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Random field solutions to linear SPDEs driven by symmetric pure jump Lévy space-time white noises

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Abstract

We study the notions of mild solution and generalized solution to a linear stochastic partial differential equation driven by a pure jump symmetric Lévy white noise, with symmetric α-stable Lévy white noise as an important special case. We identify conditions for existence of these two kinds of solutions, and, together with a new stochastic Fubini theorem, we provide conditions under which they are essentially equivalent. We apply these results to the linear stochastic heat, wave and Poisson equations driven by a symmetric α-stable Lévy white noise.

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1 Introduction

In this article, we consider a linear stochastic partial differential equation (SPDE) of the form

$$\mathcal{L}u = \dot{X}\,, \tag{1.1}$$

where $\mathcal{L}$ is a partial differential operator and $\dot{X}$ is a symmetric pure jump Lévy white noise. We study two different notions of solution to (1.1). On the one hand, from the random field approach to SPDEs, we have the concept of mild solution, which is a random field defined as the convolution of a fundamental solution of $\mathcal{L}$ with the noise. The mild solution is therefore defined as a stochastic integral, and some conditions are needed for its existence. For example, in the case of Gaussian white noise, the fundamental solution must be square integrable. The literature for the existence of mild solutions to SPDEs in the Gaussian case is already quite extensive (see [7, 13] for introductory lectures, and see [8, 6] for more advanced presentations). The case of Lévy noise has been less

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studied, but the existence of mild solutions for various equations has been considered in [1, 3], and the approach via evolution equations is considered in [17].

On the other hand, from the general theory of (deterministic) partial differential equations, we have the notion of weak solution, or solution in the sense of (Schwartz) distributions. Since the terms "weak" and "distribution" are often used with another meaning, we will instead use the term "generalized solution," in the spirit of the book [10].

In this article, we are interested in the link between the notions of mild solution and generalized solution to the linear stochastic partial differential equation (1.1). More precisely, the questions that we study are the following:

(1) When it can be defined, is a mild solution also a generalized solution?
(2) When a generalized solution exists, under what conditions can it be represented by a random field?
(3) What kinds of solution exist in the case of the stochastic heat equation, the stochastic wave equation, or the stochastic Poisson equation driven by an α-stable noise?

A question related to (2) was studied in [8, Theorem 11]. More precisely, this reference gives a necessary condition on the Green's function of the differential operator for the existence of a random field representation (see Definition 3.4) for the generalized solution to an SPDE driven by a Gaussian colored noise.

To answer the above questions, we first give a precise definition in Section 3 of the two different notions of solution to a linear SPDE. Then, in Section 4, we provide a complete answer to questions (1) and (2) in the α-stable case (Theorem 4.1). In particular, we give a necessary and sufficient condition on the fundamental solution ρ of $\mathcal{L}$ (condition **(INT)** in Section 4) for the generalized solution to have a random field representation. In this case, the mild solution is a random field representation of the generalized solution.

These results are extended to the case of symmetric pure jump Lévy white noise in Section 5 (Theorem 5.2). The restriction to *symmetric* Lévy white noise come from the fact that in the symmetric case, the stochastic integral is an isomorphism between the space of functions that are integrable with respect to the noise and their stochastic integrals with respect to this noise [18, Theorem 3.4 and Proposition 3.6]. To prove these extensions, we establish a new stochastic Fubini's theorem that applies to symmetric pure jump Lévy white noise (Theorem 5.1) and is interesting in its own right.

Finally, we address question (3) in Section 6, where we apply the above results to the case of the stochastic heat, wave equation and Poisson equations in all spatial dimensions. The main results can be found in Theorems 6.6, 6.12 and 6.13.

2 Notations and main definitions

We begin by recalling some properties of generalized functions and of Lévy white noise.

2.1 Generalized functions

We will denote by $\mathcal{D}(\mathbb{R}^d)$ the space of C^∞ compactly supported functions, and $\mathcal{D}'(\mathbb{R}^d)$ its topological dual, the space of distributions or generalized functions (as defined and studied in [22]). The evaluation of $\rho \in \mathcal{D}'(\mathbb{R}^d)$ on $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is denoted $\langle \rho, \varphi \rangle$. We suppose that $\mathcal{L}$ is a partial differential operator with adjoint $\mathcal{L}^*$ (typically, $\mathcal{L}$ may be the heat or wave operator). We consider a fundamental solution $\rho \in \mathcal{D}'(\mathbb{R}^d)$ of the operator $\mathcal{L}$, that is a solution to

$$\mathcal{L}\rho = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{2.1}$$

where δ_0 denotes the Dirac delta function. The fundamental solution is not always unique (and choosing this solution typically amounts to imposing initial and/or boundary conditions), and in the following, we fix the choice of ρ. We recall the definition [22] of the convolution between a distribution ρ and a smooth function φ with compact support:

$$\varphi * \rho(t) := \langle \rho, \varphi(t - \cdot) \rangle . \tag{2.2}$$

Note that this convolution defines a C^∞ function. Also, for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we define $\langle \check{\rho}, \varphi \rangle := \langle \rho, \check{\varphi} \rangle$, where for all $t \in \mathbb{R}^d$, $\check{\varphi}(t) := \varphi(-t)$. For any real valued function f, we will define $f_+ := \max(f, 0)$.

A generalized function usually cannot be evaluated pointwise. However, a generalized function can sometimes be represented by a true function. We first recall that a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is *locally integrable* (that is, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$) if, for all compact sets $K \subset \mathbb{R}^d$, $\int_K |f(t)| \, \mathrm{d}t < \infty$. Of course, it is sufficient to check that

$$\text{for all } n \in \mathbb{N}, \qquad \int_{[-n,n]^d} |f(t)| \, \mathrm{d}t < \infty, \tag{2.3}$$

so checking local integrability only requires checking a countable number of conditions. Condition (2.3) is equivalent to

$$\text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |f(t)| \, |\varphi(t)| \, \mathrm{d}t < \infty, \tag{2.4}$$

and in fact, the condition (2.4) only needs to be checked for a countable collection of φ (say for any sequence $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^d)$ such that $\varphi_n \geq 0$ and $\varphi_n|_{[-n,n]^d} = 1$, for all $n \in \mathbb{N}$).

A Borel function $f_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$ defines an element of $\mathcal{D}'(\mathbb{R}^d)$ via the functional

$$\varphi \mapsto \langle f_0, \varphi \rangle := \int_{\mathbb{R}^d} f_0(t) \varphi(t) \, \mathrm{d}t.$$

Conversely, we say that $\rho \in \mathcal{D}'(\mathbb{R}^d)$ is *represented by a function* if there is a Borel function $\rho_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that,

$$\text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \langle \rho, \varphi \rangle = \int_{\mathbb{R}^d} \rho_0(t) \varphi(t) \, \mathrm{d}t =: \langle \rho_0, \varphi \rangle.$$

For example, the Dirac distribution δ_0 *cannot* be represented by a function.

2.2 Lévy white noise

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\dot{X}$ be a symmetric pure jump Lévy white noise on S, where S is a Borel subset of $\mathbb{R}^d$ with positive Lebesgue measure, with characteristic triplet $(0, 0, \nu)$ (the S we have in mind are typically $S = \mathbb{R}_+ \times \mathbb{R}^d$ and $S = \mathbb{R}^d_+$). More precisely, we suppose that there is a Poisson random measure J on $S \times \mathbb{R}$ with intensity measure $\mathrm{d}s \, \nu(\mathrm{d}z)$ such that

$$X(\mathrm{d}s) := \int_{|z| \leqslant 1} z \tilde{J}(\mathrm{d}s, \, \mathrm{d}z) + \int_{|z|>1} z J(\mathrm{d}s, \, \mathrm{d}z) \, ,$$

and ν is a symmetric Lévy measure. In particular, ν is a nonnegative measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} (1 \wedge |z|^2) \nu(\mathrm{d}z) < \infty$. As usual, $\tilde{J}(\mathrm{d}s, \, \mathrm{d}z) := J(\mathrm{d}s, \, \mathrm{d}z) - \mathrm{d}s \, \nu(\mathrm{d}z)$ is the compensated Poisson random measure associated to J. This Lévy white noise is a particular example of an independently scattered random measure as introduced in [18]. For a link with other definitions of Lévy white noise, we refer the reader to [9]. In [18, Theorem 2.7], Rajput and Rosinski identified the space $L(\dot{X},S)$ of deterministic functions for which $\int_S f(s)X(ds)$ can be defined. In particular, in our framework, it turns out that

$$L(\dot{X},S)=\left\{f:S\to\mathbb{R}\text{ measurable }:\int_{S\times\mathbb{R}}\left(|f(s)z|^2\wedge 1\right)\,\mathrm{d}s\,\nu(\mathrm{d}z)<+\infty\right\}.$$

For properties of this space, we refer the reader to [18, p. 466]. In particular, it is a linear complete metric space for the norm

$$\|f\|_{\Phi_0}=\inf\left\{c>0:\int_{S\times\mathbb{R}}\min\left(1,\left|\frac{zf(s)}{c}\right|^2\right)\,\mathrm{d}s\,\nu(\mathrm{d}z)\le 1\right\}.$$

The mapping

$$f\mapsto\int_S f(s)\,X(\,\mathrm{d}s)=\langle\dot{X},f\rangle \tag{2.5}$$

from $L(\dot{X},S)$ into $L^0(\Omega)$ (the space of a.s. finite random variables with the metric of convergence in probability) is an isomorphism [18, Theorem 3.4] (this property makes use of the symmetry of $\dot{X}$: see [18, Proposition 3.6]), and the convergence $f_n\to f$ in $L(\dot{X},S)$ as $n\to+\infty$ is equivalent to

$$\int_{S\times\mathbb{R}}\left(|(f_n(s)-f(s))\,z|^2\wedge 1\right)\,\mathrm{d}s\,\nu(\mathrm{d}z)\to 0\qquad\text{as }n\to+\infty\,.$$

In particular, $f_n\to f$ in $L(\dot{X},S)$ implies $f_n\to f$ in measure on compact subsets of S [18, p.466]. In a spatio-temporal framework, we refer the reader to [4] for integrability conditions for non-deterministic integrands.

A generalized stochastic process (or generalized random field) U is a linear map from the space of test functions $\mathcal{D}(\mathbb{R}^d)$ into $L^0(\Omega)$. If, in addition, this map is continuous, then by [25, Corollary 4.2], it has a version $\tilde{U}$ (i.e. for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$, $\langle U,\varphi\rangle=\langle\tilde{U},\varphi\rangle$ a.s.) such that for almost all $\omega\in\Omega$, for any sequence $\varphi_n\to\varphi$ in $\mathcal{D}(\mathbb{R}^d)$, $\langle\tilde{U},\varphi_n\rangle(\omega)\to\langle\tilde{U},\varphi\rangle(\omega)$. That is, $\tilde{U}$ defines a random element in $\mathcal{D}'(\mathbb{R}^d)$. In this case, $\tilde{U}$ is called a *continuous generalized stochastic process,* or a *random distribution.*

3 Notions of solution to a linear SPDE

We introduce two different notions of solutions to the linear SPDE (1.1) with associated fundamental solution ρ. Notice that in this framework, we are only considering the case where the Green's function of the operator $\mathcal{L}$ is given by a shift of a fundamental solution.

3.1 Generalized solution

In the following, we will need a hypothesis on the fundamental solution ρ of the differential operator $\mathcal{L}$:

(H1) ρ is such that for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$, the convolution $\varphi*\check{\rho}$ belongs to $L(\dot{X},S)$.

The case where the noise is a symmetric α-stable noise for some $\alpha\in(0,2)$ is already quite rich, and provides some insights into the general theory. More precisely, suppose that $\dot{W}^\alpha$ is an α-stable symmetric Lévy white noise on S, with characteristic triplet $(0,0,\nu_\alpha)$, where $\nu_\alpha(\mathrm{d}z)=c_\alpha^{-1}\,|z|^{-\alpha-1}\,\mathrm{d}z$, with

$$c_\alpha=\frac{\Gamma(1+\alpha)}{\pi}\sin\frac{\alpha\pi}{2}=2\,\Gamma(-\alpha)\cos\frac{\alpha\pi}{2}.$$

The characteristic function of $\dot{W}^\alpha$ is given by

$$\mathbb{E}\left(e^{iu\dot{W}^\alpha(A)}\right) = \exp\left[-\text{Leb}_d\,(A)\,|u|^\alpha\right]\,, \qquad u\in\mathbb{R}\,,$$

for any measurable set $A \subset S$ with finite Lebesgue measure [21, Lemma 14.11]. This notion coincides with that of a symmetric α-stable random measure developed in [20, §3.3]. Since the skewness parameter β vanishes, it is well known that a function $f : \mathbb{R}^d \to \mathbb{R}$ is $\dot{W}^\alpha$-integrable if and only if $f \in L^\alpha(S)$ (see [20, §3.4]). In this framework, **(H1)** becomes:

(H1') ρ is such that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the convolution $\varphi * \check{\rho}$ belongs to $L^\alpha(S)$.

As in (44) of [8], we can then define a generalized solution to the following linear SDPE:

$$\mathcal{L}u = \dot{X}\,1_S. \tag{3.1}$$

Definition 3.1. Assume **(H1)**. The *generalized solution* to the stochastic partial differential equation (3.1) is the linear functional u_{gen} on $\mathcal{D}(\mathbb{R}^d)$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle u_{\text{gen}}, \varphi\rangle := \langle \dot{X}, \varphi * \check{\rho}\rangle\,. \tag{3.2}$$

Notice that $\langle \dot{X}, \varphi\rangle$ denotes $\int_S \varphi(s) X(ds)$, and this accounts for the 1_S in (3.1).

Remark 3.2. The generalized solution is in general not a distribution, since it may not define a continuous linear functional on $\mathcal{D}(\mathbb{R}^d)$. However, by [25, Corollary 4.2], u_{gen} does have a version in $\mathcal{D}'(\mathbb{R}^d)$ if $\varphi \mapsto \langle u_{\text{gen}}, \varphi\rangle$ from $\mathcal{D}(\mathbb{R}^d)$ to $L^0(\Omega)$ is continuous. By the isomorphism property mentioned in Section 2, this will be the case if $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ implies that $\varphi_n * \check{\rho} \to \varphi * \check{\rho}$ in $L(\dot{X}, S)$. This will occur in several examples, such as the stochastic heat equation (see Remark 6.3) and the stochastic wave equation (see Remark 6.8).

