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Inverting the coupling of the signed Gaussian free field with a loop-soup*

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Abstract

Lupu introduced a coupling between a random walk loop-soup and a Gaussian free field, where the sign of the field is constant on each cluster of loops. This coupling is a signed version of isomorphism theorems relating the square of the GFF to the occupation field of Markovian trajectories. His construction starts with a loop-soup, and by adding additional randomness samples a GFF out of it. In this article we provide the inverse construction: starting from a signed free field and using a self-interacting random walk related to this field, we construct a random walk loop-soup. Our construction relies on the previous work by Sabot and Tarrès, which inverts the coupling from the square of the GFF rather than the signed GFF itself.

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1 Introduction

The so called “isomorphism theorems” relate the square of a Gaussian free field (GFF) on an electrical network to occupation times of symmetric Markov jump processes [19, 24]. These date back to the work of Dynkin (Dynkin’s isomorphism) [6, 5, 7], and

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previously to Symanzik's identities in Euclidean Quantum Field Theory [22] and Brydges-Fröhlich-Spencer random walk representation of spin systems [2]. Here we focus on the generalized second Ray-Knight theorem [9] and on Le Jan's isomorphism [13]. The generalized second Ray-Knight theorem couples the squares of two GFFs with different, ordered, boundary conditions by adding the occupation times of independent Markovian excursions from boundary to boundary to the square with lower boundary conditions in order to obtain the square with higher boundary conditions. Le Jan's isomorphism states that the whole square of a GFF can be obtained as the occupation field of a Poisson Point Process of Markovian loops, known as loop-soup. Le Jan's isomorphism in particular implies the generalized second Ray-Knight theorem.

In [15] Lupu obtained "signed" or "polarized" versions of isomorphism theorems, where one relates to the Markovian trajectories not only the square, but also the sign of the GFF. In particular the sign of the GFF is constant on each Markovian trajectory. The construction goes through the introduction of the metric graph GFF. One first replaces each discrete edge of the electrical network by a continuous line, so as to obtain a continuous topological object, a one-dimensional simplicial complex known as metric graph or cable system, and then one interpolates the values of the GFF on the vertices by independent Brownian bridges inside the edges. This way one obtains a continuous Gaussian field. Its square can still be obtained as in Le Jan's isomorphism as an occupation field of a loop-soup of loops of the natural continuous diffusion on the metric graph. However, in this construction the sign components of the GFF are exactly the clusters of metric graph loops and the sign is chosen independently uniformly on each of them.

Lupu's isomorphism has also a purely discrete description. One enlarges the clusters of the discrete loop-soup by opening the edges not visited by loops with certain probability, and then on each enlarged cluster one chooses a sign independently uniformly.

The above couplings have also a counterpart in the Ising "world". Indeed, conditional on the absolute value of the GFF, its sign is distributed like Ising spins with coupling constants given by the absolute value. Then, the enlarged clusters in Lupu's isomorphism (discrete description), conditional on the absolute value of the GFF on the vertices, are exactly Fortuin-Kasteleyn random clusters [11], with cluster weight $q = 2$ and edge weights depending on the absolute value [18]. FK random clusters with $q = 2$ are coupled to the Ising spins through the Edwards-Sokal coupling [8], where one simply chooses the spin independently uniformly on each clusters. The discrete loop-soup in Le Jan's isomorphism is related to the random current expansion of the Ising model [18, 4, 12]. Finally [18] connected the dots and showed that there is a natural coupling between Ising random currents and FK-Ising random clusters, which is actually Lupu's isomorphism conditioned on the absolute value of the GFF. In Figure 1 we summarize all above models and the couplings and relations between them.

In this paper we deal with the inversion of Lupu's isomorphism, that is to say with retrieving the conditional law of the discrete loop-soup given a discrete Gaussian free field (both its absolute value and its sign). This extends the work of Sabot and Tarrès who in [21] gave the inversion of isomorphisms only in the case when the absolute value of the GFF was given, or equivalently its square, but not its sign. To fix the ideas, let us introduce some notations.

Let $\mathcal{G} = (V, E)$ be a connected undirected graph, with V at most countable and each vertex $x \in V$ of finite degree. We do not allow self-loops. Also in general we do not consider multiple edges, unless specified otherwise. Given $e \in E$ an edge, we will denote e_+ and e_- its end-vertices, even though e is non-oriented and one can interchange e_+ and e_-. Each edge $e \in E$ is endowed with a conductance $W_e = W_{e_-,e_+} = W_{e_+,e_-} > 0$. There may be a killing measure $\kappa = (\kappa_x)_{x\in V}$ on vertices.

We consider $(X_t)_{t\geq 0}$ the *Markov jump processes* on V which being in $x\in V$, jumps along an adjacent edge e with rate W_e. Moreover if $\kappa_x\neq 0$, the process is killed at x with rate κ_x (the process is not defined after that time). ζ will denote the time up to which X_t is defined. If $\zeta<+\infty$, then either the process has been killed by the killing measure κ (and $\kappa\not\equiv 0$) or it has gone off to infinity in finite time (and V infinite). We will assume that the process X is transient, which means, if V is finite, that $\kappa\not\equiv 0$. $\mathbb{P}_x$ will denote the law of X started from x. Let $(\mathbf{G}(x,y))_{x,y\in V}$ be the Green function of $(X_t)_{0\leq t<\zeta}$:

$$\mathbf{G}(x,y)=\mathbf{G}(y,x)=\mathbb{E}_x\left[\int_0^\zeta \mathbb{1}_{\{X_t=y\}}dt\right].$$

Let $\mathcal{E}$ be the Dirichlet form defined on functions f on V with finite support:

$$\mathcal{E}(f,f)=\sum_{x\in V}\kappa_x f(x)^2+\sum_{e\in E}W_e(f(e_+)-f(e_-))^2. \tag{1.1}$$

P_φ will be the law of $(\varphi_x)_{x\in V}$ the centred *Gaussian free field* (GFF) on V with covariance $E_\varphi[\varphi_x\varphi_y]=\mathbf{G}(x,y)$. In case V is finite, the density of P_φ is

$$\frac{1}{(2\pi)^{\frac{|V|}{2}}\sqrt{\det\mathbf{G}}}\exp\left(-\frac{1}{2}\mathcal{E}(f,f)\right)\prod_{x\in V}df_x.$$

φ under P_φ satisfies the *spatial Markov property*. If U is a subset of V and

$$\partial U=\{x\in U|\exists y\in V\setminus U,\ x \text{ and } y \text{ joined by an edge } e\in E\},$$

then $(\varphi_x)_{x\in V\setminus U}$ conditional on $(\varphi_y)_{y\in U}$ has same law as conditional on $(\varphi_y)_{y\in\partial U}$.

Given $x_0\in V$ and $a\in\mathbb{R}$, $P_\varphi^{\{x_0\},a}$ will denote the law of the GFF φ conditioned to equal a in x_0. Note that if the killing measure κ is supported in x_0, the law $P_\varphi^{\{x_0\},a}$ does not depend on κ and in this case one can as well take $\kappa=0$.

We will denote by $(\ell_x(t))_{x\in V,t\in[0,\zeta]}$ the family of local times of X:

$$\ell_x(t)=\int_0^t \mathbb{1}_{\{X_s=x\}}ds.$$

For all $x\in V$, $u>0$, let

$$\tau_u^x=\inf\{t\geq 0;\ell_x(t)>u\}.$$

Recall the generalized second Ray-Knight theorem on discrete graphs by Eisenbaum, Kaspi, Marcus, Rosen and Shi [9] (see also [19, 24]).

Theorem 1 (Generalized second Ray-Knight theorem). *For any $u>0$ and $x_0\in V$,*

$$\left(\ell_x(\tau_u^{x_0})+\frac{1}{2}\varphi_x^2\right)_{x\in V}\ \textit{under}\ \mathbb{P}_{x_0}(\cdot|\tau_u^{x_0}<\zeta)\otimes P_\varphi^{\{x_0\},0}$$

has the same law as

$$\left(\frac{1}{2}\varphi_x^2\right)_{x\in V}\ \textit{under}\ P_\varphi^{\{x_0\},\sqrt{2u}}.$$

Sabot and Tarrès showed in [21] that the so-called "magnetized" reverse Vertex-Reinforced Jump Process (VRJP) provides an inversion of the generalized second Ray-Knight theorem, in the sense that it enables to retrieve the law of $(\ell_x(\tau_u^{x_0}),\varphi_x^2)_{x\in V}$ conditional on $\left(\ell_x(\tau_u^{x_0})+\frac{1}{2}\varphi_x^2\right)_{x\in V}$. The jump rates of that latter process are a product of a first factor accountable for a self-repulsion (reverse VRJP) and a second one which can be interpreted as a ratio of two-point functions of the Ising model associated to time-evolving coupling constants (magnetization). The process introduced in [21] also inverts the Le Jan's isomorphism [13] given the square of a GFF.

However in [21] the link with the Ising model is only implicit, and a natural question is whether Ray-Knight inversion can be described in a simpler form if we enlarge the state space of the dynamics, and in particular include the "hidden" spin variables.

The answer is positive, and goes through a "signed" extension of the Ray-Knight isomorphism following Lupu's approach [15], which couples the sign of the GFF to the Markovian path. The Ray-Knight inversion will turn out to take a rather simple form in Theorem 9 of the present paper, where it will be defined not only through the spin variables but also random currents associated to the field though an extra Poisson Point Process. Further, in Theorem 11 we will describe how the process we construct inverts Lupu's "signed" isomorphism between a loop-soup and a discrete GFF.

The paper is organized as follows.

In Section 2 we recall some background on loop-soup isomorphisms and on related couplings and state and prove a signed version of the generalized second Ray-Knight theorem. We begin in Section 2.1 by a statement of Le Jan's isomorphism which couples the square of the Gaussian Free Field to the loop-soups, and recall how the generalized second Ray-Knight theorem can be seen as its corollary: for more details see [14]. In Section 2.2 we state Lupu's isomorphism which extends Le Jan's isomorphism and couples the sign of the GFF to the loop-soups, using a metric graph extension of both the GFF and the Markov process. Lupu's isomorphism yields an interesting realization of the Edwards-Sokal FK-Ising - spin Ising coupling, and provides as well a "Current+Bernoulli=FK" coupling lemma [18], which occur in the relationship between the discrete and metric graph versions. We briefly recall these couplings in Sections 2.3 and 2.4, as they are implicit in this paper. In Section 2.5 we state and prove the generalized second Ray-Knight "version" of Lupu's isomorphism, which we aim to invert. In Section 2.6 we present a diagram which summarizes the models and the couplings.

Section 3 is devoted to the statements of inversions of these isomorphisms. We state in Section 3.1 a signed version of the inversion of the generalized second Ray-Knight theorem through an extra Poisson Point Process, namely Theorem 9. In Section 3.2 we provide a discrete-time description of the process, whereas in Section 3.3 we yield an alternative version of that process through jump rates, which can be seen as an annealed version of the first one. The annealed process of Section 3.3 is a reverse VRJP (self-repelling) which evolves on a subgraph of $\mathcal{G}$ which itself shrinks over time. These subgraphs can be interpreted as FK-Ising random clusters associated to time-evolving, decreasing, edge weights. In Section 3.4 we deduce a signed inversion of Lupu's isomorphism for loop-soups.

Finally Section 4 is devoted to the proof of Theorem 9: Section 4.1 deals with the case of a finite graph without killing measure, and Section 4.2 deduces the proof in the general case.

2 Le Jan's and Lupu's isomorphisms

2.1 Loop-soups and Le Jan's isomorphism

The *loop measure* associated to the Markov jump process $(X_t)_{0\le t<\zeta}$ is defined as follows. Let $\mathbb{P}^t_{x,y}$ be the bridge probability measure from x to y in time t (conditional on $t<\zeta$). Let $p_t(x,y)$ be the transition probabilities of $(X_t)_{0\le t<\zeta}$.

Let μ_{loop} be the measure on time-parametrized nearest-neighbor based loops (i.e. loops with a starting site)

$$\mu_{\text{loop}} = \sum_{x \in V} \int_{t>0} \mathbb{P}^t_{x,x} p_t(x,x) \frac{dt}{t}.$$

The loops will be considered here up to a rotation of parametrisation (with the corresponding pushforward measure induced by μ_{loop}), that is to say a loop $(\gamma(t))_{0 \le t \le t_\gamma}$ will be the same as $(\gamma(T+t))_{0 \le t \le t_\gamma - T} \circ (\gamma(T+t-t_\gamma))_{t_\gamma - T \le t \le t_\gamma}$ for all $T \in (0, t_\gamma)$, where $\circ$ denotes the concatenation of paths. A *loop-soup* of intensity $\alpha > 0$, denoted $\mathcal{L}_\alpha$, is a Poisson random measure of intensity $\alpha \mu_{\text{loop}}$. We see it as a random collection of loops in $\mathcal{G}$. Observe that a.s. above each vertex $x \in V$, $\mathcal{L}_\alpha$ contains infinitely many trivial "loops" reduced to the vertex x. There are also with positive probability non-trivial loop that visit several vertices.

Let $L_.(\mathcal{L}_\alpha)$ be the *occupation field* of $\mathcal{L}_\alpha$ on V i.e., for all $x \in V$,

$$L_x(\mathcal{L}_\alpha) = \sum_{(\gamma(t))_{0 \le t \le t_\gamma} \in \mathcal{L}_\alpha} \int_0^{t_\gamma} \mathbb{1}_{\{\gamma(t) = x\}} dt.$$

In [13] Le Jan shows that for transient Markov jump processes, $L_x(\mathcal{L}_\alpha) < +\infty$ for all $x \in V$ a.s. For $\alpha = 1/2$ he identifies the law of $L_.(\mathcal{L}_\alpha)$:

Theorem 2 (Le Jan's isomorphism). *$L_.(\mathcal{L}_{1/2}) = \left(L_x(\mathcal{L}_{1/2})\right)_{x \in V}$ has the same law as $\frac{1}{2}\varphi^2 = \left(\frac{1}{2}\varphi_x^2\right)_{x \in V}$ under P_φ.*

Let us briefly recall how Le Jan's isomorphism enables one to retrieve the generalized second Ray-Knight theorem stated in Section 1: for more details, see for instance [14]. We assume that κ is supported by x_0: the general case can be dealt with by an argument similar to the proof of Proposition 4.6. Let $D = V \setminus \{x_0\}$, and note that the isomorphism in particular implies that $L_.(\mathcal{L}_{1/2})$ conditional on $L_{x_0}(\mathcal{L}_{1/2}) = u$ has the same law as $\varphi^2/2$ conditional on $\varphi_{x_0}^2/2 = u$.

On the one hand, given the classical energy decomposition, we have $\varphi = \varphi^D + \varphi_{x_0}$, with φ^D the GFF associated to the restriction of $\mathcal{E}$ to D, where φ^D and φ_{x_0} are independent. Now $\varphi^2/2$ conditional on $\varphi_{x_0}^2/2 = u$ has the law of $(\varphi^D + \eta\sqrt{2u})^2/2$, where η is the sign of φ_{x_0}, which is independent of φ^D. But φ^D is symmetric, so that the latter also has the law of $(\varphi^D + \sqrt{2u})^2/2$.

On the other hand, the loop-soup $\mathcal{L}_{1/2}$ can be decomposed into the two independent loop-soups $\mathcal{L}^D_{1/2}$ contained in D and $\mathcal{L}^{(x_0)}_{1/2}$ hitting x_0. Now $L_.(\mathcal{L}^D_{1/2})$ has the law of $(\varphi^D)^2/2$ and $L_.(\mathcal{L}^{(x_0)}_{1/2})$ conditional on $L_{x_0}(\mathcal{L}^{(x_0)}_{1/2}) = u$ has the law of the occupation field of the Markov chain $\ell(\tau_u^{x_0})$ under $\mathbb{P}_{x_0}(\cdot | \tau_u^{x_0} < \zeta)$, which enables us to conclude.

