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Exponentially slow mixing in the mean-field Swendsen–Wang dynamics

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Abstract. Swendsen–Wang dynamics for the Potts model was proposed in the late 1980's as an alternative to single-site heat-bath dynamics, in which global updates allow this MCMC sampler to switch between metastable states and ideally mix faster. Gore and Jerrum (*J. Stat. Phys.* **97** (1999) 67–86) found that this dynamics may in fact exhibit slow mixing: they showed that, for the Potts model with $q \geq 3$ colors on the complete graph on n vertices at the critical point $\beta_c(q)$, Swendsen–Wang dynamics has $t_{\mathrm{MIX}} \geq \exp(c\sqrt{n})$. Galanis *et al.* (In *Proc. of the 19th International Workshop on Randomization and Computation (RANDOM 2015)* (2015) 815–828) showed that $t_{\mathrm{MIX}} \geq \exp(cn^{1/3})$ throughout the critical window (β_s, β_S) around β_c, and Blanca and Sinclair (In *Proc. of the 19th International Workshop on Randomization and Computation (RANDOM 2015)* (2015) 528–543) established that $t_{\mathrm{MIX}} \geq \exp(c\sqrt{n})$ in the critical window for the corresponding mean-field FK model, which implied the same bound for Swendsen–Wang via known comparison estimates. In both cases, an upper bound of $t_{\mathrm{MIX}} \leq \exp(c'n)$ was known. Here we show that the mixing time is truly exponential in n: namely, $t_{\mathrm{MIX}} \geq \exp(cn)$ for Swendsen–Wang dynamics when $q \geq 3$ and $\beta \in (\beta_s, \beta_S)$, and the same bound holds for the related MCMC samplers for the mean-field FK model when $q > 2$.

Résumé. La dynamique de Swendsen–Wang a été proposée à la fin des années 1980 comme une alternative à la dynamique du bain-de-chaleur à un site, dans laquelle des mises à jour globales permettent à cet algorithme MCMC de passer plus vite d'un état métastable à un état de mélange idéal. Gore et Jerrum (*J. Stat. Phys.* **97** (1999) 67–86) ont trouvé que cette dynamique peut en fait montrer un mélange lent: ils ont montré, pour le modèle de Potts à $q \geq 3$ couleurs sur le graphe complet sur n sommets au point critique $\beta_c(q)$, que la dynamique de Swendsen–Wang vérifie $t_{\mathrm{MIX}} \geq \exp(c\sqrt{n})$. Galanis *et al.* (In *Proc. of the 19th International Workshop on Randomization and Computation (RANDOM 2015)* (2015) 815–828) a montré que $t_{\mathrm{MIX}} \geq \exp(cn^{1/3})$ dans toute la fenêtre critique (β_s, β_S) autour de β_c, et Blanca et Sinclair (In *Proc. of the 19th International Workshop on Randomization and Computation (RANDOM 2015)* (2015) 528–543) ont établit que $t_{\mathrm{MIX}} \geq \exp(c\sqrt{n})$ dans la fenêtre critique pour le modèle de champs moyen FK, ce qui implique la même borne pour Swendsen–Wang grâce des estimées de comparaison connues. Dans les deux cas, une borne supérieure de $t_{\mathrm{MIX}} \leq \exp(c'n)$ était connue. Dans cet article, nous montrons que le temps de mélange est vraiment exponentiel en n: plus précisément, $t_{\mathrm{MIX}} \geq \exp(cn)$ pour la dynamique de Swendsen–Wang quand $q \geq 3$ et $\beta \in (\beta_s, \beta_S)$, et la même borne est vraie pour l'algorithme MCMC associé pour le modèle de champs moyen FK quand $q > 2$.

MSC: 60K35; 82B20; 82B27; 82C20

Keywords: Potts model; Swendsen–Wang; Mixing time; FK model; Random graphs

1. Introduction

The mean-field q-state Potts model is a canonical statistical physics model extending the Curie–Weiss Ising model ($q = 2$) to $q \in \mathbb{N}$ possible states; for $q \geq 3$, it is one of the simplest models to exhibit a discontinuous (first-order) phase transition. Formally, the mean-field q-state Potts model with parameter β is a probability distribution $\mu_{n,\beta,q}$ over $\{1, \ldots, q\}^n$, given by $\mu_{n,\beta,q}(\sigma) \propto \exp(\frac{\beta}{n} H(\sigma))$, where $H(\sigma) = \sum_{i<j} \mathbf{1}\{\sigma_i = \sigma_j\}$. The model exhibits a phase transition at $\beta = \beta_c(q)$ from a disordered phase ($\beta < \beta_c$), where the sizes of all q color classes concentrate around n/q, to an ordered phase ($\beta > \beta_c$), where there is typically one color class of size $a_\beta n$ for $a_\beta > 1/q$ (see §2).

As a means of overcoming low-temperature bottlenecks in the energy landscape (dominant color classes), Swendsen and Wang [20] introduced a non-local reversible Markov chain, relying on the random cluster (FK) representation of the Potts model. The mean-field FK model is the generalization of $\mathcal{G}(n,p)$ – the Erdős–Rényi random graph – parametrized by $(p=\frac{\lambda}{n},q)$, in which the probability of a graph $G=(V,E)$, identified with $\omega\in\Omega_{\mathrm{RC}}:=\{0,1\}^{\binom{n}{2}}$, is given by $\pi_{n,p,q}(\omega)\propto p^{|E|}(1-p)^{\binom{n}{2}-|E|}q^{k(G)}$, where $k(G)$ is the number of connected components of G (clusters of ω).

Via the Edwards–Sokal coupling [9] of the q-state Potts model at inverse temperature β/n and the FK model with parameters (p,q) with $p=1-e^{-\beta/n}$, the mean-field Swendsen–Wang dynamics can be formulated as follows: consider a mean-field Potts configuration σ with $V_1,\ldots,V_q$ being the sets of vertices $V_i=\{x:\sigma_x=i\}$. An update of the dynamics, started from σ, first samples, independently for every $i=1,\ldots,q$, a configuration $G_i\sim\mathcal{G}(|V_i|,p)$ on the subgraph of V_i, forming an FK configuration ω as the union of the G_i's; then, it assigns an i.i.d. color $X_C\sim\mathrm{Uni}(\{1,\ldots,q\})$ to each cluster C in ω, and for every $x\in C$, sets $\sigma'_x=X_C$ in the new state σ' of the Markov chain.

As apparent from the second (coloring) stage of the Swendsen–Wang algorithm, it can seamlessly jump between the q ordered low-temperature metastable states where one color is dominant. It was thus expected that this MCMC sampler would converge quickly to equilibrium at all temperatures; e.g., its total variation mixing time t_{MIX}, formally defined in §2, would be at most polynomial in the system size for all $\beta>0$.

Indeed, at $q=2$ (the Ising model) Cooper, Dyer, Frieze and Rue [7] proved that, on the complete graph, Swendsen–Wang has $t_{\mathrm{MIX}}=O(\sqrt{n})$ at all β (it was later shown in [18] that $t_{\mathrm{MIX}}\asymp n^{1/4}$ at β_c while $t_{\mathrm{MIX}}=O(\log n)$ at $\beta\neq\beta_c$), and Guo and Jerrum [15] recently showed that for *any* n-vertex graph and all β, Swendsen–Wang has $t_{\mathrm{MIX}}=n^{O(1)}$ (this is in contrast to single-site dynamics, where $t_{\mathrm{MIX}}\geq\exp(cn)$ at low temperature [8]).

Countering this intuition, however, Gore and Jerrum [14] found in 1999 that, for any $q\geq 3$, the Swendsen–Wang dynamics for the mean-field q-state Potts model has $t_{\mathrm{MIX}}\geq\exp(c\sqrt{n})$ for some $c(q)>0$ at its critical point $\beta_c(q)$. This is a consequence of the discontinuity of the phase transition of the mean-field Potts model for $q\geq 3$, where at $\beta_c(q)$, both the q ordered phases (with one dominant color class) and the disordered phase (with all color classes having roughly n/q sites) are metastable.

On the lattice $(\mathbb{Z}/n\mathbb{Z})^d$, the Potts model exhibits a discontinuous phase transition for some choices of q (depending on d); there it was shown in [5], following [4], that Swendsen–Wang dynamics in fact has $t_{\mathrm{MIX}}\geq\exp(cn^{d-1})$ for all q sufficiently large, suggesting that an exponential lower bound in n should also hold in mean-field, believed to approximate high-dimensional tori. (The matching upper bound of [5] applies to general graphs and translates to $t_{\mathrm{MIX}}\leq\exp(c'n)$ on the complete graph.) On $\mathbb{Z}^2$, this lower bound was extended [12] to q where the phase transition is first-order (all $q>4$).

