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MAXIMUM OF THE GINZBURG–LANDAU FIELDS

BY DAVID BELIUS[1] AND WEI WU[2]

[1]*Department of Mathematics and Computer Science, University of Basel, david.belius@unibas.ch*
[2]*Mathematics Department, NYU Shanghai, wei.wu@nyu.edu*

We study a two-dimensional massless field in a box with potential $V(\nabla\phi(\cdot))$ and zero boundary condition, where V is any symmetric and uniformly convex function. Naddaf–Spencer (*Comm. Math. Phys.* **183** (1997) 55–84) and Miller (*Comm. Math. Phys.* **308** (2011) 591–639) proved that the rescaled macroscopic averages of this field converge to a continuum Gaussian free field. In this paper, we prove that the distribution of local marginal $\phi(x)$, for any x in the bulk, has a Gaussian tail. We further characterize the leading order of the maximum and the dimension of high points of this field, thus generalizing the results of Bolthausen–Deuschel–Giacomin (*Ann. Probab.* **29** (2001) 1670–1692) and Daviaud (*Ann. Probab.* **34** (2006) 962–986) for the discrete Gaussian free field.

1. Introduction.

1.1. *Model.* This paper studies the extreme values of certain two-dimensional lattice gradient Gibbs measures (also known as the Ginzburg–Landau field). Take a nearest neighbor potential $V \in C^2(\mathbb{R})$ that satisfies

$$V(x) = V(-x), \tag{1.1}$$

$$0 < c_- \le V''(x) \le c_+ < \infty, \tag{1.2}$$

where c_-, c_+ are positive constants.

Let $D_N := [-N, N]^2 \cap \mathbb{Z}^2$ and ∂D_N consist of the vertices in D_N that are connected to $\mathbb{Z}^2 \setminus D_N$ by some edge. Set $D_N^\circ = D_N \setminus \partial D_N$. For $x, y \in \mathbb{Z}^2$ we also write $x \sim y$ if x and y are connected by an edge. The Ginzburg–Landau Gibbs measure on D_N with zero boundary condition is given by

$$d\mu_N = Z_N^{-1} \exp\Big[-\sum_{x\in D_N^\circ}\sum_{y\sim x} V\big(\phi(x)-\phi(y)\big)\Big] \prod_{x\in D_N^\circ} d\phi(x) \prod_{x\in\partial D_N} \delta_0\big(\phi(x)\big), \tag{1.3}$$

where

$$\delta_0(y) = \begin{cases} 1 & y = 0, \\ 0 & \text{else}, \end{cases}$$

and Z_N is the normalizing constant such that μ_N is a probability measure. We denote by $\mathbb{E}^{D_N,0}$ and $\mathrm{Var}^{D_N,0}$ the expectation and variance with respect to the measure μ_N. The Ginzburg–Landau model is a natural generalization of the discrete Gaussian free field (DGFF, corresponding to the case $V(x) = x^2/2$). It is no longer Gaussian in general, but still log-correlated in two dimension. In fact, one can prove that the limit

$$\lim_{N\to\infty} \frac{\mathrm{Var}^{D_N,0}\,\phi(0)}{\log N} = g \quad \text{for some } g = g(V) > 0 \tag{1.4}$$

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exists; this result is new and follows from the proof of Theorem 1.4 in this paper. For the infinite volume limit of the measure (1.3), the analogue of (1.4) was recently established in [3]. The constant $g = g(V)$ is known as the **effective stiffness** of the random surface model.

1.2. *Results*. Our main result concerns the maximum of the Ginzburg–Landau field ϕ in D_N. For potential $V(\cdot)$ satisfying (1.1) and (1.2), the well-known Brascamp–Lieb inequalities (Lemma 2.1) imply that with high probability, $\frac{\sup_{v\in D_N}\phi(v)}{\log N}$ is uniformly bounded above by a constant depending only on c_- (see [24] where a constant lower bound was also obtained, and Remark 2.3 below). We prove that this random variable in fact satisfies a law of large numbers, with more precise tail bounds given by (4.17) and (5.1) below.

THEOREM 1.1. *Let ϕ be sampled from the Gibbs measure* (1.3). *Assume the potential $V(\cdot)$ satisfies* (1.1) *and* (1.2). *Then there is a constant $g = g(V)$, such that*

$$\frac{\sup_{v\in D_N}\phi(v)}{\log N} \to 2\sqrt{g} \quad \text{in } L^2. \tag{1.5}$$

REMARK 1.2. The explicit dependence of $g(V)$ on V is not known. The same constant also appears in the covariance of the continuum Gaussian free field that emerges as the scaling limit of the measure (1.3); see [31, 44]. One can give a variational characterization of $g(V)$ (see, e.g., [7, 31]).

Theorem 1.1 is known for the discrete Gaussian free field (see [12]), but not for any other Ginzburg–Landau fields. We will summarize related results in Section 1.3 below. The upper bound of Theorem 1.1 will be proved in Section 4.2 and the lower bound in Section 5.

Our next result studies the fractal structure of the sets where the Ginzburg–Landau field ϕ is unusually high. We say that $v \in D_N$ is an η-high point for the Ginzburg–Landau field if $\phi(v) \geq 2\sqrt{g}\eta\log N$. The following theorem generalizes the dimension of the high points for Gaussian free field, obtained by Daviaud [23].

THEOREM 1.3. *Denote by $\mathcal{H}_N(\eta) = \{v \in D_N : \phi(v) \geq 2\sqrt{g}\eta\log N\}$ the set of η-high points. Then for any $\eta \in (0, 1)$,*

$$\frac{\log|\mathcal{H}_N(\eta)|}{\log N} \to 2(1-\eta^2) \tag{1.6}$$

in probability.

This result is consistent with the conjecture that the level sets of the Ginzburg–Landau model with zero boundary condition converge to CLE(4), a collection of conformally invariant random loops (see [48] for the definition of the CLE and how to construct a coupling with GFF). Theorem 1.3 will be proved in Section 5.4. The main step in the proofs of the upper bound (4.17) and the upper bound of (1.6) is the following pointwise tail bound for the Ginzburg–Landau field (1.3). Here and in the sequel of the paper, for a set $A \subset \mathbb{Z}^2$ and a point $v \in \mathbb{Z}^2$, we use $\operatorname{dist}(v, A)$ to denote the (lattice) distance from v to A.

THEOREM 1.4. *Let g be the constant as in Theorem* 1.1. *Given any $C < \infty$, we have for all $v \in D_N$, and all $0 < u < C\log\operatorname{dist}(v, \partial D_N)$,*

$$\mathbb{P}^{D_N,0}(\phi(v) \geq u) \leq \exp\biggl(-\frac{u^2}{2g\log\operatorname{dist}(v,\partial D_N)} + o(\log\operatorname{dist}(v,\partial D_N))\biggr). \tag{1.7}$$

The tail bound (1.7) was only known for a class of potentials $V(\cdot)$ that has elliptic contast at most 2 (i.e., $c_0 \le V'' \le 2c_0$, for some $c_0 > 0$) and bounded third derivative, and ϕ is the *infinite volume* limit of the Gibbs measure (1.3) (see [22]). Theorem 1.4 will be proved in Section 4.

1.3. *Historical survey.*

1.3.1. *Ginzburg–Landau fields.* The Gibbs measure (1.3) was first introduced by Brascamp, Lebowitz and Lieb, in the name of anharmonic crystals [19]. It is believed that the large scale behaviors of this class of Gibbs measures resemble that of the Gaussian free field. Rigorous mathematical studies for convex perturbations of GFF (in particular, the special example called lattice dipole gas) were initiated by the renormalization group approach of [30], and further developed by [20], which confirm its correlation function behaves like a continuous GFF in the scaling limit. Renormalization group is a powerful tool to study gradient field models, but it is only applicable in the perturbative case, that is, when the potential is given by a small perturbation of Gaussian, and thus the Hessian of the Hamiltonian is close to the standard Laplacian. The nonperturbative approach that allows one to study any convex potential V is based on the Helffer–Sjöstrand formula [32, 33] that represents the mean and covariance of such fields in terms of an elliptic operator (or, probabilisticly, a random walk in dynamic random environment). We give here an incomplete list of references that study the scaling limits of gradient field models. The classification of the gradient Gibbs states on $\mathbb{Z}^d$ were proved by Funaki and Spohn [29]. Deuschel, Giacomin and Ioffe [25] studied the large deviation principle of the macroscopic surface profile in a bounded domain, where they also introduce the random walk representation of the Helffer–Sjöstrand formula. The central limit theorem for linear functionals of the gradient fields was first established by Naddaf and Spencer [44] for the infinite volume gradient Gibbs states with zero tilt (the corresponding dynamical CLT was proved in [31]), and later by Miller [43] for the gradient fields in bounded domains. It is also proved in [42] that the level set for such gradient fields in a bounded domain (with certain Dirichlet boundary condition) converges to the chordal SLE(4), an example of the conformally-invariant random curve in the plane known as the Schramm–Loewner Evolution (for a survey on SLE see, e.g., [39]).

Nonlinear functionals of the gradient fields are much less known. With additional bounded ellipticity assumption on V, it is proved by Conlon and Spencer [22] that for the infinite gradient-Gibbs states with zero slope, there exists $C < \infty$ such that

$$\left|\log \mathbb{E}\left[e^{t(\phi(0)-\phi(x))}\right] - \frac{t^2}{2}\operatorname{Var}\left[\phi(0)-\phi(x)\right]\right| \le Ct^3 \|V'''\|_\infty.$$

Their argument is based on the Helffer–Sjöstrand formula and operator theory on weighted Hilbert space. This phenomenon is remarkable because it indicates the pointwise distribution of $\phi(0) - \phi(x)$ is nearly Gaussian, and one has to go to the large deviation regime (corresponding to $t = O(\log|x|)$) to see non-Gaussian tails. In this paper, we remove the bounded ellipticity assumption, and rely our proof on a different strategy.

1.3.2. *Extrema of log-correlated random fields.* Although the macroscopic behavior of linear functionals of the gradient fields are now well understood, finer properties of the field, such as the behavior of its maximum, remain to be clarified. Questions about the maximum fit into the wider context of the study of extrema of log-correlated random fields.

Multiscale analysis is the key to study the extrema of such random fields. The conceptually simplest cases, which already exhibit the most crucial phenomena underlying the behavior the extrema, are tree models such as Branching Brownian Motion and Branching Random

Walk. In his seminal work, Bramson introduced a truncated second moment method to study Branching Brownian Motion [13, 17]. This method has been much refined to obtain detailed results to the level of the convergence of the extremal process in Bramson's setting and for Branching Random Walk [1, 2, 6, 14, 41].

Beyond such processes, the most investigated case is the Gaussian free field. The discrete Gaussian free field is the special case $V(x) = \frac{1}{2}x^2$in the present set-up. Bolthausen, Deuschel and Giacomin [12] first showed the equivalent of our main result, which was later improved [15, 16] to

$$\sup_{x\in D_N} \phi(x) = 2\sqrt{g_0}\log N - \frac{3}{4}\sqrt{g_0}\log\log N + O(1) \quad \text{as } N \to \infty,$$

where $g_0 = 2/\pi$. Furthermore, it has been proved that the $O(1)$ term converges in law and the geometric properties of the near extrema has been studied, including the convergence of the extremal process [9–11, 27]. The equivalent of our Theorem 1.3 for the discrete Gaussian free field was proved in [23]. Some results have been generalized to a wider class of log-correlated Gaussian fields [26].

The article [8] studied the extrema of a log-correlated field that is neither Gaussian nor endowed with an exact tree structure. It constructed what can be interpreted as a sequence of regularizations of the field and from these obtained a collection of approximate branching random walks indexed by the points of the field, to which Bramson's method can be applied (regularization also plays an important role in problems connected to the continuum Gaussian free field [28, 34, 46, 47]).

[37] adapted this approach to the Gaussian free field, with the regularizations given by harmonic averages on concentric boxes ("local projections"). It also describes a "K-level coarse-graining" which is a particularly streamlined version of the multiscale argument that provides leading order estimates for the maximum from minimal technical inputs. Subsequently, versions of it has been used to study the extrema of many cases of non-Gaussian log-correlated random fields [4, 5, 21, 38, 45].

1.4. *Proof strategy.* To prove the tail bound (1.7), the estimates (1.5) for the maximum and (1.6) for the high points we adapt the aforementioned local projections and K-level coarse-graining of [37]. Namely, we consider the harmonic averages over circles of the field around each point, as a process indexed by the side-length of the box, and use the first moment method to obtain an upper bound for the maximum and a truncated second moment argument involving the average process to get a lower bound for the maximum. These average processes are expected to evolve similarly to branching random walks as one varies the side-length of the box at dyadic scales. For the Gaussian free field, Gaussian orthogonal decomposition implies the increments of such harmonic averages are independent, making the random walk approximation fairly straightforward. This fails for the general gradient field models studied in this paper. In fact, one of the main contributions of this paper is to prove the asymptotic decoupling of these increments (Theorem 4.3). We apply the useful tool from [43], that gives an approximate harmonic coupling of the Ginzburg–Landau field on a bounded domain with different boundary conditions. Inspired by the K-level coarse-graining of [37] we exploit that for the level of accuracy we seek in the present paper, it is enough to consider the behavior of the approximate random walks over a relatively small number of large increments, corresponding to a small number of scales (only finitely many in the case of the Gaussian free field; for technical reasons, we use a slowly growing number of increments). The approximate harmonic coupling allows us to show that each increment of the harmonic average, conditioned on the Ginzburg–Landau field outside, is distributed not far from a Gaussian, after discarding a thin layers between each scale. This gives the pointwise tail bound for the Ginzburg–Landau field, and thus also the upper bound in Theorem 1.1. A similar argument via the truncated second moment method gives the two-point tail bounds needed to obtain the lower bound in Theorem 1.1.

1.5. *Open question.* We finish the Introduction with a corresponding open question for dimer models. A (uniform) dimer model on $\mathbb{Z}^2$ can be thought of as an integer valued random surface $h(v)$, $v \in \mathbb{Z}^2$. It is an integrable model with determinantal structure. It is shown in [35] and [36] that the height fluctuation $h(0) - h(v)$ has logarithmic variance and, moreover, the rescaled height function converges weakly to GFF. A main conjecture in this field is that the level sets of the height function converges to CLE(4). Still, it would be very interesting to prove the maximum of the dimer height function satisfies Theorem 1.1. The method in the present paper does not apply directly because the harmonic coupling (see Section 2.3) have not yet been established for the dimer model.

2. Tools.

2.1. *Brascamp–Lieb inequality.* One can bound the variances and exponential moments with respect to the Ginzburg–Landau measure by those with respect to the Gaussian measure, using the following Brascamp–Lieb inequality. Let ϕ be sampled from the Gibbs measure (1.3), with a nearest-neighbor potential $V \in C^2(\mathbb{R})$ that satisfies $\inf_{x\in\mathbb{R}} V''(x) \geq c_- > 0$. Given $f \in \mathbb{R}^{D_N}$, we define

$$\langle \phi, f\rangle := \sum_{x\in D_N} \phi(x) f(x).$$

LEMMA 2.1 (Brascamp–Lieb inequalities [18]). *Let $\mathbb{E}^{D_N,0}_{\mathrm{DGFF}}$ and $\mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}$ denote the expectation and variance with respect to the discrete GFF measure* (*i.e.*, (1.3) *with* $V(x) = x^2/2$). *Then, for any $f \in \mathbb{R}^{D_N}$,*

$$\mathrm{Var}^{D_N,0}\langle\phi, f\rangle \leq c_-^{-1} \mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}\langle\phi, f\rangle, \tag{2.1}$$

$$\begin{aligned} &\mathbb{E}^{D_N,0}\big(\langle\phi, f\rangle - \mathbb{E}^{D_N,0}\langle\phi, f\rangle\big)^{2k} \\ &\quad\leq c_-^{-k}\mathbb{E}^{D_N,0}_{\mathrm{DGFF}}\big(\langle\phi, f\rangle - \mathbb{E}^{D_N,0}_{\mathrm{DGFF}}\langle\phi, f\rangle\big)^{2k} \quad \text{for } k \in \mathbb{N}, \end{aligned} \tag{2.2}$$

$$\mathbb{E}^{D_N,0}\big[\exp\big(\langle\phi, f\rangle - \mathbb{E}^{D_N,0}\langle\phi, f\rangle\big)\big] \leq \exp\bigg(\frac{1}{2}c_-^{-1} \mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}\langle\phi, f\rangle\bigg). \tag{2.3}$$

The Brascamp–Lieb inequalities can be used to show the following a priori tail bound for ϕ.

