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Convergence of the empirical spectral distribution of Gaussian matrix-valued processes*

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Abstract

For a given normalized Gaussian symmetric matrix-valued process $Y^{(n)}$, we consider the process of its eigenvalues $\{(\lambda_1^{(n)}(t),\dots,\lambda_n^{(n)}(t));t\geq 0\}$ as well as its corresponding process of empirical spectral measures $\mu^{(n)}=(\mu_t^{(n)};t\geq 0)$. Under some mild conditions on the covariance function associated to $Y^{(n)}$, we prove that the process $\mu^{(n)}$ converges in probability to a deterministic limit μ, in the topology of uniform convergence over compact sets. We show that the process μ is characterized by its Cauchy transform, which is a rescaling of the solution of a Burgers' equation. Our results extend those of Rogers and Shi [14] for the free Brownian motion and Pardo et al. [12] for the non-commutative fractional Brownian motion when $H>1/2$ whose arguments use strongly the non-collision of the eigenvalues. Our methodology does not require the latter property and in particular explains the remaining case of the non-commutative fractional Brownian motion for $H<1/2$ which, up to our knowledge, was unknown.

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1 Introduction

Let us consider a family of independent centered Gaussian processes $\{X_{i,j};i,j\in\mathbb{N}\}$ defined in a probability space $(\Omega,\mathcal{F},\mathbb{P})$, with common covariance function here denoted by $R(s,t)$, for $s,t\geq 0$. That is to say, the Gaussian processes $X_{i,j}:=(X_{i,j}(t);t\geq 0)$ are independent with zero mean and covariance given by

$$\mathbb{E}\left[X_{i,j}(s)X_{i,j}(t)\right]=R(s,t),\qquad \text{for}\qquad s,t\geq 0,$$

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where $R(s,t)$ is a non-negative definite covariance function. For $n \in \mathbb{N}$, we also consider the renormalized symmetric Gaussian matrix-valued process $Y^{(n)}(t) := [Y_{i,j}^{(n)}(t)]_{1\leq i,j\leq n}$, for $t \geq 0$, defined as follows

$$Y_{i,j}^{(n)}(t) := \begin{cases} \dfrac{1}{\sqrt{n}} X_{i,j}(t) + A_{i,j}^{(n)} & \text{if } i < j, \\ \dfrac{\sqrt{2}}{\sqrt{n}} X_{i,i}(t) + A_{i,i}^{(n)} & \text{if } i = j, \end{cases}$$

where the $A_{i,j}^{(n)}$ are the coefficients of a deterministic symmetric matrix $A^n = [A_{i,j}^n]_{1\leq i,j\leq n}$. Let us denote the n-dimensional process of eigenvalues of $Y^{(n)}$ by $(\lambda_1^{(n)}(t), \cdots, \lambda_n^{(n)}(t))$, for $t \geq 0$. We also denote by $\mathtt{Pr}(\mathbb{R})$ for the space of probability measures on $\mathbb{R}$ endowed with the topology of weak convergence and let $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$ be the space of continuous functions from $\mathbb{R}_+$ into $\mathtt{Pr}(\mathbb{R})$, endowed with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$.

In this manuscript, we are interested in the asymptotic behaviour of the $\mathtt{Pr}(\mathbb{R})$-valued process of empirical distributions $\{\mu^{(n)}; n \geq 1\}$, defined by $\mu^{(n)} := (\mu_t^{(n)}, t \geq 0)$ where

$$\mu_t^{(n)} := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \quad t \geq 0,$$

and δ_x denotes the Dirac measure centered at x. In particular, we aim to determine the limit in probability of the process $\mu^{(n)}$, viewed as an element of the space $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$ of $\mathtt{Pr}(\mathbb{R})$-valued stochastic processes with continuous trajectories.

This problem has been studied before in the framework of interacting particles by Rogers and Shi [14] and Cépa and Lépingle in [5], when the $X_{i,j}$'s are standard Brownian motions. We also refer to Cabanal-Duvillard and Guionnet [4] for the case of Hermitian Brownian motion where the latter case is included. The authors in [5, 14] proved that $\mu^{(n)}$ converges, as n tends to infinity, to a deterministic process whose Cauchy transforms are given by the solution of a Burgers' equation. More recently, Pardo et al. [12] extended the previous result to the case where the $X_{i,j}$'s are fractional Brownian motions with Hurst (or self-similar) index $H > 1/2$. We briefly describe the main ideas presented in all these manuscripts where the no-collision of the eigenvalues is crucial. Let $\mathcal{C}^r(\mathbb{R})$ denote the set of real-valued functions with continuous derivatives of order r, and let us introduce the subset

$$\mathcal{C}_b^r(\mathbb{R}) := \left\{ f \in \mathcal{C}^r(\mathbb{R}) \middle| \sum_{i=1}^{r} \sup_{x\in\mathbb{R}} \left| f^{(r)}(x) \right| < \infty \right\}. \tag{1.1}$$

In the Brownian case, the main idea for determining the asymptotic behaviour of $\mu^{(n)}$ consists, first, in characterizing the process of its eigenvalues $(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ as the unique strong solution of a system of stochastic differential equations. Then one can prove that for every $f \in \mathcal{C}_b^3(\mathbb{R})$, the process

$$\langle \mu_t^{(n)}, f \rangle := \int f(x) \mu_t^{(n)}(\mathrm{d}x), \qquad t \geq 0, \tag{1.2}$$

satisfies a stochastic differential equation which converges, as n tends to infinity, to a deterministic differential equation with a given initial condition. Using the characterization of $\langle \mu_t^{(n)}, f \rangle$ as the solution of an initial value problem and a suitable approximation argument, one can also show that the Cauchy transform of $\mu_t^{(n)}$, defined by

$$G_t^{(n)}(z) = \int (x - z)^{-1} \mu_t^{(n)}(\mathrm{d}x),$$

for z belonging to the upper half plane $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \Im(z) > 0\}$, converges in probability to a function $F_t(z)$, which can be characterized as the unique solution, absolutely continuous over t and holomorphic over $z \in \mathbb{C}_+$, of the complex deterministic Burgers' equation

$$\begin{aligned} \frac{\partial}{\partial \tau} F_\tau(z) &= F_\tau(z) \frac{\partial}{\partial z} F_\tau(z), \\ F_0(z) &= \int_{\mathbb{R}} \frac{1}{x - z} \mu_0(\mathrm{d}x), \end{aligned} \tag{1.3}$$

for $\tau \geq 0$ and $z \in \mathbb{C}_+$. The complex Burgers equation has previously appeared in free harmonic analysis in the work of Voiculescu [16], in connection to the free Brownian motion with a given starting law. We refer the reader to [1, Lemma 4.3.15] for a proof of the existence and uniqueness of the system (1.3). Notice that in the case where $A^{(n)}$ is the zero matrix, we have that μ_0 is the Dirac measure centered at zero and the solution of the system (1.3) is given by

$$F_\tau(z) = \frac{1}{2\tau} \left(\sqrt{z^2 - 4\tau} - z \right), \qquad \tau \geq 0, \ z \in \mathbb{C}_+, \tag{1.4}$$

under the convention that the square root is taken with respect to a branch for which the logarithm defined in $\mathbb{C} \backslash [0, \infty)$, so that the image of $\mathbb{C}_+$ under the mapping $z \mapsto z^2 - 4\tau$ lies in the domain of the square root function (such convention will be adopted as well for the rest of the paper). Since the unique measure μ_τ associated to the Cauchy transform (1.4) is a semicircle distribution with variance t, the above convergence gives a dynamical version of Wigner's theorem. A key ingredient in the argument presented above, consists in using the well known fact that for any fixed $n \in \mathbb{N}$, the eigenvalues $(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ never collide, in other words, the trajectories of $\lambda_i^{(n)}$ and $\lambda_j^{(n)}$ never intersect for any $1 \leq i, j \leq n$, and satisfy the following non-colliding diffusion equation

$$\lambda_i^{(n)}(t) = \lambda_i^{(n)}(0) + \sqrt{2} W_t^i + \sum_{j \neq i} \int_0^t \frac{1}{\lambda_i^{(n)}(s) - \lambda_j^{(n)}(s)} \mathrm{d}s, \tag{1.5}$$

where $W^1, \ldots, W^n$ are independent one-dimensional standard Brownian motions. For further details we refer Anderson et al. [1] and Cabanal-Duvillard and Guionnet [4].

The case where the $X_{i,j}$'s are fractional Brownian motions of Hurst parameter $H \in (1/2, 1)$ was handled in [12] using Young integrals and Malliavin calculus techniques. In particular, it was shown that its eigenvalues $(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ satisfy a Young integral equation which in turn induces a Skorokhod integral equation for $\langle \mu_t^{(n)}, f \rangle$, when $f \in \mathcal{C}_b^3(\mathbb{R})$. Then by taking limits as n tends to infinity in this equation and using some estimations based on Malliavin calculus techniques, one can prove that $\langle \mu_t^{(n)}, f \rangle$ converges to the solution of a deterministic differential equation which implicitly characterizes the limit process. Similarly to the Brownian case, the well-posedness of the stochastic Young integral equation for $(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ requires the non-collision of the eigenvalues of $Y^{(n)}$, which was proved by Nualart and Perez-Abreu in [11].

As we said before, the previous arguments rely heavily on the fact that the eigenvalues of a fractional Brownian motion with Hurst parameter $H \geq 1/2$ never collide and that a suitable Itô or stochastic Young integral equation for $(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ can be formulated. In the case where the $X_{i,j}$'s are general Gaussian processes, these two properties may not hold and a more refined treatment of the problem is required. Indeed, the non-collision of the eigenvalues for Gaussian processes with highly rough paths, is still an open problem. In addition, if the trajectories of $Y_{i,j}^{(n)}$ are too rough, it is not possible to formulate a stochastic differential equation for its eigenvalues neither in the Itô or

Young integral sense. In other words an extended version of the Skorokhod integral is required and consequently the estimations based on Malliavin calculus are harder to handle, since the extended Skorokhod integration doesn't have a clear analogue of Meyers' inequality which is required for characterizing the limiting object.

In the present manuscript, we show that under some mild conditions on the covariance function associated to $Y^{(n)}$, the process $\langle \mu_t^{(n)}, f\rangle$, for $t \geq 0$, satisfies a Skorokhod stochastic differential equation (see Lemma 3.1) defined in the extended domain of the divergence (see Section 2.1 for a proper definition). In particular, we prove that the Skorokhod stochastic differential equation makes sense even in the presence of collision of the eigenvalues. Then we prove a tightness property for the sequence of processes $\{\mu^{(n)}; n \geq 1\}$ using similar arguments as those presented in [12]. It is important to note that due to the lack of a clear analogue of Meyers' inequalities for the extended Skorokhod integral deducing the limiting object is not straightforward, in fact we need completely different estimates and techniques to those used in [12].

