3.8 and 3.9 with $\mathbb{P} = \mathbb{P}^w$. More specifically, define a sequence of events:

$$E_j = E_j(\boldsymbol{L}_{j-1}) := \{\exists \boldsymbol{L}_j \in \mathfrak{L}_j(\boldsymbol{L}_{j-1}) : |\boldsymbol{L}_j| > f_j r^{b_j}\}, \quad j \in [M],$$

where for each $j \in [M]$, on event $E_j(\boldsymbol{L}_{j-1})$, we pick any suitable layer $\boldsymbol{L}_j \in \mathfrak{L}_j(\boldsymbol{L}_{j-1})$ with more than $f_j r^{b_j}$ points. If $\boldsymbol{L}_M$ is well-defined, that is, on the event $\cap_{j=1}^M E_j$, let

$$E_{M+1} = E_{M+1}(\boldsymbol{L}_M) := \Big\{\left|\mathfrak{C}_0 \cap B^{\rho(\omega^-)}_{(c_2+c_4^M+1)r}(\boldsymbol{L}_M)\right| < r^\chi\Big\}.$$

Let $c_3 = 2(M+1)c_4^M + c_2 + 1$ and note that

$$\bigcap_{j=1}^{M+1} E_{M+1} \subseteq \Big\{|\mathfrak{C}_0 \cap B^{\rho(\omega^- + \delta_{l_0})}_{c_3 r}| \ge r^\chi\Big\}.$$

We finish the proof by combining Lemmas 3.7, 3.8 and 3.9 together with the fact that $\chi > 1$ to get:

$$-\log \mathbb{P}^u\big\{|\mathfrak{C}_0 \cap B^{\rho(\omega^- + \delta_{l_0})}_{c_3 r}| < r^\chi\big\} \ge -\sum_{j=1}^{M+1} \log \mathbb{P}^w\left[E_j^c \Big| \bigcap_{k=1}^{j-1} E_k\right] \in \Omega(r),$$

where an intersection over an empty set of index equals $\mathcal{M}$. □

Proof of ***Lemma 3.5***. As $r \uparrow \infty$, the ratio of the length of the prospective highway to the size of the patches also goes to infinity, thus for r big enough, if a cylinder intersects a point in $\mathfrak{C}_0'$ and another in $\mathfrak{C}_x'$, those are the only two points in $\mathfrak{C}_0$ and $\mathfrak{C}_x$ that are intersected by that cylinder. Also, if a cylinder intersects $\mathbf{0}$ or $\boldsymbol{x}$ and one of the points in

$\mathfrak{C}'_0$, then there is at most one point in $\mathfrak{C}'_x$ that could be intersected by the same cylinder. Formally, if $l \in \mathcal{L}\{\mathbf{0}\} \cap \mathcal{L}(\mathfrak{C}'_0)$, $\boldsymbol{a} := \mathrm{Cyl}(l) \cap \mathfrak{C}'_0$ and $b_0(\boldsymbol{a}) := \mathrm{Cyl}(l) \cap \mathfrak{C}'_x$, then $b_0(\boldsymbol{a})$ contains at most 1 point for r big enough. Similarly, if $l \in \mathcal{L}\{\boldsymbol{x}\} \cap \mathcal{L}(\mathfrak{C}'_0)$, $\boldsymbol{a} := \mathrm{Cyl}(l) \cap \mathfrak{C}'_0$ and $b_x(\boldsymbol{a}) := \mathrm{Cyl}(l) \cap \mathfrak{C}'_x$, then $\left|b_x(\boldsymbol{a})\right| \le 1$.

