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Asymptotic expansion of Skorohod integrals*

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Abstract

Asymptotic expansion of the distribution of a perturbation Z_n of a Skorohod integral jointly with a reference variable X_n is derived. We introduce a second-order interpolation formula in frequency domain to expand a characteristic functional and combine it with the scheme developed in the martingale expansion. The second-order interpolation and Fourier inversion give asymptotic expansion of the expectation $E[f(Z_n, X_n)]$ for differentiable functions f and also measurable functions f. In the latter case, the interpolation method connects the two non-degeneracies of variables for finite n and ∞. Random symbols are used for expressing the asymptotic expansion formula. Quasi tangent, quasi torsion and modified quasi torsion are introduced in this paper. We identify these random symbols for a certain quadratic form of a fractional Brownian motion and for a quadratic from of a fractional Brownian motion with random weights. For a quadratic form of a Brownian motion with random weights, we observe that our formula reproduces the formula originally obtained by the martingale expansion.

Keywords: asymptotic expansion; Skorohod integral; interpolation; random symbol; quasi tangent; quasi torsion; modified quasi torsion; Malliavain covariance; quadratic form; fractional Brownian motion.
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1 Introduction

Asymptotic expansion of distributions is one of the fundamentals of theoretical statistics. Its applications spread over higher order approximation of probability distributions, theory of higher order asymptotic efficiency of estimators, prediction, information criteria for model selection, saddle point approximation, bootstrap and resampling methods,

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information geometry and so on. Bhattacharya and Rao [1] give an excellent exposition of the probabilistic aspects of asymptotic expansion for independent variables, while we omit a huge amount of literature on statistical applications of asymptotic expansion methods. Asymptotic expansion has a long history even for dependent models. A celebrated paper Götze and Hipp [5] is a compilation of the studies of asymptotic expansion for Markovian or nearly Markovian chains with mixing property. It was followed by Götze and Hipp [6], that applied their result to time series models more explicitly.

For functionals in stochastic analysis, there are two ways: martingale approach and mixing approach. Yoshida [27, 28] gave asymptotic expansions for martingales with the Malliavin calculus on Wiener/Wiener-Poisson space and applied them to an ergodic diffusion, a volatility estimation over a finite time interval in central limit case, and a stochastic regression model with an explanatory process having long memory. Regularity of the distribution is critical to validate an asymptotic expansion. Thus, naturally the Malliavin calculus was used there to ensure a decay rate of characteristic type functionals. In connection, though the regularity problem does not occur there, Mykland [12] is a pioneering work on expansion of moments for a smooth function of a martingale. The mixing approach is more efficient if a sufficiently fast mixing property is available. Kusuoka and Yoshida [10] and Yoshida [29] developed asymptotic expansions for ϵ-Markov processes possessing a mixing property. The Malliavin calculus was used to estimate certain conditional characteristic functionals defined locally in time with the assistance of support theorems. See e.g. Yoshida [32] for an overview and references therein.

In the last three decades, along with the developments in statistics for high frequency data, stable limit theorems have attracted a lot of attention. Estimation of volatility from high frequency data under finite time horizon typically becomes non-ergodic statistics. Then, the asymptotic expansion of functionals of increments of stochastic processes is once again an issue after recent tremendous progresses in limit theorems in this area. Even though big data is available, the problem of microstructure noise motivates the use of asymptotic expansion. For example, Yoshida [31] extended [27] to martingales with mixed Gaussian limit [an updated version is arXiv:1210.3680 (2012)], Podolskij and Yoshida [25] derived a distributional asymptotic expansion of the p-variation of a diffusion process and Podolskij, Veliyev and Yoshida [24] gave an Edgeworth expansion for the pre-averaging estimator for a diffusion process sampled under microstructure noise.

Beyond semimartingales theory, special attention has been focused in recent years on limit theorems for objects in the Malliavin calculus. Nualart and Peccati [20] established the fourth moment theorem and characterized the central limit theorem for a sequence of multiple stochastic integrals of a fixed order. Nualart and Oriz-Latorre [19] extended the result in Nualart and Peccati [20]. Peccati and Tudor [22] presented necessary and sufficient conditions for the central limit theorem for vectors of multiple stochastic integrals and showed that componentwise convergence implies joint convergence. Nourdin, Nualart and Peccati [14] introduced an interpolation technique and proved quantitative stable limit theorems where the limit distribution is a mixture of Gaussian distributions. Power variation, stable convergence and Berry-Esseen type inequality are also in the scope of this trend.

Nourdin and Peccati [16] showed the asymptotic behavior of a weighted power variation processes associated with the so-called iterated Brownian motion. Corcuera, Nualart and Woerner [3] gave a mixture type central limit theorem for the power variation of a stochastic integral with respect to a fractional Brownian motion. Nourdin, Nualart and Tudor [15] derived central and non-central limit theorems for certain weighted power variations of the fractional Brownian motion. Nourdin [13] showed various asymptotic behavior of weighted quadratic and cubic variations of a fractional Brownian motion having a small Hurst index.

By connecting a martingale approach and deforming a nesting condition, Peccati and Taqqu [21] showed stable convergence of multiple Wiener-Itô integrals. Nourdin and Nualart [15] proved a central limit theorem for a sequence of multiple Skorohod integrals and applied it to renormalized weighted Hermite variations of the fractional Brownian motion. Related to stable convergence are Harnett and Nualart [7, 8] on weak convergence of the Stratonovich integral with respect to a class of Gaussian processes.

Based on Stein's method, among many others, Nourdin and Peccati [17] presented a Berry-Esseen bound for multiple Wiener-Itô integrals, and Edan and Víquez [4] obtained central limit theorems with Wiener-Poisson space. Kusuoka and Tudor [11] proposed Stein's method for invariant measures of diffusions. An advantage of Stein's method is that it provides fairly explicit error bounds of approximation. The interpolation method recently introduced by Nourdin, Nualart and Peccati [14] keeps this merit.

After observing these developments, the aim of this paper is to derive asymptotic expansions for Skorohod integrals by means of the Malliavin calculus. It is worth recalling the terminology in the martingale expansion of [31] though our discussion will be apart from the martingale theory. For a sequence of continuous martingales $M^n = \{M^n_t, t \in [0,1]\}$, denote by $C^n = \langle M^n \rangle$ the quadratic variation of M^n. When $C_n := C^n_1 \to^p C_\infty$ as $n \to \infty$, stable convergence of $M_n := M^n_1$ to a mixed normal limit $M_\infty \sim N(0, C_\infty)$ usually takes place even if C_∞ is random. More precisely an evaluation of the gap $C_n - C_\infty$ is necessary to go up to an asymptotic expansion. The variable $\overset{\circ}{C}_n = r_n^{-1}(C_n - C_\infty)$ is called *tangent*, where r_n is a positive number tending to zero as $n \to \infty$. The effect of $\overset{\circ}{C}_n$ appears in the first order asymptotic expansion, and it gives everything in the classical case of constant C_∞. On the other hand, if C_∞ is random, as it is the case of non-ergodic statistics, then the exponential local martingale $e^n_t(\mathsf{z}) = \exp\big(\mathsf{i}\mathsf{z}M^n + 2^{-1}\mathsf{z}^2 C^n_t\big)$ is no longer a local martingale under the transformed measure $\exp\big(-2^{-1}\mathsf{z}^2 C_\infty\big)dP/E[\exp\big(-2^{-1}\mathsf{z}^2 C_\infty\big)]$. This effect remains in the asymptotic expansion, called the *torsion* the exponential martingale suffers from. Two random symbols $\underline{\sigma}$ and $\overline{\sigma}$ are defined for tangent and torsion, respectively, and the asymptotic expansion formula is given in terms of the Gaussian density $\phi(z; 0, C_\infty)$ with variance C_∞ and the adjoint operation of $\underline{\sigma}$ and $\overline{\sigma}$. In this article, we will make an expansion formula for the Skorohod integral $M_n = \delta(u_n)$ of u_n in a similar way through certain random symbols. However, since we do not have any self-evident martingale structure, we introduce new random symbols called *quasi tangent*, *quasi torsion* and *modified quasi torsion* defined only by Malliavin derivatives of functionals.

We will take a Fourier analytic approach. It is because the asymptotic expansion formula we will obtain is a perturbation of a Gaussian density and the actions of random symbols are simply expressed, as it was the case in classical theory. Moreover, if we extend such a result, the limit is possibly related to infinitely divisible distributions even if their mixture appears, and then formulation by random symbols seems natural from an operational point of view. We use an interpolation method in the frequency domain and expand the characteristic function of the interpolation. The second-order interpolation is provided to relate the distribution of M_n with the random symbols that determine the expansion formula.

In this paper, we combine the interpolation method and the scheme originating from martingale expansion. Non-degeneracy of distributions plays an essential role to validate the asymptotic expansion of the expectation $E[f(M_n)]$ for measurable functions f. For that, it is necessary to connect the two non-degeneracies of M_n and M_∞. The interpolation method serves as a homotopy between the two random functions, just as the interpolation along the real time t was used in the martingale expansion [31]. We require only a local non-degeneracy of M_n, not the full non-degeneracy. There is a big difference between them. For the latter, we need large deviation estimates and that plot often fails in practice. In parametric estimation, we quite often meet a situation where the estimator is not defined on the whole Ω but defined locally as a smooth functional. Then localization is inevitable.

Finally, related to this article, we mention a recent work by Tudor and Yoshida [26] on asymptotic expansion of multiple stochastic integrals.

The organization of this paper is as follows. We will work with the variable Z_n defined in Section 2 as a perturbation of a Skorohod integral $M_n = \delta(u_n)$ since such a stochastic expansion appears when statistical estimators are considered. A reference variable X_n is also considered. This formulation is natural because Studentization is common in non-ergodic statistics, and also because the principal part of the normalized estimator is often expressed as the ratio of a Skorohod integral and Fisher information. Section 2 introduces the interpolation method and an expansion of a characteristic type functional along the interpolation, as well as notion of quasi tangent, quasi tosion and modified quasi torsion. Section 3 gives asymptotic expansion of $E[f(Z_n, X_n)]$ for differentiable functions f. We compute the random symbols for a functional of a fractional Brownian motion in Section 4. Since the Skorohod integral generalizes the Itô integral, our formula should reproduce the same formula as that of [31] if applied to the quadratic form of a Brownian motion with random weights. We will see this in Section 5 but the derivation is more complicated than the direct use of the martingale expansion for the double Itô integrals. In Section 6, the random symbols are computed for a quadratic form of a fractional Brownian motion with random weights. Finally, Section 7 validates asymptotic expansions of $E[f(Z_n, X_n)]$ for measurable functions f. As mentioned above, we carry out this task by using two non-degeneracies of the Malliavin covariances, with the help of the interpolation.

2 Second-order interpolation formula in frequency domain

2.1 Perturbation of a Skorohod integral

Given a probability space $(\Omega, \mathcal{F}, P)$, we consider an isonormal Gaussian process $\mathbb{W} = \{\mathbb{W}(h), h \in \mathfrak{H}\}$ on a real separable Hilbert space $\mathfrak{H}$. For any Hilbert space E, any real number $p \geq 1$ and any integer $k \geq 1$, we denote by $\mathbb{D}^{k,p}(\mathsf{E})$ the Sobolev space of E-valued random variables which are k times differentiable in the sense of Malliavin calculus and the derivatives up to order k have finite moments of order p. We denote by D the derivative operator in the framework of Malliavin calculus. Its adjoint, denoted by δ, is called the divergence or the Skorokod integral. We refer to Nualart [18] for a detailed account on Malliavin calculus. We simply write $\mathbb{D}^{s,p}$ for $\mathbb{D}^{s,p}(\mathbb{R})$. Moreover we write $\mathbb{D}^{s,\infty}(\mathsf{E}) = \cap_{p\geq 1}\mathbb{D}^{s,p}(\mathsf{E})$.

For $n \in \mathbb{N}$, suppose that $u_n \in \mathbb{D}^{1,p}(\mathfrak{H} \otimes \mathbb{R}^{\mathsf{d}})$ for some $p \geq 2$, i.e., $u_n = (u_n^i)_{i=1}^{\mathsf{d}}$ with each $u_n^i \in \mathbb{D}^{1,\mathsf{p}}(\mathfrak{H})$, $i = 1, \ldots, \mathsf{d}$. Let $M_n = \delta(u_n)$, where $\delta(u_n) = (\delta(u_n^i))_{i=1}^{\mathsf{d}}$. Consider random vectors W_n ($n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$) and $N_n : \Omega \to \mathbb{R}^{\mathsf{d}}$ ($n \in \mathbb{N}$). For a sequence of positive numbers $(r_n)_{n\in\mathbb{N}}$ tending to 0 as $n \to \infty$, we will consider a perturbation Z_n of M_n given by

$$Z_n = M_n + W_n + r_n N_n.$$

Let $G_\infty : \Omega \to \mathbb{R}^{\mathsf{d}} \otimes_+ \mathbb{R}^{\mathsf{d}}$ be a random matrix, where $\mathbb{R}^{\mathsf{d}} \otimes_+ \mathbb{R}^{\mathsf{d}}$ is the set of $\mathsf{d} \times \mathsf{d}$ nonnegative symmetric matrices. The random matrix G_∞ will be the asymptotic random variance matrix of Z_n.

A reference variable is denoted by $X_n : \Omega \to \mathbb{R}^{\mathsf{d}_1}$, $n \in \overline{\mathbb{N}}$. Such a reference variable appears in various situations. For example, in the context of estimation of a parameter

θ in the diffusion coefficient of a stochastic differential equation, a quasi maximum likelihood estimator $\hat{\theta}_n$ is used. Then the principal part of $\hat{\theta}_n - \theta$ is usually $\Gamma_n^{-1}\Delta_n$ for the score function $\Delta_n = \partial_\theta \ell_n(\theta)$ and the observed information $\Gamma_n = -\partial_\theta^2 \ell_n(\theta)$. In this situation, naturally, $Z_n = \Delta_n$ and $X_n = \Gamma_n$. The principal part of the Studentized variable $\Gamma_n^{1/2}\big(\hat{\theta}_n - \theta\big)$ is also a function of Z_n and X_n. We may unify the writing by choosing Z_n and X_n in this way though it is also possible to replace each principal part by a Skorohod integral of a different integrand. In this paper, we are interested in a higher-order approximation of the joint distribution of (Z_n, X_n).

For a tensor $T = (T_{i_1,\ldots,i_k})_{i_1,\ldots,i_k}$, we write

$$T[u_1,\ldots,u_k] \;=\; T[u_1 \otimes \cdots \otimes u_k] \;=\; \sum_{i_1,\ldots,i_k} T_{i_1,\ldots,i_k} u_1^{i_1}\cdots u_k^{i_k}$$

for $u_1 = (u_1^{i_1})_{i_1}$, ..., $u_k = (u_k^{i_k})_{i_k}$. Brackets [ ] stand for multilinear mappings. We simply denote $u^{\otimes r} = u \otimes \cdots \otimes u$ (r times).

2.2 Interpolation and expansion

We will consider the situation where $X_n \to^p X_\infty$ and $W_n \to^p W_\infty$ as $n \to \infty$. Then it is natural to consider an interpolation between X_n and X_∞, and between W_n and W_∞. Define $W_n(\theta)$ and $X_n(\theta)$ by $W_n(\theta) = \theta W_n + (1-\theta)W_\infty$ and $X_n(\theta) = \theta X_n + (1-\theta)X_\infty$, respectively, for $\theta \in [0,1]$. Moreover, the tangent variables are defined by $\mathring{W}_n = r_n^{-1}(W_n - W_\infty)$ and $\mathring{X}_n = r_n^{-1}(X_n - X_\infty)$. We will construct an interpolation like that of Nourdin, Nualart and Peccati [14] but in the frequency domain. Define $\lambda_n(\theta;\mathsf{z},\mathsf{x})$ by

$$\lambda_n(\theta;\mathsf{z},\mathsf{x}) \;=\; \theta M_n[\mathtt{i}\mathsf{z}] + 2^{-1}(1-\theta^2)G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}] + W_n(\theta)[\mathtt{i}\mathsf{z}] + \theta r_n N_n[\mathtt{i}\mathsf{z}] + X_n(\theta)[\mathtt{i}\mathsf{x}] \tag{2.1}$$

for $\theta \in [0,1]$, $\mathsf{z} \in \mathbb{R}^{\mathsf{d}}$ and $\mathsf{x} \in \mathbb{R}^{\mathsf{d}_1}$. In particular,

$$\lambda_n(0;\mathsf{z},\mathsf{x}) \;=\; 2^{-1}G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}] + W_\infty[\mathtt{i}\mathsf{z}] + X_\infty[\mathtt{i}\mathsf{x}] \;=:\; \lambda_\infty(0;\mathsf{z},\mathsf{x}),$$

$$\lambda_n(1;\mathsf{z},\mathsf{x}) \;=\; Z_n[\mathtt{i}\mathsf{z}] + X_n[\mathtt{i}\mathsf{x}]$$

and

$$\big|e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\big| \;\le\; 1.$$

Let $\check{\mathsf{d}} = \mathsf{d} + \mathsf{d}_1$.

To discuss the joint distribution of (Z_n, X_n), it is natural to consider the characteristic function $E\big[\exp\big(\lambda_n(1;\mathsf{z},\mathsf{x})\big)\big]$ of (Z_n, X_n), though a truncated version will be treated later. Under the conditions specified later, M_n converges in distribution to the mixed normal distribution $N_{\mathsf{d}}(0, G_\infty)$ with the covariance matrix G_∞ and (M_n, W_n, X_n) converges in distribution, as well as $(W_n, X_n) \to^p (W_\infty, X_\infty)$. Thus $E\big[\exp\big(\lambda_n(1;\mathsf{z},\mathsf{x})\big)\big]$ should converge to $E\big[\exp\big(\lambda_\infty(0;\mathsf{z},\mathsf{x})\big)\big]$.

Remark 2.1. More generally, for $\gamma_i \in C^1([0,1];[0,1])$ such that $\gamma_i(0) = 0$ and $\gamma_i(1) = 1$ ($i = 0,1,2,3$), we can consider an interpolation

$$\begin{aligned}\lambda_n(\theta;\mathsf{z},\mathsf{x}) \;=\;& \gamma_0(\theta)M_n[\mathtt{i}\mathsf{z}] + 2^{-1}(1-\gamma_0(\theta)^2)G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}] \\ &+ W_n(\gamma_1(\theta))[\mathtt{i}\mathsf{z}] + \gamma_2(\theta)\, r_n N_n[\mathtt{i}\mathsf{z}] + X_n(\gamma_3(\theta))[\mathtt{i}\mathsf{x}]\end{aligned}$$

for $\theta \in [0,1]$. However, it turns out that the derived formula does not depend on a choice of γ_i. So we will take the identity function for γ_i, i.e., (2.1) as $\lambda_n(\theta;\mathsf{z},\mathsf{x})$.

Remark 2.2. We could start with the decomposition

$$Z_n \quad = \quad M_n + W_\infty + r_n \tilde{N}_n$$

of Z_n, by taking $\tilde{N}_n = \overset{\circ}{W}_n + N_n$. This decomposition would be expected to slightly simplify the presentation but the complexity would be the same because, as a matter of fact, $\overset{\circ}{W}_n$ and N_n will always be treated as a set like in $\check{G}_n^{(1)}$ and $\hat{G}_n^{(1)}$ defined below.

Consider a sequence $\psi_n \in \mathbb{D}^{1,p_1}(\mathbb{R})$, and for a while we suppose that $u_n \in \mathbb{D}^{2,p}(\mathfrak{H} \otimes \mathbb{R}^{\mathsf{d}})$, $G_\infty \in \mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}} \otimes_+ \mathbb{R}^{\mathsf{d}})$, $W_n, W_\infty, N_n \in \mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}})$ and $X_n, X_\infty \in \mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}_1})$ with $2p^{-1} + p_1^{-1} \leq 1$. We abuse the notation $\mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}} \otimes_+ \mathbb{R}^{\mathsf{d}})$ though $\mathbb{R}^{\mathsf{d}} \otimes_+ \mathbb{R}^{\mathsf{d}}$ is not a Hilbert space. In the special case $\psi_n \equiv 1$, we let $p_1 = \infty$. The functional ψ_n has two roles. It will serve as a generic variable when we express the θ-derivative of φ_n by φ_n itself. Moreover, it will denote a truncation functional that ensures local non-degeneracy of (Z_n, X_n) in Malliavin's sense.

We write

$$\varphi_n(\theta; \psi_n) = \varphi_n(\theta, \mathsf{z}, \mathsf{x}; \psi_n) \quad = \quad E\big[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\psi_n\big].$$

The random matrix G_n ($\mathsf{d} \times \mathsf{d}$) is defined by

$$G_n[\mathtt{iz}_1, \mathtt{iz}_2] \quad = \quad \langle DM_n[\mathtt{iz}_1], u_n[\mathtt{iz}_2]\rangle_{\mathfrak{H}} \quad (\mathsf{z}_1, \mathsf{z}_2 \in \mathbb{R}^{\mathsf{d}}).$$

Remark 2.3. The variable G_n is different from $C_n = \langle M^n \rangle_T$ in the martingale expansion since, in general,

$$\langle DM_n, u_n\rangle_{\mathfrak{H}} \quad \neq \quad \langle u_n, u_n\rangle_{\mathfrak{H}}.$$

That is, $\overset{\circ}{G}_n = r_n^{-1}(G_n - G_\infty)$ is not necessarily the tangent variable $\overset{\circ}{C}_n$, and the sequences $\overset{\circ}{G}_n$ and $\overset{\circ}{C}_n$ have different limits, in general. However, the limit G_∞ of G_n may coincide with the limit C_∞ of C_n. In short, we may have $G_\infty = C_\infty$, however, in general, $\overset{\circ}{G}_\infty \neq \overset{\circ}{C}_\infty$. In particular, $\overset{\circ}{G}_n$ may converge to 0.

The random tensor

$$\mathtt{qTan}[\mathtt{iz}_1, \mathtt{iz}_2] \quad = \quad r_n^{-1}\Big(\big\langle DM_n[\mathtt{iz}_1], u_n[\mathtt{iz}_2]\big\rangle_{\mathfrak{H}} - G_\infty[\mathtt{iz}_1, \mathtt{iz}_2]\Big)$$

for $(\mathtt{iz}_1, \mathtt{iz}_2) \in (\mathtt{i}\mathbb{R}^{\mathsf{d}})^2$ is called the **quasi tangent** (q-tangent), and the random tensor

$$\mathtt{qTor}[\mathtt{iz}_1, \mathtt{iz}_2, \mathtt{iz}_3] \quad = \quad r_n^{-1}\Big\langle D\big\langle DM_n[\mathtt{iz}_1], u_n[\mathtt{iz}_2]\big\rangle_{\mathfrak{H}}, u_n[\mathtt{iz}_3]\Big\rangle_{\mathfrak{H}}$$

for $(\mathtt{iz}_1, \mathtt{iz}_2, \mathtt{iz}_3) \in (\mathtt{i}\mathbb{R}^{\mathsf{d}})^3$ when $u_n \in \mathbb{D}^{3,p}(\mathfrak{H} \otimes \mathbb{R}^{\mathsf{d}})$, is called the **quasi torsion** (q-torsion). Moreover, we call the random tensor

$$\mathtt{mqTor}[\mathtt{iz}_1, \mathtt{iz}_2, \mathtt{iz}_3] \quad = \quad r_n^{-1}\big\langle DG_\infty[\mathtt{iz}_1, \mathtt{iz}_2], u_n[\mathtt{iz}_3]\big\rangle_{\mathfrak{H}}$$

for $(\mathtt{iz}_1, \mathtt{iz}_2, \mathtt{iz}_3) \in (\mathtt{i}\mathbb{R}^{\mathsf{d}})^3$ the **modified quasi torsion** (modified q-torsion). Then

$$\big\langle D\,\mathtt{qTan}[(\mathtt{iz})^{\otimes 2}], \mathtt{u_n}[\mathtt{iz}]\big\rangle_{\mathfrak{H}} \quad = \quad \mathtt{qTor}[(\mathtt{iz})^{\otimes 3}] - \mathtt{mqTor}[(\mathtt{iz})^{\otimes 3}].$$

Let

$$\Psi(\mathsf{z}, \mathsf{x}) \quad = \quad \exp\Big(2^{-1}G_\infty[(\mathtt{iz})^{\otimes 2}] + W_\infty[\mathtt{iz}] + X_\infty[\mathtt{ix}]\Big) \;\equiv\; e^{\lambda_\infty(0;\mathsf{z},\mathsf{x})}. \tag{2.2}$$

Then

$$\begin{aligned} E\big[\exp\big(Z_n[\mathtt{iz}]+X_n[\mathtt{ix}]\big)\psi_n\big]-E[\Psi(\mathsf{z},\mathsf{x})\psi_n] &= \varphi_n(1;\psi_n)-\varphi_n(0;\psi_n) \\ &= \int_0^1 \partial_\theta\varphi_n(\theta;\psi_n)d\theta. \end{aligned}$$

The derivative of $\varphi_n(\theta;\psi_n)$ is computed as follows

$$\begin{aligned} \partial_\theta\varphi_n(\theta;\psi_n) &= E\bigg[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\bigg\{\delta(u_n[\mathtt{iz}])-\theta G_\infty[(\mathtt{iz})^{\otimes 2}] \\ &\qquad +r_n\,\mathring{W}_n\,[\mathtt{iz}]+r_nN_n[\mathtt{iz}]+r_n\,\mathring{X}_n\,[\mathtt{ix}]\bigg\}\psi_n\bigg] \\ &= E\bigg[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\bigg\{\delta(u_n[\mathtt{iz}])-\theta G_\infty[(\mathtt{iz})^{\otimes 2}]+\check{G}^{(1)}_n(\mathsf{z},\mathsf{x})\bigg\}\psi_n\bigg], \end{aligned}$$

where

$$\check{G}^{(1)}_n(\mathsf{z},\mathsf{x}) = r_n\,\mathring{W}_n\,[\mathtt{iz}]+r_nN_n[\mathtt{iz}]+r_n\,\mathring{X}_n\,[\mathtt{ix}].$$

Applying the duality relationship between the Skorohod integral δ and the derivative operator D (we also call this duality relationship integration by parts (IBP) formula), yields

$$\begin{aligned} \partial_\theta\varphi_n(\theta;\psi_n) &= E\bigg[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\bigg\{\theta\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}+2^{-1}(1-\theta^2)\langle DG_\infty[(\mathtt{iz})^{\otimes 2}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} \\ &\qquad +\langle DW_n(\theta)[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}+\theta r_n\langle DN_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} \\ &\qquad +\langle DX_n(\theta)[\mathtt{ix}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\bigg\}\psi_n\bigg] \\ &\quad +E\big[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\langle D\psi_n,u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big] \\ &\quad +E\bigg[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\left\{-\theta G_\infty[(\mathtt{iz})^{\otimes 2}]+\check{G}^{(1)}_n(\mathsf{z},\mathsf{x})\right\}\psi_n\bigg]. \end{aligned}$$

This expression can be written as

$$\begin{aligned} \partial_\theta\varphi_n(\theta;\psi_n) &= E\big[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\langle D\psi_n,u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big] \\ &\quad +\theta E\bigg[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\bigg(\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}-G_\infty[(\mathtt{iz})^{\otimes 2}]\bigg)\psi_n\bigg] \\ &\quad +2^{-1}(1-\theta^2)E\bigg[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}\langle DG_\infty[(\mathtt{iz})^{\otimes 2}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\psi_n\bigg] \\ &\quad +E\big[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}G^{(1)}_n(\theta;\mathsf{z},\mathsf{x})\psi_n\big] \\ &= \varphi_n\big(\theta;\langle D\psi_n,u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big) \\ &\quad +\theta\varphi_n\Big(\theta;\big(\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}-G_\infty[(\mathtt{iz})^{\otimes 2}]\big)\psi_n\Big) \\ &\quad +2^{-1}(1-\theta^2)\varphi_n\Big(\theta;\langle DG_\infty[(\mathtt{iz})^{\otimes 2}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\psi_n\Big) \\ &\quad +\varphi_n\big(\theta;G^{(1)}_n(\theta;\mathsf{z},\mathsf{x})\psi_n\big), \end{aligned}$$

where

$$G^{(1)}_n(\theta;\mathsf{z},\mathsf{x}) = \hat{G}^{(1)}_n(\theta;\mathsf{z},\mathsf{x})+\check{G}^{(1)}_n(\mathsf{z},\mathsf{x}) \tag{2.3}$$

with

$$\hat{G}_n^{(1)}(\theta;\mathsf{z},\mathsf{x}) = \langle DW_\infty[\mathtt{iz}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} + \langle DX_\infty[\mathtt{ix}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} + \theta r_n\langle D\overset{\circ}{W}_n[\mathtt{iz}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} + \theta r_n\langle DN_n[\mathtt{iz}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} + \theta r_n\langle D\overset{\circ}{X}_n[\mathtt{ix}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}.$$

Let

$$G_n^{(2)}(\mathsf{z}) = \langle DM_n[\mathtt{iz}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} - G_\infty[(\mathtt{iz})^{\otimes 2}] = r_n\,\mathtt{qTan}[(\mathtt{iz})^{\otimes 2}] \tag{2.4}$$

and

$$G_n^{(3)}(\mathsf{z}) = \langle DG_\infty[(\mathtt{iz})^{\otimes 2}], u_n[\mathtt{iz}]\rangle_{\mathfrak{H}} = r_n\,\mathtt{mqTor}[(\mathtt{iz})^{\otimes 3}]. \tag{2.5}$$

Thus, we obtained the following lemma.