2.2 Lupu's isomorphism

As in [15], we consider the *metric graph* $\tilde{\mathcal{G}}$ (also known as *cable system*) associated to $\mathcal{G}$. As a topological space, it is obtained by replacing each discrete edge e by a continuous compact line interval I_e. If two edges e and e' share a common extremity, the corresponding endpoints of I_e and $I_{e'}$ are identified. So $\tilde{\mathcal{G}}$ is a one-dimensional simplicial complex, with 0-cells corresponding to vertices in V, and 1-cells to edges in E. V is naturally identified to a subset of $\tilde{\mathcal{G}}$. We further endow $\tilde{\mathcal{G}}$ with a metric by setting the length of each I_e to be equal to $\frac{1}{2}W_e^{-1}$. We also consider the Radon measure $\tilde{m}$ on $\tilde{\mathcal{G}}$, such that its restriction to each I_e is a one-dimensional Lebesgue measure of total mass $\frac{1}{2}W_e^{-1}$ (i.e. the length measure).

One can define a standard Brownian motion $B_t^{\tilde{\mathcal{G}}}$ on $\tilde{\mathcal{G}}$ as follows. Inside each I_e, $B_t^{\tilde{\mathcal{G}}}$ behaves as a standard one-dimensional Brownian motion. Upon reaching a vertex, $B_t^{\tilde{\mathcal{G}}}$

performs immediately independent Brownian excursions inside each adjacent edge, with equal rate for each direction, and that until eventually traversing one of the edge and reaching a neighbor vertex on the other side. See Section 2 of [15] for more details. Alternatively, $B_t^{\tilde{\mathcal{G}}}$ can be defined as the symmetric Markov process associated to the Dirichlet form

$$\tilde{\mathcal{E}}(f,f) = \frac{1}{2}\int_{x\in\tilde{\mathcal{G}}} f'(x)^2 d\tilde{m}(x),$$

where $f : \tilde{\mathcal{G}} \to \mathbb{R}$ is continuous and $\mathcal{C}^1$ inside each of the I_e. For more on Dirichlet forms and associated Markov processes, see [10]. $B_t^{\tilde{\mathcal{G}}}$ admits a family of Brownian local times $\tilde{\ell}_x(t)$, continuous in (x,t), such that for every bounded measurable function $f : \tilde{\mathcal{G}} \to \mathbb{R}$ and every $t \geq 0$,

$$\int_0^t f(B_s^{\tilde{\mathcal{G}}})ds = \int_{x\in\tilde{\mathcal{G}}} f(x)\tilde{\ell}_x(t)d\tilde{m}(x).$$

Let be $\tilde{\zeta}$ the first time when either $B_t^{\tilde{\mathcal{G}}}$ explodes to infinity (if possible), or

$$\sum_{x\in V} \kappa_x \tilde{\ell}_x(t)$$

hits an independent exponential r.v. of mean 1 (if $\kappa \not\equiv 0$). If neither of these happens, $\tilde{\zeta} = +\infty$.

One can deterministically recover the Markov jump process $(X_t)_{0\leq t<\zeta}$ out of $(B_t^{\tilde{\mathcal{G}}})_{0\leq t<\tilde{\zeta}}$ Let be

$$A_V(t) = \sum_{x\in V} \tilde{\ell}_x(t), \qquad A_V^{-1}(t) = \inf\{s \geq 0 | A_V(s) > t\}.$$

Then, if $B_0^{\tilde{\mathcal{G}}} \in V$, $B^{\tilde{\mathcal{G}}}_{A_V^{-1}(t)}$ is a Markov jump process X_t on V with jump rates $(W_e)_{e\in E}$. Moreover, $A_V(\tilde{\zeta}) = \zeta$ and $\ell_x(t) = \tilde{\ell}_x(A_V^{-1}(t))$ for $x \in V$.

In [15] Lupu introduces a measure $\tilde{\mu}_{\text{loop}}$ on time-parametrized continuous loops on $\tilde{\mathcal{G}}$, associated to the Brownian motion $(B_t^{\tilde{\mathcal{G}}})_{0\leq t<\tilde{\zeta}}$. $\tilde{\mathcal{L}}_\alpha$ will denote the Poisson Point Process of loops of intensity $\alpha\tilde{\mu}_{\text{loop}}$. The discrete-space loops $\mathcal{L}_\alpha$ can be deterministically obtained from $\tilde{\mathcal{L}}_\alpha$ by taking the print of the latter on V, using the time-change A_V^{-1}. Note that $\tilde{\mathcal{L}}_\alpha$ contains loops that do not visit V and are entirely contained in one of the I_e. These do not contribute to $\mathcal{L}_\alpha$. $\tilde{\mathcal{L}}_\alpha$ has an occupation field $(L_x(\tilde{\mathcal{L}}_\alpha))_{x\in\tilde{\mathcal{G}}}$, which is a sum over loops in $\tilde{\mathcal{L}}_\alpha$ of Brownian local times in x of the loops. Moreover,

$$L_x(\tilde{\mathcal{L}}_\alpha) = L_x(\mathcal{L}_\alpha),\ \forall x \in V.$$

Similarly, the GFF φ on $\mathcal{G}$ with law P_φ can be extended to a GFF $\tilde{\varphi}$ on $\tilde{\mathcal{G}}$ as follows. Given $e \in E$, one considers inside I_e a conditionally independent Brownian bridge, actually a bridge of a $\sqrt{2}\times$ *standard Brownian motion*, of length $\frac{1}{2}W_e^{-1}$, with end-values φ_{e_-} and φ_{e_+}. This provides a continuous Gaussian field on the metric graph which still satisfies the spatial Markov property.

Lupu introduced in [15] an isomorphism linking the GFF $\tilde{\varphi}$ and the loop-soup $\tilde{\mathcal{L}}_{1/2}$ on the metric graph $\tilde{\mathcal{G}}$. This one uses the clusters of $\tilde{\mathcal{L}}_{1/2}$. First of all, *cluster* is an equivalence class of loops in $\tilde{\mathcal{L}}_{1/2}$, where two loops $\tilde{\gamma}$ and $\tilde{\gamma}'$ belong to the same cluster if there is a finite chain $\tilde{\gamma}_0, \dots, \tilde{\gamma}_k$ of loops in $\tilde{\mathcal{L}}_{1/2}$ such that $\tilde{\gamma}_0 = \gamma$, $\tilde{\gamma}_k = \gamma'$, and for $i \in \{1, \dots, k\}$ the loops $\tilde{\gamma}_{i-1}$ and $\tilde{\gamma}_i$ (their ranges) have a non-empty intersection. By extension, a cluster will also be a (connected) subset of $\tilde{\mathcal{G}}$ obtained as a union over a cluster of loops of ranges of the loops. Note that if $\mathcal{L}_{1/2}$ is obtained as a print of $\tilde{\mathcal{L}}_{1/2}$ on V, each cluster of $\mathcal{L}_{1/2}$ is contained in a cluster of $\tilde{\mathcal{L}}_{1/2}$, but in general a cluster $\tilde{\mathcal{L}}_{1/2}$ may correspond to several different clusters of $\mathcal{L}_{1/2}$ merged together. This is because there are more connections at the level of the metric graph.

Theorem 3 (**Lupu's isomorphism**). *There is a coupling between the Poisson ensemble of loops $\tilde{\mathcal{L}}_{1/2}$ and $(\tilde{\varphi}_y)_{y\in\tilde{\mathcal{G}}}$ defined above, such that the two following constraints hold:*

- *For all $y \in \tilde{\mathcal{G}}$, $L_y(\tilde{\mathcal{L}}_{1/2}) = \frac{1}{2}\tilde{\varphi}_y^2$;*
- *the zero set of $\tilde{\varphi}$, $\{x \in \tilde{\mathcal{G}} | \tilde{\varphi}_x = 0\}$, is the set of points in $\tilde{\mathcal{G}}$ not visited by any loop of $\tilde{\mathcal{L}}_{1/2}$;*
- *the clusters of loops of $\tilde{\mathcal{L}}_{1/2}$ are exactly the sign clusters of $(\tilde{\varphi}_y)_{y\in\tilde{\mathcal{G}}}$.*

Conditional on $(|\tilde{\varphi}_y|)_{y\in\tilde{\mathcal{G}}}$, the sign of $\tilde{\varphi}$ on each of its connected components is distributed independently and uniformly in $\{-1, +1\}$.

Lupu's isomorphism and the idea of using metric graphs were applied in [16] to show that on the discrete half-plane $\mathbb{Z} \times \mathbb{N}$, the scaling limits of outermost boundaries of clusters of loops in loop-soups are the Conformal Loop Ensembles CLE.

Let $\mathcal{O}(\tilde{\varphi})$ (resp. $\mathcal{O}(\tilde{\mathcal{L}}_{1/2})$) be the set of edges $e \in E$ such that $\tilde{\varphi}$ (resp. $\tilde{\mathcal{L}}_{1/2}$) does not touch 0 on I_e, in other words such that all the edge-interval I_e remains in the same sign cluster of $\tilde{\varphi}$ (resp. loop cluster of $\tilde{\mathcal{L}}_{1/2}$). $\mathcal{O}(\tilde{\varphi})$ and $\mathcal{O}(\tilde{\mathcal{L}}_{1/2})$ have same law. Let $\mathcal{O}(\mathcal{L}_{1/2})$ be the set of edges $e \in E$ that are crossed (i.e. endpoints visited consecutively) by loops in $\mathcal{L}_{1/2}$ (obtained as the print on V of $\tilde{\mathcal{L}}_{1/2}$).

In order to translate Lupu's isomorphism back onto the initial graph $\mathcal{G}$, one needs to describe on one hand the distribution of $\mathcal{O}(\tilde{\varphi})$ conditional on the values of $(\varphi_x)_{x\in V}$, and on the other hand the distribution of $\mathcal{O}(\tilde{\mathcal{L}}_{1/2})$ conditional on $\mathcal{L}_{1/2}$ and the cluster of loops $\mathcal{O}(\mathcal{L}_{1/2})$ on the discrete graph $\mathcal{G}$. These two distributions are described respectively in Sections 2.3 and 2.4, and provide realisations of the Edwards-Sokal FK-Ising - spin Ising coupling and of the "Current+Bernoulli=FK" coupling lemma [18].

2.3 The FK-Ising distribution of $\mathcal{O}(\tilde{\varphi})$ conditional on $|\varphi|$

Lemma 2.1. *Conditional on $(\varphi_x)_{x\in V}$, $(\mathbb{1}_{\{e\in\mathcal{O}(\tilde{\varphi})\}})_{e\in E}$ is a family of independent random variables and*

$$\mathbb{P}\left(e \notin \mathcal{O}(\tilde{\varphi}) | \varphi\right) = \begin{cases} 1 & \text{if } \varphi_{e_-}\varphi_{e_+} < 0, \\ \exp\left(-2W_e\varphi_{e_-}\varphi_{e_+}\right) & \text{if } \varphi_{e_-}\varphi_{e_+} > 0. \end{cases}$$

Proof. Conditional on $(\varphi_x)_{x\in V}$, the metric graph GFF $(\tilde{\varphi}_y)_{y\in\tilde{\mathcal{G}}}$ is constructed by adding independent Brownian bridges on each edge, so that $(\mathbb{1}_{\{e\in\mathcal{O}(\tilde{\varphi})\}})_{e\in E}$ are conditionally independent random variables, and it follows from the reflection principle for one-dimensional Brownian motion that, if $\varphi_{e_-}\varphi_{e_+} > 0$, then

$$\mathbb{P}\left(e \notin \mathcal{O}(\tilde{\varphi}) | \varphi, \varphi_{e_-}\varphi_{e_+} > 0\right) = \frac{\exp\left(-\frac{1}{2}W_e(\varphi_{e_-} + \varphi_{e_+})^2\right)}{\exp\left(-\frac{1}{2}W_e(\varphi_{e_-} - \varphi_{e_+})^2\right)} = \exp\left(-2W_e\varphi_{e_-}\varphi_{e_+}\right). \qquad \square$$

Let us now recall how the conditional probability in Lemma 2.1 yields a realization of the FK-Ising coupling.

Assume V is finite. Let $(J_e)_{e\in E}$ be a family of positive weights. A *(spin) Ising model* on V with interaction constants $(J_e)_{e\in E}$ is a probability on configuration of spins $(\sigma_x)_{x\in V} \in \{+1, -1\}^V$ such that

$$\mathbb{P}_J^{\mathrm{Isg}}((\sigma_x)_{x\in V}) = \frac{1}{\mathcal{Z}_J^{\mathrm{Isg}}} \exp\left(\sum_{e\in E} J_e \sigma_{e_-}\sigma_{e_+}\right).$$

An *FK-Ising random cluster model* [11] with edge weights $(1 - e^{-2J_e})_{e\in E}$ is a random configuration of open (value 1) and closed edges (value 0) such that

$$\mathbb{P}_J^{\mathrm{FK-Isg}}((\omega_e)_{e\in E}) = \frac{1}{\mathcal{Z}_J^{\mathrm{FK-Isg}}} 2^{\#\text{clusters}} \prod_{e\in E} (1 - e^{-2J_e})^{\omega_e} (e^{-2J_e})^{1-\omega_e},$$

where "#clusters" denotes the number of clusters created by open edges.

The Edwards-Sokal [8] coupling between FK-Ising and spin Ising reads as follows:

Theorem 4 (Edwards-Sokal coupling). *Given an FK-Ising model, sample on each cluster an independent uniformly distributed spin. The spins are then distributed according to the Ising model. Conversely, given a spin configuration $\hat{\sigma}$ following the Ising distribution, consider each edge e, such that $\hat{\sigma}_{e_-}\hat{\sigma}_{e_+} < 0$, closed, and each edge e, such that $\hat{\sigma}_{e_-}\hat{\sigma}_{e_+} > 0$ open with probability $1 - e^{-2J_e}$. Then the open edges are distributed according to the FK-Ising model. The two couplings between FK-Ising and spin Ising are the same.*

Consider the GFF φ on $\mathcal{G}$ distributed according to P_φ. Let $J_e(|\varphi|)$ be the random interaction constants

$$J_e(|\varphi|) = W_e|\varphi_{e_-}\varphi_{e_+}|.$$

Conditional on $|\varphi|$, $(\mathrm{sign}(\varphi_x))_{x\in V}$ follows an Ising distribution with interaction constants $(J_e(|\varphi|))_{e\in E}$: indeed, the Dirichlet form (1.1) can be written as

$$\mathcal{E}(\varphi,\varphi) = \sum_{x\in V}\kappa_x\varphi_x^2 + \sum_{x\in V}W_x\varphi_x^2 - 2\sum_{e\in E}J_e(|\varphi|)\,\mathrm{sign}(\varphi_{e_+})\,\mathrm{sign}(\varphi_{e_-}), \tag{2.1}$$

where

$$W_x = \sum_{\substack{y\in V\\ y\sim x}} W_{x,y},$$

$y \sim x$ meaning that x and y are joined by an edge. Similarly, when φ distributed according to $P_\varphi^{\{x_0\},\sqrt{2u}}$ has boundary condition $\sqrt{2u} \geq 0$ on x_0, then $(\mathrm{sign}(\varphi_x))_{x\in V}$ has an Ising distribution with interaction $(J_e(|\varphi|))_{e\in E}$ and conditioned on $\sigma_{x_0} = +1$.

Now, conditional on φ, $\mathcal{O}(\tilde{\varphi})$ has FK-Ising distribution with weights $(1 - e^{-2J_e(|\varphi|)})_{e\in E}$. Indeed, the probability for $e \in \mathcal{O}(\tilde{\varphi})$ conditional on φ is $1 - e^{-2J_e(|\varphi|)}$, by Lemma 2.1, as in Theorem 4.