LEMMA 2.2. *There is a positive constant c_{BL} such that*

$$\mathbb{P}^{D_N,0}\big(\phi(v) \geq u\big) \leq e^{-c_{\mathrm{BL}}\frac{u^2}{\mathrm{dist}(v,\partial D_N)}} \quad \text{for } v \in D_N. \tag{2.4}$$

PROOF. By Chebyshev's inequality,

$$\mathbb{P}^{D_N,0}\big(\phi(v) \geq u\big) \leq e^{-tu}\mathbb{E}^{D_N,0} \exp\big(t\phi(v)\big).$$

Applying the Brascamp–Lieb inequality with $f = \delta_v$, and using the fact that (see the Green's function asymptotics in [40])

$$\mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}\phi(v) = G_{D_N}(v, v) = \sqrt{2/\pi} \log \mathrm{dist}(v, \partial D_N) + O(1),$$

we have

$$\mathbb{P}^{D_N,0}\big(\phi(v) \geq u\big) \leq \exp\left(-tu + \frac{t^2}{2} c_1 \log \operatorname{dist}(v, \partial D_N)\right).$$

Optimizing over t then yields the result. $\square$

REMARK 2.3. By a union bound over the $(2N+1)^2$ points of D_N and take $u \gg \sqrt{1/c_{\mathrm{BL}}} \log N$, so that the right-hand side of (2.4) is $\ll N^{-2}$, one obtains an upper bound of $\sqrt{1/c_{\mathrm{BL}}} \log N$ for the maximum of $\phi(v)$. This is an upper bound of the right order, but the constant in front of $\log N$ is larger than the "true" one $2\sqrt{g}$.

2.2. *The Helffer–Sjöstrand representation.* In this section, we summarize the idea of [44] (which was in turn inspired by the works [33]), that the variance of functions with respect to the Ginzburg–Landau field can be written in terms of the Helffer–Sjöstrand operator. The Helffer–Sjöstrand representation was used crucially in [44] to prove a central limit theorem for the statistics of $\nabla\phi$.

It is known that the finite volume measure (1.3) is invariant under the Langevin-dynamics

$$\begin{cases} d\phi_t(x) = \sum_{y \sim x} V'\big(\phi_t(y) - \phi_t(x)\big)\, dt + \sqrt{2}\, dB_t(x) & x \in D_N^\circ, \\ \phi_t(x) = 0 & x \in \partial D_N, \end{cases} \tag{2.5}$$

where $\{B_t(x) : x \in D_N^\circ\}$ is a family of independent Brownian motions. Let $\omega_x : D_N \to \mathbb{R}$ be defined by $\omega_x(y) = 1_{x=y}$. The infinitesimal generator of this process is the operator Δ_ϕ defined by

$$\Delta_\phi F(\phi) := \sum_{x \in D_N^\circ} \partial_x^2 F(\phi) - \sum_{x \in D_N^\circ} \sum_{y \sim x} V'\big(\phi(y) - \phi(x)\big) \partial_x F(\phi),$$

where

$$\partial_x F(\phi) := \lim_{h \to 0} \frac{1}{h}\big(F(\phi + h\omega_x) - F(\phi)\big).$$

Define the Helffer–Sjöstrand operator $\mathcal{L} := -\Delta_\phi + \nabla^* V''(\nabla\phi)\nabla$. Probabilistically, $\mathcal{L}$ is the generator for the Markov process (X_t, ϕ_t), where ϕ_t is the Langevin dynamics (2.5) and X_t is a continuous time random walk in D_N with jump rates $V''(\phi_t(y) - \phi_t(x))$, stopped when hitting the boundary.

The following representation of the variance is obtained in [44] (see also [25]).

PROPOSITION 2.4 (Helffer–Sjöstrand representation). *For all F such that*

$$\mathbb{E}^{D_N,0}\left[F(\phi)^2 + \sum_{x \in D_N^\circ} \big(\partial_x F(\phi)\big)^2\right] < \infty,$$

we have

$$\operatorname{Var}^{D_N,0}\big[F(\phi)\big] = \langle \partial F, \mathcal{L}^{-1} \partial F\rangle,$$

where $\langle \partial F, \mathcal{L}^{-1}\partial F\rangle := \sum_{x,y \in D_N^\circ} \mathbb{E}^{D_N,0}[\partial_x F \mathcal{L}_{xy}^{-1} \partial_y F]$.

If we consider a linear statistics of ϕ, and take $F(\phi) = \sum_{x \in D_N} \rho(x)\phi(x)$ for some test function ρ, then the above proposition implies

$$\operatorname{Var}^{D_N,0}\left[\sum_{x \in D_N} \rho(x)\phi(x)\right] = \langle \rho, \mathcal{L}^{-1}\rho\rangle. \tag{2.6}$$

So that the variance of a linear statistics is given by a bilinear quadratic form.

2.3. *Approximate harmonic coupling*. By definition, the Ginzburg–Landau measures satisfy the domain Markov property: conditioned on the values on the boundary of a domain, the field inside the domain is again a gradient field with boundary condition given by the conditioned values. For the discrete GFF, there is in addition a nice orthogonal decomposition. More precisely, the conditioned field inside the domain is the discrete harmonic extension of the boundary value to the whole domain plus an *independent* copy of a *zero boundary* discrete GFF.

While this exact decomposition does not carry over to general Ginzburg–Landau measures, the next result due to Jason Miller (see [43]), provides an approximate version. For $D \subset \mathbb{Z}^2$, define the Ginzburg–Landau measure on D with Dirichlet boundary condition f by

$$d\mu_D^f = Z_D^{-1} \exp\left[-\sum_{x \in D^\circ} \sum_{y \sim x} V(\phi(x) - \phi(y))\right] \prod_{x \in D^\circ} d\phi(x) \prod_{x \in \partial D} \delta_0(\phi(x) - f(x)). \tag{2.7}$$

THEOREM 2.5 (Theorem 1.2 in [43]). *Let $D \subset \mathbb{Z}^2$ be a simply connected domain of diameter R, and denote $D^r = \{v \in D : \operatorname{dist}(v, \partial D) > r\}$. Let Λ be such that $f : \partial D \to \mathbb{R}$ satisfies $\max_{x \in \partial D} |f(x)| \le \Lambda |\log R|^\Lambda$. Let ϕ be sampled from the measure (2.7) with zero boundary condition, and ϕ^f be sampled from the measure (2.7) with boundary condition f. Then there exist constants $c, \gamma, \delta \in (0, 1)$, that only depend on V, so that if $r > cR^\gamma$ then the following holds. There exists a coupling (ϕ, ϕ^f), such that if $\hat{\phi} : D^r \to \mathbb{R}$ is discrete harmonic with $\hat{\phi}|_{\partial D^r} = (\phi^f - \phi)|_{\partial D^r}$, then*

$$\mathbb{P}(\phi^f = \phi + \hat{\phi} \text{ in } D^r) \ge 1 - c(\Lambda) R^{-\delta}.$$

An immediate application of Theorem 2.5 shows that the mean of a Ginzburg–Landau field at one point in the bulk is approximately (discrete) harmonic.

THEOREM 2.6 (Theorem 1.3 in [43]). *Suppose the same conditions in Theorem 2.5 holds. Let ϕ^f, c, γ, δ, D^r be defined as in Theorem 2.5. For all $r > cR^\gamma$, and discrete harmonic function $\hat{\phi} : D^r \to \mathbb{R}$ with $\hat{\phi}|_{\partial D^r} = \mathbb{E}\phi^f|_{\partial D^r}$, then*

$$\max_{v \in D^r} |\mathbb{E}\phi^f(v) - \hat{\phi}(v)| \le c'(\Lambda) R^{-\delta}.$$

Theorem 2.5 allows to compare a Ginzburg–Landau field with nonzero boundary condition with one that has zero boundary condition. Since Theorem 2.5 requires that the function f is not too large, we introduce the "good" event

$$\mathcal{G}(c) = \left\{\phi : \max_{v \in D} |\phi(v)| < c(\log R)^2\right\},$$

which is typical since even using only Brascamp–Lieb one has that $\max_{v \in D} |\phi(v)| \le O(\log R)$ with high probability. Indeed, we have the following.

LEMMA 2.7. *There is some $c_1 = c_1(c) > 0$, such that $\mathbb{P}^{D,0}(\mathcal{G}(c)^c) \le \exp(-c_1(\log R)^3)$.*

PROOF. By the union bound,

$$\mathbb{P}^{D,0}(\mathcal{G}^c) \le \sum_{v \in D} \mathbb{P}^{D,0}(|\phi(v)| > c(\log R)^2).$$

We apply Lemma 2.2, to obtain

$$\mathbb{P}^{D,0}(|\phi(v)| > c(\log R)^2) \le \exp(-(4C)^{-1}(\log R)^3 + O(\log R)^2),$$

for some $C < \infty$, and summing over $v \in D$ then completes the proof. $\square$

We will use repeatedly the following consequence of Theorem 2.5. It applies to functions ρ such that the integral of ρ against a harmonic function is always zero.

LEMMA 2.8. *There exist constants $\delta, \gamma > 0$ such that for any simply connected $D \subset \mathbb{Z}^2$ of diameter R, any $r > R^\gamma$ and any $\rho : D \to \mathbb{R}$ supported on D^r that satisfies $\sum_{x \in D^r} \rho(x) f(x) = 0$ for all functions f harmonic in D^r, and $\frac{1}{R}\sum_{y \in D} |\rho(y)| < \infty$, we have for R large enough,*

$$\left|\mathbb{E}^{D,f}\left[\exp\left(R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right)1_{\mathcal{G}}\right] - \mathbb{E}^{D,0}\left[\exp\left(R^{-1}\sum_{x\in D}\rho(x)\phi(x)\right)1_{\mathcal{G}}\right]\right|$$
$$\leq 2\exp\left(c\,\mathrm{Var}^{D,0}_{\mathrm{DGFF}}\left(R^{-1}\sum_{x\in D}\rho(x)\phi(x)\right)\right)R^{-\delta},$$

for some $c < \infty$.

REMARK 2.9. This lemma is useful if $\mathrm{Var}^{D,0}_{\mathrm{DGFF}}(R^{-1}\sum_{x\in D}\rho(x)\phi(x)) \ll \delta \log R$.

PROOF. Applying Theorem 2.5, there is an event $\mathcal{C}$ with $\mathbb{P}(\mathcal{C}^c) \leq R^{-\delta_0}$, where δ_0 is the constant δ in Theorem 2.5, such that on $\mathcal{C}$ we have $\phi^f - \phi = \hat{\phi}$ in D^r. Therefore, on $\mathcal{C}$

$$\sum_{x\in D}\rho(x)\phi^f(x) = \sum_{x\in D^r}\rho(x)\phi^f(x) = \sum_{x\in D^r}\rho(x)\phi(x) + \sum_{x\in D^r}\rho(x)\hat{\phi}(x)$$
$$= \sum_{x\in D^r}\rho(x)\phi(x) = \sum_{x\in D}\rho(x)\phi(x),$$

where the first and the last equality follows from the fact that ρ is supported in D^r. On $\mathcal{C}^c$ we apply Hölder's inequality to obtain

$$\mathbb{E}^{D,f}\left[\exp\left(R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right)1_{\mathcal{G}\cap\mathcal{C}^c}\right]$$
$$\leq \mathbb{P}(\mathcal{C}^c)^{1/2}\mathbb{E}^{D,f}\left[\exp\left(2R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right)\right]^{1/2}$$
$$\leq R^{-\delta_0/2}\mathbb{E}^{D,f}\left[\exp\left(2R^{-1}\sum_{x\in D}\rho(x)\phi^f(x) - \mathbb{E}^{D,f}\left[2R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right]\right)\right]^{1/2} \tag{2.8}$$
$$\times\exp\left(\mathbb{E}^{D,f}\left[R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right]\right).$$

By the Brascamp–Lieb inequality (2.3), there exist some $c < \infty$, such that

$$\mathbb{E}^{D,f}\left[\exp\left(2R^{-1}\sum_{x\in D}\rho(x)\phi^f(x) - \mathbb{E}^{D,f}\left[2R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right]\right)\right]$$
$$\leq \exp\left(c\,\mathrm{Var}^{D,f}_{\mathrm{DGFF}}\left(R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right)\right). \tag{2.9}$$

On the other hand, applying Theorem 2.6 yields

$$\left|\mathbb{E}^{D,f}\left[R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right]\right| = \left|\mathbb{E}^{D,f}\left[R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right] - R^{-1}\sum_{x\in D}\rho(x)\hat{\phi}(x)\right|$$
$$\leq \|\mathbb{E}^{D,f}\phi^f - \hat{\phi}\|_{L^\infty(D^r)}\frac{1}{R}\sum_{x\in D}|\rho(x)| \leq CR^{-\delta_0}. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we have for R large enough

$$\mathbb{E}^{D,f}\left[\exp\left(R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right)1_{\mathcal{G}\cap\mathcal{C}^c}\right] \leq C\exp\left(c\operatorname{Var}_{\mathrm{DGFF}}\left(R^{-1}\sum_{x\in D}\rho(x)\phi^f(x)\right)\right)R^{-\delta_0/2}.$$

And similarly,

$$\mathbb{E}^{D,0}\left[\exp\left(R^{-1}\sum_{x\in D}\rho(x)\phi(x)\right)1_{\mathcal{G}\cap\mathcal{C}^c}\right] \leq C\exp\left(c\operatorname{Var}_{\mathrm{DGFF}}\left(R^{-1}\sum_{x\in D}\rho(x)\phi(x)\right)\right)R^{-\delta_0/2}.$$

Since the variance of linear functionals of Gaussian free field does not depend on boundary conditions, we complete the proof. □

2.4. *Central limit theorem.* We now state the central limit theorem for macroscopic averages of ϕ, proved in [43] as a consequence of Theorem A in [44] and Theorem 2.5 stated above.

Let $D\subset\mathbb{R}^2$ be a smooth simply connected domain. Before stating the central limit theorem, we give the definition of the (continuum) a-Gaussian Free Field (a-GFF) h in D with zero boundary condition, where a is a 2×2 positive definite matrix. The a-GFF in D is the standard Gaussian in $H_0^1(D)$, such that for any $f\in H_0^1(D)$, $\int_D \nabla h\cdot\nabla f$ is a Gaussian random variable with mean 0 and variance $\int_D \nabla f\cdot a\nabla f$. The a-GFF arise naturally as the scaling limit of the fluctuation of the Ginzburg–Landau field (2.7) with general boundary data; see [43]. In this paper, we only use the following central limit theorem for the measure with zero boundary condition (1.3), and a becomes to a diagonal matrix (or a scalar) due to the rotational symmetry.