Lemma 2.4. Suppose that $u_n \in \mathbb{D}^{2,p}(\mathfrak{H}\otimes\mathbb{R}^{\mathsf{d}})$, $G_\infty \in \mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}}\otimes_+\mathbb{R}^{\mathsf{d}})$, $W_n, W_\infty, N_n \in \mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}})$, $X_n, X_\infty \in \mathbb{D}^{1,p}(\mathbb{R}^{\mathsf{d}_1})$ and $\psi_n \in \mathbb{D}^{1,p_1}(\mathbb{R})$, $2p^{-1}+p_1^{-1}\le 1$. Then

$$\partial_\theta\varphi_n(\theta;\psi_n) = \varphi_n\big(\theta;\langle D\psi_n, u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big) + \theta\varphi_n\big(\theta; G_n^{(2)}(\mathsf{z})\psi_n\big) + 2^{-1}(1-\theta^2)\varphi_n\big(\theta; G_n^{(3)}(\mathsf{z})\psi_n\big) + \varphi_n\big(\theta; G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n\big). \tag{2.6}$$

In order to establish a second-order interpolation formula we need to further expand the last three summands in the right-hand side of (2.6). To do this, we will denote by $\mathcal{G}_n$ any one of the terms $G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$, $G_n^{(2)}(\mathsf{z})$ and $G^{(3)}(\mathsf{z})$. Suppose, in addition, that $G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n$, $G_n^{(2)}(\mathsf{z})\psi_n$, $G^{(3)}(\mathsf{z})\psi_n$, $\langle D(G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n), u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}$, $\langle D(G_n^{(2)}(\theta;\mathsf{z})\psi_n), u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}$ and $\langle D(G_n^{(3)}(\theta;\mathsf{z})\psi_n), u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}$ are in $\mathbb{D}^{1,p_2}(\mathbb{R})$ with $p_2=(3p^{-1}+p_1^{-1})^{-1}$ with p such that $5p^{-1}+p_1^{-1}\le 1$. Then, by Lemma 2.4, we have

$$\begin{aligned}
&\varphi_n\big(\theta;\mathcal{G}_n\psi_n\big) - \varphi_n\big(0;\mathcal{G}_n\psi_n\big)\\
&= \int_0^\theta \partial_{\theta_1}\varphi_n\big(\theta_1;\mathcal{G}_n\psi_n\big)d\theta_1\\
&= \int_0^\theta \Big\{\varphi_n\big(\theta_1;\langle D(\mathcal{G}_n\psi_n), u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big) + \theta_1\varphi_n\big(\theta_1; G_n^{(2)}(\mathsf{z})\mathcal{G}_n\psi_n\big)\\
&\quad + 2^{-1}(1-\theta_1^2)\varphi_n\big(\theta_1; G_n^{(3)}(\mathsf{z})\mathcal{G}_n\psi_n\big) + \varphi_n\big(\theta_1; G_n^{(1)}(\theta_1;\mathsf{z},\mathsf{x})\mathcal{G}_n\psi_n\big)\Big\}d\theta_1\\
&= \int_0^\theta \varphi_n\big(\theta_1;\langle D(\mathcal{G}_n\psi_n), u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)d\theta_1 + R_n^{(1)}(\theta;\mathsf{z},\mathsf{x},\mathcal{G}_n),
\end{aligned}$$

where

$$R_n^{(1)}(\theta;\mathsf{z},\mathsf{x},\mathcal{G}_n) = \int_0^\theta \Big\{\theta_1\varphi_n\big(\theta_1; G_n^{(2)}(\mathsf{z})\mathcal{G}_n\psi_n\big) + 2^{-1}(1-\theta_1^2)\varphi_n\big(\theta_1; G_n^{(3)}(\mathsf{z})\mathcal{G}_n\psi_n\big) + \varphi_n\big(\theta_1; G_n^{(1)}(\theta_1;\mathsf{z},\mathsf{x})\mathcal{G}_n\psi_n\big)\Big\}d\theta_1.$$

Therefore, once again by Lemma 2.4, we obtain

$$\varphi_n\big(\theta;\mathcal{G}_n\psi_n\big) - \varphi_n\big(0;\mathcal{G}_n\psi_n\big) = \theta\varphi_n\big(0;\langle D(\mathcal{G}_n\psi_n), u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big) + R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},\mathcal{G}_n), \tag{2.7}$$

where

$$
\begin{aligned}
R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},\mathcal{G}_n) &= \int_0^\theta\int_0^{\theta_1}\partial_{\theta_2}\varphi_n\big(\theta_2;\langle D(\mathcal{G}_n\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)d\theta_2d\theta_1+R_n^{(1)}(\theta;\mathsf{z},\mathsf{x},\mathcal{G}_n)\\
&= \int_0^\theta\int_0^{\theta_1}\Big\{\varphi_n\big(\theta_2;\langle D\big(\langle D(\mathcal{G}_n\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\quad+\theta_2\varphi_n\big(\theta_2;G_n^{(2)}(\mathsf{z})\langle D(\mathcal{G}_n\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\quad+2^{-1}(1-\theta_2^2)\varphi_n\big(\theta_2;G_n^{(3)}(\mathsf{z})\langle D(\mathcal{G}_n\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\quad+\varphi_n\big(\theta_2;G_n^{(1)}(\theta_2;\mathsf{z},\mathsf{x})\langle D(\mathcal{G}_n\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\Big\}d\theta_2d\theta_1+R_n^{(1)}(\theta;\mathsf{z},\mathsf{x},\mathcal{G}_n).
\end{aligned}
$$

By (2.6) and (2.7), we can write

$$
\begin{aligned}
&\partial_\theta\varphi_n(\theta;\psi_n)\\
=\ &\varphi_n\big(\theta;\langle D\psi_n,u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&+\theta\Big\{\varphi_n\big(0;G_n^{(2)}(\mathsf{z})\psi_n\big)+\theta\varphi_n\big(0;\langle D(G_n^{(2)}(\mathsf{z})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)+R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},G_n^{(2)}(\mathsf{z}))\Big\}\\
&+2^{-1}(1-\theta^2)\Big\{\varphi_n\big(0;G_n^{(3)}(\mathsf{z})\psi_n\big)+\theta\varphi_n\big(0;\langle D(G_n^{(3)}(\mathsf{z})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\qquad+R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},G_n^{(3)}(\mathsf{z}))\Big\}\\
&+\Big\{\varphi_n\big(0;G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n\big)+\theta\varphi_n\big(0;\langle D(G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\quad+R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},G_n^{(1)}(\theta;\mathsf{z},\mathsf{x}))\Big\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
R_n^{(3)}(\mathsf{z},\mathsf{x}) &= \int_0^1\Big\{\varphi_n\big(\theta;\big(1+U_n(\theta;\mathsf{z},\mathsf{x})\theta^2G_n^{(2)}(\mathsf{z})\big)\langle D\psi_n,u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)+\theta R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},G_n^{(2)}(\mathsf{z}))\\
&\quad+2^{-1}(1-\theta^2)R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},G_n^{(3)}(\mathsf{z}))+R_n^{(2)}(\theta;\mathsf{z},\mathsf{x},G_n^{(1)}(\theta;\mathsf{z},\mathsf{x}))\\
&\quad+2^{-1}(1-\theta^2)\theta\varphi_n\big(0;\langle D(G_n^{(3)}(\mathsf{z})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\quad+\theta\varphi_n\big(0;\langle D(\hat{G}_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\Big\}d\theta,
\end{aligned}
$$

where $U_n(\theta;\mathsf{z},\mathsf{x})=\exp\{\lambda_n(0;\mathsf{z},\mathsf{x})-\lambda_n(\theta;\mathsf{z},\mathsf{x})\}$. By definition, $|U_n(\theta;\mathsf{z},\mathsf{x})|\le 1$. Then, integrating $\partial_\theta\varphi_n(\theta;\psi_n)$ and using the expression for $R_n^{(3)}(\mathsf{z},\mathsf{x})$ and the decomposition (2.3), yields

$$
\begin{aligned}
\int_0^1\partial_\theta\varphi_n(\theta;\psi_n)d\theta &= \frac{1}{2}\varphi_n\big(0;G_n^{(2)}(\mathsf{z})\psi_n\big)\\
&\quad+\frac{1}{3}\varphi_n\big(0;\langle DG_n^{(2)}(\mathsf{z}),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\psi_n\big)+\frac{1}{3}\varphi_n\big(0;G_n^{(3)}(\mathsf{z})\psi_n\big)\\
&\quad+\int_0^1\varphi_n\big(0;G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n\big)d\theta+R_n^{(3)}(\mathsf{z},\mathsf{x})\\
&\quad+\int_0^1\theta\varphi_n\big(0;\langle D(\check{G}_n^{(1)}(\mathsf{z},\mathsf{x})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)d\theta\\
&= \frac{1}{2}\varphi_n\big(0;G_n^{(2)}(\mathsf{z})\psi_n\big)+\frac{1}{3}\varphi_n\big(0;\big\langle D\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n\big)\\
&\quad+\int_0^1\varphi_n\big(0;G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})\psi_n\big)d\theta+\frac{1}{2}\varphi_n\big(0;\langle D(\check{G}_n^{(1)}(\mathsf{z},\mathsf{x})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\
&\quad+R_n^{(3)}(\mathsf{z},\mathsf{x}).
\end{aligned}
$$

Consequently,

$$\begin{aligned}\varphi_n(1;\psi_n)-\varphi_n(0;\psi_n) &= \frac{1}{2}\varphi_n\big(0;\big(DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}-G_\infty[(\mathtt{iz})^{\otimes 2}]\big)\psi_n\big)\\ &\quad+\frac{1}{3}\varphi_n\big(0;\big\langle D\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n\big)\\ &\quad+\varphi_n\bigg(0;\int_0^1 G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})d\theta\psi_n\bigg)\\ &\quad+\frac{1}{2}\varphi_n\big(0;\langle D(\check{G}_n^{(1)}(\mathsf{z},\mathsf{x})\psi_n),u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big)\\ &\quad+R_n^{(3)}(\mathsf{z},\mathsf{x}).\end{aligned}$$

Using the definition of $\Psi(\mathsf{z},\mathsf{x})$ given in (2.2) and the decomposition of $G_n^{(1)}$ given in (2.3), we obtain

$$\begin{aligned}\varphi_n(1;\psi_n)-\varphi_n(0;\psi_n) &= \frac{1}{3}E\Big[\Psi(\mathsf{z},\mathsf{x})\Big\langle D\big\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}}\psi_n\Big]\\ &\quad+\frac{1}{2}E\Big[\Psi(\mathsf{z},\mathsf{x})\Big(\big\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}-G_\infty[(\mathtt{iz})^{\otimes 2}]\Big)\psi_n\Big]\\ &\quad+E\Big[\Psi(\mathsf{z},\mathsf{x})\Big\{\big\langle DW_\infty[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}+\big\langle DX_\infty[\mathtt{ix}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\\ &\qquad\qquad+r_n\,\mathring{W}_n\,[\mathtt{iz}]+r_nN_n[\mathtt{iz}]+r_n\,\mathring{X}_n\,[\mathtt{ix}]\\ &\qquad\qquad+r_n\langle D\,\mathring{W}_n\,[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}+r_n\langle DN_n[\mathtt{iz}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\\ &\qquad\qquad+r_n\langle D\,\mathring{X}_n\,[\mathtt{ix}],u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\Big\}\psi_n\Big]\\ &\quad+R_n^{(3)}(\mathsf{z},\mathsf{x})+R_n^{(4)}(\mathsf{z},\mathsf{x}),\end{aligned}$$

where

$$R_n^{(4)}(\mathsf{z},\mathsf{x}) = \frac{1}{2}\varphi_n\big(0;\check{G}_n^{(1)}(\mathsf{z},\mathsf{x})\langle D\psi_n,u_n[\mathtt{iz}]\rangle_{\mathfrak{H}}\big).$$

Suppose that there are random symbols $\mathfrak{S}^{(3,0)}$, $\mathfrak{S}_0^{(2,0)}$, $\mathfrak{S}^{(2,0)}$, $\mathfrak{S}^{(1,1)}$, $\mathfrak{S}^{(1,0)}$, $\mathfrak{S}^{(0,1)}$, $\mathfrak{S}_1^{(2,0)}$ and $\mathfrak{S}_1^{(1,1)}$, i.e., they are polynomials in $(\mathtt{iz},\mathtt{ix})$ with coefficients in $L^1(\Omega)$. Let

$$R_n^{(5)}(\mathsf{z},\mathsf{x})=r_nE\Big[\Psi(\mathsf{z},\mathsf{x})\,3^{-1}\mathsf{qTor}[(\mathtt{iz})^{\otimes 3}]\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix})\big],$$

$$R_n^{(6)}(\mathsf{z},\mathsf{x})=r_nE\Big[\Psi(\mathsf{z},\mathsf{x})\,2^{-1}\mathsf{qTan}[(\mathtt{iz})^{\otimes 2}]\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}_0^{(2,0)}(\mathtt{iz},\mathtt{ix})\big],$$

$$R_n^{(7)}(\mathsf{z},\mathsf{x})=E\Big[\Psi(\mathsf{z},\mathsf{x})\big\langle DW_\infty[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(2,0)}(\mathtt{iz},\mathtt{ix})\big],$$

$$R_n^{(8)}(\mathsf{z},\mathsf{x})=E\Big[\Psi(\mathsf{z},\mathsf{x})\big\langle DX_\infty[\mathtt{ix}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(1,1)}(\mathtt{iz},\mathtt{ix})\big],$$

$$R_n^{(9)}(\mathsf{z},\mathsf{x})=r_nE\Big[\Psi(\mathsf{z},\mathsf{x})\Big\{\,\mathring{W}_n\,[\mathtt{iz}]+N_n[\mathtt{iz}]\Big\}\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(1,0)}(\mathtt{iz},\mathtt{ix})\big],$$

$$R_n^{(10)}(\mathsf{z},\mathsf{x})=r_nE\Big[\Psi(\mathsf{z},\mathsf{x})\,\mathring{X}_n\,[\mathtt{ix}]\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(0,1)}(\mathtt{iz},\mathtt{ix})\big],$$

$$\begin{aligned}R_n^{(11)}(\mathsf{z},\mathsf{x})&=r_nE\Big[\Psi(\mathsf{z},\mathsf{x})\big\langle\{D\,\mathring{W}_n\,[\mathtt{iz}]+DN_n[\mathtt{iz}]\},u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n\Big]\\ &\quad-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}_1^{(2,0)}(\mathtt{iz},\mathtt{ix})\big],\end{aligned}$$

$$R_n^{(12)}(\mathsf{z},\mathsf{x})=r_nE\Big[\Psi(\mathsf{z},\mathsf{x})\big\langle D\,\mathring{X}_n\,[\mathtt{ix}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n\Big]-r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}_1^{(1,1)}(\mathtt{iz},\mathtt{ix})\big],$$

and

$$R_n(\mathsf{z},\mathsf{x}) \quad = \quad \sum_{i=3}^{12} R_n^{(i)}(\mathsf{z},\mathsf{x}).$$

Remark 2.5. (i) We expect

$$E\big[\Psi(\mathsf{z},\mathsf{x})\, 3^{-1}\mathsf{qTor}[(\mathtt{iz})^{\otimes 3}]\psi_n\big] - E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix})\big] \to 0.$$

However this does not mean that

$$E\big[\big|3^{-1}\mathsf{qTor}[(\mathtt{iz})^{\otimes 3}]\psi_n - \mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix})\big|\big] \to 0.$$

In fact, $\mathfrak{S}^{(3,0)}$ is not necessarily of third order in z, as we will see in this paper. (ii) In the multi-dimensional case, we can go with qTan defined as the symmetric version of the present qTan. The results will be essentially the same though we will not describe them.

Define the random symbol $\mathfrak{S}(\mathtt{iz},\mathtt{ix})$ by

$$\begin{aligned}\mathfrak{S}(\mathtt{iz},\mathtt{ix}) \quad = \quad & \mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix}) + \mathfrak{S}_0^{(2,0)}(\mathtt{iz},\mathtt{ix}) + \mathfrak{S}^{(2,0)}(\mathtt{iz},\mathtt{ix}) \\ & +\mathfrak{S}^{(1,1)}(\mathtt{iz},\mathtt{ix}) + \mathfrak{S}^{(1,0)}(\mathtt{iz},\mathtt{ix}) + \mathfrak{S}^{(0,1)}(\mathtt{iz},\mathtt{ix}) \\ & +\mathfrak{S}_1^{(2,0)}(\mathtt{iz},\mathtt{ix}) + \mathfrak{S}_1^{(1,1)}(\mathtt{iz},\mathtt{ix}).\end{aligned}$$

In applications of the asymptotic expansion, we are expecting that the residual terms $R_n^{(5)}(\mathsf{z},\mathsf{x})$ and $R_n^{(6)}(\mathsf{z},\mathsf{x})$ are of order $o(r_n)$ together with other residual terms. Then the quasi torsion is approximated by $\mathfrak{S}^{(3,0)}$ in the sense explained in Remark 2.5. Similarly the quasi tangent is approximated by $\mathfrak{S}_0^{(2,0)}$ in this sense. The random symbol $\mathfrak{S}^{(3,0)}$ is often identified by using the integration-by-parts formula. The modified quasi torsion is not involved in the random symbol $\mathfrak{S}$ though related with the quasi torsion and the quasi tangent.

We will assume the following hypothesis:

[A] $u_n \in \mathbb{D}^{4,p}(\mathfrak{H}\otimes\mathbb{R}^{\mathsf{d}})$, $G_\infty \in \mathbb{D}^{3,p}(\mathbb{R}^{\mathsf{d}}\otimes_+\mathbb{R}^{\mathsf{d}})$, $W_n, W_\infty, N_n \in \mathbb{D}^{3,p}(\mathbb{R}^{\mathsf{d}})$, $X_n, X_\infty \in \mathbb{D}^{3,p}(\mathbb{R}^{\mathsf{d}_1})$ and $\psi_n \in \mathbb{D}^{2,p_1}(\mathbb{R})$ for some p and p_1 satisfying $5p^{-1}+p_1^{-1}\le 1$.

From the above argument, we obtain the following second-order interpolation formula.

Proposition 2.6. Under Condition $[A]$,

$$\varphi_n(1;\psi_n) \quad = \quad \varphi_n(0;\psi_n) + r_n E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}(\mathtt{iz},\mathtt{ix})\big] + R_n(\mathsf{z},\mathsf{x}). \tag{2.8}$$

3 Asymptotic expansion for differentiable functions

3.1 Expansion formula for $E[f(Z_n,X_n)]$

Let $f\in\mathcal{S}(\mathbb{R}^{\check{\mathsf{d}}})$, the set of rapidly decreasing smooth functions. Let $\varsigma(\mathtt{iz},\mathtt{ix}) = \sum_{k,m} c_{k,m}[(\mathtt{iz})^{\otimes k}\otimes(\mathtt{ix})^{\otimes m}]$ (finite sum) be a random symbol with L^1 coefficients $c_{k,m}$. The factor ψ_n, if exists, can be included in ς. Let $\zeta\sim N_{\mathsf{d}}(0,I_{\mathsf{d}})$ be a random vector independent of $\mathcal{F}$, defined on an extended probability space, if necesssary. We denote by $\hat{f}$ the Fourier transform of f:

$$\hat{f}(\mathsf{z},\mathsf{x}) \quad = \quad \int_{\mathbb{R}^{\check{\mathsf{d}}}} f(z,x)e^{-\mathtt{i}z\cdot\mathsf{z}-\mathtt{i}x\cdot\mathsf{x}}dzdx.$$

Then, we can write

$$\begin{aligned}&(2\pi)^{-\check{\mathsf{d}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}} \hat{f}(\mathsf{z},\mathsf{x})\varphi_n(\theta;\varsigma(\mathtt{iz},\mathtt{ix}))d\mathsf{z}d\mathsf{x} \\ = \quad & E\left[\varsigma(\partial_z,\partial_x)f\left(\theta M_n+\sqrt{1-\theta^2}G_\infty^{1/2}\zeta+W_n(\theta)+\theta r_nN_n, X_n(\theta)\right)\right].\end{aligned} \tag{3.1}$$

In particular, for $\theta=0$, we obtain

$$\begin{aligned}\rho_n^{(8)}(f) &:= (2\pi)^{-\check{\mathsf{d}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}}\hat{f}(\mathsf{z},\mathsf{x})R_n^{(8)}(\mathsf{z},\mathsf{x})d\mathsf{z}d\mathsf{x}\\ &= E\big[\psi_n\big\langle DX_\infty[\partial_x],u_n[\partial_z]\big\rangle_{\mathfrak{H}}f\big(G_\infty^{1/2}\zeta+W_\infty,X_\infty\big)\big]\\ &\quad -r_nE\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)f\big(G_\infty^{1/2}\zeta+W_\infty,X_\infty\big)\big] \end{aligned}\tag{3.2}$$

and similar formulas for

$$\rho_n^{(i)}(f) := (2\pi)^{-\check{\mathsf{d}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}}\hat{f}(\mathsf{z},\mathsf{x})R_n^{(i)}(\mathsf{z},\mathsf{x})d\mathsf{z}d\mathsf{x}\tag{3.3}$$

for $i=5,6,7,9$, 10, 11 and 12. For $i=3,4$, we define $\rho_n^{(i)}$ by (3.3). Then

$$\sum_{i=3}^{12}\rho_n^{(i)}(f) = (2\pi)^{-\check{\mathsf{d}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}}\hat{f}(\mathsf{z},\mathsf{x})R_n(\mathsf{z},\mathsf{x})d\mathsf{z}d\mathsf{x}.\tag{3.4}$$

Let

$$\rho_n^{(2)}(f) = E[f(Z_n,X_n)(1-\psi_n)]-E[f(G_\infty^{1/2}\zeta+W_\infty,X_\infty)(1-\psi_n)]\tag{3.5}$$

and let

$$\rho_n^{(1)}(f) = \sum_{i=2}^{12}\rho_n^{(i)}(f).\tag{3.6}$$

Applying the second-order interpolation formula (2.8), we can write

$$\begin{aligned}E[f(Z_n,X_n)] &= E[f(Z_n,X_n)(1-\psi_n)]+E[f(Z_n,X_n)\psi_n]\\ &= E[f(Z_n,X_n)(1-\psi_n)]+(2\pi)^{-\check{\mathsf{d}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}}\hat{f}(\mathsf{z},\mathsf{x})\varphi_n(1;\psi_n)d\mathsf{z}d\mathsf{x}\\ &= E[f(Z_n,X_n)(1-\psi_n)]\\ &\quad +(2\pi)^{-\check{\mathsf{d}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}}\hat{f}(\mathsf{z},\mathsf{x})\Big[\varphi_n(0;\psi_n)+r_nE\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big]+R_n(\mathsf{z},\mathsf{x})\Big]d\mathsf{z}d\mathsf{x}.\end{aligned}$$

Then, using (3.1) and (3.4) leads to

$$\begin{aligned}E[f(Z_n,X_n)] &= E[f(Z_n,X_n)(1-\psi_n)]+E[f\big(G_\infty^{1/2}\zeta+W_\infty,X_\infty\big)\psi_n]\\ &\quad +r_nE[\mathfrak{S}(\partial_z,\partial_x)f\big(G_\infty^{1/2}\zeta+W_\infty,X_\infty\big)]+\sum_{i=3}^{12}\rho_n^{(i)}(f).\end{aligned}$$

Finally, taking into account the definitions of $\rho_n^{(2)}(f)$ and $\rho_n^{(1)}(f)$ given in (3.5) and (3.6), respectively, we get

$$\begin{aligned}E[f(Z_n,X_n)] &= E[f\big(G_\infty^{1/2}\zeta+W_\infty,X_\infty\big)]\\ &\quad +r_nE[\mathfrak{S}(\partial_z,\partial_x)f\big(G_\infty^{1/2}\zeta+W_\infty,X_\infty\big)]+\rho_n^{(1)}(f)\end{aligned}\tag{3.7}$$

for $f\in\mathcal{S}(\mathbb{R}^{\check{\mathsf{d}}})$.

Let $\zeta_n(\theta)=\big(\theta M_n+\sqrt{1-\theta^2}G_\infty^{1/2}\zeta+W_n(\theta)+\theta r_nN_n,X_n(\theta)\big)$. In particular, $\zeta_n(0)=(G_\infty^{1/2}\zeta+W_\infty,X_\infty)=:\zeta_\infty(0)$. Let $\eth_z=\mathtt{i}^{-1}\partial_z$ and $\eth_x=\mathtt{i}^{-1}\partial_x$. Among the summands in (3.6), $\rho_n^{(2)}(f)$ is given by (3.5). A precise expression of $\rho_n^{(3)}(f)$ is as follows

$$\begin{aligned}\rho_n^{(3)}(f) = \int_0^1\Big\{&E\big[\langle D\psi_n,u_n[\partial_z]\rangle_{\mathfrak{H}}\big(f(\zeta_n(\theta))+\theta^2G_n^{(2)}(\eth_z)f(\zeta_\infty(0))\big)\big]\\ &+\theta\rho_n^{(3,1)}(f;\theta)+2^{-1}(1-\theta^2)\rho_n^{(3,2)}(f;\theta)\\ &+\rho_n^{(3,3)}(f;\theta)+2^{-1}(1-\theta^2)\theta E\big[\langle D(G_n^{(3)}(\eth_z)\psi_n),u_n[\partial_z]\rangle_{\mathfrak{H}}f(\zeta_\infty(0))\big]\\ &+\theta E\big[\langle D(\hat{G}_n^{(1)}(\theta;\eth_z,\eth_x)\psi_n),u_n[\partial_z]\rangle_{\mathfrak{H}}f(\zeta_\infty(0))\big]\Big\}d\theta,\end{aligned}$$

where

$$
\begin{aligned}
&\rho_n^{(3,1)}(f;\theta)\\
&= \int_0^\theta \int_0^{\theta_1} \bigg\{ E\bigg[\bigg\langle D\bigg(\big\langle D(G_n^{(2)}(\not\partial_z)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}}\bigg), u_n[\partial_z]\bigg\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\bigg]\\
&\qquad +\theta_2 E\big[G_n^{(2)}(\not\partial_z)\big\langle D(G_n^{(2)}(\not\partial_z)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\\
&\qquad +2^{-1}(1-\theta_2^2) E\big[G_n^{(3)}(\not\partial_z)\big\langle D(G_n^{(2)}(\not\partial_z)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\\
&\qquad +E\big[G_n^{(1)}(\theta_2;\not\partial_z,\not\partial_x)\langle D(G_n^{(2)}(\not\partial_z)\psi_n), u_n[\partial_z]\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\bigg\} d\theta_2 d\theta_1\\
&\quad + \int_0^\theta \bigg\{\theta_1 E\big[\psi_n G_n^{(2)}(\not\partial_z) G_n^{(2)}(\not\partial_z) f(\zeta_n(\theta_1))\big]\\
&\qquad +2^{-1}(1-\theta_1^2) E\big[\psi_n G_n^{(3)}(\not\partial_z) G_n^{(2)}(\not\partial_z) f(\zeta_n(\theta_1))\big]\\
&\qquad +E\big[\psi_n G_n^{(1)}(\theta_1;\not\partial_z,\not\partial_x) G_n^{(2)}(\not\partial_z) f(\zeta_n(\theta_1))\big]\bigg\} d\theta_1,
\end{aligned}
$$

$$
\begin{aligned}
&\rho_n^{(3,2)}(f;\theta)\\
&= \int_0^\theta \int_0^{\theta_1} \bigg\{ E\bigg[\bigg\langle D\bigg(\big\langle D(G_n^{(3)}(\not\partial_z)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}}\bigg), u_n[\partial_z]\bigg\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\bigg]\\
&\qquad +\theta_2 E\big[G_n^{(2)}(\not\partial_z)\big\langle D(G_n^{(3)}(\not\partial_z)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\\
&\qquad +2^{-1}(1-\theta_2^2) E\big[G_n^{(3)}(\not\partial_z)\big\langle D(G_n^{(3)}(\not\partial_z)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\\
&\qquad +E\big[G_n^{(1)}(\theta_2;\not\partial_z,\not\partial_x)\langle D(G_n^{(3)}(\not\partial_z)\psi_n), u_n[\partial_z]\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\bigg\} d\theta_2 d\theta_1\\
&\quad + \int_0^\theta \bigg\{\theta_1 E\big[\psi_n G_n^{(2)}(\not\partial_z) G_n^{(3)}(\not\partial_z) f(\zeta_n(\theta_1))\big]\\
&\qquad +2^{-1}(1-\theta_1^2) E\big[\psi_n G_n^{(3)}(\not\partial_z) G_n^{(3)}(\not\partial_z) f(\zeta_n(\theta_1))\big]\\
&\qquad +E\big[\psi_n G_n^{(1)}(\theta_1;\not\partial_z,\not\partial_x) G_n^{(3)}(\not\partial_z) f(\zeta_n(\theta_1))\big]\bigg\} d\theta_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho_n^{(3,3)}(f;\theta)\\
&= \int_0^\theta \int_0^{\theta_1} \bigg\{ E\bigg[\bigg\langle D\bigg(\big\langle D(G_n^{(1)}(\theta;\not\partial_z,\not\partial_x)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}}\bigg), u_n[\partial_z]\bigg\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\bigg]\\
&\qquad +\theta_2 E\big[G_n^{(2)}(\not\partial_z)\big\langle D(G_n^{(1)}(\theta;\not\partial_z,\not\partial_x)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\\
&\qquad +2^{-1}(1-\theta_2^2) E\big[G_n^{(3)}(\not\partial_z)\big\langle D(G_n^{(1)}(\theta;\not\partial_z,\not\partial_x)\psi_n), u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\\
&\qquad +E\big[G_n^{(1)}(\theta_2;\not\partial_z,\not\partial_x)\langle D(G_n^{(1)}(\theta;\not\partial_z,\not\partial_x)\psi_n), u_n[\partial_z]\rangle_{\mathfrak{H}} f(\zeta_n(\theta_2))\big]\bigg\} d\theta_2 d\theta_1\\
&\quad + \int_0^\theta \bigg\{\theta_1 E\big[\psi_n G_n^{(2)}(\not\partial_z) G_n^{(1)}(\theta;\not\partial_z,\not\partial_x) f(\zeta_n(\theta_1))\big]\\
&\qquad +2^{-1}(1-\theta_1^2) E\big[\psi_n G_n^{(3)}(\not\partial_z) G_n^{(1)}(\theta;\not\partial_z,\not\partial_x) f(\zeta_n(\theta_1))\big]\\
&\qquad +E\big[\psi_n G_n^{(1)}(\theta;\not\partial_z,\not\partial_x) G_n^{(1)}(\theta_1;\not\partial_z,\not\partial_x) f(\zeta_n(\theta_1))\big]\bigg\} d\theta_1.
\end{aligned}
$$

For $i = 4, \dots, 12$, a precise expression for $\rho_n^{(i)}(f)$ is given below

$$\begin{aligned}\rho_n^{(4)}(f) &= r_n\Big\{E\big[2^{-1}\overset{\circ}{W}_n[\partial_z]\langle D\psi_n, u_n[\partial_z]\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big]\Big\} \\ &\quad +E\big[2^{-1}N_n[\partial_z]\langle D\psi_n, u_n[\partial_z]\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big]\Big\} \\ &\quad +E\big[2^{-1}\overset{\circ}{X}_n[\partial_x]\langle D\psi_n, u_n[\partial_z]\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big]\Big\},\end{aligned}$$

$$\begin{aligned}\rho_n^{(5)}(f) &= r_n\Big\{E\Big[3^{-1}\psi_n r_n^{-1}\Big\langle D\big\langle DM_n[\partial_z], u_n[\partial_z]\big\rangle_{\mathfrak{H}}, u_n[\partial_z]\Big\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\Big] \\ &\quad -E\big[\mathfrak{S}^{(3,0)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},\end{aligned}$$

$$\begin{aligned}\rho_n^{(6)}(f) &= r_n\Big\{E\big[2^{-1}\psi_n r_n^{-1}\big(\big\langle DM_n[\partial_z], u_n[\partial_z]\big\rangle_{\mathfrak{H}} - G_\infty[(\partial_z)^2]\big) f(\zeta_\infty(0))\big] \\ &\quad -E\big[\mathfrak{S}_0^{(2,0)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},\end{aligned}$$

$$\rho_n^{(7)}(f) = r_n\Big\{E\big[\psi_n r_n^{-1}\big\langle DW_\infty[\mathtt{iz}], u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(2,0)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},$$

$$\rho_n^{(8)}(f) = r_n\Big\{E\big[\psi_n r_n^{-1}\big\langle DX_\infty[\partial_x], u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},$$

$$\rho_n^{(9)}(f) = r_n\Big\{E\big[\psi_n\,(\overset{\circ}{W}_n[\partial_z] + N_n[\partial_z]) f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(1,0)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},$$

$$\rho_n^{(10)}(f) = r_n\Big\{E\big[\psi_n\,\overset{\circ}{X}_n[\partial_x] f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(0,1)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},$$

$$\begin{aligned}\rho_n^{(11)}(f) &= r_n\Big\{E\big[\psi_n\big\langle D\overset{\circ}{W}_n[\partial_z] + DN_n[\partial_z], u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big] \\ &\quad -E\big[\mathfrak{S}_1^{(2,0)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\},\end{aligned}$$

and

$$\rho_n^{(12)}(f) = r_n\Big\{E\big[\psi_n\big\langle D\overset{\circ}{X}_n[\partial_x], u_n[\partial_z]\big\rangle_{\mathfrak{H}} f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}_1^{(1,1)}(\partial_z,\partial_x) f(\zeta_\infty(0))\big]\Big\}.$$

Let $\beta = \max\{7, \beta_0\}$, where β_0 is the degree in (z,x) of the random symbol $\mathfrak{S}$.

Theorem 3.1. Suppose that Condition $[A]$ is satisfied. Then

$$\begin{aligned}E[f(Z_n, X_n)] &= E[f(G_\infty^{1/2}\zeta + W_\infty, X_\infty)] \\ &\quad + r_n E[\mathfrak{S}(\partial_z,\partial_x) f(G_\infty^{1/2}\zeta + W_\infty, X_\infty)] + \rho_n^{(1)}(f)\end{aligned}$$

for $f \in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$.

Proof. Since $\mathcal{S}(\mathbb{R}^{\check{\mathsf{d}}}) \ni f \mapsto E[f(Z_n,X_n)]$, $E[f(G_\infty^{1/2}\zeta+W_\infty,X_\infty)]$, $E[\mathfrak{S}(\partial_z,\partial_x)f(G_\infty^{1/2}\zeta+W_\infty,X_\infty)]$ and $\rho_n^{(1)}(f)$ are expressed as a sum of bounded signed measures applied to the derivatives of f, Equation (3.7) holds for all $f\in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$ with the same expression for $\rho_n^{(1)}(f)$. [On each bounded ball, take a sequence $f_k\in\mathcal{S}(\mathbb{R}^{\check{\mathsf{d}}})$ that converges to f in C_b^β. The differentiability condition can usually be relaxed since not all orders of monomials appear in $\mathfrak{S}$.] □

The effect of the quasi torsion appears in $\mathfrak{S}^{(3,0)}$ and that of the quasi tangent does in $\mathfrak{S}_0^{(2,0)}$. The modified quasi torsion does not appear in the random symbol $\mathfrak{S}$ but in $\rho_n^{(1)}(f)$ as $G_n^{(3)}(\mathsf{z})$. It is often useful to recognize the modified quasi torsion in computations since the derivative of the quasi tangent relates it to the quasi torsion.

In order to prove $\rho_n^{(1)}(f)=o(r_n)$, for a sequence of random variables A_n targeting at A_∞, we can apply either stable convergence $A_n\to^{d_s}A_\infty$, L^1-convergence $\|A_n-A_\infty\|_1\to0$, or the integration-by-parts formula to evaluate the error terms of the form $E[A_nf(\zeta_n(\theta))]-E[A_\infty f(\zeta_n(\theta))]$ whether $A_\infty=0$ or not. We will present an estimate for $\rho_n^{(1)}(f)$ in Section 3.2.

3.2 Estimate of $\rho_n^{(1)}(f)$

Define the following random symbols

$$
\begin{aligned}
\mathfrak{S}_n^{(3,0)}(\mathtt{iz}) &= \mathfrak{S}_n^{(3,0)}(\mathtt{iz},\mathtt{ix})\\
&= \frac13 r_n^{-1}\Big\langle D\big\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}} \equiv \frac13\mathsf{qTor}[(\mathtt{iz})^{\otimes3}],\\
\mathfrak{S}_{0,n}^{(2,0)}(\mathtt{iz}) &= \mathfrak{S}_{0,n}^{(2,0)}(\mathtt{iz},\mathtt{ix}) = \frac12 r_n^{-1}G_n^{(2)}(\mathsf{z})\\
&= \frac12 r_n^{-1}\Big(\big\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}-G_\infty[(\mathtt{iz})^2]\Big) \equiv \frac12\mathsf{qTan}[(\mathtt{iz})^{\otimes2}],\\
\mathfrak{S}_n^{(2,0)}(\mathtt{iz}) &= \mathfrak{S}_n^{(2,0)}(\mathtt{iz},\mathtt{ix}) = r_n^{-1}\big\langle DW_\infty[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},\\
\mathfrak{S}_n^{(1,1)}(\mathtt{iz},\mathtt{ix}) &= r_n^{-1}\big\langle DX_\infty[\mathtt{ix}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},\\
\mathfrak{S}_n^{(1,0)}(\mathtt{iz}) &= \mathfrak{S}_n^{(1,0)}(\mathtt{iz},\mathtt{ix}) = \mathring{W}_n[\mathtt{iz}]+N_n[\mathtt{iz}],\\
\mathfrak{S}_n^{(0,1)}(\mathtt{ix}) &= \mathfrak{S}_n^{(0,1)}(\mathtt{iz},\mathtt{ix}) = \mathring{X}_n[\mathtt{ix}],\\
\mathfrak{S}_{1,n}^{(2,0)}(\mathtt{iz}) &= \mathfrak{S}_{1,n}^{(2,0)}(\mathtt{iz},\mathtt{ix}) = \Big\langle D\mathring{W}_n[\mathtt{iz}]+DN_n[\mathtt{iz}],u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}},\\
\mathfrak{S}_{1,n}^{(1,1)}(\mathtt{iz},\mathtt{ix}) &= \Big\langle D\mathring{X}_n[\mathtt{ix}],u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}}.
\end{aligned}
$$

Remark 3.2. As mentioned in Remark 2.5, the order of the random symbol $\mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix})$ appearing as the limit of the corresponding sequence $\mathfrak{S}_n^{(3,0)}(\mathtt{iz},\mathtt{ix})$ does not necessarily coincide with that of the latter because $\mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix})$ is determined by the action of $\mathfrak{S}_n^{(3,0)}(\mathtt{iz},\mathtt{ix})$ to $\Psi(\mathsf{z},\mathsf{x})$ under expectation. It is also the case for other symbols.