Note that, given that $\mathcal{O}(\tilde{\varphi})$ has FK-Ising distribution, the fact that the sign on its connected components is distributed independently and uniformly in $\{-1, 1\}$ can be seen either as a consequence of Theorem 4, or from Theorem 3.

Given $\varphi = (\varphi_x)_{x\in V}$ on the discrete graph $\mathcal{G}$, we introduce in Definition 2.1 the random set of edges which has the distribution of $\mathcal{O}(\tilde{\varphi})$ conditional on $\varphi = (\varphi_x)_{x\in V}$.

Definition 2.1. *We let $\mathcal{O}(\varphi)$ be a random set of edges which has the distribution of $\mathcal{O}(\tilde{\varphi})$ conditional on $\varphi = (\varphi_x)_{x\in V}$ given by Lemma 2.1.*

2.4 Distribution of $\mathcal{O}(\tilde{\mathcal{L}}_{1/2})$ conditional on $\mathcal{L}_{1/2}$

The distribution of $\mathcal{O}(\tilde{\mathcal{L}}_{1/2})$ conditional on $\mathcal{L}_{1/2}$ can be retrieved by Corollary 3.6 in [15], which reads as follows.

Lemma 2.2 (Corollary 3.6 in [15])**.** *Conditional on $\mathcal{L}_{1/2}$, the events $e \notin \mathcal{O}(\tilde{\mathcal{L}}_{1/2})$, for $e \in E \setminus \mathcal{O}(\mathcal{L}_{1/2})$, are independent and have probability*

$$\exp\left(-2W_e\sqrt{L_{e_+}(\mathcal{L}_{1/2})L_{e_-}(\mathcal{L}_{1/2})}\right). \tag{2.2}$$

This result gives rise, together with Theorem 3, to the following discrete version of Lupu's isomorphism, which is stated without any recourse to the metric graph induced by $\mathcal{G}$.

Definition 2.2. *Let $(\omega_e)_{e\in E} \in \{0,1\}^E$ be a percolation defined as follows: conditional on $\mathcal{L}_{1/2}$, the random variables $(\omega_e)_{e\in E}$ are independent, and ω_e equals 0 with conditional probability given by* (2.2).

Let $\mathcal{O}_+(\mathcal{L}_{1/2})$ be the set of edges:

$$\mathcal{O}_+(\mathcal{L}_{1/2}) = \mathcal{O}(\mathcal{L}_{1/2}) \cup \{e \in E | \omega_e = 1\}.$$

Theorem 5 (**Discrete version of Lupu's isomorphism**, Theorem 1 bis in [15])**.** *Given a loop-soup $\mathcal{L}_{1/2}$, let $\mathcal{O}_+(\mathcal{L}_{1/2})$ be as in Definition 2.2. Let $(\sigma_x)_{x\in V} \in \{-1,+1\}^V$ be random spins taking constant values on clusters induced by $\mathcal{O}_+(\mathcal{L}_{1/2})$ ($\sigma_{e_-} = \sigma_{e_+}$ if $e \in \mathcal{O}_+(\mathcal{L}_{1/2})$) and such that the values on each cluster, conditional on $\mathcal{L}_{1/2}$ and $\mathcal{O}_+(\mathcal{L}_{1/2})$, are independent and uniformly distributed. Then*

$$\left(\sigma_x \sqrt{2L_x(\mathcal{L}_{1/2})}\right)_{x\in V}$$

is a Gaussian free field distributed according to P_φ.

Theorem 5 induces the following coupling between FK-Ising and random currents.

If V is finite, a *random current model* [1, 4] on $\mathcal{G}$ with weights $(J_e)_{e\in E}$ is a random assignment to each edge e of a non-negative integer $\hat{n}_e$ such that for all $x \in V$,

$$\sum_{\substack{e\in E \\ e \text{ adjacent to } x}} \hat{n}_e$$

is even, which is called the *parity condition*. The probability of a configuration $(n_e)_{e\in E}$ satisfying the parity condition is

$$\mathbb{P}_J^{\mathrm{RC}}(\forall e \in E, \hat{n}_e = n_e) = \frac{1}{\mathcal{Z}_J^{\mathrm{RC}}} \prod_{e\in E} \frac{(J_e)^{n_e}}{n_e!},$$

where actually $\mathcal{Z}_J^{\mathrm{RC}} = \mathcal{Z}_J^{\mathrm{Isg}}$. Let

$$\mathcal{O}(\hat{n}) = \{e \in E | \hat{n}_e > 0\}.$$

The open edges in $\mathcal{O}(\hat{n})$ induce clusters on the graph $\mathcal{G}$.

Given a loop-soup $\mathcal{L}_\alpha$, we denote by $\mathsf{N}_e(\mathcal{L}_\alpha)$ the number of times the loops in $\mathcal{L}_\alpha$ cross the nonoriented edge $e \in E$. The transience of the Markov jump process X implies that $\mathsf{N}_e(\mathcal{L}_\alpha)$ is a.s. finite for all $e \in E$. If $\alpha = 1/2$, we have the following identity (see for instance [25, 12]):

Theorem 6 (**Loop-soup and random current**)**.** *Assume V is finite and consider the loop-soup $\mathcal{L}_{1/2}$. Conditional on the occupation field $(L_x(\mathcal{L}_{1/2}))_{x\in V}$, $(\mathsf{N}_e(\mathcal{L}_{1/2}))_{e\in E}$ is distributed as a random current with weights $\left(2W_e\sqrt{L_{e_-}(\mathcal{L}_{1/2})L_{e_+}(\mathcal{L}_{1/2})}\right)_{e\in E}$. If φ is the GFF on $\mathcal{G}$ given by Le Jan's or Lupu's isomorphism, then these weights are $(J_e(|\varphi|))_{e\in E}$.*

Conditional on the occupation field $(L_x(\mathcal{L}_{1/2}))_{x\in V}$, $\mathcal{O}(\mathcal{L}_{1/2})$ are the edges occupied by a random current and $\mathcal{O}_+(\mathcal{L}_{1/2})$ the edges occupied by FK-Ising. Lemma 2.1 and Theorem 5 imply the following coupling, as noted by Lupu and Werner in [18].

Theorem 7 (**Random current and FK-Ising coupling**)**.** *Assume V is finite. Let $\hat{n}$ be a random current on $\mathcal{G}$ with weights $(J_e)_{e\in E}$. Let $(\omega_e)_{e\in E} \in \{0,1\}^E$ be an independent percolation, each edge being opened (value 1) independently with probability $1 - e^{-J_e}$. Then*

$$\mathcal{O}(\hat{n}) \cup \{e \in E | \omega_e = 1\}$$

is distributed like the open edges in an FK-Ising with weights $(1 - e^{-2J_e})_{e\in E}$.

2.5 Generalized second Ray-Knight "version" of Lupu's isomorphism

We are now in a position to state the coupled version of the second Ray-Knight theorem.

Theorem 8. *Let $x_0 \in V$. Let be $(\varphi^{(0)}_x)_{x\in V}$ with distribution $P^{\{x_0\},0}_\varphi$, and define $\mathcal{O}(\varphi^{(0)})$ as in Definition 2.1. Let X be an independent Markov jump process started from x_0.*

Fix $u > 0$. If $\tau^{x_0}_u < \zeta$, we let $\mathcal{O}_u$ be the random subset of E which contains $\mathcal{O}(\varphi^{(0)})$, the edges used by the path $(X_t)_{0\le t\le \tau^{x_0}_u}$, and additional edges e opened conditionally independently with probability

$$1 - e^{W_e|\varphi^{(0)}_{e_-}\varphi^{(0)}_{e_+}| - W_e\sqrt{(\varphi^{(0)2}_{e_-}+2\ell_{e_-}(\tau^{x_0}_u))(\varphi^{(0)2}_{e_+}+2\ell_{e_+}(\tau^{x_0}_u))}}.$$

We let $\sigma \in \{-1,+1\}^V$ be random spins sampled uniformly independently on each cluster induced by $\mathcal{O}_u$, pinned at x_0, i.e. $\sigma_{x_0} = 1$, and define

$$\varphi^{(u)}_x := \sigma_x\sqrt{\varphi^{(0)2}_x + 2\ell_x(\tau^{x_0}_u)}.$$

Then, conditional on $\tau^{x_0}_u < \zeta$, $\varphi^{(u)}$ has distribution $P^{\{x_0\},\sqrt{2u}}_\varphi$, and $\mathcal{O}_u$ has distribution $\mathcal{O}(\varphi^{(u)})$ conditional on $\varphi^{(u)}$.

Remark 2.3. *One consequence of that coupling is that the path $(X_s)_{s\le\tau^{x_0}_u}$ stays in the positive connected component of x_0 for $\varphi^{(u)}$. This yields a coupling between the range of the Markov chain and the sign component of x_0 inside a GFF $P^{\{x_0\},\sqrt{2u}}_\varphi$.*

Proof of Theorem 8. The proof is based on [15]. Let $D = V \setminus \{x_0\}$, and let $\tilde{\mathcal{L}}_{1/2}$ be the loop-soup of intensity $1/2$ on the metric graph $\tilde{\mathcal{G}}$, which we decompose into $\tilde{\mathcal{L}}^{(x_0)}_{1/2}$ (resp. $\tilde{\mathcal{L}}^D_{1/2}$) the loop-soup hitting (resp. not hitting) x_0, which are independent. We let $\mathcal{L}_{1/2}$ and $\mathcal{L}^{(x_0)}_{1/2}$ (resp. $\mathcal{L}^D_{1/2}$) be the prints of these loop-soups on V (resp. on $D = V \setminus \{x_0\}$). We condition on $L_{x_0}(\mathcal{L}_{1/2}) = u$.

Theorem 3 implies (recall also Definition 2.1) that we can couple $\tilde{\mathcal{L}}^D_{1/2}$ with $\varphi^{(0)}$ so that $L_x(\mathcal{L}^D_{1/2}) = \varphi^{(0)2}_x/2$ for all $x \in V$, and $\mathcal{O}(\tilde{\mathcal{L}}_{1/2}) = \mathcal{O}(\varphi^{(0)})$.

Define $\varphi^{(u)} = (\varphi^{(u)}_x)_{x\in V}$ from $\tilde{\mathcal{L}}_{1/2}$ by, for all $x \in V$,

$$|\varphi^{(u)}_x| = \sqrt{2L_x(\mathcal{L}_{1/2})}$$

and $\varphi^{(u)}_x = \sigma_x|\varphi^{(u)}_x|$, where $\sigma \in \{-1,+1\}^V$ are random spins sampled uniformly independently on each cluster induced by $\mathcal{O}(\tilde{\mathcal{L}}_{1/2})$, pinned at x_0, i.e. $\sigma_{x_0} = 1$. Then, by Theorem 3, $\varphi^{(u)}$ has distribution $P^{\{x_0\},\sqrt{2u}}_\varphi$.

For all $x \in V$, we have

$$L_x(\tilde{\mathcal{L}}_{1/2}) = \frac{1}{2}\varphi^{(0)2}_x + L_x(\mathcal{L}^{(x_0)}_{1/2}).$$

On the other hand, conditional on $L_\cdot(\mathcal{L}_{1/2})$,

$$\begin{aligned}
\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}_{1/2}) \,|\, e \notin \mathcal{O}(\tilde{\mathcal{L}}^D_{1/2}) \cup \mathcal{O}(\mathcal{L}_{1/2})) &= \frac{\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}_{1/2}))}{\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}^D_{1/2}) \cup \mathcal{O}(\mathcal{L}_{1/2}))}\\
= \frac{\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}_{1/2}) \,|\, e \notin \mathcal{O}(\mathcal{L}_{1/2}))}{\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}^D_{1/2}) \,|\, e \notin \mathcal{O}(\mathcal{L}_{1/2}))} &= \frac{\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}_{1/2}) \,|\, e \notin \mathcal{O}(\mathcal{L}_{1/2}))}{\mathbb{P}(e \notin \mathcal{O}(\tilde{\mathcal{L}}^D_{1/2}) \,|\, e \notin \mathcal{O}(\mathcal{L}^D_{1/2}))}\\
= \exp\left(-W_e\sqrt{L_{e_-}(\mathcal{L}_{1/2})L_{e_+}(\mathcal{L}_{1/2})} + W_e\sqrt{L_{e_-}(\mathcal{L}^D_{1/2})L_{e_+}(\mathcal{L}^D_{1/2})}\right),
\end{aligned}$$

where we use in the third equality that the event $e \notin \mathcal{O}(\tilde{\mathcal{L}}^D_{1/2})$ is measurable with respect to the σ-field generated by $\tilde{\mathcal{L}}^D_{1/2}$, which is independent of $\tilde{\mathcal{L}}^{(x_0)}_{1/2}$, and where we use Lemma 2.2 in the fourth equality, for $\tilde{\mathcal{L}}_{1/2}$ and for $\tilde{\mathcal{L}}^D_{1/2}$.

We conclude the proof by observing that $\mathcal{L}^{(x_0)}_{1/2}$ conditional on $L_{x_0}(\mathcal{L}^{(x_0)}_{1/2}) = u$ has the law of the occupation field of the Markov chain $\ell(\tau^{x_0}_u)$ under $\mathbb{P}_{x_0}(\cdot|\tau^{x_0}_u < \zeta)$. ☐

2.6 A diagram to summarize the models and the couplings and relations

Next diagram summarizes the preceding.

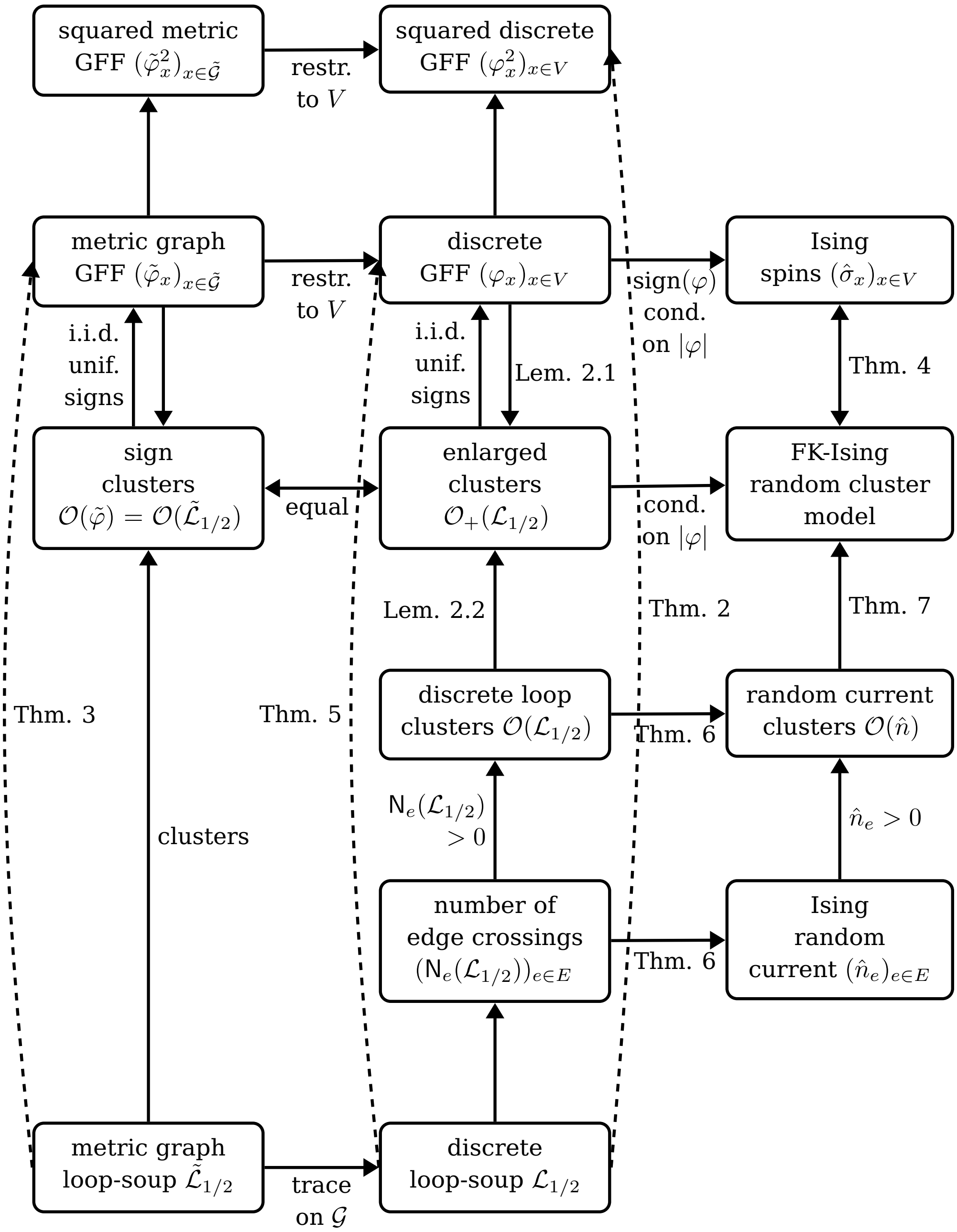

Figure 1: Couplings between discrete and metric graph GFFs, loop-soups, and their relations to the spin Ising, the FK-Ising and the Ising random current. The first column corresponds to metric graph GFF, the second to discrete GFF and the third to Ising. Straightforward links are not labeled.