For the Glauber dynamics of the mean-field Potts model, when $q\geq 3$, the mixing time for all β was characterized in [8], where it was shown that, in discrete-time, t_{MIX} has order $n\log n$ at $\beta<\beta_s$, order $n^{4/3}$ at $\beta=\beta_s$, and finally $t_{\mathrm{MIX}}\geq\exp(cn)$ at $\beta>\beta_s$, where β_s is the spinodal point corresponding to the onset of q ordered metastable phases. Recently, Galanis, Štefankovic and Vigoda [10] analyzed the mixing time of the analogous mean-field Swendsen–Wang dynamics, finding it to mix in polynomial time[1] both at high temperature and – unlike Glauber dynamics – at low temperatures, for all β outside a critical window (β_s,β_S) around β_c, where the critical point β_S (mirroring the spinodal point β_s) marks the disappearance of metastability of the disordered phase.

For $\beta\in(\beta_s,\beta_S)$, Swendsen–Wang was shown in [10] to slow down to $t_{\mathrm{MIX}}\geq\exp(cn^{1/3})$. The related Glauber dynamics for the mean-field FK model (see precise definitions in §2) with $q>2$ was shown by Blanca and Sinclair [1] to have $t_{\mathrm{MIX}}\geq\exp(c\sqrt{n})$ whenever $\lambda=np$ is in the critical window (λ_s,λ_S); this implied, via comparison results of Ullrich [21], that Swendsen–Wang has $t_{\mathrm{MIX}}\geq\exp(c\sqrt{n})$ throughout $\beta\in(\beta_s,\beta_S)$ (extending the lower bound at $\beta=\beta_c$ due to Gore and Jerrum).

The fact that three significant papers, over a period of almost twenty years, all presented a lower bound no better than $\exp(c\sqrt{n})$, left open the possibility that this is the true order of the mixing time inside the critical window.

We show that the mixing time of the mean-field Swendsen–Wang dynamics is truly exponential in n at criticality, similar to the single-site Glauber dynamics (see Figure 1).

Theorem 1. *Let $q\geq 3$ be a fixed integer, and consider the Swendsen–Wang dynamics for the q-state mean-field Potts model on n vertices at inverse temperature $\beta\in(\beta_s,\beta_S)$. There exists some $c(\beta,q)>0$ such that, for all n large enough, $t_{\mathrm{MIX}}\geq\exp(cn)$.*

The case of non-integer q (the mean-field FK model) is more delicate: the analogue of Swendsen–Wang in this setting is Chayes–Machta dynamics [6]; we nonetheless are able to obtain an analogous, exponential in n, lower bound on the mixing time. As in [1], comparison results of [21] extend the result to heat-bath Glauber dynamics.

Theorem 2. *Fix $q>2$, and consider Glauber dynamics for the mean-field FK model on n vertices with parameters $(p=\frac{\lambda}{n},q)$ where $\lambda\in(\lambda_s,\lambda_S)$. There exists $c(\lambda,q)>0$ such that $t_{\mathrm{MIX}}\geq\exp(cn)$ for large enough n. The same holds for Chayes–Machta dynamics.*

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[1] It was shown in that work that $t_{\mathrm{MIX}}=O(\log n)$ for $\beta\notin[\beta_s,\beta_S)$, whereas $t_{\mathrm{MIX}}\asymp n^{1/3}$ at $\beta=\beta_s$.

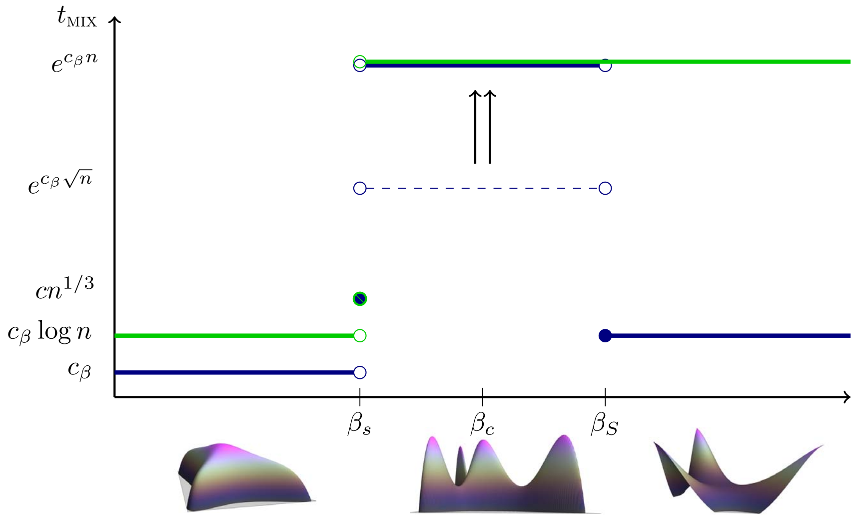

Fig. 1. Mixing times of continuous–time mean-field Potts Glauber (green) and Swendsen–Wang (blue) dynamics when $q > 2$; the dashed line represents the previous lower bound [1,10,14] for $\beta \in (\beta_s, \beta_S)$.

To outline our approach for proving Theorems 1–2, we first sketch the argument of [14], thereafter adapted to $\beta \in [\beta_s, \beta_S)$ in [10] and to the FK model in [1]. Starting from a Potts configuration where each color class has $\frac{n}{q} \pm \varepsilon n$ vertices, since $\beta < \beta_S$, for small enough ε, this corresponds to subcritical Erdős-Rényi random graphs in the first stage of the Swendsen–Wang dynamics. The exponential tail of component sizes in this regime shows that, for a sequence $k = k(n)$, with probability at least $1 - n\exp(-ck)$, no cluster in the edge configuration we obtain is larger than k; on this event, the component sizes $\mathscr{L}_i$ satisfy $\sum_i \mathscr{L}_i^2 \le k \sum \mathscr{L}_i = nk$, thus by Hoeffding's inequality, with probability $1 - O(\exp[-\varepsilon^2 n/(2k)])$, every new color class will have $n/q \pm \varepsilon n$ vertices, and in particular no dominant color class would emerge. In this argument, choosing $k \asymp \sqrt{n}$ balances the two probability estimates to $1 - \exp(-c\sqrt{n})$. However, at $\beta \ge \beta_c$, the Potts model does admit a dominant color class with positive (uniformly bounded away from 0) probability; thus the mixing time is at least $\exp(c\sqrt{n})$.

This argument extends to $\beta \in (\beta_s, \beta_c)$: there, one bounds the probability that the largest color class is smaller by εn than its mean, or one of the other color classes is larger by εn than its mean; these probabilities are then bounded using concentration of the giant component in a supercritical random graph and the discrete duality principle.

In order to improve this lower bound into $\exp(cn)$ per Theorem 1, instead of looking at the size of the largest component after the $\mathcal{G}(n, p)$ stage of the dynamics, we consider S_M, the set of vertices in connected components of size larger than M. We show that, whenever the $\mathcal{G}(n, p)$ stage is subcritical and M is sufficiently large, the probability that $|S_M| > \rho n$ is at most $\exp(-c\rho n)$. Moreover, given $|S_M| \le \rho n$, Hoeffding's inequality implies, following the second stage of the dynamics, all the new color classes will have $\frac{n}{q} \pm \varepsilon n$ vertices except with probability $\exp[-(\varepsilon - \rho)^2 n/(2M)]$, yielding $t_{\mathrm{MIX}} \ge \exp(cn)$.

The proof of Theorem 2 also relies on this random graph estimate, but is more involved. Since a step of Chayes–Machta dynamics only resamples a random proper subset of the configuration in each step, we cannot define an analogous set of configurations which is hard to escape uniformly over all initial states in the set. Instead, we use a spectral approach and bound the conductance of an analogous set of FK configurations, showing that it has an exponentially decaying bottleneck ratio under $\pi_{n,p,q}$. In order to obtain such equilibrium estimates under $\pi_{n,p,q}$, we recursively apply a fundamental lemma of Bollobás, Grimmett and Janson [3] to reduce equilibrium estimates under $\pi_{n,p,q}$ to random graph estimates for $\mathcal{G}(m, p)$ for appropriately chosen m.