For $\mathfrak{H}$-valued tensors $\mathsf{S}=(\mathsf{S}_i)$ and $\mathsf{T}=(\mathsf{T}_j)$, $\langle\mathsf{S},\mathsf{T}\rangle_{\mathfrak{H}}$ denotes the tensor with components $(\langle\mathsf{S}_i,\mathsf{T}_j\rangle_{\mathfrak{H}})_{i,j}$. In the following condition, A and B denote dummy variables.

[B] (i) $u_n\in\mathbb{D}^{4,p}(\mathfrak{H}\otimes\mathbb{R}^{\mathsf{d}})$, $G_\infty\in\mathbb{D}^{3,p}(\mathbb{R}^{\mathsf{d}}\otimes_+\mathbb{R}^{\mathsf{d}})$, $W_n,W_\infty,N_n\in\mathbb{D}^{3,p}(\mathbb{R}^{\mathsf{d}})$, $X_n,X_\infty\in\mathbb{D}^{3,p}(\mathbb{R}^{\mathsf{d}_1})$ and $\psi_n\in\mathbb{D}^{2,p_1}(\mathbb{R})$ for some p and p_1 satisfying $5p^{-1}+p_1^{-1}\le1$.

(ii) The following estimates hold:

$$\|u_n\|_{1,p} = O(1) \tag{3.8}$$

$$\sum_{k=2,3} \|G_n^{(k)}\|_{p/2} = O(r_n) \tag{3.9}$$

$$\|\big\langle DG_n^{(2)}, u_n\big\rangle_{\mathfrak{H}}\|_{p/3} = O(r_n) \tag{3.10}$$

$$\|\big\langle DG_n^{(3)}, u_n\big\rangle_{\mathfrak{H}}\|_{p/3} = o(r_n) \tag{3.11}$$

$$\sum_{k=2,3} \Big\|\Big\langle D\big\langle DG_n^{(k)}, u_n\big\rangle_{\mathfrak{H}}, u_n\Big\rangle_{\mathfrak{H}}\Big\|_{p/4} = o(r_n) \tag{3.12}$$

$$\sum_{\mathsf{A}=W_\infty[\mathsf{z}],X_\infty[\mathsf{x}]} \big\|\langle D\mathsf{A}, u_n\rangle_{\mathfrak{H}}\big\|_p = O(r_n) \tag{3.13}$$

$$\sum_{\mathsf{A}=W_\infty,X_\infty} \Big\|\Big\langle D\langle D\mathsf{A}, u_n[\mathsf{z}]\rangle_{\mathfrak{H}}, u_n[\mathsf{z}]\Big\rangle_{\mathfrak{H}}\Big\|_{p/3} = o(r_n) \tag{3.14}$$

$$\sum_{\mathsf{A}=W_\infty,X_\infty} \Big\|\Big\langle D\big\langle D\langle D\mathsf{A}, u_n\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}}, u_n\Big\rangle_{\mathfrak{H}}\Big\|_{p/4} = o(r_n) \tag{3.15}$$

$$\|\overset{\circ}{W}_n\|_{3,p} + \|N_n\|_{3,p} + \|\overset{\circ}{X}_n\|_{3,p} = O(1) \tag{3.16}$$

$$\sum_{\mathsf{B}=\overset{\circ}{W}_n,N_n,\overset{\circ}{X}_n} \Big\|\big\langle D\langle D\mathsf{B}, u_n\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}}\Big\|_{p/3} = o(1) \tag{3.17}$$

$$\sum_{\mathsf{B}=\overset{\circ}{W}_n,N_n,\overset{\circ}{X}_n} \Big\|\Big\langle D\big\langle D\langle D\mathsf{B}, u_n\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}}, u_n\Big\rangle_{\mathfrak{H}}\Big\|_{p/4} = o(1) \tag{3.18}$$

$$\|1-\psi_n\|_{2,p_1} = o(r_n) \tag{3.19}$$

(iii) For every $\mathsf{z}\in\mathbb{R}^{\mathsf{d}}$ and $\mathsf{x}\in\mathbb{R}^{\mathsf{d}_1}$, it holds that

$$\lim_{n\to\infty} E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}_n(\mathtt{iz},\mathtt{ix})\psi_n\big] = E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}(\mathtt{iz},\mathtt{ix})\big]$$

for $(\mathfrak{T}_n,\mathfrak{T}) = (\mathfrak{S}_n^{(3,0)},\mathfrak{S}^{(3,0)})$, $(\mathfrak{S}_{0,n}^{(2,0)},\mathfrak{S}_0^{(2,0)})$, $(\mathfrak{S}_n^{(2,0)},\mathfrak{S}^{(2,0)})$, $(\mathfrak{S}_n^{(1,1)},\mathfrak{S}^{(1,1)})$, $(\mathfrak{S}_n^{(1,0)},\mathfrak{S}^{(1,0)})$, $(\mathfrak{S}_n^{(0,1)},\mathfrak{S}^{(0,1)})$, $(\mathfrak{S}_{1,n}^{(2,0)},\mathfrak{S}_1^{(2,0)})$ and $(\mathfrak{S}_{1,n}^{(1,1)},\mathfrak{S}_1^{(1,1)})$.

We say that a family of random symbols $\{\varsigma^\lambda(\mathtt{iz},\mathtt{ix})=\sum_{k,m}c^\lambda_{k,m}[(\mathtt{iz})^{\otimes k}\otimes(\mathtt{ix})^{\otimes m}];\ \lambda\in\Lambda\}$ is uniformly integrable (u.i.) if the degrees of polynomials are bounded and the family $\{c^\lambda_{k,m};\ \lambda\in\Lambda\}$ of tensor-valued random variables is uniformly integrable for every (k,m). Recall that $\beta=\max\{7,\beta_0\}$, where β_0 is the degree in (z,x) of the random symbol $\mathfrak{S}$.

Theorem 3.3. Suppose that Condition $[B]$ is fulfilled. Then $\rho_n^{(1)}(f) = o(r_n)$ for every $f \in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$. Moreover,

$$\sup_{f\in\mathcal{B}} |\rho_n^{(1)}(f)| \quad = \quad o(r_n)$$

for every bounded set $\mathcal{B}$ in $C_b^{\beta+1}(\mathbb{R}^{\check{\mathsf{d}}})$.

Proof. The sequence $\{\mathfrak{S}_n^{(3,0)}; n \in \mathbb{N}\}$, is u.i. from (3.10) and (3.9) for $k=3$. $\{\mathfrak{S}_{0,n}^{(2,0)}; n \in \mathbb{N}\}$ is u.i. from (3.9) for $k=2$. $\{\mathfrak{S}_n^{(2,0)}; n \in \mathbb{N}\}$ and $\{\mathfrak{S}_n^{(1,1)}; n \in \mathbb{N}\}$ are u.i. from (3.13). $\{\mathfrak{S}_n^{(1,0)}; n \in \mathbb{N}\}$ and $\{\mathfrak{S}_n^{(0,1)}; n \in \mathbb{N}\}$ are u.i. from (3.16). $\{\mathfrak{S}_{1,n}^{(2,0)}; n \in \mathbb{N}\}$ and $\{\mathfrak{S}_{1,n}^{(1,1)}; n \in \mathbb{N}\}$ are u.i. from (3.8) and (3.16).

We consider first the case of $\rho_n^{(8)}(f)$, given by

$$\rho_n^{(8)}(f) = r_n\left\{E\big[\psi_n\mathfrak{S}_n^{(1,1)}(\partial_z,\partial_x)f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)f(\zeta_\infty(0))\big]\right\}.$$

By $[B]$ (iii) and the formula (3.2), we obtain

$$\lim_{n\to\infty} E\big[\psi_n\mathfrak{S}_n^{(1,1)}(\partial_z,\partial_x)f(\zeta_\infty(0))\big] \quad = \quad E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)f(\zeta_\infty(0))\big] \tag{3.20}$$

for $f \in \mathcal{S}(\mathbb{R}^{\check{\mathsf{d}}})$. Let $\chi : \mathbb{R}^{\check{\mathsf{d}}} \to [0,1]$ be a smooth function with a compact support. Since for $f \in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$, the function χf is uniformly approximated in $C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$ by some function in $\mathcal{S}(\mathbb{R}^{\check{\mathsf{d}}})$, we have

$$\lim_{n\to\infty}\left|E\big[\psi_n\mathfrak{S}_n^{(1,1)}(\partial_z,\partial_x)(\chi f)(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)(\chi f)(\zeta_\infty(0))\big]\right| \quad = \quad 0 \tag{3.21}$$

for $f \in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$. Let $\mathcal{B}^k$ be a bounded set in $C_b^k(\mathbb{R}^{\check{\mathsf{d}}})$. Let $\epsilon > 0$. Due to the uniformly integrability of the family $\{\mathfrak{S}_n^{(1,1)}; n \in \mathbb{N}\}$, we can write

$$\begin{aligned} &\sup_{f\in\mathcal{B}^\beta, n\in\mathbb{N}}\left|E\big[\psi_n\mathfrak{S}_n^{(1,1)}(\partial_z,\partial_x)((1-\chi)f)(\zeta_\infty(0))\big]\right| \\ &\quad + \sup_{f\in\mathcal{B}^\beta}\left|E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)((1-\chi)f)(\zeta_\infty(0))\big]\right| \;<\; \epsilon \end{aligned} \tag{3.22}$$

if we choose χ satisfying $\chi = 1$ on a sufficiently large compact set. Combining (3.21) and (3.22), we obtain

$$\lim_{n\to\infty}\left|E\big[\psi_n\mathfrak{S}_n^{(1,1)}(\partial_z,\partial_x)f(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)f(\zeta_\infty(0))\big]\right| \quad = \quad 0 \tag{3.23}$$

for $f \in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$. Moreover, we have

$$\lim_{n\to\infty}\sup_{f\in\mathcal{B}^{\beta+1}}\left|E\big[\psi_n\mathfrak{S}_n^{(1,1)}(\partial_z,\partial_x)(\chi f)(\zeta_\infty(0))\big] - E\big[\mathfrak{S}^{(1,1)}(\partial_z,\partial_x)(\chi f)(\zeta_\infty(0))\big]\right| \quad = \quad 0, \tag{3.24}$$

since the functionals in $|\ |$ are equi-continuous in f in $C^\beta(\mathbb{R}^{\check{\mathsf{d}}})$ and $\mathcal{B}^{\beta+1}$ is relatively compact in $\mathcal{B}^\beta$ if the domain is restricted to supp(χ). Therefore we showed that $\rho_n^{(8)}(f) = o(r_n)$ for every $f \in C_b^\beta(\mathbb{R}^{\check{\mathsf{d}}})$, and that $\sup_{f\in\mathcal{B}^{\beta+1}} |\rho_n^{(8)}(f)| = o(r_n)$. Similarly, we obtain the same estimate for $\rho_n^{(i)}(f)$ for $i = 5, \ldots, 12$.

Moreover, we have $\sup_{f\in\mathcal{B}^0} |\rho_n^{(2)}(f)| = o(r_n)$ from (3.19), and $\sup_{f\in\mathcal{B}^2} |\rho_n^{(4)}(f)| = o(r_n)$ from (3.8), (3.16) and (3.19).

The tensor $\hat{G}_n^{(1)}(\theta,\cdot,\cdot)$ is denoted by $\hat{G}_n^{(1)}(\theta)$. The estimate $\|\hat{G}_n^{(1)}(\theta)\|_{p/2}=O(r_n)$ follows from (3.13), (3.16) and (3.8), and the estimate $\|\check{G}_n^{(1)}\|_{p/2}=O(r_n)$ follows from (3.16). Therefore

$$\|G_n^{(1)}(\theta)\|_{p/2} \quad = \quad O(r_n). \tag{3.25}$$

We obtain

$$\big\|\big\langle D\hat{G}_n^{(1)}(\theta),u_n\big\rangle_{\mathfrak{H}}\big\|_{p/3} \quad = \quad o(r_n) \tag{3.26}$$

from (3.14) and (3.17) for every θ (or $\theta=0,1$), and

$$\big\|\big\langle D\check{G}_n^{(1)},u_n\big\rangle_{\mathfrak{H}}\big\|_{p/2} \quad = \quad O(r_n) \tag{3.27}$$

from (3.16). Moreover, we have

$$\left\|\left\langle D\big\langle D\hat{G}_n^{(1)}(\theta),u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\right\|_{p/4} \quad = \quad o(r_n)$$

by (3.15) and (3.18), and

$$\left\|\left\langle D\big\langle D\check{G}_n^{(1)},u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\right\|_{p/3} \quad = \quad o(r_n)$$

by (3.17), so that

$$\left\|\left\langle D\big\langle DG_n^{(1)}(\theta),u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\right\|_{p/4} \quad = \quad o(r_n) \tag{3.28}$$

Let us investigate the order of $\rho_n^{(3)}(f)$. Denote by $\rho[i]$ ($i=1,\dots,24$) the 24 linear functionals of f appearing in the expression of $\rho_n^{(3)}(f)$. Though not explicitly mentioned in what follows, Condition (3.19) is used every time in estimation of $\rho[i]$ to ensure either $\|\psi_n\|_{p_1}=O(1)$, $\|D\psi_n\|_{p_1}=o(r_n)$ or $\|D^2\psi_n\|_{p_1}=o(r_n)$. The estimate $\rho[1]=o(r_n)$ follows from (3.8) and (3.9) (and (3.19)). The term $\rho[2]$ corresponds to the first term in the expression of $\rho_n^{(3,1)}$; then $\rho[2]=o(r_n)$ from (3.12), (3.9), (3.10) and (3.8) with the aid of the Leibniz rule; $\rho[3]=O(r_n^2)$ from (3.9), (3.10) and (3.8); $\rho[4]=O(r_n^2)$ from (3.9), (3.10) and (3.8); $\rho[5]=O(r_n^2)$ from (3.25), (3.9), (3.10) and (3.8); $\rho[6]=O(r_n^2)$ from (3.9); $\rho[7]=O(r_n^2)$ from (3.9); $\rho[8]=O(r_n^2)$ from (3.25) and (3.9); $\rho[9]=o(r_n)+o(r_n)o(r_n)+O(r_n)o(r_n)=o(r_n)$ from (3.12), (3.11), (3.8) and (3.9); $\rho[10]=o(r_n^2)+o(r_n^3)=o(r_n^2)$ from (3.11), (3.9) and (3.8); $\rho[11]=o(r_n^2)+o(r_n^3)=o(r_n^2)$ from (3.11), (3.9) and (3.8); $\rho[12]=o(r_n^2)+o(r_n^3)=o(r_n^2)$ from (3.25), (3.11), (3.9) and (3.8); $\rho[13]=O(r_n^2)$ from (3.9); $\rho[14]=O(r_n^2)$ from (3.9); $\rho[15]=O(r_n^2)$ from (3.25) and (3.9); $\rho[16]=o(r_n)+O(r_n)o(r_n)+O(r_n)o(r_n)=o(r_n)$ from (3.28), (3.26), (3.27), (3.25) and (3.8); $\rho[17]=O(r_n^2)+o(r_n^3)=O(r_n^2)$ from (3.9), (3.26), (3.27), (3.25) and (3.8); $\rho[18]=O(r_n^2)+o(r_n^3)=O(r_n^2)$ from (3.9), (3.26), (3.27), (3.25) and (3.8); $\rho[19]=O(r_n^2)+o(r_n^3)=O(r_n^2)$ from (3.25), (3.26), (3.27) and (3.8); $\rho[20]=O(r_n^2)$ from (3.9) and (3.25); $\rho[21]=O(r_n^2)$ from (3.9) and (3.25); $\rho[22]=O(r_n^2)$ from (3.25); $\rho[23]=o(r_n)+o(r_n^2)=o(r_n)$ from (3.11), (3.9) and (3.8); $\rho[24]=o(r_n)+o(r_n^2)=o(r_n)$ from (3.26) and (3.8). We remark that some $\rho[i]$'s involve a product of five L^p variables besides ψ_n or its derivative, and that the seventh derivative of f appears in $\rho[11]$. These estimates give $\sup_{f\in\mathcal{B}^7}|\rho_n^{(3)}(f)|=o(r_n)$. This completes the proof. □

Remark 3.4. Corollary 3.2 of Nourdin, Nualart and Peccati [14] gives stable convergence of the Skorohod integral $M_n=\delta(u_n)$ under the conditions (i) $\langle DM_n,u_n\rangle_{\mathfrak{H}}\to G_\infty$

in L^1, (ii) $\langle u_n, h\rangle_{\mathfrak{H}} \to 0$ in L^1 for every $h \in \mathfrak{H}$, and (iii) $\langle DG_\infty, u_n\rangle_{\mathfrak{H}} \to 0$ in L^1. Theorem 3.3 entails the mixed normal limit theorem for Z_n and the joint convergence of (Z_n, X_n). Indeed, (3.9) gives the conditions (i) and (iii). The stable convergence was proved in [14] by taking $X_\infty = \mathbb{W}(h)$ for $h \in \mathfrak{H}$ in the present context. Condition (3.13) gives (ii) when $X_\infty = \mathbb{W}(h)$.

4 A functional of a fractional Brownian motion

In this section, we shall consider a functional of a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ on the time interval $[0,1]$. The fBm is a centered Gaussian process $B = \{B_t, t \in [0,1]\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with covariance function

$$R_H(t,s) \;=\; E[B_sB_t] \;=\; \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The process B is a standard Brownian motion for $H = \frac{1}{2}$. Denote by $\mathcal{E}$ the set of step functions on $[0,1]$. Then it is possible to introduce an inner product in $\mathcal{E}$ such that

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\right\rangle_{\mathfrak{H}} = E[B_sB_t].$$

Let $\|\cdot\|_{\mathfrak{H}} = \langle\cdot,\cdot\rangle_{\mathfrak{H}}^{1/2}$. Hilbert space $\mathfrak{H}$ is defined as the closure of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathfrak{H}}$.

It is known that the mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by B in $L^2 = L^2(\Omega,\mathcal{F},P)$. We denote this isometry by $\phi \mapsto B(\phi)$. The process $\{B(\phi), \phi \in \mathfrak{H}\}$ is an isonormal Gaussian process. We refer to [18] for a detailed account on the basic properties of the fBm. Assume again that $\mathcal{F}$ is the σ-field generated by B.

In the case $H > \frac{1}{2}$, the space $\mathfrak{H}$ contains the linear space $|\mathfrak{H}|$ of all measurable functions $\varphi : [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1\int_0^1 |\varphi(s)||\varphi(t)||t-s|^{2H-2}dsdt < \infty.$$

In this case, the inner product $\langle \varphi, \phi\rangle_{\mathfrak{H}}$ is represented by

$$\langle \varphi, \phi\rangle_{\mathfrak{H}} = H(2H-1)\int_0^1\int_0^1 \varphi(s)\phi(t)|t-s|^{2H-2}dsdt \tag{4.1}$$

for $\varphi, \phi \in |\mathfrak{H}|$. Furthermore, $L^{\frac{1}{H}}([0,1])$ is continuously embedded into $\mathfrak{H}$. The following lemma provides useful formulas for the inner product in the Hilbert space $\mathfrak{H}$.

Lemma 4.1. (i) Let $H \neq \frac{1}{2}$. For any piecewise continuous function φ on $[0,1]$, the inner product $\langle \varphi, \mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}}$ is given by

$$\begin{aligned}\langle \varphi, \mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}} &= \int_0^1 \varphi(t)\frac{\partial R_H}{\partial t}(t,s)dt \\ &= \int_0^1 \varphi(t)H\big\{t^{2H-1} - |t-s|^{2H-1}\mathrm{sign}(t-s)\big\}dt. \end{aligned}\tag{4.2}$$

(ii) Let $H \neq \frac{1}{2}$. For any piecewise continuous function φ on $[0,1]$ and $\psi \in C_b^1([0,1])$,

$$\begin{aligned}\langle \varphi, \psi\rangle_{\mathfrak{H}} &= \int_0^1 \varphi(t)\left\{\frac{\partial R_H}{\partial t}(t,1)\psi(1) - \int_0^1 \frac{\partial R_H}{\partial t}(t,s)\psi'(s)ds\right\}dt \\ &= \int_0^1 \varphi(t)H\left\{t^{2H-1}\psi(0) + (1-t)^{2H-1}\psi(1)\right. \\ &\qquad \left. + \int_0^1 |t-s|^{2H-1}\mathrm{sign}(t-s)\psi'(s)ds\right\}dt. \end{aligned}\tag{4.3}$$

(iii) Let $0 < H < 1/2$, $0 < a < b < 1$. Then for any piecewise continuous function φ on $[0,1]$,

$$\left|\langle\varphi, \mathbf{1}_{[a,b]}\rangle_{\mathfrak{H}}\right| \quad \le \quad \|\varphi\|_\infty (b-a)^{2H}.$$

Proof. Approximating φ by step functions we can derive (i). For (ii), using (i) we can write

$$\begin{aligned}\langle\varphi,\psi\rangle_{\mathfrak{H}} &= \left\langle \varphi, \psi(1)\mathbf{1}_{[0,1]} - \int_0^1 \mathbf{1}_{[0,s]}\psi'(s)ds \right\rangle_{\mathfrak{H}} \\ &= \psi(1)\int_0^1 \varphi(t)\frac{\partial R_H}{\partial t}(t,1)dt - \int_0^1\int_0^1 \varphi(t)\frac{\partial R_H}{\partial t}(t,s)\psi'(s)dsdt.\end{aligned}$$

Simple calculus with (i) gives (iii). □

In what follows, we shall write

$$c_H = \sqrt{H\Gamma(2H)}, \quad H \in (0,1). \tag{4.4}$$

Notice that $c_{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ and for $H > 1/2$, $c_H = \sqrt{H(2H-1)\Gamma(2H-1)}$.

We will consider the sequence of random variables $Z_n = \delta(u_n)$, $n \ge 1$, where

$$u_n(t) = n^H t^n B_t \mathbf{1}_{[0,1]}(t).$$

The following provides the convergence in law of the sequence Z_n. For the Brownian motion case, this result goes back to Peccati and Yor [23]. For $H > \frac{1}{2}$, it was proved by Peccati and Tudor [21] and a rate of convergence in the total variation distance was established in [14]. For $\frac{1}{4} < H < \frac{1}{2}$, the convergence in law of Z_n is a consequence of our asymptotic expansion proved below. The process u_n belongs to the domain of δ only if $H > \frac{1}{4}$. Indeed, if $H \le \frac{1}{4}$, the process B_t does not belong to $L^2(\Omega;\mathfrak{H})$ (see Cheridito and Nualart [2], Proposition 3.2).

Proposition 4.2. The sequence Z_n converges stably in law to $\zeta\sqrt{G_\infty}$, where $G_\infty = c_H^2 B_1^2$ and ζ is a $N(0,1)$ random variable, independent of $\{B_t, t \in [0,1]\}$.

In the setting of Section 2, the variables are now $Z_n = M_n$, $W_n = W_\infty = 0$, $X_n = X_\infty = 0$, $\psi_n = 1$ and $G_\infty = c_H^2 B_1^2$. We are interested in investigating the asymptotic behavior of the three basic terms: modified quasi torsion, quasi tangent and quasi torsion. We denote by C_H a generic constant depending on H, that may vary in different lines.

Consider first the case of the modified quasi torsion.

(i) Case $H = \frac{1}{2}$. We have $G_\infty = \frac{1}{2}B_1^2$ and $D_t G_\infty = B_1 \mathbf{1}_{[0,1]}(t)$. Therefore,

$$G_n^{(3)} \quad = \quad \langle DG_\infty, u_n\rangle_{\mathfrak{H}} = \sqrt{n} B_1 \int_0^1 t^n B_t dt.$$

As a consequence, taking $r_n = n^{-1/2}$, we obtain

$$\mathsf{mqTor} \quad = \quad \sqrt{n}G_n^{(3)} \xrightarrow{L^p} B_1^2, \tag{4.5}$$

for all $p \ge 2$.

(ii) Case $H \ne \frac{1}{2}$. We have $G_\infty = c_H^2 B_1^2$, where c_H is the constant defined in (4.4) and $D_t G_\infty = 2c_H^2 B_1 \mathbf{1}_{[0,1]}(t)$. Applying (4.2) yields

$$\begin{aligned}\langle DG_\infty, u_n\rangle_{\mathfrak{H}} &= 2c_H^2 n^H B_1 \langle t^n B_t \mathbf{1}_{[0,1]}(t), \mathbf{1}_{[0,1]}(t)\rangle_{\mathfrak{H}} \\ &= 2c_H^2 n^H B_1 \int_0^1 B_t t^n H\big\{t^{2H-1} + (1-t)^{2H-1}\big\}dt.\end{aligned}$$

Using the decomposition $B_1 B_t = B_1^2 + B_1(B_t - B_1)$, we can write

$$\begin{aligned}\langle DG_\infty, u_n\rangle_{\mathfrak{H}} &= 2c_H^2 n^H B_1^2 \int_0^1 t^n H\{t^{2H-1} + (1-t)^{2H-1}\}dt \\ &\quad + 2c_H^2 n^H B_1 \int_0^1 t^n (B_t - B_1) H\{t^{2H-1} + (1-t)^{2H-1}\}dt \\ &= 2c_H^2 B_1^2 H\left\{n^{H-1} + n^H \frac{\Gamma(n+1)\Gamma(2H)}{\Gamma(n+1+2H)}\right\} + R_n,\end{aligned}$$

where $\|R_n\|_p = O(n^{-1}) + O(n^{-2H})$ for all $p \geq 2$. As a consequence, we obtain the following results:
If $H > \frac{1}{2}$ we take $r_n = n^{H-1}$ and

$$\mathsf{mqTor} = r_n^{-1} G_n^{(3)} \xrightarrow{L^p} 2H^2\Gamma(2H)B_1^2. \tag{4.6}$$

If $H < \frac{1}{2}$ we take $r_n = n^{-H}$ and

$$\mathsf{mqTor} = r_n^{-1} G_n^{(3)} \xrightarrow{L^p} 2H^2\Gamma(2H)^2 B_1^2. \tag{4.7}$$

Notice that the limit is discontinuous at $H = \frac{1}{2}$. With these preliminaries, we can now proceed to deduce the asymptotic expansion for $E[f(Z_n)]$ with classification for H.

4.1 Brownian motion case

We will analyze the asymptotic behavior of the quasi tangent and the quasi torsion, which are the main ingredients in the asymptotic expansions.

4.1.1 Quasi tangent

Let us now establish the asymptotic behavior of the quasi tangent, defined by

$$\mathsf{qTan} = \sqrt{n}G_n^{(2)} = \sqrt{n}\left(\langle DZ_n, u_n\rangle_{\mathfrak{H}} - G_\infty\right).$$

We have, for $s \in [0,1]$,

$$D_s Z_n = \sqrt{n} s^n B_s + \sqrt{n}\int_s^1 t^n dB_t.$$

Therefore,

$$\langle DZ_n, u_n\rangle_{\mathfrak{H}} = n\int_0^1 s^{2n} B_s^2 ds + n\int_0^1 s^n B_s \left(\int_s^1 t^n dB_t\right) ds.$$

Then,

$$\begin{aligned}G_n^{(2)} &= \langle DZ_n, u_n\rangle_{\mathfrak{H}} - \frac{1}{2}B_1^2 \\ &= n\int_0^1 s^{2n}(B_s^2 - B_1^2)ds + B_1^2\left(\int_0^1 n s^{2n} ds - \frac{1}{2}\right) \\ &\quad + n\int_0^1 s^n (B_s - B_1)\left(\int_s^1 t^n dB_t\right) ds + nB_1 \int_0^1 s^n \left(\int_s^1 t^n dB_t\right) ds.\end{aligned}$$

Using the decomposition $(B_s^2 - B_1^2) = (B_s - B_1)^2 - 2B_1(B_1 - B_s)$, yields

$$\begin{aligned}G_n^{(2)} &= n\int_0^1 s^{2n}(B_s - B_1)^2 ds - 2nB_1\int_0^1 s^{2n}(B_1 - B_s)ds - \frac{B_1^2}{4n+2} \\ &\quad - n\int_0^1 s^n (B_1 - B_s)\left(\int_s^1 t^n dB_t\right) ds + B_1 \frac{n}{n+1}\int_0^1 t^{2n+1} dB_t \\ &= Z_{n,1} + Z_{n,2} + Z_{n,3},\end{aligned}$$

where

$$\begin{aligned} Z_{n,1} &= n\int_0^1 s^{2n}[(B_1-B_s)^2-(1-s)]ds \\ &\quad -n\int_0^1 s^n\left[(B_1-B_s)\left(\int_s^1 t^n dB_t\right)-\frac{1-s^{n+1}}{n+1}\right]ds, \\ Z_{n,2} &= -\frac{n^2}{2(n+1)^2(2n+1)}-\frac{B_1^2}{4n+2}, \\ Z_{n,3} &= -\frac{n}{(n+1)(2n+1)}B_1\int_0^1 t^{2n+1}dB_t. \end{aligned}$$

The term $Z_{n,1}$ belongs to the second Wiener chaos and it can be written as a double stochastic integral:

$$Z_{n,1} = \int_0^1 2ns^{2n}\left(\int_{[s,1]^2} dB_r dB_u\right)ds-\int_0^1 ns^n\left(\int_{[s,1]^2}(r^n+u^n)dB_r dB_u\right)ds = I_2(f_n),$$

where

$$f_n(r,u) = \frac{n}{2n+1}\min(r,u)^{2n+1}-\frac{n}{2(n+1)}\min(r,u)^{n+1}(r^n+u^n).$$

It is not difficult to check that $n^2\|f_n\|^2_{\mathfrak{H}^{\otimes 2}}$ converges to a constant as n tends to infinity. Therefore, $\|Z_{n,1}\|_2=O(n^{-1})$. Clearly, we also have $\|Z_{n,2}\|_2=O(n^{-1})$. Finally, $\|Z_{n,3}\|_2=O(n^{-3/2})$. Consequently, $\|\sqrt{n}G_n^{(2)}\|_2=O(n^{-1/2})$, and hence the effect of the quasi tangent disappears in the limit, that is, $\mathfrak{S}_0^{(2,0)}=0$.

4.1.2 Quasi torsion

Let us first recall the definition of the quasi torsion

$$\mathsf{qTor}=\sqrt{n}\langle D\langle DZ_n,u_n\rangle_{\mathfrak{H}},u_n\rangle_{\mathfrak{H}}=\sqrt{n}\langle DG_n^{(2)},u_n\rangle_{\mathfrak{H}}+\sqrt{n}\langle DG_\infty,u_n\rangle_{\mathfrak{H}}.$$

Since $G_n^{(2)}$ is in the second chaos, it follows that $\sqrt{n}\|\|DG_n^{(2)}\|_{\mathfrak{H}}\|_2=O(n^{-1/2})$ from $\sqrt{n}\|G_n^{(2)}\|_2=O(n^{-1/2})$ in Section 4.1.1. Therefore, from (4.5) we deduce

$$\sqrt{n}\langle D\langle DZ_n,u_n\rangle_{\mathfrak{H}},u_n\rangle_{\mathfrak{H}} \quad \stackrel{L^p}{\to} \quad B_1^2$$

for all $p\geq 2$. Thus we obtain $\mathfrak{S}^{(3,0)}=3^{-1}B_1^2$.

4.1.3 Asymptotic expansion

From the computations in Sections 4.1.1 and 4.1.2, we deduce that conditions (3.8), (3.9), (3.10), (3.11) and (3.12) are satisfied for all $p\geq 2$. Thus, taking $\psi_n=1$, assumption [B] holds and we can apply Theorems 3.1 and 3.3. In this way, we obtain

$$\begin{aligned} E[f(Z_n)] &= E[f(G_\infty^{1/2}\zeta)]+\frac{1}{\sqrt{n}}E[\mathfrak{S}^{(3,0)}(\partial_z^3)f(G_\infty^{1/2}\zeta)]+\rho_n^{(1)}(f) \\ &= E[f(2^{-1/2}|B_1|\zeta)]+\frac{1}{3\sqrt{n}}E[B_1^2f^{(3)}(2^{-1/2}|B_1|\zeta)]+\rho_n^{(1)}(f) \qquad (4.8) \end{aligned}$$

for $f\in C_b^7(\mathbb{R})$, where $\zeta\sim N(0,1)$ is independent of B_1, and $\rho_n^{(1)}(f)=o(n^{-\frac{1}{2}})$.

4.2 Fractional Brownian motion. Case $H>\frac{1}{2}$

Recall that in that case $r_n=n^{H-1}$.