Remark 3.3. When u_{gen} has a version in $\mathcal{D}'(\mathbb{R}^d)$, then the functional u_{gen} is a solution to (3.1) in the weak sense: indeed, for $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \mathcal{L}u_{\text{gen}}, \varphi\rangle = \langle u_{\text{gen}}, \mathcal{L}^*\varphi\rangle = \langle \dot{X}, (\mathcal{L}^*\varphi) * \check{\rho}\rangle\,.$$

Also, by (2.1),

$$(\mathcal{L}^*\varphi) * \check{\rho}(t) = \langle \check{\rho}, \mathcal{L}^*\varphi(t-\cdot)\rangle = \langle \rho, \mathcal{L}^*\varphi(t+\cdot)\rangle = \langle \mathcal{L}\rho, \varphi(t+\cdot)\rangle = \langle \delta_0, \varphi(t+\cdot)\rangle = \varphi(t)\,.$$

Therefore, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \mathcal{L}u_{\text{gen}}, \varphi\rangle = \langle \dot{X}, \varphi\rangle\,.$$

Definition 3.4. We say that a generalized stochastic process u has a random field representation if there exists a jointly measurable random field $(Y_t)_{t\in\mathbb{R}^d}$ such that Y has almost surely locally integrable sample paths and for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle u, \varphi\rangle = \int_{\mathbb{R}^d} Y_t\,\varphi(t)\,\mathrm{d}t \qquad \text{a.s.} \tag{3.3}$$

The generalized stochastic processes that have a random field representation are exactly those which can be evaluated pointwise.

3.2 Mild solution

Generalized solutions are a useful generalization of classical solutions to a partial differential equation. However, non-linear operations on generalized functions are in general difficult to define, and we are often interested in finding solutions that can be evaluated pointwise. One type of solution that is often used in the SPDE literature is the notion of mild solution. In order to be able to define a mild solution to (3.1), we will need another hypothesis on the fundamental solution ρ:

(H2) ρ is represented by a function ρ_0 such that, for Leb_d-a.a. $t \in \mathbb{R}^d$,

$$\rho_0(t - \cdot) \in L(\dot{X}, S). \tag{3.4}$$

Again, in the case where the noise is a symmetric α-stable noise for some $\alpha \in (0, 2)$ **(H2)** becomes:

(H2') ρ is represented by a function ρ_0 such that, for Leb_d-a.a. $t \in \mathbb{R}^d$,

$$\rho_0(t - \cdot) \in L^\alpha(S). \tag{3.5}$$

We will see in Remark 4.3 below that in certain special cases, such as $S = \mathbb{R}_+ \times \mathbb{R}^{d-1}$, then: (i) if (3.5) holds for a.a. $t \in S$, then it holds for all $t \in S$; and (ii) **(H2')** implies **(H1')**.

Definition 3.5. Under hypothesis **(H2)**, the *mild solution* of (3.1) is the random field $(u_{\text{mild}}(t),\, t \in \mathbb{R}^d)$ defined as follows: for those $t \in \mathbb{R}^d$ which satisfy (3.4) (respectively (3.5) in the α-stable case),

$$u_{\text{mild}}(t) := \langle \dot{X}, \rho_0(t - \cdot) \rangle, \tag{3.6}$$

and for those t such that $\rho(t - \cdot) \notin L(\dot{X}, S)$, we set $u_{\text{mild}}(t) = 0$.

Recall that $u = (u_t)$ and $\tilde{u} = (\tilde{u}_t)$ are *versions* (also called *modifications*) of each other if, for all $t \in \mathbb{R}^d$, $u(t) = \tilde{u}(t)$ a.s. (where the null set may depend on t). The random field u_{mild} defined in (3.6) has a jointly measurable version. This is a consequence of the following proposition, whose proof is based on a result of [5].

Proposition 3.6. Let $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function such that for Lebesgue-a.a. $t \in \mathbb{R}^n$, $f(t, \cdot) \in L(\dot{X}, S)$. For any $t \in \mathbb{R}^n$, let

$$u(t) = \begin{cases} \langle \dot{X}, f(t, \cdot) \rangle, & \text{if } f(t, \cdot) \in L(\dot{X}, S), \\ 0, & \text{otherwise.} \end{cases}$$

Then the random field u has a jointly measurable version.

Proof. Let $A \subset \mathbb{R}^n$ be a Borel null set such that for $t \in \mathbb{R}^n \setminus A$, $f(t, \cdot) \in L(\dot{X}, S)$. Set $g(t, \cdot) = f(t, \cdot) 1_{A^c}(t)$. Then $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is measurable and for all $t \in \mathbb{R}^n$, $u(t) = \langle \dot{X}, g(t, \cdot) \rangle$ a.s. By [2, p. 925], u has a measurable modification. $\square$

3.3 Possible relationships between mild and generalized solutions

We point out that the generalized and mild solutions depend on the choice of the fundamental solution ρ. Therefore, we fix a fundamental solution to the operator $\mathcal{L}$, and then it makes sense to study *the* generalized solution and *the* mild solution to (3.1) (under **(H1)** and **(H2)**, respectively), as defined in (3.2) and (3.6).

When it exists, the mild solution is always a random field. It may turn out that the generalized solution has a random field representation, and we can then ask if this representation is the mild solution.

In general, if u_{mild} has locally integrable sample paths, then for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} u_{\mathrm{mild}}(t)\varphi(t)\,\mathrm{d}t$ is well-defined, and

$$\langle u_{\mathrm{mild}}, \varphi\rangle := \int_{\mathbb{R}^d} u_{\mathrm{mild}}(t)\,\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} \langle \dot{X}, \rho(t-\cdot)\rangle\,\varphi(t)\,\mathrm{d}t\,.$$

If we can exchange the stochastic integral and the Lebesgue integral, then we get

$$\langle u_{\mathrm{mild}}, \varphi\rangle = \langle \dot{X}, \int_{\mathbb{R}^d} \rho(t-\cdot)\varphi(t)\,\mathrm{d}t\rangle = \langle \dot{X}, \varphi * \check{\rho}\rangle = \langle u_{\mathrm{gen}}, \varphi\rangle\,,$$

and u_{mild} will be a random field representation of u_{gen}.

In order to make the steps above rigorous, we need to know when u_{mild} has locally integrable sample paths, and we need a stochastic Fubini's theorem. We will consider the α-stable case in Section 4 and the general case in Section 5.

We recall that in the case of a Gaussian noise, that can be spatially correlated, this type of question has already been investigated under slightly different assumptions in [8, Theorem 11]. Transposed to our framework, this theorem implies that in the case of an SPDE driven by Gaussian white noise (in space and time), if the generalized solution has a random field representation, then the fundamental solution of this SPDE is necessarily represented by a square integrable function. Here, we first extend this kind of statement to the setting of symmetric α-stable Lévy white noises.

4 The α-stable case

Fix $\alpha \in (0,2)$ and let $\dot{W}^\alpha$ be a symmetric α-stable noise as in Section 3.1. We consider the linear SPDE

$$\mathcal{L}u = \dot{W}^\alpha\, 1_S. \tag{4.1}$$

We fix a fundamental solution ρ of $\mathcal{L}$. We are interested in determining if u_{gen}, as defined in (3.2) with $\dot{X}$ replaced by $\dot{W}^\alpha$, has a random field representation, and if u_{mild}, as defined in (3.6) with $\dot{X}$ replaced by $\dot{W}^\alpha$, is a random field representation of u_{gen}. It will turn out that condition **(INT)** below is a necessary and sufficient condition for this.

Notation. For any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, let $\mu_\varphi(\mathrm{d}t) = \varphi(t)\,\mathrm{d}t$ and $\mu_{|\varphi|}(\mathrm{d}t) = |\varphi(t)|\,\mathrm{d}t$.

Consider the following condition.

Condition (INT). The fundamental solution ρ is represented by a function ρ_0, and either

(i) $\alpha > 1$ and for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left(\int_S |\rho_0(t-s)|^\alpha\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \mu_{|\varphi|}(\mathrm{d}t) < +\infty\,, \tag{4.2}$$

or

(ii) $\alpha = 1$ and ρ_0 is such that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \mu_{|\varphi|}(\mathrm{d}t) \int_S \mathrm{d}s\, |\rho_0(t-s)| \left[1 + \log_+\left(\frac{|\rho_0(t-s)| \int_{\mathbb{R}^d} \mu_{|\varphi|}(\mathrm{d}r) \int_S \mathrm{d}v |\rho_0(r-v)|}{\left(\int_S |\rho_0(t-v)|\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} |\rho_0(r-s)| \mu_{|\varphi|}(\mathrm{d}r)\right)}\right)\right] < +\infty. \tag{4.3}$$

or

(iii) $\alpha < 1$ and ρ_0 is such that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_S \left(\int_{\mathbb{R}^d} |\rho_0(t-s)| \mu_{|\varphi|}(\mathrm{d}t) \right)^\alpha \mathrm{d}s < +\infty \,. \tag{4.4}$$

The main result of this section is the following.

Theorem 4.1. Let $\alpha \in (0,2)$. The following three conditions are equivalent.

(1) Condition **(INT)** holds;
(2) **(H2')** holds and the jointly measurable version of u_{mild} has locally integrable sample paths;
(3) **(H1')** holds and u_{gen} has a random field representation.

Under any one of these three equivalent conditions, u_{mild} is a random field representation of u_{gen}.

Proof of Theorem 4.1. **(1) implies (2).** Suppose that ρ is represented by a function ρ_0 satisfying the properties in Condition **(INT)**. When $\alpha > 1$, since the condition in (4.2) can be written $\int_{\mathbb{R}^d} \|\rho_0(t-\cdot)\|_{L^\alpha(S)}\, \mu_{|\varphi|}(\,\mathrm{d}t) < \infty$, this condition implies that for a.a. $t \in \mathbb{R}^d$, $\rho_0(t-\cdot) \in L^\alpha(S)$, that is, **(H2')** holds. When $\alpha = 1$, (4.3) clearly implies that

$$\int_{\mathbb{R}^d} \|\rho(t-\cdot)\|_{L^1(S)}\, \mu_{|\varphi|}(\,\mathrm{d}t) < \infty, \tag{4.5}$$

which clearly implies that for a.a. $t \in \mathbb{R}^d$, $\rho(t-\cdot) \in L^1(S)$, that is, **(H2')** holds. When $\alpha < 1$, then by Remark 4.3(3) below, (4.4) also implies that **(H2')** holds. Therefore, for $\alpha \in (0,2)$, u_{mild} is well-defined.

According to [20, Theorem 11.3.2] (which uses the symmetry of $\dot{W}^\alpha$ when $\alpha = 1$), Condition **(INT)** implies that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, a.s.,

$$\int_{\mathbb{R}^d} |u_{\text{mild}}(t)|\, \mu_{|\varphi|}(\mathrm{d}t) < +\infty. \tag{4.6}$$

As discussed in Subsection 2.1, we can find a single null set such that (4.6) holds simultaneously for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Therefore, the sample paths of u_{mild} are almost surely locally integrable. This proves (2).

(2) implies (3). Local integrability of the sample paths of u_{mild} implies that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, (4.6) holds. According to the stochastic Fubini theorem in [20, Theorem 11.4.1], this implies that

$$\int_{\mathbb{R}^d} \rho_0(t-\cdot)\, \mu_{|\varphi|}(\mathrm{d}t) \in L^\alpha(S),$$

or, equivalently, $|\varphi| * \check{\rho}_0 \in L^\alpha(S)$. This implies that **(H1')** holds. By the same stochastic Fubini theorem, for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u_{\text{mild}}(t)\varphi(t)\,\mathrm{d}t = \int_S \left(\int_{\mathbb{R}^d} \rho(t-s)\varphi(t)\,\mathrm{d}t \right) \dot{W}^\alpha(\mathrm{d}s) = \langle \dot{W}_\alpha, \varphi * \check{\rho} \rangle \qquad \text{a.s.} \tag{4.7}$$

Therefore, for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle u_{\text{mild}}, \varphi \rangle = \langle \dot{W}_\alpha, \varphi * \check{\rho} \rangle =: \langle u_{\text{gen}}, \varphi \rangle \qquad \text{a.s.} \tag{4.8}$$

Therefore, $u_{\text{mild}} = u_{\text{gen}}$ in the sense of generalized stochastic processes, that is, u_{mild} is a random field representation of u_{gen}.

(3) implies (1). Let (Y_t) be a random field representation of u_{gen}, that is, (3.3) holds. According to Definition 3.4, there exists a set $\tilde{\Omega} \subset \Omega$ of probability one such that for all $\omega \in \tilde{\Omega}$, the function $t \mapsto Y_t(\omega)$ is locally integrable. Without loss of generality, we can suppose that $\Omega = \tilde{\Omega}$. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be such that $\varphi \geqslant 0$, $\operatorname{supp}\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi = 1$.