3 Inversion of the signed isomorphism

In [21], Sabot and Tarrès give a new proof of the generalized second Ray-Knight theorem together with a construction that inverts the coupling between the square of a GFF conditional its value at a vertex x_0 and the excursions of the jump process X from and to x_0. In this paper we are interested in inverting the coupling of Theorem 8 with the signed GFF: more precisely, we want to describe the law of $(X_t)_{0\le t\le \tau_u^{x_0}}$ conditional on $\varphi^{(u)}$.

We present in Section 3.1 an inversion involving an extra Poisson process. We provide in Section 3.2 a discrete-time description of the process and in Section 3.3 an alternative description via jump rates. Section 3.4 is dedicated to a signed inversion of Le Jan's isomorphism for loop-soups.

3.1 A description via an extra Poisson point process

Let $(\check{\varphi}_x)_{x\in V}$ be a real function on V such that $\check{\varphi}_{x_0} = +\sqrt{2u}$ for some $u > 0$. Set

$$\check{\Phi}_x = |\check{\varphi}_x|, \ \ \sigma_x = \operatorname{sign}(\check{\varphi}_x).$$

We define a self-interacting process $(\check{X}_t, (\check{n}_e(t))_{e\in E})$ living on $V \times \mathbb{N}^E$ as follows. The process $\check{X}$ starts at $\check{X}(0) = x_0$. For $t \ge 0$, we set

$$\check{\Phi}_x(t) = \sqrt{(\check{\Phi}_x)^2 - 2\check{\ell}_x(t)}, \ \forall x \in V, \qquad J_e(\check{\Phi}(t)) = W_e \check{\Phi}_{e_-}(t)\check{\Phi}_{e_+}(t), \ \forall e \in E.$$

where $\check{\ell}_x(t) = \int_0^t \mathbb{1}_{\{\check{X}_s = x\}} ds$ is the local time of the process $\check{X}$ up to time t. Let $(N_e(v))_{v\ge 0}$ be an independent Poisson Point Processes on $\mathbb{R}_+$ with intensity 1, for each edge $e \in E$. We set

$$\check{n}_e(t) = \begin{cases} N_e(2J_e(\check{\Phi}(t))), & \text{if } \sigma_{e_-}\sigma_{e_+} = +1, \\ 0, & \text{if } \sigma_{e_-}\sigma_{e_+} = -1. \end{cases}$$

Given $(n_e)_{e\in E} \in \mathbb{N}^E$ non-negative integer weights on edges, we will denote

$$\mathcal{C}(n) = \{e \in E | n_e > 0\}.$$

We consider the edges in $\mathcal{C}(n)$ as "open", and they naturally induce clusters. So $\mathcal{C}(\check{n}(t)) \subset E$ denotes the configuration of edges such that $\check{n}_e(t) > 0$. As time increases, the interaction parameters $J_e(\check{\Phi}(t))$ decreases for the edges neighboring $\check{X}_t$, and at some random times $\check{n}_e(t)$ may drop by 1. The process $(\check{X}_t)_{t\ge 0}$ is defined as the process that jumps only at the times when one of the $\check{n}_e(t)$ drops by 1, as follows:

- if $\check{n}_e(t)$ decreases by 1 at time t, but does not create a new cluster in $\mathcal{C}(\check{n}(t))$, then $\check{X}_t$ crosses the edge e with probability $1/2$ or does not move with probability $1/2$;
- if $\check{n}_e(t)$ decreases by 1 at time t, and does create a new cluster in $\mathcal{C}(\check{n}(t))$, then $\check{X}_t$ moves/or stays with probability 1 on the unique extremity of e which is in the cluster of the origin x_0 in the new configuration.

We set

$$\check{T} := \inf\{t \ge 0 | \exists x \in V, \text{ s. t. } \check{\Phi}_x(t) = 0\},$$

clearly, the process is well-defined up to time $\check{T}$.

Proposition 3.1. *For all $0 \le t \le \check{T}$, $\check{X}_t$ is in the connected component of x_0 of the configuration $\mathcal{C}(\check{n}(t))$. If V is finite, the process ends at x_0, i.e. $\check{X}_{\check{T}} = x_0$.*

Theorem 9. *Assume that V is finite. With the notation of Theorem 8, conditional on $\varphi^{(u)} = \check{\varphi}$, $(X_t)_{t \leq \tau_u^{x_0}}$ has the law of $(\check{X}_{\check{T}-t})_{0 \leq t \leq \check{T}}$.*

Moreover, conditional on $\varphi^{(u)} = \check{\varphi}$, $(\varphi^{(0)}, \mathcal{O}(\varphi^{(0)}))$ has the law of $(\sigma' \check{\Phi}(\check{T}), \mathcal{C}(\check{n}(\check{T})))$ where $(\sigma'_x)_{x \in V} \in \{-1,+1\}^V$ are random spins sampled uniformly independently on each cluster induced by $\mathcal{C}(\check{n}(\check{T}))$, with the condition that $\sigma'_{x_0} = +1$.

If V is infinite, then $P_\varphi^{\{x_0\},\sqrt{2u}}$-a.s., $\check{X}_t$ (with the initial condition $\check{\varphi} = \varphi^{(u)}$) ends at x_0, i.e. $\check{T} < +\infty$ and $\check{X}_{\check{T}} = x_0$. All previous conclusions for the finite case still hold.

Remark 3.2. *For the process above, one could also take $(\check{\Phi}_x)_{x \in V}$ deterministic and $(\sigma_x)_{x \in V}$ random, distributed as an Ising model with interaction constants $J_e(\check{\Phi}(0))$. Then the process $(\check{X}_t, \check{\Phi}(t))_{0 \leq t \leq \check{T}}$, averaged out by the law of $(\sigma_x)_{x \in V}$ and the evolution of $(\check{n}_e(t))_{e \in E}$, is exactly the same as the process introduced in [21], inverting the Ray-Knight identity for the square of the GFF (without the sign). Indeed, both processes (here and in [21]), when we average out by $\check{\Phi}(0) = \check{\Phi}$ random, distributed as $|\varphi^{(u)}|$ under $P_\varphi^{\{x_0\},\sqrt{2u}}$, give us in law $\left(X_{\tau_u^{x_0}-t}, \sqrt{\varphi_x^{(0)2} + 2\ell_x(\tau_u^{x_0}) - 2\ell_x(t)}\right)_{x \in V, 0 \leq t \leq \tau_u^{x_0}}$ under $\mathbb{P}_{x_0}(\cdot | \tau_u^{x_0} < \zeta) \otimes P_\varphi^{\{x_0\},0}$. To conclude we use the fact that the law of $(\check{X}_t, \check{\Phi}(t))_{0 \leq t \leq \check{T}}$ is continuous with respect to $\check{\Phi}(0)$.*

3.2 Discrete time description of the process

We give a discrete time description of the process $(\check{X}_t, (\check{n}_e(t))_{e \in E})$ that appears in the previous section. Let $t_0 = 0$ and $0 < t_1 < \cdots < t_j$ be the stopping times when one of the stacks $\check{n}_e(t)$ decreases by 1, where t_j is the time when one of the stacks is completely depleted. It is elementary to check the following:

Proposition 3.3. *The discrete time process $(\check{X}_{t_i}, (\check{n}_e(t_i))_{e \in E})_{0 \leq i \leq j}$ is a stopped Markov process. The transition from time $i-1$ to i is the following:*

- *first chose e an edge adjacent to the vertex $\check{X}_{t_{i-1}}$ according to a probability proportional to $\check{n}_e(t_{i-1})$;*
- *decrease the stack $\check{n}_e(t_{i-1})$ by 1;*
- *if decreasing $\check{n}_e(t_{i-1})$ by 1 does not create a new cluster in $\mathcal{C}(\check{n}(t_{i-1}))$, then $\check{X}_{t_{i-1}}$ crosses the edge e with probability $1/2$ or does not move with probability $1/2$;*
- *if decreasing $\check{n}_e(t_{i-1})$ by 1 does create a new cluster in $\mathcal{C}(\check{n}(t_{i-1}))$, then $\check{X}_{t_{i-1}}$ moves/or stays with probability 1 on the unique extremity of e which is in the cluster of the origin x_0 in the new configuration.*

3.3 An alternative description via jump rates

We provide an alternative description of the process $(\check{X}_t, \mathcal{C}(\check{n}(t)))$ that appears in Section 3.1. We will use this description in [17] by passing it to a fine mesh limit to obtain a process inverting the Ray-Knight identity for a Brownian motion on $\mathbb{R}$.

We will denote $\check{\mathcal{C}}(t) = \mathcal{C}(\check{n}(t))$, since below we will not have access to the knowledge of $\check{n}(t)$, only to that of $\mathcal{C}(\check{n}(t))$.

Proposition 3.4. *The process $(\check{X}_t, \check{\mathcal{C}}(t))$ defined in Section 3.1 can be alternatively described by its jump rates: conditional on its past at time t, if $\check{X}_t = x$, $y \sim x$ and $\{x,y\} \in \check{\mathcal{C}}(t)$, then*

(1) $\check{X}$ jumps to y without modification of $\check{\mathcal{C}}(t)$ at rate

$$W_{x,y} \frac{\check{\Phi}_y(t)}{\check{\Phi}_x(t)};$$

(2) the edge $\{x,y\}$ is closed in $\check{\mathcal{C}}(t)$ at rate

$$2W_{x,y}\frac{\check\Phi_y(t)}{\check\Phi_x(t)}\left(e^{2W_{x,y}\check\Phi_x(t)\check\Phi_y(t)}-1\right)^{-1}$$

and, conditional on that last event:

- if y is connected to x in the configuration $\check{\mathcal{C}}(t)\setminus\{x,y\}$, then $\check X$ instantaneously jumps to y with probability $1/2$ and stays at x with probability $1/2$;

- otherwise $\check X_t$ moves/or stays with probability 1 on the unique extremity of $\{x,y\}$ which is in the cluster of the origin x_0 in the new configuration.

Remark 3.5. *It is clear from this description that the joint process $(\check X_t,\check{\mathcal{C}}(t),\check\Phi(t))$ is Markov process, and well defined up to the time*

$$\check T := \inf\{t\ge 0 | \exists x\in V,\ \text{s.t. } \check\Phi_x(t)=0\}.$$

Remark 3.6. *One can also retrieve the process in Section 3.1 from the representation in Proposition 3.4 as follows. Consider the representation of Proposition 3.4 on the graph where each edge e is replaced by a large number N of parallel edges with conductance W_e/N. Consider now $\check n^{(N)}_{x,y}(t)$ the number of parallel edges that are open in the configuration $\check{\mathcal{C}}^{(N)}(t)$ between x and y. Then, when $N\to\infty$, $(\check n^{(N)}(t))_{t\ge0}$, converges in law to $(\check n(t))_{t\ge0}$, defined in Section 3.1. We will not detail this, but roughly, this corresponds to approximation of Poisson r.v. by binomial r.v.*

Proof of Proposition 3.4. Here $(\check{\mathcal{F}}_t)_{t\ge0}$ will denote the natural filtration of

$$(\check X_{t\wedge\check T},\check{\mathcal{C}}(t\wedge\check T))_{t\ge0}.$$

Assume $\check X_t=x$, fix $y\sim x$ and let $e=\{x,y\}$. Recall that $\{x,y\}\in\check{\mathcal{C}}(t)$ iff $\check n_e(t)\ge1$. Let be

$$J^x_e(t,\Delta t)=W_{x,y}\sqrt{\check\Phi_x(t)^2-2\Delta t}\,\check\Phi_y(t),$$

which is the value of $J_e(\Phi(t+\Delta t))$ on the event

$$\{X_t=x,\ n(t+\Delta t)=n(t),\ W_{x,y}\sqrt{\check\Phi_x(t)^2-2\Delta t}\,\check\Phi_y(t)\ge0\}.$$

Let us first prove (1):

$$\begin{aligned}
&\mathbb{P}\left(\text{On } [t,t+\Delta t]\ \check X \text{ first jumps from } x \text{ to } y \text{ without modifying } \check{\mathcal{C}} | \check X_t=x,\ e\in\check{\mathcal{C}}(t),\ \check{\mathcal{F}}_t\right)\\
&=\frac12\mathbb{P}\left(\exists s\in[t,t+\Delta t],\check n(t)-\check n(s)=\delta_e,\ \check n_e(s)\ge1\,|\,\check X_t=x,\ \check n_e(t)\ge1,\ \check{\mathcal{F}}_t\right)+o(\Delta t)\\
&=\frac12\mathbb{P}\left(N_e(2J_e(\check\Phi(t))\ge2,\ N_e(2J_e(\check\Phi(t))-N_e(2J^x_e(t,\Delta t))\ge1\,|\,\check X_t=x,\ \check n_e(t)\ge1,\ \check{\mathcal{F}}_t\right)\\
&\quad+o(\Delta t)\\
&=\frac12\mathbb{P}\left(N_e(2J^x_e(t,\Delta t))\ge1,\ N_e(2J_e(\check\Phi(t))-N_e(2J^x_e(t,\Delta t))\ge1\,|\,\check X_t=x,\ \check n_e(t)\ge1,\ \check{\mathcal{F}}_t\right)\\
&\quad+o(\Delta t)\\
&=\frac12\frac{1-e^{-2J^x_e(t,\Delta t)}}{1-e^{-2J_e(\check\Phi(t))}}(1-e^{-2(J_e(\check\Phi(t))-J^x_e(t,\Delta t))})+o(\Delta t)\\
&=J_e(\check\Phi(t))-J^x_e(t,\Delta t)+o(\Delta t)=W_{xy}\frac{\check\Phi_y(t)}{\check\Phi_x(t)}\Delta t+o(\Delta t).
\end{aligned}$$

Similarly, (2) follows from the following computation:

$$\begin{aligned}
&\mathbb{P}\left(\text{On } [t, t+\Delta t] \text{ first the edge } e \text{ is closed in } \check{\mathcal{C}} \,|\, \check{X}_t = x,\, e \in \check{\mathcal{C}}(t),\, \check{\mathcal{F}}_t\right) \\
&= \mathbb{P}\left(N_e(2J_e^x(t,\Delta t)) = 0,\, N_e(2J_e(\check{\Phi}(t)) = 1 \,|\, \check{X}_t = x,\, \check{n}_e(t) \geq 1,\, \check{\mathcal{F}}_t\right) + o(\Delta t) \\
&= \frac{e^{-2J_e^x(t,\Delta t)} 2(J_e(\check{\Phi}(t)) - J_e^x(t,\Delta t)) e^{-2(J_e(\check{\Phi}(t)) - J_e^x(t,\Delta t))}}{1 - e^{-2J_e(\check{\Phi}(t))}} + o(\Delta t) \\
&= \frac{2(J_e(\check{\Phi}(t)) - J_e^x(t,\Delta t))}{e^{2J_e(\check{\Phi}(t))} - 1} + o(\Delta t) \\
&= 2W_{x,y} \frac{\check{\Phi}_y(t)}{\check{\Phi}_x(t)} \left(e^{2W_{x,y}\check{\Phi}_x(t)\check{\Phi}_y(t)} - 1\right)^{-1} + o(\Delta t). \qquad \square
\end{aligned}$$

We easily deduce from the Proposition 3.4 and Theorem 12 the following alternative inversion of the coupling in Theorem 8.