THEOREM 2.10. *Let $D\subset\mathbb{R}^2$ be a piecewise smooth, simply connected domain, and $D^{(N)}=D\cap\frac{1}{N}\mathbb{Z}^2$. Let ϕ be sampled from the Ginzburg–Landau measure on $D^{(N)}$ with zero boundary condition. Suppose that the sequence of functions $\rho_N: D^{(N)}\to\mathbb{R}$ satisfies*

$$\sum_{x\in D^{(N)}}\rho_N(x)H(x)=0 \quad \text{for any harmonic function } H: D^{(N)}\to\mathbb{R}. \tag{2.11}$$

Also assume there exist some $\rho\in C_0^\infty(D)$, such that

$$\int_D \rho(x)H(x)\,dx=0 \quad \text{for any harmonic function } H: D\to\mathbb{R}, \tag{2.12}$$

and $N\|\rho_N-\rho\|_{L^\infty(D)}\to 0$ as $N\to\infty$. Then the linear functional

$$N^{-1}\sum_{x\in D^{(N)}}\rho_N(x)\phi(x)$$

converges in L^{2k}, $k\in\mathbb{N}$, to the random variable

$$\int_D h(x)\rho(x)\,dx,$$

where h is the a-GFF on D with zero boundary condition, for some $a(V)=\bar{a}(V)I$ that satisfies $c_-\leq\bar{a}\leq c_+$.

We will apply Theorem 2.10 with ρ_N defined in terms of the harmonic measure of a discrete cube (see (3.3) and comments thereafter) and ρ will be the corresponding quantity defined using the harmonic measure of the standard Brownian motion.

PROOF. If $\rho_N = \rho$ for all $N \geq 1$, this is a consequence of Theorem 1.1 in [43]. It suffices to show that if ρ_N converges to ρ sufficiently fast, the linear statistics converges to the same limit. This can be shown by a direct comparison of their variance using the Helffer–Sjöstrand representation (Proposition 2.4).

In fact, applying (2.6) and use bilinearity we have

$$\begin{aligned}
&\mathrm{Var}\Big[N^{-1}\sum_{x\in D^{(N)}}\rho_N(x)\phi(x)\Big]-\mathrm{Var}\Big[N^{-1}\sum_{x\in D^{(N)}}\rho(x)\phi(x)\Big] \\
&\quad=\Big\langle\frac{1}{N}(\rho_N-\rho),\mathcal{L}^{-1}\frac{1}{N}\rho\Big\rangle+\Big\langle\frac{1}{N}\rho_N,\mathcal{L}^{-1}\frac{1}{N}(\rho_N-\rho)\Big\rangle,
\end{aligned}\tag{2.13}$$

where we recall $\mathcal{L} := -\Delta_\phi + \nabla^* V''(\nabla\phi)\nabla$ is the Helffer–Sjöstrand operator. The uniform convexity assumption of V implies that $\mathcal{L} \geq c_-\Delta$, and therefore $\nabla^*\mathcal{L}^{-1}\nabla \leq c_-^{-1}\nabla^*\Delta^{-1}\nabla$ is a bounded operator $\mathbb{Z}^2 \to \mathbb{Z}^2$.

Notice that (2.11) and (2.12) implies we can write $\frac{1}{N}\rho_N = \nabla g_N$ and $\frac{1}{N}\rho = \nabla g$ for some g, g_N, and $N\|\rho_N - \rho\|_{L^\infty(D)} \to 0$ suggests that one may take g, g_N so that $\|g_N - g\|_{L^\infty(D^{(N)})} = o(\frac{1}{N})$. Substitute them into (2.13) and we obtain

$$\begin{aligned}
&\Big|\mathrm{Var}\Big[N^{-1}\sum_{x\in D^{(N)}}\rho_N(x)\phi(x)\Big]-\mathrm{Var}\Big[N^{-1}\sum_{x\in D^{(N)}}\rho(x)\phi(x)\Big]\Big| \\
&\quad\leq\big|\langle g_N-g,\nabla^*\mathcal{L}^{-1}\nabla g\rangle\big|+\big|\langle g_N,\nabla^*\mathcal{L}^{-1}\nabla(g_N-g)\rangle\big| \\
&\quad\leq C\|g_N-g\|_{L^2(D^{(N)})}\big(\|g\|_{L^2(D^{(N)})}+\|g\|_{L^2(D^{(N)})}\big).
\end{aligned}$$

Since $\|g_N - g\|_{L^\infty(D^{(N)})} = o(\frac{1}{N})$, we have $\|g_N - g\|_{L^2(D^{(N)})} \to 0$ as $N \to \infty$, and this completes the proof for $k = 1$. The general case $k \geq 2$ follows from combining the above argument for $k = 1$ with the exponential Brascamp–Lieb inequality (2.3). □

For the rest of the paper, we will only apply the convergence of the second moment (i.e., $k = 1$ result) in Theorem 2.10.

3. Harmonic averages. Our method to prove Theorem 1.4 is built upon Theorem 2.5 and a detailed study of the harmonic average of the Ginzburg–Landau field. Given $B \subset \mathbb{Z}^2$, $v \in B$ and $y \in \partial B$, we denote by $a_B(v, \cdot)$ the harmonic measure on ∂B seen from v. In other words, let S^x denote the simple random walk starting at x, and $\tau_{\partial B} = \inf\{t > 0 : S^x[t] \in \partial B\}$, we have

$$a_B(x, y) = \mathbb{P}\big(S^x[\tau_{\partial B}] = y\big).$$

Given $v \in \mathbb{Z}^2$ and $R > r > 0$, let $B_R(v) = \{y \in \mathbb{Z}^2 : |v_1 - y_1| \leq R, |v_2 - y_2| \leq R\}$, and $A_{r,R}(v) := B_R(v) \setminus B_r(v)$. Define the circle average of the Ginzburg–Landau field with radius R at v by

$$C_R(v, \phi) = \sum_{y\in\partial B_R(v)} a_{B_R(v)}(v, y)\phi(y). \tag{3.1}$$

For each $\varepsilon, R > 0$, such that $(1 + \varepsilon)R < \mathrm{dist}(v, \partial D_N)$, we take a nonnegative smooth radial function $f_\varepsilon \in C_c^\infty([1 - \varepsilon, 1 + \varepsilon])$ such that $f_\varepsilon(1 - s) = f_\varepsilon(1 + s)$ for $s \in [0, \varepsilon]$ and $\int_{1-\varepsilon}^{1+\varepsilon} f_\varepsilon(s)\,ds = 1$. We further define

$$X_R(v, \phi) = \sum_{r=(1-\varepsilon)R}^{(1+\varepsilon)R} f_\varepsilon(r/R)C_r(v, \phi). \tag{3.2}$$

The crucial object that we use below is the increment of the harmonic average process X. For $v \in D_N$, $(1+\varepsilon)^{-1}\operatorname{dist}(v, \partial D_N) > R_1 > R_2 > 0$, we would like to study the increment

$$\begin{aligned} X_{R_2}(v,\phi) - X_{R_1}(v,\phi) \\ = \left(\sum_{r=(1-\varepsilon)R_2}^{(1+\varepsilon)R_2} f_\varepsilon(r/R_2) - \sum_{r=(1-\varepsilon)R_1}^{(1+\varepsilon)R_1} f_\varepsilon(r/R_1)\right) \sum_{y\in\partial B_r(v)} a_{B_r(v)}(v,y)\phi(y). \end{aligned}$$

This can be written as $\sum_{y\in D_N} \rho_N(v,y)\phi(y)$, where we define

$$\rho_{R_1,R_2}(v,y) = \left[f_\varepsilon\left(\frac{|v-y|}{R_2}\right) - f_\varepsilon\left(\frac{|v-y|}{R_1}\right)\right] a_{B_{|v-y|}(v)}(v,y). \tag{3.3}$$

The definition of ρ_{R_1,R_2} depends on R_1, R_2. Later we will take R_1, R_2 at some scale $r(N)$, where $r(N)$ grows to infinity as a power of N. In such situation, we will simply denote ρ_{R_1,R_2} as $\rho_{r(N)}$ or ρ_N, to emphasize its dependence on N.

LEMMA 3.1. *For any discrete harmonic function h in D_N, we have $\sum_{y\in D_N} \rho_N(v,y) \times h(y) = 0$.*

PROOF. Suppose h is define up to ∂D_N, and $h|_{\partial D_N} = H$. We conclude the proof by showing for $i = 1, 2$,

$$\sum_{r=(1-\varepsilon)R_i}^{(1+\varepsilon)R_i} f_\varepsilon(r/R_i) \sum_{y\in\partial B_r(v)} a_{B_r(v)}(v,y)h(y) = h(v).$$

Indeed, since h is harmonic,

$$h(y) = \sum_{z\in\partial D_N} a_{D_N}(y,z)H(z).$$

Using the fact that

$$\sum_{y\in\partial B_r(v)} a_{B_r(v)}(v,y)a_{D_N}(y,z) = a_{D_N}(v,z),$$

we obtain

$$\begin{aligned} \sum_{r=(1-\varepsilon)R_i}^{(1+\varepsilon)R_i} f_\varepsilon(r/R_i) \sum_{y\in\partial B_r(v)} a_{B_r(v)}(v,y)h(y) &= \sum_{r=(1-\varepsilon)R_i}^{(1+\varepsilon)R_i} f_\varepsilon(r/R_i) \sum_{z\in\partial D_N} a_{D_N}(v,z)H(z) \\ &= h(v). \end{aligned}$$

□

The following result is a consequence of Theorem 2.5 and the lemma above.

LEMMA 3.2. *Suppose the same conditions in Theorem 2.5 holds. Given $v \in D_N$, $R_1 > R_2 > 0$, $\varepsilon > 0$ such that $(1+2\varepsilon)R_1 < \operatorname{dist}(v,\partial D_N)$, $(1+2\varepsilon)R_2 < (1-2\varepsilon)R_1$. Let δ be the constant from Theorem 2.5. Let ϕ^f be sampled from Ginzburg–Landau field (2.7), and ϕ^0 be sampled from the zero boundary Ginzburg–Landau field on D_N. Then, on an event with probability $1 - O(R_1^{-\delta})$, we have*

$$X_{R_2}(v,\phi^f) - X_{R_1}(v,\phi^f) = X_{R_2}(v,\phi^0) - X_{R_1}(v,\phi^0).$$

We sometimes omit the dependence of X on v and ϕ when it is clear from the context.

We are mostly concerned with large deviation estimates and, therefore, with moment generating functions. Thus we will use Proposition 3.3 below, which gives a Gaussian limit of the moment generating function of macroscopic observables.

Now fix $v=0$. Given $\varepsilon>0$ fixed in the definition (3.2) and $r>0$, take $\varepsilon_1=\varepsilon^{1/4}$, and note that we can write

$$X_{(1+\varepsilon_1)r}(0,\phi)=\sum_{y\in D_N}\rho_{r,+}(y)\phi(y),$$

$$X_{(1-\varepsilon_1)r}(0,\phi)=\sum_{y\in D_N}\rho_{r,-}(y)\phi(y).$$

Let $A_{r_1,r_2}=B_{r_2}(0)\setminus B_{r_1}(0)$. Note that $\rho_{r,+}$ and $\rho_{r,-}$ are supported on annuli $A_{(1+\varepsilon_1-\varepsilon)r,(1+\varepsilon_1+\varepsilon)r}$ and $A_{(1-\varepsilon_1-\varepsilon)r,(1-\varepsilon_1+\varepsilon)r}$, respectively, and let $\rho_r=\rho_{r,+}-\rho_{r,-}$. We further notice that as $r\to\infty$, the rescaled harmonic measure

$$ra_{B_r(0)}(0,\cdot)\to h(\cdot),$$

where h is the Poisson kernel for a unit square in $\mathbb{R}^2$; see [40], Chapter 8.1 for an explicit formula. Thus as $r\to\infty$ and $y/r\to x$, $r\rho_{r,+}$, $r\rho_{r,-}$ converge respectively to the smooth functions

$$f^+(x)=\frac{h(x)}{|x|}f_\varepsilon(|x|),\qquad x\in A_{(1+\varepsilon_1-\varepsilon),(1+\varepsilon_1+\varepsilon)},$$

$$f^-(x)=\frac{h(x)}{|x|}f_\varepsilon(|x|),\qquad x\in A_{(1-\varepsilon_1-\varepsilon),(1-\varepsilon_1+\varepsilon)}.$$

To simplify the notation below, we write $A_+:=A_{(1+\varepsilon_1-\varepsilon),(1+\varepsilon_1+\varepsilon)}$ and $A_-:=A_{(1-\varepsilon_1-\varepsilon),(1-\varepsilon_1+\varepsilon)}$. We further denote $f=f^+-f^-$. The following estimate is proved by combining Theorem 2.10 with the Brascamp–Lieb inequality.

PROPOSITION 3.3. *Let $D=[-1,1]^2$, $v=0$, and fix some $t\in(0,\infty)$. Then for any $\varepsilon_1>0$ small enough, depending on t, and $r=r(N)$ such that $N/4<(1-\varepsilon_1)r<(1+\varepsilon_1)r<N$, we have that*

$$\begin{aligned}&\log\mathbb{E}^{D_N,0}\big[\exp\big(t(X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r})\big)\big]\\&\quad=\frac{t^2}{2}\int_{A_+\cup A_-}f(x)g_{a,D}(x,y)f(y)\,dx\,dy+f_1(\varepsilon_1,r,t)t^2+f_2(\varepsilon_1,t)t^4,\end{aligned}\tag{3.4}$$

where $a(V)=\bar{a}(V)I$ is defined in Theorem 2.10, $g_{a,D}(x,\cdot)$ is the Dirichlet Green's function that solves the PDE

$$\begin{cases}\nabla^*\cdot a\nabla u=\delta(x) & \text{in } D,\\ u=0 & \text{on } \partial D.\end{cases}$$

And there exists $C<\infty$, such that $|f_2(\varepsilon_1,t)|\le C\varepsilon_1^2$, and $f_1(\varepsilon_1,r,t)/\varepsilon_1\to0$ as $N\to\infty$. Moreover, there exists $g=g(V)$, such that

$$\begin{aligned}&\log\mathbb{E}^{D_N,0}\big[\exp\big(t(X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r})\big)\big]\\&\quad=\frac{t^2}{2}g\log\frac{1+\varepsilon_1}{1-\varepsilon_1}+\hat{f}_1(\varepsilon_1,r,t)t^2+f_2(\varepsilon_1,t)\big(t^2+t^4\big),\end{aligned}\tag{3.5}$$

where $\hat{f}_1(\varepsilon_1,r,t)/\varepsilon_1\to0$ as $N\to\infty$.

PROOF. We first show

$$\mathbb{E}^{D_N,0}\big[\exp\big(t(X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r})\big)\big]=\frac{t^2}{2}\operatorname{Var}^{D_N,0}[X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r}]+f_2(\varepsilon_1,t)t^4. \tag{3.6}$$

Indeed, we can expand $\mathbb{E}^{D_N,0}[\exp(t(X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r}))]$ into the Taylor series of t, and bound the higher moments. Use the fact that the distribution of ϕ is symmetric, we can write

$$\begin{aligned}
&\mathbb{E}^{D_N,0}\big[\exp\big(t(X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r})\big)\big]\\
&\quad=\mathbb{E}^{D_N,0}\Big[\exp\Big(t\sum_{x\in D_N}\phi(x)\rho_r(x)\Big)\Big]\\
&\quad=1+\frac{t^2}{2}\operatorname{Var}^{D_N,0}\Big[\sum_{x\in D_N}\phi(x)\rho_r(x)\Big]+\sum_{k=2}^{\infty}\frac{t^{2k}}{(2k)!}\mathbb{E}^{D_N,0}\Big|\sum_{x\in D_N}\phi(x)\rho_r(x)\Big|^{2k}.
\end{aligned}$$

We now claim

$$\sum_{k=2}^{\infty}\frac{t^{2k}}{(2k)!}\mathbb{E}^{D_N,0}\Big|\sum_{x\in D_N}\phi(x)\rho_r(x)\Big|^{2k}=O(\varepsilon_1^2)t^4. \tag{3.7}$$

By the Brascamp–Lieb inequality for even moments (2.2), we have

$$\mathbb{E}^{D_N,0}\Big|\sum_{x\in D_N}\phi(x)\rho_r(x)\Big|^{2k}\le c_-^{-k}\mathbb{E}^{D_N,0}_{\mathrm{DGFF}}\Big|\sum_{x\in D_N}\phi(x)\rho_r(x)\Big|^{2k}\le(2k-1)!!c_-^{-k}\varepsilon_1^{2k}. \tag{3.8}$$

By taking ε_1 small enough such that $\varepsilon_1 t^2<1$, summing over k yields (3.7), and thus concludes (3.6).