For each $t \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we define $\varphi_n^t(\cdot) = n^d \varphi(n(\cdot - t))$. Let $Z_t^n(\omega) := \langle Y(\omega), \varphi_n^t \rangle$. Then

$$Z_t^n(\omega) = \int_{\mathbb{R}^d} Y_s(\omega) n^d \varphi(n(s-t))\, \mathrm{d}s = \int_{\mathbb{R}^d} Y_{r+t}(\omega) n^d \varphi(nr)\, \mathrm{d}r\,. \tag{4.9}$$

Define $f(t,s,\omega) := (t+s,\omega)$. The function f is measurable as a map from $(\mathbb{R}^d \times \mathbb{R}^d \times \Omega, B(\mathbb{R}^d) \otimes B(\mathbb{R}^d) \otimes \mathcal{F})$ to $(\mathbb{R}^d \times \Omega, B(\mathbb{R}^d) \otimes \mathcal{F})$, and $Y_{r+t}(\omega) = Y \circ f(r,t,\omega)$. Since Y is a jointly measurable process, by Fubini's theorem, we deduce from the second equality in (4.9) that Z^n is a jointly measurable process. We define the set

$$A = \left\{ (t,\omega) : \langle Y(\omega), \varphi_n^t \rangle \to Y_t(\omega) \text{ as } n \to +\infty \right\}\,.$$

We can write

$$A = \bigcap_{k \in \mathbb{N}*} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geqslant N} \left\{ (t,\omega) : |Z_t^n(\omega) - Y_t(\omega)| \leqslant \frac{1}{k} \right\}\,,$$

and since Z^n and Y are both jointly measurable processes, $A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$. By Lebesgue's differentiation theorem (see [26, Chapter 7, Exercise 2]), for any $\omega \in \Omega$, $\int_{\mathbb{R}^d} \mathbb{1}_{(t,\omega) \in A^c}\, \mathrm{d}t = 0$. Then, by Fubini's theorem, $(\mathrm{Leb}_d \times \mathbb{P})(A^c) = 0$. Therefore, there is a non random set $\tilde{A} \subset \mathbb{R}^d$ such that $\mathrm{Leb}_d(\tilde{A}) = 0$ and for all $t \notin \tilde{A}$, $\mathbb{P}\left\{\omega : (t,\omega) \in A^c\right\} = 0$, that is,

$$\mathbb{P}\left\{ \langle Y, \varphi_n^t \rangle \to Y_t \text{ as } n \to +\infty \right\} = 1. \tag{4.10}$$

By [20, Proposition 3.4.1], for any $f \in L^\alpha(S)$,

$$\mathbb{E}\left(e^{i \langle \dot{W}^\alpha, f \rangle} \right) = e^{-\|f\|^\alpha_{L^\alpha(S)}}\,. \tag{4.11}$$

Therefore, by (3.2) and (3.3), for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\mathbb{E}\left(e^{i \langle u_{\mathrm{gen}}, \varphi \rangle} \right) = e^{-\|\varphi * \check{\rho}\|^\alpha_{L^\alpha}} = \mathbb{E}\left(\exp\left(i \int_{\mathbb{R}^d} Y_s \varphi(s)\, \mathrm{d}s \right) \right)\,. \tag{4.12}$$

Let $t_0 \in \tilde{A}^c$. Then

$$\langle Y, \varphi_n^{t_0} \rangle \to Y_{t_0} \qquad \text{a.s. as } n \to +\infty. \tag{4.13}$$

We define $\rho_n^{t_0} = \varphi_n^{t_0} * \check{\rho} \in L^\alpha(S)$ by **(H1')**. By (4.12) and (4.10), for $n,m \in \mathbb{N}$,

$$e^{-\|\rho_n^{t_0} - \rho_m^{t_0}\|^\alpha_{L^\alpha}} = \mathbb{E}\left(\exp\left(i \int_{\mathbb{R}^d} Y_s \left(\varphi_n^{t_0}(s) - \varphi_m^{t_0}(s) \right)\, \mathrm{d}s \right) \right) \to 1 \qquad \text{as } n,m \to +\infty\,. \tag{4.14}$$

We deduce that $(\rho_n^{t_0})_{n \geqslant 1}$ is a Cauchy sequence in $L^\alpha(S)$. By completeness of this space, there is a function $g^{t_0} \in L^\alpha(S)$ such that

$$\rho_n^{t_0} \to g^{t_0} \qquad \text{in } L^\alpha(S) \text{ as } n \to +\infty\,. \tag{4.15}$$

Furthermore, we know from the theory of generalized functions that $\varphi_n^{t_0} \to \delta_{t_0}$ in $\mathcal{D}'(\mathbb{R}^d)$ as $n \to +\infty$. Therefore,

$$\rho_n^{t_0} \to \delta_{t_0} * \check{\rho}\,, \qquad \text{in } \mathcal{D}'(\mathbb{R}^d) \text{ as } n \to +\infty\,. \tag{4.16}$$

From (4.15) and (4.16), we would like to deduce that the function g^{t_0} represents $\delta_{t_0} * \check{\rho}$. If this is true, then it will mean that $s \mapsto \rho(t_0 - s)$ can be considered as a function in $L^\alpha(S)$. However, in order to prove this equality, it suffices to show that for any $\theta \in \mathcal{D}(S)$, $\langle \delta_{t_0} * \check{\rho}, \theta \rangle = \langle g^{t_0}, \theta \rangle$.

If $\alpha \in (0,2)$ is arbitrary, we deduce from (4.11) and (4.14) that $\langle \dot{W}^\alpha, \rho_n^{t_0} - g^{t_0} \rangle \to 0$ in law as $n \to +\infty$, and by [12, Lemma 4.7], the convergence is also in probability. By almost sure linearity, we deduce that $\langle \dot W^\alpha, \rho_n^{t_0}\rangle \to \langle \dot W^\alpha, g^{t_0}\rangle$ in probability as $n\to+\infty$. By uniqueness of the limit, and since $\langle \dot W^\alpha, \rho_n^{t_0}\rangle = \langle u_{\mathrm{gen}}, \varphi_n^{t_0}\rangle = \langle Y, \varphi_n^{t_0}\rangle$, it follows from (4.13) that

$$\text{for any } t_0\in \tilde A^c\,, \qquad Y_{t_0} = \langle \dot W^\alpha, g^{t_0}\rangle \qquad \text{a.s.} \tag{4.17}$$

For any $(t,s)\in\mathbb{R}^d\times S$, let

$$g(t,s) = \limsup_{n\to+\infty}\, \rho_n^t(s)\,. \tag{4.18}$$

For fixed $n\in\mathbb{N}$, $\rho_n^t(s) = \varphi_n^t * \check\rho = \langle \check\rho, \varphi_n^t(s-\cdot)\rangle = \langle \check\rho, \varphi_n^0(s-\cdot-t)\rangle$, so $(s,t)\mapsto \rho_n^t(s)$ is continuous. Hence $(t,s)\mapsto g(t,s)$ is measurable, and by (4.15), for $t\in\tilde A^c$, $g(t,\cdot) = g^t(\cdot)$ almost everywhere. Therefore, for $t_0\in\tilde A^c$,

$$g(t_0,\cdot)\in L^\alpha(S), \tag{4.19}$$

and

$$Y_{t_0} = \langle \dot W^\alpha, g(t_0,\cdot)\rangle\,, \qquad \text{a.s.}, \tag{4.20}$$

where the “a.s.” depends on t_0. Let $(\tilde u_t,\, t\in\mathbb{R}^d)$ be a jointly measurable version of $\langle \dot W^\alpha, g(t,\cdot)\rangle$ (which exists by Proposition 3.6). By (4.20),

$$\text{for } t_0\in\tilde A^c, \qquad Y_{t_0} = \tilde u_{t_0} \text{ a.s.} \tag{4.21}$$

Since both (Y_t) and $(\tilde u_t)$ are jointly measurable, $(\mathrm{Leb}_d\times\mathbb{P})\{(t,\omega): Y_t(\omega)\neq \tilde u_t(\omega)\}=0$. By Fubini's theorem, there is a $\mathbb{P}$-null set N_0 such that for $\omega\notin N_0$,

$$Y_t(\omega) = \tilde u_t(\omega), \qquad \text{for a.a. } t\in\mathbb{R}^d. \tag{4.22}$$

Let $\psi\in\mathcal{D}(\mathbb{R}^d)$. Then $\mu_\psi(\mathrm{d}t) := \psi(t)\,\mathrm{d}t$ is a finite signed measure, that we can decompose into positive and negative parts μ_ψ^+ and μ_ψ^-. Since Y is almost surely locally integrable,

$$\int_{\mathbb{R}^d} |Y_t|\mu_\psi^+(\mathrm{d}t) < +\infty\,, \qquad \text{and} \qquad \int_{\mathbb{R}^d} |Y_t|\mu_\psi^-(\mathrm{d}t) < +\infty \qquad \text{a.s.}$$

By [20, Theorem 11.3.2], if $\alpha>1$, we get

$$\int_{\mathbb{R}^d}\left(\int_S |g(t,s)|^\alpha\,\mathrm{d}s\right)^{\frac{1}{\alpha}} |\psi(t)|\,\mathrm{d}t < +\infty\,, \tag{4.23}$$

if $\alpha=1$, we get

$$\int_{\mathbb{R}^d}\mathrm{d}t\int_S \mathrm{d}s\,|g(t,s)\psi(t)|\left[1+\log_+\left(\frac{|g(t,s)|\int_{\mathbb{R}^d}\int_S |g(r,v)|\,\mathrm{d}v|\psi(r)|\,\mathrm{d}r}{\left(\int_S |g(t,v)|\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d}|g(r,s)\psi(r)|\,\mathrm{d}r\right)}\right)\right] < +\infty\,, \tag{4.24}$$

and if $\alpha<1$, we get

$$\int_S\left(\int_{\mathbb{R}^d} |g(t,s)\psi(t)|\,\mathrm{d}t\right)^\alpha \mathrm{d}s < +\infty\,. \tag{4.25}$$

If $\alpha=1$ or $\alpha<1$, then (4.24) and (4.25) imply that there is a Lebesgue-null set N_ψ such that, for $s\in S\setminus N_\psi$, $\int_{\mathbb{R}^d}|g(t,s)\psi(t)|\,\mathrm{d}t<\infty$. Let $(\psi_n)\subset\mathcal{D}(\mathbb{R}^d)$ be such that $1_{[-n,n]^d}\le\psi_n$, for all $n\in\mathbb{N}$. For $s\in S\setminus\cup_{n\in\mathbb{N}}N_{\psi_n}$, we have that for all $n\in\mathbb{N}$, $\int_{[-n,n]^d}|g(t,s)|\,\mathrm{d}t<\infty$, that is, $t\mapsto g(t,s)$ is locally integrable. When $\alpha>1$, by the generalized Minkowsky inequality (see [23, A.1]) and by (4.23),

$$\left(\int_S\left|\int_{\mathbb{R}^d}|g(t,s)\psi(t)|\,\mathrm{d}t\right|^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} \leqslant \int_{\mathbb{R}^d}\left(\int_S |g(t,s)|^\alpha\,\mathrm{d}s\right)^{\frac{1}{\alpha}}|\psi(t)|\,\mathrm{d}t < +\infty\,.$$

In particular, we see that also in the case $\alpha > 1$, for almost all $s \in S$, $t \mapsto g(t,s)$ is locally integrable (and therefore represents a distribution).

By (4.22), for all $\alpha \in (0,2)$,

$$\int_{\mathbb{R}^d} Y_t \, \mu_\psi(\mathrm{d}t) = \int_{\mathbb{R}^d} \tilde{u}(t)\psi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} \langle \dot{W}^\alpha, g(t,\cdot)\rangle \, \psi(t)\,\mathrm{d}t, \tag{4.26}$$

where $\langle \dot{W}^\alpha, g(t,\cdot)\rangle$ is the jointly measurable version of this process. By [20, Theorem 11.4.1], we can exchange the stochastic integral with the Lebesgue integral in (4.26), to conclude that

$$\langle Y, \psi\rangle = \langle \dot{W}^\alpha, \int_{\mathbb{R}^d} \psi(t) g(t,\cdot)\,\mathrm{d}t\rangle \qquad \text{a.s.} \tag{4.27}$$

We define $\int_{\mathbb{R}^d} \psi(t)g(t,s)\,\mathrm{d}t =: (\psi \circledast g)(s)$ (this operation on ψ and g is not commutative). From (4.27) and (3.3), we get

$$\langle \dot{W}^\alpha, \psi \circledast g - \psi * \check{\rho}\rangle = \langle \dot{W}^\alpha, \psi \circledast g\rangle - \langle \dot{W}^\alpha, \psi * \check{\rho}\rangle = \langle Y, \psi\rangle - \langle u_{\text{gen}}, \psi\rangle = 0 \text{ a.s.}, \tag{4.28}$$

and by (4.11), we deduce that $\|\psi \circledast g - \psi * \check{\rho}\|_{L^\alpha(S)} = 0$. Then, for any $\psi \in \mathcal{D}(\mathbb{R}^d)$, there is a set B_ψ such that $\text{Leb}_d(B_\psi) = 0$ and for any $s \in S \setminus B_\psi$, $(\psi \circledast g)(s) = (\psi * \check{\rho})(s)$. Since $\mathcal{D}(\mathbb{R}^d)$ is separable, there is a countable dense subset $D \subset \mathcal{D}(\mathbb{R}^d)$. Let

$$B = \bigcup_{\psi \in D} B_\psi \,.$$

Then $\text{Leb}_d(B) = 0$ and, for all $s \in S \setminus B$, for all $\psi \in D$,

$$\langle g(\cdot,s), \psi\rangle = \psi \circledast g(s) = \psi * \check{\rho}(s) = \langle \rho, \psi(s+\cdot)\rangle = \langle \delta_s * \rho, \psi\rangle \,,$$

where we have used (2.2). Since two distributions equal on a dense subset of $\mathcal{D}(\mathbb{R}^d)$ are equal (by continuity), we get that for all $s \in S \setminus B$, $g(\cdot,s) = \delta_s * \rho$ in $\mathcal{D}'(\mathbb{R}^d)$. Then, $\rho = \delta_{-s} * g(\cdot,s)$ in $\mathcal{D}'(S)$, and $\delta_{-s} * g(\cdot,s)$ is in fact a function ρ_0 depending only on the $t \in \mathbb{R}^d$ variable, i.e., ρ_0 represents ρ. Further, for almost all $t \in \mathbb{R}^d$, $\rho_0(t) = (\delta_{-s} * g(\cdot,s))(t) = g(t+s,s)$ which does not depend on s. Then, for almost all $(t,s) \in \mathbb{R}^d \times S$, $g(t,s) = \rho_0(t-s)$. By (4.19), we deduce that for almost all $t \in \mathbb{R}^d$, $\rho_0(t-\cdot) \in L^\alpha(S)$, so **(H2')** holds and $u_{\text{mild}} = \langle \dot{W}^\alpha, \rho_0(t_0-\cdot)\rangle$ is well-defined. Also, from (4.23), (4.24) and (4.25), we get that (4.2)–(4.4) hold. This proves (1).