4.2.1 Quasi tangent

We are going to establish the convergence in law of the tangent and show that it does not contribute to the asymptotic expansion. We have, for $s \in [0,1]$,

$$D_s Z_n = n^H s^n B_s + n^H \int_s^1 t^n dB_t.$$

Therefore,

$$\langle DZ_n, u_n \rangle_{\mathfrak{H}} = \|u_n\|_{\mathfrak{H}}^2 + n^H \left\langle u_n, \int_\cdot^1 t^n dB_t \right\rangle_{\mathfrak{H}} =: \|u_n\|_{\mathfrak{H}}^2 + \Phi_n, \tag{4.9}$$

and the quasi tangent qTan is given by

$$\mathsf{qTan} = n^{1-H} G_n^{(2)} = n^{1-H} \left(\|u_n\|_{\mathfrak{H}}^2 - c_H^2 B_1^2 + \Phi_n \right).$$

Let

$$\begin{aligned}
\sigma_{H,1}^2 \quad = \quad & 2H^2(2H-1)^2 \int_{[0,1]^4} |\log y_1 - \log x_1|^{2H-2} |\log y_2 - \log x_2|^{2H-2} \\
& \times (|\log y_1|^{2H} + |\log y_2|^{2H} - |\log y_1 - \log y_2|^{2H}) dx_1 dx_2 dy_1 dy_2
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\sigma_{H,2}^2 \quad = \quad & H^3(2H-1)^3 \int_{[0,1]^2} |1-s_1|^{2H-2} |1-s_2|^{2H-2} ds_1 ds_2 \\
& \times \int_{[0,1]^2} |\log x_1 - \log x_2|^{2H-2} dx_1 dx_2.
\end{aligned} \tag{4.11}$$

Proposition 4.3. For the term $\|u_n\|_{\mathfrak{H}}^2$ we have

$$n^H (\|u_n\|_{\mathfrak{H}}^2 - c_H^2 B_1^2) \xrightarrow{\mathcal{L}} \sigma_{H,1} B_1 \zeta \tag{4.12}$$

where ζ is a $N(0,1)$-random variable independent of B. On the other hand,

$$n^{1-H} \Phi_n \xrightarrow{\mathcal{L}} \sigma_{H,2} B_1 \zeta, \tag{4.13}$$

where ζ is a $N(0,1)$-random variable independent of B. As a consequence, taking into account that $H > \frac{1}{2}$, we obtain

$$\mathsf{qTan} = n^{1-H} G_n^{(2)} \xrightarrow{\mathcal{L}} \sigma_{H,2} B_1 \zeta. \tag{4.14}$$

Proof. We first show (4.12). We can write

$$\begin{aligned}
\|u_n\|_{\mathfrak{H}}^2 - c_H^2 B_1^2 \quad & = \quad H(2H-1) \left(n^{2H} \int_0^1 \int_0^1 t^n s^n B_t B_s |t-s|^{2H-2} ds dt - \Gamma(2H-1) B_1^2 \right) \\
& = \quad H(2H-1) n^{2H} \int_0^1 \int_0^1 t^n s^n [B_t B_s - B_1^2] |t-s|^{2H-2} ds dt \\
& \qquad + H(2H-1) \left(n^{2H} \int_0^1 \int_0^1 t^n s^n |t-s|^{2H-2} ds dt - \Gamma(2H-1) \right) B_1^2 \\
& =: \quad A_{n,1} + A_{n,2} B_1^2.
\end{aligned}$$

The term $A_{n,2}$ is a deterministic term bounded by Cn^{-1}, therefore it does not contribute to the limit. In order to handle the term $A_{n,1}$ we make the decomposition

$$B_sB_t - B_1^2 = B_1(B_t - B_1) + (B_s - B_1)(B_t - B_1) + B_1(B_s - B_1).$$

We claim that the product $(B_s - B_1)(B_t - B_1)$ does not contribute to the limit of $n^H A_{n,1}$. In fact,

$$\begin{aligned} H(2H-1)n^{2H} \int_0^1 \int_0^1 s^n t^n |B_s - B_1||B_t - B_1||t-s|^{2H-2} ds dt = n^{2H} \|s^n |B_s - B_1|\|_{\mathfrak{H}}^2 \\ \le C_H n^{2H} \|s^n |B_s - B_1|\|_{L^{1/H}}^2 = C_H n^{2H} \left(\int_0^1 s^{\frac{n}{H}} |B_s - B_1|^{\frac{1}{H}} ds \right)^{2H}. \end{aligned}$$

By Minkowski's inequality, the expectation of this quantity is estimated as follows

$$\begin{aligned} & C_H n^{2H} \left\| \int_0^1 s^{\frac{n}{H}} |B_s - B_1|^{\frac{1}{H}} ds \right\|_{L^{2H}(\Omega)}^{2H} \le C_H n^{2H} \left(\int_0^1 s^{\frac{n}{H}} \left\| |B_s - B_1|^{\frac{1}{H}} \right\|_{L^{2H}(\Omega)} ds \right)^{2H} \\ \le \quad & C_H n^{2H} \left(\int_0^1 s^{\frac{n}{H}} (1-s) ds \right)^{2H} \le C'_H n^{-2H}. \end{aligned}$$

Therefore, it suffices to consider the term

$$\tilde{A}_{n,1} = 2B_1 H(2H-1) n^{2H} \int_0^1 \int_0^1 t^n s^n (B_s - B_1) |t-s|^{2H-2} ds dt =: B_1 A_{n,3},$$

and to show that $n^H A_{n,3}$ converges in law to a Gaussian random variable with mean zero and variance $\sigma_{H,1}^2$ independent of $\{B_t, t \in [0,1]\}$. This is a consequence of the following two facts:

(i) $E(n^{2H} A_{n,3}^2) \to \sigma_{H,1}^2$.
(ii) $E(n^H A_{n,3} B_t) \to 0$, for any $t \in [0,1]$.

The proof of (i) is based on the computation of the limit of the following quantity

$$\begin{aligned} & 4H^2(2H-1)^2 n^{6H} \int_{[0,1]^4} s_1^n t_1^n s_2^n t_2^n \\ & \quad \times E[(B_{t_1} - B_1)(B_{t_2} - B_1)] |t_1 - s_1|^{2H-2} |t_2 - s_2|^{2H-2} ds_1 ds_2 dt_1 dt_2 \\ & = 2H^2(2H-1)^2 n^{6H} \int_{[0,1]^4} s_1^n t_1^n s_2^n t_2^n \\ & \quad \times (|1-t_1|^{2H} + |1-t_2|^{2H} - |t_1 - t_2|^{2H}) |t_1 - s_1|^{2H-2} |t_2 - s_2|^{2H-2} ds_1 ds_2 dt_1 dt_2. \end{aligned}$$

This limit can be evaluated using the change of variables $s_1^{n+1} = x_1$, $s_2^{n+1} = x_2$, $t_1^{n+1} = y_1$ and $t_2^{n+1} = y_2$, which leads to the representation (4.10) of $\sigma_{H,1}^2$. The proof of (ii) can be done in a similar way. This concludes the proof of (4.12).

For (4.13), we can write

$$\begin{aligned} \Phi_n &= H(2H-1)n^{2H} \int_0^1 \int_0^1 t^n B_t \left(\int_s^1 \theta^n dB_\theta \right) |t-s|^{2H-2} ds dt \\ &= H(2H-1)n^{2H} \int_0^1 \int_0^1 t^n (B_t - B_1) \left(\int_s^1 \theta^n dB_\theta \right) |t-s|^{2H-2} ds dt \\ & \qquad + H(2H-1) B_1 n^{2H} \int_0^1 \int_0^1 t^n \left(\int_s^1 \theta^n dB_\theta \right) |t-s|^{2H-2} ds dt \\ &=: \Phi_{n,1} + \Phi_{n,2}. \end{aligned} \tag{4.15}$$

We first show that $\Phi_{n,1}$ does not contribute to the limit:

$$\begin{aligned} |\Phi_{n,1}| &= n^{2H}\left\langle t^n(B_t-B_1), \int_\cdot^1 \theta^n dB_\theta\right\rangle_{\mathfrak{H}} \\ &\le n^{2H}\|t^n(B_t-B_1)\|_{\mathfrak{H}}\left\|\int_\cdot^1 \theta^n dB_\theta\right\|_{\mathfrak{H}} \\ &\le C_H n^{2H}\|t^n(B_t-B_1)\|_{L^{\frac{1}{H}}}\left\|\int_\cdot^1 \theta^n dB_\theta\right\|_{L^{\frac{1}{H}}} \\ &= C_H n^{2H}\left[\int_0^1 t^{\frac{n}{H}}|B_t-B_1|^{\frac{1}{H}}dt\int_0^1 |\int_s^1 \theta^n dB_\theta|^{\frac{1}{H}}ds\right]^H. \end{aligned}$$

Then, taking expectation and using Minkowski's inequality, we get

$$\begin{aligned} E[|\Phi_{n,1}|^2] &\le C_H n^{4H}\left\|\int_0^1 t^{\frac{n}{H}}|B_t-B_1|^{\frac{1}{H}}dt\right\|_{L^{4H}(\Omega)}^{2H}\left\|\int_0^1\left|\int_s^1 \theta^n dB_\theta\right|^{\frac{1}{H}}ds\right\|_{L^{4H}(\Omega)}^{2H} \\ &\le C_H n^{4H}\left(\int_0^1 t^{\frac{n}{H}}(1-t)dt\right)^{2H}\left(\int_0^1\left\|\int_s^1 \theta^n dB_\theta\right\|_{L^4(\Omega)}^{\frac{1}{H}}ds\right)^{2H} \\ &\le C_H\left(\int_0^1 \|\theta^n\mathbf{1}_{[s,1]}\|_{\mathfrak{H}}^{\frac{1}{H}}ds\right)^{2H} \\ &\le C_H\left(\|\theta^n\|_{L^{\frac{1}{H}}}^{\frac{1}{H}}\right)^{2H} \\ &\le C_H n^{-2H}, \end{aligned}$$

and $n^{1-H}\|\Phi_{n,1}\|_2$ converges to zero as n tends to infinity. Finally, it suffices to consider the term

$$\Phi_{n,2} = B_1\tilde{\Phi}_{n,2},$$

where

$$\tilde{\Phi}_{n,2} = H(2H-1)n^{2H}\int_0^1\int_0^1 t^n\left(\int_s^1 \theta^n dB_\theta\right)|t-s|^{2H-2}dsdt. \tag{4.16}$$

We claim that $n^{1-H}\tilde{\Phi}_{n,2}$ converges in law to a Gaussian random variable with zero mean and variance $\sigma_{H,2}^2$ independent of $\{B_t, t\in[0,1]\}$. This is a consequence of the following two facts:

(i) $E(n^{2-2H}\tilde{\Phi}_{n,2}^2)\to\sigma_{H,2}^2$.

(ii) $E(n^{1-H}\tilde{\Phi}_{n,2}B_t)\to 0$, for any $t\in[0,1]$.

We first show (i):

$$\begin{aligned} E(n^{2-2H}\tilde{\Phi}_{n,2}^2) &= H^2(2H-1)^2 n^{2+2H}\int_{[0,1]^4} t_1^n t_2^n E\left[\left(\int_{s_1}^1 \theta^n dB_\theta\right)\left(\int_{s_2}^1 \theta^n dB_\theta\right)\right] \\ &\quad\times|t_1-s_1|^{2H-2}|t_2-s_2|^{2H-2}ds_1dt_1ds_2dt_2 \\ &= H^3(2H-1)^3 n^{2+2H}\int_{[0,1]^4} t_1^n t_2^n\int_{s_1}^1\int_{s_2}^1 \theta_1^n\theta_2^n|\theta_1-\theta_2|^{2H-2}d\theta_1d\theta_2 \\ &\quad\times|t_1-s_1|^{2H-2}|t_2-s_2|^{2H-2}ds_1dt_1ds_2dt_2. \end{aligned}$$

Using the change of variables $t_1^{n+1}=y_1$, $t_2^{n+1}=y_2$, $\theta_1^{n+1}=x_1$ and $\theta_2^{n+1}=x_2$, we can show that this quantity converges to $\sigma_{H,2}^2$ given in (4.11). The proof of (ii) can be done in a similar way. □

In spite of the preceding proposition, the quasi tangent does not contribute to the asymptotic expansion derived in the last section. In fact, the convergence (4.14), together with uniform integrability, gives $\lim_{n\to\infty} E[\Psi(\mathsf{z})\,\mathsf{qTan}] = 0$, that is, $\mathfrak{S}_0^{(2,0)} = 0$. More strongly, using the duality relationship between the Skorohod integral and the derivative operator (IBP formula), we can show this fact directly for $\Phi_{n,2}$ as follows:

$$
\begin{aligned}
& n^{1-H}\left| E\left[\Psi(\mathsf{z}) n^{2H} B_1 \int_0^1 \int_0^1 t^n \left(\int_s^1 \theta^n dB_\theta\right) |t-s|^{2H-2} ds dt\right] \right| \\
&= n^{1+H}\left| E\left[\int_0^1 \int_0^1 t^n |t-s|^{2H-2} \left\langle D_\theta \left(B_1 \exp(-\frac{1}{2}\mathsf{z}^2 c_H^2 B_1^2)\right), \mathbf{1}_{[s,1]}(\theta)\theta^n \right\rangle_{\mathfrak{H}} ds dt\right] \right| \\
&\le C n^{H-1}.
\end{aligned}
$$

By (4.14), qTan never converges to zero in probability. Thus $D\mathsf{qTan}$ potentially has some effect at the rate n^{1-H}.

4.2.2 Quasi torsion

By (4.9), we have

$$
\begin{aligned}
\mathsf{qTor} &= n^{1-H}\big\langle D\langle DZ_n, u_n\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}} \\
&= n^{1-H}\big(\big\langle D\big(\|u_n\|_{\mathfrak{H}}^2 - c_H^2 B_1^2\big), u_n\big\rangle_{\mathfrak{H}} + \langle D\Phi_n, u_n\rangle_{\mathfrak{H}} + \langle DG_\infty, u_n\rangle_{\mathfrak{H}}\big).
\end{aligned}
$$

Notice that

$$
\|\|u_n\|_{\mathfrak{H}}\|_2 \quad \le \quad C_H n^H \|\|t^n B_t\|_{L^{1/H}}\|_2 \le C'_H,
$$

which implies that $\|\|u_n\|_{\mathfrak{H}}\|_p$ is uniformly bounded for any $p \ge 2$. On the other hand, the computations in the previous section imply $\|\|D\left(\|u_n\|_{\mathfrak{H}}^2 - c_H^2 B_1^2\right)\|_{\mathfrak{H}}\|_p = O(n^{-H})$ for any $p \ge 2$. Therefore, $\|n^{1-H}\big\langle D\big(\|u_n\|_{\mathfrak{H}}^2 - c_H^2 B_1^2\big), u_n\big\rangle_{\mathfrak{H}}\|_p = O(n^{1-2H})$ and this term does not contribute to the limit.

Consider the term $\langle D\Phi_n, u_n\rangle_{\mathfrak{H}}$. Using the decomposition (4.15), we can write

$$
\langle D\Phi_n, u_n\rangle_{\mathfrak{H}} = \langle D\Phi_{n,1}, u_n\rangle_{\mathfrak{H}} + \langle D\Phi_{n,2}, u_n\rangle_{\mathfrak{H}}.
$$

The term $\langle D\Phi_{n,1}, u_n\rangle_{\mathfrak{H}}$ does not contribute to the limit since $\Phi_{n,1}$ is in the second chaos and $\|n^{1-H}\Phi_{n,1}\|_2 \to 0$. As for $\langle D\Phi_{n,2}, u_n\rangle_{\mathfrak{H}}$, we can write

$$
\langle D\Phi_{n,2}, u_n\rangle_{\mathfrak{H}} = \langle DB_1, u_n\rangle_{\mathfrak{H}}\tilde{\Phi}_{n,2} + B_1\langle D\tilde{\Phi}_{n,2}, u_n\rangle_{\mathfrak{H}},
$$

where $\tilde{\Phi}_{n,2}$ is defined in (4.16). The term $\langle DB_1, u_n\rangle_{\mathfrak{H}}\tilde{\Phi}_{n,2}$ does not contribute to the limit at the rate n^{1-H} in L^p, $p \ge 2$ due to the computations in the previous section. On the other hand, for the second term we can write

$$
\begin{aligned}
n^{1-H}B_1\langle D\tilde{\Phi}_{n,2}, u_n\rangle_{\mathfrak{H}} &= n^{1+H}H(2H-1)B_1\int_0^1\int_0^1 t^n|t-s|^{2H-2}\langle u_n(\xi), \theta^n\mathbf{1}_{[s,1]}(\theta)\rangle_{\mathfrak{H}} ds dt \\
&= n^{1+2H}H^2(2H-1)^2 B_1 \\
&\quad \times \int_{[0,1]^4} t^n|t-s|^{2H-2}B_\xi \xi^n\theta^n|\xi-\theta|^{2H-2}\mathbf{1}_{[s,1]}(\theta) d\xi d\theta ds dt \\
&= C_n B_1^2 + H^2(2H-1)^2\Delta_n,
\end{aligned}
$$

where

$$
C_n = n^{1+2H}H^2(2H-1)^2\int_{[0,1]^4} t^n|t-s|^{2H-2}\xi^n\theta^n|\xi-\theta|^{2H-2}\mathbf{1}_{[s,1]}(\theta) d\xi d\theta ds dt,
$$

and

$$\Delta_n = n^{1+2H} B_1 \int_{[0,1]^4} t^n |t-s|^{2H-2} (B_\xi - B_1) \xi^n \theta^n |\xi - \theta|^{2H-2} \mathbf{1}_{[s,1]}(\theta) d\xi d\theta ds dt.$$

With the change of variables $t^{n+1} = x$, $\theta^{n+1} = y$, $\xi^{n+1} = z$, we obtain

$$\begin{aligned} \lim_{n\to\infty} C_n &= H^2(2H-1) \int_{[0,1]^2} |\log y - \log z|^{2H-2} dy dz \\ &= H^2(2H-1)\Gamma(2H-1) \;=\; c_H^2 H. \end{aligned}$$

On the other hand, it is easy to check that $\|\Delta_n\|_p \le Cn^{-1}$, so this term does not contribute to the limit. In conclusion, taking into account (4.6), the quasi torsion $\mathsf{qTor} = n^{1-H}\langle D\langle DZ_n, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}}$ converges in L^p to $3c_H^2 H B_1^2$ for all $p \ge 2$. In other words, $\mathfrak{S}^{(3,0)} = c_H^2 H B_1^2$ for $H > 1/2$, which is discontinuous at $H = 1/2$. In this way, we obtain the expansion

$$E[f(Z_n)] \;=\; E[f(c_H|B_1|\zeta)] + n^{H-1} E[c_H^2 H B_1^2 f^{(3)}(c_H|B_1|\zeta)] + \rho_n^{(1)}(f), \qquad (4.17)$$

for $f \in C_b^7(\mathbb{R})$, where $\zeta \sim N(0,1)$ is independent of B_1, Again, from the computations in Sections 4.2.1 and 4.2.2, we deduce that conditions (3.8), (3.9), (3.10), (3.11) and (3.12) are satisfied for all $p \ge 2$. Thus, taking $\psi_n = 1$, assumption [B] holds and by Theorem 3.3 $\rho_n^{(1)}(f) = o(n^{H-1})$.

4.3 Fractional Brownian motion. Case $\frac{1}{4} < H < \frac{1}{2}$

Let $c_{n,H} = \frac{\Gamma(2H+1)\Gamma(n)}{\Gamma(n+2H+1)}$. We need the following preliminary result.

Lemma 4.4. The norm $\|t^n\|_{\mathfrak{H}}^2$ is given by

$$\|t^n\|_{\mathfrak{H}}^2 = \frac{n^2 + 2nH}{2n + 2H} c_{n,H}.$$

Proof. We can write

$$\begin{aligned} \|t^n\|_{\mathfrak{H}}^2 &= E\left[\left(\int_0^1 t^n dB_t\right)^2\right] \\ &= E\left[\left(B_1 - \int_0^1 n t^{n-1} B_t dt\right)^2\right] \\ &= 1 - 2\left(\frac{n+H}{n+2H} - \frac{n}{2} c_{n,H}\right) + n^2 \int_0^1 \int_0^1 t^{n-1} s^{n-1} E(B_t B_s) ds dt. \end{aligned}$$

We have

$$\begin{aligned} n^2 \int_0^1 \int_0^1 t^{n-1} s^{n-1} E(B_t B_s) ds dt &= \frac{n^2}{2} \int_0^1 \int_0^1 t^{n-1} s^{n-1} (t^{2H} + s^{2H} - |t-s|^{2H}) ds dt \\ &= \frac{n}{n+2H} - \frac{n^2}{2n+2H} c_{n,H} \end{aligned}$$

Therefore

$$\|t^n\|_{\mathfrak{H}}^2 = \frac{n^2 + 2nH}{2n + 2H} c_{n,H} = \frac{n}{2(n+H)} \frac{\Gamma(2H+1)\Gamma(n)}{\Gamma(n+2H)}. \qquad \square$$

As a consequence,

$$\lim_{n\to\infty} n^{2H} \|t^n\|_{\mathfrak{H}}^2 \;=\; \frac{1}{2}\Gamma(2H+1) = c_H^2,$$

where c_H is the constant introduced in (4.4).

4.3.1 Quasi tangent

Recall that $r_n = n^{-H}$ and the quasi tangent is defined by

$$\mathsf{qTan} = n^H G_n^{(2)} = n^H \left(\langle DZ_n, u_n \rangle_{\mathfrak{H}} - G_\infty \right).$$

We know that

$$\langle DZ_n, u_n \rangle_{\mathfrak{H}} = \|u_n\|_{\mathfrak{H}}^2 + n^{2H} \langle t^n B_t, \int_t^1 s^n dB_s \rangle_{\mathfrak{H}}. \tag{4.18}$$

The inner product in the Hilbert space $\mathfrak{H}$ is more involved than in the case $H > \frac{1}{2}$, and it is convenient to rewrite the stochastic integral $\int_t^1 s^n dB_s$ using integration by parts:

$$\int_t^1 s^n dB_s = B_1 - t^n B_t - n \int_t^1 B_s s^{n-1} ds. \tag{4.19}$$

Substituting (4.19) into (4.18) yields

$$\begin{aligned}
\langle DZ_n, u_n \rangle_{\mathfrak{H}} &= \|u_n\|_{\mathfrak{H}}^2 + n^{2H} \langle t^n B_t, B_1 - t^n B_t - n \int_t^1 B_s s^{n-1} ds \rangle_{\mathfrak{H}} \\
&= n^{2H} \langle t^n B_t, B_1 - n \int_t^1 B_s s^{n-1} ds \rangle_{\mathfrak{H}} \\
&= n^{2H} B_1 \langle t^n B_t, 1 \rangle_{\mathfrak{H}} - n^{2H+1} \int_0^1 \langle t^n B_t, \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} B_s ds.
\end{aligned}$$

Putting $B_t = (B_t - B_1) + B_1$ and $B_t B_s = (B_t - B_1)(B_s - B_1) + B_1(B_t - B_1) + B_1(B_s - B_1) + B_1^2$, we obtain

$$\begin{aligned}
\langle DZ_n, u_n \rangle_{\mathfrak{H}} &= n^{2H} B_1^2 \left(\langle t^n, 1 \rangle_{\mathfrak{H}} - n \int_0^1 \langle t^n, \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} ds \right) \\
&\quad + n^{2H} B_1 \langle t^n (B_t - B_1), 1 \rangle_{\mathfrak{H}} \\
&\quad - n^{2H+1} B_1 \int_0^1 \langle t^n (B_t - B_1), \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} ds \\
&\quad - n^{2H+1} B_1 \int_0^1 \langle t^n, \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} (B_s - B_1) ds \\
&\quad - n^{2H+1} \int_0^1 \langle t^n (B_t - B_1), \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} (B_s - B_1) ds.
\end{aligned}$$

Actually, we can combine the second and third terms and last two terms as follows:

$$\begin{aligned}
\langle DZ_n, u_n \rangle_{\mathfrak{H}} &= n^{2H} B_1^2 \left(\langle t^n, 1 \rangle_{\mathfrak{H}} - n \int_0^1 \langle t^n, \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} ds \right) \\
&\quad + n^{2H} B_1 \langle t^n (B_t - B_1), t^n \rangle_{\mathfrak{H}} \\
&\quad - n^{2H+1} \int_0^1 \langle t^n B_t, \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} (B_s - B_1) ds \\
&=: \sum_{i=1}^3 A_{i,n}.
\end{aligned}$$

The dominant term in the limit will be $A_{1,n}$, which can be expressed as

$$A_{1,n} = n^{2H} B_1^2 \left(\langle t^n, 1 \rangle_{\mathfrak{H}} - n \int_0^1 \langle t^n, \mathbf{1}_{[0,s]}(t) \rangle_{\mathfrak{H}} s^{n-1} ds \right) = n^{2H} B_1^2 \|t^n\|_{\mathfrak{H}}^2,$$

and converges to $G_\infty = H\Gamma(2H)B_1^2$ as n tends to infinity by Lemma 4.4. It is not difficult to check, using formulas (4.2) and (4.3), that the other two terms converge to zero in L^2 as n tends to infinity.

We are going to show that the quasi tangent does not contribute to the asymptotic expansion. From the preceding computations, we deduce

$$\mathsf{qTan} = n^H B_1^2[n^{2H}\|t^n\|_{\mathfrak{H}}^2 - c_H^2] + \sum_{i=2}^{3} n^H A_{i,n}.$$

We examine each term of this expression as follows:

(i) For the first term, by Lemma 4.4, we have

$$\tilde{A}_{1,n} := n^H B_1^2[n^{2H}\|t^n\|_{\mathfrak{H}}^2 - c_H^2] = B_1^2 n^H \frac{1}{2}\Gamma(2H+1)\left[\frac{n^{2H+1}}{(n+H)}\frac{\Gamma(n)}{\Gamma(n+2H)} - \mathbf{1}\right],$$

which converges to zero.

(ii) For the second term, by Lemma 4.1 (ii), we have

$$\begin{aligned} n^H A_{2.n} &= n^{3H} B_1 \langle t^n(B_t - B_1), t^n\rangle_{\mathfrak{H}} \\ &= n^{3H} H B_1 \int_0^1 t^n (B_t - B_1)(1-t)^{2H-1} dt \\ &\quad + n^{3H} H B_1 \int_0^1 t^n (B_t - B_1) \int_0^1 |t-s|^{2H-1}\mathrm{sign}(t-s) n s^{n-1} ds dt. \end{aligned} \tag{4.20}$$

We claim that

$$\lim_{n\to\infty} n^H E[\Psi(z) A_{2.n}] = 0. \tag{4.21}$$

In fact, integrating by parts, the factor $B_t - B_1$ produces a term of the form $|t-1|^{2H}$ due to Lemma 4.1 (iii), and then we have

$$\int_0^1 t^n (1-t)^{4H-1} dt \lesssim n^{-4H},$$

hence the first term on the right-hand side of (4.20) converges to 0. For two sequences of numbers a_n and b_n, $a_n \lesssim b_n$ means that there exists a positive constant C independent of n such that $a_n \le C b_n$ for large $n \in \mathbb{N}$. For the second term we apply the integration-by-parts formula to $B_t - B_1$ as well as Lemma 4.1 (iii) and Lemma 4.5 below to obtain the bound

$$\int_0^1 t^n (1-t)^{2H} \int_0^1 |t-s|^{2H-1} n s^{n-1} ds dt \quad = \quad O(n^{-4H}).$$

Therefore the second term on the right-hand side of (4.20) converges to 0, which proves (4.21).

Lemma 4.5. Let $\alpha, \beta, \mu \in (-1,\infty)$ and $\nu \in [0,\infty)$. Let

$$B(\alpha,\beta,\mu,\nu) \quad = \quad B(\mu+\nu+\beta+2, \alpha+1)B(\beta+1,\nu+1) + \frac{1}{\beta+1}B(\mu+1,\alpha+\beta+2).$$

Then

(i) $\int_0^1 \int_0^1 t^\mu (1-t)^\alpha |t-s|^\beta s^\nu dsdt \le B(\alpha,\beta,\mu,\nu)$.

(ii) For fixed α and β, it holds that

$$B(\alpha,\beta,\mu,\nu) \quad \sim \quad \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\mu+\nu)^{\alpha+1}\nu^{\beta+1}} + \frac{\Gamma(\alpha+\beta+2)}{(\beta+1)\mu^{\alpha+\beta+2}}$$

as $\mu,\nu \to \infty$.

Proof. First,

$$\begin{aligned}\int_0^1 t^\mu (1-t)^\alpha \int_0^t (t-s)^\beta s^\nu dsdt &= \int_0^1 t^{\mu+\nu+\beta+1}(1-t)^\alpha dt \int_0^1 (1-v)^\beta v^\nu dv \\ &= B(\mu+\nu+\beta+2,\alpha+1)B(\beta+1,\nu+1).\end{aligned}$$

Next,

$$\begin{aligned}\int_0^1 t^\mu (1-t)^\alpha \int_t^1 (s-t)^\beta s^\nu dsdt &= \int_0^1 t^\mu (1-t)^\alpha \Big\{ \frac{(1-t)^{\beta+1}}{\beta+1} - \int_t^1 \frac{(s-t)^{\beta+1}}{\beta+1} \nu s^{\nu-1} ds \Big\} dt \\ &\le \frac{1}{\beta+1}\int_0^1 t^\mu (1-t)^{\alpha+\beta+1} dt \\ &= \frac{1}{\beta+1} B(\mu+1,\alpha+\beta+2).\end{aligned}$$

Property (i) follows from these inequalities and (ii) is obvious. □

(iii) The third term also does not produce contribution. By Lemma 4.1 (i) and Lemma 4.1 (iii) after integration-by-parts in $B_s - B_1$, we estimate $n^H E[\Psi(z)A_{3,n}]$ by

$$\begin{aligned} & Cn^{3H+1}\int_0^1 \int_0^1 t^n (t^{2H-1} + |t-s|^{2H-1}) s^{n-1}(1-s)^{2H} dsdt \\ \lesssim\ & n^{3H+1}\big\{B(2H,0,n-1,n+2H-1) + B(2H,2H-1,n-1,n)\big\} \\ \lesssim\ & n^{H-1} + n^{-H}.\end{aligned}$$

In this way, we have proved that qTan has no contribution in the limit, that is, $\mathfrak{S}_0^{(2,0)} = 0$.

4.3.2 Quasi torsion

The quasi torsion can be written as

$$\mathsf{qTor} = n^H \langle D\langle DZ_n, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}} = n^H(\langle DG_n^{(2)}, u_n\rangle_{\mathfrak{H}} + \langle DG_\infty, u_n\rangle_{\mathfrak{H}}).$$

Let us show that $n^H\langle DG_n^{(2)}, u_n\rangle_{\mathfrak{H}}$ does not contribute to the asymptotic expansion. First,

$$\big\|\langle D\tilde{A}_{1,n}, u_n\rangle_{\mathfrak{H}}\big\|_p \quad = \quad \big\|\langle D(B_1^2), u_n\rangle_{\mathfrak{H}}\big\|_p \times o(1) \ = \ o(1)$$

for $p \ge 2$. We have, uniformly in s,

$$\big\|\big\langle DB_s, u_n\big\rangle_{\mathfrak{H}}\big\|_p \quad = \quad \Big\| n^H \int_0^1 HB_t t^n \{t^{2H-1} - |t-s|^{2H-1}\mathrm{sign}(t-s)\} dt \Big\|_p \ = \ O(n^{H-1}). \tag{4.22}$$

Therefore by (4.20), $\left\|\langle n^H DA_{2,n}, u_n\rangle_{\mathfrak{H}}\right\|_p = O(n^{4H-2}) = o(1)$. For the term $A_{3,n}$ we can write for any $p \geq 2$, using Lemma 4.1 (i), (4.22) and Lemma 4.5,

$$\begin{aligned}\left\|n^H\langle DA_{3,n}, u_n\rangle_{\mathfrak{H}}\right\|_p &\leq C \sup_{s\in[0,t]}\left\|\langle DB_s, u_n\rangle_{\mathfrak{H}}\right\|_p \\ &\quad \times n^{3H+1}\int_0^1\int_0^1 t^n(t^{2H-1}+|t-s|^{2H-1})s^{n-1}dsdt \\ &= O(n^{4H-2})+O(n^{2H-1}) = o(1).\end{aligned}$$

Thus we have proved that $\left\|n^H\langle DG_n^{(2)}, u_n\rangle_{\mathfrak{H}}\right\|_p = o(1)$.

Therefore, taking into account (4.7), the quasi torsion qTor converges in L^p to $2H\Gamma(2H)c_H^2B_1^2$ for all $p \geq 2$. In other words, $\mathfrak{S}^{(3,0)} = \frac{2}{3}H\Gamma(2H)c_H^2B_1^2$ for $H < 1/2$.

4.3.3 Asymptotic expansion

In this way, we obtain the expansion

$$E[f(Z_n)] = E\big[f(c_H|B_1|\zeta)\big] + n^{-H}E\left[\frac{2}{3}Hc_H^2\Gamma(2H)B_1^2f^{(3)}(c_H|B_1|\zeta)\right] + \rho_n^{(1)}(f), \tag{4.23}$$

for $f \in C_b^3(\mathbb{R})$, where $\zeta \sim N(0,1)$ is independent of B_1, Again, from the computations in Sections 4.2.1 and 4.2.2, we deduce that conditions (3.8), (3.9), (3.10), (3.11) and (3.12) are satisfied for all $p \geq 2$. Thus, taking $\psi_n = 1$, assumption [B] holds and by Theorem 3.3 $\rho_n^{(1)}(f) = o(n^{-H})$.

5 Quadratic form of a Brownian motion with predictable weights

In this section, we consider a quadratic form of a Brownian motion with predictable weights and show that the asymptotic expansion formula for the Skorohod integral reproduces the results obtained in [31, 30].