Our main result requires the following assumptions on the covariance function R:

(H1) For every $T > 0$, the mapping $s \mapsto R(s,s)$ is continuously differentiable in $(0,\infty)$, continuous at zero and $\frac{d}{ds}R(s,s)$ has finitely many zeros in $(0,T]$. In addition, there exists $\alpha > 1$ such that for all $t \in [0,T]$, the mapping $s \mapsto R(s,t)$ is absolutely continuous on $[0,T]$, and

$$\sup_{0\leq t\leq T}\int_0^T \left|\frac{\partial R}{\partial s}(s,t)\right|^{\alpha} \mathrm{d}s < \infty.$$

(H2) There exist constants $\kappa, \gamma > 0$, such that for every $s,t > 0$,

$$R(s,s) - 2R(s,t) + R(t,t) \leq \kappa\,|t-s|^{\gamma}.$$

Theorem 1.1. *Assume that the covariance function R satisfies conditions **(H1)** and **(H2)**, and $\mu_0^{(n)}$ converges weakly to a probability measure μ_0. Then the family of measure-valued processes $\{\mu^{(n)} : n \geq 1\}$ converges weakly in $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$ to the unique (deterministic) continuous probability-measure valued function $(\mu_t; t \geq 0)$, satisfying*

$$\big\langle \mu_t, f\big\rangle = \big\langle \mu_0, f\big\rangle + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x)-f'(y)}{x-y}\frac{\mathrm{d}}{\mathrm{d}s}(R(s,s))\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mathrm{d}s, \tag{1.6}$$

for each $t \geq 0$ and $f \in \mathcal{C}_b^3$. Moreover, its Cauchy transform $G_t(z)$, satisfies

$$G_t(z) = F_{R(t,t)}(z), \qquad \text{for } z \in \mathbb{C}_+ \text{ and } t \geq 0, \tag{1.7}$$

where $F_\tau(z)$, for $\tau \geq 0$, is the unique solution of the system (1.3).

We note that the term $\frac{f'(x)-f'(y)}{x-y}$ when $x = y$, in the integral of the right-hand side of (1.6) is understood as $f''(x)$.

Another important observation is related to the fractional Brownian motion. Recall that its covariance function satisfies

$$R(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \tag{1.8}$$

for $H \in (0,1)$. Such covariance function clearly satisfies conditions **(H1)** and **(H2)** and consequently, Theorem 1.1 generalizes the results previously proved for the Brownian motion in [5] and [14], and for the fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ in [12].

Finally, we recall that in the case that $\mu_0 = \delta_0$, the unique solution to (1.3) is given by (1.4). The latter implies that

$$G_t(z) = \frac{1}{2R(t,t)}\left(\sqrt{z^2 - 4R(t,t)} - z\right) \qquad t \geq 0,\ z \in \mathbb{C}_+.$$

Hence, for each $t \geq 0$, μ_t is a semicircle distribution with variance $R(t,t)$.

The remainder of this manuscript is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus and Skorohod integration. In particular, we introduce the extended domain of the divergence. Section 3 is devoted to the proof of the Skorokhod stochastic differential equation for the process of eigenvalues associated to $Y^{(n)}$. The tightness property is proved in Section 4 and, finally in Section 5, the convergence in law of the sequence $\{\mu^{(n)}; n \geq 1\}$ is given.

2 Preliminaries on Malliavin calculus and Skorokhod integral

Let $d \geq 1$ and $T > 0$ be fixed. We denote by $X = ((X_t^1, \ldots, X_t^d); t \in [0,T])$ a d-dimensional continuous Gaussian process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose covariance satisfies

$$\mathbb{E}\left[X_s^i X_t^j\right] = \delta_{i,j} R(s,t), \qquad s,t \in [0,T],$$

for some positive definite covariance function R. Denote by $\mathscr{E}$ the space of step functions on $[0,T]$. We define in $\mathscr{E}$ the scalar product

$$\left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \right\rangle_{\mathfrak{H}} := \mathbb{E}\left[X_s^1 X_t^1\right] \qquad \text{for} \qquad s,t \in [0,T].$$

Let $\mathfrak{H}$ be the Hilbert space obtained by taking the completion of $\mathscr{E}$ with respect to this product. For every $1 \leq i \leq n$ fixed, the mapping $\mathbb{1}_{[0,t]} \mapsto X_t^i$ can be extended to linear isometry between $\mathfrak{H}$ and the Gaussian subspace of $L^2(\Omega)$ generated by the process $(X_t^i, t \geq 0)$. We will denote this isometry by $X^i(h)$, for $h \in \mathfrak{H}$.

If $f \in \mathfrak{H}^d$ is of the form $f = (f_1, \ldots, f_d)$, we set $X(f) := \sum_{i=1}^d X^i(f_i)$. The mapping $f \mapsto X(f)$ is a linear isometry between $\mathfrak{H}^d$ and the Gaussian subspace of $L^2(\Omega)$ generated by X. Let $\mathscr{S}$ denote the set of all cylindrical random variables of the form

$$F = g(X(h_1), \ldots, X(h_m)),$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $h_j \in \mathscr{E}^d$. The Malliavin derivative of F with respect to X, is the element of $L^2(\Omega; \mathfrak{H}^d)$, defined by

$$DF = \sum_{i=1}^m \frac{\partial g}{\partial x_i}(X(h_1), \ldots, X(h_m)) h_i. \tag{2.1}$$

For $p \geq 1$, the set $\mathbb{D}^{1,p}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{1,p}}$, defined by

$$\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|DF\|_{\mathfrak{H}^d}^p\right]\right)^{\frac{1}{p}}.$$

The operator D can be consistently extended to the set $\mathbb{D}^{1,p}$. We denote by δ the adjoint of the operator D, also called the divergence operator. A random element $u \in L^2(\Omega; \mathfrak{H}^d)$ belongs to the domain of δ, denoted by $\operatorname{Dom}\delta$, if and only if satisfies

$$\left|\mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right]\right| \leq C_u \mathbb{E}\left[F^2\right]^{\frac{1}{2}}, \text{ for every } F \in \mathbb{D}^{1,2},$$

where C_u is a constant only depending on u. If $u \in \mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right],$$

which holds for every $F \in \mathbb{D}^{1,2}$. We will make use of the notation

$$\sum_{i=1}^{d} \int_0^t u_s^i \delta X_s^i := \delta(u\mathbb{1}_{[0,t]}), \tag{2.2}$$

for $u \in L^2(\Omega; \mathfrak{H}^d)$ of the form $u_t = (u_t^1, \ldots, u_t^d)$.

In the case where X is a d-dimensional Brownian motion, i.e. its covariance function is given by $R(s,t) = s \wedge t$ and $\mathfrak{H} = L^2[0,T]$, the random variable (2.2) is an extension of the Itô integral. Motivated by this fact, we may interpret $\sum_{i=1}^d \int_0^t u_s^i \delta X_s^i$ as the stochastic integral of the process u. Nevertheless, the space $\mathfrak{H}$ turns out to be too small for this purpose. Indeed, in [6] it was shown that in the case where X is a fractional Brownian motion with Hurst parameter $0 < H < \frac{1}{4}$, that is to say its covariance function is of the form (1.8), the trajectories of X do not belong to the space $\mathfrak{H}$, and in particular, non-trivial processes of the form $(f(u_s); s \in [0,T])$, with $f : \mathbb{R} \to \mathbb{R}$, might not belong to the domain of δ. In order to overcome this difficulty, we extend the domain of δ by following the approach presented in [10] (see also [6]). The main idea for extending the domain of δ, consists on extending the definition of $\langle \varphi, \psi\rangle_{\mathfrak{H}}$ to the case where $\varphi \in L^\beta[0,T]$ for some $\beta > 1$, and ψ belongs to the space $\mathscr{E}$ of step functions over $[0,T]$.

In the sequel, we will assume that there exists a constant $\alpha > 1$ such that the following condition holds. Let β be the conjugate of α, defined by $\beta := \alpha/(\alpha-1)$. For any pair of functions $\varphi \in L^\beta[0,T]$ and $\psi \in \mathscr{E}$ of the form $\psi = \sum_{j=1}^m c_j \mathbb{1}_{[0,t_j]}$, we define

$$\langle \psi, \varphi\rangle_{\mathfrak{H}} := \sum_{j=1}^{m} c_j \int_0^T \varphi(s) \frac{\partial R}{\partial s}(s, t_j)\mathrm{d}s. \tag{2.3}$$

This expression is well defined since

$$\left|\left\langle \mathbb{1}_{[0,t]}, \varphi\right\rangle_{\mathfrak{H}}\right| = \left|\int_0^T \varphi_s \frac{\partial R}{\partial s}(s,t)\mathrm{d}s\right| \leq \|\varphi\|_{L^\beta[0,T]} \sup_{0\leq t\leq T} \left(\int_0^T \left|\frac{\partial R}{\partial s}(s,t)\right|^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} < \infty,$$

and coincides with the inner product in $\mathfrak{H}$ in the case where $\varphi \in \mathscr{E}$. Indeed, for $\varphi \in \mathscr{E}$ of the form $\varphi = \sum_{i=1}^n a_i \mathbb{1}_{[0,t_i]}$, we have

$$\left\langle \mathbb{1}_{[0,t]}, \varphi\right\rangle_{\mathfrak{H}} = \sum_{i=1}^{n} a_i R(t_i, t) = \sum_{i=1}^{n} a_i \int_0^{t_i} \frac{\partial R}{\partial s}(s,t)\mathrm{d}s = \int_0^T \varphi(s)\frac{\partial R}{\partial s}(s,t)\mathrm{d}s.$$

We define the extended domain of the divergence as follows.

Definition 2.1. *Let $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$ be the bilinear function defined by* (2.3)*. We say that a stochastic process $u \in L^1(\Omega; L^\beta[0,T])$ belongs to the extended domain of the divergence* $\mathrm{Dom}^*\delta$ *if there exists $p > 1$, such that*

$$\left|\mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right]\right| \leq C_u \left\|F\right\|_{L^p(\Omega)},$$

for any smooth random variable $F \in \mathscr{S}$, where C_u is some constant depending on u. In this case, $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}^d}\right]. \tag{2.4}$$

It is important to note that for a general covariance function $R(s,t)$ and $\beta > 1$, the domains $\mathrm{Dom}^*\delta$ and $\mathrm{Dom}\ \delta$ are not necessarily comparable (see Section 3 in [10] for further details about this fact).

The next result is a multidimensional version of Itô's formula for the Skorokhod integral and for functions that are smooth only on a dense open subset of the Euclidean space and satisfy some extra regularity conditions. In the sequel, for every $i \in \{1,\dots,d\}$, the map $\gamma_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ denotes the orthogonal projection onto the hyperplane $P_i := \{(x_1,\dots,x_d) \mid x_i = 0\}$. We use as well the following notation: for every real function $h : \mathcal{D} \subset \mathbb{R}^d \to \mathbb{R}$, we define

$$\mathbf{1}_{\mathcal{D}}(x)h(x) := \begin{cases} h(x) & \text{if } x \in \mathcal{D}, \\ 0 & \text{if } x \in \mathbb{R}^d \backslash \mathcal{D}, \end{cases}$$

for every $x \in \mathbb{R}^d$.