Finally, still assuming (3), by (4.20), for $t_0 \in \tilde{A}^c$, $Y_{t_0} = \langle \dot{W}^\alpha, \rho_0(t_0-\cdot)\rangle = u_{\text{mild}}(t_0) = \tilde{u}_{t_0}$, a.s. By Fubini's theorem, a.s., $Y_t = \tilde{u}_t$ for a.a. $t \in \mathbb{R}^d$. We conclude that $(\tilde{u}_t)$ is a jointly measurable version of u_{mild}, $(\tilde{u}_t)$ a.s. has locally integrable sample paths, and $(\tilde{u}_t)$ is a random field representation of u_{gen} (since this was the case of (Y_t)). This proves the last statement of Theorem 4.1. □

Remark 4.2. For the implication "(3) implies (1)" in the proof of Theorem 4.1, in the case $\alpha \geqslant 1$, there is a different way of deducing from the considerations that lead to (4.16) that g^{t_0} represents $\delta_{t_0} * \check{\rho}$. Indeed, when $\alpha \geqslant 1$, using the notation introduced in the proof, by Hölder's inequality,

$$|\langle g^{t_0} - \rho_n^{t_0}, \theta\rangle| \leqslant \int_S \left|g^{t_0}(s) - \rho_n^{t_0}(s)\right| |\theta(s)|\,\mathrm{d}s \leqslant \|g^{t_0} - \rho_n^{t_0}\|_{L^\alpha(S)} \|\theta\|_{L^{\frac{\alpha}{\alpha-1}}(S)} \,.$$

Passing to the limit as $n \to +\infty$, we get that for all $t_0 \in \tilde{A}^c$, $\delta_{t_0} * \check{\rho} = g^{t_0} \in L^\alpha(S)$ in $\mathcal{D}'(S)$. Then, in the sense of distributions, $\check{\rho} = \delta_{-t_0} * \delta_{t_0} * \check{\rho} = \delta_{-t_0} * g^{t_0}$. Therefore, ρ is represented by the function $t \in \mathbb{R}^d \mapsto g^{t_0}(t_0 - t) =: \rho_0(t)$, which therefore does not depend on t_0, and is such that for almost all $t \in \mathbb{R}^d$, $\delta_t * \check{\rho} = \rho_0(t - \cdot) \in L^\alpha(S)$, so **(H2')** holds. Also, for any $t \in \tilde{A}^c$, by (3.3),

$$\langle Y, \varphi_n^t \rangle = \langle u_{\text{gen}}, \varphi_n^t \rangle = \langle \dot{W}^\alpha, \varphi_n^t * \check{\rho} \rangle = \langle \dot{W}^\alpha, \rho_n^t \rangle \,,$$

and $\langle Y, \varphi_n^t \rangle \to Y_t$ almost surely as $n \to +\infty$ by (4.13), and

$$\rho_n^t \to g^t = \delta_t * \check{\rho} = \rho_0(t - \cdot) \,, \qquad \text{in } L^\alpha(S) \text{ as } n \to +\infty \,. \tag{4.29}$$

Therefore, $\langle \dot{W}^\alpha, \rho_n^t - \rho_0(t - \cdot) \rangle \to 0$ in law by (4.11), hence in probability, that is, $\langle \dot{W}^\alpha, \rho_n^t \rangle \to \langle \dot{W}^\alpha, \rho_0(t - \cdot) \rangle$ in probability as $n \to +\infty$. Therefore, using (4.13), we see that $Y_t = \langle \dot{W}^\alpha, \rho_0(t - \cdot) \rangle$ a.s. Since we used Hölder's inequality, this method does not work in the case $\alpha < 1$, and does not imply (4.2), (4.3) or (4.4), which is why the proof of Theorem 4.1 uses a different argument that works for any $\alpha \in (0, 2)$.

Remark 4.3. It is not difficult to check the following statements:

(1) For $\alpha \in (1, 2)$, we have:

   (a) Condition (4.2) is equivalent to local integrability of the function $t \mapsto \|\rho_0(t - \cdot)\|_{L^\alpha(S)}$, that is,
   $$[t \mapsto \|\rho_0(t - \cdot)\|_{L^\alpha(S)}] \in L^1_{\text{loc}}(\mathbb{R}^d) \,. \tag{4.30}$$

   (b) If $S = \mathbb{R}^d_+$, and $\rho_0(t) = 0$ for all $t \in \mathbb{R}^d \setminus \mathbb{R}^d_+$, then (4.2) is equivalent to $\rho_0 \in L^\alpha_{\text{loc}}(\mathbb{R}^d_+)$.

   (c) If $S = \mathbb{R}_+ \times \mathbb{R}^{d-1}$, and $\rho_0(t, x) = 0$ if $t < 0$, then (4.2) is equivalent to
   $$\text{for all } t > 0, \qquad \int_0^t \int_{\mathbb{R}^{d-1}} |\rho_0(s, y)|^\alpha \, \mathrm{d}s \, \mathrm{d}y < +\infty \,.$$
   In particular, when (3.5) holds for a.a. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$, then in fact, it holds for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$. In addition, **(H2')** implies **(H1')**.

(2) For $\alpha \in (0, 1)$, we have:

   (a) Condition (4.4) can be written $|\check{\rho}_0| * |\varphi| \in L^\alpha(S)$, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, which differs from **(H1')** only because of the presence of the absolute values and because ρ_0 is a function. Further, condition (4.4) is equivalent to:
   $$\text{for any compact set } K \subset \mathbb{R}^d, \qquad \int_S \left(\int_K |\rho_0(t - s)| \, \mathrm{d}t \right)^\alpha \, \mathrm{d}s < +\infty \,.$$

   (b) If $S = \mathbb{R}^d_+$, and $\rho_0(t) = 0$ for all $t \in \mathbb{R}^d \setminus \mathbb{R}^d_+$, then (4.4) is equivalent to $\rho_0 \in L^1_{\text{loc}}(\mathbb{R}^d_+)$.

(3) When $\alpha \in (0, 1)$, (4.4) implies (4.2).

Indeed, the equivalence of (4.2) and (4.30) occurs for the same reason as the equivalence between (2.3) and (2.4). For (1)(b), if $S = \mathbb{R}^d_+$ and $\rho_0(t) = 0$ for all $t \in \mathbb{R}^d \setminus \mathbb{R}^d_+$, then for any $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{[-n,n]^d} \mathrm{d}t \, \|\rho_0(t - \cdot)\|_{L^\alpha(S)} &= \int_{[-n,n]^d} \mathrm{d}t \left(\int_{\mathbb{R}^d_+} \mathrm{d}s \, |\rho_0(t - s)|^\alpha \right)^{\frac{1}{\alpha}} \\
&= \int_{[0,n]^d} \mathrm{d}t \left(\int_{[0,t]} \mathrm{d}r \, |\rho_0(r)|^\alpha \right)^{\frac{1}{\alpha}} \\
&= \int_{[0,n]^d} \mathrm{d}t \, \|\rho_0 1_{[0,t]}\|_{L^\alpha(S)},
\end{aligned}$$

where we have used the change of variables $r = t - s$ and the notation $[0,t] = [0,t_1] \times \cdots \times [0,t_d]$ if $t = (t_1, \ldots, t_d)$. Since $\|\rho_0 1_{[0,t]}\|_{L^\alpha(S)}$ is a nondecreasing function of each coordinate of t, we conclude that $[t \mapsto \|\rho_0(t - \cdot)\|_{L^\alpha(S)}] \in L^1_{\text{loc}}(\mathbb{R}^d)$ if and only if $\|\rho_0 1_{[0,t]}\|_{L^\alpha(S)} < \infty$, for all $t \in \mathbb{R}^d_+$, that is, $\rho_0 \in L^\alpha_{\text{loc}}(\mathbb{R}^d_+)$.

The first two statements in property (1)(c) are checked in a similar way. For the implication **(H2')** implies **(H1')**, notice that for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, by Minkowski's inequality for integrals,

$$\begin{aligned}\|\varphi * \check{\rho}\|_{L^\alpha(\mathbb{R}_+ \times \mathbb{R}^{d-1})} &\leq \int_{\mathbb{R}^d} \mathrm{d}t\, |\varphi(t)|\, \|\rho(t - \cdot)\|_{L^\alpha(\mathbb{R}_+ \times \mathbb{R}^{d-1})} \\ &= \int_{\mathbb{R}^d} \mathrm{d}t\, |\varphi(t)|\, \|\rho(\cdot)\|_{L^\alpha([0,t] \times \mathbb{R}^{d-1})} \leq C \|\rho(\cdot)\|_{L^\alpha([0,T] \times \mathbb{R}^{d-1})} < \infty,\end{aligned}$$

if $\operatorname{supp}(\varphi) \subset [0,T] \times \mathbb{R}^{d-1}$.

For (2)(a), it is clear that

$$\int_S \left(\int_{\mathbb{R}^d} |\rho_0(t - s)| \mu_{|\varphi|}(\mathrm{d}t) \right)^\alpha \mathrm{d}s = \|\, |\check{\rho}_0| * |\varphi|\, \|^\alpha_{L^\alpha(S)}.$$

The second statement (2)(a) is checked in the same way as (1)(a). For (2)(b), in the case $d = 1$ and $S = \mathbb{R}_+$, with $\rho_0(t) = 0$ for all $t \in \mathbb{R} \setminus \mathbb{R}_+$ and $K = [-n, n]$,

$$\begin{aligned}\int_S \left(\int_K |\rho_0(t - s)|\, \mathrm{d}t \right)^\alpha \mathrm{d}s &= \int_0^n \left(\int_s^n |\rho_0(t - s)|\ \mathrm{d}t \right)^\alpha \mathrm{d}s \\ &= \int_0^n \left(\int_0^r |\rho_0(u)|\ \mathrm{d}u \right)^\alpha \mathrm{d}r,\end{aligned}$$

where we have used the changes of variables $u = t - s$ (s fixed), then $r = n - s$. Since the inner integral is an increasing function of r, we get $\rho_0 \in L^1_{\text{loc}}(\mathbb{R}_+)$. The case $d > 1$ is checked in a similar way.

For (3), we can write

$$\begin{aligned}\int_{\mathbb{R}^d} \|\rho_0(t - \cdot)\|_{L^\alpha(S)}\, \mu_{|\varphi|}(\,\mathrm{d}t) &= \int_{\mathbb{R}^d} \left(\int_S |\rho_0(t - s)|^\alpha\, \mathrm{d}s \right)^{\frac{1}{\alpha}} \mu_{|\varphi|}(\,\mathrm{d}t) \\ &= \left\| \int_S |\rho_0(\cdot - s)|^\alpha\, \mathrm{d}s \right\|^{\frac{1}{\alpha}}_{L^{1/\alpha}(\mathbb{R}^d,\, \mu_{|\varphi|})},\end{aligned}$$

then apply Minkowski's inequality for integrals (since $1/\alpha > 1$) to bound this above by

$$\left(\int_S \|\, |\rho_0(\cdot - s)|^\alpha \|_{L^{1/\alpha}(\mathbb{R}^d,\, \mu_{|\varphi|})}\, \mathrm{d}s \right)^{\frac{1}{\alpha}} = \left(\int_S \left(\int_{\mathbb{R}^d} |\rho_0(t - s)\,| \mu_{|\varphi|}(\,\mathrm{d}t) \right)^\alpha \mathrm{d}s \right)^{\frac{1}{\alpha}} < \infty \tag{4.31}$$

by (4.4). Therefore, (4.2) holds.

Corollary 4.4. Suppose $\alpha \in (0,1)$, **(H1')** holds and ρ is represented by a function ρ_0 such that $\rho_0 \geq 0$. Then the three equivalent conditions of Theorem 4.1 hold and u_{mild} is a random field representation of u_{gen}.

Proof. According to (2)(a) in Remark 4.3, for $\alpha \in (0,1)$, when ρ is represented by a nonnegative function ρ_0, then **(H1')** and condition **(INT)** are equivalent. So the conclusion follows from Theorem 4.1. □

5 Case of symmetric pure jump Lévy noise

In this section, we suppose that the driving noise $\dot{X}$ is a pure jump symmetric Lévy white noise, that is, a Lévy white noise with characteristic triplet $(0, 0, \nu)$, where ν is a symmetric Lévy measure.

We consider the linear SPDE (3.1). We fix a fundamental solution ρ of $\mathcal{L}$. As in Section 4, we are interested in determining if u_{gen}, as defined in (3.2), has a random field representation, and if u_{mild}, as defined in (3.6), is a random field representation of u_{gen}. As we mentioned in Section 3.3, this requires a stochastic Fubini's theorem, which we now present.

5.1 A stochastic Fubini theorem

Using the isomorphism (2.5), we will extend the stochastic Fubini theorem of [20, Theorem 11.4.1] to the noise $\dot{X}$. Other stochastic Fubini theorems for L^0-valued random measures already exist in the literature, but they use a different set of hypotheses. For instance, [15, Corollary 1] deals with stochastic integrands, but integration of non-deterministic processes with respect to Lévy white noises relies on a space-time framework, in which the time component is critical for the definition of predictable processes. Other stochastic Fubini theorems, such as [14, Theorem 3.1], impose integrability and/or regularity assumptions on the function $f(s,t)$ instead of a condition such as (5.1) below.

Theorem 5.1. Let $\dot{X}$ be a symmetric pure jump Lévy white noise on $S \subset \mathbb{R}^d$, with characteristic triplet $(0, 0, \nu)$. Let $f : S \times \mathbb{R}^n \mapsto \mathbb{R}$ be measurable and such that for any $t \in \mathbb{R}^n$, $f(\cdot, t) \in L(\dot{X}, S)$, and let μ be a finite (nonnegative) measure on $\mathbb{R}^n$. Suppose that

$$\int_{\mathbb{R}^n} \left| \langle \dot{X}, f(\cdot, t) \rangle \right| \mu(\mathrm{d}t) < +\infty\,, \qquad \text{a.s.} \tag{5.1}$$

Then, for almost all $s \in S$, $f(s, \cdot) \in L^1(\mu)$, and the function $\mu \circledast f : s \mapsto \int_{\mathbb{R}^n} f(s,t)\mu(\mathrm{d}t)$ is in $L(\dot{X}, S)$, and

$$\int_{\mathbb{R}^n} \langle \dot{X}, f(\cdot, t) \rangle \mu(\mathrm{d}t) = \langle \dot{X}, \mu \circledast f \rangle \qquad \text{a.s.} \tag{5.2}$$

(We emphasize that the $\circledast$ operation is not commutative. In particular, it involves a measure and a measurable function whose roles are not interchangeable.)