5.1 Quadratic form with random weights and $\mathfrak{H}$-derivatives

For a one-dimensional standard Brownian motion $B = \{B_t, t \in [0,1]\}$, let

$$Z_n = \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^t dB_sdB_t = \sqrt{n}\sum_{j=1}^n 2^{-1}a_{t_{j-1}}\big\{(B_{t_j}-B_{t_{j-1}})^2 - n^{-1}\big\},$$

where $t_j = j/n$, $a_t = a(B_t)$ and a is an infinitely differentiable function with derivatives of moderate growth (g has moderate growth if $|g(x)| \leq c\exp(c|x|^\alpha)$ for some constant $c > 0$ and $0 \leq \alpha < 2$). That is, $Z_n = \delta(u_n)$ with

$$u_n(t) = \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}(B_t - B_{t_{j-1}})\mathbf{1}_{I_j}(t),$$

where $I_j = [t_{j-1}, t_j)$. In this situation $Z_n = M_n$ and we have $W_n = W_\infty = 0$, $N_n = 0$, $X_n = X_\infty = 0$ and $\psi = 1$. We make $r_n = n^{-1/2}$.

Let

$$q_j = (B_{t_j} - B_{t_{j-1}})^2 - n^{-1} = 2\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^t dB_sdB_t.$$

We have

$$\begin{aligned}\langle DZ_n, u_n\rangle_{\mathfrak{H}} &= \int_0^1 \left\{\sqrt{n}\sum_{j=1}^n 2^{-1}D_t a_{t_{j-1}}\, q_j + \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\mathbf{1}_{I_j}(t)\right\} \\ &\quad \times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \qquad (5.1)\end{aligned}$$

and

$$\begin{aligned}&D_s\langle DZ_n, u_n\rangle_{\mathfrak{H}} \\ &= \int_0^1 \left\{\sqrt{n}\sum_{j=1}^n 2^{-1}D_sD_t a_{t_{j-1}}\, q_j + \sqrt{n}\sum_{j=1}^n D_t a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\mathbf{1}_{I_j}(s)\right. \\ &\quad \left. +\sqrt{n}\sum_{j=1}^n D_s a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\mathbf{1}_{I_j}(t) + \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}\mathbf{1}_{I_j}(s)\mathbf{1}_{I_j}(t)\right\} \\ &\quad \times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \\ &\quad +\int_0^1 \left\{\sqrt{n}\sum_{j=1}^n 2^{-1}D_t a_{t_{j-1}}\, q_j + \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\mathbf{1}_{I_j}(t)\right\} \\ &\quad \times\left\{\sqrt{n}\sum_{k=1}^n D_s a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t) + \sqrt{n}\sum_{k=1}^n a_{t_{k-1}}\mathbf{1}_{[t_{k-1},t)}(s)\mathbf{1}_{I_k}(t)\right\}dt. \qquad (5.2)\end{aligned}$$

It is known that in this example, $G_\infty = \frac{1}{2}\int_0^1 a_t^2 dt$. Then,

$$D_sG_\infty = \int_0^1 (D_s a_t)a_t dt.$$

5.2 Quasi torsion

We shall study the asymptotic behavior of the eight terms appearing in the expression of $\langle D\langle DZ_n, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}}$ corresponding to (5.2).

(i) The first term is

$$\begin{aligned}\mathcal{I}_1 &= \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n D_sD_t a_{t_{j-1}} q_j \times \sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \\ &\quad \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s - B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.\end{aligned}$$

We investigate the rate of $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1]$. The factor $n^{1.5}$ comes from three $\sqrt{n}$. It suffices to consider the terms for which $k\vee\ell < j$; otherwise the term vanishes due to $D_sD_t a_{t_{j-1}}$. The number of terms in the sum $\sum_j$ is of order n^1. The number of terms in the sum $\sum_{k.\ell}$ for $k=\ell$ and $k\vee\ell<j$ is $O(n^1)$, and each $B_t - B_{t_{k-1}}$ ($= B_t - B_{t_{\ell-1}}$) or its $\mathfrak{H}$-derivative contributes $O(n^{-0.5})$ in L^p-norm. By the IBP formula for q_j we get a factor n^{-2}. So that the partial sum in $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1]$ for $k=\ell$ is $O(n^{-1.5})$ since both ds and dt-integrals give $O(n^{-1})$. For the partial sum in $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1]$ for $k\neq\ell$ is also $O(n^{-1.5})$, since the consecutive IBP formulas (i.e., duality) for $B_t - B_{t_{k-1}}$ and $B_s - B_{t_{\ell-1}}$ gives the rate $O(n^{-2})$. Thus, we obtain $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1] = O(n^{-1.5})$, or $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1] = O(n^{-1})$. This means $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1]$ is negligible in the expansion. Table 1 summarizes how the orders of the partial sums were obtained.

Table 1: $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_1]$

n	$\sum_j(k\vee\ell<j)$	$\sum_{k,\ell}(k=\ell)$	ds	dt	IBP(q_j)	$B_t-B_{t_{k-1}}$	$B_s-B_{t_{\ell-1}}$	order
1.5	1	1	-1	-1	-2	-0.5	-0.5	-1.5
		$\sum_{k,\ell}(k\neq\ell)$	ds	dt	IBP(q_j)	IBP($B_t-B_{t_{k-1}}$)	IBP($B_s-B_{t_{\ell-1}}$)	order
		2	-1	-1	-2	-1	-1	-1.5

(ii) The second term can be written as

$$\begin{aligned}\mathcal{I}_2 &= \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n D_t a_{t_{j-1}}(B_{t_j}-B_{t_{j-1}})\mathbf{1}_{I_j}(s)\\ &\quad\times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t-B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt\\ &\quad\times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s-B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.\end{aligned}$$

Only terms with $k<j=\ell$ remain due to the product $\mathbf{1}_{I_k}(t)D_t a_{t_{j-1}}\mathbf{1}_{I_j}(s)\mathbf{1}_{I_\ell}(s)$. Table 2 shows $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2]=O(n^{-0.5})$, as explained more precisely below.

Table 2: $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2]$

n	$\sum_{j,\ell}(j=\ell)$	$\sum_k(k<\ell)$	ds	dt	IBP($B_t-B_{t_{k-1}}$)	$(B_s-B_{t_{\ell-1}})(B_{t_j}-B_{t_{j-1}})$	order
1.5	1	1	-1	-1	-1	-1	-0.5

The contribution of $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2]$ is evaluated as follows. $A_n\equiv^a B_n$ means $A_n-B_n=o(1)$ as $n\to\infty$. By Itô's formula,

$$\begin{aligned}&(B_{t_j}-B_{t_{j-1}})(B_s-B_{t_{j-1}})\\ &\quad=(B_{t_j}-B_s)(B_s-B_{t_{j-1}})+2\int_{t_{j-1}}^s\int_{t_{j-1}}^t dB_r dB_t+(s-t_{j-1})\end{aligned}\tag{5.3}$$

for $s\in I_j$. As already mentioned, only the terms with $k<j=\ell$ contribute the result. Applying the IBP formula for the first two terms of the right-hand side of (5.3), we obtain

$$\begin{aligned}\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2] &= E\Big[\Psi(\mathsf{z},\mathsf{x})\int_0^1\int_0^1 n^2\sum_{j=1}^n a_{t_{j-1}}D_t a_{t_{j-1}}(B_{t_j}-B_{t_{j-1}})(B_s-B_{t_{j-1}})\mathbf{1}_{I_j}(s)\\ &\qquad\times\sum_{k:k<j}a_{t_{k-1}}(B_t-B_{t_{k-1}})\mathbf{1}_{I_k}(t)dtds\Big]\\ &\equiv^a E\Big[\Psi(\mathsf{z},\mathsf{x})\int_0^1\int_0^1 n^2\sum_{j=1}^n a_{t_{j-1}}D_t a_{t_{j-1}}(s-t_{j-1})\mathbf{1}_{I_j}(s)\\ &\qquad\times\sum_{k:k<j}a_{t_{k-1}}(B_t-B_{t_{k-1}})\mathbf{1}_{I_k}(t)dtds\Big].\end{aligned}$$

The IBP formula for $B_t - B_{t_{k-1}} = \delta(\mathbf{1}_{[t_{k-1},t]})$ yields

$$\begin{aligned}\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2] &\equiv^a \sum_j \sum_{k:k<j} \int_{s\in I_j}\int_{t\in I_k} n^2(s-t_{j-1}) \\ &\quad \times E\bigg[\int_{r\in[t_{k-1},t]} D_r\big\{\Psi(\mathsf{z},\mathsf{x})(D_t a_{t_{j-1}})a_{t_{j-1}}a_{t_{k-1}}\big\}dr\bigg]dtds \\ &\equiv^a \int_0^1 \frac{1}{2}\int_0^s \frac{1}{2}E\Big[D_t\big\{\Psi(\mathsf{z},\mathsf{x})(D_t a_s)a_s a_t\big\}\Big]dtds,\end{aligned}$$

where

$$D_t\big\{\Psi(\mathsf{z},\mathsf{x})(D_t a_s)a_s a_t\big\} = \lim_{r\uparrow t} D_r\big\{\Psi(\mathsf{z},\mathsf{x})(D_t a_s)a_s a_t\big\}.$$

Therefore,

$$\begin{aligned}\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2] &\equiv^a \frac{1}{4}\int_0^1\int_0^s E\Big[(D_t\Psi(\mathsf{z},\mathsf{x}))(D_t a_s)a_s a_t + \Psi(\mathsf{z},\mathsf{x})(D_t D_t a_s)a_s a_t \\ &\quad +\Psi(\mathsf{z},\mathsf{x})(D_t a_s)^2 a_t\Big]dtds. \qquad (5.4)\end{aligned}$$

(iii) The third term is given by

$$\begin{aligned}\mathcal{I}_3 &= \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n D_s a_{t_{j-1}}(B_{t_j}-B_{t_{j-1}})\mathbf{1}_{I_j}(t) \\ &\quad \times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \\ &\quad \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s - B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.\end{aligned}$$

By symmetry, it is easy to see $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_3] = \sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2]$, and hence the limit is the same as (5.4).

(iv) Consider now the fourth term given by

$$\begin{aligned}\mathcal{I}_4 &= \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}\mathbf{1}_{I_j}(s)\mathbf{1}_{I_j}(t)\times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \\ &\quad \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s - B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.\end{aligned}$$

Table 3 suggests that $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_4]$ remains.

Table 3: $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_4]$

n	$\sum_{j,k,\ell}(j=k=\ell)$	ds	dt	$(B_t - B_{t_{k-1}})(B_s - B_{t_{\ell-1}})$	order
1.5	1	−1	−1	−1	−0.5

The contribution of this term is given by

$$\begin{aligned}\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_4] &\equiv^a E\bigg[\Psi(\mathsf{z},\mathsf{x})\int_0^1\int_0^1 n^2\sum_{j=1}^n a_{t_{j-1}}^3\big((t\wedge s)-t_{j-1}\big)\mathbf{1}_{I_j}(s)\mathbf{1}_{I_j}(t)dtds\bigg] \\ &\equiv^a \frac{1}{3}\int_0^1 E[\Psi(\mathsf{z},\mathsf{x})a_t^3]dt.\end{aligned}$$

(v) The fifth term is

$$\mathcal{I}_5 = \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n 2^{-1}D_t a_{t_{j-1}} q_j \times \sqrt{n}\sum_{k=1}^n D_s a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s - B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.$$

Notice that $\mathcal{I}_5$ looks like $\mathcal{I}_1$ but they are slightly different from each other. Only the terms satisfying $\ell < k < j$ remain due to $D_t a_{t_{j-1}}$ and $D_s a_{t_{k-1}}$. According to Table 4, we

Table 4: $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_5]$

n	$\sum_{j,k,\ell}(\ell<k<j)$	ds	dt	IBP(q_j)	IBP($B_t - B_{t_{k-1}}$)	IBP($B_s - B_{t_{\ell-1}}$)	order
1.5	3	-1	-1	-2	-1	-1	-1.5

see $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_5] = O(n^{-1})$ and is negligible.

(vi) The sixth term is given by

$$\mathcal{I}_5 = \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n 2^{-1}D_t a_{t_{j-1}} q_j \times \sqrt{n}\sum_{k=1}^n a_{t_{k-1}}\mathbf{1}_{[t_{k-1},t]}(s)\mathbf{1}_{I_k}(t)dt \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s - B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.$$

Thanks to the product $\mathbf{1}_{[t_{k-1},t]}(s)\mathbf{1}_{I_k}(t)\mathbf{1}_{I_\ell}(s)D_t a_{t_{j-1}}$, only the terms satisfying $k = \ell < j$ remain. By Table 5 below $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_6] = O(n^{-1})$ and this term is negligible.

Table 5: $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_6]$

n	$\sum_j$	$\sum_{k,\ell}(\ell=k<j)$	ds	dt	IBP(q_j)	IBP($B_s - B_{t_{\ell-1}}$)	order
1.5	1	1	-1	-1	-2	-1	-1.5

(vii) Consider the seventh term given by

$$\mathcal{I}_7 = \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\mathbf{1}_{I_j}(t) \times\sqrt{n}\sum_{k=1}^n D_s a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s - B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.$$

Due to the product $\mathbf{1}_{I_\ell}(s)D_s a_{t_{k-1}}\mathbf{1}_{I_j}(t)\mathbf{1}_{I_k}(t)$, only the terms satisfying $\ell < k = j$ contribute to the sum. Then it turns out that $\mathcal{I}_7$ is the same as $\mathcal{I}_2$. Therefore $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_7] = \sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_2]$ and the limit is given by (5.4).

(viii) Finally, the last term is

$$\mathcal{I}_8 = \int_0^1\int_0^1 \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}(B_{t_j}-B_{t_{j-1}})\mathbf{1}_{I_j}(t)\times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}\mathbf{1}_{[t_{k-1},t]}(s)\mathbf{1}_{I_k}(t)dt \times\sqrt{n}\sum_{\ell=1}^n a_{t_{\ell-1}}(B_s-B_{t_{\ell-1}})\mathbf{1}_{I_\ell}(s)ds.$$

It suffices to consider the case $j=k=\ell$. Table 6 says that $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_8]$ contributes to the limit.

Table 6: $E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_8]$

n	$\sum_{j,k,\ell}(j=k=\ell)$	ds	dt	$(B_{t_j}-B_{t_{j-1}})(B_s-B_{t_{\ell-1}})$	order
1.5	1	-1	-1	-1	-0.5

More precisely, following a procedure quite similar to that of $\mathcal{I}_4$, we obtain

$$\begin{aligned}\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathcal{I}_8] &\equiv^a E\left[\Psi(\mathsf{z},\mathsf{x})\int_0^1\int_0^1 n^2\sum_{j=1}^n a_{t_{j-1}}^3\big(s-t_{j-1}\big)\mathbf{1}_{[t_{j-1},t]}(s)\mathbf{1}_{I_j}(t)dtds\right]\\ &\equiv^a \frac{1}{6}\int_0^1 E\big[\Psi(\mathsf{z},\mathsf{x})a_t^3\big]dt.\end{aligned}$$

Now, from (i)-(viii) and

$$\Psi(\mathsf{z},\mathsf{x}) = \exp\left(\frac{1}{4}\int_0^1 a_s^2ds(\mathtt{i}\mathsf{z})^2\right),$$

we obtain

$$\begin{aligned}&E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}^{(3,0)}(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big]\\ =\ &\lim_{n\to\infty}\frac{\sqrt{n}}{3}E\left[\Psi(\mathsf{z},\mathsf{x})\left\langle D\big\langle DM_n,u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\psi_n(\mathtt{i}\mathsf{z})^3\right]\\ =\ &\frac{1}{3}\left\{\frac{3}{4}\int_{s=0}^1\int_{t=0}^s E\Big[(D_t\Psi(\mathsf{z},\mathsf{x}))(D_ta_s)a_sa_t\right.\\ &+\Psi(\mathsf{z},\mathsf{x})(D_tD_ta_s)a_sa_t+\Psi(\mathsf{z},\mathsf{x})(D_ta_s)^2a_t\Big]dtds(\mathtt{i}\mathsf{z})^3\\ &\left.+\left(\frac{1}{3}+\frac{1}{6}\right)\int_0^1 E[\Psi(\mathsf{z},\mathsf{x})a_t^3]dt(\mathtt{i}\mathsf{z})^3\right\}\\ =\ &\frac{1}{8}E\left[\Psi(\mathsf{z},\mathsf{x})\int_0^1 a_t\left(\int_t^1(D_ta_s)a_sds\right)^2dt\right](\mathtt{i}\mathsf{z})^5\\ &+\frac{1}{4}E\left[\Psi(\mathsf{z},\mathsf{x})\int_0^1 a_t\int_t^1\big\{(D_tD_ta_s)a_s+(D_ta_s)^2\big\}dsdt\right](\mathtt{i}\mathsf{z})^3\\ &+\frac{1}{6}E\left[\Psi(\mathsf{z},\mathsf{x})\int_0^1 a_t^3dt\right](\mathtt{i}\mathsf{z})^3.\end{aligned}$$

It should be remarked that the three terms on the right-hand side of the above equation correspond to $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_1$ of [31], pp. 917–918, respectively. We remark that two random symbols with the same adjoint action are considered equivalent.

Obviously $\mathfrak{S}^{(2,0)} = \mathfrak{S}^{(1,1)} = \mathfrak{S}^{(1,0)} = \mathfrak{S}^{(0,1)} = 0$ in the present situation, and moreover we will show that $\mathfrak{S}_0^{(2,0)} = 0$ in Section 5.3. Consequently,

$$\begin{aligned}\mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix}) &= \frac{1}{8}\int_0^1 a_t\bigg(\int_t^1 (D_t a_s)a_s ds\bigg)^2 dt(\mathtt{iz})^5 \\ &\quad +\frac{1}{4}\int_0^1 a_t\int_t^1 \big\{(D_tD_ta_s)a_s + (D_ta_s)^2\big\}dsdt(\mathtt{iz})^3 + \frac{1}{6}\int_0^1 a_t^3 dt(\mathtt{iz})^3\end{aligned}$$

and

$$r_n^{-1}\big(\mathfrak{S}_n(\mathtt{iz},\mathtt{ix}) - 1\big) \quad = \quad \mathfrak{S}^{(3,0)}(\mathtt{iz},\mathtt{ix}).$$

This random symbol is equivalent to the full random symbol $\sigma(\mathtt{iz},\mathtt{ix})$ of [31], p. 918 with a replaced by $a/2$. For the quadratic form of a Brownian motion, $\mathfrak{S}^{(3,0)}$ provides both the adapted random symbol and the anticipative random symbol, in other words, the quasi torsion includes the tangent as well as the torsion. In this way, we found that the quasi torsion reproduces the asymptotic expansion of the quadratic form of a Brownian motion.

5.3 Quasi tangent

For the quasi tangent, we have

$$\begin{aligned}\langle DZ_n, u_n\rangle_{\mathfrak{H}} - G_\infty &= \int_0^1 \bigg\{\sqrt{n}\sum_{j=1}^n 2^{-1}D_t a_{t_{j-1}}q_j + \sqrt{n}\sum_{j=1}^n a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\mathbf{1}_{I_j}(t)\bigg\} \\ &\qquad \times\sqrt{n}\sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt - \frac{1}{2}\int_0^1 a_t^2 dt \\ &= \mathfrak{G}_1 + \mathfrak{G}_2 + \mathfrak{G}_3,\end{aligned}$$

where

$$\mathfrak{G}_1 = \int_0^1 n\sum_{j=1}^n 2^{-1}D_t a_{t_{j-1}}q_j \times \sum_{k=1}^n a_{t_{k-1}}(B_t - B_{t_{k-1}})\mathbf{1}_{I_k}(t)dt,$$

$$\mathfrak{G}_2 = \int_0^1 \bigg\{n\sum_{j=1}^n a_{t_{j-1}}^2(B_t - B_{t_{j-1}})^2\mathbf{1}_{I_j}(t) - 2^{-1}a_t^2\bigg\}dt,$$

and

$$\mathfrak{G}_3 = \int_0^1 n\sum_{j=1}^n a_{t_{j-1}}^2(B_{t_j} - B_t)(B_t - B_{t_{j-1}})\mathbf{1}_{I_j}(t)dt.$$

We shall investigate these terms.

(i) For $\mathfrak{G}_1$, Table 7 shows $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_1] = O(n^{-0.5})$ and it is negligible in the asymptotic expansion.

(ii) For $\mathfrak{G}_2$, applying Itô's formula, we have $\mathfrak{G}_2 = \mathfrak{G}_2' + \mathfrak{G}_2''$, where

$$\mathfrak{G}_2' = \sum_{j=1}^n \int_{I_j} na_{t_{j-1}}^2 \int_{t_{j-1}}^t \int_{t_{j-1}}^s 2dB_r dB_s\, dt$$

Table 7: $E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_1]$

n	$\sum_j$	$\sum_k (k<j)$	dt	IBP by q_j	IBP by $B_t - B_{t_{k-1}}$	order
1	1	1	-1	-2	-1	-1

and

$$\mathfrak{G}_2'' \;=\; \frac{1}{2}\sum_{j=1}^{n}\int_{I_j}\left(a_{t_{j-1}}^2 - a_t^2\right)\mathbf{1}_{I_j}(t)dt.$$

Table 8 shows $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_2'] = O(n^{-0.5})$ and it is negligible in the asymptotic expansion. We should remark that $\sqrt{n}\mathfrak{G}_2'$ is $\overset{\circ}{C}_n$ of [31] and it has non-trivial limit distribution though the expectation $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_2']$ asymptotically vanishes. We see $\mathfrak{G}_2''$ is of $O(n^{-1})$ in L^2, and it is also negligible. Consequently, $\mathfrak{G}_2$ is negligible in the asymptotic expansion.

Table 8: $E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_2']$

n	$\sum_j$	dt	IBP by $\int_{t_{j-1}}^t\int_{t_{j-1}}^s dB_r dB_s$	order
1	1	-1	-2	-1

(iii) Finally, for $\mathfrak{G}_3$ Table 9 shows $\sqrt{n}E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_3] = O(n^{-0.5})$ and we can neglect it.

Table 9: $E[\Psi(\mathsf{z},\mathsf{x})\mathfrak{G}_3]$

n	$\sum_j$	dt	IBP by $B_{t_j} - B_t$	IBP by $B_t - B_{t_{k-1}}$	order
1	1	-1	-1	-1	-1

As a consequence of these observations, $\mathfrak{S}_0^{(2,0)} = 0$, i.e., the quasi tangent has no effect in the asymptotic expansion. However, the effect of the tangent already appeared in that of the quasi torsion.

6 Quadratic form of a fractional Brownian motion with random weights

6.1 Weighted quadratic variation

Let $B = \{B_t, t \in [0,1]\}$ be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{3}{4})$. We are interested in the following sequence of weighted quadratic variations:

$$Z_n = n^{2H-\frac{1}{2}}\sum_{j=1}^{n} a_{t_{j-1}}((\Delta B_{j,n})^2 - n^{-2H}),$$

where $t_j = j/n$, $a_t = a(B_t)$ and a is a function such that a and all its derivatives up to some order N have moderate growth. We use the notation $\Delta B_{j,n} = B_{j/n} - B_{(j-1)/n}$. It is known (see, for instance [15, 14]) that for this example the limit variance G_∞ is given by

$$G_\infty = 2c_H^2\int_0^1 a(B_s)^2 ds,$$

where

$$c_H^2 \quad = \quad \sum_{k=-\infty}^{\infty} \rho_H(k)^2 \tag{6.1}$$

$$\rho_H(k) \quad = \quad \frac{1}{2}\big(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\big).$$

Set $I_j = [t_{j-1}, t_j)$. Then,

$$a_{t_{j-1}}(\Delta_{j,n}B)^2 = \delta(a_{t_{j-1}}\Delta B_{j,n}\mathbf{1}_{I_j}) + a_{t_{j-1}}n^{-2H} + \Delta B_{j,n}\langle Da_{t_{j-1}}, \mathbf{1}_{I_j}\rangle_{\mathfrak{H}}.$$

Therefore, we obtain the decomposition

$$Z_n = \delta(u_n) + r_n N_n =: M_n + r_n N_n,$$

where

$$u_n(t) = n^{2H-\frac{1}{2}} \sum_{j=1}^{n} a_{t_{j-1}}\Delta B_{j,n}\mathbf{1}_{I_j}(t)$$

and

$$r_n N_n = n^{2H-\frac{1}{2}} \sum_{j=1}^{n} \Delta B_{j,n}\langle Da_{t_{j-1}}, \mathbf{1}_{I_j}\rangle_{\mathfrak{H}}.$$

Set

$$\Psi(\mathsf{z}) = \exp\left(-\mathsf{z}^2 c_H^2 \int_0^1 a^2(B_s)ds\right). \tag{6.2}$$

In this example, we take $W_n = W_\infty = 0$, $X_n = X_\infty = 0$ and $\psi = 1$. We are going to study the quasi torsion and the quasi tangent of the Skorohod integral $M_n = \delta(u_n)$. In this example there will be also a contribution to the asymptotic expansion coming from the perturbation term N_n. The scaling factor r_n will be taken as

$$r_n \quad = \quad \begin{cases} n^{2H-\frac{3}{2}} & \text{when } H \in (\frac{1}{2}, \frac{3}{4}), \\ n^{\frac{1}{2}-2H} & \text{when } H \in \left(\frac{1}{4}, \frac{1}{2}\right). \end{cases}$$

This choice of r_n is motivated by the rate of convergence

$$|E[\varphi(Z_n) - E[\varphi(\zeta G_\infty^{1/2})]| \quad \leq \quad C_{a,H} \max_{1\leq i\leq 5} \|\varphi^{(i)}\|_\infty r_n$$

obtained in [14] for $\varphi \in C_b^5(\mathbb{R})$, where ζ is $N(0,1)$.

6.2 Quasi torsion

We recall that

$$\mathsf{qTor} \quad = \quad r_n^{-1}\langle D\langle DM_n, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}}.$$

Set $q_j = (\Delta B_{j,n})^2 - n^{-2H} = I_2(\mathbf{1}_{I_j}^{\otimes 2})$. We have

$$\begin{aligned} \langle DM_n, u_n\rangle_{\mathfrak{H}} \quad &= \quad n^{4H-1}\left\langle \sum_{j=1}^{n}\left[(Da_{t_{j-1}})q_j + 2a_{t_{j-1}}\Delta B_{j,n}\mathbf{1}_{I_j}\right], \sum_{k=1}^{n} a_{t_{k-1}}\Delta B_{k,n}\mathbf{1}_{I_k}\right\rangle_{\mathfrak{H}} \\ &\quad -r_n\langle DN_n, u_n\rangle_{\mathfrak{H}} \\ &=: \quad \Phi_{n,1} - r_n\langle DN_n, u_n\rangle_{\mathfrak{H}}. \end{aligned}$$

(A) We first study the contribution of the term $\langle D\Phi_{n,1}, u_n\rangle_{\mathfrak{H}}$ in the asymptotic expansion. We have

$$\begin{aligned}\langle D\Phi_{n,1}, u_n\rangle_{\mathfrak{H}} &= n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} \Big[(D_rD_sa_{t_{j-1}})q_j + 2(D_sa_{t_{j-1}})\Delta B_{j,n}\mathbf{1}_{I_j}(r) \\ &\qquad +2(D_ra_{t_{j-1}})\Delta B_{j,n}\mathbf{1}_{I_j}(s) + 2a_{t_{j-1}}\mathbf{1}_{I_j}(s)\mathbf{1}_{I_j}(r)\Big] \\ &\qquad *[a_{t_{k-1}}\Delta B_{k,n}\mathbf{1}_{I_k}(s)] * [a_{t_{\ell-1}}\Delta B_{\ell,n}\mathbf{1}_{I_\ell}(r)] \\ &\quad +n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} \Big[(D_sa_{t_{j-1}})q_j + 2a_{t_{j-1}}\Delta B_{j,n}\mathbf{1}_{I_j}(s)\Big] \\ &\qquad *\Big[(D_ra_{t_{k-1}})\Delta B_{k,n}\mathbf{1}_{I_k}(s) + a_{t_{k-1}}\mathbf{1}_{I_k}(r)\mathbf{1}_{I_k}(s)\Big] \\ &\qquad *[a_{t_{\ell-1}}\Delta B_{\ell,n}\mathbf{1}_{I_\ell}(r)],\end{aligned}$$

where the product $A * B$ means that whenever we found repeated variables in A and B, we compute the corresponding inner product in $\mathfrak{H}$. We have a total of eight terms, that we denote by

$$\langle D\Phi_{n,1}, u_n\rangle_{\mathfrak{H}} = \sum_{i=1}^{8} \mathcal{I}_i.$$

We are interested in the asymptotic behavior of $r_n^{-1}E[\Psi(\mathrm{z})\mathcal{I}_i]$ for $i = 1, \dots, 8$.

(i) The first term is

$$\mathcal{I}_1 = n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a''_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} q_j \Delta B_{k,n}\Delta B_{\ell,n}\alpha_{t_{j-1},k}\alpha_{t_{j-1},\ell},$$

where we have used the notation $\alpha_{t,k} = \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{I_k}\rangle_{\mathfrak{H}}$. We can make the decomposition

$$\Delta B_{k,n}\Delta B_{\ell,n} = I_2(\mathbf{1}_{I_k}\otimes \mathbf{1}_{I_\ell}) + \beta_{k,\ell},$$

where $\beta_{k,\ell} = \langle \mathbf{1}_{I_k}, \mathbf{1}_{I_\ell}\rangle_{\mathfrak{H}}$. Integrating by parts shows that the contribution of $I_2(\mathbf{1}_{I_k}\otimes \mathbf{1}_{I_\ell})$ is of order lower than that of $\beta_{k,\ell}$. In this way, it suffices to consider the term

$$\mathcal{I}_{1,0} = n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a''_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} q_j \beta_{k,\ell}\alpha_{t_{j-1},k}\alpha_{t_{j-1},\ell}.$$

In this case, the factors of the above expressions have the following orders of convergence:

- First factor: $n^{6H-\frac{3}{2}}$
- The IPB formula for q_j produces a factor $n^{-2(2H\wedge 1)}$, due to part (a) of Lemma 6.2 below.
- The terms $|\alpha_{t_{j-1},k}\alpha_{t_{j-1},\ell}|$ are bounded by $C_H n^{-2(2H\wedge 1)}$ by part (a) of Lemma 6.2 below.
- $\sum_{k,\ell=1}^{n} |\beta_{k,\ell}| \le C_H n^{(1-2H)\vee 0}$ due to part (c) of Lemma 6.2.
- Finally we get a factor n from the sum in j.

Therefore, the order of this term is $n^{6H-\frac{1}{2}-4(2H\wedge 1)+(1-2H)\vee 0}$. For $H > \frac{1}{2}$ this gives $n^{6H-\frac{9}{2}}$ and for $H < \frac{1}{2}$, we obtain the order $n^{\frac{1}{2}-4H}$. In both cases, when $H \to \frac{1}{2}$, we obtain $n^{-1.5}$ as in the Brownian motion case. We remark that $6H - \frac{9}{2} < 2H - \frac{3}{2}$ if $\frac{1}{2} < H < \frac{3}{4}$, and $\frac{1}{2} - 4H < \frac{1}{2} - 2H$ if $\frac{1}{4} < H < \frac{1}{2}$. Therefore, this term will not contribute to the limit.

(ii) The second term is equal to

$$\mathcal{I}_2 = 2n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a_{t_{k-1}} a'_{t_{j-1}} a_{t_{\ell-1}} \Delta B_{j,n} \Delta B_{k,n} \Delta B_{\ell,n} \beta_{j,\ell} \alpha_{t_{j-1},k},$$

where $\beta_{j,\ell} = \langle \mathbf{1}_{I_j}, \mathbf{1}_{I_\ell} \rangle_{\mathfrak{H}}$. As before, we can replace the product $\Delta B_{j,n} \Delta B_{\ell,n}$ by $\beta_{j,\ell}$, and we have to deal with the term

$$\mathcal{I}_{2,0} = 2n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a_{t_{k-1}} a'_{t_{j-1}} a_{t_{\ell-1}} \Delta B_{k,n} \beta_{j,\ell}^2 \alpha_{t_{j-1},k}.$$

We get the following contributions:

- First factor: $n^{6H-\frac{3}{2}}$
- The IPB formula for $\Delta B_{k,n}$ produces a factor $n^{-(2H\wedge 1)}$, due to part (a) of Lemma 6.2 below.
- $\sum_{k=1}^{n} |\alpha_{t_{j-1},k}| \leq C_H$ due to part (b) of Lemma 6.2, and $\sum_{j,\ell=1}^{n} \beta_{j,\ell}^2 \leq C_H n^{1-4H}$ by part (d) of Lemma 6.2.

Therefore, the order of this term is $n^{2H-\frac{1}{2}-(2H\wedge 1)} = n^{-[(\frac{3}{2}-2H)\wedge\frac{1}{2}]}$, which in the Brownian case gives $n^{-\frac{1}{2}}$. From this result we deduce that this term will not contribute to the asymptotic expansion if $H < \frac{1}{2}$. The contribution of $n^{\frac{3}{2}-2H} E[\Psi(\mathsf{z})\mathcal{I}_2]$ when $H > \frac{1}{2}$ is evaluated as follows.