2. Preliminaries

Notation

Throughout this paper, we use the notation $f \lesssim g$ for two sequences $f(n), g(n)$ to denote $f = O(g)$, and let $f \asymp g$ denote $f \lesssim g \lesssim f$. We will consider these models on the complete graph on n vertices, $G = (V, E) = (\{1, \dots, n\}, \{ij\}_{1 \le i < j \le n})$.

Denote by $\pi_{n,p,q}$ the FK measure on the complete graph on n vertices with parameters p, q, and by $\mu_{n,p,q}$, the Potts measure with β such that $p = 1 - e^{-\beta/n}$. The FK model with $q = 1$ corresponds precisely to the Erdős–Rényi random

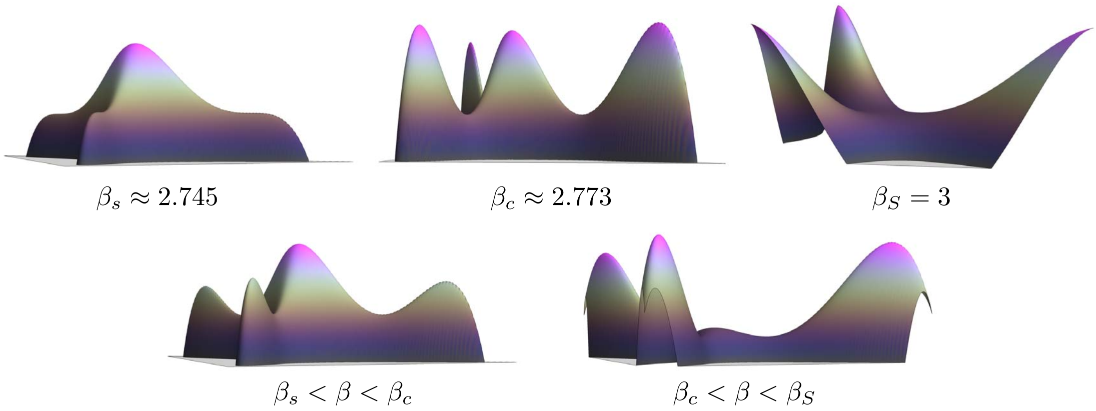

Fig. 2. The free energy landscape of the 3-state Potts model in the metastability window $\beta_s \le \beta \le \beta_S$. The three outer peaks correspond to the ordered phases; middle peak corresponds to the disordered phase.

graph $\mathcal{G}(n, p)$ and we use the shortened notation $\pi_{n,p} = \pi_{n,p,1}$. We occasionally use $\mathcal{G}(n, p, q)$ to denote the mean-field FK model given by $\pi_{n,p,q}$. Finally, we use the notation $\restriction_F$ to denote the restriction to some subset $F \subset E$, e.g., $\pi_{n,p}(\cdot\restriction_{\{e\}})$ is the marginal of $\pi_{n,p}$ on $\{e\}$.

For any FK configuration $\omega \in \{0,1\}^E$, enumerate the clusters of ω in decreasing size $\mathcal{C}_1, \mathcal{C}_2, \ldots$ and let $\mathscr{L}_i = |\mathcal{C}_i|$. For a vertex x let, $\mathcal{C}_x$ denote the cluster to which x belongs.

We re-parametrize the FK and Potts models by λ instead of p and β via the relations $p = \lambda/n$ and $\lambda/n = 1 - e^{-\beta/n}$, as this is the scaling at which $\mathcal{G}(n, p, q)$ undergoes a phase transition. For $q \le 2$, let $\lambda_s = \lambda_c = \lambda_S = q$, and for $q > 2$, let

$$\lambda_s = \min_{z \ge 0}\left\{z + \frac{qz}{e^z - 1}\right\}, \qquad \lambda_c = \frac{2(q-1)\log(q-1)}{q-2}, \qquad \lambda_S = q,$$

so that for $q > 2$, we have $\lambda_s < \lambda_c < \lambda_S$ (see e.g., [1,10]). The critical points λ_s, λ_S correspond to the emergence and disappearance of metastability of the ordered phase and disordered phase, respectively; namely, above λ_s, the free energy has a local maximum corresponding to the ordered phase (where there is a giant component in the FK configuration), and below λ_S, it has a local maximum corresponding to the disordered phase (where there is no giant component). At the critical $\lambda = \lambda_c$, the ordered and disordered phases have the same free energy (see Figure 2). These two critical points can also have the following alternative interpreation [10]: λ_s corresponds to the first uniqueness/non-uniqueness threshold of the Δ-regular infinite tree, and λ_S should correspond to a second uniqueness/non-uniqueness threshold of the Δ-regular tree with periodic boundary conditions.

The FK and Potts phase transitions

The following give a description of the static phase transition undergone by the mean-field FK and Potts models respectively. Let $\Theta_r = \Theta_r(\lambda, q)$ be the largest solution of $e^{-\lambda x} = 1 - \frac{qx}{1+(q-1)x}$ so $\Theta_r = \frac{q-2}{q-1}$ when $\lambda = \lambda_c$.

Proposition 2.1 ([3, Thms. 2.1–2.2], [19, Thm. 19]). *Consider the n-vertex mean-field FK model with parameters (p, q) with $p = \lambda/n$; if $\lambda < \lambda_c(q)$, for every $\varepsilon > 0$, we have $\lim_{n\to\infty} \pi_{n,p,q}(\mathscr{L}_1 \le \varepsilon n) = 1$ whereas if $\lambda > \lambda_c(q)$, for every $\varepsilon > 0$, we have $\lim_{n\to\infty} \pi_{n,p,q}(\mathscr{L}_1 \ge (\Theta_r - \varepsilon)n) = 1$. If $\lambda = \lambda_c(q)$, there exists $\gamma(q) \in (0,1)$ so that for all $\varepsilon > 0$, $\lim_{n\to\infty} \pi_{n,p,q}(\mathscr{L}_1 \le \varepsilon n) \ge \gamma$ and $\lim_{n\to\infty} \pi_{n,p,q}(\mathscr{L}_1 \ge (\Theta_r - \varepsilon)n) \ge 1 - \gamma$.*

Corollary 2.2. *Consider the mean-field Potts model parametrized by q and $p = \lambda/n = 1 - e^{-\beta/n}$. If $\lambda < \lambda_c(q)$, for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mu_{n,p,q}\left(\sigma : \max_{r=1,\ldots,q}\left|\frac{1}{n}\sum_{i\le n} \mathbf{1}\{\sigma_i = r\} - \frac{1}{q}\right| < \varepsilon\right) = 1,$$

and if $\lambda > \lambda_c(q)$, then there exists $m_\lambda(q) > q^{-1}$ such that for sufficiently small $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu_{n,p,q}\left(\sigma : \max_{r=2,\ldots,q}\left\{\left|\frac{1}{n}\sum_{i\le n} \mathbf{1}\{\sigma_i = 1\} - m_\lambda\right|, \left|\frac{1}{n}\sum_{i\le n} \mathbf{1}\{\sigma_i = r\} - \frac{1 - m_\lambda}{q-1}\right|\right\} < \varepsilon\right) = \frac{1}{q}.$$

If $q > 2$ and $\lambda = \lambda_c(q)$, there exists $m_c(q) > q^{-1}$ and $\gamma(q) \in (0,1)$ so that for all sufficiently small $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu_{n,p,q}\left(\max_{r=1,\dots,q}\left|\frac{1}{n}\sum_{i\le n}\mathbf{1}\{\sigma_i = r\} - \frac{1}{q}\right| < \varepsilon\right) \ge \gamma, \quad \textit{and}$$

$$\lim_{n\to\infty} \mu_{n,p,q}\left(\max_{r=2,\dots,q}\left\{\left|\frac{1}{n}\sum_{i\le n}\mathbf{1}\{\sigma_i = 1\} - m_c\right|, \left|\frac{1}{n}\sum_{i\le n}\mathbf{1}\{\sigma_i = r\} - \frac{1-m_c}{q-1}\right|\right\} < \varepsilon\right) \ge \frac{1-\gamma}{q}.$$

Cluster dynamics

Swendsen–Wang dynamics for the q-state Potts model on $G = (V, E)$ with parameter β, such that $p = 1 - e^{-\beta/n}$, is the following discrete-time reversible Markov chain. From a Potts configuration σ on G, generate a new state σ' as follows.