Proof of Theorem 5.1. The main probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. Since μ is a finite measure, we can suppose without loss of generality that it is a probability measure on $\mathbb{R}^n$. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space, and $(T_i)_{i \geqslant 1}$ be a sequence of i.i.d. random vectors on this space with law μ. We write $\mathbb{E}'$ for the expectation with respect to the probability measure $\mathbb{P}'$. In this framework, (5.1) is equivalent to

$$\mathbb{E}' \left(\left| \langle \dot{X}, f(\cdot, T_1) \rangle \right| \right) < +\infty \qquad \mathbb{P} - \text{a.s.}$$

(we are using the jointly measurable version of $\langle \dot{X}, f(\cdot, t) \rangle$ provided by Proposition 3.6). More precisely, there is a set $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$, and for any $\omega \in \Omega_1$,

$$\mathbb{E}' \left(\left| \langle \dot{X}, f(\cdot, T_1) \rangle(\omega) \right| \right) < +\infty\,.$$

By the strong law of large numbers, for any $\omega \in \Omega_1$, there is a set $\Omega_1'(\omega) \subset \Omega'$ such that $\mathbb{P}'(\Omega_1'(\omega)) = 1$ and for any $\omega' \in \Omega_1'(\omega)$,

$$\frac{1}{n} \sum_{i=1}^{n} \langle \dot{X}, f(\cdot, T_i(\omega')) \rangle(\omega) \to \mathbb{E}' \left(\langle \dot{X}, f(\cdot, T_1) \rangle(\omega) \right) \qquad \text{as } n \to +\infty\,. \tag{5.3}$$

We define

$$A = \{(\omega, \omega') \in \Omega \times \Omega' : (5.3) \text{ occurs}\} \ .$$

Then $A \in \mathcal{F} \times \mathcal{F}'$. For $\omega \in \Omega$, let

$$A_\omega = \{\omega' \in \Omega' : (\omega, \omega') \in A\} \ .$$

Then, for any $\omega \in \Omega_1$, $\mathbb{P}'(A_\omega) = 1$, and we deduce from Fubini's theorem that $\mathbb{P} \times \mathbb{P}'(A) = 1$.

For any $n \in \mathbb{N}$, $s \in S$ and $\omega' \in \Omega'$, we set $f_n(s, \omega') = \frac{1}{n} \sum_{i=1}^{n} f(s, T_i(\omega'))$. Then $f_n(\cdot, \omega') \in L(\dot{X}, S)$ since this is a vector space. For $\mathbb{P}'$-a.e. $\omega' \in \Omega'$, there is a set $\Omega_n(\omega') \subset \Omega$ such that $\mathbb{P}(\Omega_n(\omega')) = 1$ and for any $\omega \in \Omega_n(\omega')$,

$$\frac{1}{n} \sum_{i=1}^{n} \langle \dot{X}, f(\cdot, T_i(\omega')) \rangle(\omega) = \langle \dot{X}, f_n(\cdot, \omega') \rangle(\omega) \, . \tag{5.4}$$

For these $\omega' \in \Omega'$, the set $\Omega_\infty(\omega') = \bigcap_{n=1}^{+\infty} \Omega_n(\omega')$ is such that $\mathbb{P}(\Omega_\infty(\omega')) = 1$ and for any $\omega \in \Omega_\infty(\omega')$, (5.4) holds for all $n \in \mathbb{N}$. We define

$$B = \{(\omega, \omega') \in \Omega \times \Omega' : (5.4) \text{ occurs for all } n \in \mathbb{N}\} \ .$$

Then $B \in \mathcal{F} \times \mathcal{F}'$, and for $\omega' \in \Omega'$, let

$$B^{\omega'} = \{\omega \in \Omega : (\omega, \omega') \in B\} \ .$$

For $\mathbb{P}'$-a.e. $\omega' \in \Omega'$, $\mathbb{P}\left(B^{\omega'}\right) = 1$, and we deduce from Fubini's theorem that $(\mathbb{P} \times \mathbb{P}')(B) = 1$. Therefore,

$$\int_{\Omega'} \left(\int_{\Omega} \mathbb{1}_{(\omega,\omega') \in A \cap B} \, \mathbb{P}(\mathrm{d}\omega) \right) \mathbb{P}'(\mathrm{d}\omega') = \int_{\Omega} \left(\int_{\Omega'} \mathbb{1}_{(\omega,\omega') \in A \cap B} \, \mathbb{P}'(\mathrm{d}\omega') \right) \mathbb{P}(\mathrm{d}\omega) = 1 \, . \tag{5.5}$$

Let $\omega' \in \Omega$. We define

$$(A \cap B)^{\omega'} = \{\omega \in \Omega : (\omega, \omega') \in A \cap B\} \ .$$

From (5.5), for $\mathbb{P}'$-almost all $\omega' \in \Omega'$, $\mathbb{P}\left((A \cap B)^{\omega'}\right) = 1$. In other words, for $\mathbb{P}'$-almost all $\omega' \in \Omega'$,

$$\frac{1}{n} \sum_{i=1}^{n} \langle \dot{X}, f(\cdot, T_i(\omega')) \rangle(\omega) = \langle \dot{X}, f_n(\cdot, \omega') \rangle(\omega) \to \mathbb{E}'\left(\langle \dot{X}, f(\cdot, T_1) \rangle(\omega) \right) \qquad \text{as } n \to +\infty \, ,$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. In particular, for $\mathbb{P}'$-almost all $\omega' \in \Omega$, the sequence of random variables $(\langle \dot{X}, f_n(\cdot, \omega') \rangle)_{n \geqslant 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a Cauchy sequence in probability. By $\mathbb{P}$-a.s. linearity of $\dot{X}$ and the isomorphism property in (2.5), we deduce that $(f_n(\cdot, \omega'))_{n \geqslant 1}$ is a Cauchy sequence in $L(\dot{X}, S)$. By completeness of $L(\dot{X}, S)$, for $\mathbb{P}'$-almost all $\omega' \in \Omega'$, there is a function $\tilde{f}(\cdot, \omega') \in L(\dot{X}, S)$ such that $f_n(\cdot, \omega') \to \tilde{f}(\cdot, \omega')$ as $n \to +\infty$ in $L(\dot{X}, S)$. By (5.1) and [19, Theorem 6] (which also uses the symmetry of $\dot{X}$), for almost every $s \in S$, $\int_{\mathbb{R}^d} |f(s, t)| \mu(\mathrm{d}t) < +\infty$, that is $\mathbb{E}'(|f(s, T_1)|) < +\infty$. By the strong law of large numbers, we deduce that for almost all $s \in S$, there is a set Ω'_s such that $\mathbb{P}'(\Omega'_s) = 1$ and for any $\omega' \in \Omega'_s$,

$$\frac{1}{n} \sum_{i=1}^{n} f(s, T_i(\omega')) = f_n(s, \omega') \to \mathbb{E}'(f(s, T_1)) = \mu \circledast f(s) \qquad \text{as } n \to +\infty. \tag{5.6}$$

Let $C = \{(s,\omega') \in S \times \Omega' : \text{(5.6) holds}\}$, for $s \in S$, $C_s = \{\omega' \in \Omega' : (s,\omega') \in C\}$, and for $\omega' \in \Omega'$, $C^{\omega'} = \{s \in S : (s,\omega') \in C\}$. Since for almost all $s \in S$, $\mathbb{P}'(C_s) = 1$, Fubini's theorem implies that $(\mathrm{Leb}_d \times \mathbb{P}')(C) = 1$. We deduce that for almost all $\omega' \in \Omega'$, (5.6) holds for Leb_d-almost every $s \in S$. We can then drop the dependence in ω', so that there is a sequence $(t_i)_{i\geqslant 1}$ of deterministic times (for $\mathbb{P}$) in $\mathbb{R}^n$ such that

$$\frac{1}{n}\sum_{i=1}^{n} f(s,t_i) \to \mu \circledast f(s) \qquad \text{a.e. in } s \text{ as } n \to +\infty, \tag{5.7}$$

$$\frac{1}{n}\sum_{i=1}^{n}\langle \dot{X}, f(\cdot,t_i)\rangle = \langle \dot{X}, \frac{1}{n}\sum_{i=1}^{n} f(\cdot,t_i)\rangle \to \int_{\mathbb{R}^d}\langle \dot{X}, f(\cdot,t)\rangle \mu(\mathrm{d}t) \qquad \mathbb{P}-\text{a.s.}, \tag{5.8}$$

as $n \to +\infty$, and

$$\frac{1}{n}\sum_{i=1}^{n} f(\cdot,t_i) \to \tilde{f}(\cdot) \qquad \text{in } L(\dot{X},S) \text{ as } n \to +\infty. \tag{5.9}$$

Since convergence in $L(\dot{X},S)$ implies convergence almost everywhere along a subsequence (see [18, p. 466]), by uniqueness of the limit we get from (5.7) and (5.9) that $\mu \circledast f = \tilde{f}$ almost everywhere (and hence $\tilde{f}$ does not depend on ω'), and $\frac{1}{n}\sum_{i=1}^{n} f(\cdot,t_i) \to \mu \circledast f$ in $L(\dot{X},S)$. Therefore,

$$\langle \dot{X}, \frac{1}{n}\sum_{i=1}^{n} f(\cdot,t_i)\rangle \to \langle \dot{X}, \mu \circledast f\rangle \qquad \text{as } n \to +\infty\,, \tag{5.10}$$

in $\mathbb{P}$-probability. By uniqueness of the limit, gathering (5.8) and (5.10), we deduce that $\mathbb{P}$-almost surely, (5.2) holds. □

5.2 Relationships between mild and generalized solutions

In this section, we aim to answer the following question. Suppose that **(H1)** is satisfied. Then, the generalized solution u_{gen} of (3.1) can be defined as in Definition 3.2. Suppose also that the generalized solution has a random field representation Y. Then, is **(H2)** satisfied? If so, then the mild solution u_{mild} can be defined as in (3.6). In this case, is Y the mild solution? These questions are answered in the next theorem.

Theorem 5.2. Consider the following three conditions.

(1) ρ is represented by a function $\rho_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$;
(2) **(H2)** holds and the jointly measurable version of u_{mild} has locally integrable sample paths;
(3) **(H1)** holds and u_{gen} has a random field representation.

Then (2) implies (1), and (2) and (3) are equivalent. If either (2) or (3) holds, then u_{mild} is a random field representation of u_{gen}.

Proof. **(2) implies (1).** Condition **(H2)** implies that ρ is represented by a function ρ_0 such that for a.a. $t \in \mathbb{R}^d$, $\rho_0(t-\cdot) \in L(\dot{X},S)$, and $u_{\text{mild}}(t) = \int_S \rho_0(t-s)X(ds)$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. By (2), the jointly measurable version of u_{mild} is such that

$$\int_{\mathbb{R}^d} |\varphi(t)|\,|u_{\text{mild}}(t)|\,\mathrm{d}t < \infty, \qquad \text{a.s.}$$

By [19, Theorem 6], there is a Lebesgue-null set $N_\varphi \subset S$ such that for $s \in S \setminus N_\varphi$,

$$\int_{\mathbb{R}^d} |\rho_0(t-s)|\,|\varphi(t)|\,\mathrm{d}t < \infty. \tag{5.11}$$

Set $N = \cup_{n\in\mathbb{N}} N_{\varphi_n}$, where $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ is such that $1_{[-n,n]^d}(t) \le \varphi_n(t)$, for all $t \in \mathbb{R}^d$. Then for $s \in S \setminus N$,

$$\text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |\rho_0(t-s)|\,|\varphi(t)|\;\mathrm{d}t < \infty,$$

or equivalently, for all $n \in \mathbb{N}$, $\int_{[-n,n]^d} dt\,|\rho_0(t-s)| < \infty$. Fix $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_d) \in S \setminus N$ and do the change of variables $r = t - \tilde{s}$. Then for all $n \in \mathbb{N}$,

$$\int_{[-n-\tilde{s}_1, n-\tilde{s}_1]\times\cdots\times[-n-\tilde{s}_d, n-\tilde{s}_d]} |\rho_0(r)|\;\mathrm{d}r < \infty,$$

and this implies that $\rho_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$, proving (1).

(2) implies (3). Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, and let $\mu^+_\varphi(\mathrm{d}t) := \varphi_+(t)\,\mathrm{d}t$ and $\mu^-_\varphi(\mathrm{d}t) := \varphi_-(t)\,\mathrm{d}t$, where $\varphi_+ = \max(\varphi, 0)$ and $\varphi_- = \max(-\varphi, 0)$ are, respectively, the positive and negative parts of φ. These two measures are finite, and are the positive and negative parts of the signed measure $\mu_\varphi(\mathrm{d}t) := \varphi(t)\,\mathrm{d}t$. By (2), u_{mild} has almost surely locally integrable sample paths, therefore,

$$\int_{\mathbb{R}^d} |\langle \dot{X}, \rho(t-\cdot)\rangle|\,\mu^\pm_\varphi(\mathrm{d}t) = \int_{\mathbb{R}^d} |u_{\text{mild}}(t)|\;\mu^\pm_\varphi(\mathrm{d}t) < +\infty \qquad \text{a.s.}$$

We can now apply Theorem 5.1 separately with the positive and negative part of μ_φ, and recombine them. Let $f(s,t) = \rho(t-s)$. Notice that $\mu_\varphi \circledast f = \varphi * \check{\rho}$, and since $\mu_\varphi \circledast f$ is $\dot{X}$-integrable by Theorem 5.1, we see that **(H1)** holds. Using (3.6), Theorem 5.1 and (3.2) yields

$$\int_{\mathbb{R}^d} u_{\text{mild}}(t)\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} \langle \dot{X}, \rho(t-\cdot)\rangle \mu_\varphi(\mathrm{d}t) = \langle \dot{X}, \varphi * \check{\rho}\rangle = \langle u_{\text{gen}}, \varphi\rangle\,, \qquad \text{a.s.},$$

which proves that u_{mild} is a random field representation of u_{gen}, therefore (3) holds.