To prove (3.4), it suffices to obtain the asymptotic variance of $X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r}$. The Brascamp–Lieb inequality implies $\operatorname{Var}^{D_N,0}[X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r}]\le C\log\frac{1+\varepsilon_1}{1-\varepsilon_1}$ for all $N\ge 1$. Notice that from Lemma 3.1 and standard harmonic measure estimates (see, e.g., [40], Chapter 8.1) $|ra_{B_r(0)}(0,\cdot)-h(\cdot)|=O(1/r^2)$, we see that the spatial average $\sum_{x\in D_N}\phi(x)\rho_r(x)=r^{-1}\sum_{x\in D_N}\phi(x)r\rho_r(x)$satisfies the conditions of Theorem 2.10. Apply Theorem 2.10, and note that $r\rho_r(x)\to f(x)$ as $r\to\infty$, we see that there exists a positive definite 2×2 matrix $a(V)=\bar{a}(V)I$, such that

$$\begin{aligned}
&\operatorname{Var}^{D_N,0}[X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r}]\\
&\quad=\operatorname{Var}^{D_N,0}\Big[r^{-1}\sum_{x\in D_N}\phi(x)r\rho_r(x)\Big]\\
&\quad=\operatorname{Var}^{D}_{a\text{-GFF}}\Big[\int_{A_+}f(x)h(x)\,dx+\int_{A_-}f(x)h(x)\,dx\Big]+f_1(\varepsilon_1,r,t),
\end{aligned}$$

where $f_1(\varepsilon_1,r,t)/\varepsilon_1\to 0$ as $N\to\infty$. Then the definition of the a-GFF implies

$$\begin{aligned}
&\operatorname{Var}^{D}_{a\text{-GFF}}\Big[\int_{A_+}f(x)h(x)\,dx+\int_{A_-}f(x)h(x)\,dx\Big]\\
&\quad=\int_{A_+\cup A_-}f(x)g_{a,D}(x,y)f(y)\,dx\,dy,
\end{aligned} \tag{3.9}$$

which concludes (3.4).

To obtain (3.5), we further claim that there exists $g=g(\bar{a})$, such that

$$\operatorname{Var}^{D_N,0}[X_{(1-\varepsilon_1)r}-X_{(1+\varepsilon_1)r}]=g\log\frac{1+\varepsilon_1}{1-\varepsilon_1}+\hat{f}_1(\varepsilon_1,r)+O(\varepsilon_1^2). \tag{3.10}$$

This can be proved by an explicit evaluation of the integral (3.9). Instead, we give a proof here using comparison to the standard discrete GFF. Since $g_{a,D}(x,y) = \bar{a}^{-1}\Delta_D^{-1}(x,y)$, where Δ_D is the standard Dirichlet Laplacian in D, we can conclude (3.10) by showing

$$\operatorname{Var}_{\mathrm{DGFF}}^{D_N,0}[X_{(1-\varepsilon_1)r} - X_{(1+\varepsilon_1)r}] = \frac{2}{\pi}\log\frac{1+\varepsilon_1}{1-\varepsilon_1} + O(\varepsilon_1^2), \tag{3.11}$$

since the left-hand side converge as $N \to \infty$ to $\int_{A_+\cup A_-} f(x)\Delta_D^{-1}(x,y)f(y)\,dx\,dy$, which only differs from (3.9) by a multiplicative constant. This then follows from an explicit computation: let $R = (1+\varepsilon_1+\varepsilon_1^4)r$, using the Gibbs–Markov property of discrete GFF, we have

$$\operatorname{Var}_{\mathrm{DGFF}}^{D_N,0}[X_{(1-\varepsilon_1)r} - X_{(1+\varepsilon_1)r}] = \operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r} - X_{(1+\varepsilon_1)r}].$$

Since $\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1+\varepsilon_1)r}] \le C\varepsilon_1^4$, the right-hand side equals to

$$\begin{aligned}
&\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r}] + \operatorname{Cov}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r}, X_{(1+\varepsilon_1)r}] + O(\varepsilon_1^4)\\
&\quad = \operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r}] + O(\varepsilon_1^2),
\end{aligned}$$

where we apply Cauchy–Schwarz to bound the covariance. To compute $\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r}]$, again using the Gibbs–Markov property, which implies for any $N/4 < r < N$,

$$\begin{aligned}
\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[C_r(0,\phi)] &= \operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[\phi(0)] - \operatorname{Var}_{\mathrm{DGFF}}^{D_r,0}[\phi(0)]\\
&= \frac{2}{\pi}\log\frac{R}{r} + O(1/N).
\end{aligned}$$

Here, we applied the standard Green's function asymptotics (see, e.g., [40]) to obtain the last line. Take a weighted sum over f_ε (with $\varepsilon = \varepsilon_1^4$), we have

$$\begin{aligned}
\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r}] = \operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}\Bigg[&C_{(1-\varepsilon_1)r}(0,\phi)\\
&- \sum_{r_1=-\varepsilon r}^{\varepsilon r} f_\varepsilon\left(1+\frac{r_1}{(1-\varepsilon_1)r}\right)\big(C_{(1-\varepsilon_1)r}(0,\phi) - C_{(1-\varepsilon_1)r+r_1}(0,\phi)\big)\Bigg].
\end{aligned} \tag{3.12}$$

Again, the Gibbs–Markov property implies for any $N/4 < r_1 < r_2 < R$,

$$\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[C_{r_1}(0,\phi) - C_{r_2}(0,\phi)] = \operatorname{Var}_{\mathrm{DGFF}}^{D_{r_2},0}[C_{r_1}(0,\phi)] = \frac{2}{\pi}\log\frac{r_2}{r_1} + O(1/N).$$

Substitute into the right-hand side of (3.12), we conclude that

$$\operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[X_{(1-\varepsilon_1)r}] = \operatorname{Var}_{\mathrm{DGFF}}^{D_R,0}[C_{(1-\varepsilon_1)r}(0,\phi)] + O(\varepsilon_1^2) = \frac{2}{\pi}\log\frac{1+\varepsilon_1}{1-\varepsilon_1} + O(\varepsilon_1^2).$$

This yields (3.10). □

4. Pointwise distribution for Ginzburg–Landau field. The main result of this section is the Gaussian tail for the Ginzburg–Landau field at one site (Theorem 1.4). To prove this, we will employ a multiscale decomposition argument to obtain the approximate Gaussian asymptotics of moment generating function of the harmonic average process.

We first introduce the proper scales in order to carry out the inductive argument. Given any $v \in D_N$, $\varepsilon > 0$ and $c \in (0,1)$, denote by $\Delta = \operatorname{dist}(v, \partial D_N)$ and $M = M(c) = (1-c)\log\Delta/\log(1+\varepsilon)$. Define the sequence of numbers $\{r_k\}_{k=1}^\infty$, $\{r_{k,+}\}_{k=0}^\infty$ and $\{r_{k,-}\}_{k=0}^\infty$ by

$$\begin{aligned}
r_k &= (1+\varepsilon)^{-k}\Delta,\\
r_{k,+} &= (1+\varepsilon^3)r_k,\\
r_{k,-} &= (1-\varepsilon^3)r_k.
\end{aligned} \tag{4.1}$$

We also define

$$X_{r_{k,+}}(v) = \sum_{r=(1-\varepsilon^4)r_{k,+}}^{(1+\varepsilon^4)r_{k,+}} f_{\varepsilon^4}\left(\frac{r}{r_{k,+}}\right) C_r(v,\phi),$$

$$X_{r_{k,-}}(v) = \sum_{r=(1-\varepsilon^4)r_{k,-}}^{(1+\varepsilon^4)r_{k,-}} f_{\varepsilon^4}\left(\frac{r}{r_{k,-}}\right) C_r(v,\phi),$$

where C_r is defined in (3.1), and f_{ε^4} is the smooth function defined just below (3.1).

For $r > 0$, denote by $\mathbb{P}^{r,0}$ the law of the Ginzburg–Landau field in $B_r(v)$ with zero boundary condition (and denote by $\mathbb{E}^{r,0}$ the corresponding expectation). The basic building block of all our large deviation estimates is the following.

THEOREM 4.1. *There exists $g = g(V)$, such that given $C > 0$, $c \in (0, 1)$ we have for all $v \in D_N$ and $|t| \le C$,*

$$\log \mathbb{E}^{D_N,0}\big[\exp\big(tX_{r_{M,+}}(v)\big)\big] = \frac{t^2}{2}(1-c)g \log \Delta + o_\Delta(\log \Delta)\big(t^2 + t^4\big) + O(1),$$

where the $O(1)$ term depends on C and c.

REMARK 4.2. The proof of Theorem 4.1 also yields

$$\log \mathbb{E}^{D_N,0}\big[\exp\big(tX_{r_{M,+}}(v) - tX_{r_{0,-}}(v)\big)\big] = \frac{t^2}{2}(1-c)g \log \Delta + o_\Delta(\log \Delta)\big(t^2 + t^4\big) + O(1).$$

This will be used to prove Theorem 5.8 below.

Roughly speaking, this theorem indicates that as long as the last scale r_M satisfies $r_M > \Delta^c$, for some $c > 0$, the harmonic average $X_{r_{M,+}}$ is nearly Gaussian with mean zero and variance $g \log \frac{N}{r_M}$. To prove this theorem, we will first prove the following decoupling result. We denote

$$\begin{aligned} W_j &= \exp\big(t(X_{r_{j,+}} - X_{r_{j-1,-}})\big), \\ Y_j &= \exp\big(t(X_{r_{j,-}} - X_{r_{j,+}})\big), \\ Z_j &= \exp\big(t(X_{r_{j,+}})\big). \end{aligned}$$

Here, W_j encodes the distribution of the increment of the harmonic average process $X_\cdot$, Y_j are introduced to make the W_j's decouple, and we will show they have little influence on the large deviation estimates.

THEOREM 4.3. *Given $C > 0$, $c \in (0, 1)$ and $C_1 < \infty$ we have for all $v \in D_N$ and $|t| \le C$,*

$$\begin{aligned} \log \mathbb{E}^{D_N,0}\big[\exp(tX_{r_{M,+}})\big] = {}& \sum_{j=1}^{M} \log \mathbb{E}^{r_{j-1},0}[W_j] + \log \mathbb{E}^{D_N,0}\big[\exp(tX_{r_{0,-}})\big] \\ & + t^2 f(\varepsilon) \log \Delta + O(1), \end{aligned}$$

where $|f(\varepsilon)| \le C_1\varepsilon^2/\log(1+\varepsilon)$, *and the* $O(1)$ *term depends on* C *and constants from Lemma* 2.5. *More precisely, we have for and* $k = 1, \dots, M$,

$$\begin{aligned}
&\log \mathbb{E}^{D_N,0}\big[\exp(tX_{r_{k,+}})\big] \\
&\quad = \sum_{j=1}^{k} \log \mathbb{E}^{r_{j-1},0}[W_j] + \log \mathbb{E}^{D_N,0}\big[\exp(tX_{r_{0,-}})\big] \\
&\qquad + t^2 O\bigg(\frac{\varepsilon^2}{\log(1+\varepsilon)}\bigg)\log\frac{\Delta}{r_k} + O\Bigg(\sum_{j=1}^{k-1} r_j^{-\delta}\Bigg).
\end{aligned} \tag{4.2}$$

Notice that $\sum_{j=1}^{k} r_j^{-\delta}$ is a geometric sum, and is thus $O(r_k^{-\delta})$.

PROOF OF THEOREM 4.1. Applying Proposition 3.3 (in particular, (3.5)), we see that there exists $g = g(V)$, such that as $\Delta \to \infty$,

$$\log \mathbb{E}^{r_{j-1},0}[W_j] = \frac{t^2}{2} g \log \frac{r_{j-1}}{r_j} + o_\Delta(1) \log \frac{r_{j-1}}{r_j} t^2 + O(\varepsilon^2)(t^2 + t^4).$$

Summing over j and applying Theorem 4.3, we have

$$\begin{aligned}
\log \mathbb{E}^{D_N,0}\big[\exp(tX_{r_{M,+}})\big] &= \frac{t^2}{2}(1-c) g \log \Delta + o_\Delta(\log \Delta) t^2 + (t^2 + t^4) O\bigg(\frac{\varepsilon^2}{\log(1+\varepsilon)}\bigg) \log \Delta \\
&\quad + t^2 f(\varepsilon) \log \Delta + O(1).
\end{aligned}$$

Since $|t| \le C$, sending $\varepsilon \to 0$ we conclude Theorem 4.1. $\square$

4.1. *Proof of Theorem* 4.3. We write $X_{r_{M,+}}$ as a telescoping sum

$$X_{r_{M,+}} = (X_{r_{M,+}} - X_{r_{M-1,-}}) + (X_{r_{M-1,-}} - X_{r_{M-2,+}}) + \cdots (X_{r_{1,+}} - X_{r_{0,-}}) + X_{r_{0,-}},$$

and, therefore,

$$\begin{aligned}
Z_M = e^{tX_{r_{M,+}}} &= e^{tX_{r_{0,-}}} \prod_{j=1}^{M} \exp\big(t(X_{r_{j,+}} - X_{r_{j-1,-}})\big) \prod_{j=1}^{M-1} \exp\big(t(X_{r_{j,-}} - X_{r_{j,+}})\big) \\
&= W_M Y_{M-1} W_{M-1} \cdots Y_1 W_1 \exp(tX_{r_{0,-}}).
\end{aligned}$$

Notice that

$$Z_k = W_k Y_{k-1} Z_{k-1} = W_k Z_{k-1} + W_k(Y_{k-1} - 1) Z_{k-1}.$$

Since $Z_{k-1} = W_{k-1} Y_{k-2} Z_{k-2}$, by iterating we obtain

$$\begin{aligned}
Z_k &= \sum_{m=1}^{k-1} W_{m+1} Z_m \prod_{j=m+2}^{k} \big(W_j(Y_{j-1} - 1)\big) + Z_1 \prod_{j=2}^{k} W_j(Y_{j-1} - 1) \\
&= W_k Z_{k-1} + E_Y^{(k)},
\end{aligned} \tag{4.3}$$

where

$$E_Y^{(k)} = \sum_{m=1}^{k-2} W_{m+1} Z_m \prod_{j=m+2}^{k} \big(W_j(Y_{j-1} - 1)\big) + Z_1 \prod_{j=2}^{k} W_j(Y_{j-1} - 1). \tag{4.4}$$

We will show that the main contribution to $\log \mathbb{E}^{D_N,0}[Z_k]$ is the first term in the summation (4.3), that is, $\log \mathbb{E}^{D_N,0}[W_k Z_{k-1}]$, and that the other terms are negligible. We denote by $\mathcal{F}_k =$

$\sigma(\phi(x) : x \in D_N \setminus B_{r_k}(v))$, and take $\mathcal{G} = \{\max_{x\in B_\Delta(v)} |\phi(x)| \le (\log\Delta)^2\}$. Recall that by Lemma 2.7, $\mathbb{P}(\mathcal{G}^c) \le \exp(-c_1(\log\Delta)^3)$ for some $c_1 > 0$.