Theorem 2.2. *Assume that R satisfies **(H1)**. Consider a function $F : \mathbb{R}^d \to \mathbb{R}$, with $d > 1$, satisfying the following conditions:*

1. *There exists a measurable set $M \subset \mathbb{R}^d$, with Lebesgue measure zero, such that F is twice continuously differentiable in $\mathcal{D} := \mathbb{R}^d \backslash M$ and $\gamma_i(M)$ has measure zero with respect to the Lebesgue measure in P_i.*
2. *There exist constants $C > 0$ and $N > 0$, such that for all $x \in \mathcal{D}$ and $i \in \{1,\dots,d\}$,*

$$|F(x)| + \left|\frac{\partial F}{\partial x_i}(x)\right| \leq C(1 + |x|^N). \tag{2.5}$$

3. *The random variable $|\frac{\partial^2 F}{\partial x_i^2}(X_s)|$ has finite expectation, and there exists a deterministic constant $C > 0$, such that with probability one,*

$$\left|\sum_{i=1}^{d} \frac{\partial^2 F}{\partial x_i^2}(X_s)\right| \leq C, \tag{2.6}$$

Then, the process $u_s = (u_s^1,\dots,u_s^d)$ defined by $u_s^i := \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial F}{\partial x_i}(X_s)\mathbf{1}_{[0,t]}(s)$, belongs to $\mathrm{Dom}^*\delta$, and

$$F(X_t) = F(X_0) + \sum_{i=1}^{d}\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial F}{\partial x_i}(X_s)\delta X_s^i + \frac{1}{2}\sum_{i=1}^{d}\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial^2 F}{\partial x_i^2}(X_s)\frac{\mathrm{d}R(s,s)}{\mathrm{d}s}\mathrm{d}s, \tag{2.7}$$

for every $t \in [0,T]$.

It is well known from classical papers by Cheridito-Nualart, Decreusefond & al. (see [6], [7] and [8]) and others, that some regularity is required for the extended Skorokhod integral in (2.7) to make sense. For instance, it would seem natural to require F to be twice continuously differentiable in $\mathbb{R}^d$ with polynomial growth. Nevertheless, for the purposes of our application, F will be closely related to the function that associates a symmetric matrix to its vector of eigenvalues and will be only differentiable in a dense subset of the space of symmetric matrices. This motivates the use of conditions (1),(2) and (3) in Theorem 2.2, rather than a (more classical) smoothness condition on F over $\mathbb{R}^d$. Before proving this result, we provide some interesting remarks.

(i) For every $T > 0$, with probability one, the random set $I_T = \{s \in [0,T] \mid X_s \in M\}$ has Lebesgue measure $|I_T|$ equal to zero, since

$$\mathbb{E}\left[|I_T|\right] = \mathbb{E}\left[\int_0^T \mathbf{1}_{\{X_s \in M\}}\mathrm{d}s\right] = \int_0^T \mathbb{P}[X_s \in M]\mathrm{d}s = 0,$$

where the last equality follows from the fact that M has Lebesgue measure zero and X_s has a Gaussian distribution. As a consequence, with probability one the trajectories of $\mathbf{1}_{\mathcal{D}}(X_s)$ are Lebesgue almost everywhere equal to one, which allows us to rewrite equation (2.7) as follows

$$F(X_t) = F(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(X_s)\delta X_s^i + \frac{1}{2}\sum_{i=1}^{d}\int_0^t \frac{\partial^2 F}{\partial x_i^2}(X_s)\frac{\mathrm{d}R(s,s)}{\mathrm{d}s}\mathrm{d}s,$$

with the understanding that, although the integrands might be undefined for some values of s, they are well defined Lebesgue almost everywhere. Nevertheless, we will use the notation (2.7), in order to avoid confusion.

(ii) A version of the previous result was first presented in [11, Theorem 3.1] for $d \geq 1$, where the condition (1) was replaced by the weaker condition that f is differentiable in an open dense set $\mathcal{D}$ of $\mathbb{R}^d$. Unfortunately, this result is false, as we can verify by taking $d = 1$, $f(x) = |x|$, and covariance $R(s,t) = s \wedge t$, which corresponds to the standard Brownian motion. Under these conditions, the third term appearing in the right hand side of (2.7) must be replaced by the local time of the Brownian motion. In order for the result to hold, we require the more restrictive condition (1) instead of the differentiability of f over an open dense set.

(iii) Condition (3) is slightly different than the one presented in [11, Theorem 3.1]. This alternative condition is crucial for providing a Skorohod integral equation for $\langle \mu_t^{(n)}, f\rangle$, since in the application to random matrices that we address in this paper, the function $\sum_{i=1}^{d} \frac{\partial^2 F}{\partial x_i^2}$ is smooth and bounded, unlike the individual components $\frac{\partial^2 F}{\partial x_i^2}$, which are considerably more erratic (see Section 3 for details).

The proof that we present below is based on similar arguments as those used in [11, Theorem 3.1], but some modifications and additional techniques are required.

Proof of Theorem 2.2. Let $Y \in \mathscr{S}$ be of the form

$$Y = \tilde{g}(V(h_1), \ldots, V(h_q)),$$

for $h_i = (h_i^1, \ldots, h_i^d)$, with $h_i^l \in \mathscr{E}$ and $\tilde{g} : \mathbb{R}^{qd} \to \mathbb{R}$ infinitely differentiable with compact support. Since each h_i^l is a step function of the form

$$h_i^l(x) = \sum_{j=1}^{r} a_{i,j}^l \mathbf{1}_{[0,s_{i,j}^l)}(x),$$

for some $r \in \mathbb{N}$, $a_{i,j}^l \in \mathbb{R}$ and $s_{i,j}^l \in \mathbb{R}_+$ for $1 \leq j \leq r$, we deduce that there exist $m \in \mathbb{N}$ and $0 < t_1 < \cdots < t_m$, such that

$$Y = g(X_{t_1}, \ldots, X_{t_m}),$$

for some $g : \mathbb{R}^{md} \to \mathbb{R}$ infinitely differentiable with compact support. Using the chain rule for D, we obtain

$$DY = \sum_{i=1}^{m}\sum_{j=1}^{d} \frac{\partial g}{\partial y_{i,j}}(X_{t_1}, \ldots, X_{t_m}) DX_{t_i}^j,$$

where $\frac{\partial g}{\partial y_{i,j}}(v_1, \ldots, v_m)$ denotes the partial derivative of g with respect to the j-th component of v_i, evaluated at $(v_1, \ldots, v_m)$. By condition (2.5) and the way we choose the process u_s, the inner product $\left\langle \mathbf{1}_{[0,t_i]}, u^j \right\rangle_{\mathfrak{H}}$ is well defined and satisfies

$$\left\langle \mathbf{1}_{[0,t_i]}, u^j \right\rangle_{\mathfrak{H}} = \left\langle \mathbf{1}_{[0,t_i]}, \mathbf{1}_{\mathcal{D}}(X_\cdot)\frac{\partial F}{\partial x_j}(X_\cdot)\mathbf{1}_{[0,t]}(\cdot)\right\rangle_{\mathfrak{H}} = \int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial F}{\partial x_j}(X_s)\frac{\partial R}{\partial s}(s,t_i)\mathrm{d}s.$$

Hence, using the fact that $\left\langle DX^j_{t_i}, u\right\rangle_{\mathfrak{H}^d} = \left\langle \mathbf{1}_{[0,t_i]}, u^j\right\rangle_{\mathfrak{H}}$, we get

$$\langle DY, u\rangle_{\mathfrak{H}^d} = \sum_{i=1}^m \sum_{j=1}^d \frac{\partial g}{\partial y_{i,j}}(X_{t_1}, \dots, X_{t_m}) \int_0^t \mathbf{1}_{\mathcal{D}}(X_s) \frac{\partial F}{\partial x_j}(X_s) \frac{\partial R}{\partial s}(s, t_i) \mathrm{d}s.$$

Using the previous expression as well as (2.5), we deduce that $\langle DY, u\rangle_{\mathfrak{H}^d}$ is integrable. Indeed, since g is compactly supported, we can use (2.5) to obtain a constant $C > 0$ such that

$$\mathbb{E}\left[\left|\langle DY, u\rangle_{\mathfrak{H}^d}\right|\right] \leq C \sum_{i=1}^m \sum_{j=1}^d \int_0^T \mathbb{E}\left[(1 + |X_s|^N)\right] \left|\frac{\partial R}{\partial s}(s, t_i)\right| \mathrm{d}s. \tag{2.8}$$

Moreover, since X_s is Gaussian and the mapping $s \mapsto R(s, s)$ is continuous in $[0, T]$,

$$\mathbb{E}\left[|X_s|^N\right] \leq N!! R(s, s)^{\frac{N}{2}} \leq N!! \sup_{0 \leq s \leq T} R(s, s)^{\frac{N}{2}} < \infty,$$

where $N!!$ denotes the double factorial of N. The integrability of $\left|\langle DY, u\rangle_{\mathfrak{H}^d}\right|$ then follows from (2.8) and condition **(H1)**. As a consequence, we can write

$$\begin{aligned}
\mathbb{E}\left[\langle DY, u\rangle_{\mathfrak{H}^d}\right] &= \sum_{i=1}^m \sum_{j=1}^d \int_0^t \mathbb{E}\left[\frac{\partial g}{\partial y_{i,j}}(X_{t_1}, \dots, X_{t_m}) \mathbf{1}_{\mathcal{D}}(X_s) \frac{\partial F}{\partial x_j}(X_s)\right] \frac{\partial R}{\partial s}(s, t_i) \mathrm{d}s \\
&= \sum_{i=1}^m \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^{dm}} \int_{\mathbb{R}^d} \frac{\partial g}{\partial y_{i,j}}(y) \mathbf{1}_{\mathcal{D}}(x) \frac{\partial F}{\partial x_j}(x) \frac{\partial R}{\partial s}(s, t_i) f_s(x, y) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s,
\end{aligned} \tag{2.9}$$

where $f_s : \mathbb{R}^{d(m+1)} \to \mathbb{R}_+$ denotes the join density of the Gaussian vector $(X_s, X_{t_1}, \dots X_{t_m})$. Let x be of the form $x = (x_1, \dots, x_d)$. Since $\gamma_j(M)$ has measure zero in P_j, we deduce that for every $y \in \mathbb{R}^{md}$, $s > 0$ and $j \geq 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{D}}(x) \frac{\partial F}{\partial x_j}(x) f_s(x, y) \mathrm{d}x &= \int_{P_j} \int_{\mathbb{R}} \mathbf{1}_{\mathcal{D}}(x) \frac{\partial F}{\partial x_j}(x) f_s(x, y) \mathrm{d}x_j \prod_{i \neq j} \mathrm{d}x_i \\
&= \int_{\gamma_j(M)^c} \int_{\mathbb{R}} \mathbf{1}_{\mathcal{D}}(x) \frac{\partial F}{\partial x_j}(x) f_s(x, y) \mathrm{d}x_j \prod_{i \neq j} \mathrm{d}x_i.
\end{aligned} \tag{2.10}$$

By condition (1), for every $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \gamma_i(M)^c$, the mapping

$$t \mapsto F(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)$$

is differentiable in $\mathbb{R}$, and hence, using the polynomial growth of F and $\frac{\partial F}{\partial x_j}$, we can remove the term $\mathbf{1}_{\mathcal{D}}(x)$ in the right hand side of (2.10), and integrate by parts the variable x_i, in order to deduce