Theorem 10. *With the notation of Theorem 8, conditional on $(\varphi^{(u)}, \mathcal{O}_u)$, $(X_t)_{t \leq \tau_u^{x_0}}$ has the law of self-interacting process $(\check{X}_{\check{T}-t})_{0 \leq t \leq \check{T}}$ defined by jump rates of Proposition 3.4 starting with*

$$\check{\Phi}_x = \sqrt{(\varphi_x^{(0)})^2 + 2\ell_x(\tau_u^{x_0})} \text{ and } \check{\mathcal{C}}(0) = \mathcal{O}_u.$$

Moreover $(\varphi^{(0)}, \mathcal{O}(\varphi^{(0)}))$ has the same law as $(\sigma'\check{\Phi}(\check{T}), \check{\mathcal{C}}(\check{T}))$ where $(\sigma'_x)_{x \in V}$ is a configuration of signs obtained by picking a sign at random independently on each connected component of $\check{\mathcal{C}}(\check{T})$, with the condition that the component of x_0 has a + sign.

3.4 Inversion of Lupu's isomorphism for loop-soup

Let us first recall how the loops in $\mathcal{L}_\alpha$ are connected to the excursions of the jump process X. We refer to [13] for details. $\mathbf{G}$ is the Green's function. $L_{x_0}(\mathcal{L}_\alpha)$ follows a $\Gamma(\alpha, \mathbf{G}(x_0, x_0))$ distribution, that is to say $L_{x_0}(\mathcal{L}_\alpha)/\mathbf{G}(x_0, x_0)$ follows a Gamma distribution $\Gamma(\alpha, 1)$ with density

$$\mathbb{1}_{\{r>0\}} \frac{1}{\Gamma(\alpha)} r^{\alpha-1} e^{-r} dr.$$

As a process in α, where one drops independent loops as the intensity parameter α increases, $(L_{x_0}(\mathcal{L}_\alpha)/\mathbf{G}(x_0, x_0))_{\alpha \geq 0}$ is a pure jump Gamma subordinator with Lévy measure

$$d\Lambda(r) = \mathbb{1}_{\{r>0\}} \frac{1}{\Gamma(\alpha)} \frac{e^{-r}}{r} dr.$$

Given such a Gamma subordinator $(R(\alpha))_{\alpha \geq 0}$ with $R(0) = 0$, then for all $\alpha > 0$, the marginal $R(\alpha)$ follows a $\Gamma(\alpha, 1)$ distribution. Moreover, the normalized family of jump sizes

$$\left(\frac{R(a) - R(a^-)}{R(\alpha)}\right)_{0 \leq a \leq \alpha, R(a) \neq R(a^-)}$$

is independent of $R(\alpha)$ and has the law of a Poisson-Dirichlet partition $PD(0, \alpha)$ of $[0, 1]$. The above may be taken as a definition of $PD(0, \alpha)$. It is a random infinite countable family of positive reals summing to 1. For more on Poisson-Dirichlet partitions, we refer to [20].

Proposition 3.7 (From excursions to loops)**.** *Let $\alpha > 0$ and $x_0 \in V$. $L_{x_0}(\mathcal{L}_\alpha)$ is distributed according to a Gamma $\Gamma(\alpha, \mathbf{G}(x_0, x_0))$ law, where $\mathbf{G}$ is the Green's function. Let $u > 0$, and consider the path $(X_t)_{0 \leq t \leq \tau_u^{x_0}}$ conditional on $\tau_u^{x_0} < \zeta$. Let $(Y_j)_{j \geq 1}$ be an independent Poisson-Dirichlet partition $PD(0, \alpha)$ of $[0, 1]$ (so that $\sum_{j \geq 1} Y_j = 1$). Let $S_0 = 0$ and*

$$S_j = \sum_{i=1}^{j} Y_i.$$

Let $\tau_j = \tau^{x_0}_{uS_j}$. Consider the family of paths

$$\left((X_{\tau_{j-1}+t})_{0\leq t\leq \tau_j-\tau_{j-1}}\right)_{j\geq 1}.$$

It is a countable family of loops rooted in x_0. It has the same law as the family of all the loops in $\mathcal{L}_\alpha$ that visit x_0, conditional on $L_{x_0}(\mathcal{L}_\alpha) = u$.

Next we describe how to invert the discrete version for Lupu's isomorphism Theorem 5 for the loop-soup in the same way as in Theorem 9. The idea is to define an arbitrary order on the vertices, $(x_i)_{1\leq i\leq |V|}$. Then, starting from a signed GFF, run the inverting process introduced previously starting from x_1, up to exhausting the field in x_1. This would produce a path from x_1 to x_1, which is conditionally distributed like all the loops in $\mathcal{L}_{1/2}$ that visit x_1, glued together. By partitioning this path according to the procedure described in Proposition 3.7, one recovers all the loops visiting x_1. Then one continues with the remaining field, which is 0 in x_1 and has smaller FK-Ising clusters than the initial one, and runs the inverting process starting from x_2, in order to get all the loops that visit x_2 but not x_1. Then one iterates. At each step, one gets all the loops that visit x_i, but none of $x_1, \dots, x_{i-1}$. In what follows we describe this more formally.

Let $(\check{\varphi}_x)_{x\in V}$ be a real function on V. Set

$$\check{\Phi}_x = |\check{\varphi}_x|, \quad \sigma_x = \operatorname{sign}(\check{\varphi}_x).$$

Let $(x_i)_{1\leq i\leq |V|}$ be an enumeration of V (which may be infinite). We define by induction on i the self interacting processes $((\check{X}_{i,t})_{1\leq i\leq |V|}, (\check{n}_e(t))_{e\in E})$. $\check{T}_i$ will denote the end-time for $\check{X}_{i,t}$, and $\check{T}_i^+ = \sum_{1\leq j\leq i} \check{T}_j$. By definition, $\check{T}_0^+ = 0$. $L_x(t)$ will denote

$$L_x(t) := \sum_{1\leq i\leq |V|} \check{\ell}_x(i, 0\vee(t-\check{T}_i^+)),$$

where $\check{\ell}_x(i,t)$ are the occupation times for $\check{X}_{i,t}$. For $t\geq 0$, we set

$$\check{\Phi}_x(t) = \sqrt{(\check{\Phi}_x)^2 - 2L_x(t)},\ \forall x\in V, \qquad J_e(\check{\Phi}(t)) = W_e\check{\Phi}_{e_-}(t)\check{\Phi}_{e_+}(t),\ \forall e\in E.$$

The end-times $\check{T}_i$ are defined by induction as

$$\check{T}_i = \inf\{t\geq 0 | \check{\Phi}_{\check{X}_{i,t}}(t+\check{T}^+_{i-1}) = 0\}.$$

Let $(N_e(v))_{v\geq 0}$ be independent Poisson Point Processes on $\mathbb{R}_+$ with intensity 1, for each edge $e\in E$. We set

$$\check{n}_e(t) = \begin{cases} N_e(2J_e(\check{\Phi}(t))), & \text{if } \sigma_{e_-}\sigma_{e_+} = +1,\\ 0, & \text{if } \sigma_{e_-}\sigma_{e_+} = -1.\end{cases}$$

We also denote by $\mathcal{C}(\check{n}(t))\subset E$ the configuration of edges such that $\check{n}_e(t) > 0$. $\check{X}_{i,t}$ starts at x_i. For $t\in[\check{T}^+_{i-1}, \check{T}^+_i]$,

- if $\check{n}_e(t)$ decreases by 1 at time t, but does not create a new cluster in $\mathcal{C}(\check{n}(t))$, then $\check{X}_{i,t-\check{T}^+_{i-1}}$ crosses the edge e with probability $1/2$ or does not move with probability $1/2$;
- if $\check{n}_e(t)$ decreases by 1 at time t, and does create a new cluster in $\mathcal{C}(\check{n}(t))$, then $\check{X}_{i,t-\check{T}^+_{i-1}}$ moves/or stays with probability 1 on the unique extremity of e which is in the cluster of the origin x_i in the new configuration.

By induction, using Theorem 9, we deduce the following.

Theorem 11. *Let φ be a GFF on $\mathcal{G}$ with the law P_φ. If one sets $\check{\varphi} = \varphi$ in the preceding construction, then for all $i \in \{1, \ldots, |V|\}$, $\check{T}_i < +\infty$, $\check{X}_{i,\check{T}_i} = x_i$ and the path $(\check{X}_{i,t})_{t \leq \check{T}_i}$ has the same law as a concatenation in x_i of all the loops in a loop-soup $\mathcal{L}_{1/2}$ that visit x_i, but none of the $x_1, \ldots, x_{i-1}$. To retrieve the loops out of each path $(\check{X}_{i,t})_{t \leq \check{T}_i}$, one has to partition it according to a Poisson-Dirichlet partition as in Proposition 3.7. The coupling between the GFF φ and the loop-soup obtained from $((\check{X}_{i,t})_{1 \leq i \leq |V|}, (\check{n}_e(t))_{e \in E})$ is the same as in Theorem 5.*

Remark 3.8. *One could consider the discrete time version of the procedure described in Theorem 11, and look only at the total number of times $\hat{n}_e$ an edge e is visited by the trajectories constructed, without distinguishing the directions. Than $(\hat{n}_e)_{e \in E}$ is a current, and its conditional distribution given $\check{\mathcal{C}}(0)$ is the same as the conditional distribution of a random current given an FK-Ising cluster when both are coupled as in Theorem 7.*

Corollary 3.9. *Let φ be a GFF on $\mathcal{G}$ with the law P_φ. Let $(\mathfrak{n}_e(\varphi))_{e \in E}$ be a family of r.v., distributed conditional on ϕ as independent Poisson r.v., each one with mean $W_e \varphi_{e_-} \varphi_{e_+}$ if $\varphi_{e_-} \varphi_{e_+} > 0$, 0 otherwise. One can couple φ, $(\mathfrak{n}_e(\varphi))_{e \in E}$ and a loop-soup $\mathcal{L}_{1/2}$ such that the coupling between $\mathcal{L}_{1/2}$ and φ is that of Theorem 5, and moreover, a.s. for every $e \in E$, $\mathsf{N}_e(\mathcal{L}_{1/2}) \in \{\mathfrak{n}_e(\varphi) - 1, \mathfrak{n}_e(\varphi), \mathfrak{n}_e(\varphi) + 1\}$.*

Proof. We use the construction of Theorem 11. Conditional on φ, we have independent Poisson stacks with mean $\mathbb{1}_{\{\varphi_{e_-} \varphi_{e_+} > 0\}} 2 W_e \varphi_{e_-} \varphi_{e_+}$ and each time we unpile a stack, we chose with probability $1/2$ to jump (which gives a Poisson r.v. with mean $\mathbb{1}_{\{\varphi_{e_-} \varphi_{e_+} > 0\}} W_e \varphi_{e_-} \varphi_{e_+}$), except possibly when the stack is reduced to 1, when our choice might be constrained (which gives ± 1). $\square$

4 Proof of theorem 9

4.1 Case of finite graph without killing measure

Here we will assume that V is finite and that the killing measure $\kappa \equiv 0$.

In order to prove Theorem 9, we first enlarge the state space of the process $(X_t)_{t \geq 0}$. We define a process $(X_t, (n_e(t)))_{t \geq 0}$ living on the space $V \times \mathbb{N}^E$ as follows. Let $\varphi^{(0)} \sim P_\varphi^{\{x_0\},0}$ be a GFF pinned at x_0. Let $\sigma_x = \text{sign}(\varphi_x^{(0)})$ be the signs of the GFF with the convention that $\sigma_{x_0} = +1$. The process $(X_t)_{t \geq 0}$ is as usual the Markov Jump process starting at x_0 with jump rates $(W_e)_{e \in E}$. We set

$$\Phi_x = |\varphi_x^{(0)}|, \ \ \Phi_x(t) = \sqrt{\Phi_x^2 + 2\ell_x(t)}, \ \forall x \in V, \quad J_e(\Phi(t)) = W_e \Phi_{e_-}(t) \Phi_{e_+}(t), \ \forall e \in E. \tag{4.1}$$

The initial values $(n_e(0))$ are choosen independently on each edge with distribution

$$n_e(0) \sim \begin{cases} 0 & \text{if } \sigma_{e_-} \sigma_{e_+} = -1, \\ \mathcal{P}(2 J_e(\Phi)) & \text{if } \sigma_{e_-} \sigma_{e_+} = +1, \end{cases} \tag{4.2}$$

where $\mathcal{P}(2J_e(\Phi))$ is a Poisson random variable with parameter $2J_e(\Phi)$. Let $((N_e(v))_{v \geq 0})_{e \in E}$ be independent Poisson point processes on $\mathbb{R}_+$ with intensity 1. We define the process $(n_e(t))$ by

$$n_e(t) = n_e(0) + N_e(J_e(\Phi(t))) - N_e(J_e(\Phi)) + K_e(t),$$

where $K_e(t)$ is the number of crossings of the edge e by the Markov jump process X before time t.

Remark 4.1. *Note that compared to the process defined in Section 3.1, the speed of the Poisson process is related to $J_e(\Phi(t))$ and not $2J_e(\Phi(t))$.*

We recall that with our notations,

$$\mathcal{C}(n(t)) = \{e \in E | n_e(t) > 0\}.$$

Recall also that $\tau_u^{x_0} = \inf\{t \geq 0 | \ell_{x_0}(t) = u\}$ for $u > 0$. . We define $\varphi^{(u)}$ by

$$\varphi_x^{(u)} = \sigma_x \Phi(\tau_u^{x_0}),\ \forall x \in V,$$

where $(\sigma_x)_{x\in V} \in \{-1,+1\}^V$ are random spins sampled uniformly independently on each cluster induced by $\mathcal{C}(n(\tau_u^{x_0}))$ with the condition that $\sigma_{x_0} = +1$.