Define a measure μ_n on $[0,1]^2$ by

$$\mu_n \quad = \quad \sum_{j,\ell=1}^{n} n^{-1+4H} \beta_{j,\ell}^2 \, \delta_{(t_{j-1},t_{\ell-1})}.$$

Then

$$\int_{[0,1]^2} \varphi(t,s)\mu_n(dt,ds) \quad \to \quad c_H^2 \int_{[0,1]} \varphi(t,t)dt \tag{6.3}$$

as $n \to \infty$ for every $\varphi \in C([0,1]^2)$, where c_H^2 is defined in (6.1). Notice that

$$\sup_{\tau\in[0,1],1\leq k\leq n} \left| \int_0^\tau \left(n \int_{I_k} |r-s|^{2H-2} dr - |t_{k-1}-s|^{2H-2} \right) ds \right| \leq C n^{1-2H} \to 0 \tag{6.4}$$

as n tends to infinity. We can write

$$\begin{aligned}
& n^{\frac{3}{2}-2H} E[\Psi(\mathsf{z})\mathcal{I}_2] \\
= \; & 2n^{4H} E\left[\Psi(\mathsf{z}) \sum_{j,k,\ell=1}^{n} a_{t_{k-1}} a'_{t_{j-1}} a_{t_{\ell-1}} \beta_{j,\ell}^2 \Delta B_{k,n} \alpha_{t_{j-1},k} \right] + o(1) \\
= \; & 2n \sum_{k=1}^{n} \int_{[0,1]^2} \mu_n(dt,dt') E\big[\big\langle D\{\Psi(\mathsf{z}) a_{t_{k-1}} a'_t a_{t'}\}, \mathbf{1}_{I_k} \big\rangle_{\mathfrak{H}} \big] \alpha_{t,k} + o(1) \\
= \; & 2\alpha_H^2 n^{-1} \sum_{k=1}^{n} \int_{[0,1]^2} \mu_n(dt,dt') E\left[\int_0^1 \int_{t_{k-1}}^{t_k} D_{s'}\{\Psi(\mathsf{z}) a_{t_{k-1}} a'_t a_{t'}\} |r-s'|^{2H-2} dr ds' \right] \\
& \times \int_{[0,1]} \mathbf{1}_{[0,t]}(s) |t_{k-1}-s|^{2H-2} ds + o(1)
\end{aligned}$$

Using (6.4) and (6.3), we obtain

$$\begin{aligned}
& n^{\frac{3}{2}-2H} E[\Psi(\mathsf{z})\mathcal{I}_2] \\
= \;& 2\alpha_H^2 \int_{[0,1]^2} \mu_n(dt,dt')\bigg\{ \int_{[0,1]} dr E\bigg[\int_{[0,1]} D_{s'}\{\Psi(\mathsf{z}) a_r a_t' a_{t'}\} |r-s'|^{2H-2} ds'\bigg] \\
& \times \int_{[0,1]} \mathbf{1}_{[0,t]}(s)|r-s|^{2H-2} ds\bigg\} + o(1) \\
\to \;& 2\alpha_H^2 c_H^2 \int_{[0,1]} dt \bigg\{ \int_{[0,1]} dr E\bigg[\int_{[0,1]} D_{s'}\{\Psi(\mathsf{z}) a_r a_t' a_t\} |r-s'|^{2H-2} ds'\bigg] \\
& \times \int_{[0,1]} \mathbf{1}_{[0,t]}(s)|r-s|^{2H-2} ds\bigg\} \\
= \;& C_{H,3} \int_{[0,1]^4} E\left[D_{s'}\left(\Psi(\mathsf{z}) a(B_r) D_s(a^2(B_t))\right)\right] |r-s|^{2H-2}|r-s'|^{2H-2} ds ds' dr dt,
\end{aligned}$$

where $\alpha_H = H(2H-1)$ and

$$C_{H,3} \quad = \quad \alpha_H^2 \sum_{j=-\infty}^{+\infty} \rho_H(j)^2.$$

(iii) By symmetriy, the third term is analogous to the second one and produces the same contribution.

(iv) The fourth term is given by

$$\mathcal{I}_4 = 2n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} \beta_{j,k}\beta_{j,\ell} \Delta B_{k,n} \Delta B_{\ell,n}.$$

We can make the decomposition

$$\Delta B_{k,n} \Delta B_{\ell,n} = I_2(\mathbf{1}_{I_k} \otimes \mathbf{1}_{I_\ell}) + \beta_{k,\ell}.$$

By integration by parts the contribution of $I_2(\mathbf{1}_{I_k} \otimes \mathbf{1}_{I_\ell})$ is of lower order than that of $\beta_{k,\ell}$. In this way, it suffices to consider the term

$$\mathcal{I}_{4,0} = 2n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} \beta_{j,k}\beta_{j,\ell}\beta_{k,\ell}.$$

From part (f) of Lemma 6.2, we see that this term is bounded in L^p by $n^{2H-\frac{3}{2}}$ if $H > \frac{1}{2}$ and by $n^{-\frac{1}{2}}$ if $H < \frac{1}{2}$. This clearly implies that

$$\lim_{n\to\infty} n^{\frac{1}{2}-2H} \mathcal{I}_4 \quad = \quad 0 \tag{6.5}$$

if $H < \frac{1}{2}$, in L^p for all $p \geq 2$. On the other hand, we claim that, if $H > \frac{1}{2}$ we also have the following convergence is L^p for all $p \geq 2$, and, as a consequence, this term produces no contribution.

$$\lim_{n\to\infty} n^{\frac{3}{2}-2H} \mathcal{I}_4 \quad = \quad 0. \tag{6.6}$$

Proof of (6.6): We need to show that

$$\lim_{n\to\infty} n^{4H} \sum_{j,k,\ell=1}^{n} a_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} \beta_{j,k}\beta_{j,\ell}\beta_{k,\ell} \quad = \quad 0$$

in L^p. We can write

$$\begin{aligned} & n^{4H} \sum_{j,k,\ell=1}^{n} a_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} \beta_{j,k} \beta_{j,\ell} \beta_{k,\ell} \\ & = n^{-2H} \sum_{j,k,\ell=1}^{n} a(B_{\frac{j-1}{n}}) a(B_{\frac{k-1}{n}}) a(B_{\frac{\ell-1}{n}}) \rho_H(j-k) \rho_H(j-\ell) \rho_H(k-\ell) \\ & = n^{-2H} \sum_{\substack{1\le j\le n \\ 1\le j+i\le n \\ 1\le j+h\le n}} a(B_{\frac{j-1}{n}}) a(B_{\frac{j+i-1}{n}}) a(B_{\frac{j+h-1}{n}}) \rho_H(i) \rho_H(h) \rho_H(i-h) \end{aligned}$$

and it suffices to show that

$$\lim_{n\to\infty} n^{1-2H} \sum_{j,h:1\le i<h\le n} \left|\rho_H(i)\rho_H(h)\rho_H(i-h)\right| \quad = \quad 0. \tag{6.7}$$

By the inequality $\sup_{i\ge 1} |\rho(i)|/i^{2H-2} < \infty$, we have

$$\begin{aligned} & n^{1-2H} \sum_{j,h:1\le i<h\le n} \left|\rho_H(i)\rho_H(h)\rho_H(i-h)\right| \\ \lesssim \quad & n^{1-2H} \sum_{j,h:1\le i<h\le n} i^{2H-2} h^{2H-2} (h-i)^{2H-2} \\ = \quad & n^{1-2H} \sum_{h=1}^{n} h^{6H-5} \sum_{i=1}^{h-1} (i/h)^{2H-2} (1-i/h)^{2H-2}/h \\ \lesssim \quad & B(2H-1, 2H-1) n^{1-2H+(6H-4)_+}. \end{aligned}$$

The last term is $O(n^{4H-3})$ when $2/3 \le H < 3/4$, and $O(n^{1-2H})$ when $1/2 < H < 2/3$. This gives (6.7) and hence (6.6).

(v) The fifth term is

$$\mathcal{I}_5 = n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a'_{t_{j-1}} a'_{t_{k-1}} a_{t_{\ell-1}} q_j \alpha_{t_{j-1},k} \alpha_{t_{k-1},\ell} \Delta B_{k,n} \Delta B_{\ell,n}.$$

As before, we replace $\Delta B_{k,n}\Delta B_{\ell,n}$ by $\beta_{k,\ell}$ and it suffices to study the term

$$\mathcal{I}_{5,0} = n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a'_{t_{j-1}} a'_{t_{k-1}} a_{t_{\ell-1}} q_j \alpha_{t_{j-1},k} \alpha_{t_{k-1},\ell} \beta_{k,\ell}.$$

We get the following contributions:

- The first factor $n^{6H-\frac{3}{2}}$.
- Integration by parts for q_j produces $n^{-2(2H\wedge 1)}$.
- $|\alpha_{t_{j-1},k}\alpha_{t_{k-1},\ell}|$ is bounded by $C_H n^{-2(2H\wedge 1)}$, due to Lemma 6.2 (a).
- $\sum_{k,\ell} |\beta_{k,\ell}| \le C_H n^{(1-2H)\vee 0}$ due to Lemma 6.2 (c).
- A factor n comes from the sum on j.

All together gives the order $n^{6H-\frac{1}{2}-4(2H\wedge 1)+(1-2H)\vee 0}$, which does not produce any contribution.

(vi) The sixth term is

$$\mathcal{I}_6 = n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a'_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} q_j \alpha_{t_{j-1},k} \beta_{k,\ell} \Delta B_{\ell,n}.$$

We get the following contributions:

- The first factor $n^{6H-\frac{3}{2}}$.
- Integration by parts for q_j and $\Delta B_{\ell,n}$ produces $n^{-3(2H\wedge 1)}$.
- $|\alpha_{t_{j-1},k}|$ is bounded by $C_H n^{-(2H\wedge 1)}$, due to Lemma 6.2 (a).
- $\sum_{k,\ell} |\beta_{k,\ell}| \le C_H n^{(1-2H)\vee 0}$, due to Lemma 6.2 (c).
- A factor n comes from the sum on j.

We get the same order as for the fifth term and no contribution.

(vii) The seventh term is given by

$$\mathcal{I}_7 = 2n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a_{t_{j-1}} a'_{t_{k-1}} a_{t_{\ell-1}} \Delta B_{j,n} \Delta B_{k,n} \Delta B_{\ell,n} \alpha_{t_{k-1},\ell} \beta_{j,k}.$$

Its contribution is the same as that of (ii); replace indices j, k and ℓ by k, ℓ and j, respectively.

(viii) The eighth term is given by

$$\mathcal{I}_8 = 2n^{6H-\frac{3}{2}} \sum_{j,k,\ell=1}^{n} a_{t_{j-1}} a_{t_{k-1}} a_{t_{\ell-1}} \Delta B_{j,n} \Delta B_{\ell,n} \beta_{j,k} \beta_{k,\ell}.$$

Its contribution is the same as that of term (iv) ; exchange j and k.

(B) One can show by a similar argument that

$$\lim_{n\to\infty} \langle D\langle DN_n, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}} = 0,$$

in L^p, for all $p \ge 2$.

In conclusion, we obtain the following results on the random symbol $\mathfrak{S}^3(\mathtt{iz})$ for $\mathfrak{S}^{(3,0)}(\mathtt{iz}, \mathtt{ix})$:

Case $H > \frac{1}{2}$ We have proved that

$$\begin{aligned} E[\Psi(\mathsf{z})\mathfrak{S}^3(\mathtt{iz})] &= \lim_{n\to\infty} \frac{1}{3} n^{\frac{3}{2}-2H} E\left[\Psi(\mathsf{z}) \left\langle D\langle DM_n, u_n\rangle_{\mathfrak{H}}, u_n\right\rangle_{\mathfrak{H}} (\mathtt{iz})^3\right] \\ &= C_{H,3}(\mathtt{iz})^3 \int_{[0,1]^4} E\left[D_{s'}\left(\Psi(\mathsf{z}) a(B_r) D_s(a^2(B_t))\right)\right] \\ &\qquad \times |r-s|^{2H-2}|r-s'|^{2H-2} ds ds' dr dt. \end{aligned}$$

Taking into account that $\alpha_H \int_0^t |r-s|^{2H-2} ds = \frac{\partial R_H}{\partial r}(t,r)$, we can write the above expression as follows:

$$\begin{aligned} &E[\Psi(\mathsf{z})\mathfrak{S}^3(\mathtt{iz})] \\ &= c_H^4 (\mathtt{iz})^5 \int_{[0,1]^3} E\left[\Psi(\mathsf{z})(a^2)'(B_\theta) a(B_r)(a^2)'(B_t)\right] \frac{\partial R_H}{\partial r}(\theta, r) \frac{\partial R_H}{\partial r}(t,r) dt d\theta dr \\ &\quad + H c_H^2 (\mathtt{iz})^3 \int_{[0,1]^2} E\left[\Psi(\mathsf{z}) a'(B_r)(a^2)'(B_t)\right] r^{2H-1} \frac{\partial R_H}{\partial r}(t,r) dt dr \\ &\quad + c_H^2 (\mathtt{iz})^3 \int_{[0,1]^2} E\left[\Psi(\mathsf{z}) a(B_r)(a^2)''(B_t)\right] \left(\frac{\partial R_H}{\partial r}(t,r)\right)^2 dt dr. \end{aligned}$$

Case $H < \frac{1}{2}$ We have obtained, that in this case, $\mathfrak{S}^3(\mathtt{iz}) = 0$.

6.3 Quasi tangent

For the quasi tangent we have

$$\begin{aligned}\langle DM_n, u_n\rangle_{\mathfrak{H}} - G_\infty &= n^{4H-1}\left\langle \sum_{j=1}^n \left[(Da_{t_{j-1}})q_j + 2a_{t_{j-1}}\Delta B_{j,n}\mathbf{1}_{I_j}\right], \sum_{k=1}^n a_{t_{k-1}}\Delta B_{k,n}\mathbf{1}_{I_k}\right\rangle_{\mathfrak{H}} \\ &\quad -2c_H^2\int_0^1 a^2(B_s)ds - r_n\langle DN_n, u_n\rangle_{\mathfrak{H}} \\ &= G_{n,1} + G_{n,2} + G_{n,3} + G_{n,4},\end{aligned}$$

where

$$G_{n,1} = n^{4H-1}\sum_{j,k=1}^n \langle (Da_{t_{j-1}})q_j, a_{t_{k-1}}\Delta B_{k,n}\mathbf{1}_{I_k}\rangle_{\mathfrak{H}} = n^{4H-1}\sum_{j,k=1}^n a'_{t_{j-1}}a_{t_{k-1}}\alpha_{t_{j-1},k}q_j\Delta B_{k,n},$$

$$G_{n,2} = 2n^{4H-1}\sum_{j,k=1}^n a_{t_{j-1}}a_{t_{k-1}}[\Delta B_{j,n}\Delta B_{k,n} - \beta_{j,k}]\beta_{j,k},$$

$$G_{n,3} = 2n^{4H-1}\sum_{j,k=1}^n a_{t_{j-1}}a_{t_{k-1}}\beta_{j,k}^2 - 2c_H^2\int_0^1 a^2(B_s)ds$$

and

$$G_{n,4} = -r_n\langle DN_n, u_n\rangle_{\mathfrak{H}}.$$

Integrating by parts the terms q_j and $\Delta B_{k,n}$ and using that, by Lemma 6.2 (b), $\sum_{k=1}^n |\alpha_{t_{j-1},k}|$ is bounded by a constant not depending on n, we obtain:

$$|E[\Psi(\mathsf{z})G_{n,1}]| \le Cn^{4H-1+1-3(2H\wedge 1)} = Cn^{4H-3(2H\wedge 1)}.$$

For $H > \frac{1}{2}$, we get $4H-3$ which is faster than $2H-\frac{3}{2}$ because $H < 3/4$, and for $H < \frac{1}{2}$, we obtain $-2H$ which is faster than $\frac{1}{2} - 2H$.

For the term $G_{n,2}$, by using integration-by-parts to $\Delta B_{j,n}\Delta B_{k,n} - \beta_{j,k}$ and Lemma 6.2 (c), we get

$$|E[\Psi(\mathsf{z})G_{n,2}]| \le Cn^{4H-1-2(2H\wedge 1)+(1-2H)\vee 0},$$

which produces the same rates as before.

It is possible to show that $G_{n,4}$ has no contribution to the limit.

Finally,

$$\begin{aligned}G_{n,3} &= 2n^{4H-1}\sum_{j,k=1}^n a_{t_{j-1}}a_{t_{k-1}}\beta_{j,k}^2 - 2c_H^2\int_0^1 a^2(B_s)ds \\ &= \frac{2}{n}\sum_{j,k=1}^n a(B_{t_{j-1}})a(B_{t_{k-1}})\rho_H^2(j-k) - 2c_H^2\int_0^1 a^2(B_s)ds \\ &= \frac{2}{n}\sum_{1\le j\le n, 1\le j+i\le n} a(B_{t_{j-1}})a(B_{t_{j+i-1}})\rho_H^2(i) - 2c_H^2\int_0^1 a^2(B_s)ds.\end{aligned}$$

Replacing $a(B_{t_{j+i-1}})$ by $a(B_{t_{j-1}})$ with integration-by-parts, we have the error

$$
\begin{aligned}
& E\left[\Psi(\mathsf{z})\left\{\frac{1}{n}\sum_{1\le j\le n,1\le j+i\le n} a(B_{t_{j-1}})\big(a(B_{t_{j+i-1}})-a(B_{t_{j-1}})\big)\rho_H^2(i)\right\}\right] \\
= \;& \frac{1}{n}\sum_{1\le j\le n,1\le j+i\le n} E\left[\left\langle D\left\{\Psi(\mathsf{z})a(B_{t_{j-1}})\int_0^1 a'\big(B_{t_{j-1}}+\theta(B_{t_{j+i-1}}-B_{t_{j-1}})\big)d\theta\right\},\right.\right. \\
& \qquad\qquad \left.\left. \mathbf{1}_{[t_{j-1},t_{j+i-1}]}\right\rangle\right]\rho_H^2(i) \\
\lesssim \;& \sum_{i=1}^n (i/n)^{(2H)\wedge 1}\rho_H^2(i) \\
\lesssim \;& n^{-2H\wedge 1+(2H\wedge 1+2(2H-2)+1)_+}.
\end{aligned}
$$

In the case $1/2 < H < 3/4$, this converges to zero at the rate n^{4H-3}, that is faster than $n^{2H-3/2}$, and in the case $1/4 < H < 1/2$, at the rate n^{-2H}, that is faster than $n^{\frac{1}{2}-2H}$. Let $\epsilon\in(0,1)$. Divide the sum $\sum_{1\le j\le n}$ into $\sum_{n^\epsilon+1\le j\le n-n^\epsilon}$ and the rest. Since the convergence $c_H^2=\lim_{I\uparrow\mathbb{Z}}\sum_{i\in I}\rho(i)^2$ is monotone, the order of the L^p-norm of

$$
\frac{1}{n}\sum_{1\le j\le n,1\le j+i\le n} a(B_{t_{j-1}})^2\rho_H(i)^2-\frac{1}{n}\sum_{1\le j\le n} a(B_{t_{j-1}})^2 c_H^2
$$

is not greater than

$$
\left(c_H^2-\sum_{-n^\epsilon\le i\le n^\epsilon}\rho_H(i)^2\right)+n^{-1+\epsilon}\quad\lesssim\quad n^{(4H-3)\epsilon}+n^{-1+\epsilon}.
$$

Then, in case $H\in(1/2,3/4)$, we can find $\epsilon\in(1/2,1)$ such that $n^{-(3-4H)\epsilon}+n^{-1+\epsilon}=o(n^{2H-3/2})$. In case $H\in(1/4,1/2)$, it is possible to find $\epsilon\in(0,1/2)$ such that $n^{-(3-4H)\epsilon}+n^{-1+\epsilon}=o(n^{\frac{1}{2}-2H})$. Once again by the Taylor formula and integration-by-parts, we see

$$
E\left[\Psi(\mathsf{z})\left\{\frac{1}{n}\sum_{1\le j\le n} a(B_{t_{j-1}})^2-\int_0^1 a^2(B_s)ds\right\}\right]\quad=\quad O(n^{-(2H)\wedge 1}),
$$

which is $o(n^{2H-3/2})$ for $H>1/2$, and $o(n^{\frac{1}{2}-2H})$ for $H\in(1/4,1/2)$. Therefore, $G_{n,3}$ has no contribution.

In conclusion, the quasi tangent has no effect in the asymptotic expansion, that is, $\mathfrak{G}_0^2=\mathfrak{S}_0^{(2,0)}=0$.

6.4 Perturbation term

In this subsection we study the contribution of the term N_n to the asymptotic expansion. More precisely we will compute the random symbol $\mathfrak{G}^1(\mathtt{iz})$ for $\mathfrak{S}^{(1,0)}$. The action of this symbol is defined by

$$
E[\Psi(\mathsf{z})\mathfrak{G}^1(\mathtt{iz})]=\lim_{n\to\infty}E[\Psi(z)N_n(\mathtt{iz})].
$$

(i) Case $H>\frac{1}{2}$. We recall that

$$
N_n\quad=\quad n\sum_{j=1}^n \Delta B_{j,n}\langle Da_{t_{j-1}},\mathbf{1}_{I_j}\rangle_{\mathfrak{H}}.
$$

Integrating by parts yields

$$\begin{aligned} E[\Psi(z)N_n(\mathtt{iz})] &= n\sum_{j=1}^n E[\langle D\langle \Psi(\mathsf{z})Da_{t_{j-1}}, \mathbf{1}_{I_j}\rangle_{\mathfrak{H}}, \mathbf{1}_{I_j}\rangle_{\mathfrak{H}}(\mathtt{iz}) \\ &= n\sum_{j=1}^n E[\Psi(\mathsf{z})a''(B_{t_{j-1}})]\alpha^2_{t_{j-1},j}(\mathtt{iz}) \\ &\quad +\frac{n}{2}c_H^2\sum_{j=1}^n\int_0^1 E[\Psi(\mathsf{z})(a^2)'(B_s)a'_{t_{j-1}}]\alpha_{t_{j-1},j}\alpha_{s,j}ds(\mathtt{iz})^3 \\ &=: G_{1,n}+G_{2,n}. \end{aligned}$$

We know that

$$\alpha^2_{t_{j-1},j} = \frac{1}{4}n^{-4H}(j^{2H}-(j-1)^{2H}-1)^2. \tag{6.8}$$

Expanding the square $(j^{2H}-(j-1)^{2H}-1)^2$ it turns out that the terms $-2(j^{2H}-(j-1)^{2H})$ and 1 do not contribute to the limit. So, for $G_{1,n}$ it suffices to consider the term

$$\begin{aligned} &\frac{1}{4}n^{1-4H}\sum_{j=1}^n E[\Psi(\mathsf{z})a''(B_{t_{j-1}})](j^{2H}-(j-1)^{2H})^2 \\ &= H^2 n^{1-4H}\sum_{j=1}^n E[\Psi(\mathsf{z})a''(B_{t_{j-1}})]j^{2(2H-1)} + o(1) \\ &= H^2 E\left[\Psi(\mathsf{z})\int_0^1 a''(B_s)s^{4H-2}ds\right] + o(1). \end{aligned}$$

For $G_{2,n}$, we have

$$\begin{aligned} & n\sum_{j=1}^n\int_0^1 E[\Psi(\mathsf{z})(a^2)'_s a'_{t_{j-1}}]\alpha_{t_{j-1},j}\alpha_{s,j}ds \\ =\ & H^2(2H-1)\sum_{j=1}^n\int_0^1 E[\Psi(\mathsf{z})(a^2)'_s a'_{t_{j-1}}](j/n)^{2H-1}\left(\int_{t=t_{j-1}}^{t_j}\int_{r=0}^s |t-r|^{2H-2}drdt\right)ds \\ & +o(1) \\ =\ & H^2(2H-1)\int_0^1\int_0^1 E[\Psi(\mathsf{z})(a^2)'_s a'_t]t^{2H-1}\left(\int_0^s |t-r|^{2H-2}dr\right)dtds + o(1) \\ =\ & H^2\int_0^1\int_0^1 E[\Psi(\mathsf{z})(a^2)'_s a'_t]t^{2H-1}\left\{t^{2H-1}+|s-t|^{2H-1}\mathrm{sign}(s-t)\right\}dtds + o(1) \end{aligned}$$

and hence

$$G_{2,n} = \frac{1}{2}Hc_H^2\int_0^1\int_0^1 E[\Psi(\mathsf{z})(a^2)'_s a'_t]t^{2H-1}\frac{\partial R_H}{\partial t}(t,s)dtds\ (\mathtt{iz})^3 + o(1).$$

Therefore, we have proved that

$$\begin{aligned} E[\Psi(\mathsf{z})\mathfrak{S}^1(\mathtt{iz})] &= H^2E\left[\Psi(\mathsf{z})\int_0^1 a''(B_s)s^{4H-2}ds\right](\mathtt{iz}) \\ &\quad +\frac{1}{2}Hc_H^2E\left[\Psi(\mathsf{z})\int_0^1\int_0^1(a^2)'(B_s)a'(B_t)t^{2H-1}\frac{\partial R_H}{\partial t}(t,s)dtds\right](\mathtt{iz})^3. \end{aligned}$$

(ii) Case $H < \frac{1}{2}$. In this case,

$$N_n \quad = \quad n^{4H-1} \sum_{j=1}^{n} \Delta B_{j,n} \langle D a_{t_{j-1}}, \mathbf{1}_{I_j} \rangle_{\mathfrak{H}}.$$

Re-defining $G_{1,n}$ and $G_{2,n}$ by replacing the factor n by n^{4H-1}, we proceed as before, but in that case the dominating term in $\alpha^2_{t_{j-1},j}$ in $G_{1,n}$ is the constant 1 and we obtain

$$E[\Psi(\mathsf{z})\mathfrak{G}^1(\mathtt{i}\mathsf{z})] \quad = \quad E\left[\Psi(\mathsf{z})\,\frac{1}{4}(\mathtt{i}\mathsf{z})\int_0^1 a''(B_s)ds\right].$$

Indeed, the term $G_{2,n}$ converges to zero because, by Lemma 6.2 (a) and (b), we can write

$$|G_{2,n}| \quad \lesssim \quad n^{2H-1} \sup_{s\in[0,1]} \sum_{j=1}^{n} |\alpha_{s,j}| \le n^{2H-1} C_H.$$

We already know

$$\lim_{n\to\infty} E[\Psi(z)\langle DN_n, u_n\rangle_{\mathfrak{H}}] = 0.$$

This means $\mathfrak{G}^2_1 = \mathfrak{S}^{(2,0)}_1 = 0$.

Remark 6.1. The functional $\Psi(\mathsf{z})$ should be replaced by $\Psi(\mathsf{z})\psi_n$ when we need a truncation ψ_n. However, the above arguments are essentially unchanged because $\|1-\psi_n\|_{\ell,p}$ would converge to zero much faster than the total error we found.

The following lemma is in Nourdin, Nualart and Peccati [14].

Lemma 6.2. Let $0 < H < 1$ and $n \ge 1$. We have, for some constant C_H,

- (a) $|\alpha_{t,k}| \leqslant n^{-(2H\wedge 1)}$ for any $t \in [0,1]$ and $k = 1, \dots, n$.
- (b) $\sup_{t\in[0,1]} \sum_{k=1}^n |\alpha_{t,k}| \le C_H$.
- (c) $\sum_{k,j=1}^n |\beta_{j,k}| \le C_H n^{(1-2H)\vee 0}$.
- (d) If $H < \frac{3}{4}$, then $\sum_{k,j=1}^n \beta^2_{j,k} \le C_H n^{1-4H}$.
- (e) $\sum_{k,j=1}^n |\beta_{k,l}\beta_{j,l}| \le C_H n^{-(4H\wedge 2)}$ for any $l = 1, \dots, n$.
- (f) If $H < \frac{3}{4}$, then $\sum_{k,j=1}^n |\beta_{k,l}\beta_{j,l}\beta_{j,k}| \le C_H n^{-4H-(2H\wedge 1)}$ for any $l = 1, \dots, n$.

7 Asymptotic expansion for measurable functions

Let $\ell = \check{\mathsf{d}} + 8$ and denote by β_x the maximum degree in x of $\mathfrak{S}$. We denote by σ_F the Malliavin covariance matrix of a multivariate functional F and write $\Delta_F = \det \sigma_F$. Let $\mathsf{d}_2 = (\ell + \beta_x - 7) \vee \big(2[(\mathsf{d}_1+2)/2] + 2[(\beta_x+1)/2]\big)$, where $[x]$ is the maximum integer not larger than x. We consider the following condition.

[C] (i) $u_n \in \mathbb{D}^{\ell+1,\infty}(\mathfrak{H}\otimes\mathbb{R}^{\mathsf{d}})$, $G_\infty \in \mathbb{D}^{(\ell+1)\vee \mathsf{d}_2,\infty}(\mathbb{R}^{\mathsf{d}}\otimes_+\mathbb{R}^{\mathsf{d}})$, $W_n, N_n \in \mathbb{D}^{\ell,\infty}(\mathbb{R}^{\mathsf{d}})$, $W_\infty \in \mathbb{D}^{\ell\vee \mathsf{d}_2,\infty}(\mathbb{R}^{\mathsf{d}})$, $X_n \in \mathbb{D}^{\ell,\infty}(\mathbb{R}^{\mathsf{d}_1})$, $X_\infty \in \mathbb{D}^{\ell\vee(\mathsf{d}_2+1),\infty}(\mathbb{R}^{\mathsf{d}_1})$.

(ii) For every $p > 1$, the following estimates hold:

$$\|u_n\|_{\ell,p} \quad = \quad O(1) \tag{7.1}$$

$$\|G^{(2)}_n\|_{\ell-2,p} \quad = \quad O(r_n) \tag{7.2}$$

$$\|G^{(3)}_n\|_{\ell-2,p} \quad = \quad O(r_n) \tag{7.3}$$

$$\|\langle DG_n^{(3)}, u_n\rangle_{\mathfrak{H}}\|_{\ell-1,p} \;=\; o(r_n) \tag{7.4}$$

$$\big\|\langle D\langle DG_n^{(2)}, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}}\big\|_{\ell-3,p} \;=\; o(r_n) \tag{7.5}$$

$$\sum_{\mathsf{A}=W_\infty,X_\infty}\big\|\langle D\mathsf{A}, u_n\rangle_{\mathfrak{H}}\big\|_{\ell-3,p} \;=\; O(r_n) \tag{7.6}$$

$$\sum_{\mathsf{A}=W_\infty,X_\infty}\big\|\langle D\langle D\mathsf{A}, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}}\big\|_{\ell-2,p} \;=\; o(r_n) \tag{7.7}$$

$$\|\overset{\circ}{W}_n\|_{\ell-1,p} + \|N_n\|_{\ell-1,p} + \|\overset{\circ}{X}_n\|_{\ell-1,p} \;=\; O(1) \tag{7.8}$$

$$\sum_{\mathsf{B}=\overset{\circ}{W}_n,N_n,\overset{\circ}{X}_n}\big\|\langle D\langle D\mathsf{B}, u_n\rangle_{\mathfrak{H}}, u_n\rangle_{\mathfrak{H}}\big\|_{\ell-2,p} \;=\; o(1) \tag{7.9}$$

(iii) For each pair $(\mathfrak{T}_n,\mathfrak{T}) = (\mathfrak{S}_n^{(3,0)},\mathfrak{S}^{(3,0)})$, $(\mathfrak{S}_{0,n}^{(2,0)},\mathfrak{S}_0^{(2,0)})$, $(\mathfrak{S}_n^{(2,0)},\mathfrak{S}^{(2,0)})$, $(\mathfrak{S}_n^{(1,1)},\mathfrak{S}^{(1,1)})$, $(\mathfrak{S}_n^{(1,0)},\mathfrak{S}^{(1,0)})$, $(\mathfrak{S}_n^{(0,1)},\mathfrak{S}^{(0,1)})$, $(\mathfrak{S}_{1,n}^{(2,0)},\mathfrak{S}_1^{(2,0)})$ and $(\mathfrak{S}_{1,n}^{(1,1)},\mathfrak{S}_1^{(1,1)})$, the following conditions are satisfied.

- **(a)** $\mathfrak{T}$ is a polynomial random symbol the coefficients of which are in $\mathbb{D}^{\check{\mathsf{d}}+\beta_x+1,1+} = \bigcup_{p>1}\mathbb{D}^{\check{\mathsf{d}}+\beta_x+1,p}$.
- **(b)** For some $p>1$, there exists a polynomial random symbol $\bar{\mathfrak{T}}_n$ that has L^p coefficients and the same degree as $\mathfrak{T}$,

$$E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}_n(\mathtt{iz},\mathtt{ix})\big] \;=\; E\big[\Psi(\mathsf{z},\mathsf{x})\bar{\mathfrak{T}}_n(\mathtt{iz},\mathtt{ix})\big]$$

and $\bar{\mathfrak{T}}_n\to\mathfrak{T}$ in L^p.

(iv)
- **(a)** $\det G_\infty^{-1}\in L^{\infty-}$.
- **(b)** There exist $c\in(-1,0)\cup(0,1)$ and $\kappa>0$ such that

$$P\big[\Delta_{(cM_n+W_\infty,X_\infty)} < s_n\big] \;=\; O(r_n^{1+\kappa})$$

for some positive random variables $s_n\in\mathbb{D}^{\ell-2,\infty}$ satisfying $\sup_{n\in\mathbb{N}}(\|s_n^{-1}\|_p + \|s_n\|_{\ell-2,p})<\infty$ for every $p>1$.
- **(c)** There exists $\kappa_1>0$ such that

$$\sum_{\mathsf{A}=DW_\infty,DX_\infty}\big\|\langle DM_n,\mathsf{A}\rangle_{\mathfrak{H}}\big\|_p \;=\; O(r_n^{\kappa_1})$$

for every $p>1$.

Remark 7.1. (i) In [C] (i), the index $\ell+1$ of u_n comes from (7.5), and $\ell+1$ of G_∞ comes from $D^{\ell-1}G_\infty$ in (7.4). The index ℓ in (7.1) is for $D^{\check{\mathsf{d}}+6}\psi_n$; $\ell-2$ in (7.2) for (7.18); $\ell-2$ in (7.3) for $R[11]$ defined later; $\ell-1$ in (7.4) for (7.19); $\ell-3$ in (7.5) for $R[2]$; $\ell-1$ in (7.6) for (7.22); $\ell-2$ in (7.7) for (7.20); $\ell-1$ in (7.8) is for application of IBP $\check{\mathsf{d}}+6$ times; $\ell-2$ in (7.9) is for (7.21). (ii) Intuitively, [C] (iv) (c) is a kind of orthogonality between M_n and (W_∞,X_∞). It is natural because M_n converges stably in most statistical problems. We will give a slightly different formulation of the problem later. (iii) The degree of $\mathfrak{T}_n$ and $\mathfrak{T}$ may be different. That $\bar{\mathfrak{T}}_n\to\mathfrak{T}$ in L^p means L^p convergence of all the random coefficients. (iv) Condition [C] (iv) (b) ensures non-degeneracy of X_∞, that is, $\Delta_{X_\infty}^{-1}\in L^{\infty-}$.