(3) implies (2). Let (Y_t) be a random field representation of u_{gen}, that is, (3.3) holds. We use the same notations as in the proof of "(3) implies (1)" in Theorem 4.1. By the same reasoning as in the proof of Theorem 4.1, there is a non random set $\tilde{A} \subset \mathbb{R}^d$ such that $\text{Leb}_d(\tilde{A}) = 0$ and for all $t \notin \tilde{A}$, $\mathbb{P}\left\{\langle Y, \varphi^t_n\rangle \to Y_t \text{ as } n \to +\infty\right\} = 1$. Then, we define $\rho^{t_0}_n = \varphi^{t_0}_n * \check{\rho} \in L(\dot{X}, S)$ (by **(H1)**). For $n, m \in \mathbb{N}$ and $t_0 \notin \tilde{A}$,

$$\mathbb{E}\left(e^{i\langle \dot{X}, \rho^{t_0}_n - \rho^{t_0}_m\rangle}\right) = \mathbb{E}\left(e^{i\int_{\mathbb{R}^d} Y_s\left(\varphi^{t_0}_n(s) - \varphi^{t_0}_m(s)\right)\mathrm{d}s}\right) \to 1 \qquad \text{as } n, m \to +\infty\,.$$

We deduce that $\langle \dot{X}, \rho^{t_0}_n - \rho^{t_0}_m\rangle$ converges to zero in law, hence in probability. Using the isomorphism in (2.5), we see that the sequence $(\rho^{t_0}_n)_{n\in\mathbb{N}}$ is Cauchy in $L(\dot{X}, S)$. This space is complete, therefore there is a function g^{t_0} such that $\rho^{t_0}_n \to g^{t_0}$ in $L(\dot{X}, S)$. For any $(t,s) \in \mathbb{R}^d \times S$, let

$$g(t,s) = \limsup_{n\to+\infty} \rho^t_n(s)\,.$$

Then $(t,s) \mapsto g(t,s)$ is measurable, and for $t \in \tilde{A}^c$, $g(t,\cdot) = g^t(\cdot)$ almost everywhere, hence $g(t,\cdot) \in L(\dot{X}, S)$. As in (4.17) and (4.20), we get that

$$\text{for all } t_0 \in \tilde{A}^c, \qquad Y_{t_0} = \langle \dot{X}, g^{t_0}\rangle = \langle \dot{X}, g(t_0,\cdot)\rangle, \qquad \text{a.s.} \tag{5.12}$$

Let $(\tilde{u}_t,\, t \in \mathbb{R}^d)$ be a jointly measurable version of $\langle \dot{X}, g(t_0,\cdot)\rangle$ (which exists by Proposition 3.6). By (5.12), for $t_0 \in \tilde{A}^c$, $Y_{t_0} = \tilde{u}_{t_0}$ a.s. As in (4.22), we can find a $\mathbb{P}$-null set N_0 such that

$$\text{for } \omega \in N^c_0, \qquad Y_t(\omega) = \tilde{u}_t(\omega) \text{ for a.a. } t \in \mathbb{R}^d. \tag{5.13}$$

Also, since Y has almost surely locally integrable sample paths, for any $\psi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |Y_t|\, \mu_\psi(\mathrm{d}t) = \int_{\mathbb{R}^d} \left|\langle \dot{X}, g^t\rangle\right| \mu_\psi(\mathrm{d}t) < +\infty \qquad \text{a.s.}\,,$$

where $\mu_\psi(\mathrm{d}t) = |\psi(t)|\,\mathrm{d}t$. By Theorem 5.1, for a.a. $s \in S$, $\int_{\mathbb{R}^d} |g(t,s)|\,|\psi(t)|\,\mathrm{d}t < \infty$, $\psi \circledast g \in L(\dot{X}, S)$ and as in (4.26)–(4.27),

$$\int_{\mathbb{R}^d} Y_t\, \mu_\psi(\,\mathrm{d}t) = \int_{\mathbb{R}^d} \tilde{u}_t\, \psi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} \langle \dot{X}, g^t\rangle \psi(t)\,\mathrm{d}t = \langle \dot{X}, \psi \circledast g\rangle \qquad \text{a.s.}$$

Therefore, for any $\psi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \dot{X}, \psi \circledast g\rangle = \int_{\mathbb{R}^d} \langle \dot{X}, g^t\rangle \psi(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} Y_t \psi(t)\,\mathrm{d}t = \langle \dot{X}, \check{\rho} * \psi\rangle \qquad \text{a.s.},$$

where the last equality is by Definitions 3.4 and 3.1. Therefore, for almost every $s \in S$, $\psi \circledast g(s) = \psi * \check{\rho}(s)$. Then, as in the proof of Theorem 4.1 after (4.28), we deduce that ρ is represented by the function $\rho_0 = \delta_{-s} * g(\cdot, s)$ and $g(t,s) = \rho_0(t-s)$ for a.a. $(t,s) \in \mathbb{R}^d \times S$. Therefore, for a.a. $t \in \mathbb{R}^d$, $\rho_0(t - \cdot) \in L(\dot{X}, S)$, which means that **(H2)** holds and u_{mild} is well-defined. By (5.12) and the lines that follow (5.12), for all $t_0 \in \tilde{A}^c$, $Y_{t_0} = \tilde{u}_{t_0} = \langle \dot{X}, \rho_0(t_0 - \cdot)\rangle = u_{\text{mild}}(t_0)$ a.s., therefore $(\tilde{u}_t,\, t \in \mathbb{R}^d)$ is a version of u_{mild}. By Fubini's theorem a.s., $\tilde{u}_t = Y_t$ for a.a. $t \in \mathbb{R}^d$, therefore, $(\tilde{u}_t)$ a.s. has locally integrable sample paths. This proves (2).

In addition, still assuming (3), (5.13) implies that $(\tilde{u}_t)$ is a random field representation of u_{gen} (since this was the case of (Y_t)). This proves the last statement in Theorem 5.2. □

Remark 5.3. Contrary to the α-stable case (Theorem 4.1), where the condition (1) was equivalent to conditions (2) and (3) there, in Theorem 5.2, condition (1) is only shown to be necessary for conditions (2) and (3).

6 Examples

In this section, we give some examples to which Theorem 4.1 applies. We focus on three well-known SPDEs: the linear stochastic heat, wave and Poisson equations, in all spatial dimensions $d \geq 1$. We focus on the case of a symmetric α-stable noise, in order to determine the range of $\alpha \in (0,2)$ and $d \geq 1$ which correspond to the different cases.

6.1 The stochastic heat equation

Let $\dot{W}^\alpha$ be an α-stable symmetric Lévy white noise on $S = \mathbb{R}_+ \times \mathbb{R}^d$. The heat operator $\mathcal{H}$ in dimension d is the constant coefficient partial differential operator given by

$$\mathcal{H} = \frac{\partial}{\partial t} - \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}\,.$$

A fundamental solution $\rho_{\mathcal{H}}$ for this operator (with support in S) is given by the nonnegative function

$$\rho_{\mathcal{H}}(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right) \mathbb{1}_{t>0}\,.$$

We consider the following Cauchy problem

$$\begin{cases} \mathcal{H}u = \dot{W}^\alpha, \\ u(0,\cdot) = 0. \end{cases} \tag{6.1}$$

6.1.1 Existence of a generalized solution

We are interested in the generalized solution of this equation associated with the fundamental solution $\rho_{\mathcal{H}}$.

Proposition 6.1. For any $\alpha \in (0,2)$ and $d \geqslant 1$, **(H1')** holds, so the generalized solution to the linear stochastic heat equation driven by a symmetric α-stable Lévy white noise is well defined.

Proof. Fix $\alpha \in (0,2)$. We begin by checking **(H1')**. We must show that for all $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$, the convolution $\varphi * \check{\rho}_{\mathcal{H}}$ belongs to $L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$. Fix $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$. For $(t,x) \in \mathbb{R} \times \mathbb{R}^d$,

$$\varphi * \check{\rho}_{\mathcal{H}}(t,x) = \int_t^{+\infty} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right) \varphi(s,y)\,. \tag{6.2}$$

Since φ has compact support in $\mathbb{R} \times \mathbb{R}^d$, we fix $T > 0$ and $K \subset \mathbb{R}^d$ compact such that $\operatorname{supp} \varphi \subset [-T/2, T/2] \times K$. Then we see from (6.2) that for any $t \geqslant T/2$ and $x \in \mathbb{R}^d$, $\varphi * \check{\rho}_{\mathcal{H}}(t,x) = 0$. Therefore, we need only to check that $\varphi * \check{\rho}_{\mathcal{H}} \in L^\alpha([0,T/2] \times \mathbb{R}^d)$. The function $\varphi * \check{\rho}_{\mathcal{H}}$ is smooth, so we only need to check integrability for x in a neighborhood of infinity. Clearly, for $t \in [0, T/2]$,

$$|\varphi * \check{\rho}_{\mathcal{H}}(t,x)| \leqslant |\varphi| * \check{\rho}_{\mathcal{H}}(t,x) = \mathbb{1}_{t \leqslant T/2} \int_t^{T/2} \mathrm{d}s \int_K \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right) |\varphi(s,y)|\,.$$

The function $s \mapsto (4\pi(s-t))^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4(s-t)})$ is increasing on $]t, t + \frac{1}{2d}|x-y|^2]$ (and decreasing on $[t + \frac{1}{2d}|x-y|^2, \infty[$). Suppose that $K \subset B(0,r)$, the Euclidean ball centered at 0 with radius r. Choose R large enough so that $T < \frac{1}{2d}(R-r)^2$. For $|x| > R$ and $y \in K$, we have $T < t + \frac{1}{2d}|x-y|^2$, therefore

$$|\varphi * \check{\rho}_{\mathcal{H}}(t,x)| \leq \mathbb{1}_{t \leqslant T/2} \|\varphi\|_\infty \int_t^{T/2} \mathrm{d}s \int_K \mathrm{d}y \frac{1}{(4\pi(T-t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x|^2}{4(T-t)}\right)\,.$$

Since $|x-y| \geq |x| - r$, this is bounded above by

$$\mathbb{1}_{t \leqslant T/2} \frac{\|\varphi\|_\infty}{(4\pi)^{\frac{d}{2}}} \left(\frac{T}{2}\right)^{-\frac{d}{2}+1} \int_K \mathrm{d}y \exp\left(-\frac{(|x|-r)^2}{4(T-t)}\right)\,.$$

Then, using the inequality $(a-b)^2 \geqslant \frac{1}{2}a^2 - b^2$, we obtain

$$|\varphi * \check{\rho}_{\mathcal{H}}(t,x)| \leqslant \mathbb{1}_{t \leqslant T/2} \frac{\|\varphi\|_\infty}{(4\pi)^{\frac{d}{2}}} \left(\frac{T}{2}\right)^{-\frac{d}{2}+1} \exp\left(-\frac{|x|^2}{8(T-t)}\right) \int_K \mathrm{d}y \exp\left(\frac{r^2}{2T}\right)\,.$$

We deduce that for $t \in \mathbb{R}_+$ and $|x| > R$,

$$|\varphi * \check{\rho}_{\mathcal{H}}(t,x)| \leqslant c_{T,K} \|\varphi\|_\infty \mathbb{1}_{t \leqslant T/2} \exp\left(-\frac{|x|^2}{8T}\right)\,, \tag{6.3}$$

where $c_{T,K}$ is a constant that depends only on the support of φ. From (6.3), we deduce that $\varphi * \check{\rho}_{\mathcal{H}}$ has compact support in the time variable (uniformly with respect to the space variable), and has rapid decay in the space variable. Therefore $\varphi * \check{\rho}_{\mathcal{H}} \in L^\alpha([0,T] \times \mathbb{R}^d)$.

It follows that **(H1')** holds, therefore, the linear stochastic heat equation driven by symmetric α-stable Lévy white noise always has a generalized solution u_{gen} defined by

$$\langle u_{\text{gen}}, \varphi \rangle := \langle \dot{W}^\alpha, \varphi * \check{\rho}_{\mathcal{H}} \rangle\,, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^{d+1}). \tag{6.4}$$

□

Remark 6.2. The previous proof is still valid if we formally replace α by 2, and therefore the same result is true in the Gaussian case.

Remark 6.3. From (6.3), we get that

$$\|\varphi * \check{\rho}_{\mathcal{H}}\|_{L^\alpha([0,T]\times\mathbb{R}^d)} \leqslant C\|\varphi\|_\infty ,$$

for some constant C that depends on the support of φ. Therefore, if φ_n is a sequence of test functions in $\mathcal{D}(\mathbb{R}^{d+1})$ such that $\varphi_n \to 0$ in $\mathcal{D}(\mathbb{R}^{d+1})$, then

$$\mathbb{E}\left[e^{i\xi\langle u_{\text{gen}},\varphi_n\rangle}\right] = e^{-|\xi|^\alpha \|\varphi_n * \check{\rho}_{\mathcal{H}}\|^\alpha_{L^\alpha([0,T]\times\mathbb{R}^d)}} \to 1 , \qquad \text{as } n \to +\infty .$$

Therefore, $\langle u_{\text{gen}}, \varphi_n\rangle \to 0$ in law as $n \to +\infty$, and since convergence in law to a constant is equivalent to the convergence in probability to this constant, we deduce that $\langle u_{\text{gen}}, \varphi_n\rangle \to 0$ in probability as $n \to +\infty$. Therefore, u_{gen} defines a linear functional on $\mathcal{D}(\mathbb{R}^{d+1})$ that is continuous in probability. The space $\mathcal{D}(\mathbb{R}^{d+1})$ is nuclear (see [24, p. 510]), so by [25, Corollary 4.2], u_{gen} has a version in $\mathcal{D}'(\mathbb{R}^{d+1})$.