(1) Introduce auxiliary edge variables $\omega \in \{0,1\}^E$ as follows: for every $e = xy \in E$, set $\omega(e) = 0$ if $\sigma_x \neq \sigma_y$ and independently sample $\omega(e) \sim \mathrm{Ber}(p)$ if $\sigma_x = \sigma_y$.
(2) For every connected component of the resulting ω, reassign the cluster, collectively, an i.i.d. color in $1, \dots, q$, to obtain the new configuration σ'.

Chayes–Machta dynamics for the FK model on $G = (V, E)$ with parameters (p, q), for $q \ge 1$ and $p = \lambda/n$, is the following discrete-time reversible Markov chain: From an FK configuration $\omega \in \Omega_{\mathrm{RC}}$ on G, generate a new state $\omega' \in \Omega_{\mathrm{RC}}$ as follows.

(1) Assign each cluster C of ω an auxiliary i.i.d. variable $X_C \sim \mathrm{Bernoulli}(1/q)$.
(2) For every $e = xy$, if x and y belong to *active* clusters ($X_C = 1$), independently sample $\omega'(e) \sim \mathrm{Ber}(\lambda/n)$, and otherwise, set $\omega'(e) = \omega(e)$.

Variants of Chayes–Machta dynamics with $1 \le k \le \lfloor q \rfloor$ "active colors" have also been studied, with numerical evidence for $k = \lfloor q \rfloor$ being the most efficient choice; see [11].

Glauber dynamics for the FK model

Swendsen–Wang dynamics is closely related to the FK model; much of the analysis of Swendsen–Wang dynamics on general graphs has been via the Glauber dynamics for the corresponding FK model. Discrete-time Glauber dynamics [13] for the FK model on $G = (V, E)$ with $p = \lambda/n$ is as follows: select an edge $e = xy$ in E uniformly at random and update $\omega(e)$ according to $\pi_{n,p,q}(\cdot\restriction_{\{e\}} \mid \omega\restriction_{G-\{e\}})$.

Size of largest component and drift functions

For $\lambda > 1$, let θ_λ be the unique positive root of $e^{-\lambda x} = 1 - x$. Recall the following tail estimates for $\mathscr{L}_1$ in $\mathcal{G}(n, p)$.

Fact 2.3 (e.g., cf. [16, p. 109]). *Consider $\mathcal{G}(n, p)$ with $pn = \lambda < 1$. Then for any x,*

$$\pi_{n,p}\big(|\mathcal{C}_x| \ge k\big) \le e^{-\frac{(1-\lambda)^2 k}{2}}.$$

In particular, $\pi_{n,p,1}(\mathscr{L}_1 \ge k) \le n\exp(-(1-\lambda)^2 k/2)$.

Proposition 2.4 ([18, Lemma 5.4]). *Consider $\mathcal{G}(n, p)$ with $np = \lambda > 1$. There exists $c(\lambda) > 0$ such that for every $\varepsilon > 0$,*

$$\pi_{n,p,1}\big(|\mathscr{L}_1 - \theta_\lambda n| \ge \varepsilon n\big) \lesssim e^{-c\varepsilon^2 n}.$$

For the proof of Theorem 1, following [10,14] define the drift function for the average size of the largest color class of the Swendsen–Wang dynamics

$$F_\lambda(z) = \begin{Bmatrix} z\theta_{\lambda z} + \frac{1}{q}(1 - z\theta_{\lambda z}) & \text{for } z > 1/\lambda \\ \frac{1}{q} & \text{for } z \le 1/\lambda \end{Bmatrix}.$$

(The expression for $F_\lambda(z)$ when $z > 1/\lambda$ is as such since in the $\mathcal{G}(n, p)$ step of the largest color class–of size zn–there will be a macroscopic component of size $z\theta_{\lambda z}$ contributing to the largest color class at the next time step, in addition to $\frac{1}{q}$

of the remaining vertices). The function $F_\lambda(z)$ has, for some values of λ a second fixed point besides $\frac{1}{q}$, which we denote by $a_\lambda > 1/q$, which solves (for a short explanation see [10, Eq. (4)–(5)])

$$\log \frac{(q-1)a_\lambda}{1-a_\lambda} = \lambda\left(a_\lambda - \frac{1-a_\lambda}{q-1}\right).$$

Proposition 2.5 ([10, Lemma 5]). *If $\lambda > \lambda_s$, the fixed point a_λ is such that $\lambda a_\lambda > 1$ and if $b_\lambda = \frac{1-a_\lambda}{q-1}$, we have $\lambda b_\lambda < 1$. Moreover, if $q > 2$ and $\lambda > \lambda_s$, a_λ is a Jacobian attractive fixed point of $F_\lambda(z)$ so that $|F'(a_\lambda)| < 1$.*

Similarly to the above, we can define the function f given by

$$f(\theta) = \theta_{\lambda(1+(q-1)\theta)/q},$$

which governs the mean drift of the size of the giant component in Chayes–Machta dynamics. We can also define Θ_r to be the largest solution to $e^{-\lambda x} = 1 - \frac{qx}{1+(q-1)x}$. Following [1], let $\Theta_{\min}(\lambda, q) = \max\{0, (q-\lambda)/(\lambda(q-1))\}$, observe that if $\lambda < \lambda_S$, $\lambda(\Theta_{\min} + q^{-1}(1-\Theta_{\min})) = 1$, and define the drift function $g(\theta) = f(\theta) - \theta$.

Proposition 2.6 ([1, Lemma 2.14]). *When $q > 2$ and $\lambda > \lambda_s$, the drift function g has two roots, $\Theta^* < \Theta_r$ in $(\Theta_{\min}, 1]$; moreover, g is strictly positive on (Θ^*, Θ_r).*

Mixing time and spectral gap

In this section, we introduce the quantities of interest regarding the time for the Swendsen–Wang and Glauber dynamics to reach equilibrium. Consider a Markov chain with finite state space Ω and transition matrix P reversible with respect to π. For two measures ν, π, define their total variation distance by

$$\|\nu - \pi\|_{\mathrm{TV}} = \sup_{A \subset \Omega} \left|\nu(A) - \pi(A)\right| = \frac{1}{2}\|\nu - \pi\|_{\ell^1}.$$

Then the mixing time of P is defined as

$$t_{\mathrm{MIX}} = \inf\left\{t : \max_{X_0 \in \Omega} \left\| P^t(X_0, \cdot) - \pi \right\|_{\mathrm{TV}} < 1/(2e)\right\}.$$

A related quantity that is sometimes easier to work with is the spectral gap of P; Since P is reversible with respect to π, we can enumerate its spectrum from largest to smallest as $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_{|\Omega|}$; then the spectral gap of P is defined as $\mathtt{gap} = 1 - \max\{\lambda_2, -\lambda_{|\Omega|}\}$. The following is a standard comparison between the spectral gap and the mixing time of a Markov chain with transition matrix P (see e.g., [17]):

$$\mathtt{gap}^{-1} - 1 \le t_{\mathrm{MIX}} \le \log(2e/\pi_{\min})\mathtt{gap}^{-1}. \tag{2.1}$$

Spectral gap comparisons

The following comparison inequalities between the aforementioned Markov chains are due to Ullrich.

Proposition 2.7 ([21]). *Let $q \ge 2$ be integer. Let $\mathtt{gap}_{\mathrm{RC}}$ be the spectral gap of Glauber dynamics FK model on a graph $G = (V, E)$ and let $\mathtt{gap}_{\mathrm{SW}}$ be the spectral gap of Swendsen–Wang. Then*

$$(1 - p + p/q)\mathtt{gap}_{\mathrm{RC}} \le \mathtt{gap}_{\mathrm{SW}} \le 8\mathtt{gap}_{\mathrm{RC}}|E|\log|E|. \tag{2.2}$$

The proof of (2.2) further extends to all real $q > 1$, whence

$$\mathtt{gap}_{\mathrm{RC}} \lesssim \mathtt{gap}_{\mathrm{CM}} \lesssim \mathtt{gap}_{\mathrm{RC}}|E|\log|E|, \tag{2.3}$$

as was observed (and further generalized) by Blanca and Sinclair [1, §5], where $\mathtt{gap}_{\mathrm{CM}}$ is the spectral gap of Chayes–Machta dynamics.