Remark 7.2. Condition [C] (iii) is a sufficient condition for the forthcoming results. We can replace [C] (iii) by

[C] (iii)$^\flat$ For the pairs of polynomial random symbols $(\mathfrak{T}_n,\mathfrak{T})=(\mathfrak{S}_n^{(3,0)},\mathfrak{S}^{(3,0)})$, $(\mathfrak{S}_{0,n}^{(2,0)},\mathfrak{S}_0^{(2,0)})$, $(\mathfrak{S}_n^{(2,0)},\mathfrak{S}^{(2,0)})$, $(\mathfrak{S}_n^{(1,1)},\mathfrak{S}^{(1,1)})$, $(\mathfrak{S}_n^{(1,0)},\mathfrak{S}^{(1,0)})$, $(\mathfrak{S}_n^{(0,1)},\mathfrak{S}^{(0,1)})$, $(\mathfrak{S}_{1,n}^{(2,0)},\mathfrak{S}_1^{(2,0)})$ and $(\mathfrak{S}_{1,n}^{(1,1)},\mathfrak{S}_1^{(1,1)})$, the coefficients of $\mathfrak{T}$ are in $\mathbb{D}^{\check{\mathsf{d}}+\beta_x+1,1+}$, and

$$\lim_{n\to\infty}\eth^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}_n(\mathtt{iz},\mathtt{ix})\big] \quad=\quad \eth^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}(\mathtt{iz},\mathtt{ix})\big]$$

for every $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $\alpha\in\mathbb{Z}_+^{\check{\mathsf{d}}}$.

Define ξ_n by

$$\xi_n \quad=\quad \frac{3s_n}{2s_n+12\Delta_n}+\frac{e_n}{s_n^2}+\frac{f_n}{\Delta_{X_\infty}^2} \tag{7.10}$$

for $\Delta_n=\Delta_{(cM_n+W_\infty,X_\infty)}$. The functionals e_n and f_n will be specified later. Let $\psi\in C^\infty(\mathbb{R};[0,1])$ such that $\psi(x)=1$ for $|x|\le 1/2$ and $\psi(x)=0$ for $|x|\ge 1$. Let $\psi_n=\psi(\xi_n)$. Then $\sup_{n\in\mathbb{N}}\|\psi_n\|_{\ell-2,p}<\infty$ for every $p>1$.

Denote by $\phi(z;\mu,\Sigma)$ the density function of the normal distribution with mean vector μ and covariance matrix Σ. We write $\mathfrak{S}_n=1+r_n\mathfrak{S}$. Define the function $p_n(z,x)$ by

$$p_n(z,x) \quad=\quad E\bigg[\mathfrak{S}_n(\partial_z,\partial_x)^*\Big\{\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\Big\}\bigg],$$

where $\delta_x(X_\infty)$ is Watanabe's delta function, i.e., the pull-back of the delta function δ_x by X_∞. See [9] for the notion of generalized Wiener functionals and Watanabe's delta function. The operation of the adjoint $\varsigma(\partial_z,\partial_x)^*$ for a random polynomial symbol $\varsigma(\mathtt{iz},\mathtt{ix})=\sum_\alpha c_\alpha(\mathtt{iz},\mathtt{ix})^\alpha$ is defined by

$$E\bigg[\varsigma(\partial_z,\partial_x)^*\Big\{\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\Big\}\bigg] \quad=\quad \sum_\alpha(-\partial_z,-\partial_x)^\alpha E\Big[c_\alpha\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\Big].$$

The function $p_n(z,x)$ is well defined under [C].

Given positive numbers M and γ, denote by $\mathcal{E}(M,\gamma)$ the set of measurable functions $f:\mathbb{R}^{\check{\mathsf{d}}}\to\mathbb{R}$ satisfying $|f(z,x)|\le M(1+|z|+|x|)^\gamma$ for all $(z,x)\in\mathbb{R}^{\check{\mathsf{d}}}$. We intend to approximate the joint distribution of (Z_n,X_n) by the density function $p_n(z,x)$. The error of the approximation is evaluated by the supremum of

$$\Delta_n(f) \quad=\quad \bigg|E\big[f(Z_n,X_n)\big]-\int_{\mathbb{R}^{\check{\mathsf{d}}}}f(z,x)p_n(z,x)dzdx\bigg|$$

in $f\in\mathcal{E}(M,\gamma)$.

For $\check{Z}_n=(Z_n,X_n)$, we write $\check{Z}_n^\alpha=Z_n^{\alpha_1}X_n^{\alpha_2}$ for $\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^{\mathsf{d}}\times\mathbb{Z}_+^{\mathsf{d}_1}=\mathbb{Z}_+^{\check{\mathsf{d}}}$. Define $\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})$ by

$$\hat{g}_n^\alpha(\mathsf{z},\mathsf{x}) \quad=\quad E\big[\psi_n\check{Z}_n^\alpha\exp\big(Z_n[\mathtt{iz}]+X_n[\mathtt{ix}]\big)\big]$$

for $\mathsf{z}\in\mathbb{R}^{\mathsf{d}}$ and $\mathsf{x}\in\mathbb{R}^{\mathsf{d}_1}$. Define $g_n^\alpha(z,x)$ by

$$g_n^\alpha(z,x) \quad=\quad \frac{1}{(2\pi)^{\check{\mathsf{d}}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}}\exp\big(-z[\mathtt{iz}]-x[\mathtt{ix}]\big)\,\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})d\mathsf{z}d\mathsf{x}$$

if the integral exists.

In the notation of Section 2.2,

$$\hat{g}_n^\alpha(\mathsf{z},\mathsf{x}) \quad=\quad E\big[e^{\lambda_n(1;\mathsf{z},\mathsf{x})}\psi_n\check{Z}_n^\alpha\big] \;=\; \varphi_n\big(1,\mathsf{z},\mathsf{x};\psi_n\check{Z}_n^\alpha\big) \;=\; \varphi_n\big(1;\psi_n\check{Z}_n^\alpha\big).$$

Moreover,

$$\hat{g}_n^\alpha(\mathsf{z},\mathsf{x}) \;=\; \eth^\alpha \hat{g}_n^0(\mathsf{z},\mathsf{x}) \;=\; \eth^\alpha \varphi_n(1;\psi_n)$$

for $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$, where $\eth^\alpha = \eth_{\mathsf{z}}^{\alpha_1}\eth_{\mathsf{x}}^{\alpha_2}$.

Let

$$\begin{aligned} h_n^0(z,x) \;&=\; E\big[\psi_n \phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\big] \\ &\quad +r_n E\big[\mathfrak{S}(\partial_z,\partial_x)^*\big\{\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\big\}\big] \end{aligned}$$

and let

$$h_n^\alpha(z,x) \;=\; (z,x)^\alpha h_n^0(z,x)$$

for $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$. It holds that $\sup_{(z,x)\in\mathbb{R}^{\check{\mathsf{d}}}} \big|h_n^\alpha(z,x)\big| < \infty$ for any $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$ and $n \in \mathbb{N}$. Let

$$\hat{h}_n^\alpha(\mathsf{z},\mathsf{x}) \;=\; \int_{\mathbb{R}^{\check{\mathsf{d}}}} e^{z[\mathtt{iz}]+x[\mathtt{ix}]} h_n^\alpha(z,x)\,dzdx.$$

Then

$$\hat{h}_n^\alpha(\mathsf{z},\mathsf{x}) \;=\; \eth^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\psi_n\big] + r_n \eth^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{S}(\mathtt{iz},\mathtt{ix})\big].$$

Let $\Lambda_n(d) = \{u \in \mathbb{R}^d; |u| \le r_n^{-q}\}$, where $q \in (0,1/2)$.

Lemma 7.3. Suppose that $[C]$ is fulfilled. Then

(a) For each $(\mathsf{z},\mathsf{x}) \in \mathbb{R}^{\check{\mathsf{d}}}$ and $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$,

$$\hat{g}_n^\alpha(\mathsf{z},\mathsf{x}) - \hat{h}_n^\alpha(\mathsf{z},\mathsf{x}) \;=\; o(r_n). \tag{7.11}$$

(b) For every $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$,

$$\sup_n \sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1} r_n^{-1}\big|\hat{g}_n^\alpha(\mathsf{z},\mathsf{x}) - \hat{h}_n^\alpha(\mathsf{z},\mathsf{x})\big| \;<\; \infty. \tag{7.12}$$

Proof. First we estimate

$$\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi) \;=\; E\big[e^{\lambda_n(\theta,\mathsf{z},\mathsf{x})}\Xi\big]$$

for $\Xi \in \mathbb{D}^{\mathsf{k},\infty} = \cap_{p>1}\mathbb{D}^{\mathsf{k},p}$, $\mathsf{k} \le \check{\mathsf{d}}+6$. Let

$$\check{M}_n(\theta) \;=\; \big(M_n + \theta^{-1}W_n(\theta) + r_n N_n, X_n(\theta)\big)$$

for $\theta \in (0,1]$. Suppose that there exists $\theta_0 \in [0,1)$ such that

$$\sup_{\theta\in(\theta_0,1]} E\Big[\Delta_{\check{M}_n(\theta)}^{-p}\mathbf{1}_{\{\sum_{j=0}^{\mathsf{k}}\|D^j\Xi\|_{\mathfrak{H}^{\otimes j}}>0\}}\Big] \;<\; \infty \tag{7.13}$$

and that

$$\sup_{\theta\in(0,\theta_0]} E\Big[\Delta_{X_n(\theta)}^{-p}\mathbf{1}_{\{\sum_{j=0}^{\mathsf{k}}\|D^j\Xi\|_{\mathfrak{H}^{\otimes j}}>0\}}\Big] \;<\; \infty \tag{7.14}$$

for every $p > 1$.

By definition,

$$\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi) \;=\; E\big[e^{\check{M}_n(\theta)[\theta\mathtt{iz},\mathtt{ix}]}e^{2^{-1}(1-\theta^2)G_\infty[(\mathtt{iz})^{\otimes 2}]}\,\Xi\big].$$

Use $M_n, W_n, W_\infty, N_n \in \mathbb{D}^{\mathsf{k}+1,\infty}(\mathbb{R}^{\mathsf{d}})$, $X_n, X_\infty \in \mathbb{D}^{\mathsf{k}+1,\infty}(\mathbb{R}^{\mathsf{d}_1})$, and $\det G_\infty^{-1} \in L^{\infty-}$, then with the IBP-formula k-times with respect to $\check{M}_n(\theta)$, we obtain

$$\big|\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi)\big| \quad\leq\quad \big|(\theta\mathsf{z},\mathsf{x})\big|^{-\mathsf{k}}E\bigg[\exp\bigg(-\frac{1}{2}(1-\theta^2)G_\infty[\mathsf{z}^{\otimes 2}]\bigg)\times\big|A_n(\theta,\mathsf{z};\Xi)\big|\bigg]$$

for $\theta\in(\theta_0,1]$, the functional $A_n(\theta,\mathsf{z};\Xi)$ is linear in Ξ, and the expectation on the right-hand side is dominated by a polynomial in

$$\|G_\infty^{-1}\|_p,\ \|G_\infty\|_{\mathsf{k},p},\ \|\check{M}_n(\theta)\|_{\mathsf{k}+1,p},\ \|\Delta_{\check{M}_n(\theta)}^{-1}\mathbf{1}_{\{\sum_{j=0}^{\mathsf{k}}\|D^j\Xi\|_{\mathfrak{H}^{\otimes j}}>0\}}\|_p,\ \|\Xi\|_{\mathsf{k},p}$$

for some $p>1$ uniformly in $\theta\in(0,1]$, $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $n\in\mathbb{N}$. For it, we may add an independent Gaussian variable to $\check{M}_n(\theta)$ and shrink its variance after integration-by-parts. Here we remark that

$$\sup_{\theta,\mathsf{z}}\bigg[\exp\bigg(-\frac{1}{2}(1-\theta^2)G_\infty[\mathsf{z}^{\otimes 2}]\bigg)\times\|(1-\theta^2)D^{j_1}G_\infty[\mathsf{z}^{\otimes 2}]\|_{\mathfrak{H}^{\otimes j_1}}\cdots$$
$$\cdots\|(1-\theta^2)D^{j_m}G_\infty[\mathsf{z}^{\otimes 2}]\|_{\mathfrak{H}^{\otimes j_m}}\bigg]\in L^{\infty-},$$

which is a consequence of L^p integrability of G_∞^{-1} for sufficiently large p. [This estimate is possible only when z appears with factor $1-\theta^2$. Otherwise, even though the non-degeneracy of G_∞ is used, the factor $(1-\theta^2)^{-1}$ would appear and the estimation failed for θ near 1.] Therefore

$$\sup_{\theta\in(\theta_0,1]}\big|\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi)\big| \quad\lesssim\quad \big|(\mathsf{z},\mathsf{x})\big|^{-\mathsf{k}}$$

uniformly in $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $n\in\mathbb{N}$.

For $\theta\in(0,\theta_0)$, we use nondegeneracy of $X_n(\theta)$. Applying ingeration-by-parts with respect to $X_n(\theta)$ to

$$\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi) \quad=\quad E\bigg[e^{X_n(\theta)[\mathtt{i}\mathsf{x}]}\exp\bigg(2^{-1}(1-\theta^2)G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}]$$
$$+\theta M_n[\mathtt{i}\mathsf{z}]+W_n(\theta)[\mathtt{i}\mathsf{z}]+\theta r_n N[\mathtt{i}\mathsf{z}]\bigg)\,\Xi\bigg],$$

we obain

$$(\mathtt{i}\mathsf{x})^{\alpha_2}\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi) \quad=\quad E\big[e^{\lambda_n(\theta;\mathsf{z},\mathsf{x})}B_{n,\alpha_2}(\theta,\mathsf{z};\Xi)\big]$$

for some functional $B_{n,\alpha_2}(\theta,\mathsf{z};\Xi)$ for $\theta\in(0,\theta_0)$ and $\alpha_2\in\mathbb{Z}_+^{\mathsf{d}_1}$ with $|\alpha_2|=\mathsf{k}$. For every $L>0$, the L^1-norm of the functional $B_{n,\alpha_2}(\theta,\mathsf{z};\Xi)$ is dominated by a polynomial of

$$\|X_n(\theta)\|_{\mathsf{k}+1,p},\quad \|\Delta_{X_n(\theta)}^{-1}\mathbf{1}_{\{\sum_{j=0}^{\mathsf{k}}\|D^j\Xi\|_{\mathfrak{H}^{\otimes j}}>0\}}\|_p,\quad \|G_\infty^{-1}\|_p,\quad \|G_\infty\|_{\mathsf{k},p},$$
$$\|M_n\|_{\mathsf{k},p},\quad \|W_n(\theta)\|_{\mathsf{k},p},\quad \|N_n\|_{\mathsf{k},p},\quad \|\Xi\|_{\mathsf{k},p},\quad (1+|\mathsf{z}|)^{-L}$$

uniformly in $\theta\in(0,\theta_0)$ and $n\in\mathbb{N}$, if we take a sufficiently large p. Therefore we have

$$\big|\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi)\big| \quad\lesssim\quad |(\mathsf{z},\mathsf{x})|^{-\mathsf{k}}$$

uniformly in $\theta\in(0,\theta_0)$, $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $n\in\mathbb{N}$. Consequently, we obtained

$$\sup_{\theta\in(0,1]}\big|\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi)\big| \quad\lesssim\quad \big|(\mathsf{z},\mathsf{x})\big|^{-\mathsf{k}} \tag{7.15}$$

uniformly in $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $n\in\mathbb{N}$, under the assumptions (7.13) and (7.14).

For $\theta\geq|c|$,

$$\Delta_{\check{M}_n(\theta)} \;=\; \Delta_{(M_n+\theta^{-1}W_\infty,X_\infty)}+r_nd_n^*(\theta),$$

where $\{d_n^*(\theta);\theta\in[|c|,1],n\in\mathbb{N}\}$ is a family of functionals bounded in $\mathbb{D}^{\check{\mathsf{d}}+6,\infty}$. Moreover,

$$\begin{aligned}
&\Delta_{(M_n+\theta^{-1}W_\infty,X_\infty)}\\
=\;&\theta^{-2\mathsf{d}}\det\left[\begin{array}{cc}\langle\theta DM_n,\theta DM_n\rangle_{\mathfrak{H}}+\langle DW_\infty,DW_\infty\rangle_{\mathfrak{H}} & \langle DX_\infty,DW_\infty\rangle_{\mathfrak{H}}\\ \langle DW_\infty,DX_\infty\rangle_{\mathfrak{H}} & \langle DX_\infty,DX_\infty\rangle_{\mathfrak{H}}\end{array}\right]+r_n^{(1\wedge\kappa_1)/2}\dot{d}_n(\theta)\\
\geq\;&\det\left[\begin{array}{cc}\langle cDM_n,cDM_n\rangle_{\mathfrak{H}}+\langle DW_\infty,DW_\infty\rangle_{\mathfrak{H}} & \langle DX_\infty,DW_\infty\rangle_{\mathfrak{H}}\\ \langle DW_\infty,DX_\infty\rangle_{\mathfrak{H}} & \langle DX_\infty,DX_\infty\rangle_{\mathfrak{H}}\end{array}\right]+r_n^{(1\wedge\kappa_1)/2}\dot{d}_n(\theta)\\
=\;&\Delta_{(cM_n+W_\infty,X_\infty)}+r_n^{(1\wedge\kappa_1)/2}\tilde{d}_n(\theta)
\end{aligned}$$

for $\theta\in[|c|,1]$, where $\dot{d}_n(\theta)$ and $\tilde{d}_n(\theta)$ are functionals in $\mathbb{D}^{\check{\mathsf{d}}+6,\infty}$ such that

$$\sup_{\theta\in[0,1],n\in\mathbb{N}} r_n^{-(1\wedge\kappa_1)/2}\Big\{\|\dot{d}_n(\theta)\|_{\check{\mathsf{d}}+6,p}+\|\tilde{d}_n(\theta)\|_{\check{\mathsf{d}}+6,p}\Big\} \;<\; \infty$$

for every $p>1$. Consequently,

$$\Delta_{\check{M}_n(\theta)} \;\geq\; \Delta_{(cM_n+W_\infty,X_\infty)}+r_n^{(1\wedge\kappa_1)/2}d_n^{**}(\theta) \tag{7.16}$$

for $\theta\in[|c|,1]$ and $n\in\mathbb{N}$. The functional $d_n^{**}(\theta)$ is defined by $d_n^*(\theta)$ and $\tilde{d}_n(\theta)$. We define e_n as the sum of squares of the coefficient of the polynomial $d_n^{**}(\theta)$ in θ and θ^{-1}. Then $\|e_n\|_{\check{\mathsf{d}}+6,p}=O(r_n^{1\wedge\kappa_1})$ for every $p>1$. On the other hand, we have an expansion

$$\Delta_{X_n(\theta)} \;=\; \Delta_{X_\infty}\big(1+r_n^{1/2}\Delta_{X_\infty}^{-1}\hat{d}_n(\theta)\big)$$

with a functional $\hat{d}_n(\theta)$ such that all coefficients of this polynomial in θ are of $O(r_n^{1/2})$ in $L^{\infty-}$. Let f_n be the sum of squares of the coefficients of $\hat{d}_n(\theta)$. By [C] (iv) (b) and the definition of ψ_n, considering the event $\{\xi_n>1/2\}\supset\{\psi_n<1\}$, we have

$$\|1-\psi_n\|_{\check{\mathsf{d}}+6,1+2^{-1}\kappa}=O(r_n^{p_1}) \tag{7.17}$$

for $p_1=(1+\kappa)/(1+2^{-1}\kappa)>1$. If $\xi_n\leq1$, then $\inf_{\theta\in[|c|,1]}\Delta_{\check{M}_n(\theta)}\geq s_n/13$ and $\inf_{\theta\in[0,1]}\Delta_{X_n(\theta)}\geq\Delta_{X_\infty}/2$ for large n. Thus the conditions (7.13) and (7.14) are ensured and hence the estimate (7.15) is available for various functionals Ξ having a factor related to ψ_n, as we will see below. Condition (7.1) gives $L^{\infty-}$-boundedness of $\check{\mathsf{d}}+6$ derivatives of σ_{M_n}.

Condition (7.2) with (7.1) implies

$$\|\big\langle DG_n^{(2)},u_n\big\rangle_{\mathfrak{H}}\|_{\check{\mathsf{d}}+5,p} \;=\; O(r_n) \tag{7.18}$$

[This is the only place where the L^p boundedness of $D^{\ell-2}G_n^{(2)}$ is required. That is, we will only need that $\|G_n^{(2)}\|_{\ell-3,p}=O(r_n)$ and (7.18) in what follows.] Condition (7.4) implies

$$\bigg\|\bigg\langle D\Big(\big\langle DG_n^{(3)},u_n\big\rangle_{\mathfrak{H}}\Big),u_n\bigg\rangle_{\mathfrak{H}}\bigg\|_{\check{\mathsf{d}}+6,p} \;=\; o(r_n) \tag{7.19}$$

for every $p>1$ under (7.1). Condition (7.7) implies

$$\sum_{\mathsf{A}=W_\infty,X_\infty}\bigg\|\bigg\langle D\Big(\big\langle D\langle D\mathsf{A},u_n\rangle_{\mathfrak{H}},u_n\big\rangle_{\mathfrak{H}}\Big),u_n\bigg\rangle_{\mathfrak{H}}\bigg\|_{\check{\mathsf{d}}+5,p} \;=\; o(r_n) \tag{7.20}$$

under (7.1). Moreover, Condition (7.9) implies

$$\sum_{\mathsf{B}=\mathring{W}_n,N_n,\mathring{X}_n} \left\| \left\langle D\big\langle D\langle D\mathsf{B},u_n\rangle_{\mathfrak{H}},u_n\big\rangle_{\mathfrak{H}},u_n \right\rangle_{\mathfrak{H}} \right\|_{\check{\mathsf{d}}+5,p} \;=\; o(1) \tag{7.21}$$

under (7.1).

The estimate $\|\hat{G}^{(1)}_n(\theta)\|_{\check{\mathsf{d}}+5,p} = O(r_n)$ for every $p>1$ follows from (7.6), (7.8) and (7.1). The estimate $\|\check{G}^{(1)}_n\|_{\check{\mathsf{d}}+5,p} = O(r_n)$ for every $p>1$ follows from (7.8), therefore

$$\|G^{(1)}_n(\theta)\|_{\check{\mathsf{d}}+5,p} \;=\; O(r_n) \tag{7.22}$$

for every $p>1$. We obtain

$$\big\|\big\langle D\hat{G}^{(1)}_n(\theta),u_n\big\rangle_{\mathfrak{H}}\big\|_{\check{\mathsf{d}}+5,p} \;=\; o(r_n) \tag{7.23}$$

for every $p>1$ from (7.7) and (7.9). Estimate

$$\big\|\big\langle D\check{G}^{(1)}_n,u_n\big\rangle_{\mathfrak{H}}\big\|_{\check{\mathsf{d}}+5,p} \;=\; O(r_n) \tag{7.24}$$

for every $p>1$ follows from (7.8).

We have

$$\left\|\left\langle D\big\langle D\hat{G}^{(1)}_n(\theta),u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\right\|_{\check{\mathsf{d}}+5,p} \;=\; o(r_n)$$

by (7.20) and (7.21). Follows the estimate

$$\left\|\left\langle D\big\langle D\check{G}^{(1)}_n,u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\right\|_{\check{\mathsf{d}}+5,p} \;=\; o(r_n)$$

from (7.9), so that

$$\left\|\left\langle D\big\langle DG^{(1)}_n(\theta),u_n\big\rangle_{\mathfrak{H}},u_n\right\rangle_{\mathfrak{H}}\right\|_{\check{\mathsf{d}}+5,p} \;=\; o(r_n). \tag{7.25}$$

Since $\varphi_n(0;\psi_n) = E[\Psi(\mathsf{z},\mathsf{x})\psi_n]$, Proposition 2.6 gives

$$\hat{g}^{\alpha}_n(\mathsf{z},\mathsf{x}) - \hat{h}^{\alpha}_n(\mathsf{z},\mathsf{x}) \;=\; \eth^{\alpha}R_n(\mathsf{z},\mathsf{x}) \;=\; \sum_{i=3}^{12}\eth^{\alpha}R^{(i)}_n(\mathsf{z},\mathsf{x}).$$

We shall show

$$\sup_n \sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1} r_n^{-1}\big|\eth^{\alpha}R^{(i)}_n(\mathsf{z},\mathsf{x})\big| \;<\; \infty \tag{7.26}$$

for $i=3,...,12$.

We remind the representation of $R^{(3)}_n(\mathsf{z},\mathsf{x}) = \rho^{(3)}_n(f)$ with $f(z,x) = \exp(z[\mathtt{iz}]+x[\mathtt{ix}])$. There appear 24 terms in this expression and we name them $R[i]$ ($i=1,...,24$). We will repeatedly use the inequality (7.15) based on integration-by-parts (IBP) to estimate $\eth^{\alpha}R[i]$. It should be noted that the factors $(1-\theta^2)G_\infty[\mathtt{iz},\cdot]$ and their Malliavin derivatives come out but they are controlled by $\exp\big(2^{-1}(1-\theta^2)G_\infty[(\mathtt{iz})^{\otimes 2}]\big)$ if non-degeneracy of G_∞ is used, as already mentioned. Estimates of $R[i]$ are as follows.

- $R[1]$. The estimate $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^{\alpha}R[1]| = o(r_n)$ follows from $\check{\mathsf{d}}+4$ times IBP, (7.17), (7.1) and (7.2).

- $R[9]$. There are three components $R[9,j]$ $(j=1,2,3)$ of $R[9]$ corresponding to the decomposition

$$\begin{aligned}\Big\langle D\big\langle D(G(\mathsf{z})\psi_n),u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}} &= \Big\langle D\big\langle DG(\mathsf{z}),u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}}\psi_n\\ &\quad+2\big\langle DG(\mathsf{z}),u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\big\langle D\psi_n,u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\\ &\quad+G(\mathsf{z})\Big\langle D\big\langle D\psi_n,u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}}\end{aligned}\tag{7.27}$$

  for $G=G_n^{(3)}$. The estimate $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[9,1]|=o(r_n)$ follows from $\check{\mathsf{d}}+6$ IBP, (7.19) and (7.17). Since $\|\big\langle DG_n^{(3)}(\mathsf{z}),u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\|_{\check{\mathsf{d}}+4,p}\lesssim o(r_n)|\mathsf{z}|^4$ by (7.4) and $|(\mathsf{z},\mathsf{x})|\le r_n^{-q}\le r_n^{-1/2}$, we may deal with $|(\mathsf{z},\mathsf{x})|^3$ for $R[9,2]$. Apply $\check{\mathsf{d}}+4$ IBP, (7.17) and (7.1) to obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[9,2]|=O(r_n^{p_1})$. Similarly, we use $\|G_n^{(3)}(\mathsf{z})\|_{\check{\mathsf{d}}+4,p}\lesssim O(r_n)|\mathsf{z}|^3$ by (7.3), $\check{\mathsf{d}}+4$ IBP, (7.17) and (7.1) to obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[9,3]|=O(r_n^{p_1})$. Thus, $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[9]|=o(r_n)$.
- $R[2]$. We take a way similar to $R[9]$ to show $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[2]|=o(r_n)$. $R[2,j]$ $(j=1,2,3)$ are defined by (7.27) for $G=G_n^{(2)}$. Apply $\check{\mathsf{d}}+5$ IBP to $R[2,1]$ with (7.5) and (7.17). $\check{\mathsf{d}}+3$ IBP to $R[2,2]$ with (7.18), (7.17) and (7.1). $\check{\mathsf{d}}+3$ IBP to $R[2,3]$ with (7.2), (7.17) and (7.1).
- $R[16]$. In the same way as for $R[2]$, we can show $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[16]|=o(r_n)$. In this case, decomposing $R[16]$ into $R[16,j]$ $(j=1,2,3)$ by (7.27) for $G=G_n^{(1)}$, we apply $\check{\mathsf{d}}+5$ IBP to $R[16,1]$ with (7.25) and (7.17). $\check{\mathsf{d}}+3$ IBP to $R[16,2]$ with (7.23), (7.24), (7.17) and (7.1). $\check{\mathsf{d}}+3$ IBP to $R[16,3]$ with (7.22), (7.17) and (7.1).
- $R[23]$. There are two terms $R[23,i]$ $(i=1,2)$ for the decomposition

$$F\big\langle D\big(G\psi_n\big),u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}} = F\big\langle DG,u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\psi_n+\ FG\big\langle D\psi_n,u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}}\tag{7.28}$$

  for $F=1$ and $G=G_n^{(3)}(\mathsf{z})$. Apply $\check{\mathsf{d}}+5$ IBP, (7.4) and (7.17) to $R[23,1]$, and $\check{\mathsf{d}}+3$ IBP, (7.3) and (7.17) to $R[23,2]$. Then we obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[23]|=o(r_n)$.
- $R[24]$. There are two terms $R[24,i]$ $(i=1,2)$ according to (7.28) for $F=1$ and $G=\hat{G}_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$. To $R[24,1]$, use $\check{\mathsf{d}}+4$ IBP with (7.23) and (7.17). To $R[24,2]$, $\check{\mathsf{d}}+2$ IBP with (7.22) and (7.17). Then $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[24]|=o(r_n)$.
- $R[11]$. By the decomposition (7.28) for $F=G=G_n^{(3)}(\mathsf{z})$, we have two terms $R[11,i]$ $(i=1,2)$ as the components of $R[11]$. The factor $|(\mathsf{z},\mathsf{x})|^2$ is canceled by r_n^{2q}. Apply $\check{\mathsf{d}}+6$ IBP with (7.4) and (7.17) to $R[11,1]$. Apply $\check{\mathsf{d}}+4$ IBP with (7.3) and (7.17) to $R[11,2]$. Then we have $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[11]|=o(r_n)$.
- $R[10]$. This case is similar to $R[11]$. There appear two terms $R[10,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(2)}(\mathsf{z})$ and $G=G_n^{(3)}(\mathsf{z})$. Then we obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[10]|=o(r_n)$ by applying $\check{\mathsf{d}}+5$ IBP with (7.4), (7.2) and (7.17) to $R[10,1]$, and by $\check{\mathsf{d}}+3$ IBP with (7.2), (7.3) and (7.17) to $R[10,2]$.
- $R[12]$. This case is similar to $R[10]$. There appear two terms $R[10,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$ and $G=G_n^{(3)}(\mathsf{z})$. Then $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[12]|=o(r_n)$. For that, apply $\check{\mathsf{d}}+5$ IBP with (7.4), (7.22) and (7.17) to $R[12,1]$, and by $\check{\mathsf{d}}+3$ IBP with (7.22), (7.3) and (7.17) to $R[12,2]$.