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{\partial F}{\partial x_j}(x) f_s(x, y) \mathrm{d}x &= \int_{\gamma_j(M)^c} \int_{\mathbb{R}} \frac{\partial F}{\partial x_j}(x) f_s(x, y) \mathrm{d}x_j \prod_{i \neq j} \mathrm{d}x_i \\
&= -\int_{\gamma_j(M)^c} \int_{\mathbb{R}} F(x) \frac{\partial f_s}{\partial x_j}(x, y) \mathrm{d}x_j \prod_{i \neq j} \mathrm{d}x_i \\
&= -\int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{D}}(x) F(x) \frac{\partial f_s}{\partial x_j}(x, y) \mathrm{d}x.
\end{aligned} \tag{2.11}$$

From (2.9) and (2.11), we conclude that

$$\begin{aligned}\mathbb{E}\left[\langle DY,u\rangle_{\mathfrak{H}^d}\right] &= -\sum_{i=1}^{m}\sum_{j=1}^{d}\int_0^t\int_{\mathbb{R}^{dm}}\int_{\mathbb{R}^d}\frac{\partial g}{\partial y_{i,j}}(y)\mathbf{1}_{\mathcal{D}}(x)F(x)\frac{\partial R}{\partial s}(s,t_i)\frac{\partial f_s}{\partial x_j}(x,y)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &= \sum_{i=1}^{m}\sum_{j=1}^{d}\int_0^t\int_{\mathbb{R}^{dm}}\int_{\mathbb{R}^d} g(y)\mathbf{1}_{\mathcal{D}}(x)F(x)\frac{\partial R}{\partial s}(s,t_i)\frac{\partial^2 f_s}{\partial y_{i,j}\partial x_j}(x,y)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{2.12}$$

Similarly, using relation (2.6), as well as the fact that g is compactly supported, we have that for every $1\le j\le d$, the random variable

$$Y\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\sum_{j=1}^{d}\frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s \tag{2.13}$$

is integrable. Indeed, by (2.6), there exists a constant $C>0$ such that

$$\begin{aligned}\mathbb{E}\left[\left|Y\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\sum_{j=1}^{d}\frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right|\right] &\le C\int_0^T \mathbb{E}\left[\left|\sum_{j=1}^{d}\frac{\partial^2 F}{\partial x_j^2}(X_s)\right|\right]\left|\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\right|\mathrm{d}s\\ &\le C\int_0^T (1+R(s,s)^{-1+\delta})\left|\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\right|\mathrm{d}s.\end{aligned} \tag{2.14}$$

By condition **(H1)**, there exist $L\in\mathbb{N}$ and $0=T_1<\cdots<T_L=T$, such that $R(s,s)$ is monotone in $[T_i,T_{i+1}]$ for all $1\le i\le L-1$. Hence,

$$\begin{aligned}\int_0^T (1+R(s,s)^{-1+\delta})\left|\frac{\mathrm{d}R(s,s)}{\mathrm{d}s}\right|\mathrm{d}s &= \sum_{i=1}^{L-1}\left|\int_{T_i}^{T_{i+1}}(1+R(s,s)^{-1+\delta})\frac{\mathrm{d}R(s,s)}{\mathrm{d}s}\mathrm{d}s\right|\\ &= \sum_{i=1}^{L-1}\big|R(T_{i+1},T_{i+1})-R(T_i,T_i)\\ &\quad+\frac{1}{\delta}(R(T_{i+1},T_{i+1})^{\delta}-R(T_i,T_i)^{\delta})\big|.\end{aligned} \tag{2.15}$$

Therefore, by (2.14), the random variable in (2.13) is integrable, as required. Proceeding as in the proof of (2.11) and using the fact that $\mathbb{E}\left[\left|\frac{\partial^2 F}{\partial x_j^2}(X_s)\right|\right]<\infty$ for all $s>0$, we can show that for all $y\in\mathbb{R}^{md}$ and $s>0$,

$$\int_{\mathbb{R}^d}\mathbf{1}_{\mathcal{D}}(x)\frac{\partial^2 F}{\partial x_j^2}(x)f_s(x,y)\mathrm{d}x=\int_{\mathbb{R}^d}\mathbf{1}_{\mathcal{D}}(x)F(x)\frac{\partial^2 f_s}{\partial x_j^2}(x,y)\mathrm{d}x,$$

and consequently,

$$\begin{aligned}&\mathbb{E}\left[Y\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\sum_{j=1}^{d}\frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right]\\ &\qquad=\int_0^t\int_{\mathbb{R}^{dm}}\int_{\mathbb{R}^d} g(y)\mathbf{1}_{\mathcal{D}}(x)\sum_{j=1}^{d}\frac{\partial^2 F}{\partial x_j^2}(x)f_s(x,y)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &\qquad=\int_0^t\int_{\mathbb{R}^{dm}}\int_{\mathbb{R}^d} g(y)\mathbf{1}_{\mathcal{D}}(x)\sum_{j=1}^{d}F(x)\frac{\partial^2 f_s}{\partial x_j^2}(x,y)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{2.16}$$

In addition, by (2.5), the random variable $YF(X_t)$ is integrable for every $t \geq 0$, and

$$\begin{aligned}\mathbb{E}\left[YF(X_t) - YF(X_0)\right] &= \int_{\mathbb{R}^{dm}} \int_{\mathbb{R}^d} g(y)F(x)f_t(x,y)\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^{dm}} \int_{\mathbb{R}^d} g(y)F(x)f_0(x,y)\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^t \int_{\mathbb{R}^{dm}} \int_{\mathbb{R}^d} g(y)F(x)\frac{\partial f_s}{\partial s}(x,y)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^{dm}} \int_{\mathbb{R}^d} g(y)\mathbf{1}_{\mathcal{D}}(x)F(x)\frac{\partial f_s}{\partial s}(x,y)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{2.17}$$

where the last identity follows from the fact that $\mathbb{R}\backslash\mathcal{D}$ has Lebesgue measure zero. Finally, if $\Sigma(s)$ denotes the covariance function of $(X_s, X_{t_1}, \ldots, X_{t_m})$, by the Fourier inversion formula we have that for every $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{dm}$,

$$\begin{aligned}\frac{\partial f_s}{\partial s}(x,y) &= \frac{1}{2\pi}\frac{\partial}{\partial s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dm}} e^{\mathbf{i}\langle(\xi,\eta),(x,y)\rangle}e^{-\frac{1}{2}(\xi,\eta)^*\Sigma(s)(\xi,\eta)}\mathrm{d}\eta\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dm}} e^{\mathbf{i}\langle(\xi,\eta),(x,y)\rangle}e^{-\frac{1}{2}(\xi,\eta)^*\Sigma(s)(\xi,\eta)}\big(-\frac{1}{2}(\eta,\xi)^*\frac{\partial}{\partial s}\Sigma(s)(\eta,\xi)\big)\mathrm{d}\eta\mathrm{d}\xi.\end{aligned}$$

A further application of the Fourier inversion formula to the right hand side of the previous identity, leads to

$$\frac{\partial f_s}{\partial s}(x,y) = \sum_{i=1}^m\sum_{j=1}^d \frac{\partial R}{\partial s}(s,t_i)\frac{\partial^2 f_s}{\partial y_{i,j}\partial x_j}(x,y) + \frac{1}{2}\sum_{j=1}^d \frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\frac{\partial^2 f_s}{\partial x_j^2}(x,y). \tag{2.18}$$

From (2.12),(2.16),(2.17) and (2.18), we get that

$$\mathbb{E}\left[YF(X_t) - YF(X_0)\right] = \mathbb{E}\left[\langle DY, u\rangle_{\mathfrak{H}^d}\right] + \frac{1}{2}\mathbb{E}\left[Y\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\sum_{j=1}^d \frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right]. \tag{2.19}$$

Next we use (2.19) to prove that u belongs to the extended domain of the divergence $\mathrm{Dom}^*\delta$. Using Hölder inequality and Minkowski inequality, we have that

$$\begin{aligned}&\left|\mathbb{E}\left[Y\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\sum_{j=1}^d \frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right]\right| \\ &\qquad \leq \|Y\|_{L^{\frac{p}{p-1}}(\Omega)}\left\|\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\sum_{j=1}^d \frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right\|_{L^p(\Omega)} \\ &\qquad \leq \|Y\|_{L^{\frac{p}{p-1}}(\Omega)}\int_0^t \left\|\sum_{j=1}^d \frac{\partial^2 F}{\partial x_j^2}(X_s)\right\|_{L^p(\Omega)}\left|\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\right|\mathrm{d}s.\end{aligned}$$

In addition, using (2.6), we get

$$\int_0^t \left\|\frac{\partial^2 F}{\partial x_j^2}(X_s)\right\|_{L^p(\Omega)}\left|\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\right|\mathrm{d}s \leq C\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\right|\mathrm{d}s < \infty. \tag{2.20}$$

Therefore, using Hölder's inequality, as well as (2.19) and (2.20), we deduce that

there exists a constant $K_1 > 0$ such that

$$\begin{aligned}\left|\mathbb{E}\left[\langle DY, u\rangle_{\mathfrak{H}^d}\right]\right| &\leq \left|\mathbb{E}\left[YF(X_0)\right]\right| + \frac{1}{2}\left|\mathbb{E}\left[Y\int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial^2 F}{\partial x_j^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right]\right| + \left|\mathbb{E}\left[YF(X_t)\right]\right| \\ &\leq K_1\left(\|Y\|_{L^1(\Omega)} + \|Y\|_{L^{\frac{p}{p-1}}(\Omega)} + \mathbb{E}\left[|Y|\left(1+|X_t|^N\right)\right]\right) \\ &\leq K_1\left(\|Y\|_{L^1(\Omega)} + \|Y\|_{L^{\frac{p}{p-1}}(\Omega)} + \|Y\|_{L^{\frac{p}{p-1}}(\Omega)}\left\|1+|X_t|^N\right\|_{L^p(\Omega)}\right),\end{aligned}$$

and hence, there exists a constant $K_2 > 0$ such that

$$\left|\mathbb{E}\left[\langle DY, u\rangle_{\mathfrak{H}^d}\right]\right| \leq K_2\|Y\|_{L^{\frac{p}{p-1}}(\Omega)}. \tag{2.21}$$