Lemma 4.2. *The random family* $(\varphi^{(0)}, \mathcal{C}(n(0)), \varphi^{(u)}, \mathcal{C}(n(\tau_u^{x_0})))$ *thus defined has the same distribution as* $(\varphi^{(0)}, \mathcal{O}(\varphi^{(0)}), \varphi^{(u)}, \mathcal{O}_u)$ *defined in Theorem 8.*

Proof. It is clear from construction, that $\mathcal{C}(n(0))$ has the same law as $\mathcal{O}(\varphi^{(0)})$ (cf Definition 2.1), the FK-Ising configuration coupled with the signs of $\varphi^{(0)}$ as in Theorem 4. Indeed, for each edge $e \in E$ such that $\varphi_{e_-}^{(0)}\varphi_{e_+}^{(0)} > 0$, the probability that $n_e(0) > 0$ is $1 - e^{-2J_e(\Phi)}$. Moreover, conditional on $\mathcal{C}(n(0)) = \mathcal{O}(\varphi^{(0)})$, $\mathcal{C}(n(\tau_u^{x_0}))$ has the same law as $\mathcal{O}_u$ defined in Theorem 8. Indeed, $\mathcal{C}(n(\tau_u^{x_0}))$ is the union of the set $\mathcal{C}(n(0))$, the set of edges crossed by the process $(X_u)_{u\leq\tau_u^{x_0}}$, and the additional edges such that $N_e(J_e(\tau_u^{x_0})) - N_e(J_e(\Phi)) > 0$. Clearly $N_e(J_e(\tau_u^{x_0})) - N(J_e(\Phi)) > 0$ independently with probability $1 - e^{-(J_e(\Phi(\tau_u^{x_0})) - J_e(\Phi))}$ which coincides with the probability given in Theorem 8. □

We will prove the following theorem that, together with Lemma 4.2, contains the statements of both Theorem 8 and 9.

Theorem 12. *The random field* $\varphi^{(u)}$ *is a GFF distributed according to* $P_\varphi^{\{x_0\},\sqrt{2u}}$*. Moreover, conditional on* $\varphi^{(u)} = \check{\varphi}$*, the process*

$$(X_t, (n_e(t))_{e\in E})_{t\leq\tau_u^{x_0}}$$

has the law of the process $(\check{X}_{\check{T}-t}, (\check{n}_e(\check{T}-t))_{e\in E})_{t\leq\check{T}}$ *described in Section 3.1.*

Proof. Before proceeding to the proof of the theorem, we will briefly outline our method. We have a Markov process $(X_t, \Phi(t), n(t))$, and would like to find another Markov process $(\check{X}_t, \check{\Phi}(t), \check{n}(t))$ such that the latter has the law of $(X_{\tau_u^{x_0}-t}, \Phi(\tau_u^{x_0} - t), n(\tau_u^{x_0} - t))$ in the particular case when the entrance (initial) distribution of $(X_0, \Phi(0), n(0))$ is given by $(x_0, |\varphi^{(0)}|, n(0))$, $n(0)$ given by (4.2). To this end, we will first introduce an intermediate Markov process $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ which will correspond to $(X_{\tau_u^{x_0}-t}, \Phi(\tau_u^{x_0} - t), n(\tau_u^{x_0} - t))$ in case when the entrance distribution of $(X_0, \Phi(0), n(0))$ is not the one we are interested in, but given by $X_0 = x_0$ and $(\Phi(0), n(0))$ following the product measure (with infinite total mass)

$$\sum_{n\in\mathbb{N}^E} \int d\Phi F(\Phi, n).$$

We do that because $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ is simpler, and in particular $\bar{X}_t$ is a Markov jump process with jump rates $(W_e)_{e\in E}$ (just as X_t) which does not interact with $(\bar{\Phi}(t), \bar{n}(t))$. In this way $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ and $(\check{X}_t, \check{\Phi}(t), \check{n}(t))$ correspond to $(X_{\tau_u^{x_0}-t}, \Phi(\tau_u^{x_0}-t), n(\tau_u^{x_0}-t))$ for two different entrance distributions of $(X_0, \Phi(0), n(0))$, and we show that $(\check{X}_t, \check{\Phi}(t), \check{n}(t))$ is absolutely continuous with respect to $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ and identify the corresponding Radon-Nikodym derivative $\bar{M}_{t\wedge\bar{T}}/\bar{M}_0$. Out of this we further identify $(\check{X}_t, \check{\Phi}(t), \check{n}(t))$ as a Doob's h-transform (see [3], Chapter 11) of $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$. Then we obtain the infinitesimal generator of $(\check{X}_t, \check{\Phi}(t), \check{n}(t))$ as a conjugate of the infinitesimal generator of $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$.

Our proof is cut in four steps. In step 1 we simply give an explicit formula for the law of $(|\varphi^{(0)}|, n(0))$, $n(0)$ given by (4.2). In step 2 we introduce $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ and prove the time reversal for the product entrance distribution of $(X_0, \Phi(0), n(0))$. In step 3 we identify the Radon-Nikodym derivative of $(\check{X}_t, \check{\Phi}(t), \check{n}(t))$ with respect to $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$. In step 4 we describe $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ as a Doob's h-transform of $(\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ and give its the infinitesimal generator, which is that of the process introduced in Section 3.1.

Step 1: We start by a simple lemma.

Lemma 4.3. *The distribution of $(\Phi := |\varphi^{(0)}|, n(0))$ is given by the following formula for any bounded measurable test function h:*

$$\mathbb{E}\left(h(\Phi, n(0))\right) = \sum_{n\in\mathbb{N}^E} \int_{\mathbb{R}_+^{V\setminus\{x_0\}}} d\Phi h(\Phi,n) e^{-\frac{1}{2}\sum_{x\in V} W_x\Phi_x^2 - \sum_{e\in E} J_e(\Phi)} \left(\prod_{e\in E}\frac{(2J_e(\Phi))^{n_e}}{n_e!}\right) 2^{\#c.\mathcal{C}(n)-1}.$$

where the integral is on the set $\{(\Phi_x)_{x\in V}\in\mathbb{R}_+^V | \forall x\neq x_0,\ \Phi_x>0,\ \Phi_{x_0}=0\}$, and

$$d\Phi = \frac{\prod_{x\in V\setminus\{x_0\}} d\Phi_x}{\sqrt{2\pi}^{|V|-1}},$$

and $\#c.\mathcal{C}(n)$ is the number of clusters induced by the edges such that $n_e>0$.

Proof. Indeed, by construction, summing on possible signs of $\varphi^{(0)}$, we have

$$\mathbb{E}\left(h(\Phi, n(0))\right) = \sum_{\substack{\sigma\in\{\pm1\}^V\\ \sigma_{x_0}=+1}} \sum_{\substack{n\in\mathbb{N}^E\\ n\ll\sigma}} \int_{\mathbb{R}_+^{V\setminus\{x_0\}}} d\Phi h(\Phi,n) e^{-\frac{1}{2}\mathcal{E}(\sigma\Phi,\sigma\Phi)} \left(\prod_{\substack{e\in E\\ \sigma_{e_-}\sigma_{e_+}=+1}} \frac{e^{-2J_e(\Phi)}(2J_e(\Phi))^{n_e}}{n_e!}\right), \quad (4.3)$$

where $n\ll\sigma$ means that n_e vanishes on the edges such that $\sigma_{e_-}\sigma_{e_+}=-1$. Since we have, similarly to (2.1),

$$\begin{aligned}\frac{1}{2}\mathcal{E}(\sigma\Phi,\sigma\Phi) &= \frac{1}{2}\sum_{x\in V} W_x\Phi_x^2 - \sum_{e\in E} J_e(\Phi)\sigma_{e_-}\sigma_{e_+}\\ &= \frac{1}{2}\sum_{x\in V} W_x\Phi_x^2 + \sum_{e\in E} J_e(\Phi) - \sum_{\substack{e\in E\\ \sigma_{e_-}\sigma_{e_+}=+1}} 2J_e(\Phi),\end{aligned}$$

we deduce that the integrand in (4.3) is equal to

$$\begin{aligned} & h(\Phi,n)e^{-\frac{1}{2}\mathcal{E}(\sigma\Phi,\sigma\Phi)}\left(\prod_{\substack{e\in E\\ \sigma_{e_-}\sigma_{e_+}=+1}} \frac{e^{-2J_e(\Phi)}(2J_e(\Phi))^{n_e}}{n_e!}\right)\\ =\ & h(\Phi,n)e^{-\frac{1}{2}\mathcal{E}(\sigma\Phi,\sigma\Phi)} e^{-\sum_{e\in E,\ \sigma_{e_-}\sigma_{e_+}=+1} 2J_e(\Phi)} \left(\prod_{e\in E}\frac{(2J_e(\Phi))^{n_e}}{n_e!}\right)\\ =\ & h(\Phi,n)e^{-\frac{1}{2}\sum_{x\in V} W_x\Phi_x^2 - \sum_{e\in E} J_e(\Phi)} \left(\prod_{e\in E}\frac{(2J_e(\Phi))^{n_e}}{n_e!}\right),\end{aligned}$$

where we used in the first equality that $n_e = 0$ on the edges such that $\sigma_{e_-}\sigma_{e_+} = -1$. Thus,

$$\mathbb{E}\left(h(\Phi, n(0))\right) = \sum_{\substack{\sigma\in\{\pm1\}^V\\ \sigma_{x_0}=+1}} \sum_{\substack{n\in\mathbb{N}^E\\ n\ll\sigma}} \int_{\mathbb{R}_+^{V\setminus\{x_0\}}} d\Phi h(\Phi,n) e^{-\frac{1}{2}\sum_{x\in V} W_x\Phi_x^2 - \sum_{e\in E} J_e(\Phi)} \left(\prod_{e\in E} \frac{(2J_e(\Phi))^{n_e}}{n_e!}\right).$$

Reversing the sum on σ and n and summing on the number of possible signs which are constant on clusters induced by the configuration of edges $\{e \in E | n_e > 0\}$, we deduce Lemma 4.3. □

Step 2: We denote by $Z_t = (X_t, \Phi(t), n(t))$ the process defined previously and by E_{x_0,Φ,n_0} its law with initial condition (x_0, Φ, n).

We now introduce a process $\bar{Z}_t$, which is a "time reversal" of the process Z_t. This process will be related to the process defined in Section 3.1 in Step 4, Lemma 4.5.

For $(\bar{n}_e)_{e\in E} \in \mathbb{N}^E$ and $(\bar{\Phi}_x)_{x\in V}$ such that

$$\bar{\Phi}_{x_0} = u,\ \forall x \neq x_0,\ \bar{\Phi}_x > 0,$$

we define the process $\bar{Z}_t = (\bar{X}_t, \bar{\Phi}(t), \bar{n}(t))$ with values in $V \times \mathbb{R}_+^V \times \mathbb{Z}^E$ as follows. The process $\bar{X}_t$ is a Markov jump process with jump rates $(W_e)_{e\in E}$ (so that $\bar{X} \stackrel{\text{law}}{=} X$), and $\bar{\Phi}(t)$, $\bar{n}(t)$ are defined by

$$\bar{\Phi}_x(t) = \sqrt{\bar{\Phi}_x^2 - 2\bar{\ell}_x(t)},\ \forall x \in V, \tag{4.4}$$

where $\bar{\ell}_x(t)$ is the local time of the process $\check{X}$ up to time t,

$$\bar{n}_e(t) = \bar{n}_e - \left(N_e(J_e(\bar{\Phi})) - N_e(J_e(\bar{\Phi}(t)))\right) - \bar{K}_e(t), \tag{4.5}$$

where $((N_e(v))_{v\geq 0})_{e\in E}$ are independent Poisson point process on $\mathbb{R}_+$ with intensity 1 for each edge e, and $\bar{K}_e(t)$ is the number of crossings of the edge e by the process $\bar{X}$ before time t. We set

$$\bar{Z}_t = (\bar{X}_t, (\bar{\Phi}_x(t)), (\bar{n}_e(t))). \tag{4.6}$$

This process is well-defined up to time

$$\bar{T} = \inf\left\{t \geq 0 | \exists x \in V\ \bar{\Phi}_x(t) = 0\right\}.$$

We denote by $\bar{E}_{x_0,\bar{\Phi},\bar{n}}$ its law. Clearly $\bar{Z}_t = (\bar{X}_t, \bar{\Phi}(t), \bar{n}_e(t))$ is a Markov process, we will later on make explicit its generator.

We have the following change of variable lemma.

Lemma 4.4. *For all bounded measurable test functions F, G, H*

$$\sum_{n\in\mathbb{N}^E} \int d\Phi F(\Phi,n) E_{x_0,\Phi,n}\left(G((Z_{\tau_u^{x_0}-t})_{0\leq t\leq \tau_u^{x_0}}) H(\Phi(\tau_u^{x_0}), n(\tau_u^{x_0}))\right) =$$

$$\sum_{\bar{n}\in\mathbb{N}^E} \int d\bar{\Phi} H(\bar{\Phi},\bar{n}) \bar{E}_{x_0,\bar{\Phi},\bar{n}}\Big(\mathbb{1}_{\{\bar{X}_{\bar{T}}=x_0,\ \forall e\in E,\ \bar{n}_e(\bar{T})\geq 0\}}$$

$$G((\bar{Z}_t)_{t\leq\bar{T}}) F(\bar{\Phi}(\bar{T}), \bar{n}(\bar{T})) \prod_{x\in V\setminus\{x_0\}} \frac{\bar{\Phi}_x}{\bar{\Phi}_x(\bar{T})}\Big),$$

where the integral on the l.h.s. is on the set $\{(\Phi_x)_{x\in V}\in\mathbb{R}_+^V|\Phi_{x_0}=0\}$ *with*

$$d\Phi=\frac{\prod_{x\in V\setminus\{x_0\}}d\Phi_x}{\sqrt{2\pi}^{|V|-1}}$$

and the integral on the r.h.s. is on the set $\{(\bar\Phi_x)_{x\in V}\in\mathbb{R}_+^V|\bar\Phi_{x_0}=u\}$ *with*

$$d\bar\Phi=\frac{\prod_{x\in V\setminus\{x_0\}}d\bar\Phi_x}{\sqrt{2\pi}^{|V|-1}}.$$

Proof. We start from the left-hand side, i.e. the process $(X_t,n_e(t))_{0\le t\le\tau_u^{x_0}}$. We define

$$\bar X_t=X_{\tau_u^{x_0}-t},\qquad \bar n_e(t)=n_e(\tau_u^{x_0}-t),$$

and

$$\bar\Phi_x=\Phi_x(\tau_u^{x_0}),\qquad \bar\Phi_x(t)=\Phi_x(\tau_u^{x_0}-t),$$

The law of the processes such defined will later be identified with the law of the processes $(\bar X_t,\bar\Phi(t),\bar n(t))$ defined at the beginning of step 2, see (4.4) and (4.5). We also set

$$\bar K_e(t)=K_e(\tau_u^{x_0})-K_e(t),$$

which is also the number of crossings of the edge e by the process $\bar X$, between time 0 and t. With these notations we clearly have

$$\bar\Phi_x(t)=\sqrt{\bar\Phi_x^2-2\bar\ell_x(t)},$$

where $\bar\ell_x(t)=\int_0^t\mathbb{1}_{\{\bar X_s=x\}}ds$ is the local time of $\bar X$ at time t, and

$$\bar n_e(t)=\bar n_e(0)+(N_e(J_e(\bar\Phi(t)))-N_e(J_e(\bar\Phi(0))))-\bar K_e(t).$$

By time reversal, the law of $(\bar X_t)_{0\le s\le\bar\tau_u^{x_0}}$ is the same as the law of the Markov Jump process $(X_t)_{0\le t\le\tau_u^{x_0}}$, where $\bar\tau_u^{x_0}=\inf\{t\ge0|\bar\ell_{x_0}(t)=u\}$. Hence, we see that up to the time

$$\bar T=\inf\{t\ge0|\exists x\ \bar\Phi_x(t)=0\},$$

the process $(\bar X_t,(\bar\Phi_x(t))_{x\in V},(\bar n_e(t))_{e\in E})_{t\le\bar T}$ has the same law as the process defined at the beginning of step 2.