3. Slow mixing of Swendsen–Wang dynamics

Towards the proof of Theorem 1, we first establish some preliminary estimates. For $\omega \in \Omega_{\mathrm{RC}}$, we will frequently be interested in bounding the following quantity:

$$S_M := S_M(\omega) = \{x \in V : |\mathcal{C}_x| > M\}.$$

The bottlenecks in the proofs of Theorems 1–2 both rely on the following estimate.

Lemma 3.1. *Consider $\omega \sim \mathcal{G}(n,p)$ with $np = \lambda < 1$. There exists $c(\lambda) > 0$ such that for every $\rho > 0$, there exists $M_0(\lambda, \rho)$ such that for every $M \ge M_0$,*

$$\pi_{n,p}\big(|S_M| \ge \rho n\big) \lesssim e^{-c\rho n}.$$

Proof. Recall that by Fact 2.3, there exists $c_1(\lambda) > 0$ such that $\pi_{n,p}(|\mathcal{C}_x| \ge k) \le e^{-c_1 k}$ for all k. Moreover, conditioned on other clusters, the remaining graph is distributed as $\mathcal{G}(m, \lambda/n)$ for $m \le n$, so that for any ℓ vertices $y_1, \dots, y_\ell$,

$$\pi_{n,p}\left(|\mathcal{C}_x| \ge k \,\middle|\, \mathcal{C}_{y_1}, \dots, \mathcal{C}_{y_\ell}, \mathcal{C}_x \cap \left(\bigcup_{i=1}^{\ell} \mathcal{C}_{y_i}\right) = \varnothing\right) \le e^{-c_1 k}. \tag{3.1}$$

Let Y_M be the number of clusters with at least M vertices. Then by revealing clusters sequentially, e.g., beginning with revealing $\mathcal{C}_x$, then $\mathcal{C}_y$ for some $y \notin \mathcal{C}_x$ and so on, we see from (3.1), that $Y_M \preceq \mathrm{Bin}(n, e^{-c_1 M})$. Thus, by Azuma–Hoeffding inequality,

$$\pi_{n,p}\big(Y_M \ge n e^{-c_1 M} + t\big) \le e^{-t^2/(2n)}.$$

Now let

$$K = \big(e^{-c_1 M} + M^{-1}\big)n;$$

plugging into the Azuma–Hoeffding bound, we obtain

$$\pi_{n,p}\big(|S_M| \ge \rho n\big) \le \pi_{n,p}\left(\sum_{i=1}^{K} \mathscr{L}_i \ge \rho n\right) + e^{-n/(2M^2)}. \tag{3.2}$$

In order to bound the right-hand side above, fix vertices $x_1, \dots, x_K$; by (3.1), the law of $|\bigcup_{i=1}^{K} \mathcal{C}_{x_i}|$ is dominated by the sum of K i.i.d. random variables $Z_1, \dots, Z_K$, where, for some $a(\lambda), b(\lambda), \nu(\lambda) > 0$ (independent of M and n), Z_1 is sub-exponential with parameters (ν, b) and mean a. This domination is evident if one exposes the clusters $\mathcal{C}_{x_i}$ and sums $|\mathcal{C}_{x_i}|$ sequentially, noting that if $x_j \in \mathcal{C}_{x_i}$ for some $i < j$, the additional contribution to the sum is zero, and otherwise the exponential tail of (3.1) applies.

By the definition of K, for any sufficiently large M (depending on ρ), $K\mathbb{E}[Z_i] = Ka \le \rho n/2$. By a union bound and symmetry, we have

$$\begin{aligned}\pi_{n,p}\left(\sum_{i=1}^{K} \mathscr{L}_i \ge \rho n\right) &\le \binom{n}{K} \pi_{n,p}\left(\left|\bigcup_{i=1}^{K} \mathcal{C}_{x_i}\right| \ge \rho n\right) \\ &\le \left(\frac{en}{K}\right)^{K} \pi_{n,p}\left(\sum_{i=1}^{K} Z_i \ge K\mathbb{E}[Z_i] + \rho n/2\right).\end{aligned}$$

Moreover, $\sum_{i=1}^{K} Z_i$ is also sub-exponential with parameters $(K\nu, b)$. Therefore, there exists $c_2(\lambda) > 0$ so that for all $\rho > 0$, there exists $M_0(\lambda, \rho)$ such that for all $M \ge M_0$,

$$\pi_{n,p}\left(\sum_{i=1}^{K} \mathscr{L}_i \ge \rho n\right) \le \left(\frac{e}{e^{-c_1 M} + M^{-1}}\right)^{(e^{-c_1 M} + M^{-1})n} e^{-\frac{\rho n}{4b}} \lesssim e^{-c_2 \rho n}.$$

Plugging this bound in to (3.2) concludes the proof. □

In the coloring stage of the Swendsen–Wang dynamics, the following simple application of a Chernoff–Hoeffding inequality proves useful.

Lemma 3.2. *Consider an FK realization ω on n vertices and suppose $|S_M(\omega)| \le \varepsilon n$ for some $M > 0$. Independently color each cluster of ω collectively red with probability $r \in [0,1]$, and let R be the set of all red vertices. For all $\delta > 0$,*

$$\mathbb{P}\big(\big||R| - rn\big| \ge (\varepsilon + \delta)n\big) \le 2\exp\left(-\frac{\delta^2 n}{2M}\right).$$

Proof. We consider $\mathbb{P}(|R| \ge (r+\varepsilon+\delta)n)$ and $\mathbb{P}(|R| \le (r-\varepsilon-\delta)n)$ separately. To bound the former, it suffices to prove an upper bound on

$$\begin{aligned}\mathbb{P}\big(|R \cup S_M| \ge (r+\varepsilon+\delta)n\big) &= \mathbb{P}\big(|R - S_M| \ge (r+\varepsilon+\delta)n - |S_M|\big)\\ &\le \mathbb{P}\big(|R - S_M| \ge (r+\delta)n\big),\end{aligned}$$

which by Hoeffding's inequality satisfies

$$\mathbb{P}\big(|R - S_M| - r\big(n - |S_M|\big) \ge \delta n + r|S_M|\big) \le e^{-\frac{(\delta n + r|S_M|)^2}{2M(n-|S_M|)}} \le e^{-\delta^2 n/(2M)}.$$

Similarly bounding $\mathbb{P}(|R| \le (r-\varepsilon-\delta)n) \le \mathbb{P}(|R - S_M| \le (r-\delta)n)$ by Hoeffding's inequality and combining the two via a union bound concludes the proof. □

We prove Theorem 1 for $q > 2$ separately for λ that is below, above and at λ_c.

3.1. *The supercritical regime: Proof of Theorem 1 for $np = \lambda \in (\lambda_c, \lambda_S)$*

To prove Theorem 1 for $\lambda \in (\lambda_c, \lambda_S)$, let $\rho > 0$, and define the set of configurations,

$$A_\rho = \left\{\sigma \in \{1,\dots,q\}^n : \max_{r=1,\dots,q}\left|\sum_{i=1}^n \mathbf{1}\{\sigma_i = r\} - \frac{n}{q}\right| < \rho n\right\}.$$

Now consider the Markov chain $(X_t)_{t\ge 0}$ given by the Swendsen–Wang dynamics and let $v_t = (v_t^1,\dots,v_t^q)$ be the corresponding vector counting the number of sites in each state in X_t. We need the following claim.