From (2.19) and (2.21), it follows that u belongs to the extended domain of the divergence $\mathrm{Dom}^*\delta$, and

$$F(X_t) - F(X_0) = \sum_{i=1}^d \int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial F}{\partial x_i}(X_s)\delta X_s^i + \frac{1}{2}\sum_{i=1}^d \int_0^t \mathbf{1}_{\mathcal{D}}(X_s)\frac{\partial^2 F}{\partial x_i^2}(X_s)\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s,$$

as required. □

3 Stochastic Evolution of the eigenvalues of a matrix-valued Gaussian process

We first recall some notation. Consider a family of independent and identically distributed centered Gaussian processes $\{X_{i,j}; i,j \in \mathbb{N}\}$ defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will assume that the covariance function $R(s,t) := \mathbb{E}\left[X_{1,1}(s)X_{1,1}(t)\right]$ satisfies the hypotheses **(H1)** and **(H2)**. Consider as well a sequence of deterministic symmetric matrices $A^{(n)} = [A_{i,j}^{(n)}]_{1\leq i,j\leq n}$, with ordered eigenvalues $\lambda_1^{(n)}(0) \geq \cdots \geq \lambda_n^{(n)}(0)$ and spectral empirical distribution

$$\mu_0^{(n)} := \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i^{(n)}(0)} \qquad \text{satisfying} \qquad \mu_0^{(n)} \xrightarrow{\mathcal{L}} \mu_0 \qquad \text{as } n \to \infty,$$

for some probability law μ_0 and where $\xrightarrow{\mathcal{L}}$ means convergence in law. Let $Y^{(n)} = [Y_{i,j}^{(n)}]_{1\leq i,j\leq n}$ be the renormalized symmetric Gaussian matrix of dimension $n \times n$, given by

$$Y_{i,j}^{(n)}(t) := \begin{cases} \dfrac{1}{\sqrt{n}}X_{i,j}(t) + A_{i,j}^{(n)} & \text{if } i < j, \\ \dfrac{\sqrt{2}}{\sqrt{n}}X_{i,i}(t) + A_{i,i}^{(n)} & \text{if } i = j, \end{cases} \tag{3.1}$$

Denote by $\lambda_1^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t)$ the ordered eigenvalues of $Y^{(n)}(t)$, and by $\mu^{(n)} := (\mu_t^{(n)}, t \geq 0)$ the corresponding empirical measure process

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j^{(n)}(t)}.$$

For a given probability measure ν, and a ν-integrable function f, we use the notation $\langle \nu, f\rangle := \int f(x)\nu(\mathrm{d}x)$. In particular,

$$\langle \mu_t^{(n)}, f\rangle = \frac{1}{n}\sum_{i=1}^n f(\lambda_i^{(n)}(t)). \tag{3.2}$$

From [11, Lemma 5.1], it follows that for every $i = 1, \dots, n$, there exists a function $\Phi_i^n : \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{R}$, which is infinitely differentiable in an open subset $G \subset \mathbb{R}^{\frac{n(n+1)}{2}}$, with $|G^c| = 0$, such that $\lambda_i^{(n)}(t) = \Phi_i^n(Y^{(n)}(t))$. Moreover, every element $X \in G$, viewed as an $n \times n$ matrix, has a factorization of the form $Z^{(n)} = U^{(n)}\mathtt{D}(U^{(n)})^*$, where $\mathtt{D}$ is a diagonal matrix with entries $\mathtt{D}_{i,i} = \lambda_i^{(n)}$ such that $\lambda_1^{(n)} > \cdots > \lambda_n^{(n)}$, $U^{(n)}$ is an orthogonal matrix with $U_{i,i}^{(n)} > 0$ for all i, $U_{i,j}^{(n)} \neq 0$ and all the minors of $U^{(n)}$ have non zero determinants. In addition, for any $k \leq h$, we have

$$\frac{\partial \Phi_i^n}{\partial y_{k,h}}(Z^{(n)}) = 2U_{i,k}^{(n)}U_{i,h}^{(n)}\mathbf{1}_{\{k\neq h\}} + \sqrt{2}(U_{i,k}^{(n)})^2\mathbf{1}_{\{k=h\}}, \tag{3.3}$$

$$\frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}(Z^{(n)}) = 2\sum_{j\neq i}\frac{\left|U_{i,k}^{(n)}U_{j,h}^{(n)} + U_{i,h}^{(n)}U_{j,k}^{(n)}\right|^2}{\lambda_i^{(n)} - \lambda_j^{(n)}}\mathbf{1}_{\{k\neq h\}} + 4\sum_{j\neq i}\frac{\left|U_{i,k}^{(n)}U_{j,k}^{(n)}\right|^2}{\lambda_i^{(n)} - \lambda_j^{(n)}}\mathbf{1}_{\{k=h\}}. \tag{3.4}$$

Using the orthogonality of the columns of $U^{(n)}$, we deduce from (3.3) and (3.4) that

$$\sum_{k\leq h}\frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}(Z^{(n)}) = \sum_{j\neq i}\frac{2}{\lambda_i^{(n)} - \lambda_j^{(n)}} \qquad \text{and} \qquad \sum_{k\leq h}\left(\frac{\partial \Phi_i^n}{\partial y_{k,h}}(Z^{(n)})\right)^2 = 2. \tag{3.5}$$

Using Lemma 2.2, we can prove the following result, which describes the time evolution of the eigenvalues $(\lambda_1^{(n)}(t), \dots, \lambda_n^{(n)}(t))$ in terms of the Skorohod integral.

Lemma 3.1. *For every $f \in \mathcal{C}_b^2(\mathbb{R})$ and $t \geq 0$, we have*

$$\begin{aligned}\left\langle \mu_t^{(n)}, f\right\rangle = \left\langle \mu_0^{(n)}, f\right\rangle &+ \frac{1}{2n^2}\sum_{i=1}^n \int_0^t f''(\Phi_i^n(Y^{(n)}(s)))\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s \\ &+ \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^n\sum_{k\leq h}\int_0^t f'(\Phi_i^n(Y^{(n)}(s)))\mathbf{1}_G(Y^{(n)}(s))\frac{\partial \Phi_i^n}{\partial y_{k,l}}(Y^{(n)}(s))\delta X_{k,h}(s) \\ &+ \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\frac{f'(x) - f'(y)}{x-y}\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mu_s^{(n)}(\mathrm{d}x)\mu_s^{(n)}(\mathrm{d}y)\mathrm{d}s.\end{aligned} \tag{3.6}$$

Proof. For simplicity, we introduce $n^{-1/2}X^{(n)} := Y^{(n)} - A^{(n)}$ and we write

$$\langle \mu_t^{(n)}, f\rangle = \frac{1}{n}\sum_{i=1}^n f(\lambda_i^{(n)}(t)) = \frac{1}{n}\sum_{i=1}^n f(\Phi_i^n(Y^{(n)}(t))) =: F_n(n^{-1/2}X^{(n)}(t)), \tag{3.7}$$

where $F_n(B)$, for $B \in \mathbb{R}^{\frac{n(n+1)}{2}}$, is such that

$$F_n(B) := \frac{1}{n}\sum_{i=1}^n f(\Phi_i^n(A^{(n)} + B)).$$

Next we show that the right hand side of (3.7) satisfies the conditions of Theorem 2.2 for a suitable choice of $\mathcal{D}$ and M.

Observe that Φ^n is infinitely differentiable in the set of symmetric matrices whose characteristic polynomials do not have multiple roots, or equivalently, the matrices without multiple eigenvalues. As a consequence, the mapping $x \mapsto \Phi(x + A^{(n)})$ is differentiable in the complement of

$$M_{A^{(n)}} := \{x \in \mathbb{R}^{\frac{n(n+1)}{2}} \mid p(x + A^{(n)}) = 0\}, \tag{3.8}$$

where $p : \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{R}$ denotes the discriminant of the matrix induced by x, defined by $p(x) = \prod_{i\neq j}(\Phi_i^n(x) - \Phi_j^n(x))^2$. It is well known that p is a polynomial in the entries of x (see

[1, Appendix A.4] for a proof of this fact) and consequently, $M_{A^{(n)}}$ is an algebraic variety. Moreover, by a result by Von Neumann and Wigner (see [9]), $M_{A^{(n)}}$ has codimension 2, namely, the maximal dimension of the tangent vector spaces at the non-singular points of $M_{A^{(n)}}$ is equal to $\frac{n(n+1)}{2} - 2$. As a consequence, the projection $\gamma_j(M_{A^{(n)}})$ is a variety of codimension at least 1 embedded in $P_j := \{(x_1, \ldots, x_d) \mid x_j = 0\}$, and thus $\gamma_j(M_{A^{(n)}})$ has Lebesgue measure zero on P_j. From here we conclude that condition (1) in Theorem 2.2 holds for $\mathcal{D} := \mathbb{R}^{\frac{n(n+1)}{2}} \backslash M_A^{(n)}$.

Next we prove condition (2.5). First we observe for every $B \in G$ that the partial derivative of $F_n(n^{-1/2}B)$ with respect to the (k,h)-th component, denoted by $\frac{\partial}{\partial y_{k,h}} F_n(n^{-1/2}B)$, is given by

$$\frac{\partial}{\partial y_{k,h}} F_n(n^{-1/2}B) = \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} f'(\Phi_i^n(n^{-1/2}B + A^{(n)})) \frac{\partial \Phi_i^n}{\partial y_{k,h}}(n^{-1/2}B + A^{(n)}), \tag{3.9}$$

Hence, using the fact that $\left|\frac{\partial \Phi_i^n}{\partial y_{k,h}}\right| \leq 2$ (see equation (3.3)), we get

$$\left| \frac{\partial}{\partial y_{k,h}} F_n(n^{-1/2}B) \right| \leq \frac{1}{n^{\frac{3}{2}}} \|f'\|_\infty \sum_{i=1}^{n} \left| \frac{\partial \Phi_i^n}{\partial y_{k,h}}(n^{-1/2}B + A^{(n)}) \right| \leq \frac{2}{\sqrt{n}} \|f'\|_\infty .$$

Using the previous inequality, we conclude that condition (2.5) holds.

To prove condition (2.6) in Theorem 2.2, we see that for every $1 \leq k \leq h \leq n$ and $t > 0$ fixed,

$$\begin{aligned} \frac{\partial^2}{\partial y_{k,h}^2} F_n(n^{-1/2}X^{(n)}(t)) = \frac{1}{n^2} \sum_{i=1}^{n} f''(\Phi_i^n(Y^{(n)}(t))) \left(\frac{\partial \Phi_i^n}{\partial y_{k,h}}(Y^{(n)}(t)) \right)^2 \\ + \frac{1}{n^2} \sum_{i=1}^{n} f'(\Phi_i^n(Y^{(n)}(t))) \frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}(Y^{(n)}(t)), \end{aligned} \tag{3.10}$$

and hence, using relations $\left|\frac{\partial \Phi_i^n}{\partial y_{k,h}}\right| \leq 2$ and $\left|\frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}\right| \leq \sum_{i \neq j} \frac{4}{|\lambda_i^{(n)} - \lambda_j^{(n)}|}$ (see equation (3.4)), we obtain

$$\begin{aligned} \mathbb{E}\left[\left| \frac{\partial^2}{\partial y_{k,h}^2} F_n(n^{-1/2}X^{(n)}(t)) \right| \right] &\leq \frac{4}{n} \|f''\|_\infty + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}\left[\left| \frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}(Y^{(n)}(t)) \right| \right] \\ &\leq \frac{4}{n} \|f''\|_\infty + \frac{4}{n^2} \sum_{i \neq j} \mathbb{E}\left[|\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)|^{-1} \right]. \end{aligned} \tag{3.11}$$

To show the right hand side is finite we proceed as follows. For $x \in \mathbb{R}^{\frac{n(n+1)}{2}}$, let $\phi_\varepsilon(x)$ denote the Gaussian kernel given by

$$\phi_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{n(n+1)}{4}} e^{-\frac{1}{2\varepsilon}\|x\|^2}.$$