We can write

$$\mathbb{E}^{D_N,0}[Z_k] = \mathbb{E}^{D_N,0}[Z_k 1_{\mathcal{G}}] + \mathbb{E}^{D_N,0}[Z_k 1_{\mathcal{G}^c}].$$

Since $|t| \le C$, apply Hölder and the exponential Brascamp–Lieb inequality,

$$\begin{aligned}\mathbb{E}^{D_N,0}[Z_k 1_{\mathcal{G}^c}] &\le \big(\mathbb{E}^{D_N,0}[Z_k^2]\big)^{1/2}\mathbb{P}^{D_N,0}(\mathcal{G}^c)^{1/2}\\ &\le \exp\big(2c_-^{-1}t^2 \operatorname{Var}^{D_N,0}_{\mathrm{DGFF}}[X_{r_{M,+}}]\big)\mathbb{P}^{D_N,0}(\mathcal{G}^c)^{1/2}\\ &\le \exp\bigg(Ct^2\log\Delta - \frac{c_1}{2}(\log\Delta)^3\bigg) \le \exp\bigg(-\frac{c_1}{4}(\log\Delta)^3\bigg),\end{aligned}\tag{4.5}$$

which is negligible.

In order to prove (4.2), we set up a joint induction for:

- There exists an absolute constant $C_1 < \infty$, such that for all $k \ge 0$,

$$\begin{aligned}&\log\mathbb{E}^{D_N,0}[Z_k 1_{\mathcal{G}}]\\ &\quad= \sum_{j=1}^{k}\log\mathbb{E}^{r_{j-1},0}[W_j 1_{\mathcal{G}}] + \log\mathbb{E}^{D_N,0}\big[\exp(tX_{r_{0,-}})1_{\mathcal{G}}\big] + t^2F_k + R_{k-1},\end{aligned}\tag{4.6}$$

  where $|F_k| \le C_1 k\varepsilon^2 = C_1\frac{\varepsilon^2}{\log(1+\varepsilon)}\log\frac{\Delta}{r_k}$, and $|R_{k-1}| \le C_1\sum_{j=1}^{k-1} r_j^{-\delta}$.
- There exists an absolute constant $C_2 < \infty$, such that for all $k \ge 2$,

$$\mathbb{E}^{D_N,0}\big[E_Y^{(k)}1_{\mathcal{G}}\big] \le C_2\varepsilon^2\mathbb{E}^{D_N,0}[Z_{k-2}1_{\mathcal{G}}].\tag{4.7}$$

Notice that (4.6) implies (4.2), since for all $k \ge 1$,

$$\mathbb{E}^{r_{k-1},0}[W_k 1_{\mathcal{G}^c}] \le \exp\bigg(-\frac{c_1}{4}(\log r_{k-1})^3\bigg),\tag{4.8}$$

and similar bound hold for $\mathbb{E}^{D_N,0}[\exp(tX_{r_{0,-}})1_{\mathcal{G}^c}]$. Clearly, the base case $k = 0$ for (4.6) is trivial.

Now assume both (4.6) and (4.7) hold up to $k-1$. Let us first show the desired bound for $\mathbb{E}^{D_N,0}[E_Y^{(k)}1_{\mathcal{G}}]$. For $m = 1,\dots,k-2$, using the Markov property and Cauchy–Schwarz, we conclude that each term in the first summand of (4.4) (multiplied by $1_{\mathcal{G}}$) can be bounded by

$$\begin{aligned}&\mathbb{E}^{D_N,0}\Bigg[W_{m+1}Z_m\prod_{j=m+2}^{k}\big(W_j(Y_{j-1}-1)\big)1_{\mathcal{G}}\Bigg]\\ &\quad= \mathbb{E}^{D_N,0}\Bigg[\mathbb{E}\Bigg[W_{m+1}\prod_{j=m+2}^{k}\big(W_j(Y_{j-1}-1)\big)1_{\mathcal{G}}\Bigg|\mathcal{F}_m\Bigg]Z_m 1_{\mathcal{G}}\Bigg]\\ &\quad\le \mathbb{E}^{D_N,0}\Bigg[\mathbb{E}\Bigg[\prod_{j=m+1}^{k}W_j^2 1_{\mathcal{G}}\Bigg|\mathcal{F}_m\Bigg]^{1/2}\mathbb{E}\Bigg[\prod_{j=m+1}^{k-1}(Y_j-1)^2 1_{\mathcal{G}}\Bigg|\mathcal{F}_m\Bigg]^{1/2}Z_m 1_{\mathcal{G}}\Bigg].\end{aligned}\tag{4.9}$$

We now claim that there exist constants $C_3, C_4 < \infty$, such that for $|t| \le C$,

$$\mathbb{E}^{r_{j-1,-},0}\big[(Y_j-1)^2\big] \le C_3\varepsilon^4,\tag{4.10}$$

and

$$\mathbb{E}^{r_{j-1},0}\big[W_j^2\big] \le \exp\big(4c_-^{-1}t^2\operatorname{Var}^{r_{j-1},0}_{\mathrm{DGFF}}[X_{r_{j,+}} - X_{r_{j-1,-}}]\big) \le C_4.\tag{4.11}$$

Indeed, using the Taylor expansion we can write

$$\mathbb{E}^{r_{j-1,-},0}\big[(Y_j-1)^2\big]=\mathbb{E}^{r_{j-1,-},0}\bigg[\bigg(\sum_{k\ge 1}\frac{t^k}{k!}(X_{r_{j,-}}-X_{r_{j,+}})^k\bigg)^2\bigg]$$
$$\le\sum_{k\ge 1}\mathbb{E}^{r_{j-1,-},0}\bigg[\sum_{j=1}^{2k}\frac{2}{j!(2k-j)!}t^{2k}(X_{r_{j,-}}-X_{r_{j,+}})^{2k}\bigg].$$

Using the identity

$$\sum_{j=1}^{2k}\frac{1}{j!(2k-j)!}=\frac{1}{(2k)!}2^{2k},$$

and the Brascamp–Lieb inequality (2.2) combined with Wick's theorem,

$$\mathbb{E}^{r_{j-1,-},0}\big[(X_{r_{j,-}}-X_{r_{j,+}})^{2k}\big]\le c_-^{-k}\mathbb{E}_{\mathrm{DGFF}}^{r_{j-1,-},0}\big[(X_{r_{j,-}}-X_{r_{j,+}})^{2k}\big]$$
$$\le c_-^{-k}\frac{(2k)!}{k!2^k}\big(\mathbb{E}_{\mathrm{DGFF}}^{r_{j-1,-},0}\big[(X_{r_{j,-}}-X_{r_{j,+}})^{2}\big]\big)^k,$$

we obtain

$$\mathbb{E}^{r_{j-1,-},0}\big[(Y_j-1)^2\big]\le\sum_{k\ge 1}\frac{t^{2k}2^{k+1}}{k!}c_-^{-k}\big(\mathbb{E}_{\mathrm{DGFF}}^{r_{j-1,-},0}\big[(X_{r_{j,-}}-X_{r_{j,+}})^{2}\big]\big)^k$$
$$\le C't^2\mathbb{E}_{\mathrm{DGFF}}^{r_{j-1,-},0}\big[(X_{r_{j,-}}-X_{r_{j,+}})^{2}\big]$$

for some $C'<\infty$. A similar computation as (3.11) using the Gibbs–Markov property then yields

$$\mathrm{Var}_{\mathrm{DGFF}}^{r_{j-1,-},0}[X_{r_{j,-}}-X_{r_{j,+}}]\le C''\log\frac{r_{j,+}}{r_{j,-}}\le C''\varepsilon^4. \tag{4.12}$$

This verifies (4.10). (4.11) follows directly from the exponential Brascamp–Lieb inequality (2.3).

We then use (4.10) and (4.11) to obtain an upper bound of (4.9). Let $\mathcal{F}_k^-=\sigma(\phi(x):x\in D_N\setminus B_{r_k,-}(v))$. Again use the Markov property

$$\mathbb{E}\bigg[\prod_{j=m+1}^{k-1}(Y_j-1)^2 1_{\mathcal{G}}\Big|\mathcal{F}_m\bigg]$$
$$=\mathbb{E}\bigg[\mathbb{E}\big[(Y_{k-1}-1)^2 1_{\mathcal{G}}|\mathcal{F}_{k-2}^-\big]\prod_{j=m+1}^{k-2}(Y_j-1)^2 1_{\mathcal{G}}\Big|\mathcal{F}_m\bigg].$$

We now use the fact that $r_{k-2,-}\ge r_M\ge\Delta^c$ and, therefore, on the event $\mathcal{G}$,

$$\max_{x\in\partial B_{r_{k-2,-}}(v)}|\phi(x)|\le(\log\Delta)^2\le\bigg(\frac{1}{c}\log r_{k-2,-}\bigg)^2. \tag{4.13}$$

Applying Theorem 2.5 (to any realization of $\phi|_{\partial B_{r_{k-2,-}}}$ that satisfy (4.13)), Cauchy–Schwarz and the Brascamp–Lieb inequality, we conclude there is some $C_3'<\infty$ and $\delta>0$, such that with probability one,

$$\big|\mathbb{E}\big[(Y_{k-1}-1)^2 1_{\mathcal{G}}|\mathcal{F}_{k-2}^-\big]-\mathbb{E}^{r_{k-2,-},0}\big[(Y_{k-1}-1)^2 1_{\mathcal{G}}\big]\big|\le C_3'\varepsilon^4 r_{k-2}^{-\delta},$$

for some $\delta > 0$. Thus

$$\begin{aligned}&\mathbb{E}\Bigg[\prod_{j=m+1}^{k-1}(Y_j-1)^2 1_{\mathcal{G}}\Big|\mathcal{F}_m\Bigg]\\&\quad=\big(\mathbb{E}^{r_{k-2,-},0}\big[(Y_{k-1}-1)^2 1_{\mathcal{G}}\big]+O\big(\varepsilon^4 r_{k-2}^{-\delta}\big)\big)\mathbb{E}\Bigg[\prod_{j=m+1}^{k-2}(Y_j-1)^2 1_{\mathcal{G}}\Big|\mathcal{F}_m\Bigg],\end{aligned}$$

here we also use the fact that $\prod_{j=m+1}^{k-2}(Y_j-1)^2\geq 0$. By iterating this for $j\geq m+1$, applying the bound (4.10) and notice $\sum_j r_{j-2}^{-\delta}<\infty$, we conclude there exist absolute constants $C_2', C_3''<\infty$, such that with probability one,

$$\mathbb{E}\Bigg[\prod_{j=m+1}^{k-1}(Y_j-1)^2 1_{\mathcal{G}}\Big|\mathcal{F}_m\Bigg]\leq C_2'\big(C_3''\varepsilon^4\big)^{k-m-1}.$$

Similarly, there exists $C_4'<\infty$, such that with probability one,

$$\mathbb{E}\Bigg[\prod_{j=m+1}^{k}W_j^2 1_{\mathcal{G}}\Big|\mathcal{F}_m\Bigg]\leq C_2'\big(C_4'\big)^{k-m-1}.$$

Substitute these bounds into (4.9), we have for some $C_5<\infty$,

$$\mathbb{E}^{D_N,0}\Bigg[W_{m+1}Z_m\prod_{j=m+2}^{k}\big(W_j(Y_{j-1}-1)\big)1_{\mathcal{G}}\Bigg]\leq C_2'\big(C_5\varepsilon^2\big)^{k-m-1}\mathbb{E}^{D_N,0}[Z_m 1_{\mathcal{G}}].\tag{4.14}$$

By the induction hypothesis (4.6) for $m\leq k-2$,

$$\begin{aligned}&\log\mathbb{E}^{D_N,0}[Z_m 1_{\mathcal{G}}]-\log\mathbb{E}^{D_N,0}[Z_{k-2}1_{\mathcal{G}}]\\&\quad\leq-\sum_{j=m+1}^{k-2}\log\mathbb{E}^{r_{j-1},0}[W_j 1_{\mathcal{G}}]+t^2|F_{k-2}-F_m|+|R_{m-1}|,\end{aligned}$$

where $|F_{k-2}-F_m|\leq C_1(k-2-m)\varepsilon^2=C_1\frac{\varepsilon^2}{\log(1+\varepsilon)}\log\frac{r_m}{r_{k-2}}$, $|R_{m-1}|\leq C_1 r_{m-1}^{-\delta}$. Applying Proposition 3.3 (and use the smallness of $\mathbb{E}^{r_{j-1},0}[W_j 1_{\mathcal{G}^c}]$) to evaluate $\log\mathbb{E}^{r_{j-1},0}[W_j 1_{\mathcal{G}}]$ as $\Delta\to\infty$, the right-hand side is bounded above by

$$-\frac{t^2}{2}g\log\frac{r_m}{r_{k-2}}+t^2 o_\Delta(1)\log\frac{r_m}{r_{k-2}}+O\bigg(\frac{\varepsilon^2}{\log(1+\varepsilon)}\bigg)\log\frac{r_m}{r_{k-2}}+O\big(r_{m-1}^{-\delta}\big).$$

For ε sufficiently small, this is bounded by $O(r_{m-1}^{-\delta})$, and we have $\mathbb{E}^{D_N,0}[Z_m 1_{\mathcal{G}}]\leq 2\mathbb{E}^{D_N,0}[Z_{k-2}1_{\mathcal{G}}]$ for all $m\leq k-2$. This concludes that for some absolute constant $C_6<\infty$, (4.14) is bounded by

$$C_6\varepsilon^{2(k-m-1)}\mathbb{E}^{D_N,0}[Z_{k-2}1_{\mathcal{G}}].$$

Summing over m, we then have

$$\mathbb{E}^{D_N,0}\Bigg[\sum_{m=1}^{k-2}W_{m+1}Z_m\prod_{j=m+2}^{k}\big(W_j(Y_{j-1}-1)\big)1_{\mathcal{G}}\Bigg]\leq C_7\varepsilon^2\mathbb{E}^{D_N,0}[Z_{k-2}1_{\mathcal{G}}],$$

for some $C_7<\infty$. A similar argument yields

$$\mathbb{E}^{D_N,0}\Bigg[Z_1\prod_{j=2}^{k}W_j(Y_{j-1}-1)1_{\mathcal{G}}\Bigg]\leq C_7'\varepsilon^2\mathbb{E}^{D_N,0}[Z_{k-2}1_{\mathcal{G}}].$$

This completes the proof of (4.7) for k (with $C_2=C_7+C_7'$).

We now move to the proof of (4.6). We first show that there exists $C_0 < \infty$, such that

$$\log \mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}] = \log \mathbb{E}^{D_N,0}[Z_{k-1} 1_{\mathcal{G}}] + \log \mathbb{E}^{r_{k-1},0}[W_k 1_{\mathcal{G}}] + R'_{k-1}, \tag{4.15}$$

where $|R'_{k-1}| \le C_0 r_{k-1}^{-\delta}$. Using Markov property,

$$\mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}] = \mathbb{E}^{D_N,0}\big[Z_{k-1} 1_{\mathcal{G}} \mathbb{E}[W_k 1_{\mathcal{G}} | \mathcal{F}_{k-1}]\big].$$

Apply Lemma 2.8 (to any realization of $\phi|_{\partial B_{r_{k-1}}}$ such that $\max_{x \in \partial B_{r_{k-1}}} \phi(x) \le (\frac{1}{c} \times \log r_{k-1})^2$) to obtain with probability one, there is some $C_0 < \infty$ and $\delta > 0$, such that

$$\begin{aligned} \big|\mathbb{E}[W_k 1_{\mathcal{G}} | \mathcal{F}_{k-1}] - \mathbb{E}^{r_{k-1},0}[W_k 1_{\mathcal{G}}]\big| &\le 2 \exp\big(c \operatorname{Var}_{\mathrm{DGFF}}^{r_{k-1},0} W_k\big) r_{k-1}^{-\delta} \\ &\le C_0 r_{k-1}^{-\delta}. \end{aligned}$$

Therefore,

$$\big|\mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}] - \mathbb{E}^{D_N,0}[Z_{k-1} 1_{\mathcal{G}}] \mathbb{E}^{r_{k-1},0}[W_k 1_{\mathcal{G}}]\big| \le C_0 r_{k-1}^{-\delta} \mathbb{E}^{D_N,0}[Z_{k-1} 1_{\mathcal{G}}].$$

This yields (4.15).