Claim 3.3. *Consider Swendsen–Wang dynamics with $p = \lambda/n = 1 - e^{-\beta/n}$ for $\lambda < \lambda_S$; there exists $\rho_0(\lambda,q)$, $c(\rho,\lambda,q), C(\lambda,q) > 0$ such that that for every $\rho < \rho_0$*

$$\max_{X_0 \in A_\rho} \mathbb{P}_{X_0}(X_1 \notin A_\rho) \le Ce^{-cn}. \tag{3.3}$$

Proof. Consider a fixed $X_0 \in A_\rho$. In the $\mathcal{G}(n,p)$ step of the Swendsen–Wang dynamics, we consider the color components separately. For each of the q colored components a new edge configuration is sampled according to $\mathcal{G}(v_0^i, \lambda/n)$ where $i = 1,\dots,q$; call the edge configuration we obtain ω_1^i and note that by definition of Swendsen–Wang dynamics, the clusters of $\{\omega_1^i\}_{i=1}^q$ will all be disconnected. Then since $\|v_0 - (\frac{n}{q},\dots,\frac{n}{q})\|_\infty < \rho n$ and $\lambda < \lambda_S = q$, if $\rho < \lambda^{-1} - q^{-1} =: \rho_0$, every colored component is sub-critical in the $\mathcal{G}(n,p)$ step. Thus, for all $i = 1,\dots,q$, by Lemma 3.1, for some $c(\lambda) > 0$, if $\rho < \lambda^{-1} - q^{-1}$, for every $M \ge M_0(\lambda,\rho)$ and every $\delta > 0$,

$$\mathbb{P}_{X_0}\big(\big|S_M\big(\omega_1^i\big)\big| \ge \delta n\big) = \pi_{v_0^i,p}\big(|S_M| \ge \delta n\big) \lesssim e^{-c\delta n}.$$

Union bounding over the q different such components, we obtain

$$\mathbb{P}_{X_0}\left(\bigcup_{i=1}^q \big\{\big|S_M\big(\omega_1^i\big)\big| \ge \delta n\big\}\right) \lesssim e^{-c\delta n}.$$

In that case, if $\delta = \frac{\rho}{2q}$ and ω_1 is the edge configuration induced on the whole graph after the $\mathcal{G}(n,p)$ step of the dynamics, there exists $c(\lambda, q) > 0$ so that for $M \geq M_0(\lambda, \rho)$,

$$\mathbb{P}_{X_0}\left(\left|S_M(\omega_1)\right| \geq \frac{\rho n}{2}\right) \lesssim e^{-c\rho n}.$$

We can then split up

$$\mathbb{P}_{X_0}(X_1 \notin A_\rho) \leq \mathbb{P}_{X_0}\left(\left|S_M(\omega_1)\right| \geq \frac{\rho n}{2}\right) + \mathbb{P}_{X_0}\left(X_1 \notin A_\rho \;\middle|\; \left|S_M(\omega_1)\right| < \frac{\rho n}{2}\right),$$

and consider the coloring step of the Swendsen–Wang dynamics. Then we obtain

$$\mathbb{P}_{X_0}\left(X_1 \notin A_\rho \;\middle|\; \left|S_M(\omega_1)\right| < \frac{\rho n}{2}\right) \leq \mathbb{P}_{X_0}\left(\bigcup_{i=1}^{q}\left\{\left|v_1^i - \frac{n}{q}\right| \geq \rho n\right\} \;\middle|\; \left|S_M(\omega_1)\right| < \frac{\rho n}{2}\right).$$

By an application of Lemma 3.2 with $\varepsilon = \delta = \frac{\rho}{2}$ and a union bound, the above is, for every $\rho < \lambda^{-1} - q^{-1}$ and $M \geq M_0(\lambda, \rho)$, bounded above by

$$2q \exp\bigl(-\rho^2 n/(8M)\bigr).$$

Since all the above estimates were uniform in $X_0 \in A_\rho$, we obtain the desired. □

By Corollary 2.2, since β is such that $np = \lambda > \lambda_c$, for every small $\rho > 0$, we have $\mu_{n,p,q}(A_\rho^c) > \frac{1}{2}$. If X_0 is such that $v_0 = (\frac{n}{q}, \ldots, \frac{n}{q})$, clearly $X_0 \in A_\rho$, and by Claim 3.3 and a union bound, since $\lambda < \lambda_S$, there exists $c(\rho, \lambda, q) > 0$ such that for every $\rho < \rho_0$,

$$\mathbb{P}_{X_0}\left(\bigcup_{t \leq e^{cn/2}}\left\{X_t \in A_\rho^c\right\}\right) \lesssim e^{-cn/2}.$$

The definition of total variation mixing time then implies $t_{\text{MIX}} \geq e^{cn/2}$ as desired.

3.2. *The subcritical regime*: *Proof of Theorem 1 for $np = \lambda \in (\lambda_s, \lambda_c)$*

We first prove the following consequence of Lemma 3.1.

Lemma 3.4. *Consider $\mathcal{G}(n,p)$ with $np = \lambda > 1$. There exist $c(\lambda), c'(\lambda) > 0$ such that for every $\rho > 0$ and $\varepsilon > 0$ sufficiently small and for every $M \geq M_0(\lambda, \rho)$, we have*

$$\pi_{n,p}\bigl(\{|\mathscr{L}_1 - n\theta_\lambda| \geq \varepsilon n)\} \cup \{|S_M - \mathcal{C}_1| \geq \rho n\}\bigr) \lesssim e^{-c\rho n} + e^{-c'\varepsilon^2 n}.$$

Proof. By a union bound, rewrite the left-hand side above as

$$\begin{aligned}&\pi_{n,p}\bigl(\{|\mathscr{L}_1 - n\theta_\lambda| \geq \varepsilon n\} \cup \{|S_M - \mathcal{C}_1| \geq \rho n\}\bigr)\\ &\quad\leq \pi_{n,p}\bigl(|S_M - \mathcal{C}_1| \geq \rho n \mid |\mathscr{L}_1 - n\theta_\lambda| < \varepsilon n\bigr) + \pi_{n,p}\bigl(|\mathscr{L}_1 - n\theta_\lambda| \geq \varepsilon n\bigr).\end{aligned}$$

Since $\lambda > 1$, by Proposition 2.4, we have that $\pi_{n,p}(|\mathscr{L}_1 - \theta_\lambda| \geq \varepsilon n) \leq e^{-c\varepsilon^2 n}$ for some $c(\lambda) > 0$. We now suppose that $\mathscr{L}_1 \geq (\theta_\lambda - \varepsilon)n$ and appeal to a precise form of the *discrete duality principle* (see, e.g.,[2, §6.3] and [16, §5.6]). Observe that conditioning on $\mathcal{C}_1$ (if there are multiple largest clusters of the same size, pick the one with the smallest vertex label), the remaining graph is distributed as $\mathcal{G}(n - \mathscr{L}_1, p)$ conditional on the event that its largest component has size at most $\mathscr{L}_1$ and has no component of size exactly $\mathscr{L}_1$ with smaller vertex label than $\mathcal{C}_1$.

Since $n - \mathscr{L}_1 \leq (1 - \theta_\lambda + \varepsilon)n$, the random graph $\mathcal{G}(n - \mathscr{L}_1, p)$ is subcritical for all small ε (this is the essence of the duality principle for branching processes; we again refer the reader to [2, §6.3] and [16, §5.6] for further details on this). Thus the probability of it having a cluster of size at least $(\theta_\lambda - \varepsilon)n \leq \mathscr{L}_1$ is at most $ne^{-c(\theta_\lambda - \varepsilon)n}$ (see Fact 2.3). The conditioning on the largest cluster size of $\mathcal{G}(n - \mathscr{L}_1, p)$ is negligible, and it suffices to compute probabilities under

$\mathcal{G}(n-\mathscr{L}_1, p)$. In that case, given $n-\mathscr{L}_1 \le (1-\theta_\lambda+\varepsilon)n$ and therefore subcriticality of $\mathcal{G}(n-\mathscr{L}_1, p)$, by Lemma 3.1, there exists $c(\lambda)>0$ such that for every $\rho>0$, there exists $M_0(\lambda,\rho)>0$ so that for $M \ge M_0$, we have

$$\pi_{n-\mathscr{L}_1,p}\big(|S_M| \ge \rho n\big) \le e^{-c\rho n};$$

combined with the union bound, this implies the desired. □

The proof of Theorem 1 for $\lambda \in (\lambda_s, \lambda_c)$ is a slight modification of the proof for $\lambda \in (\lambda_c, \lambda_S)$. Recall the definitions of θ_λ, a_λ and b_λ from §2. Fix $\lambda > \lambda_s$. In decreasing order, let the number of vertices in each color class of σ be $v^1, \dots, v^q$ and let

$$A'_\rho = \left\{\sigma \in \{1,\dots,q\}^n : \left|v^1 - a_\lambda n\right| \le \rho n, v^2 \le \frac{n-v^1}{q-1} + \rho n\right\}.$$

By Corollary 2.2, since $\lambda < \lambda_c$, for sufficiently small ρ, we have $\mu_{n,p,q}(A'_\rho) \le \frac{1}{2}$. Therefore, it suffices by definition of total variation mixing to prove the following.