We can easily check that there exists constants $C > 0$ and $\sigma > 0$ only depending on $n, A^{(n)}$ and $R(t,t)$ such that, after identifying $A^{(n)}$ as an element of $\mathbb{R}^{\frac{n(n+1)}{2}}$,

$$\phi_{R(t,t)}(n^{-1/2}(y - A^{(n)})) \leq C\phi_{\sigma^2}(x),$$

and consequently,

$$\begin{aligned}\mathbb{E}\left[|\lambda_i^{(n)}(t)-\lambda_j^{(n)}(t)|^{-1}\right] &= \int_{\mathbb{R}^{\frac{n(n+1)}{2}}} |\Phi_i^n(n^{-1/2}x+A^{(n)})-\Phi_j^n(n^{-1/2}x+A^{(n)})|^{-1}\phi_{R(t,t)}(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^{\frac{n(n+1)}{2}}} |\Phi_i^n(y)-\Phi_j^n(y)|^{-1}\phi_{R(t,t)}(n^{-1/2}(y-A^{(n)}))\mathrm{d}x \\ &\leq C\int_{\mathbb{R}^{\frac{n(n+1)}{2}}} |\Phi_i^n(x)-\Phi_j^n(x)|^{-1}\phi_{\sigma^2}(x)\mathrm{d}x \\ &= C\sigma^{\frac{n(n+1)}{2}}\int_{\mathbb{R}^{\frac{n(n+1)}{2}}} |\Phi_i^n(x)-\Phi_j^n(x)|^{-1}\phi_1(x)\mathrm{d}x. \end{aligned} \tag{3.12}$$

Similarly to [11, Equation (5.6)], we can use the joint density of the eigenvalues of a standard GOE of dimension n, to deduce that the right hand side of (3.12) is finite. Hence, from (3.11) we conclude that

$$\mathbb{E}\left[\left|\frac{\partial^2}{\partial y_{k,h}^2}F_n(n^{-1/2}X^{(n)}(t))\right|\right]<\infty.$$

Moreover, by (3.5) and (3.10), we have

$$\begin{aligned}\sum_{k\leq h}\frac{\partial^2}{\partial y_{k,h}^2}F_n(n^{-1/2}X^{(n)}(t)) &= \frac{2}{n^2}\sum_{i=1}^n f''(\Phi_i^n(Y^{(n)}(t))) + \frac{2}{n^2}\sum_{i\neq j}\frac{f'(\Phi_i^n(Y^{(n)}(t)))}{\Phi_i^n(Y^{(n)}(t))-\Phi_j^n(Y^{(n)}(t))} \\ &= \frac{2}{n^2}\sum_{i=1}^n f''(\Phi_i^n(Y^{(n)}(t))) + \frac{1}{n^2}\sum_{i\neq j}\frac{f'(\Phi_i^n(Y^{(n)}(t)))-f'(\Phi_j^n(Y^{(n)}(t)))}{\Phi_i^n(Y^{(n)}(t))-\Phi_j^n(Y^{(n)}(t))},\end{aligned}$$

where we have used

$$\sum_{i\neq j}\frac{f'(\Phi_i^n(Y^{(n)}(t)))}{\Phi_i^n(Y^{(n)}(t))-\Phi_j^n(Y^{(n)}(t))} = -\sum_{i\neq j}\frac{f'(\Phi_j^n(Y^{(n)}(t)))}{\Phi_i^n(Y^{(n)}(t))-\Phi_j^n(Y^{(n)}(t))}.$$

Thus, by the mean value theorem, we conclude that

$$\left|\sum_{k\leq h}\frac{\partial^2}{\partial y_{k,h}^2}F_n(n^{-1/2}X^{(n)}(t))\right| \leq \frac{4}{n}\left\|f''\right\|_\infty,$$

which proves relation (2.6). Therefore, the right hand side of (3.7) satisfies the conditions of Theorem 2.2. As a consequence,

$$\begin{aligned}\langle\mu_t^{(n)},f\rangle-\langle\mu_0^{(n)},f\rangle = &\sum_{1\leq k\leq h\leq n}\int_0^t \mathbf{1}_{\mathcal{D}}(X^{(n)}(s))\frac{\partial F_n}{\partial y_{k,h}}(n^{-1/2}X^{(n)}(s))\delta X_{k,h}(s) \\ &+\frac{1}{2}\sum_{1\leq k\leq h\leq n}\int_0^t \mathbf{1}_{\mathcal{D}}(X^{(n)}(s))\frac{\partial^2 F_n}{\partial y_{k,h}^2}(n^{-1/2}X^{(n)}(s))\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s.\end{aligned} \tag{3.13}$$

Moreover, by Remark (i) after Theorem 2.2, the indicators $\mathbf{1}_{\mathcal{D}}(X^{(n)}(s))$ can be replaced by $\mathbf{1}_G(X^{(n)}(s))$, which leads to

$$\begin{aligned}\langle\mu_t^{(n)},f\rangle-\langle\mu_0^{(n)},f\rangle = &\sum_{1\leq k\leq h\leq n}\int_0^t \mathbf{1}_{G}(X^{(n)}(s))\frac{\partial F_n}{\partial y_{k,h}}(n^{-1/2}X^{(n)}(s))\delta X_{k,h}(s) \\ &+\frac{1}{2}\sum_{1\leq k\leq h\leq n}\int_0^t \mathbf{1}_{G}(X^{(n)}(s))\frac{\partial^2 F_n}{\partial y_{k,h}^2}(n^{-1/2}X^{(n)}(s))\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s.\end{aligned} \tag{3.14}$$

From relations (3.5) and (3.10), we deduce that

$$\begin{aligned}
\mathbf{1}_G(X^{(n)}(t)) &\sum_{1\le k\le h\le n} \frac{\partial^2}{\partial y_{k,h}^2} F_n(n^{-1/2}X^{(n)}(t)) \\
&= \frac{\mathbf{1}_G(X^{(n)}(t))}{n^2} \sum_{i=1}^n f''(\Phi_i^n(Y^{(n)}(t))) \sum_{1\le k\le h\le n} \left(\frac{\partial \Phi_i^n}{\partial y_{k,h}}(Y^{(n)}(t))\right)^2 \\
&+ \frac{\mathbf{1}_G(X^{(n)}(t))}{n^2} \sum_{i=1}^n f'(\Phi_i^n(Y^{(n)}(t))) \sum_{1\le k\le h\le n} \frac{\partial^2 \Phi_i^n}{\partial y_{k,h}^2}(Y^{(n)}(t)) \qquad (3.15) \\
&= \frac{2\mathbf{1}_G(X^{(n)}(t))}{n^2} \left(\sum_{i=1}^n f''(\Phi_i^n(Y^{(n)}(t))) + \sum_{i=1}^n \sum_{j\neq i} \frac{f'(\Phi_i^n(Y^{(n)}(t)))}{\lambda_i^{(n)} - \lambda_j^{(n)}} \right).
\end{aligned}$$

Combining (3.9), (3.14) and (3.15), we get

$$\begin{aligned}
\left\langle \mu_t^{(n)}, f \right\rangle &= \left\langle \mu_0^{(n)}, f \right\rangle + \frac{1}{n^2} \sum_{i=1}^n \int_0^t f''(\Phi_i^n(Y^{(n)}(s))) \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mathrm{d}s \\
&+ \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^n \sum_{k\le h} \int_0^t f'(\Phi_i^n(Y^{(n)}(s))) \mathbf{1}_G(Y^{(n)}(s)) \frac{\partial \Phi_i^n}{\partial y_{k,l}}(Y^{(n)}(s)) \delta X_{k,h}(s) \\
&+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{x\neq y\}} \frac{f'(x) - f'(y)}{x - y} \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mu_s^{(n)}(\mathrm{d}x) \mu_s^{(n)}(\mathrm{d}y) \mathrm{d}s.
\end{aligned}$$

Equation (3.6) then follows from the fact that for every $s > 0$,

$$\frac{1}{n^2} \sum_{i=1}^n f''(\Phi_i^n(Y^{(n)}(s))) = \int_{\mathbb{R}^2} \mathbf{1}_{\{x=y\}} f''(x) \mu_s^n(dx) \mu_s^n(dy). \qquad \square$$

4 Tightness of the family of laws $\{\mu^{(n)}, n \ge 1\}$.

In order to prove tightness for the family $\{\mu^{(n)}, n \ge 1\}$, we follow the approach presented in [12]. Namely, we show that for every test function f belonging to the set $\mathcal{C}^1(\mathbb{R})$, the process $\langle \mu_t^{(n)}, f \rangle$ satisfies the Billingsley criteria.

Proposition 4.1. *Assume that $R(s,t)$ satisfies hypothesis **(H2)**. Then, almost surely, the family of measures $\{\mu^{(n)}, n \ge 1\}$ is tight in the space $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$.*

Proof. We follow the same argument as in [12, Proposition 1]. It suffices to prove that for every bounded function $f \in \mathcal{C}^1(\mathbb{R})$ with bounded derivative, the process $\{(\langle \mu_t^{(n)}, f \rangle, t \ge 0), n \ge 1\}$ is tight. To show this, we observe that, since $\mu_0^{(n)}$ converges weakly, by Billingsley's criteria (see [3, Theorem 12.3]), it is enough to show that there exist constants $C, p > 0$ and $q > 1$, independent of n, such that for every $0 \le t_1 \le t_2$,

$$\mathbb{E}\left[\left| \langle \mu_{t_1}^{(n)}, f \rangle - \langle \mu_{t_2}^{(n)}, f \rangle \right|^p \right] \le C \left| t_2 - t_1 \right|^q. \qquad (4.1)$$

To prove (4.1) we proceed as follows. By the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left| \langle \mu_{t_1}^{(n)}, f \rangle - \langle \mu_{t_2}^{(n)}, f \rangle \right| &= \left| \frac{1}{n} \sum_{i=1}^n f(\lambda_i^{(n)}(t_2)) - f(\lambda_i^{(n)}(t_1)) \right| \\
&\le \left| \frac{1}{n} \sum_{i=1}^n \left(f(\lambda_i^{(n)}(t_2)) - f(\lambda_i^{(n)}(t_1)) \right)^2 \right|^{\frac{1}{2}} \\
&\le \|f'\|_\infty \left| \sum_{i=1}^n \frac{1}{n} \left(\lambda_i^{(n)}(t_2) - \lambda_i^{(n)}(t_1) \right)^2 \right|^{\frac{1}{2}}.
\end{aligned}$$

Hence, using the Hoffman-Wielandt inequality (see [1, Lemma 2.1.19]), as well as the symmetry of $Y^{(n)}(t)$, we deduce that for every $1 \leq j \leq n$,

$$\begin{aligned}\left|\langle \mu_{t_1}^{(n)}, f\rangle - \langle \mu_{t_2}^{(n)}, f\rangle\right| &\leq \|f'\|_\infty \left(\frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i^{(n)}(t_2) - \lambda_i^{(n)}(t_1)\right)^2\right)^{\frac{1}{2}} \\ &\leq \|f'\|_\infty \left(\frac{1}{n} Tr\Big(Y^{(n)}(t_2) - Y^{(n)}(t_1)\Big)^2\right)^{\frac{1}{2}} \\ &= \|f'\|_\infty \left(\frac{1}{n}\sum_{i,k=1}^{n}\left(Y_{i,k}^{(n)}(t_2) - Y_{i,k}^{(n)}(t_1)\right)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