- $R[4]$. There appear two terms $R[4,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(3)}(\mathsf{z})$ and $G=G_n^{(2)}(\mathsf{z})$. We obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[4,1]|=O(r_n^{2(1-q)})$ by applying $\check{\mathsf{d}}+5$ IBP with (7.18) and (7.17). In this case, the factor $|(\mathsf{z},\mathsf{x})|^2$ is evaluated by r_n^{-2q}. Apply $\check{\mathsf{d}}+3$ IBP with (7.2), (7.3) and (7.17) to $R[4,2]$ to obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[4,2]|=O(r_n^{p_1})$, therefore $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[4]|$ $=O(r_n^{p_2})$, where $p_2=p_1\wedge\{2(1-q)\}$.
- $R[3]$. This is similar to the case $R[4]$. There appear two terms $R[3,i]$ $(i=1,2)$ by (7.28) for $F=G=G_n^{(2)}(\mathsf{z})$. We obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[3]|=O(r_n^{p_2})$ by applying $\check{\mathsf{d}}+4$ IBP with (7.18) and (7.17) to $R[3,1]$, and by $\check{\mathsf{d}}+2$ IBP with (7.2) and (7.17) to $R[3,2]$.
- $R[5]$. Similar to the case $R[3]$. There appear two terms $R[5,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$ and $G=G_n^{(2)}(\mathsf{z})$. We obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[5]|=O(r_n^{p_2})$ by applying $\check{\mathsf{d}}+4$ IBP with (7.18), (7.22) and (7.17) to $R[3,1]$, and by $\check{\mathsf{d}}+2$ IBP with (7.2), (7.22) and (7.17) to $R[3,2]$.
- $R[18]$. Similar to $R[4]$. There are two terms $R[18,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(3)}(\mathsf{z})$ and $G=G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$. Then $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[18]|=O(r_n^{p_2})$ follows from $\check{\mathsf{d}}+5$ IBP with (7.23), (7.24), (7.3) and (7.17) to $R[18,1]$, and also $\check{\mathsf{d}}+3$ IBP with (7.22), (7.3) and (7.17) to $R[18,2]$.
- $R[17]$. Similar to $R[18]$. There are two terms $R[17,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(2)}(\mathsf{z})$ and $G=G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$. Then $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[17]|=O(r_n^{p_2})$ follows from $\check{\mathsf{d}}+4$ IBP with (7.23), (7.24), (7.2) and (7.17) to $R[17,1]$, as well as $\check{\mathsf{d}}+2$ IBP with (7.22), (7.2) and (7.17) to $R[17,2]$.
- $R[19]$. Similar to $R[17]$. Two terms $R[19,i]$ $(i=1,2)$ by (7.28) for $F=G_n^{(1)}(\theta';\mathsf{z},\mathsf{x})$ and $G=G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})$. Then $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[19]|=O(r_n^{p_2})$, which follows from $\check{\mathsf{d}}+4$ IBP with (7.23), (7.24), (7.22) and (7.17) to $R[19,1]$, as well as $\check{\mathsf{d}}+2$ IBP with (7.22) and (7.17) to $R[19,2]$.
- $R[14]$. One factor $|(\mathsf{z},\mathsf{x})|$ is cancelled by $r_n^{1/2}$ offered by $\big(G_n^{(3)}(\mathsf{z})\big)^2$. We apply $\check{\mathsf{d}}+6$ IBP with (7.3) and (7.17) to obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[14]|=O(r_n^{3/2})$. [Another way of estimating is to cancel the factor $|(\mathsf{z},\mathsf{x})|^2$ by r_n^{2q} taken from $\big(G_n^{(3)}(\mathsf{z})\big)^2$ before applying less order of IBP formulas. This is the case in the following estimates though we adopted the same way as for $R[14]$.]
- $R[13]$. Similar to $R[14]$. Cancelling one factor $|(\mathsf{z},\mathsf{x})|$ by $r_n^{1/2}$ in $G_n^{(2)}(\mathsf{z})G_n^{(3)}(\mathsf{z})$, we apply $\check{\mathsf{d}}+5$ IBP with (7.2), (7.3) and (7.17) to show $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[13]|=O(r_n^{3/2})$.
- $R[7]$. It is essentially the same as $R[13]$. Therefore $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[7]|=O(r_n^{3/2})$.
- $R[15]$. Similar to $R[14]$. Cancelling one factor $|(\mathsf{z},\mathsf{x})|$ by $r_n^{1/2}$ in $G_n^{(1)}(\theta;\mathsf{z},\mathsf{x})G_n^{(3)}(\mathsf{z})$, we apply $\check{\mathsf{d}}+5$ IBP with (7.22), (7.3) and (7.17) to show$\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[15]|=O(r_n^{3/2})$.
- $R[21]$. Essentially same as $R[15]$, therefore $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[21]|=O(r_n^{3/2})$.
- $R[6]$. One factor cancellation and $\check{\mathsf{d}}+4$ IBP with (7.2) and (7.17) give $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[6]|=O(r_n^{3/2})$.
- $R[8]$. Similarly to $R[6]$, $\check{\mathsf{d}}+4$ IBP with (7.22), (7.2) and (7.17) gives $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[8]|=O(r_n^{3/2})$.
- $R[20]$. This is essentially equivalent to $R[8]$. $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[20]|=O(r_n^{3/2})$.

- $R[22]$. Apply $\check{\mathsf{d}}+4$ IBP with (7.22) and (7.17) to obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}|\eth^\alpha R[22]| = O(r_n^{3/2})$.

In conclusion,

$$\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}\big|\eth^\alpha R_n^{(i)}(\mathsf{z},\mathsf{x})\big| \;=\; o(r_n) \tag{7.29}$$

for $i=3$ and in particular (7.26) is valid for $i=3$.

It is possible to obtain $\sup_{(\mathsf{z},\mathsf{x})\in\Lambda_n(\check{\mathsf{d}})}|(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}\big|\eth^\alpha R_n^{(i)}(\mathsf{z},\mathsf{x})\big| = O(r_n^{p_1})$, and (7.29), hence (7.26) for $i=4$ with $\check{\mathsf{d}}+3$ IBP, (7.8) and (7.17). Each $R_n^{(i)}(\mathsf{z},\mathsf{x})$ $(i=5,...,12)$ is a difference of two terms. We will estimate each term separately. Apply $\check{\mathsf{d}}+\beta_x+1$ IBP with respect to X_∞ to ς and use $\exp\big(2^{-1}G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}]\big)$ with non-degeneracy of G_∞ (without IBP) to obtain

$$\sup_{(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}\big|E\big[\Psi(\mathsf{z},\mathsf{x})\varsigma(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big]\big| \;<\; \infty$$

for $\varsigma=\mathfrak{S}^{(3,0)},\mathfrak{S}_0^{(2,0)},\mathfrak{S}^{(2,0)},\mathfrak{S}^{(1,1)},\mathfrak{S}^{(1,0)},\mathfrak{S}^{(0,1)},\mathfrak{S}_1^{(2,0)}$ and $\mathfrak{S}_1^{(1,1)}$. Remark that we need $X_\infty\in\mathbb{D}^{\check{\mathsf{d}}+\beta_x+2,\infty}(\mathbb{R}^{\mathsf{d}_1})$, $W_\infty\in\mathbb{D}^{\check{\mathsf{d}}+\beta_x+1,\infty}(\mathbb{R}^{\mathsf{d}})$ and $G_\infty\in\mathbb{D}^{\check{\mathsf{d}}+\beta_x+1,\infty}(\mathbb{R}^{\mathsf{d}}\otimes_+\mathbb{R}^{\mathsf{d}})$ in this procedure. To estimate each first term, $\check{\mathsf{d}}+4=\ell-4$ IBP with respect to X_∞ is sufficient because the degree of each random symbol is not greater than three. For that, Conditions (7.1), (7.2), (7.3), (7.6) and (7.8) work together with the factor $\exp\big(2^{-1}G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}]\big)$ with non-degeneracy of G_∞. In this way, we obtain

$$\sup_n \sup_{(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+1}\big|E\big[\Psi(\mathsf{z},\mathsf{x})\varsigma_n(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big]\big| \;<\; \infty$$

for $\varsigma_n=\mathfrak{S}_n^{(3,0)},\mathfrak{S}_{0,n}^{(2,0)},\mathfrak{S}_n^{(2,0)},\mathfrak{S}_n^{(1,1)},\mathfrak{S}_n^{(1,0)},\mathfrak{S}_n^{(0,1)},\mathfrak{S}_{1,n}^{(2,0)}$ and $\mathfrak{S}_{1,n}^{(1,1)}$. Consequently, (7.26) was verified for $i=5,...,12$. Thus, (7.12) was proved.

Furthermore, [C] (iii)$^\flat$ and (7.17) gives $\eth^\alpha R_n^{(i)}(\mathsf{z},\mathsf{x})=o(r_n)$ for $i=5,...,12$, and then we obtain (7.11). Now it suffices to show that [C] (iii) implies [C] (iii)$^\flat$. For $\eta>0$, let

$$F_n^\eta(\mathsf{z}',\mathsf{x}') \;=\; E\big[\Psi(\mathsf{z}',\mathsf{x}')\psi\big(\eta(|G_\infty|+|W_\infty|+|X_\infty|)\big)\bar{\mathfrak{T}}_n(\mathtt{i}\mathsf{z}',\mathtt{i}\mathsf{x}')\big] \qquad ((\mathsf{z}',\mathsf{x}')\in\mathbb{C}^{\check{\mathsf{d}}})$$

for $n\in\mathbb{N}\cup\{\infty\}$, where $\bar{\mathfrak{T}}_\infty=\mathfrak{T}$. F_n^η are analytic functions of $(\mathsf{z}',\mathsf{x}')$ for $\eta>0$ and $n\in\mathbb{N}\cup\{\infty\}$. Let

$$F_n^0(\mathsf{z},\mathsf{x}) \;=\; E\big[\Psi(\mathsf{z},\mathsf{x})\bar{\mathfrak{T}}_n(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big] \qquad ((\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}})$$

for $n\in\mathbb{N}\cup\{\infty\}$. Then $\eth^\alpha F_n^\eta$ on $\mathbb{R}^{\check{\mathsf{d}}}$ is explicitly expressed by

$$\eth^\alpha F_n^\eta(\mathsf{z},\mathsf{x}) \;=\; E\big[\eth^\alpha\{\Psi(\mathsf{z},\mathsf{x})\bar{\mathfrak{T}}_n(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\}\psi\big(\eta(|G_\infty|+|W_\infty|+|X_\infty|)\big)\big]$$

for $\eta\geq 0$, $n\in\mathbb{N}\cup\{\infty\}$ and $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$. Remark that the differential operator $\eth^\alpha$ is in the real domain, so that this equation is valid even for $\eta=0$. On the other hand, $\partial^\alpha_{(\mathsf{z}',\mathsf{x}')}F_n^0(\mathsf{z}',\mathsf{x}')$ is not defined. Fix $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $\alpha\in\mathbb{Z}_+^{\check{\mathsf{d}}}$. Let $\epsilon>0$. Then there exists $\eta>0$ such that

$$\sup_{n\in\mathbb{N}\cup\{\infty\}}\sum_{a=0,\alpha}|\eth^a F_n^\eta(\mathsf{z},\mathsf{x})-\eth^a F_n^0(\mathsf{z},\mathsf{x})| \;<\; \epsilon \tag{7.30}$$

For $\eta>0$, the gap $F_n^\eta(\mathsf{z}',\mathsf{x}')-F_\infty^\eta(\mathsf{z}',\mathsf{x}')\to 0$ locally uniformly as $n\to\infty$ because the coefficients of $\bar{\mathfrak{T}}_n-\mathfrak{T}$ converge to zero in L^p for some $p>1$. Cauchy's integral formula for multivariate analytic functions ensures the convergence

$$\eth^\alpha F_n^\eta(\mathsf{z},\mathsf{x})\to\eth^\alpha F_\infty^\eta(\mathsf{z},\mathsf{x}) \qquad (n\to\infty) \tag{7.31}$$

for every $\eta > 0$. Then (7.30) and (7.31) give $\limsup_{n\to\infty}\big|\vartheta^\alpha F^0_\infty(\mathsf{z},\mathsf{x}) - \vartheta^\alpha F^0_n(\mathsf{z},\mathsf{x})\big| < 2\epsilon$, and hence

$$\lim_{n\to\infty}\vartheta^\alpha F^0_n(\mathsf{z},\mathsf{x}) \quad=\quad \vartheta^\alpha F^0_\infty(\mathsf{z},\mathsf{x}).$$

By the equality in [C] (iii) (b), we obtain

$$\lim_{n\to\infty}\vartheta^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}_n(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big] \quad=\quad \vartheta^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\mathfrak{T}(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big],$$

that is [C] (iii)$^\flat$. This completes the proof of Lemma 7.3. □

The following is a slightly different set of conditions.

[C$^\natural$] (i) [C] (i) holds.

(ii) (7.1), (7.2), (7.3), (7.6) and (7.8) hold for every $p > 1$. Furthermore, there exists a positive constant κ such that the following estimates hold:

$$\|\big\langle DG^{(3)}_n, u_n\big\rangle_{\mathfrak{H}}\|_{\ell-1,p} \quad=\quad O(r_n^{1+\kappa}) \tag{7.32}$$

$$\big\|\big\langle D\big\langle DG^{(2)}_n, u_n\big\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}}\big\|_{\ell-3,p} \quad=\quad O(r_n^{1+\kappa}) \tag{7.33}$$

$$\sum_{\mathsf{A}=W_\infty, X_\infty}\big\|\big\langle D\langle D\mathsf{A}, u_n\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}}\big\|_{\ell-2,p} \quad=\quad O(r_n^{1+\kappa}) \tag{7.34}$$

$$\sum_{\mathsf{B}=\mathring{W}_n, N_n, \mathring{X}_n}\big\|\big\langle D\langle D\mathsf{B}, u_n\rangle_{\mathfrak{H}}, u_n\big\rangle_{\mathfrak{H}}\big\|_{\ell-2,p} \quad=\quad O(r_n^{\kappa}) \tag{7.35}$$

(iii) [C] (iii) holds.

(iv) (a) $\det G_\infty^{-1} \in L^{\infty-}$.

(b) There exists $\kappa > 0$ such that

$$P\big[\Delta_{(M_n+W_\infty, X_\infty)} < s_n\big] \quad=\quad O(r_n^{1+\kappa})$$

for some positive random variables $s_n \in \mathbb{D}^{\ell-2,\infty}$ satisfying $\sup_{n\in\mathbb{N}}(\|s_n^{-1}\|_p + \|s_n\|_{\ell-2,p}) < \infty$ for every $p > 1$.

The functional ψ_n is re-defined by $\psi_n = \psi(\xi_n)$ with

$$\xi_n \quad=\quad \frac{3s_n}{2s_n + 12\Delta_n} + \frac{e_n}{s_n^2} + \frac{f_n}{\Delta^2_{X_\infty}} \tag{7.36}$$

for $\Delta_n = \Delta_{(M_n+W_\infty, X_\infty)}$ this time. The functional f_n is defined as before, and e_n will be specified in the proof of the following lemma.

Lemma 7.4. Under $[C^\natural]$, the properties (a) and (b) of Lemma 7.3 hold true.

Proof. The plot of the proof is quite similar to that of Lemma 7.3, however some modifications are necessary. Let ϵ be a positive number. We may assume that $r_n < 1$ for all $n \in \mathbb{N}$. Instead of (7.13) and (7.14), we will use the non-degeneracy of the forms

$$\sup_{\theta\in(\sqrt{1-r_n^\epsilon}, 1]} E\bigg[\Delta^{-p}_{\check{M}_n(\theta)}\mathbf{1}_{\{\sum_{j=0}^{\mathsf{k}}|D^j\Xi|_{\mathfrak{H}^{\otimes j}}>0\}}\bigg] \quad<\quad \infty \tag{7.37}$$

and

$$\sup_{\theta\in(0,\sqrt{1-r_n^\epsilon}]} E\left[\Delta_{X_n(\theta)}^{-p}\mathbf{1}_{\{\sum_{j=0}^{\mathsf{k}}|D^j\Xi|_{\mathfrak{H}^{\otimes j}}>0\}}\right] \quad < \quad \infty \tag{7.38}$$

for every $p>1$ and a suitably differentiable functional Ξ. For the same reason as before, we have

$$\big|\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi)\big| \quad \lesssim \quad |(\mathsf{z},\mathsf{x})|^{-\mathsf{k}}$$

uniformly in $\theta\in(\sqrt{1-r_n^\epsilon},1]$, $(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}$ and $n\in\mathbb{N}$. For $\theta\in(0,\sqrt{1-r_n^\epsilon}]$, we will use $\check{\mathsf{d}}+\beta_x+1$ IBP below, just like before, and this procedure gives some power of $|\mathsf{z}|$. To cancel the power of $|\mathsf{z}|$ (including z's coming from random polynomials when we use it), we attach the factor $(1-\theta^2)$, and then a power of $(1-\theta^2)^{-1}$ appears. We can replace it by $r_n^{-\epsilon L}$, where L is a definite number. Thus what we obtained is

$$\sup_n \sup_{\theta\in(0,1)} \sup_{(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}} r_n^{\epsilon L}\big|(\mathsf{z},\mathsf{x})\big|^{\mathsf{k}}\big|\varphi_n(\theta,\mathsf{z},\mathsf{x};\Xi)\big| \quad < \quad \infty. \tag{7.39}$$

We need non-degeneracy (7.37) and (7.38) to apply the estimate (7.39). For our purposes, when the functional Ξ has ψ_n or its derivative of certain order, it is sufficient to show non-degeneracy of $\check{M}_n(\theta)$ and $X_n(\theta)$ under truncation by ψ_n with ξ_n of (7.36). We make ϵ sufficiently small. Then it is easy to see

$$\Delta_{\check{M}_n(\theta)} \quad = \quad \Delta_{(M_n+W_\infty,X_\infty)} + r_n^{\epsilon/2}d_n^{**}(\theta)$$

for some functional $d_n^{**}(\theta)$ such that $\sup_{\theta\in(\sqrt{1-r_n^\epsilon},1],n\in\mathbb{N}} r_n^{-\epsilon/2}\|d_n^{**}(\theta)\|_{\check{\mathsf{d}}+6,p}<\infty$ for every $p>1$. Define e_n as before with the coefficients of $d_n^{**}(\theta)$. Then we see (7.17) holds and $\Delta_{\check{M}_n(\theta)}$ and $\Delta_{X_n(\theta)}$ have uniform non-degeneracy under ψ_n, as before.

For proof of the lemma, it is sufficient to show (7.26) for $i=3,...,12$. We can take the same way as the proof of Lemma 7.3. Indeed, estimations of $R_n^{(i)}(\mathsf{z},\mathsf{x})$ $(i=4,...,12)$ are the same since we only use nondegneracy of G_∞ and Δ_{X_∞}. Only estimation of $R_n^{(3)}(\mathsf{z},\mathsf{x})$ is slightly different. We do the same way for estimation of $R[i]$ $(i=1,...,24)$ with (7.39), but in this situation, the bounds $o(r_n)$ that appeared in the previous proof become $O(r_n^{1+\kappa'})$ for some positive constant κ', thanks to [$C^\natural$] (ii). Taking a sufficiently small ϵ so that $\epsilon L<\kappa'$, we obtain (7.12) in the present situation. □

The definition of ψ_n varies, depending on $[C]$ or $[C^\natural]$, in the following lemma.

Lemma 7.5. Suppose that either $[C]$ or $[C^\natural]$ is fulfilled. Then, for each $m\in\mathbb{Z}_+$,

$$\sup_{(z,x)\in\mathbb{R}^{\check{\mathsf{d}}}} \big||(z,x)|^m\big(g_n^0(z,x)-h_n^0(z,x)\big)\big| \quad = \quad o(r_n)$$

as $n\to\infty$.

Proof. $\check{\mathsf{d}}+3$ times IBP provides

$$\sup_n \sup_{(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}} |(\mathsf{z},\mathsf{x})|^{\check{\mathsf{d}}+3}|\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})| \quad < \quad \infty.$$

Therefore,

$$\int_{\mathbb{R}^{\check{\mathsf{d}}}\setminus\Lambda_n(\check{\mathsf{d}})} |\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})|d\mathsf{z}d\mathsf{x} \quad = \quad O(r_n^{3q}) \tag{7.40}$$

for every $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$. We apply $\check{\mathsf{d}}+3$ times IBP with respect to X_∞ to $\eth^\alpha E[\Psi(\mathsf{z},\mathsf{x})\psi_n]$ and use $\exp\big(2^{-1}G_\infty[(\mathtt{i}\mathsf{z})^{\otimes 2}]\big)$ with non-degeneracy of G_∞ to derive

$$\sup_n \sup_{(\mathsf{z},\mathsf{x})\in\mathbb{R}^{\check{\mathsf{d}}}} \big|(\mathsf{z},\mathsf{x})\big|^{\check{\mathsf{d}}+3}\big|\eth^\alpha E[\Psi(\mathsf{z},\mathsf{x})\psi_n]\big| \quad < \quad \infty.$$

We remark that the factor x does not emerge but some product of z can newly appear though cancelled by the exponential. So

$$\int_{\mathbb{R}^{\check{\mathsf{d}}}\setminus\Lambda_n(\check{\mathsf{d}})} \big|\eth^\alpha E[\Psi(\mathsf{z},\mathsf{x})\psi_n]\big| d\mathsf{z}d\mathsf{x} \quad = \quad O(r_n^{3q})$$

for every $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$. Let ς be any random symbol, like $\mathfrak{S}^{(3,0)}$, that appears in the r_n-order term of $\mathfrak{S}_n$. We apply $\check{\mathsf{d}}+\beta_x+1$ times IBP with respect to X_∞ to $\eth^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\varsigma(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big]$, and next use the Gaussianity of Ψ in z to show

$$\int_{\mathbb{R}^{\check{\mathsf{d}}}\setminus\Lambda_n(\check{\mathsf{d}})} \big|\eth^\alpha E\big[\Psi(\mathsf{z},\mathsf{x})\varsigma(\mathtt{i}\mathsf{z},\mathtt{i}\mathsf{x})\big]\big| d\mathsf{z}d\mathsf{x} \quad = \quad O(r_n^{q})$$

for every $\alpha \in \mathbb{Z}_+^{\check{\mathsf{d}}}$. Thus

$$\int_{\mathbb{R}^{\check{\mathsf{d}}}\setminus\Lambda_n(\check{\mathsf{d}})} |\hat{h}_n^\alpha(\mathsf{z},\mathsf{x})| d\mathsf{z}d\mathsf{x} \quad = \quad O(r_n^{3q})+O(r_n^{1+q}). \tag{7.41}$$

This term becomes $o(r_n)$ if we choose $q\in(1/3,1/2)$.

Now

$$\begin{aligned}
\Delta_n^\alpha \quad &:= \quad \sup_{(z,x)\in\mathbb{R}^{\check{\mathsf{d}}}} \big|(z,x)^\alpha\big(g_n^0(z,x)-h_n^0(z,x)\big)\big| \\
&= \quad \sup_{(z,x)\in\mathbb{R}^{\check{\mathsf{d}}}} \frac{1}{(2\pi)^{\check{\mathsf{d}}}}\bigg|\int_{\mathbb{R}^{\check{\mathsf{d}}}} e^{-z[\mathtt{i}\mathsf{z}]-x[\mathtt{i}\mathsf{x}]}\big(\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})-\hat{h}_n^\alpha(\mathsf{z},\mathsf{x})\big)d\mathsf{z}d\mathsf{x}\bigg| \\
&\leq \quad \frac{1}{(2\pi)^{\check{\mathsf{d}}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}\setminus\Lambda_n(\check{\mathsf{d}})} \big|\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})\big|d\mathsf{z}d\mathsf{x} + \frac{1}{(2\pi)^{\check{\mathsf{d}}}}\int_{\mathbb{R}^{\check{\mathsf{d}}}\setminus\Lambda_n(\check{\mathsf{d}})} \big|\hat{h}_n^\alpha(\mathsf{z},\mathsf{x})\big|d\mathsf{z}d\mathsf{x} \\
&\quad + \frac{r_n}{(2\pi)^{\check{\mathsf{d}}}}\int_{\Lambda_n(\check{\mathsf{d}})} r_n^{-1}\big|\hat{g}_n^\alpha(\mathsf{z},\mathsf{x})-\hat{h}_n^\alpha(\mathsf{z},\mathsf{x})\big|d\mathsf{z}d\mathsf{x}.
\end{aligned}$$

By (7.40), (7.41) and the properties (a) and (b) provided by either Lemma 7.3 or Lemma 7.4, we obtain $\Delta_n^\alpha = o(r_n)$. □

Here is the main theorem in this section.

Theorem 7.6. Suppose that either $[C]$ or $[C^\natural]$ is fulfilled. Then, for any positive numbers M and γ,

$$\sup_{f\in\mathcal{E}(M,\gamma)} \Delta_n(f) \quad = \quad o(r_n)$$

as $n\to\infty$.

Proof. The local density g_n^0 is a continuous version of the density $(E[\psi_n|\check{Z}_n=(z,x)]dP^{\check{Z}_n})/dzdx$, admits any order of moments and

$$E\big[f(\check{Z}_n)\psi_n\big] \quad = \quad \int_{\mathbb{R}^{\check{\mathsf{d}}}} f(z,x)g_n^0(z,x)dzdx.$$

Let $p=1+\kappa/2$, where κ is the one given in [C] (iv) (b) or in [C$^\natural$] (iv) (b). Then

$$\sup_{f\in\mathcal{E}(M,\gamma)} \big|E\big[f(\check{Z}_n)\big]-E\big[f(\check{Z}_n)\psi_n\big]\big| \quad \leq \quad \sup_{f\in\mathcal{E}(M,\gamma)} \|f(\check{Z}_n)\|_{p/(p-1)}\|1-\psi_n\|_p \;=\; o(r_n).$$

For $k_1, k_2 \in \mathbb{Z}_+$, we have

$$\begin{aligned} & |z|^{k_1}|x|^{2k_2}\big|E\big[(1-\psi_n)\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\big]\big| \\ = & \big|E\big[X_\infty^{2k_2}(1-\psi_n)|z|^{k_1}\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\big]\big| \\ \leq & C(k_1,k_2)\|1-\psi_n\|_{\nu,p} \end{aligned}$$

for all $(z,x) \in \mathbb{R}^{\check{\mathsf{d}}}$, where $C(k_1,k_2)$ is a constant depending on (k_1,k_2), where $\nu = 2[1+\mathsf{d}_1/2] \leq \mathsf{d}_1+2$. Therefore,

$$\begin{aligned} & \sup_{f\in\mathcal{E}(M,\gamma)}\left|\int_{\mathbb{R}^{\check{\mathsf{d}}}} f(z,x)E\big[(1-\psi_n)\phi(z;W_\infty,G_\infty)\delta_x(X_\infty)\big]dzdx\right| \\ = & O(\|1-\psi_n\|_{\nu,p}) \;=\; o(r_n). \end{aligned}$$

This estimate makes it possible to replace p_n by h_n^0.

In this way, estimation of $\Delta_n(f)$ is reduced to

$$\begin{aligned} & \left|\int_{\mathbb{R}^{\check{\mathsf{d}}}} f(z,x)g_n^0(z,x)dzdx - \int_{\mathbb{R}^{\check{\mathsf{d}}}} f(z,x)h_n^0(z,x)dzdx\right| \\ \leq & \int_{\mathbb{R}^{\check{\mathsf{d}}}}|f(z,x)|\big(1+|(z,x)|\big)^{-m}dzdx \times \sup_{(z,x)\in\mathbb{R}^{\check{\mathsf{d}}}}\big|(1+|(z,x)|)^m\big(g_n^0(z,x)-h_n^0(z,x)\big)\big| \\ = & o(r_n) \end{aligned}$$

by Lemma 7.5 if m is chosen as $m > \check{\mathsf{d}}+\gamma$. □

It is easy to give the joint asymptotic expansion with a reference variable in the applications of the previous sections, while we do not give statements explicitly here. In statistics, the joint expansion is quite important because the reference variable will be the random Fisher information matrix, an asymptotically ancillary statistic, and so on, in the context of the non-ergodic statistics.

On the other hand, it is also possible to give a similar asymptotic expansion of $E[f(Z_n)]$ without a reference variable X_n. In fact, our result already applies to such a case if we take a variable $X_n = X_\infty \sim N(0,1)$ independent of other variables. The expansion formula is valid in particular for functions $f(z)$ of z. Integrating out x from $p_n(z,x)$, we obtain a formula $\int p_n(z,x)dx$. Formally, this formula corresponds to the case $\beta_x = 0$ and $\mathsf{d}_1 = 0$. As a matter of fact, some of differentiability conditions can be reduced due to lack of the reference variable X_n. We shall give a simplified version of Theorem 7.6 with $[C^\natural]$ but without the reference variable X_n, among several possibilities. In what follows, we will only consider the variable

$$Z_n \;=\; M_n + r_nN_n.$$

In this situation, we need the random symbols

$$\begin{aligned} \mathfrak{S}_n^{(3,0)}(\mathtt{iz}) &= \frac{1}{3}r_n^{-1}\Big\langle D\big\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}},u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}} \;\equiv\; \frac{1}{3}\mathsf{qTor}[(\mathtt{iz})^{\otimes 3}], \\ \mathfrak{S}_{0,n}^{(2,0)}(\mathtt{iz}) &= \frac{1}{2}r_n^{-1}G_n^{(2)}(\mathtt{z}) \;=\; \frac{1}{2}r_n^{-1}\Big(\big\langle DM_n[\mathtt{iz}],u_n[\mathtt{iz}]\big\rangle_{\mathfrak{H}} - G_\infty[(\mathtt{iz})^2]\Big) \;\equiv\; \frac{1}{2}\mathsf{qTan}[(\mathtt{iz})^{\otimes 2}], \\ \mathfrak{S}_n^{(1,0)}(\mathtt{iz}) &= N_n[\mathtt{iz}], \\ \mathfrak{S}_{1,n}^{(2,0)}(\mathtt{iz}) &= \Big\langle DN_n[\mathtt{iz}],u_n[\mathtt{iz}]\Big\rangle_{\mathfrak{H}}. \end{aligned}$$

Let

$$\Psi(\mathtt{z}) \;=\; \exp\big(2^{-1}G_\infty[(\mathtt{iz})^{\otimes 2}]\big).$$

We consider the following condition. Recall $\ell = \mathsf{d}+8$ when $\mathsf{d}_1 = 0$.

[D] (i) $u_n \in \mathbb{D}^{\ell+1,\infty}(\mathfrak{H}\otimes\mathbb{R}^{\mathsf{d}})$, $G_\infty \in \mathbb{D}^{\ell+1,\infty}(\mathbb{R}^{\mathsf{d}}\otimes_+\mathbb{R}^{\mathsf{d}})$, $N_n \in \mathbb{D}^{\ell,\infty}(\mathbb{R}^{\mathsf{d}})$.

(ii) There exists a positive constant κ such that the following estimates hold for every $p>1$:

$$\|u_n\|_{\ell,p} = O(1)$$

$$\|G_n^{(2)}\|_{\ell-2,p} = O(r_n)$$

$$\|G_n^{(3)}\|_{\ell-2,p} = O(r_n)$$

$$\|\big\langle DG_n^{(3)},u_n\big\rangle_{\mathfrak{H}}\|_{\ell-1,p} = O(r_n^{1+\kappa})$$

$$\Big\|\Big\langle D\big\langle DG_n^{(2)},u_n\big\rangle_{\mathfrak{H}},u_n\Big\rangle_{\mathfrak{H}}\Big\|_{\ell-3,p} = O(r_n^{1+\kappa})$$

$$\|N_n\|_{\ell-1,p} = O(1)$$

$$\Big\|\big\langle D\langle DN_n,u_n\rangle_{\mathfrak{H}},u_n\big\rangle_{\mathfrak{H}}\Big\|_{\ell-2,p} = O(r_n^{\kappa}).$$

(iii) For each pair $(\mathfrak{T}_n,\mathfrak{T}) = (\mathfrak{S}_n^{(3,0)},\mathfrak{S}^{(3,0)})$, $(\mathfrak{S}_{0,n}^{(2,0)},\mathfrak{S}_0^{(2,0)})$, $(\mathfrak{S}_n^{(1,0)},\mathfrak{S}^{(1,0)})$ and $(\mathfrak{S}_{1,n}^{(2,0)},\mathfrak{S}_1^{(2,0)})$, the following conditions are satisfied.

(a) $\mathfrak{T}$ is a polynomial random symbol the coefficients of which are in $L^{1+} = \cup_{p>1}L^p$.

(b) For some $p>1$, there exists a polynomial random symbol $\bar{\mathfrak{T}}_n$ that has L^p coefficients and the same degree as $\mathfrak{T}$,

$$E\big[\Psi(\mathsf{z})\mathfrak{T}_n(\mathtt{iz})\big] = E\big[\Psi(\mathsf{z})\bar{\mathfrak{T}}_n(\mathtt{iz})\big]$$

and $\bar{\mathfrak{T}}_n\to\mathfrak{T}$ in L^p.

(iv) **(a)** $\det G_\infty^{-1}\in L^{\infty-}$.

(b) There exists $\kappa>0$ such that

$$P\big[\Delta_{M_n}<s_n\big] = O(r_n^{1+\kappa})$$

for some positive random variables $s_n\in\mathbb{D}^{\ell-2,\infty}$ satisfying $\sup_{n\in\mathbb{N}}(\|s_n^{-1}\|_p+\|s_n\|_{\ell-2,p})<\infty$ for every $p>1$.

In the present situation, the random symbol $\mathfrak{S}$ is defined by

$$\mathfrak{S}(\mathtt{iz}) = \mathfrak{S}^{(3,0)}(\mathtt{iz})+\mathfrak{S}_0^{(2,0)}(\mathtt{iz})+\mathfrak{S}^{(1,0)}(\mathtt{iz})+\mathfrak{S}_1^{(2,0)}(\mathtt{iz}).$$

Let $\mathfrak{S}_n = 1+r_n\mathfrak{S}$ and define $\hat{p}_n(z)$ by

$$\hat{p}(z) = E\big[\mathfrak{S}_n(\partial_z)^*\phi(z;0,G_\infty)\big]$$

with naturally defined adjoint operation $\mathfrak{S}_n(\partial_z)^*$. We follow the proof of Theorem 7.6 but with

$$\varphi_n(\theta,\mathsf{z};\Xi) = E\big[e^{\lambda_n(\theta;\mathsf{z})}\Xi\big]$$

for $\varphi_n(\theta, \mathsf{z}, \mathsf{x}; \Xi)$, where

$$\lambda_n(\theta; \mathsf{z}) \quad = \quad \theta M_n[\mathtt{iz}] + 2^{-1}(1-\theta^2)G_\infty[(\mathtt{iz})^{\otimes 2}] + \theta r_n N_n[\mathtt{iz}].$$

Then, in place of (7.39), we obtain

$$\sup_n \sup_{\theta\in(0,1)} \sup_{\mathsf{z}\in\mathbb{R}^{\mathsf{d}}} r_n^{\epsilon L}|\mathsf{z}|^{\mathsf{k}}\big|\varphi_n(\theta, \mathsf{z}; \Xi)\big| \quad < \quad \infty.$$

For this estimate for $\theta \in (0, \sqrt{1-r_n^\epsilon}\,]$, only non-degeneracy of G_∞ is used. In this way, we can prove the validity of the asymptotic expansion by $\hat{p}_n$. Denote by $\hat{\mathcal{E}}(M,\gamma)$ the set of measurable functions $f : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}$ such that $|f(z)| \leq M(1+|z|)^\gamma$ for all $z \in \mathbb{R}^{\mathsf{d}}$. Let

$$\hat{\Delta}_n(f) \quad = \quad \left| E\big[f(Z_n)\big] - \int_{\mathbb{R}^{\mathsf{d}}} f(z)\hat{p}_n(z)dz \right|$$

for $f \in \hat{\mathcal{E}}(M,\gamma)$.

Theorem 7.7. Suppose that Condition $[D]$ is satisfied. Then, for any positive numbers M and γ,

$$\sup_{f\in\hat{\mathcal{E}}(M,\gamma)} \hat{\Delta}_n(f) \quad = \quad o(r_n)$$

as $n \to \infty$.

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