Then, following [21], we make the following change of variables conditional on the processes $(X_t,(n_e(t))_{e\in E})$:

$$\begin{array}{ccc}(0,+\infty)^V\times\mathbb{N}^E&\to&(0,+\infty)^V\times\mathbb{N}^E\\((\Phi_x)_{x\in V},(n_e)_{e\in E})&\mapsto&((\bar\Phi_x)_{x\in V},(\bar n_e)_{e\in E}),\end{array}$$

which is bijective onto the set

$$\{(\bar\Phi_x)_{x\in V}\in\mathbb{R}_+^V|\bar\Phi_{x_0}=\sqrt{2u},\ \forall x\neq x_0,\ \check\Phi_x>\sqrt{2\ell_x(\tau_u^{x_0})}\}$$
$$\times\{(\bar n_e)_{e\in E}\in\mathbb{N}^E|\forall e\in E,\ \bar n_e\ge K_e(\tau_u^{x_0})+(N_e(J_e(\bar\Phi(\tau_u^{x_0})))-N_e(J_e(\Phi)))\}.$$

Note that we always have $\bar\Phi_{x_0}=\sqrt{2u}$. The last conditions on $\bar\Phi$ and $\bar n_e$ are equivalent to the conditions $\bar X_{\bar T}=x_0$ and $\bar n_e(\bar T)\ge0$. The Jacobian of the change of variable is given by

$$\prod_{x\in V\setminus\{x_0\}}d\Phi_x=\left(\prod_{x\in V\setminus\{x_0\}}\frac{\bar\Phi_x}{\Phi_x}\right)\prod_{x\in V\setminus\{x_0\}}d\bar\Phi_x,$$

since

$$d\bar\Phi_x=d\sqrt{\Phi_x^2+2\ell_x(\tau_u^{x_0})}=\frac{\Phi_x}{\sqrt{\Phi_x^2+2\ell_x(\tau_u^{x_0})}}\,d\Phi_x=\frac{\Phi_x}{\bar\Phi_x}\,d\Phi_x.$$

□

Step 3: With the notations of Theorem 12, we consider the following expectation for g and h bounded measurable test functions:

$$\mathbb{E}\left(g\left(\left(X_{\tau_u^{x_0}-t}, n_e(\tau_u^{x_0}-t)\right)_{0\le t\le \tau_u^{x_0}}\right) h(\varphi^{(u)})\right). \tag{4.7}$$

By definition, we have

$$\varphi^{(u)} = \sigma\Phi(\tau_u^{x_0}),$$

where $(\sigma_x)_{x\in V} \in \{-1,+1\}^V$ are random signs sampled uniformly independently on clusters induced by $\{e\in E | n_e(\tau_u^{x_0})>0\}$ and conditioned on $\sigma_{x_0}=+1$. Hence, we define for $(\Phi_x)_{x\in V}\in\mathbb{R}_+^V$ and $(n_e)_{e\in E}\in\mathbb{N}^E$,

$$H(\Phi,n) = 2^{-\#c.\mathcal{C}(n)+1} \sum_{\substack{\sigma\in\{\pm1\}^V \\ \sigma_{x_0}=+1,\ \sigma\ll n}} h(\sigma\Phi), \tag{4.8}$$

where $\sigma\ll n$ means that the signs σ_x are constant on clusters of $\mathcal{C}(n)$. Hence, setting

$$F(\Phi,n) = e^{-\frac{1}{2}\sum_{x\in V} W_x\Phi_x^2 - \sum_{e\in E} J_e(\Phi)} \left(\prod_{e\in E} \frac{(2J_e(\Phi))^{n_e}}{n_e!}\right) 2^{\#c.\mathcal{C}(n)-1},$$

$$G\left((Z_{\tau_u^{x_0}-t})_{t\le\tau_u^{x_0}}\right) = g\left(\left(X_{\tau_u^{x_0}-t}, n_e(\tau_u^{x_0}-t)\right)_{t\le\tau_u^{x_0}}\right),$$

using Lemma 4.3 in the first equality and Lemma 4.4 in the second equality, we deduce that (4.7) is equal to

$$\begin{aligned}
&\mathbb{E}\left(G\left((Z_{\tau_u^{x_0}-t})_{0\le t\le\tau_u^{x_0}}\right) H(\Phi(\tau_u^{x_0}), n(\tau_u^{x_0})))\right) = \\
&\quad\sum_{n\in\mathbb{N}^E}\int d\Phi F(\Phi,n) E_{x_0,\Phi,n}\left(G\left((Z_{\tau_u^{x_0}-t})_{t\le\tau_u^{x_0}}\right) H\left(\Phi(\tau_u^{x_0}), n(\tau_u^{x_0})\right)\right) d\Phi = \\
&\qquad\sum_{\bar n\in\mathbb{N}^E}\int d\bar\Phi H\left(\bar\Phi,\bar n\right) \bar E_{x_0,\bar\Phi,\bar n}\Big(\mathbb{1}_{\{\bar X_{\bar T}=x_0,\ \forall e\in E\ \bar n_e(\bar T)\ge 0\}} \\
&\qquad\qquad F\left(\bar\Phi(\bar T),\bar n(\bar T)\right) G\left((\bar Z_t)_{t\le\bar T}\right) \prod_{x\in V\setminus\{x_0\}} \frac{\bar\Phi_x}{\bar\Phi_x(\bar T)}\Big),
\end{aligned} \tag{4.9}$$

with notations of Lemma 4.4.

Let $\bar{\mathcal{F}}_t = \sigma((\bar X_s)_{s\le t})$ be the filtration generated by $\bar X$. We define the $\bar{\mathcal{F}}$-adapted process $\bar M_t$, defined up to time $\bar T$ by

$$\begin{aligned}
\bar M_t = \frac{F(\bar\Phi(t),\bar n(t))}{\prod_{x\in V\setminus\{\bar X_t\}}\bar\Phi_x(t)} \mathbb{1}_{\{\bar X_t\in\mathcal{C}(x_0,\bar n)\}}\mathbb{1}_{\{\bar n_e(t)\ge0,\ \forall e\in E\}} = e^{-\frac{1}{2}\sum_{x\in V} W_x\bar\Phi_x(t)^2 - \sum_{e\in E} J_e(\bar\Phi(t))}\times \\
\times\Big(\prod_{e\in E}\frac{(2J_e(\bar\Phi(t)))^{\bar n_e(t)}}{\bar n_e(t)!}\Big)\frac{2^{\#c.\mathcal{C}(\bar n(t))-1}}{\prod_{x\in V\setminus\{\bar X_t\}}\bar\Phi_x(t)}\mathbb{1}_{\{\bar X_t\in\mathcal{C}(x_0,\bar n(t))\}}\mathbb{1}_{\{\bar n_e(t)\ge0,\ \forall e\in E\}},
\end{aligned} \tag{4.10}$$

where $\mathcal{C}(x_0,\bar n(t))$ denotes the cluster of the origin x_0 induced by the configuration $\mathcal{C}(\bar n(t))$. Note that at time $t=\bar T$, we also have

$$\bar M_{\bar T} = \frac{F(\bar\Phi(\bar T),\bar n(\bar T))}{\prod_{x\in V\setminus\{x_0\}}\bar\Phi_x(\bar T)}\mathbb{1}_{\{\bar X_{\bar T}=x_0\}}\mathbb{1}_{\{\bar n_e(t)\ge0,\ \forall e\in E\}} \tag{4.11}$$

since $\bar M_{\bar T}$ vanishes on the event where $\{\bar X_{\bar T}=x\}$, with $x\neq x_0$. Indeed, if $\bar X_{\bar T}=x\neq x_0$, then $\bar\Phi_x(\bar T)=0$ and $J_e(\bar\Phi(\bar T))=0$ for $e\in E$ such that e adjacent to x. It means that $\bar M_{\bar T}$ is equal to 0 if $\bar n_e(\bar T)>0$ for some edge e neighboring x. Thus, $\bar M_{\bar T}$ is null unless $\{x\}$ is a cluster in $\mathcal{C}(\bar{n}(\bar{T}))$. Hence, $\bar{M}_{\bar{T}} = 0$ if $x \neq x_0$ since $\bar{M}_{\bar{T}}$ contains the indicator of the event that $\bar{X}_{\bar{T}}$ and x_0 are in the same cluster.

Hence, using identities (4.9) and (4.11) we deduce that (4.7) is equal to

$$(4.7) \quad = \quad \sum_{\bar{n}\in\mathbb{N}^E} \int d\bar{\Phi} H\left(\bar{\Phi},\bar{n}\right) F\left(\bar{\Phi},\bar{n}\right) \bar{E}_{x_0,\bar{\Phi},\bar{n}} \left(\frac{\bar{M}_{\bar{T}}}{\bar{M}_0} G\left((\bar{Z}_t)_{t\leq\bar{T}}\right)\right). \tag{4.12}$$

Step 4: We denote by $\check{Z}_t = (\check{X}_t, \check{\Phi}_t, \check{n}(t))$ the process defined in Section 3.1, which is well defined up to stopping time $\check{T}$, and $\check{Z}_t^T = \check{Z}_{t\wedge\check{T}}$. We denote by $\check{E}_{x_0,\check{\Phi},\check{n}}$ the law of the process $\check{Z}$ conditional on the initial value $\check{n}(0)$, i.e. conditional on $(N_e(2J(\check{\Phi})))_{e\in E} = (\check{n}_e)_{e\in E}$. The last step of the proof goes through the following lemma.

Lemma 4.5. *i) Under $\check{E}_{x_0,\check{\Phi},\check{n}}$, $\check{X}$ ends at $\check{X}_{\check{T}} = x_0$ a.s. and $\check{n}_e(\check{T}) \geq 0$ for all $e \in E$.*

ii) Let $\bar{P}^{\leq t}_{x_0,\bar{\Phi},\bar{n}}$ and $\check{P}^{\leq t}_{x_0,\check{\Phi},\check{n}}$ be the law of the process $(\bar{Z}^T_s)_{s\leq t}$ and $(\check{Z}^T_s)_{s\leq t}$ respectively, then

$$\frac{d\check{P}^{\leq t}_{x_0,\bar{\Phi},\bar{n}}}{d\bar{P}^{\leq t}_{x_0,\check{\Phi},\check{n}}} = \frac{\bar{M}_{t\wedge\bar{T}}}{\bar{M}_0}.$$

Using this lemma we obtain that in the right-hand side of (4.12)

$$\bar{E}_{x_0,\bar{\Phi},\bar{n}} \left(\frac{\bar{M}_{\bar{T}}}{\bar{M}_0} G\left((\bar{Z}_t)_{t\leq\bar{T}}\right)\right) = \check{E}_{x_0,\bar{\Phi},\bar{n}} \left(G\left((\check{Z}_t)_{t\leq\check{T}}\right)\right).$$

Hence, we deduce, using formula (4.8) and proceeding as in Lemma 4.3, that (4.7) is equal to

$$\int_{\mathbb{R}^{V\setminus\{x_0\}}} d\bar{\varphi} e^{-\frac{1}{2}\mathcal{E}(\bar{\varphi},\bar{\varphi})} h(\bar{\varphi}) \sum_{\bar{n}\ll\bar{\varphi}} \Bigg(\prod_{\substack{e\in E\\ \bar{\varphi}_{e_-}\bar{\varphi}_{e_+}\geq 0}} \frac{e^{-2J_e(|\bar{\varphi}|)}(2J_e(|\bar{\varphi}|))^{\bar{n}_e}}{\bar{n}_e!} \Bigg) \bar{E}_{x_0,|\bar{\varphi}|,\bar{n}} \left(\frac{\bar{M}_{\bar{T}}}{\bar{M}_0} G\left((\bar{Z}_t)_{t\leq\bar{T}}\right)\right),$$

where the last integral is on the set $\{(\bar{\varphi}_x)_{x\in V} \in \mathbb{R}^V | \varphi_{x_0} = u\}$,

$$d\bar{\varphi} = \frac{\prod_{x\in V\setminus\{x_0\}} d\bar{\varphi}_x}{\sqrt{2\pi}^{|V|-1}},$$

and where $\bar{n} \ll \bar{\varphi}$ means that $\bar{n}_e = 0$ if $\bar{\varphi}_{e_-}\bar{\varphi}_{e_+} \leq 0$. Finally, we conclude that

$$\mathbb{E}\left[g\left(\left(X_{\tau_u^{x_0}-t}, n_e(\tau_u^{x_0}-t)\right)_{0\leq t\leq\tau_u^{x_0}}\right) h(\varphi^{(u)})\right] = \mathbb{E}\left[g\left(\left(\check{X}_t, \check{n}_e(t)\right)_{0\leq t\leq\check{T}}\right) h(\check{\varphi})\right],$$

where in the right-hand side $\check{\varphi} \sim P_\varphi^{\{x_0\},\sqrt{2u}}$ is a GFF and $(\check{X}_t, \check{n}(t))$ is the process defined in Section 3.1 from the GFF $\check{\varphi}$. This exactly means that $\varphi^{(u)} \sim P_\varphi^{\{x_0\},\sqrt{2u}}$ and that

$$\text{Law}\left(\left(X_{\tau_u^{x_0}-t}, n_e(\tau_u^{x_0}-t)\right)_{0\leq t\leq\tau_u^{x_0}} \;\Big|\; \varphi^{(u)} = \check{\varphi}\right) = \text{Law}\left(\left(\check{X}_t, \check{n}(t)\right)_{t\leq\check{T}}\right).$$

This concludes the proof of Theorem 12. □

Proof of Lemma 4.5. The generator of the process $\bar{Z}_t$ defined in (4.6) is given, for any bounded and $\mathcal{C}^1$ for the second component test function f, by

$$\begin{aligned}&(\bar{\mathfrak{L}}f)(x,\bar{\Phi},\bar{n}) = -\frac{1}{\bar{\Phi}_x}\Big(\frac{\partial}{\partial\bar{\Phi}_x} f\Big)(x,\bar{\Phi},\bar{n}) \\ &+ \sum_{\substack{y\in V\\ y\sim x}} \left(W_{x,y}\left(f(y,\bar{\Phi},\bar{n}-\delta_{\{x,y\}}) - f(x,\bar{\Phi},n)\right) + W_{x,y}\frac{\bar{\Phi}_y}{\bar{\Phi}_x}\left(f(x,\bar{\Phi},n-\delta_{\{x,y\}}) - f(x,\bar{\Phi},n)\right)\right).\end{aligned} \tag{4.13}$$

where $n - \delta_{\{x,y\}}$ is the value obtained by removing 1 from n at edge $\{x,y\}$. Indeed, since $\bar{\Phi}_x(t) = \sqrt{\bar{\Phi}_x(0)^2 - 2\bar{\ell}_x(t)}$, we have

$$\frac{d}{dt}\bar{\Phi}_x(t) = -\mathbb{1}_{\{\bar{X}_t = x\}}\frac{1}{\bar{\Phi}_x(t)}, \tag{4.14}$$

which explains the first term in the expression. The second term is obvious from the definition of $\bar{Z}_t$, and corresponding to the term induced by jumps of the Markov process $\bar{X}_t$. The last term corresponds to the decrease of $\bar{n}$ due to the increase in the process $N_e(J_e(\bar{\Phi})) - N_e(J_e(\bar{\Phi}(t)))$. Indeed, on the interval $[t, t+dt]$, the probability that $N_e(J_e(\bar{\Phi}(t))) - N_e(J_e(\bar{\Phi}(t+dt)))$ is equal to 1 is of order

$$-\frac{d}{dt}N_e(J_e(\bar{\Phi}(t)))dt = \mathbb{1}_{\{\bar{X}_t \text{ endpoint of } e\}}\frac{W_e\bar{\Phi}_{e_-}(t)\bar{\Phi}_{e_+}(t)}{\Phi_{\bar{X}_t}(t)^2}dt,$$

using identity (4.14).