Claim 3.5. *Consider Swendsen–Wang dynamics with $p = \lambda/n = (1-e^{-\beta/n})$ for $\lambda > \lambda_s$; there exist $\rho_0(\lambda, q)$, $c(\rho,\lambda,q), C(\lambda,q) > 0$ such that for every $\rho < \rho_0$,*

$$\max_{X_0 \in A_\rho} \mathbb{P}_{X_0}\big(X_1 \notin A'_\rho\big) \lesssim Ce^{-cn}; \tag{3.4}$$

Proof. Fix any $X_0 \in A'_\rho$ and let $(v_0^1, \dots, v_0^q)$ be its corresponding color class vector. By definition of a_λ, for some $\rho'(\lambda,q) > 0$ there exists $\gamma \in (F'(a_\lambda), 1)$ such that if $|v_0^1 - a_\lambda n| \le \rho' n$, we have $|F(v_0^1/n) - a_\lambda| < \gamma |v_0^1/n - a_\lambda|$. From now on we take $\rho < \rho'$.

Consider the $\mathcal{G}(n,p)$ step of the Swendsen–Wang dynamics. Since $\lambda > \lambda_s$, $\lambda a_\lambda > 1$ and $\lambda b_\lambda < 1$, so that for $\rho > 0$ sufficiently small, the first colored class of X_0 will be supercritical in the $\mathcal{G}(n,p)$ step and the other $q-1$ will all be subcritical; call the q random graph configurations we obtain in this step ω_1^i for $i = 1, \dots, q$. Now fix such a $\rho > 0$ and let $\varepsilon = \frac{(1-\gamma)\rho}{2(q+1)}$. By Proposition 2.4, we obtain that for some $c(\lambda) > 0$,

$$\mathbb{P}_{X_0}\big(\big|\mathscr{L}_1(\omega_1^1) - v_0^1 \theta_{\lambda v_0^1/n}\big| \ge \varepsilon n\big) \lesssim e^{-c\varepsilon^2 n}.$$

Moreover, by Lemma 3.4, we also have for some $c(\lambda) > 0$, for every $M \ge M_0(\lambda, \varepsilon)$,

$$\mathbb{P}_{X_0}\left(\big\{\big|\mathscr{L}_1(\omega_1^1) - v_0^1\theta_{\lambda v_0^1/n}\big| \ge \varepsilon n\big\} \cup \bigcup_{i=1}^{q} \big\{\big|S_M(\omega_1^i) - \mathcal{C}_1(\omega_1^1)\big| \ge \varepsilon n\big\}\right) \lesssim e^{-c\varepsilon n}.$$

On the complement of the above event, ω^1 has a single giant component of size θn for $\theta n \in (v_0^1 \theta_{\lambda v_0^1/n} - \varepsilon n, v_0^1 \theta_{\lambda v_0^1/n} + \varepsilon n)$, and $|S_M - \mathcal{C}_1| \le q\varepsilon n$. By Lemma 3.2, with probability $1 - e^{-c\theta n}$, the largest color class of X_1 will be the one containing $\mathcal{C}_1(\omega_1^1)$ so without loss of generality, we also assume that is the case.

At that stage, observe that $\mathbb{E}[v_1^1 \mid \theta] = \theta n + \frac{1}{q}(1-\theta)n$ and $\mathbb{E}[v_1^i \mid \theta] = \frac{1}{q}(1-\theta)n$ for $i \ne 1$. Then, we can see that for some $c(M,\lambda) > 0$, for every $M \ge M_0(\lambda,\varepsilon)$,

$$\mathbb{P}_{X_0}\big(\big|v_1^1 - nF\big(v_0^1/n\big)\big| \ge q\varepsilon n + \varepsilon n + \delta n\big) \lesssim e^{-c\delta^2 n} + e^{-c\varepsilon n}.$$

In particular, this follows from a union bound over the aforementioned events that

(1) the size of the giant component $|\mathcal{C}_1|$ is within εn of $v_0^1 \theta_{\lambda v_0^1/n}$ and $|S_M - \mathcal{C}_1| \le q\varepsilon n$,
(2) the largest color class of X_1 contains $\mathcal{C}_1(\omega_1^1)$,
(3) the fluctuations of the number of vertices outside $\mathcal{C}_1$, sharing the color of $\mathcal{C}_1$, are at most δn (this probability is bounded by Lemma 3.2).

By a similar bound on the other $q-1$ coloring steps, the choice $\delta = (1-\gamma)\rho/2$, and the relationship between ε and ρ, there exists $c(q, M, \lambda, \gamma) > 0$ such that for small ρ,

$$\mathbb{P}_{X_0}\left(\left\|\big(v_1^1, \dots v_1^q\big) - n\left(F\big(v_0^1/n\big), \frac{1-F(v_0^1/n)}{q-1}, \dots, \frac{1-F(v_0^1/n)}{q-1}\right)\right\|_\infty \ge (1-\gamma)\rho n\right) \lesssim e^{-c\varepsilon^2 n}.$$

By the choice of γ and the triangle inequality, this implies

$$\mathbb{P}_{X_0}\big(X_1 \notin A'_\rho\big) \le \mathbb{P}_{X_0}\big(\big\|\big(v_1^1,\dots,v_1^q\big) - (a_\lambda n, b_\lambda n,\dots,b_\lambda n)\big\|_\infty \ge \rho n\big) \lesssim e^{-c\varepsilon^2 n},$$

which by uniformity of the estimates over $X_0 \in A'_\rho$, concludes the proof. □

3.3. *The critical point: Proof of Theorem 1 for $np = \lambda = \lambda_c$*

In Corollary 2.2, for every $q > 2$, either $\gamma(q) \ge \frac{1}{2}$ in which case Claim 3.5 concludes the proof, or $1 - \gamma(q) \ge \frac{1}{2}$ in which case Claim 3.3 concludes the proof.

4. Slow mixing of Glauber dynamics for the FK model

Since for q noninteger, Chayes–Machta dynamics activates a strict subset of the vertices at a time, we will need to use a modified argument to prove Theorem 2. We instead construct a bottleneck set S and bound its bottleneck ratio.

We first recall the important relationship between bottlenecks and the spectral gap. For a Markov chain with stationary distribution π and kernel P on state space Ω, for any $A, B \subset \Omega$, define the *edge measure*

$$Q(A,B) = \sum_{\omega\in A} \pi(\omega)P(\omega,B) = \sum_{\omega\in A}\pi(\omega)\sum_{\omega'\in B} P\big(\omega,\omega'\big),$$

Then, the *Cheeger constant* of Ω is given by

$$\Phi = \min_{S\subset\Omega} \frac{Q(S,S^c)}{\pi(S)\pi(S^c)}, \quad \text{and satisfies} \quad 2\Phi \ge \mathtt{gap} \ge \Phi^2/2. \tag{4.1}$$

In order to prove the lower bound of Theorem 2, we prove such a lower bound on the inverse spectral gap of the Chayes–Machta dynamics, then using Proposition 2.7 and a standard comparison between the spectral gap and mixing time (2.1), we obtain the desired for the Glauber dynamics. Before the proof of Theorem 2, we prove some preliminary equilibrium bottleneck estimates for the mean-field FK model.

The following lemma that was fundamental to the understanding of the distribution $\pi_{n,p,q}$ in [3] is very useful for the proof of Theorem 2.

Lemma 4.1 ([3, Lemma 3.1]). *Fix $r \in [0,1]$; consider a mean-field FK realization $\omega \sim \pi_{n,p,q}$. Independently color each cluster of ω red with probability r and let R be the collection of all red vertices. Conditional on R, the subgraph $\omega\restriction_R$ is distributed as $\mathcal{G}(|R|, p, rq)$ and the subgraph $\omega\restriction_{V-R}$ is distributed as $\mathcal{G}(|V-R|, p, (1-r)q)$.*

The following corollary follows from iterating the process of Lemma 4.1 $\lfloor q\rfloor$ times.