By condition **(H2)**, we have that for all $\gamma > 0$,

$$\mathbb{E}\Big[\Big(Y_{i,k}^{(n)}(t_2) - Y_{i,k}^{(n)}(t_1)\Big)^{2(\gamma+1)/\gamma}\Big] \leq 2\frac{\kappa}{n}\left|t_2 - t_1\right|^{\frac{\gamma+1}{\gamma}},$$

for some constants $\kappa, \gamma > 0$, and consequently, by (4.2),

$$\begin{aligned}\left\|\langle \mu_{t_1}^{(n)}, f\rangle - \langle \mu_{t_2}^{(n)}, f\rangle\right\|_{L^{\frac{2\gamma+2}{\gamma}}(\Omega)} &\leq \|f'\|_\infty \mathbb{E}\left[\left(\frac{1}{n}\sum_{i,k=1}^{n}\left(Y_{i,k}^{(n)}(t_2) - Y_{i,k}^{(n)}(t_1)\right)^2\right)^{\frac{\gamma+1}{\gamma}}\right]^{\frac{\gamma}{2\gamma+2}} \\ &= \|f'\|\left\|\frac{1}{n}\sum_{i,k=1}^{n}\left(Y_{i,k}^{(n)}(t_2) - Y_{i,k}^{(n)}(t_1)\right)^2\right\|^{\frac{1}{2}}_{L^{\frac{\gamma+1}{\gamma}}(\Omega)} \\ &\leq \|f'\|\left(\frac{1}{n}\sum_{i,k=1}^{n}\left\|\left(Y_{i,k}^{(n)}(t_2) - Y_{i,k}^{(n)}(t_1)\right)^2\right\|_{L^{\frac{\gamma+1}{\gamma}}(\Omega)}\right)^{\frac{1}{2}} \\ &\leq K\|f'\|_\infty \left|t_2 - t_1\right|^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

for some universal constant $K > 0$. The latter implies,

$$\mathbb{E}\left[\left|\langle \mu_{t_1}^{(n)}, f\rangle - \langle \mu_{t_2}^{(n)}, f\rangle\right|^{\frac{2\gamma+2}{\gamma}}\right] \leq K^{\frac{2\gamma+2}{\gamma}}\|f'\|_\infty \left|t_2 - t_1\right|^{1+\frac{1}{\gamma}}.$$

Thus Billingsley's critera (4.1) holds for $p = \frac{2\gamma+1}{\gamma}$ and $q = 1 + \frac{1}{\gamma}$. The proof is now complete. □

5 Weak convergence of the empirical measure of eigenvalues

This section is devoted to the proof of Theorem 1.1. It is worth mentioning that, although some of the arguments we present are similar to [12], our estimates are very different, mainly due to the fact that we do not have an analogue for Meyers' inequality for the extended Skorohod integral.

The following Proposition is useful for the proof of Theorem 1.1. Its proof will be given at the end of this section.

Proposition 5.1. *For every $t > 0$ fixed, the random variable*

$$G_r := \frac{1}{n_r^{\frac{3}{2}}}\sum_{i=1}^{n_r}\sum_{k\leq h}\int_0^t f'(\Phi_i^{n_r}(Y^{(n_r)}(s)))\mathbf{1}_G(Y^{(n_r)}(s))\frac{\partial \Phi_i^{n_r}}{\partial y_{k,l}}(Y^{(n_r)}(s))\delta X_{k,h}(s), \tag{5.1}$$

converges to zero in $L^2(\Omega)$ as $n \to \infty$.

Proof of Theorem 1.1: By Lemma 4.1, the sequence $\{\mu^{(n)}, n \geq 1\}$ is tight, which implies that there exists a subsequence $\{\mu^{(n_r)}, r \geq 1\}$ that converges in law, in the topology of $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$, to a measure valued stochastic process $\mu = (\mu_t, t \geq 0)$. Then, if we show that μ is deterministic, we conclude that $\{\mu^{(n)}, n \geq 1\}$ converges in probability to μ. Using Proposition 5.1 together with relation (3.6), we deduce that the sequence of random variables

$$\begin{aligned}\left\langle \mu_t^{(n_r)}, f \right\rangle - \left\langle \mu_0^{(n_r)}, f \right\rangle - \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mu_s^{(n_r)}(\mathrm{d}x) \mu_s^{(n_r)}(\mathrm{d}y)\mathrm{d}s \\ - \frac{1}{2n_r^2} \sum_{i=1}^{n_r} \int_0^t f''(\Phi_i^{n_r}(Y^{(n_r)}(s))) \frac{\mathrm{d}}{\mathrm{d}s} R(s,s)\mathrm{d}s, \qquad (5.2)\end{aligned}$$

converges to zero in $L^2(\Omega)$. In particular, since $\mu^{(n_r)}$ converges in law to μ, it implies that μ satisfies the following measure-valued differential equation

$$\left\langle \mu_t, f \right\rangle = \langle \mu_0, f \rangle + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mu_s(\mathrm{d}x) \mu_s(\mathrm{d}y)\mathrm{d}s, \qquad (5.3)$$

for each $t \geq 0$ and $f \in \mathcal{C}_b^2(\mathbb{R})$. Then we can conclude that any weak limit of a subsequence $\{\mu^{(n_r)}, r \geq 1\}$ should satisfy (5.3). We now proceed to prove that μ is characterized by (1.6). In order to do so, we apply (5.3) to the sequence of functions

$$f_z(x) = \frac{1}{x - z},$$

with $z \in \mathbb{C}_+$ having its real and imaginary parts in the rational numbers, we get

$$\begin{aligned}\left\langle \mu_t, f_z \right\rangle &= \langle \mu_0, f_z \rangle + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{(x - z) + (y - z)}{(x - z)^2 (y - z)^2} \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mu_s(\mathrm{d}x) \mu_s(\mathrm{d}y)\mathrm{d}s \\ &= \langle \mu_0, f_z \rangle + \int_0^t \int_{\mathbb{R}^2} \frac{1}{(x - z)(y - z)^2} \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mu_s(\mathrm{d}x) \mu_s(\mathrm{d}y)\mathrm{d}s,\end{aligned}$$

where the last identity follows from the symmetry over the variables x and y. Therefore, using a continuity argument, we get that the Cauchy-Stieltjes transform $G_t(z) := \int_{\mathbb{R}} \frac{1}{x - z} \mu_t(dx)$, defined in the domain $\mathbb{C}^+$, satisfies the integral equation

$$\begin{aligned}G_t(z) &= G_0(z) + \int_0^t \int_{\mathbb{R}^2} \frac{1}{(x - z)(y - z)^2} \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) \mu_s(\mathrm{d}x) \mu_s(\mathrm{d}y)\mathrm{d}s \\ &= G_0(z) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} R(s,s) G_s(z) \frac{\partial}{\partial z} G_s(z)\mathrm{d}s.\end{aligned}$$

In particular,

$$G_t(z) = F_{R(t,t)}(z),$$

where $F_\tau(z)$, for $z \in \mathbb{C}_+$, is the unique solution to the Burgers' equation

$$\begin{aligned}\frac{\partial}{\partial \tau} F_\tau(z) &= F_\tau(z) \frac{\partial}{\partial z} F_\tau(z), \\ F_0(z) &= \int_{\mathbb{R}} \frac{1}{x - z} \mu_0(\mathrm{d}x),\end{aligned}$$

which completes the proof. □

Finally we prove Proposition 5.1.

Proof of Proposition 5.1. By relation (3.6), we have that

$$G_r = \langle \mu_t^{(n_r)}, f\rangle - \langle \mu_0^{(n_r)}, f\rangle - \frac{1}{2n_r^2}\sum_{i=1}^{n_r}\int_0^t f''(\Phi_i^n(Y^{(n_r)}(s)))\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s \\ - \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mu_s^{(n_r)}(\mathrm{d}x)\mu_s^{(n_r)}(\mathrm{d}y)\mathrm{d}s, \tag{5.4}$$

and consequently, we can write

$$\mathbb{E}\left[G_r^2\right] = \mathbb{E}\left[(\langle \mu_t^{(n_r)}, f\rangle - \langle \mu_0^{(n_r)}, f\rangle)G_r\right] \tag{5.5}$$
$$- \frac{1}{2n_r^2}\sum_{i=1}^{n_r}\int_0^t \mathbb{E}\left[f''(\Phi_i^{n_r}(Y^{(n_r)}(s)))G_r\right]\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s$$
$$- \frac{1}{2}\int_0^t \mathbb{E}\left[\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mu_s^{(n_r)}(\mathrm{d}x)\mu_s^{(n_r)}(\mathrm{d}y)G_r\right]\mathrm{d}s.$$

Next we bound the terms appearing in the right hand side. Using relation (5.4), as well as the fact that f' and f'' are bounded, we can easily show that for every $T>0$, there exists a constant $K_1>0$, only depending on T and the properties of $R(s,t)$, such that for every $t\in[0,T]$,

$$|G_r| \le K_1(\|f\|_\infty + \|f''\|_\infty). \tag{5.6}$$

From here we obtain

$$\left|\frac{1}{n_r^2}\sum_{i=1}^{n_r}\int_0^t \mathbb{E}\left[f''(\Phi_i^{n_r}(Y^{(n_r)}(s)))G_r\right]\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right| \\ \le \frac{K_1}{n_r}\|f''\|_\infty(\|f\|_\infty + \|f''\|_\infty)\int_0^T\left|\frac{d}{ds}R(s,s)\right|\mathrm{d}s,$$

and hence

$$\lim_{r\to\infty}\left|\frac{1}{n_r^2}\sum_{i=1}^{n_r}\int_0^t \mathbb{E}\left[f''(\Phi_i^{n_r}(Y^{(n_r)}(s)))G_r\right]\frac{\mathrm{d}}{\mathrm{d}s}R(s,s)\mathrm{d}s\right| = 0. \tag{5.7}$$

Next we notice, by the zero mean property of G_r, that

$$\mathbb{E}\left[\left(\langle \mu_t^{(n_r)}, f\rangle - \langle \mu_0^{(n_r)}, f\rangle\right)G_r\right] = \mathbb{E}\left[\langle \mu_t^{(n_r)}, f\rangle G_r\right] = \frac{1}{n_r}\sum_{i=1}^{n_r}\mathbb{E}\left[f(\Phi_i(Y^{(n_r)}(t)))G_r\right].$$