6.1.2 Existence of a mild solution

The criterion for the existence of the mild solution to the linear stochastic heat equation (6.1) is known (see [1]). However, we can also obtain this from **(H2')**.

Proposition 6.4. Condition **(H2')** holds and the mild solution to the linear stochastic heat equation driven by a symmetric α-stable noise, as defined in (3.6), exists if and only if

$$\alpha < 1 + \frac{2}{d} . \tag{6.5}$$

In this case,

$$u_{\text{mild}}(t,x) := \langle \dot{W}^\alpha, \rho_{\mathcal{H}}(t-\cdot, x-\cdot)\rangle . \tag{6.6}$$

Proof. According to Remark 4.3(1)(b), condition **(H2')** holds if and only if the following integral is finite for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$\begin{aligned}\int_{\mathbb{R}_+} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, \rho_{\mathcal{H}}(t-s,x-y)^\alpha &= \int_0^t \mathrm{d}s \frac{1}{(4\pi s)^{\alpha\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{d}y \exp\left(-\frac{\alpha|y|^2}{4s}\right) \\ &= \int_0^t \mathrm{d}s \frac{1}{(4\pi s)^{\frac{d}{2}(\alpha-1)}\,\alpha^{\frac{d}{2}}} ,\end{aligned} \tag{6.7}$$

and the last integral is finite if and only if (6.5) holds. In this case, by Definition 3.5,

$$u_{\text{mild}}(t,x) := \langle \dot{W}^\alpha, \rho_{\mathcal{H}}(t-\cdot, x-\cdot)\rangle . \qquad \square$$

6.1.3 Existence of a random field representation

We have seen in the previous subsections that for any α and d, it is possible to define the generalized solution u_{gen}, and that the mild solution u_{mild} exists if and only if $\alpha < 1 + \frac{2}{d}$. We now apply the results of Theorem 4.1 to learn more about the relations between those two notions of solution.

Proposition 6.5. Condition (6.5) is equivalent to condition **(INT)**. The generalized solution u_{gen} to the linear stochastic heat equation driven by a symmetric α-stable noise has a random field representation Y if and only if (6.5) is satisfied, and in that case, u_{mild} is a random field representation of u_{gen}.

Proof. We know from Theorem 4.1 that condition **(INT)** implies **(H2')**, therefore (6.5), as we have seen in the proof of Proposition 6.4. For the converse implication, we assume that (6.5) holds and we distinguish three cases.

If $\alpha \in \left(1, 1+\frac{d}{2}\right)$, then the condition (4.2) is immediately verified using (6.7) and Remark 4.3(1). If $\alpha < 1$, we know that **(H1')** holds by Proposition 6.1, and since $\rho_{\mathcal{H}} \geq 0$, condition **(INT)** holds by Corollary 4.4. Therefore, for $\alpha \neq 1$, (6.5) implies condition **(INT)**.

The case $\alpha = 1$ is slightly more involved, since we need to check condition (4.3). Let $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$. First, we have

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \rho_{\mathcal{H}}(t-s, x-v)\,\mathrm{d}v = t\mathbb{1}_{t>0}\,,$$

and for any $x \in \mathbb{R}_+$, $\log_+(x) \leqslant |\log(x)|$, therefore, for $t > 0$

$$\begin{aligned}
&\log_+\left(\frac{\rho_{\mathcal{H}}(t-s,x-y)\int_{\mathbb{R}^{d+1}}\int_{\mathbb{R}_+\times\mathbb{R}^d}|\rho_{\mathcal{H}}(u-v,r-w)|\,\mathrm{d}v\,\mathrm{d}w\,\mu_\varphi(\mathrm{d}u,\mathrm{d}r)}{\left(\int_{\mathbb{R}_+\times\mathbb{R}^d}\rho_{\mathcal{H}}(t-v,x-w)\,\mathrm{d}v\,\mathrm{d}w\right)\left(\int_{\mathbb{R}^{d+1}}\rho_{\mathcal{H}}(u-s,r-y)\mu_\varphi(\mathrm{d}u,\mathrm{d}r)\right)}\right)\\
&\qquad\leqslant |\log\left(\rho_{\mathcal{H}}(t-s,x-y)\right)| + \left|\log\left(\int_{\mathbb{R}^{d+1}} u\mu_\varphi(\mathrm{d}u,\mathrm{d}r)\right)\right| + |\log(t)|\\
&\qquad\quad + \left|\log\left(\check{\rho}_{\mathcal{H}} * |\varphi|(s,y)\right)\right|\,.
\end{aligned}$$

Hence, to have (4.3), we need to check the finiteness of the following integrals:

$$\begin{aligned}
I_I &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\check{\rho}_{\mathcal{H}} * |\varphi|\right)(s,y)\,\mathrm{d}s\,\mathrm{d}y\,,\\
I_2 &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\left(\check{\rho}_{\mathcal{H}}\left|\log\left(\check{\rho}_{\mathcal{H}}\right)\right|\right) * |\varphi|\right)(s,y)\,\mathrm{d}s\,\mathrm{d}y\,,\\
I_3 &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left(\int_{\mathbb{R}^{d+1}} \rho_{\mathcal{H}}(t-s,x-y)|\log(t)\varphi(t,x)|\,\mathrm{d}t\,\mathrm{d}x\right)\,\mathrm{d}s\,\mathrm{d}y\,,\\
I_4 &:= \int_{\mathbb{R}_+\times\mathbb{R}^d} \left|\log\left(\check{\rho}_{\mathcal{H}} * |\varphi|(s,y)\right)\right|\left(\check{\rho}_{\mathcal{H}} * |\varphi|\right)(s,y)\,\mathrm{d}s\,\mathrm{d}y\,.
\end{aligned}$$

The case of I_1 has already been treated after (6.3), and for I_3, we can simply permute the integrals and get

$$I_3 = \int_{\mathbb{R}^{d+1}} |t\mathbb{1}_{t>0}\log(t)\varphi(t,x)|\,\mathrm{d}t\,\mathrm{d}x < +\infty\,.$$

For I_2 and I_4, by the same considerations as for the case $\alpha \neq 1$, we need to check that for any $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$,

$$(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto |\check{\rho}_{\mathcal{H}}\log(\check{\rho}_{\mathcal{H}})| * |\varphi|(t,x)\,,$$

and

$$(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto \left(\check{\rho}_{\mathcal{H}} * |\varphi|\right)(t,x)\left|\log(\check{\rho}_{\mathcal{H}} * |\varphi|(t,x))\right|\,,$$

are in $L^1([0,T]\times\mathbb{R}^d)$ for any $T \in \mathbb{R}_+$. By (6.3), we get that $\left(\check{\rho}_{\mathcal{H}} * |\varphi|\right)|\log(\check{\rho}_{\mathcal{H}} * |\varphi|)| \in L^1(\mathbb{R}_+\times\mathbb{R}^d)$, therefore $I_4 < +\infty$. We now turn to I_2. Observe that

$$\begin{aligned}
|\check{\rho}_{\mathcal{H}}\log(\check{\rho}_{\mathcal{H}})| * |\varphi|(t,x) &= \mathbb{1}_{t\leqslant T}\int_t^T \mathrm{d}s \int_K \mathrm{d}y \frac{1}{(4\pi(s-t))^{\frac{d}{2}}}\exp\left(-\frac{|y-x|^2}{4(s-t)}\right)\\
&\qquad\times\left|-\frac{d}{2}\log\left(4\pi(s-t)\right) - \frac{|y-x|^2}{4(s-t)}\right||\varphi(s,y)|\,.
\end{aligned}$$

Again, by continuity (since $|\varphi|$ is continuous and has compact support), we are only concerned about integrability near a neighborhood of infinity ($x \to \pm\infty$, $t \in [0,T]$). Since $\log(4\pi(s-t))$ is integrable at $s=t$ and the polynomial term $|y-x|^2/(s-t)$ barely affects the decay as $x \to \pm\infty$ of $\exp\left(-|y-x|^2/(4(s-t))\right)$, we can obtain a bound similar to (6.3): see [11, Proof of Proposition 4.4.4] for details. We conclude that (4.3) holds. This completes the proof of the equivalence of (6.5) and condition **(INT)**.

The remaining statements in Proposition 6.5 are now an immediate consequence of Theorem 4.1. □

Propositions 6.1, 6.4 and 6.5 together establish the following theorem.

Theorem 6.6. The generalized solution u_{gen} to the stochastic heat equation (6.1) defined by (6.4) always exists. The mild solution u_{mild} defined by (6.6) exists if and only if

$$\alpha < 1 + \frac{2}{d}\,. \tag{6.8}$$

Furthermore, u_{gen} has a random field representation if and only if (6.8) is satisfied and in this case, u_{mild} has locally integrable sample paths and u_{mild} is a random field representation of u_{gen}.

6.2 The stochastic wave equation

We now consider the stochastic wave equation. For an overview of this SPDE in the Gaussian case, see [7]. Let $\dot{W}^\alpha$ be a symmetric α-stable Lévy white noise on $S = \mathbb{R}_+ \times \mathbb{R}^d$. The wave operator $\mathcal{O}$ in spatial dimension d is the constant coefficient partial differential operator given by

$$\mathcal{O} = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\,.$$

The fundamental solution of this operator (with support in the forward light cone) is a function only in spatial dimensions one and two. In spatial dimension one, it is given by

$$\rho_1^{\mathcal{O}}(t,x) = \frac{1}{2}\mathbb{1}_{|x|\leqslant t} \qquad \text{for all } (x,t) \in \mathbb{R}^2\,,$$

and, in spatial dimension two, by

$$\rho_2^{\mathcal{O}}(t,x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbb{1}_{|x|<t} \qquad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^2\,.$$

In dimension $d \geqslant 3$, the fundamental solution is a distribution that can be characterized by its Fourier transform in the space variable x (see [8]), but it *cannot* be represented by a function.

This fundamental solution is related to the following Cauchy problem:

$$\begin{cases} \mathcal{O}u = \dot{W}^\alpha\,, \\ u(0,\cdot) = 0\,, \\ \frac{\partial u}{\partial t}(0,\cdot) = 0\,. \end{cases} \tag{6.9}$$

6.2.1 Existence of the generalized solution

We first study the existence of the generalized solution in all dimensions $d \geqslant 1$.

Proposition 6.7. For any dimension $d \geqslant 1$ and $\alpha \in (0,2)$, **(H1')** holds, so the generalized solution u_{gen} to the linear stochastic wave equation driven by a symmetric α-stable Lévy white noise is well-defined.

Proof. We check that **(H1')** holds.

$\underline{d=1}$: We need to check that for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$, the convolution $\varphi * \check{\rho}_1^{\mathcal{O}}$ is in $L^\alpha(\mathbb{R}_+ \times \mathbb{R})$. Clearly, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\varphi * \check{\rho}_1^{\mathcal{O}}(t,x) = \int_0^{+\infty} \mathrm{d}s \int_{-s}^{s} \mathrm{d}y\, \varphi(s+t, y+x)\,,$$

and we can see from this expression that this is a smooth function with compact support, hence in $L^\alpha(\mathbb{R}_+ \times \mathbb{R})$.

$\underline{d=2}$: Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$. We will show that for all $\alpha \in (0,2)$, the function $\varphi * \check{\rho}_2^{\mathcal{O}}$ belongs to $L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$. By standard properties of the convolution, $\varphi * \check{\rho}_2^{\mathcal{O}}$ is a smooth function. Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$. Then

$$\varphi * \check{\rho}_2^{\mathcal{O}}(t,x) = \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}y \rho_2^{\mathcal{O}}(s-t, y-x)\varphi(s,y)\,.$$

Since φ has compact support and $\rho_2^{\mathcal{O}}$ has support in the set $\{(r,z) \in \mathbb{R}_+ \times \mathbb{R}^2 : |z| \leqslant r\}$, we can write

$$\varphi * \check{\rho}_2^{\mathcal{O}}(t,x) = \mathbb{1}_{t \leqslant T} \int_t^T \mathrm{d}s \int_{B_x(T-t)} \mathrm{d}y \rho_2^{\mathcal{O}}(s-t, y-x)\varphi(s,y)\,, \tag{6.10}$$

for some $T \in \mathbb{R}_+$, where $B_x(r)$ is the open ball of radius r centered at x. We see from this expression that the convolution has compact support in space and time, since if x is far enough from the support of φ, the integrand is zero. We deduce that for any $\alpha \in (0,2)$, $\varphi * \check{\rho}_2^{\mathcal{O}} \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$, and the generalized solution to the linear stochastic wave equation in dimension 2 always exists.

$\underline{d \geqslant 3}$: For any $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, the function $\varphi * \check{\rho}_d^{\mathcal{O}}$ is smooth. By the same type of considerations on the support of the convolution $\varphi * \check{\rho}_d^{\mathcal{O}}$ as in dimensions one and two, we see that this function has compact support, therefore $\varphi * \check{\rho}_d^{\mathcal{O}} \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$ for any $\alpha \in (0,2)$.

We conclude that **(H1')** holds, and therefore the generalized solution is well-defined. □

Remark 6.8. Looking at (6.10), since both φ and $\rho_d^{\mathcal{O}}$ have compact support, we see that there are $T > 0$ and a compact set K such that

$$\|\varphi * \check{\rho}_d^{\mathcal{O}}\|^\alpha_{L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)} \le \int_0^T \int_K dx\, |\varphi * \check{\rho}_d^{\mathcal{O}}(t,x)|^\alpha,$$

and from the explicit formula for $\rho_d^{\mathcal{O}}$ (see [16, p.281]), we see that

$$|\varphi * \check{\rho}_d^{\mathcal{O}}(t,x)| \le C \sup_{s \in [0,T]} \sup_{\underline{i} \le p} \|D^{(\underline{i})}\varphi(s,\cdot)\|_\infty,$$

where $\underline{i}$ is a multi-index and $D^{(\underline{i})}$ is the corresponding derivative, and $p \le \frac{d-2}{2}$. Therefore,

$$\|\varphi * \check{\rho}_d^{\mathcal{O}}\|_{L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)} \le C \| \sup_{\underline{i} \le p} D^{(\underline{i})}\varphi\|_\infty,$$

where C depends only on the support of φ. We can then deduce, as in Remark 6.3, that u_{gen} defines a linear functional on $\mathcal{D}(\mathbb{R}^{d+1})$ that is continuous in probability, and therefore u_{gen} has a version in $\mathcal{D}'(\mathbb{R}^d)$.