Finally, we prove (4.6) for k using the joint induction hypothesis for (4.6) up to $k-1$ and for (4.7) up to k, and apply (4.15). By (4.15), and the induction hypothesis for (4.6),

$$\begin{aligned} &\log \mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}] \\ &\quad = \log \mathbb{E}^{D_N,0}[Z_{k-1} 1_{\mathcal{G}}] + \log \mathbb{E}^{r_{k-1},0}[W_k 1_{\mathcal{G}}] + R'_{k-1} \\ &\quad = \sum_{j=1}^{k} \log \mathbb{E}^{r_{j-1},0}[W_j 1_{\mathcal{G}}] + \log \mathbb{E}^{D_N,0}\big[\exp\big(t(X_{r_{0,-}})\big) 1_{\mathcal{G}}\big] + t^2 F_{k-1} + R_{k-2} + R'_{k-1}. \end{aligned} \tag{4.16}$$

We may write

$$\log \mathbb{E}^{D_N,0}[Z_k 1_{\mathcal{G}}] = \log \mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}] + \log\left[1 + \frac{\mathbb{E}^{D_N,0}[E_Y^{(k)} 1_{\mathcal{G}}]}{\mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}]}\right].$$

Using the induction hypothesis for (4.7), (4.16) and the asymptotics of $\log \mathbb{E}^{r_{j-1},0}[W_j]$, we conclude for some absolute constant $C_8 < \infty$,

$$\left|\log\left[1 + \frac{\mathbb{E}^{D_N,0}[E_Y^{(k)} 1_{\mathcal{G}}]}{\mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}]}\right]\right| \le C_8 \varepsilon^2 \frac{\mathbb{E}^{D_N,0}[Z_{k-2} 1_{\mathcal{G}}]}{\mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}]} \le 2 C_8 \varepsilon^2.$$

Let $F_k = F_{k-1} + \log[1 + \frac{\mathbb{E}^{D_N,0}[E_Y^{(k)} 1_{\mathcal{G}}]}{\mathbb{E}^{D_N,0}[W_k Z_{k-1} 1_{\mathcal{G}}]}]$ and $R_{k-1} = R_{k-2} + R'_{k-1}$. Combining with (4.16), we conclude

$$\log \mathbb{E}^{D_N,0}[Z_k 1_{\mathcal{G}}] = \sum_{j=1}^{k} \log \mathbb{E}^{r_{j-1},0}[W_j 1_{\mathcal{G}}] + \log \mathbb{E}^{D_N,0}\big[\exp\big(t(X_{r_{0,-}})\big) 1_{\mathcal{G}}\big] + t^2 F_k + R_{k-1},$$

with $|F_k| \le \max\{C_1, 2C_8\} k \varepsilon^2$, $|R_{k-1}| \le C_1 \sum_{j=1}^{k-1} r_j^{-\delta}$. This completes the proof of (4.6), and also Theorem 4.3.

4.2. *Proof of upper bound.* In this section, we prove the pointwise Gaussian tail bound Theorem 1.4, and as a consequence derive the upper bound of the law of large numbers Theorem 1.1. In fact, we obtain the following tail bound for the maximum of $\phi(x)$. For the rest of the paper, $g = g(V)$ denotes the positive constant that appears in Theorem 3.3 and Theorem 4.1.

PROPOSITION 4.4. *For any $\delta > 0$, there is some $C = C(\delta) < \infty$, such that*

$$\mathbb{P}\Big(\sup_{v\in D_N} \phi(v) \geq (2\sqrt{g}+\delta)\log N\Big) \leq C(\delta) N^{-\delta/\sqrt{g}}. \tag{4.17}$$

We first give the proof of Theorem 1.4.

PROOF OF THEOREM 1.4. Given $\delta > 0$ and $v \in D_N$, take $M = M(\delta) = (1-\delta^6)\log\Delta$. Therefore,

$$\mathbb{P}^{D_N,0}\big(\phi(v) > u\big) \leq \mathbb{P}^{D_N,0}\big(X_{r_{M,+}}(v) > u - \delta\log\Delta\big) + \mathbb{P}^{D_N,0}\big(\phi(v) - X_{r_{M,+}}(v) > \delta\log\Delta\big).$$

We apply Theorem 4.1 to obtain for all bounded t,

$$\begin{aligned}\mathbb{P}^{D_N,0}(X_{r_{M,+}} > u - \delta\log\Delta) &\leq \exp\big(-t(u-\delta\log\Delta)\big)\mathbb{E}^{D_N,0}\big[\exp(tX_{r_{M,+}})\big]\\ &= \exp\Big(-t(u-\delta\log\Delta) + \frac{t^2}{2}g\big(1-\delta^6\big)\log\Delta + o(\log\Delta)\Big).\end{aligned}$$

Minimize the last display over t. Since $u \leq C\log\Delta$ the minimum is achieved at some bounded t, thus

$$\begin{aligned}\mathbb{P}^{D_N,0}(X_{r_{M,+}} > u - \delta\log\Delta) &\leq \exp\Big(-\frac{(u-\delta\log\Delta)^2}{2g(1-\delta^6)\log\Delta} + o(\log\Delta)\Big)\\ &\leq \exp\Big(-\frac{(u-\delta\log\Delta)^2}{2g\log\Delta} + o(\log\Delta)\Big).\end{aligned}$$

Apply Lemma 2.2 to obtain

$$\mathbb{P}^{D_N,0}\big(\phi(v) - X_{r_{M,+}} > \delta\log\Delta\big) \leq \exp\Big(-c_{\mathrm{BL}}\frac{(\delta\log\Delta)^2}{g\delta^6\log\Delta}\Big) = \exp\Big(-c_{\mathrm{BL}}\frac{\log\Delta}{g\delta^4}\Big).$$

Notice that for δ small enough,

$$2c_{\mathrm{BL}}\frac{\log\Delta}{g\delta^4} > \frac{(u-\delta\log\Delta)^2}{2g\log\Delta},$$

so we send $\delta \to 0$ to conclude the proof. $\square$

Finally, we show how Proposition 4.4 follows easily from Theorem 1.4.

PROOF OF PROPOSITION 4.4. If we pick γ_0 small enough then for $v \in D_N$ such that $\mathrm{dist}(v, \partial D_N) \leq N^{\gamma_0}$ we have from the Brascamp–Lieb tail bound, Lemma 2.2, that

$$P\big(\phi(v) \geq 2\sqrt{g}\log N\big) \leq \exp\Big(-c_{\mathrm{BL}}\frac{4g(\log N)^2}{\gamma_0\log N}\Big) \leq N^{-2-2\delta/\sqrt{g}}.$$

Then a union bound shows that

$$P\Big(\max_{v:\mathrm{dist}(v,\partial D_N)\leq N^{\gamma_0}} \phi(v) \geq 2\sqrt{g}\log N\Big) \leq N^{\gamma_0-1-2\delta/\sqrt{g}}.$$

Fix this γ_0 and take any $v \in D_N$ such that $\mathrm{dist}(v, \partial D_N) > N^{\gamma_0}$. Given any $\delta > 0$, applying Proposition 1.4 with $u = (2\sqrt{g}+\delta)\log N$ yields

$$\begin{aligned}P\big(\phi(v) \geq (2\sqrt{g}+\delta)\log N\big) &\leq \exp\Big(-2\frac{(\log N)^2}{\log\Delta} - \frac{2\delta}{\sqrt{g}}\frac{(\log N)^2}{\log\Delta} + o(\log N)\Big)\\ &\leq CN^{-2-2\delta/\sqrt{g}+o(1)},\end{aligned}$$

for some $C < \infty$. Therefore,

$$P\Big(\max_{v:\mathrm{dist}(v,\partial D_N)>N^{\gamma_0}} \phi(v) \geq 2\sqrt{g}\log N\Big) \leq CN^{-2\delta/\sqrt{g}+o(1)},$$

thus completing the proof of (4.17). $\square$

5. Proof of the lower bound. In this section, we prove the lower bound of the law of large numbers Theorem 1.1. In fact, we prove the following tail bound.

PROPOSITION 5.1. *For any $\delta > 0$, there is some $C = C(\delta) < \infty$, such that*

$$\mathbb{P}^{D_N,0}\Big(\sup_{v\in D_N} \phi(v) \leq (2\sqrt{g}-\delta)\log N\Big) \leq C(\delta)N^{-C\delta^{-1}}. \tag{5.1}$$

We first prove a weaker form of the lower bound in Section 5.1, and then "bootstrap" to obtain the desired lower bound in Section 5.3. Recall that $\mathbb{P}^{B,f}$ represents the law of the gradient field in $B \subset \mathbb{Z}^2$ with boundary condition f on ∂B.

5.1. *Second moment argument.* Given $B \subset \mathbb{Z}^2$, $x \in B$ and $y \in \partial B$, we recall $a_B(x, y)$ is the harmonic measure on ∂B seen from x. Also recall the harmonic averaged field $X_{r_j,+}(v)$ and $X_{r_j,-}(v)$ from the beginning of Section 4. Heuristically, the process $\{X_{r_j,+}(v)\}$ should behave like a random walk with increments of variance $g\log(1+\varepsilon)$. We make this heuristic rigorous and show the following weak lower bound.

PROPOSITION 5.2. *For all $s > 0$, there is $N_0 = N_0(s)$ such that for $N > N_0(s)$,*

$$\mathbb{P}^{D_N,0}\begin{bmatrix} \exists v \in [-0.9N, 0.9N]^2 \text{ s.t.} \\ \phi(v) - X_{r_{0,-}}(v) \geq (1-2s)2\sqrt{g}\log N \end{bmatrix} \geq N^{-22s}. \tag{5.2}$$

In fact, this probability tends to one as $N \to \infty$. This will be proved later by bootstrapping the weaker bound stated in Proposition 5.2. The proof of Proposition 5.2 is based on a second moment method studying the truncated count of the increment of the harmonic averaged process.

It suffices to prove Proposition 5.2 for small s. Given $v \in [-0.9N, 0.9N]^2$, take $c = s^3$ and $M = M(s^3) = (1-s^3)\log N/\log(1+\varepsilon)$, and define r_k and $r_{k,\pm}$ as in (4.1). Denote by $[m]$ the integer part of m. Then we have

$$\begin{aligned}
&\mathbb{P}^{D_N,0}\begin{bmatrix} \exists v \in [-0.9N, 0.9N]^2 \text{ s.t.} \\ \phi(v) - X_{r_{0,-}}(v) \geq (1-2s)2\sqrt{g}\log N \end{bmatrix} \\
&\quad\geq \mathbb{P}^{D_N,0}\begin{bmatrix} \exists v \in [-0.9N, 0.9N]^2 \text{ s.t.} \\ X_{r_{[M]},+}(v) - X_{r_{0,-}}(v) \geq \Big(1-\dfrac{3}{2}s\Big)2\sqrt{g}\log N \end{bmatrix} \\
&\qquad - \mathbb{P}^{D_N,0}\begin{bmatrix} \exists v \in [-0.9N, 0.9N]^2 \text{ s.t.} \\ \phi(v) - X_{r_{[M]},+}(v) \leq -\dfrac{s}{2}2\sqrt{g}\log N \end{bmatrix}.
\end{aligned}$$

The last term above can be bounded using the Brascamp–Lieb inequality. Indeed,

$$\mathbb{P}^{D_N,0}\begin{bmatrix} \exists v \in [-0.9N, 0.9N]^2 \text{ s.t.} \\ \phi(v) - X_{r_{[M]},+}(v) \leq -\dfrac{s}{2}2\sqrt{g}\log N \end{bmatrix}$$

$$
\begin{aligned}
&\leq \sum_{v\in[-0.9N,0.9N]^2} \mathbb{P}^{D_N,0}\left[\phi(v) - X_{r_{[M]},+}(v) \geq \frac{s}{2} 2\sqrt{g}\log N\right] \\
&\leq N^2 \exp\left(-c_{\mathrm{BL}} \frac{s^2 g(\log N)^2}{\mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}(\phi(v) - X_{r_{[M]},+}(v))}\right) \\
&\leq N^2 \exp\left(-c_{\mathrm{BL}} \frac{s^2 g(\log N)^2}{g s^3 \log N}\right) = N^{2-c's^{-1}},
\end{aligned}
\tag{5.3}
$$

for some $c' > 0$. For small s, this is much smaller than N^{-22s}. Therefore, it suffices to study $X_{r_{[M]},+}(v) - X_{r_{0,-}}(v)$.

For fixed integer $K \geq 2$ (which will be taken sufficiently large in the end), split "time" into $K_1 := [(1 - s^3)K] + 1$ intervals of size $1/K$ and consider the increments over these intervals

$$
U_m(v) = X_{r_{[\frac{mM}{K_1}],+}}(v) - X_{r_{[\frac{(m-1)M}{K_1}],-}}(v) \qquad \text{for } m = 1, \dots, K_1. \tag{5.4}
$$

Roughly speaking, when v is in the bulk of D_N, $\{U_m\}_{m=1}^{K_1}$ are the differences between the harmonic average at scale $N^{1-m/K}$ and the scale $N^{1-(m-1)/K}$. Consider the events

$$
J_m(v; s) = \left\{U_m(v) \in \left[\frac{1}{K}(1-s)2\sqrt{g}\log N, \frac{1}{K}(1+s)2\sqrt{g}\log N\right]\right\}
$$

and

$$
J(v; s) = \bigcap_{m=1,\dots,K_1} J_m(v; s).
$$

Define the counting random variable

$$
\mathcal{N}_{K_1}(s) = \sum_{v\in[-0.9N,0.9N]^2} 1_{J(v;s)}.
$$

Note that if $\mathcal{N}_{K_1}(s) \geq 1$ then there exists a $v \in [-0.9N, 0.9N]^2$ such that

$$
\sum_{m=1}^{K_1} U_m(v) \geq (1-s)(1-s^3)2\sqrt{g}\log N \geq \left(1 - \frac{5}{4}s\right)2\sqrt{g}\log N.
$$

Furthermore, since

$$
X_{r_{[M]},+}(v) - X_{r_{0,-}}(v) = \sum_{m=1}^{K_1} U_m(v) + \sum_{m=1}^{K_1} \big(X_{[mM/K_1],-}(v) - X_{[mM/K_1],+}(v)\big),
$$

and by direct computation

$$
\mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}\left[\sum_{m=1}^{K_1} \big(X_{[mM/K_1],-}(v) - X_{[mM/K_1],+}(v)\big)\right] = O(K_1),
$$

the Brascamp–Lieb tail bound Lemma 2.2 implies there exist some $c(s, K_1) > 0$, such that

$$
\mathbb{P}^{D_N,0}\left(\sum_{m=1}^{K_1} \big(X_{[mM/K_1],-}(v) - X_{[mM/K_1],+}(v)\big) > 2\sqrt{g}\frac{s}{4}\log N\right) \leq e^{-c(s,K_1)(\log N)^2}. \tag{5.5}
$$

Combining (5.3) and (5.5), Proposition 5.2 will follow from

$$
\mathbb{P}^{D_N,0}[\mathcal{N}_{K_1}(s) \geq 1] \geq N^{-22s}. \tag{5.6}
$$

We will prove the following.