Finally, we use (4.1) to obtain the statement of this lemma:

$$\begin{aligned} -\log \mathbb{P}\Big\{\omega^-\big(\mathcal{L}(\mathfrak{C}'_0) \cap \mathcal{L}(\mathfrak{C}'_x)\big) = 0\Big\} &= -\sum_{\boldsymbol{a} \in \mathfrak{C}'_0} \sum_{\boldsymbol{b} \in \mathfrak{C}'_x \setminus \{b_0(\boldsymbol{a}), b_x(\boldsymbol{a})\}} \log \mathbb{P}\Big\{\omega\big(\mathcal{L}\{\boldsymbol{a},\boldsymbol{b}\}\big) = 0\Big\} \\ &\in \Omega\Big(|\mathfrak{C}'_0|\,(|\mathfrak{C}'_x| - 2)\, R^{1-d}\Big) = \Omega(r^{2\chi}/R^{d-1}). \end{aligned}$$

This finishes the proof. □

Proof of ***Proposition 3.4***. Pick any $l_0 \in \mathcal{L}\{\mathbf{0}\}$ and $l_x \in \mathcal{L}\{\boldsymbol{x}\}$. Divide $\omega \sim \mathbb{P}^u$ into 3 i.i.d. Poisson cylinder processes of intensity $w := u/3$. We pick some $\epsilon > 0$ so that for $\chi(\epsilon) := (d-1)(1-\epsilon)$, we have $2a\chi - (d-1) > \delta$. Note that this is possible because $\delta \in (0, 1/2)$ is less than $(2a-1)(d-1)$.

Pick c_3 as in Proposition 3.6 with $\mathbb{P} = \mathbb{P}^w$. Let

$$\mathfrak{C}'_0 := \mathfrak{C}_0 \cap B^{\rho(\omega^-)}_{c_3 r}, \qquad \mathfrak{C}'_x := \mathfrak{C}_x \cap B^{\rho(\omega^-)}_{c_3 r}(\boldsymbol{x}).$$

Let $F_i := \big\{|\mathfrak{C}'_i| < r^\chi\big\}$ for $i \in \{1,2\}$. Thanks to Proposition 3.6, $cr^{1\wedge\chi} < -\log \mathbb{P}^w(F_i)$ for some c. We can similarly apply Lemma 3.5:

$$-\log \mathbb{P}^w[F | F_1^c \cap F_2^c] > c' R^{2a\chi-(d-1)}, \qquad F := \big\{\omega^-\big(\mathcal{L}(\mathfrak{C}'_0) \cap \mathcal{L}(\mathfrak{C}'_x)\big) = 0\big\}$$

for some $c' > 0$. Both c and c' can be chosen independently of l_0 and l_x. If we let $c_1 := 2(c_3+1)$, we will have $E_{l_1,l_2} \subset F_1 \cup F_2 \cup F$ and $-\log \mathbb{P}^u(E_{l_1,l_2}) > c'' R^\delta$ for some $c'' = c''(c,c')$ which does not depend on $l_0 \in \mathcal{L}\{\mathbf{0}\}$ or $l_x \in \mathcal{L}\{\boldsymbol{x}\}$. This finishes the proof. □

Proof of ***Proposition 3.2***. Take c_1 as in the Proposition 3.4. Note that given $l_0 \in \mathcal{L}\{\mathbf{0}\}$ and $l_x \in \mathcal{L}\{\boldsymbol{x}\}$, E_{l_1,l_2} is independent of $\omega(\mathcal{L}\{\mathbf{0}\})$ and $\omega\big(\mathcal{L}\{\boldsymbol{x}\}\big)$. Apply Proposition 3.4:

$$\begin{aligned} &\mathbb{P}\big[\rho(\mathbf{0},\boldsymbol{x}) > R + c_1 r \big| \mathbf{0}, \boldsymbol{x} \in \mathcal{C}\big] \\ =\,&\mathbb{P}\Big[\bigcap\big\{E_{l_0,l_x} : l_0 \in \mathcal{L}\{\mathbf{0}\}, l_x \in \mathcal{L}\{\boldsymbol{x}\},\ l_0, l_x \in \omega\big\}\Big|\omega\big(\mathcal{L}\{\mathbf{0}\}\big) > 0, \omega\big(\mathcal{L}\{\boldsymbol{x}\}\big) > 0\Big] \\ \le\,& \sup_{l_0 \in \mathcal{L}\{\mathbf{0}\}, l_x \in \mathcal{L}\{\boldsymbol{x}\}} \mathbb{P}(E_{l_0,l_x}) \in O\big(\exp\{-R^\delta\}\big). \end{aligned}$$

This finishes the proof. □

We are now ready to prove the Shape Theorem.