6.2.2 Existence of the mild solution

Proposition 6.9. For all $\alpha \in (0,2)$, **(H2')** holds if and only if $d \in \{1,2\}$, hence the mild solution to the stochastic wave equation driven by a symmetric α-stable Lévy white noise exists only in dimensions one and two.

Proof. We first check **(H2')** for $d \in \{1,2\}$.

$\underline{d=1}$: According to Remark 4.3(1)(b), we must show that for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $\rho_1^{\mathcal{O}}(t-\cdot, x-\cdot) \in L^\alpha(\mathbb{R}_+ \times \mathbb{R})$. Therefore, for any $T>0$, we need to check the finiteness of the integral

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x \rho_1^{\mathcal{O}}(t,x)^\alpha = \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x\, \frac{1}{2^\alpha} \mathbb{1}_{|x|\leqslant t} = \frac{T^2}{2^\alpha}\,. \tag{6.11}$$

We deduce that **(H2')** holds for $d=1$ and any $\alpha \in (0,2)$.

$\underline{d=2}$: Again, we must show that $\rho_2^{\mathcal{O}}(t-\cdot, x-\cdot) \in L^\alpha(\mathbb{R}_+ \times \mathbb{R})$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$. We have

$$\begin{aligned}\|\rho_2^{\mathcal{O}}(t-\cdot, x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)}^{\alpha\vee 1} &= \int_0^t \mathrm{d}s \int_{B_x(t-s)} \mathrm{d}y \frac{1}{(2\pi)^\alpha \left((t-s)^2 - |x-y|^2\right)^{\frac{\alpha}{2}}} \\ &= \frac{1}{(2\pi)^\alpha} \int_0^t \mathrm{d}s \int_{|u|\leqslant s} \mathrm{d}u \frac{1}{(s^2-|u|^2)^{\frac{\alpha}{2}}}\,.\end{aligned}$$

Changing to polar coordinates, we get

$$\|\rho_2^{\mathcal{O}}(t-\cdot, x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)}^{\alpha\vee 1} = \frac{1}{(2\pi)^{\alpha-1}} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \frac{r}{(s-r)^{\frac{\alpha}{2}}(s+r)^{\frac{\alpha}{2}}}\,.$$

This integral is finite if and only if $\frac{\alpha}{2} < 1$, that is $\alpha < 2$. We can further evaluate this integral and we get

$$\begin{aligned}\|\rho_2^{\mathcal{O}}(t-\cdot, x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)}^{\alpha\vee 1} &= \frac{1}{(2\pi)^{\alpha-1}} \int_0^t \mathrm{d}s \int_0^s \frac{\mathrm{d}r}{2} \frac{2r}{(s^2-r^2)^{\frac{\alpha}{2}}} \\ &= \frac{1}{(2\pi)^{\alpha-1}} \int_0^t \mathrm{d}s \frac{s^{2-\alpha}}{2-\alpha} = \frac{t^{3-\alpha}}{(2\pi)^{\alpha-1}(2-\alpha)(3-\alpha)}\,.\end{aligned} \tag{6.12}$$

Therefore, **(H2')** holds for $d=2$ and any $\alpha \in (0,2)$.

We conclude that there is always a mild solution to the linear stochastic wave equation with α-stable noise when $d \in \{1,2\}$.

We now consider the case $d \geqslant 3$.

$\underline{d \geqslant 3}$: Since fundamental solutions of the wave equation in dimensions $d \geqslant 3$ are not functions, **(H2')** cannot hold and there is no mild solution in such dimensions. $\square$

Remark 6.10. From this proof, we can deduce the already known result in the Gaussian case (see [7, p. 46]) that a mild solution to the linear stochastic wave equation only exists in spatial dimension one.

6.2.3 Existence of a random field representation

Proposition 6.11. Condition **(INT)** holds if and only if $d \in \{1,2\}$. Therefore, the generalized solution u_{gen} to the linear stochastic wave equation driven by a symmetric α-stable Lévy white noise has a random field representation if and only if $d \in \{1,2\}$, and in that case, u_{mild} is a random field representation of u_{gen}.

Proof. We will show that condition **(INT)** holds when $d = 1$ or $d = 2$ by considering separately the two cases, and then we will show that condition **(INT)** does not hold for $d \geqslant 3$.

<u>$d = 1$:</u> If $\alpha > 1$, it suffices to check that for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $\|\rho_1^{\mathcal{O}}(t-\cdot, x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R})} \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$, and this is the case by (6.11). If $\alpha < 1$, we know that **(H1')** holds by Proposition 6.7, and since $\rho_1^{\mathcal{O}} \geq 0$, condition **(INT)** holds by Corollary 4.4. In the case $\alpha = 1$, it is necessary to check that for any compact set $K \subset \mathbb{R}^2$,

$$\int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y\, |\rho_1^{\mathcal{O}}(t-s,x-y)| \Bigg[1 + \log_+\left(\frac{|\rho_1^{\mathcal{O}}(t-s,x-y)| \int_K \mathrm{d}u\,\mathrm{d}r \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}v\,\mathrm{d}w |\rho_1^{\mathcal{O}}(u-v,r-w)|}{\left(\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}v\,\mathrm{d}w |\rho_1^{\mathcal{O}}(t-v,x-w)|\right)\left(\int_K |\rho_1^{\mathcal{O}}(u-s,r-y)|\,\mathrm{d}u\,\mathrm{d}r\right)}\right)\Bigg] < +\infty\,. \tag{6.13}$$

The details of this calculation can be found in [11, Proof of Proposition 4.4.9].

We conclude that, for any $\alpha \in (0,2)$, condition **(INT)** holds.

<u>$d = 2$:</u> For $\alpha > 1$, by (6.12), $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \to \|\rho_2^{\mathcal{O}}(t-\cdot, x-\cdot)\|_{L^\alpha(\mathbb{R}_+\times\mathbb{R}^2)}$ does not depend on x and is continuous in the t variable, therefore (4.2) is verified. In the case where $\alpha < 1$, the same argument as in the case $d = 1$ shows that (4.4) is verified.

The case $\alpha = 1$ is again more involved, since we need to consider the expression (4.3). We must check that for any compact set $K \subset \mathbb{R}^3$,

$$\int_K \mathrm{d}t\,\mathrm{d}x \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}s\,\mathrm{d}y\, |\rho_2^{\mathcal{O}}(t-s,x-y)| \Bigg[1 + \log_+\left(\frac{|\rho_2^{\mathcal{O}}(t-s,x-y)| \int_K \mathrm{d}u\,\mathrm{d}r \int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}v\,\mathrm{d}w |\rho_2^{\mathcal{O}}(u-v,r-w)|}{\left(\int_{\mathbb{R}_+\times\mathbb{R}^2} \mathrm{d}v\,\mathrm{d}w |\rho_2^{\mathcal{O}}(t-v,x-w)|\right)\left(\int_K |\rho_2^{\mathcal{O}}(u-s,r-y)|\,\mathrm{d}u\,\mathrm{d}r\right)}\right)\Bigg] < +\infty\,. \tag{6.14}$$

For the details of this calculation, see [11, Proof of 4.4.9].

Therefore, for any $\alpha \in (0,2)$, condition **(INT)** holds.

We conclude from the above and Theorem 4.1 that when $d \in \{1,2\}$, then the generalized solution has a random field representation, the mild solution is well-defined and is a random field representation of the generalized solution.

<u>$d \geqslant 3$:</u> Condition **(INT)** cannot hold because in these dimensions, $\rho_d^{\mathcal{O}}$ is not represented by a function. By Theorem 4.1, there cannot be any random field representation of the generalized solution. □

We summarize Propositions 6.7, 6.9 and 6.11 in the following theorem.

Theorem 6.12. The generalized solution u_{gen} to the stochastic wave equation (6.9) defined by (3.2) always exists. The mild solution u_{mild} exists if and only if $d \leqslant 2$. Furthermore, a random field representation Y of the generalized solution exists if and only if $d \in \{1,2\}$, and in this case, u_{mild} is a random field representation of u_{gen}.

6.3 The stochastic Poisson equation

Let $\dot{W}^\alpha$ be an α-stable symmetric noise on $\mathbb{R}^d$. The laplacian operator Δ is given by

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}\,.$$

The fundamental solution of the Poisson operator $\mathcal{P} = -\Delta$ on $\mathbb{R}^d$ is given by

$$\rho^1_{\mathcal{P}}(x) = \frac{1}{2}|x|, \qquad x \in \mathbb{R},$$
$$\rho^2_{\mathcal{P}}(x) = \frac{1}{2\pi} \ln \frac{1}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\},$$
$$\rho^d_{\mathcal{P}}(x) = \frac{1}{C_d} \frac{1}{|x|^{d-2}}, \qquad x \in \mathbb{R}^d \setminus \{0\},\ d \geq 3,$$

where

$$C_d = \frac{2\pi^{\frac{d}{2}}(d-2)}{\Gamma(\frac{d}{2})}.$$

We consider the following SPDE in $\mathbb{R}^d$:

$$-\Delta u = \dot{W}^\alpha. \tag{6.15}$$

Theorem 6.13. (a) For $d \geq 1$ and $\alpha \in (0,2)$, **(H2')** does not hold and therefore, there is no mild solution to (6.15).

(b) Condition **(H1')** holds if and only if $d > 4$ and $\alpha \in \left(\frac{d}{d-2}, 2\right)$. For these d and α, there is a generalized solution u_{gen} to (6.15), but there is no random field representation of u_{gen}.

Proof. (a) It is immediate to check that for all $d \geq 1$ and $\alpha \in (0,2)$, $\rho^d_{\mathcal{P}} \notin L^\alpha(\mathbb{R}^d)$, therefore **(H2')** does not hold and there is no mild solution to (6.15).

(b) Turning to the generalized solution, we first examine dimensions 1 and 2.

$\underline{d=1}$: Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned}\check{\rho}^1_{\mathcal{P}} * \varphi(x) &= -\frac{1}{2}\int_{\mathbb{R}} |x-y|\varphi(y)\,dy \\ &= -\frac{1}{2}\left(x\left(\int_{-\infty}^{x} \varphi(y)\,dy - \int_{x}^{+\infty} \varphi(y)\,dy\right) + \int_{x}^{+\infty} y\varphi(y)\,dy - \int_{-\infty}^{x} y\varphi(y)\,dy\right).\end{aligned}$$

Since φ has compact support, for large enough $|x|$,

$$\check{\rho}^1_{\mathcal{P}} * \varphi(x) = -\frac{1}{2}\left(x\int_{\mathbb{R}} \varphi(y)\,dy - \int_{\mathbb{R}} y\varphi(y)\,dy\right). \tag{6.16}$$

In particular, we see that for any $\alpha \in (0,2)$, $\check{\rho}^d_{\mathcal{P}} * \varphi \notin L^\alpha(\mathbb{R})$ (unless the two integrals in (6.16) vanish), therefore **(H2')** does not hold.

$\underline{d=2}$: Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Then

$$\check{\rho}^2_{\mathcal{P}} * \varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}\,\varphi(y)\,dy.$$

Assuming that $\varphi \geq 0$ and $\varphi \not\equiv 0$, for $|x|$ large enough and ε small enough, the right-hand side is bounded below by

$$\frac{\|\varphi\|_\infty}{2}\int_{|y|\leq\varepsilon} \ln\frac{1}{|x-y|}\,dy \geq \varepsilon^2\,\frac{\|\varphi\|_\infty}{2}\,\frac{1}{\ln(|x|+\varepsilon)},$$

hence $\check{\rho}^2_{\mathcal{P}} * \varphi \notin L^\alpha(\mathbb{R}^2)$ and **(H2')** does not hold.

$\underline{d \geq 3}$: Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then

$$\check{\rho}^d_{\mathcal{P}} * \varphi(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \varphi(y)\, dy.$$

This is a C^∞-function of x, hence we only need to consider its integrability as $|x| \to \infty$. Proceeding as for the case $d = 2$, for $\varphi \geq 0$ and $\varphi \not\equiv 0$, we can bound the integral below, up to a constant, by

$$\int_{|y|\leq\varepsilon} \frac{1}{|x-y|^{d-2}}\, dy \geq \varepsilon^d \frac{1}{(|x|+\varepsilon)^{d-2}}.$$

Passing to polar coordinates, we see that $\check{\rho}^d_{\mathcal{P}} * \varphi \notin L^\alpha(\mathbb{R}^d)$ unless $\alpha(2-d)+d<0$, which is equivalent to $\alpha > \frac{d}{d-2}$. Since $\alpha < 2$, this can only occur if $d > 4$.

On the other hand, if $d > 4$ and $\alpha > \frac{d}{d-2}$ (≥ 1), then for $N > 0$, by the generalized Minkowski inequality,

$$\left[\int_{|x|>N} dx \left|\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \varphi(y)\, dy\right|^\alpha\right]^{\frac{1}{\alpha}} \leq \int_{\mathbb{R}^d} dy\, |\varphi(y)| \left|\int_{|x|>N} \frac{1}{|x-y|^{\alpha(d-2)}}\, dx\right|^{\frac{1}{\alpha}}.$$

The dx-integral only needs to be evaluated for y in a bounded set. For large enough N, the dx-integral is finite if and only if $\alpha(2-d)+d<0$, which is the case since $d>4$ and $\alpha > \frac{d}{d-2}$.

In summary, $\check{\rho}^d_{\mathcal{P}} * \varphi \in L^\alpha(\mathbb{R}^2)$ if and only if $d > 4$ and $\alpha > \frac{d}{d-2}$, proving the first part of statement (b).

For the second part, when $d > 4$ and $\alpha > \frac{d}{d-2}$, u_{gen} cannot have a random field representation, since by Theorem 4.1, this would imply that **(H2')** holds, and this is not the case by (a). □

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