LEMMA 5.3. *For all $s > 0$ and $K \geq 2/s$, we have*

$$\mathbb{E}^{D_N,0}\big[\mathcal{N}_{K_1}(s)^2\big] \leq N^{22s}\mathbb{E}^{D_N,0}\big[\mathcal{N}_{K_1}(s)\big]^2. \tag{5.7}$$

With additional work, the term N^{22s} could be replaced $(1+o(1))$, but for our purposes (5.7) is enough. Note that (5.7) is true only because $\mathcal{N}_{K_1}(s)$ is a *truncated* count of high points.

By the Paley–Zygmund inequality, Lemma 5.3 implies (5.6) and, therefore, yields Proposition 5.2.

Lemma 5.3 follows from the following estimates.

LEMMA 5.4. *For all fixed $s > 0$ and $K \geq 2$, we have*

$$\mathbb{E}^{D_N,0}\big[\mathcal{N}_{K_1}(s)\big] \geq cN^{-5s}.$$

LEMMA 5.5. *For all fixed $s > 0$ and $K \geq 2$, we have*

$$\mathbb{E}^{D_N,0}\big[\mathcal{N}_{K_1}(s)^2\big] \leq N^{\frac{2}{K}+11s}.$$

The proof of these lemmas use estimates for the joint distribution of $\{U_m\}_{m=1}^{K_1}$, proved in Section 5.2 below. Lemma 5.4 is immediate from taking union bound from the following result.

LEMMA 5.6. *For all fixed $s > 0$ and $K \geq 2$, we have that*

$$\mathbb{P}^{D_N,0}\big[J(v;s)\big] \geq cN^{-2-5s},$$

uniformly over $v \in [-0.9N, 0.9N]^2$.

PROOF. Letting $\frac{dQ}{d\mathbb{P}^{D_N,0}} = \frac{\exp(\lambda\sum_{m=1}^{K_1}U_m(v))}{\mathbb{E}^{D_N,0}[\exp(\lambda\sum_{m=1}^{K_1}U_m(v))]}$ we have

$$\begin{aligned}\mathbb{P}^{D_N,0}\big[J(v,s)\big] &= Q\big[J(v,s); e^{-\lambda\sum_{m=1}^{K_1}U_m(v)}\big]\mathbb{E}^{D_N,0}\left[\exp\left(\lambda\sum_{m=1}^{K_1}U_m(v)\right)\right]\\ &\geq Q\big[J(v,s)\big]e^{-\lambda(1+s)(1-s^3)2\sqrt{g}\log N}\mathbb{E}^{D_N,0}\left[\exp\left(\lambda\sum_{m=1}^{K_1}U_m(v)\right)\right].\end{aligned}$$

By Theorem 5.8, for all $\lambda \leq 2/\sqrt{g}$,

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda U_m(v)\right)\right] = \exp\left(\frac{1}{2}\sum_{m=1}^{K_1}\lambda^2\frac{1}{K}g\log N + o(\log N)\right). \tag{5.8}$$

Therefore,

$$\mathbb{P}^{D_N,0}\big[J(v,s)\big] \geq Q\big[J(v,s)\big]e^{\frac{1}{2}\lambda^2(1-s^3)g\log N - \lambda(1+s)(1-s^3)2\sqrt{g}\log N + o(\log N)}.$$

Setting $\lambda = 2/\sqrt{g},$ we find that

$$\mathbb{P}^{D_N,0}\big[J(v,s)\big] \geq Q\big[J(v,s)\big]e^{-2\log N - 5s\log N}.$$

It thus only remains to show that $Q[J(v)] \geq c$. Under Q, we have for each j that

$$\begin{aligned}&Q\left[\exp\left(t\left(U_j(v) - \frac{1}{K}2\sqrt{g}\log N\right)\right)\right]\\ &\quad= \frac{\mathbb{E}^{D_N,0}[\exp(\sum_{m=1}^{K_1}(\lambda + 1_{\{m=j\}}t)U_m(v)]}{\mathbb{E}^{D_N,0}[\exp(\sum_{m=1}^{K_1}\lambda U_m(v)]}\exp\left(-2t\frac{1}{K}\sqrt{g}\log N\right).\end{aligned} \tag{5.9}$$

Thus applying Theorem 5.8 (with $\max \lambda_i = 2/\sqrt{g} + 1$), we have that (5.9) equals

$$\frac{\exp(\frac{1}{2}\lambda^2(1-s^3)g\log N + \lambda t\frac{1}{K}g\log N + \frac{1}{2}t^2\frac{1}{K}g\log N + o(\log N))}{\exp(\frac{1}{2}\lambda^2(1-s^3)g\log N + o(\log N))}\exp\left(-2t\frac{1}{K}\sqrt{g}\log N\right)$$
$$= \exp\left(\frac{1}{2}t^2\frac{1}{K}g\log N + o(\log N)\right),$$

where the last equality follows because $\lambda = 2/\sqrt{g}$. Using the exponential Chebyshev inequality with $t = \pm s/\sqrt{g}$ therefore shows that

$$Q\left[\left|U_j - \frac{1}{K}\sqrt{g}\log N\right| \geq s\frac{1}{K}\sqrt{g}\log N\right] \leq \exp\left(-c\frac{s^2}{K}\log N\right),$$

for some $c > 0$. Thus $Q[J(v,s)] \geq 1 - K\exp(-c\frac{s^2}{K}\log N) \to 1$, as $N \to \infty$ for all K and s. $\square$

Lemma 5.5 will follow from the following.

LEMMA 5.7. *For all fixed $s > 0$ and $K \geq 1$, we have if $N^{1-\frac{j}{K}} \leq |v_1 - v_2| \leq N^{1-\frac{j-1}{K}}$ for some $j \in \{1,\dots,K_1\}$, then*

$$\mathbb{P}^{D_N,0}[J(v_1,s)\cap J(v_2,s)] \leq \exp\left(-2\frac{2K_1 - j}{K}\log N + 5s\frac{2K_1-j}{K}\log N\right).$$

PROOF. Note that $B_{N^{1-\frac{j}{K}}}(v_i)$ for $i = 1,2$ are disjoint, but $B_{N^{1-\frac{j-1}{K}}}(v_i)$ are not. Thus, roughly speaking, the increments $U_{j+1}(v_i)$ for $i = 1,2$ depend on disjoint regions but $U_j(v_i)$ do not. Because of this we expect $U_m(v_i)$, $i = 1,2$ to be correlated for $m = 1,\dots,j$ (and essentially perfectly correlated if $m \leq j-1$), but essentially independent for $m = j+1,\dots,K_1$. With this in mind we in fact bound

$$\mathbb{P}^{D_N,0}[J'],$$

where

$$J' = \bigcap_{m=1,\dots,K_1} J_m(v_1,s) \cap \bigcap_{m=j+1,\dots,K_1} J_m(v_2,s),$$

that is, we drop the condition on v_2 for $m = 1,\dots,j$.

Letting $\frac{dQ}{d\mathbb{P}^{D_N,0}} = \frac{\exp(\sum_{m=1}^{K_1}\lambda U_m(v_1) + \lambda\sum_{m=j+1}^{K_1}U_m(v_2)))}{\mathbb{E}^{D_N,0}[\exp(\sum_{m=1}^{K_1}\lambda U_m(v_1)+\lambda\sum_{m=j+1}^{K_1}U_m(v_2))]}$ we have

$$\begin{aligned}
&\mathbb{P}^{D_N,0}[J'] \\
&\quad\leq Q\left[J';\exp\left(-\sum_{m=1}^{K_1}\lambda U_m(v_1) - \lambda\sum_{m=j+1}^{K_1}U_m(v_2)\right)\right] \\
&\qquad\times\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda U_m(v_1) + \lambda\sum_{m=j+1}^{K_1}U_m(v_2)\right)\right] \\
&\quad\leq \exp\left(-\lambda\frac{2K_1-j}{K}(1-s)2\sqrt{g}\log N\right) \\
&\qquad\times\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda U_m(v_1) + \lambda\sum_{m=j+1}^{K_1}U_m(v_2)\right)\right].
\end{aligned}$$

By Theorem 5.8, for all $\lambda \le 2/\sqrt{g}$,

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1} \lambda U_m(v_1) + \sum_{m=j+1}^{K_1} \lambda U_m(v_2)\right))\right] \tag{5.10}$$
$$= \exp\left(\frac{1}{2}\sum_{m=1}^{K_1} \lambda^2 \frac{1}{K} g \log N + \frac{1}{2}\sum_{m=j+1}^{K_1} \lambda^2 \frac{1}{K} g \log N + o(\log N)\right).$$

Thus in fact $\mathbb{P}^{D_N,0}[J']$ is at most

$$\exp\left(\frac{1}{2}\lambda^2 \frac{2K_1 - j}{K} g \log N - \lambda \frac{2K_1 - j}{K}(1-s)2\sqrt{g}\log N + o(\log N)\right).$$

Setting $\lambda = 2/\sqrt{g},$ we find that

$$\mathbb{P}^{D_N,0}[J'] \le \exp\left(-2\frac{2K_1 - j}{K}\log N + 5s\frac{2K_1 - j}{K}\log N\right). \qquad \square$$

We can now prove the second moment estimate Lemma 5.5.

PROOF OF LEMMA 5.5. We write the second moment as

$$\mathbb{E}^{D_N,0}[\mathcal{N}_{K_1}^2] \le \sum_{v_1, v_2 \in [-0.9N, 0.9N]^2} \mathbb{P}^{D_N,0}[J(v_1, s) \cap J(v_2, s)].$$

Splitting the sum according to the distance $|v_1 - v_2|$, we get that

$$\mathbb{E}^{D_N,0}[\mathcal{N}_{K_1}^2] = \sum_{j=1}^{K_1} \sum_{N^{1-j/K} \le |v_1 - v_2| \le N^{1-(j-1)/K}} \mathbb{P}^{D_N,0}[J(v_1, s) \cap J(v_2, s)]$$
$$+ \sum_{|v_1 - v_2| \le N^{s^3}} \mathbb{P}^{D_N,0}[J(v_1, s) \cap J(v_2, s)].$$

The first summation gives the main contribution. Now using Lemma 5.7 and the fact that there are at most $N^2 \times N^{2-2(j-1)/K}$ points at distance less than $N^{1-(j-1)/K}$ we obtain an upper bound of

$$\sum_{j=1}^{K_1} N^{4-2(j-1)/K} \times N^{-2\frac{2K_1-j}{K} + 5s\frac{2K_1-j}{K}} + N^2 N^{-2+5s}$$
$$= N^4 \sum_{j=1}^{K_1} N^{-4(1-s^3)+2/K} N^{10s(1-s^3)} + \sum_{j=1}^{K_1} N^{5s}$$
$$\le [K_1 + 1] N^{\frac{2}{K}+10s},$$

which for N large enough is at most $N^{\frac{2}{K}+11s}$. $\square$

5.2. *Finite dimensional distribution of the harmonic averages*. We now state and prove a result concerns the joint distribution of the increment of the harmonic averages at mesoscopic scales. The next theorem shows approximate joint Gaussianity of $\{U_m\}_{m=1}^{K_1}$, defined in (5.4).

THEOREM 5.8. *For all bounded sequence* $\{\lambda_m\}_{m=1,\dots,K_1}$ *such that* $\max_m \lambda_m \le C$ *and* $v \in [-0.9N, 0.9N]^2$, *we have for all* K *sufficiently large*,

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1} \lambda_m U_m(v)\right)\right] = \exp\left(\frac{1}{2}\sum_{m=1}^{K_1} \lambda_m^2 \frac{1}{K} g \log N + o(\log N)\right), \tag{5.11}$$

where the $o(\log N)$ *term depends on* K, ε, C, *and the constant* δ *from Theorem* 2.5. *Also, for* $v_1, v_2 \in [-N/2, N/2]^2$ *such that for some* $j \in \{1, \dots, K_1\}$, $N^{1-\frac{j}{K}} \le |v_1 - v_2| \le N^{1-\frac{j-1}{K}}$, *and for bounded sequences* $\{\lambda_{m,i}\}_{i=1,2}$ *such that* $\max_{m,i} \lambda_{m,i} \le C$, *we have for all* K *sufficiently large*,

$$\begin{aligned}
&\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1} \lambda_{m,1} U_m(v_1) + \sum_{m=j+1}^{K_1} \lambda_{m,2} U_m(v_2)\right))\right] \\
&\quad = \exp\left(\frac{1}{2}\sum_{m=1}^{K_1} \lambda_{m,1}^2 \frac{1}{K} g \log N + \frac{1}{2}\sum_{m=j+1}^{K_1} \lambda_{m,2}^2 \frac{1}{K} g \log N + o(\log N)\right).
\end{aligned} \tag{5.12}$$

PROOF. We first prove (5.11). Recall that

$$\begin{aligned}
\mathcal{G} &= \left\{\phi : \max_{v \in D_N} |\phi(v)| < (\log N)^2\right\} \\
&= \left\{\phi : \max_{v \in D_N} |\phi(v)| < c(s)(\log r_M)^2\right\}.
\end{aligned}$$

Using the Brascamp–Lieb inequality and Lemma 2.7, it is easy to bound

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1} \lambda_m U_m(v)\right) 1_{\mathcal{G}^c}\right] = o_N(1),$$

therefore we only need to compute $\mathbb{E}^{D_N,0}[\exp(\sum_{m=1}^{K_1} \lambda_m U_m(v)) 1_{\mathcal{G}}]$.

Indeed, denote $r_{[mM/K_1]}$ as $\tilde{r}_m$, and $\mathcal{F}_m = \sigma\{\phi(v) : v \in D_N \setminus B_{\tilde{r}_m}(v)\}$, by the Markov property we have

$$\begin{aligned}
&\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1} \lambda_m U_m(v)\right) 1_{\mathcal{G}}\right] \\
&\quad = \mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1-1} \lambda_m U_m\right) 1_{\mathcal{G}} \mathbb{E}\left[e^{\lambda_{K_1} U_{K_1}} 1_{\mathcal{G}} | \mathcal{F}_{K_1-1}\right]\right].
\end{aligned}$$

By Lemma 2.8, there exist $C_1 < \infty$ and $\delta > 0$, such that

$$\begin{aligned}
\left|\mathbb{E}\left[e^{\lambda_{K_1} U_{K_1}} 1_{\mathcal{G}} | \mathcal{F}_{K_1-1}\right] - \mathbb{E}^{\tilde{r}_{K_1-1},0}\left[e^{\lambda_{K_1} U_{K_1}} 1_{\mathcal{G}}\right]\right| &\le \tilde{r}_{K_1-1}^{-\delta} \exp\left(c_1 \operatorname{Var}_{\mathrm{DGFF}}^{\tilde{r}_{K_1-1},0}(\lambda_{K_1} U_{K_1})\right) \\
&\le \tilde{r}_{K_1-1}^{-\delta} \exp\left(C^2 C_1 \frac{1}{K} \log N\right),
\end{aligned}$$

where $C = \max_m \lambda_m$. Take K large enough such that

$$C^2 C_1 \frac{1}{K} \le \frac{1}{2} \delta s^3,$$

we thus have

$$\left|\mathbb{E}\left[e^{\lambda_{K_1} U_{K_1}} 1_{\mathcal{G}} | \mathcal{F}_{K_1-1}\right] - \mathbb{E}^{\tilde{r}_{K_1-1},0}\left[e^{\lambda_{K_1} U_{K_1}} 1_{\mathcal{G}}\right]\right| \le \tilde{r}_{K_1-1}^{-\delta/2} \le \tilde{r}_{K_1-1}^{-\delta/2} \mathbb{E}^{\tilde{r}_{K_1-1},0}\left[e^{\lambda_{K_1} U_{K_1}} 1_{\mathcal{G}}\right].$$

Therefore,

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda_m U_m(v)\right)1_{\mathcal{G}}\right]$$
$$=(1+O(\tilde r_{K_1-1}^{-\delta/2}))\mathbb{E}^{\tilde r_{K_1-1},0}[e^{\lambda_{K_1}U_{K_1}}1_{\mathcal{G}}]\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1-1}\lambda_m U_m\right)1_{\mathcal{G}}\right].$$

Keep iterating then yields

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda_m U_m(v)\right)1_{\mathcal{G}}\right]=\prod_{m=1}^{K_1}(1+O(\tilde r_{m-1}^{-\delta/2}))\mathbb{E}^{\tilde r_{m-1},0}[e^{\lambda_m U_m}1_{\mathcal{G}}].$$

By Theorem 4.1 (and Remark 4.2), there exists $g=g(V)>0$, such that

$$\mathbb{E}^{\tilde r_{m-1},0}[e^{\lambda_m U_m}]=\exp\left(\frac{\lambda_m^2}{2}\frac{g}{K}\log N+o(\log N)\right),\tag{5.13}$$

and by Lemma 2.7 and the Brascamp–Lieb inequality,

$$\mathbb{E}^{\tilde r_{m-1},0}[e^{\lambda_m U_m}1_{\mathcal{G}^c}]=o_N(1).$$

Since $\sum_{m=1}^{K_1}\tilde r_{m-1}^{-\delta/2}<\infty$, this completes the proof of (5.11).