Proof of ***Theorem 3.1***. Pick c_1 and δ as in Proposition 3.2. It suffices to prove the theorem for $c > 2c_1$. Define the following sequence of events:

$$A_n := \big\{\exists \boldsymbol{x} \in \mathcal{C} \cap B_n \cap (d^{-1/2}\,\mathbb{Z}^d) : \rho(\mathbf{0},\boldsymbol{x}) > n + c_1 n^a\big\}, \qquad n \ge 1.$$

The conditional probabilities of these events under $\mathbb{P}[\cdot|\mathbf{0} \in \mathcal{C}]$ can be bounded above:

$$\mathbb{P}[A_n | \mathbf{0} \in \mathcal{C}] \le \sum_{\boldsymbol{x} \in B_n \cap (d^{-1/2}\,\mathbb{Z}^d)} \mathbb{P}\big[\rho(\mathbf{0},\boldsymbol{x}) > n + c_1 n^a \big| \mathbf{0}, \boldsymbol{x} \in \mathcal{C}\big]\, \mathbb{P}\{\boldsymbol{x} \in \mathcal{C}\}.$$

We now prove that the sum of these probabilities in n is finite. Let

$$p_n(\boldsymbol{y}) := \mathbb{P}\big[\rho(\mathbf{0},\boldsymbol{y}) > n + c_1 n^a \big| \mathbf{0}, \boldsymbol{y} \in \mathcal{C}\big], \qquad \boldsymbol{y} \in \mathbb{R}^d.$$

We now bound p_n from above uniformly on $B_n \cap (d^{-1/2}\,\mathbb{Z}^d)$. By Proposition 3.2, we know that $cn^\delta < -\log p_n(\boldsymbol{y})$ for some c and any $\boldsymbol{y} \in \partial B_n$. On the other hand, $p_n(\boldsymbol{y})$ increases in $|\boldsymbol{y}|$, because in order to connect $\mathbf{0}$ to a distant point by a polygonal path one must first connect it to points lying closer.

We have thus proved that $\mathbb{P}[A_n|\mathbf{0} \in \mathcal{C}] < cn^{d-1}\exp\{-n^\delta\}$ for some c. Thanks to the Borel-Cantelli lemma only finitely many of the events $(A_n)_{n\geq 1}$ happen $\mathbb{P}[\cdot|\mathbf{0} \in \mathcal{C}]$-almost surely. Note that if a cylinder intersects $\boldsymbol{x} \in \mathbb{R}^d$, it also intersects the point closest to $\boldsymbol{x}$ in $d^{-1/2}\,\mathbb{Z}^d$. That is, defining $\boldsymbol{y}(\boldsymbol{x}) := \operatorname{argmin}_{\boldsymbol{z}\in d^{-1/2}\,\mathbb{Z}^d}|\boldsymbol{x}-\boldsymbol{y}|$, we have $\boldsymbol{x} \in \mathcal{C} \iff \boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}) \in \mathcal{C}$. Finally, since $\rho(\boldsymbol{x},\boldsymbol{y}) \leq d^{-1/2}$, if $\rho(\mathbf{0},\boldsymbol{y}) \leq n + c_1 n^a$ and $c_1(R_0-1)^a > d^{-1/2}$,

$$\rho(\mathbf{0},\boldsymbol{x}) \leq n + c_1 n^a + d^{-1/2} < n + cn^a.$$

This finishes the proof. □

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