Consequently, using the the duality property (2.4), as well as the fact that $G_r=\delta(u_r)$, with $u_r=(u_r^{k,h}(s)\mid k\le h)$ of the form

$$u_r^{k,h}(s) := \frac{1}{n_r^{\frac{3}{2}}}\sum_{i=1}^{n_r}\mathbf{1}_{[0,t]}(s)f'(\Phi_i^{n_r}(Y^{(n_r)}(s)))\mathbf{1}_G(Y^{(n_r)}(s))\frac{\partial\Phi_i^{n_r}}{\partial y_{k,h}}(Y^{(n_r)}(s)),$$

we get that

$$\mathbb{E}\Big[\Big(\langle \mu_t^{(n_r)}, f\rangle - \langle \mu_0^{(n_r)}, f\rangle\Big)G_r\Big] \\ = \frac{1}{n_r}\sum_{i=1}^{n_r}\mathbb{E}\big[\big\langle Df(\Phi_i^{n_r}(Y^{(n_r)}(t))), u_r\big\rangle_{\mathfrak{H}^{\frac{n_r(n_r+1)}{2}}}\big] \\ = \frac{1}{n_r^3}\sum_{i,j=1}^{n_r}\sum_{k\le h}\mathbb{E}\Big[f'(\Phi_i^{n_r}(Y^{(n_r)}(t)))\mathbf{1}_G(Y^{(n_r)}(t))\frac{\partial\Phi_i^{n_r}}{\partial y_{k,h}}(Y^{(n_r)}(t)) \\ \times\int_0^t f'(\Phi_j^{n_r}(Y^{(n_r)}(s)))\mathbf{1}_G(Y^{(n_r)}(s))\frac{\partial\Phi_j^{n_r}}{\partial y_{k,h}}(Y^{(n_r)}(s))\frac{\partial R}{\partial s}(s,t)\mathrm{d}s\Big]. \tag{5.8}$$

On the other hand, by the Cauchy-Schwarz inequality and the relation (3.5), we have that for every $u, v > 0$ and $i, j \in \mathbb{N}$,

$$\left| \sum_{k \le h} \left(\frac{\partial \Phi_i^{n_r}}{\partial y_{k,h}} (Y^{(n_r)}(u)) \right) \left(\frac{\partial \Phi_j^{n_r}}{\partial y_{k,h}} (Y^{(n_r)}(v)) \right) \right| \leq \left(\sum_{k \le h} \left(\frac{\partial \Phi_i^{n_r}}{\partial y_{k,h}} (Y^{(n_r)}(u)) \right)^2 \right)^{\frac{1}{2}} \left(\sum_{k \le h} \left(\frac{\partial \Phi_j^{n_r}}{\partial y_{k,h}} (Y^{(n_r)}(v)) \right)^2 \right)^{\frac{1}{2}} = 2. \quad (5.9)$$

Hence, from (5.8) we conclude that

$$\mathbb{E}\left[\Big(\langle \mu_t^{(n_r)}, f \rangle - \langle \mu_0^{(n_r)}, f \rangle \Big) G_r \right] \leq \frac{2 \left\| f' \right\|_\infty^2}{n_r} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right| \mathrm{d}s,$$

and consequently,

$$\lim_{r \to \infty} \mathbb{E}\left[\Big(\langle \mu_t^{(n_r)}, f \rangle - \langle \mu_0^{(n_r)}, f \rangle \Big) G_r \right] = 0. \quad (5.10)$$

Finally, we handle the third term in (5.5). Using the following identity

$$\frac{f'(x) - f'(y)}{x - y} = \int_0^1 f''(\theta x + (1-\theta) y) \mathrm{d}\theta,$$

we deduce that for every $s > 0$,

$$\mathbb{E}\left[\int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s^{(n_r)}(\mathrm{d}x) \mu_s^{(n_r)}(\mathrm{d}y) G_r \right] = \frac{1}{n_r^2} \sum_{i,j=1}^n \int_0^1 \mathbb{E}\left[f''(I_{i,j}^{s,r}(\theta)) G_r \right] \mathrm{d}\theta, \quad (5.11)$$

where

$$I_{i,j}^{s,r}(\theta) := \theta \Phi_i^{n_r}(Y^{(n_r)}(s)) + (1-\theta) \Phi_j^{n_r}(Y^{(n_r)}(s)).$$

The term in the right hand side of (5.11) can be estimated as follows. Define the processes

$$\Lambda_{k,h}^{l,r}(u) := \frac{\partial \Phi_l^{n_r}}{\partial y_{k,h}} (Y^{(n_r)}(u)).$$

We can easily show that

$$Df''(I_{i,j}^{s,r}(\theta)) = \frac{\theta}{\sqrt{n}} f'''(I_{i,j}^{s,r}(\theta)) \Lambda_{k,h}^{i,r}(s) \mathbf{1}_{[0,s]} + \frac{1-\theta}{\sqrt{n}} f^{(''')}(I_{i,j}^{s,r}(\theta)) \Lambda_{k,h}^{j,r}(s) \mathbf{1}_{[0,s]}.$$

Then, using the duality relation of the Skorohod integral, as well as the expression (5.1), we obtain

$$\begin{aligned} &\mathbb{E}\left[f''(I_{i,j}^{s,r}(\theta)) G_r \right] \\ &\quad = \frac{1}{n_r^2} \sum_{l=1}^{n_r} \sum_{k \le h} \mathbb{E}\Big[\int_0^t \theta f'''(I_{i,j}^{s,r}(\theta)) f'(\Phi_i^{n_r}(Y^{(n_r)}(s))) \Lambda_{k,h}^{i,r}(s) \Lambda_{k,h}^{l,r}(u) \frac{\partial R}{\partial u}(u,s) \mathrm{d}u \Big] \\ &\quad + \frac{1}{n_r^2} \sum_{l=1}^{n_r} \sum_{k \le h} \mathbb{E}\Big[\int_0^t (1-\theta) f'''(I_{i,j}^{s,r}(\theta)) f'(\Phi_i^{n_r}(Y^{(n_r)}(s))) \Lambda_{k,h}^{j,r}(s) \Lambda_{k,h}^{l,r}(u) \frac{\partial R}{\partial u}(u,s) \mathrm{d}u \Big], \end{aligned}$$

which, by the boundedness of f''' and f', implies that there exists a constant $K_2 > 0$, only depending on f, such that

$$\begin{aligned}\left|\mathbb{E}\left[f''(I_{i,j}^{s,r}(\theta))G_r\right]\right| &\leq \frac{K_2}{n_r^2}\mathbb{E}\bigg[\int_0^t \sum_{l=1}^{n_r}\bigg|\sum_{k\leq h}\Lambda_{k,h}^{i,r}(s)\Lambda_{k,h}^{l,r}(u)\bigg|\left|\frac{\partial R}{\partial u}(u,s)\right|\mathrm{d}u\bigg] \\ &\quad + \frac{K_2}{n_r^2}\mathbb{E}\bigg[\int_0^t \sum_{l=1}^{n_r}\bigg|\sum_{k\leq h}\Lambda_{k,h}^{j,r}(s)\Lambda_{k,h}^{l,r}(u)\bigg|\left|\frac{\partial R}{\partial u}(u,s)\right|\mathrm{d}u\bigg]. \qquad (5.12)\end{aligned}$$

Using (5.9) and (5.12), we get

$$\begin{aligned}\left|\mathbb{E}\left[f''(I_{i,j}^{s,r}(\theta))G_r\right]\right| &\leq \frac{4K_2}{n_r}\int_0^t\left|\frac{\partial R}{\partial u}(u,s)\right|\mathrm{d}u \leq \frac{4K_2 t^{1-\frac{1}{\alpha}}}{n_r}\bigg(\int_0^t\left|\frac{\partial R}{\partial u}(u,s)\right|^{\alpha}\mathrm{d}u\bigg)^{\frac{1}{\alpha}} \\ &\leq \frac{4K_2 t^{1-\frac{1}{\alpha}}}{n_r}\sup_{s\in[0,t]}\bigg(\int_0^t\left|\frac{\partial R}{\partial u}(u,s)\right|^{\alpha}\mathrm{d}u\bigg)^{\frac{1}{\alpha}}.\end{aligned}$$

Using the previous identity in (5.11), we deduce that there exists a constant $K_3 > 0$, such that

$$\left|\mathbb{E}\left[\int_{\mathbb{R}^2}\mathbf{1}_{\{x\neq y\}}\frac{f'(x)-f'(y)}{x-y}\mu_s^{(n_r)}(\mathrm{d}x)\mu_s^{(n_r)}(\mathrm{d}y)G_r\right]\right| \leq \frac{K_3}{n_r}. \qquad (5.13)$$

From (5.5), (5.7), (5.10) and (5.13), we conclude that G_r converges to zero in $L^2(\Omega)$, as required. □

References

[1] Anderson, G. and Guionnet, A. and Zeitouni, O (2010). An introduction to random matrices. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge. MR-2760897

[2] Biane, P. (1997). Free Brownian motion, free stochastic calculus and random matrices. In: Free probability theory (Waterloo, ON, 1995). Amer. Math. Soc., Providence, RI. MR-1426833

[3] Billingsley, P. (1999). Convergence of probability measures (2nd ed.). John Wiley & Sons, Inc., New York. MR-1700749

[4] Cabanal Duvillard T., Guionnet A. (2001). Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies. *Annals of Probability.* **29**, 1205–1261. MR-1872742

[5] Cépa, E. and Lépingle, D. (1997). Diffusing particles with electrostatic repulsion. *Probability Theory Related Fields.* **107**, 429–449. MR-1440140

[6] Cheridito, P. and Nualart, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques* **41**, 1049–1081. MR-2172209

[7] Decreusefond, L. and Nualart, D. (2008). Hitting times for Gaussian processes. *Annals of Probability.* **36**, 319–330. MR-2370606

[8] Decreusefond, L. and Üstünel, A. S. (1999). Stochastic analysis of the fractional Brownian motion *Potential Analysis.* **10**, 177–214. MR-1677455

[9] Lax, P. (1998). On the discriminant of real symmetric matrices. *Communications on Pure and Applied Mathematics.* **51**, 1387–1396. MR-1639147

[10] Lei, P. and Nualart, D. (2012). Stochastic calculus for Gaussian processes and application to hitting times. *Communications on Stochastic Analysis.* **6**, 379–402. MR-2988698

[11] Nualart, D. and Pérez-Abreu, V. (2014). On the eigenvalue process of a matrix fractional Brownian motion. *Stochastic Processes and their Applications.* **124**, 4266–4282. MR-3264448

[12] Pardo, J. C., Pérez, J. L. and Pérez-Abreu, V. (2016). A random matrix approximation for the non-commutative fractional Brownian motion. *Journal of Theoretical Probability*. **29** (4), 1581–1598. MR-3571255

[13] Perez-Abreu V. and Tudor C. (2007). Functional Limit Theorem for Trace processes in a Dyson Brownian motion. *Communications on Stochastic Analysis*. **1** (3), 415–428. MR-2403859

[14] Rogers, L. C. G. and Shi, Z. (1993). Interacting Brownian particles and the Wigner law. *Probability Theory and Related Fields*. **95**, 555–570. MR-1217451

[15] Nualart, D. (2006). The Malliavin calculus and related topics. Probability and its Applications (New York). Springer-Verlag, Berlin MR-2200233

[16] Voiculescu D. (1993). The analogues of entropy and of Fisher's information measure in free probability theory. I. *Communications in Mathematical Physics*. **155** , 411-440. MR-1296352

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