The proof of (5.12) is very similar to that of (5.11). We define for $i=1,2$, $\mathcal{F}_{m,i}=\sigma\{\phi(v): v\in D_N\setminus B_{\tilde r_m}(v_i)\}$. Then, by the same argument,

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda_{m,1}U_m(v_1)+\sum_{m=j+1}^{K_1}\lambda_{m,2}U_m(v_2)\right)1_{\mathcal{G}}\right]$$
$$=\mathbb{E}^{D_N,0}\Bigg[\exp\left(\sum_{m=1}^{K_1-1}\lambda_{m,1}U_m(v_1)+\sum_{m=j+1}^{K_1}\lambda_{m,2}U_m(v_2)\right)$$
$$\times 1_{\mathcal{G}}\mathbb{E}[\exp(\lambda_{K_1,1}U_{K_1}(v_1))1_{\mathcal{G}}|\mathcal{F}_{K_1-1,1}]\Bigg]$$
$$=(1+O(\tilde r_{K_1-1}^{-\delta/2}))\mathbb{E}^{\tilde r_{K_1-1},0}[\exp(\lambda_{K_1,1}U_{K_1}(v_1))1_{\mathcal{G}}]$$
$$\times\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1-1}\lambda_{m,1}U_m(v_1)+\sum_{m=j+1}^{K_1}\lambda_{m,2}U_m(v_2)\right)1_{\mathcal{G}}\right].$$

Then conditioned on $\mathcal{F}_{K_1-1,2}$, apply the Markov property and Lemma 2.8, we can write the above display as

$$(1+O(\tilde r_{K_1-1}^{-\delta/2}))\mathbb{E}^{\tilde r_{K_1-1},0}[\exp(\lambda_{K_1,1}U_{K_1}(v_1))1_{\mathcal{G}}]\mathbb{E}^{\tilde r_{K_1-1},0}[\exp(\lambda_{K_1,2}U_{K_1}(v_2))1_{\mathcal{G}}]$$
$$\times\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1-1}\lambda_{m,1}U_m(v_1)+\sum_{m=j+1}^{K_1-1}\lambda_{m,2}U_m(v_2)\right)1_{\mathcal{G}}\right].$$

Keep iterating, we obtain

$$\mathbb{E}^{D_N,0}\left[\exp\left(\sum_{m=1}^{K_1}\lambda_{m,1}U_m(v_1)+\sum_{m=j+1}^{K_1}\lambda_{m,2}U_m(v_2)\right)1_{\mathcal{G}}\right]$$

$$= \prod_{m=j+1}^{K_1} \bigl(1 + O(\tilde{r}_{m-1}^{-\delta/2})\bigr)\mathbb{E}^{\tilde{r}_{m-1},0}\bigl[\exp(\lambda_{m,1}U_m(v_1))1_{\mathcal{G}}\bigr]\mathbb{E}^{\tilde{r}_{m-1},0}\bigl[\exp(\lambda_{m,2}U_m(v_2))1_{\mathcal{G}}\bigr]$$

$$\times \prod_{m=1}^{j} \bigl(1 + O(\tilde{r}_{m-1}^{-\delta/2})\bigr)\mathbb{E}^{\tilde{r}_{m-1},0}\bigl[\exp(\lambda_{m,1}U_m(v_1))1_{\mathcal{G}}\bigr].$$

Applying (5.13), we conclude the proof of (5.12). □

5.3. *Bootstrapping*. We now use Proposition 5.2 to prove the desired lower bound (5.1). Proposition 5.2 shows that the field reaches $(1-2s)2\sqrt{g}\log N$ with at least polynomially small probability. We will apply Theorem 2.5 to see that the field in different regions of $[-N,N]^2$ are essentially decoupled. Therefore, applying Proposition 5.2 in each region one can show with high probability, there is some $v \in [-N,N]^2$ such that $\phi(v) - X_{r_{0,-}}(v) \geq (1-2s)2\sqrt{g}\log N$.

To carry out this argument, tile $[-N,N]^2$ by disjoint boxes $D_1, D_2, \dots, D_m$ of side-length $N^{1-\eta}$, where $m \asymp N^\eta$, and η is a small number that will be chosen later. Let $\mathcal{B}$ be the union of all the ∂D_i.

Consider the good event

$$\mathcal{G} = \Bigl\{ \max_{v\in[-N,N]^2} \bigl|\phi(v)\bigr| \leq (\log N)^2 \Bigr\}.$$

By Lemma 2.7, we have $\mathbb{P}^{D_N,0}[\mathcal{G}^c] \preccurlyeq e^{-c(\log N)^3}$, as $N \to \infty$.

On the event $\mathcal{G}$, for $i = 1, \dots, m$, let $\bar{D}_i$ be the box concentric to D_i, but with side length $\frac{1}{2}N^{1-\eta}$. Let $R = \frac{1}{2}N^{1-\eta}$. We further define

$$\tilde{\mathcal{N}_{K}}_i = \bigl\{\forall v \in \bar{D}_i : \phi(v) - X_{R,-}(v,\phi) < (1-2s)(1-\eta)2\sqrt{g}\log N\bigr\}.$$

Now

$$\mathbb{P}^{D_N,0}[\tilde{\mathcal{N}_{K}}_i, i = 1,\dots,m; \mathcal{G}] = \mathbb{P}^{D_N,0}\bigl[\mathbb{P}[\tilde{\mathcal{N}_{K}}_i, i = 1,\dots,m|\phi(x), x \in \mathcal{B}]; \mathcal{G}\bigr].$$

Using the Gibbs property of the measure (1.3), we have the conditional decoupling

$$\mathbb{P}^{D_N,0}\bigl[\tilde{\mathcal{N}_{K}}_i, i = 1,\dots,m|\phi(x), x \in \mathcal{B}\bigr] = \prod_{i=1}^{m} \mathbb{P}^{D_i,\phi 1_{\partial D_i}}\bigl[\tilde{\mathcal{N}_{K}}_i|\phi(x), x \in \partial D_i\bigr].$$

Consider for each i the law $\mathbb{P}^{D_i,\phi 1_{\partial D_i}}$. Then on $\mathcal{G}$ we can apply Lemmas 2.5 and 3.2 to construct a coupling Q^i of a field ϕ with law $\mathbb{P}^{D_i,\phi 1_{\partial D_i}}$ and a field $\phi^{0,i}$ with law $\mathbb{P}^{D_i,0}$ such that

$$Q^i\bigl[\forall v \in \bar{D}_i : \phi(v) - X_{R,-}(v,\phi) = \phi^{0,i}(v) - X_{R,-}\bigl(v,\phi^{0,i}\bigr)\bigr] \geq 1 - N^{-\delta(1-\eta)},$$

where the constant $\delta > 0$ is from Theorem 2.5.

Thus

$$\mathbb{P}^{D_N,0}\left(\begin{matrix}\forall v \in [-0.9N, 0.9N]^2 : \\ \phi(v) - X_{R,-}(v,\phi) < (1-2s)(1-\eta)2\sqrt{g}\log N; \mathcal{G}\end{matrix}\right)$$

$$\leq \prod_{i=1}^{m} \bigl(\mathbb{P}^{D_i,0}[\tilde{\mathcal{N}_{K}}_i] + N^{-\delta(1-\eta)}\bigr)$$

$$\leq \prod_{i=1}^{m} \bigl(1 - \bigl(N^{1-\eta}\bigr)^{-21s} + N^{-\delta(1-\eta)}\bigr),$$

where we apply Proposition 5.2 to obtain the last inequality. Now let s and η be small enough, depending on δ, such that

$$21s < \delta \quad \text{and} \quad \eta > 21s/(1+21s). \tag{5.14}$$

Thus we have

$$\mathbb{P}^{D_N,0}\begin{pmatrix} \forall v \in [-0.9N, 0.9N]^2 : \\ \phi(v) - X_{R,-}(v,\phi) < (1-2s)(1-\eta)2\sqrt{g}\log N; \mathcal{G} \end{pmatrix} \preccurlyeq e^{-N^{\varepsilon_1}}, \tag{5.15}$$

for some $\varepsilon_1 > 0$.

In view of (5.14), we can take $\eta = 21s$. Then, on the complement of the event (5.15), there exists $v_1 \in [-0.9N, 0.9N]^2$ such that

$$\phi(v_1) - X_{R,-}(v_1,\phi) \geq (1-19s)2\sqrt{g}\log N.$$

Notice that (for $g_0 = 2/\pi$)

$$\mathrm{Var}^{D_N,0}_{\mathrm{DGFF}}[X_{R,-}(v_1,\phi)] = g_0\eta\log N + o(\log N) = 21sg_0\log N + o(\log N).$$

By Lemma 2.2, there exists $c_{\mathrm{BL}} > 0$, such that

$$\mathbb{P}^{D_N,0}[X_{R,-}(v_1,\phi) > s^{1/3}\log N] \leq \exp\left(-c_{\mathrm{BL}}\frac{s^{2/3}(\log N)^2}{s\log N}\right) = N^{-c_{\mathrm{BL}}s^{-1/3}}. \tag{5.16}$$

Combining (5.15) and (5.16), we see that

$$\mathbb{P}^{D_N,0}\Big[\max_{v\in[-0.9N,0.9N]^2}\phi(v) < (1-2s^{1/3})2\sqrt{g}\log N\Big] \leq N^{-c_{\mathrm{BL}}s^{-1/3}} + e^{-N^{\varepsilon_1}}.$$

And we conclude (5.1).

5.4. *High points.* We now sketch the proof of Theorem 1.3. The proof follows from the same argument as the proof of Theorem 1.1, for completeness we sketch the idea below.

It suffices to prove that for any $s > 0$,

$$\mathbb{P}^{D_N,0}\big(|\mathcal{H}_N(\eta)| > N^{2(1-\eta^2)+s}\big) = o_N(1) \quad \text{and} \tag{5.17}$$

$$\mathbb{P}^{D_N,0}\big(|\mathcal{H}_N(\eta)| < N^{2(1-\eta^2)-s}\big) = o_N(1). \tag{5.18}$$

Since

$$\begin{aligned}\mathbb{P}^{D_N,0}\big(|\mathcal{H}_N(\eta)| > N^{2(1-\eta^2)+s}\big) &\leq N^{-2(1-\eta^2)-s}\mathbb{E}\big[|\mathcal{H}_N(\eta)|\big] \\ &\leq N^{-2(1-\eta^2)-s}\sum_{v\in D_N}\mathbb{P}^{D_N,0}\big(\phi(v) \geq 2\sqrt{g}\eta\log N\big),\end{aligned}$$

the upper bound (5.17) follows directly from applying Theorem 1.4 with $u = 2\sqrt{g}\eta\log N$.

We now focus on the lower bound (5.18). Recall the definition of U_m in (5.4). For $\eta \in (0,1)$, having in mind that we aim to count the points $\{v \in D_N : \phi(v) > 2\sqrt{g}\eta\log N\}$, we look at the following truncated event such that the increments U_m are slightly higher than $2\sqrt{g}\frac{\eta}{K}\log N$:

$$J_m(v;\eta;s) = \left\{U_m(v) \in \left[(1+s)2\sqrt{g}\frac{\eta}{K}\log N, (1+2s)2\sqrt{g}\frac{\eta}{K}\log N\right]\right\}$$

and for $K_1 := [(1-s^3)K] + 1$,

$$J(v;\eta;s) = \bigcap_{m=1,\dots,K_1} J_m(v;\eta;s).$$

Also define the counting random variable

$$\mathcal{N}_{K_1}(\eta, s) = \sum_{v \in [-0.9N, 0.9N]^2} 1_{J(v;\eta;s)}.$$

By the same Brascamp–Lieb bounds as (5.3) and (5.5), to study the dimension of $\mathcal{H}_N(\eta)$, it suffices to study $\{v : J(v; \eta; s) \text{ occurs}\}$. Indeed, the same first moment computation as Lemma 5.4 and Lemma 5.6 (but instead using the change of measure $\frac{dQ}{d\mathbb{P}^{D_N,0}} = \frac{\exp(\lambda\eta \sum_{m=1}^{K_1} U_m(v))}{\mathbb{E}^{D_N,0}[\exp(\lambda\eta \sum_{m=1}^{K_1} U_m(v))]}$) yields

$$\mathbb{E}[\mathcal{N}_{K_1}(\eta, s)] \geq N^{2(1-\eta^2)-8s\eta^2},$$

and the same second moment computation as Lemma 5.5 and Lemma 5.7 yields

$$\mathbb{E}[\mathcal{N}_{K_1}^2(\eta, s)] \leq N^{4(1-\eta^2)-5s\eta^2}.$$

Therefore,

$$\mathbb{E}[\mathcal{N}_{K_1}^2(\eta, s)] \leq N^{11s\eta^2} \mathbb{E}[\mathcal{N}_{K_1}(\eta, s)]^2.$$

Applying the Payley–Zygmund inequality then yields

$$\begin{aligned}
&\mathbb{P}^{D_N,0}\Big(\big|\{v : J(v; \eta; s) \text{ occurs}\}\big| < \frac{1}{2} N^{2(1-\eta^2)-s}\Big) \\
&\leq 1 - \mathbb{P}^{D_N,0}\Big(\mathcal{N}_{K_1}(\eta, s) > \frac{1}{2}\mathbb{E}[\mathcal{N}_{K_1}(\eta, s)]\Big) \\
&\leq 1 - cN^{-11s\eta^2}.
\end{aligned}$$

But to complete the proof of (5.18) we want $\mathbb{P}^{D_N,0}(\mathcal{N}_{K_1}(\eta, s) > \frac{1}{2}\mathbb{E}[\mathcal{N}_{K_1}(\eta, s)])$ to be close to 1. This can be proved by carrying out the same bootstrapping in Section 5.3, obtaining the high probability by creating a large number (N^γ, where $\gamma = \gamma(s, \delta)$, and δ is the constant from Theorem 2.5) of essentially independent trials with success probability $N^{-11s\eta^2}$.

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The second author is also affiliated with University of Warwick.

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