Let $\check{\mathfrak{L}}$ be the generator of the Markov jump process $\check{Z}_t = (\check{X}_t, (\check{\Phi}_x(t)), (\check{n}_e(t)))$. We have that the generator is equal, for any smooth test function f, to

$$\begin{aligned}(\check{\mathfrak{L}}f)(x,\Phi,n) = &-\frac{1}{\Phi_x}(\frac{\partial}{\partial\Phi_x}f)(x,\Phi,n) \\ &+\frac{1}{2}\sum_{\substack{y\in V\\ y\sim x}}\frac{n_{x,y}}{\Phi_x^2}\mathbb{1}_{\mathcal{A}_1(x,y)}\left(f(y,\bar{\Phi},n-\delta_{\{x,y\}}) + f(x,\bar{\Phi},n-\delta_{\{x,y\}}) - 2f(x,\bar{\Phi},n)\right) \\ &+\sum_{\substack{y\in V\\ y\sim x}}\frac{n_{x,y}}{\Phi_x^2}\mathbb{1}_{\mathcal{A}_2(x,y)}\left(f(y,\bar{\Phi},n-\delta_{\{x,y\}}) - f(x,\bar{\Phi},n))\right) \\ &+\sum_{\substack{y\in V\\ y\sim x}}\frac{n_{x,y}}{\Phi_x^2}\mathbb{1}_{\mathcal{A}_3(x,y)}\left(f(x,\bar{\Phi},n-\delta_{\{x,y\}}) - f(x,\bar{\Phi},n)\right),\end{aligned} \tag{4.15}$$

where $\mathcal{A}_i(x,y)$ correspond to the following disjoint events:

- $\mathcal{A}_1(x,y)$ if the numbers of connected clusters induced by $n - \delta_{\{x,y\}}$ is the same as that of $\check{n}$;
- $\mathcal{A}_2(x,y)$ if a new cluster is created in $n - \delta_{\{x,y\}}$ compared with $\check{n}$ and if y is in the connected component of x_0 in the cluster induced by $n - \delta_{\{x,y\}}$;
- $\mathcal{A}_3(x,y)$ if a new cluster is created in $n - \delta_{\{x,y\}}$ compared with n and if x is in the connected component of x_0 in the cluster induced by $n - \delta_{\{x,y\}}$.

Indeed, conditional on the value of $\check{n}_e(t) = N_e(2J_e(\check{\Phi}(t)))$ at time t, the point process N_e on the interval $[0, 2J_e(\check{\Phi}(t))]$ has the law of $n_e(t)$ independent points with uniform distribution on $[0, 2J_e(\check{\Phi}(t))]$. Hence, the probability that a point lies in the interval $[2J_e(\check{\Phi}(t+dt)), 2J_e(\check{\Phi}(t))]$ is of order

$$-\check{n}_e(t)\frac{1}{J_e(\check{\Phi}(t))}\frac{d}{dt}J_e(\check{\Phi}(t))dt = \mathbb{1}_{\{X_t \text{ endpoint of } e\}}\;\check{n}_e(t)\frac{1}{\check{\Phi}_{X_t}(t)^2}dt.$$

We define the function

$$\begin{aligned}&\Theta(x,(\Phi_x),(n_e)) = \\ &e^{-\frac{1}{2}\sum_{x\in V}W_x\Phi_x^2 - \sum_{e\in E}J_e(\Phi)}\left(\prod_{e\in E}\frac{(2J_e(\Phi))^{n_e}}{n_e!}\right)\frac{2^{\#c.\mathcal{C}(n)-1}}{\prod_{y\in V\setminus\{x\}}\Phi_y}\mathbb{1}_{\{x\in\mathcal{C}(x_0,n),\text{ and }\forall e\in E,\ n_e\geq 0\}},\end{aligned}$$

so that

$$\bar{M}_{t\wedge\bar{T}} = \Theta(\bar{Z}_{t\wedge\bar{T}}).$$

To prove the lemma it is sufficient to prove ([3], Chapter 11) that for any bounded smooth test function f,

$$\frac{1}{\Theta}\bar{\mathfrak{L}}\left(\Theta f\right) = \check{\mathfrak{L}}\left(f\right). \tag{4.16}$$

Let us first consider the first term in (4.13). Direct computation gives

$$\left(\frac{1}{\Theta}\frac{1}{\Phi_x}\left(\frac{\partial}{\partial\Phi_x}\Theta\right)\right)(x,\Phi,n) = -W_x + \sum_{\substack{y\in V\\ y\sim x}}\left(-W_{x,y}\frac{\Phi_y}{\Phi_x} + n_{x,y}\frac{1}{\Phi_x^2}\right).$$

For the second part, remark that the indicators $\mathbb{1}_{\{x\in\mathcal{C}(x_0,n)\}}$ and $\mathbb{1}_{\{n_e\geq 0,\ \forall e\in E\}}$ imply that $\Theta(y,\Phi,n-\delta_{\{x,y\}})$ vanishes if $n_{x,y}=0$ or if $y\notin\mathcal{C}(x_0,n-\delta_{\{x,y\}})$. By inspection of the expression of Θ, we obtain for $x\sim y$,

$$\begin{aligned}\Theta(y,\Phi,n-\delta_{\{x,y\}}) &= \left(\mathbb{1}_{\{n_{x,y}>0\}}(\mathbb{1}_{\mathcal{A}_1}+2\mathbb{1}_{\mathcal{A}_2})\frac{n_{x,y}}{2J_{x,y}(\Phi)}\frac{\Phi_y}{\Phi_x}\right)\Theta(x,\Phi,n)\\ &= \left((\mathbb{1}_{\mathcal{A}_1}+2\mathbb{1}_{\mathcal{A}_2})\frac{n_{x,y}}{2W_{x,y}}\frac{1}{\Phi_x^2}\right)\Theta(x,\Phi,n).\end{aligned}$$

Similarly, for $x\sim y$,

$$\begin{aligned}\Theta(x,\Phi,n-\delta_{\{x,y\}}) &= \left(\mathbb{1}_{\{n_{x,y}>0\}}(\mathbb{1}_{\mathcal{A}_1}+2\mathbb{1}_{\mathcal{A}_3})\frac{n_{x,y}}{2J_{x,y}}\right)\Theta(x,\Phi,n)\\ &= \left((\mathbb{1}_{\mathcal{A}_1}+2\mathbb{1}_{\mathcal{A}_3})\frac{n_{x,y}}{2W_{x,y}\Phi_x\Phi_y}\right)\Theta(x,\Phi,n).\end{aligned}$$

Combining these three identities with the expression (4.13) we deduce

$$\begin{aligned}\frac{1}{\Theta}\bar{\mathfrak{L}}\left(\Theta f\right)(x,\Phi,n) &= -\frac{1}{\Phi_x}\frac{\partial}{\partial\Phi_x}f(x,\Phi,n) - \sum_{\substack{y\in V\\ y\sim x}}\left(n_{x,y}\frac{1}{\Phi_x^2}\right)f(x,\Phi,n)\\ &\quad+\sum_{\substack{y\in V\\ y\sim x}}(\mathbb{1}_{\mathcal{A}_1}+2\mathbb{1}_{\mathcal{A}_2})n_{x,y}\frac{1}{2\Phi_x^2}f(y,n-\delta_{\{x,y\}},\Phi)\\ &\quad+\sum_{\substack{y\in V\\ y\sim x}}(\mathbb{1}_{\mathcal{A}_1}+2\mathbb{1}_{\mathcal{A}_3})\frac{1}{2\Phi_x^2}f(x,n-\delta_{\{x,y\}},\Phi).\end{aligned}$$

It exactly coincides with the expression (4.15) for $\check{\mathfrak{L}}$ since $1=\mathbb{1}_{\mathcal{A}_1}+\mathbb{1}_{\mathcal{A}_2}+\mathbb{1}_{\mathcal{A}_3}$. □

4.2 General case

Proposition 4.6. *The conclusion of Theorem 9 still holds if the graph $\mathcal{G}=(V,E)$ is finite and the killing measure is non-zero ($\kappa\not\equiv 0$).*

Proof. Let h be the function on V defined as

$$h(x) = \mathbb{P}_x(X \text{ hits } x_0 \text{ before } \zeta).$$

By definition $h(x_0)=1$. Moreover, for all $x\in V\setminus\{x_0\}$,

$$-\kappa_x h(x) + \sum_{\substack{y\in V\\ y\sim x}}W_{x,y}(h(y)-h(x)) = 0.$$

Define the conductances $W^h_e := W_e h(e_-)h(e_+)$, and the corresponding jump process X^h, and the GFF $\varphi^{(0)}_h$ and $\varphi^{(u)}_h$ with conditions 0 respectively $\sqrt{2u}$ at x_0. The Theorem 9 holds for the graph $\mathcal{G}$ with conductances $(W^h_e)_{e\in E}$ and with zero killing measure. Denote

$$\ell^h_x(t) = \int_0^t \mathbb{1}_{\{X^h_s = x\}} ds, \qquad \tau^{x,h}_u = \inf\{t \geq 0 | \ell^h_{x_0}(t) \geq \sqrt{2u}\}.$$

The process $(X^h_t)_{t\leq \tau^{x_0,h}_u}$ has the same law as the process $(X_{\theta^h(t)})_{t\leq (\theta^h)^{-1}(\tau^{x_0}_u)}$, conditional on $\tau^{x_0}_u < \zeta$, after the change of time

$$d\theta^h(t) = h(X_{\theta^h(t)})^2 dt.$$

This means in particular that for the occupation times,

$$\ell^h_x(t) = h(X_{\theta^h(t)})^{-2} \ell_x(\theta^h(t)). \tag{4.17}$$

Moreover, we have the equalities in law

$$\varphi^{(0)}_h \overset{\text{law}}{=} h^{-1}\varphi^{(0)}, \qquad \varphi^{(u)}_h \overset{\text{law}}{=} h^{-1}\varphi^{(u)}.$$

Indeed, at the level of energy functions, we have:

$$\begin{aligned}
\mathcal{E}(hf,hf) &= \sum_{x\in V} \kappa_x h(x)^2 f(x)^2 + \sum_{e\in E} W_e (h(e_+)f(e_+) - h(e_-)f(e_-))^2 \\
&= \sum_{x\in V} [\kappa_x h(x)^2 f(x)^2 + \sum_{\substack{y\in V\\ y\sim x}} W_{x,y} h(y)f(y)(h(y)f(y) - h(x)f(x))] \\
&= \sum_{x\in V} [\kappa_x h(x)^2 f(x)^2 - \sum_{y\sim x} W_{x,y}(h(y) - h(x))h(x)f(x)^2] \\
&\quad - \sum_{\substack{x\in V\\ y\sim x}} W_{x,y} h(x)h(y)(f(y) - f(x))f(x) \\
&= [\kappa_{x_0} - \sum_{\substack{y\in V\\ y\sim x_0}} W_{x_0,y}(h(y) - 1)]f(x_0)^2 + \sum_{e\in E} W^h_e (h(e_+)f(e_+) - h(e_-)f(e_-))^2 \\
&= \text{Cst}(f(x_0)) + \mathcal{E}^h(f,f),
\end{aligned}$$

where $\text{Cst}(f(x_0))$ means that this term does not depend of f once the value of the function at x_0 fixed.

Let $\check{X}^h_t$ be the inverse process for the conductances $(W^h_e)_{e\in E}$ and the initial condition for the field $\varphi^{(u)}_h$, given by Theorem 9. By applying the inverse of the time change (4.17) to the process $\check{X}^h_t$, we obtain an inverse process for the conductances W_e and the field $\varphi^{(u)}$. □

Proposition 4.7. *Assume that the graph $\mathcal{G} = (V,E)$ is infinite. The killing measure κ may be non-zero. Then the conclusion of Theorem 9 holds.*

Proof. Consider an increasing sequence of connected sub-graphs $\mathcal{G}_i = (V_i, E_i)$ of $\mathcal{G}$ which converges to the whole graph. We assume that V_0 contains x_0. Let $\mathcal{G}^*_i = (V^*_i, E^*_i)$ be the graph obtained by adding to $\mathcal{G}_i$ an abstract vertex x_*, and for every vertex $x \in V_i$ connected by an edge in E_i to a $y \in V\setminus V_i$, adding an edge $\{x, x_*\}$ with a conductance

$$W_{x,x_*} = \sum_{\substack{y\in V\setminus V_i\\ y\sim x}} W_{x,y}.$$

$(X_{i,t})_{t\geq 0}$ will denote the Markov jump process on $\mathcal{G}_i^*$, started from x_0. Let ζ_i be the first hitting time of x_* or the first killing time by the measure $\kappa\mathbb{1}_{V_i}$. Let $\varphi_i^{(0)}$, $\varphi_i^{(u)}$ will denote the GFFs on $\mathcal{G}_i^*$ with condition 0 respectively $\sqrt{2u}$ at x_0, with condition 0 at x_*, and taking in account the possible killing measure $\kappa\mathbb{1}_{V_i}$. The limits in law of $\varphi_i^{(0)}$ respectively $\varphi_i^{(u)}$ are $\varphi^{(0)}$ respectively $\varphi^{(u)}$.

We consider the process $(\check{X}_{i,t},(\check{n}_{i,e}(t))_{e\in E_i^*})_{0\leq t\leq \check{T}_i}$ be the inverse process on $\mathcal{G}_i^*$, with initial field $\varphi_i^{(u)}$. $(X_{i,t})_{t\leq \tau_{i,u}^{x_0}}$, conditional on $\tau_{i,u}^{x_0}$, has the same law as $(\check{X}_{i,\check{T}_i-t})_{t\leq \check{T}_i}$. Taking the limit in law as i tends to infinity, we conclude that $(X_t)_{t\leq \tau_u^{x_0}}$, conditional on $\tau_u^{x_0}<+\infty$, has the same law as $(\check{X}_{\check{T}-t})_{t\leq \check{T}}$ on the infinite graph $\mathcal{G}$. The same for the clusters. In particular,

$$\mathbb{P}(\check{T}\leq t,\check{X}_{[0,\check{T}]}\text{ stays in }V_j)\geq \lim_{i\to+\infty}\mathbb{P}(\check{T}_i\leq t,\check{X}_{i,[0,\check{T}_i]}\text{ stays in }V_j)$$
$$=\lim_{i\to+\infty}\mathbb{P}(\tau_{i,u}^{x_0}\leq t,X_{i,[0,\tau_{i,u}^{x_0}]}\text{ stays in }V_j|\tau_{i,u}^{x_0}<\zeta_i)=\mathbb{P}(\tau_u^{x_0}\leq t,X_{[0,\tau_u^{x_0}]}\text{ stays in }V_j|\tau_u^{x_0}<\zeta),$$

where in the first two probabilities we also average by the values of the free fields. Hence

$$\mathbb{P}(\check{T}=+\infty\text{ or }\check{X}_{\check{T}}\neq x_0)=1-\lim_{\substack{t\to+\infty\\ j\to+\infty}}\mathbb{P}(\tau_u^{x_0}\leq t,X_{[0,\tau_u^{x_0}]}\text{ stays in }V_j|\tau_u^{x_0}<\zeta)=0.\qquad\square$$

Remark 4.8. *Consider $\mathcal{G}=(V,E)$ an infinite transient electrical network (with $\kappa\equiv 0$). Proposition 4.7 tells that if the inversions algorithm of Section 3 is applied to a Gaussian free field $\varphi^{(u)}$ with condition $\sqrt{2u}$ at x_0, and implicitly 0 at infinity, the algorithm terminates a.s., that is to say the inverting process $\check{X}$ does not escape to infinity. However, one could consider a Gaussian free field with positive condition $a>0$ at infinity, $\varphi^{(u,a)}$. Such a GFF is related by isomorphism not only to a loop-soup $\mathcal{L}_{1/2}$ but also to a Sznitman's random interlacement, which is a Poisson point process of paths from and to infinity, infinite in both directions of time [23, 24, 15]. If applied to $\varphi^{(u,a)}$, the algorithm would create a path which has a positive probability to escape to infinity, which would correspond to the event of having an interlacement visiting x_0.*

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