Corollary 4.2. *Consider a mean-field FK realization $\omega \sim \pi_{n,p,q}$. Independently color each cluster of ω color $r_1,\dots,r_{\lfloor q\rfloor}$ with probability q^{-1} each and r_0 otherwise. Then letting $R_0, R_1,\dots,R_{\lfloor q\rfloor}$ be the sets of vertices colored each of $r_0,\dots,r_{\lfloor q\rfloor}$, the subgraph restricted to R_i for $i = 1,\dots,\lfloor q\rfloor$ is distributed as $\mathcal{G}(|R_i|, p)$. The subgraph restricted to $R_0 := V - \bigcup_{i=1}^{\lfloor q\rfloor} R_i$ is distributed according to $\mathcal{G}(|R_0|, p, q-\lfloor q\rfloor)$. Moreover, the distributions of the $\lceil q\rceil$ color classes are (conditionally on $R_0,\dots,R_{\lfloor q\rfloor}$) independent.*

Proof. Begin by independently coloring clusters of ω color r_1 with probability q^{-1}. By Lemma 4.1, conditional on R_1, the subgraph $\omega\restriction_{R_1}$ is distributed as $\mathcal{G}(|R_1|, p)$ and the subgraph $\omega\restriction_{V-R_1}$ is conditionally independent and distributed as $\mathcal{G}(|V-R_1|, p, q-1)$. Now on $V - R_1$, the distributions of the colors is $r_2,\dots,r_{\lfloor q\rfloor}$ with probability $(q-1)^{-1}$ and r_0 otherwise. Coloring vertices in $V - R_1$ color r_2 with probability $(q-1)^{-1}$, we see that conditional on R_2 the subgraph $\omega\restriction_{R_2}$ is conditionally independent of $\omega\restriction_{R_1}$ and distributed as $\mathcal{G}(|R_2|, p)$; moreover, the remainder is conditionally independent and distributed as $\mathcal{G}(|V-R_1-R_2|, p, q-2)$ since $(1-(q-1)^{-1})(q-1) = q-2$.

Repeating $\lfloor q\rfloor$ times we obtain $\lfloor q\rfloor$ conditionally independent subgraphs $\omega\restriction_{R_i}$ distributed as $\mathcal{G}(|R_i|, p)$ and a remaining subgraph $\omega\restriction_{R_0}$ also distributed as desired. □

(Note that when q is an integer, the set R_0 is deterministically empty.) Via Lemma 4.1, we prove the following analogues of Lemmas 3.1 and 3.4 when $q < 1$.

Lemma 4.3. *Consider the mean-field FK model on n vertices with parameters (p,q) with $q<1$ and $np=\lambda<\lambda_c=q$. There exists $c(\lambda,q)>0$ such that for all $\rho>0$ sufficiently small, there exists $M_0(\lambda,\rho)>0$ such that for all $M\ge M_0$,*

$$\pi_{n,p,q}\big(|S_M|\ge\rho n\big)\lesssim e^{-c\rho n}.$$

Proof. We prove the desired using Lemma 4.1. Consider the random graph $\mathcal{G}(m,p)$ with the choice of $m=\lceil q^{-1}n\rceil$; applying Lemma 4.1 to $\mathcal{G}(m,p)$ with $r=q$, by [3, Lemma 9.1], for all $\lambda\neq q$, we have $\mathbb{P}(|R|=n)\ge\frac{C}{\sqrt{m}}$, for some $C(\lambda)>0$. Then, we can write for any event $A\subset\Omega_{\mathrm{RC}}$,

$$\sum_{l=1}^{m}\mathbb{P}_{\mathrm{col},m,\lambda}\big(|R|=l\big)\pi_{m,p}\big(\omega\restriction_R\in A\mid R,|R|=l\big)=\mathbb{E}_{\mathrm{col},m,\lambda}\big[\pi_{m,p}(\omega\restriction_R\in A\mid R)\big],\tag{4.2}$$

where $\mathbb{P}_{\mathrm{col},m,\lambda}$ is the distribution over colorings of ω, averaged over realizations of $\omega\sim\pi_{m,p}$. Letting $A=A_{\rho,M}=\{|S_M|\ge\rho n\}$, for every R the probability on the right-hand side is bounded above by $\pi_{m,p}(A_{\rho,M})$ which, by Lemma 3.1, satisfies

$$\pi_{m,p}(A_{\rho,M})\lesssim e^{-c\rho n},$$

for some $c(\lambda)>0$ and for every $\rho>0$ and every $M\ge M_0(\lambda,\rho)$. But by Lemma 4.1,

$$\pi_{m,p}\big(\omega\restriction_R\in\cdot\mid R,|R|=l\big)\overset{d}{=}\pi_{l,p,q}(\omega\in\cdot),$$

which combined with $\mathbb{P}_{\mathrm{col},m,\lambda}(|R|=n)\ge C/\sqrt{m}$ implies

$$\pi_{n,p,q}\big(|S_M|\ge\rho n\big)\lesssim\sqrt{q^{-1}n}e^{-c\rho n}.\qquad\square$$

Lemma 4.4. *Consider the mean-field FK model on n vertices with parameters (p,q) with $q<1$ and $np=\lambda>\lambda_c=q$. There exists $c(\lambda,q)>0$ such that for all $\rho>0$ sufficiently small, there exists $M_0(\lambda,\rho)>0$ such that for all $M\ge M_0$,*

$$\pi_{n,p,q}\big(|S_M-\mathcal{C}_1|\ge\rho n\big)\lesssim e^{-c\rho n}.$$

Proof. As before, consider $\mathcal{G}(m,p)$ with $m=\lceil q^{-1}n\rceil$; by Lemma 4.1 with $r=q$ and [3, Lemma 9.1], $\mathbb{P}(|R|=n)\ge C/\sqrt{m}$. Let $A=A_{\rho,M}=\{|S_M-\mathcal{C}_1|\ge\rho n\}$ in (4.2). Then observe that $\pi_{m,p}(\omega\restriction_R\in A_{\rho,M})\le\pi_{m,p}(A_{\rho,M})$ and by Lemma 3.4, $\pi_{m,p}(A_{\rho,M})\lesssim e^{-c\rho n}$. Altogether, plugging the above bounds in to (4.2) implies that there exists $c(\lambda)>0$ such that for all $\rho>0$ and all $M\ge M_0(\lambda,\rho)$,

$$\pi_{n,p,q}\big(|S_M-\mathcal{C}_1|\ge\rho n\big)\lesssim\sqrt{q^{-1}n}e^{-c\rho n}.\qquad\square$$

4.1. *The supercritical/critical regime*, $np=\lambda\in[\lambda_c,\lambda_S)$

We first prove the desired mixing time lower bound for $\lambda\in[\lambda_c,\lambda_S)$, using the following bottleneck estimate.

Lemma 4.5. *Consider the mean-field FK model on n vertices with parameters (p,q) where $q>2$ and $np=\lambda<\lambda_S$; there exists $c(\rho,M,\lambda,q)>0$ such that for all sufficiently small $\rho>0$, there exists $M_0(\lambda,\rho)$ such that for every $M\ge M_0$,*

$$\pi_{n,p,q}\left(\frac{\rho n}{2}<|S_M|<\rho n\mid|S_M|<\rho n\right)\lesssim e^{-cn}.$$

Proof. For $\rho,M>0$, define the events

$$\begin{aligned}A_{\rho,M}&=\big\{\omega\in\Omega_{\mathrm{RC}}:\big|S_M(\omega)\big|<\rho n\big\},\\B_{\rho,M}&=\left\{\omega\in\Omega_{\mathrm{RC}}:\frac{\rho n}{2}<\big|S_M(\omega)\big|<\rho n\right\}.\end{aligned}\tag{4.3}$$

In order to bound $\pi_{n,p,q}(B_{\rho,M}\mid A_{\rho,M})$, use the coloring scheme described in Corollary 4.2. Let $\mathcal{P}$ be the set of all possible partitions of $\{1,\dots,n\}$ into $\lceil q\rceil$ sets, i.e., the set of all possible colorings of FK configurations. Denote by $\mathbb{P}_{\mathrm{col}}$

the probability measure over colorings $(R_0, \dots, R_{\lfloor q \rfloor})$ averaged over $\pi_{n,p,q}$, and $\mathbb{P}_{\mathrm{col}}(\cdot \mid \mathcal{F})$ the probability measure over such colorings, averaged over $\pi_{n,p,q}(\cdot \mid \mathcal{F})$. For every $\mathbf{R} \in \mathcal{P}$,

$$\pi_{\mathbf{R}} = \begin{cases} \pi_{|R_i|,p,1} & \text{on } R_i \text{ for } i = 1, \dots, \lfloor q \rfloor \\ \pi_{|R_0|,p,q-\lfloor q \rfloor} & \text{on } V - \bigcup_{i=1}^{\lfloor q \rfloor} R_i =: R_0. \end{cases}$$

Then we can write, by Corollary 4.2,

$$\pi_{n,p,q}(B_{\rho,M} \mid A_{\rho,M}) = \sum_{\mathbf{R} \in \mathcal{P}} \mathbb{P}_{\mathrm{col}}(\mathbf{R} \mid A_{\rho,M}) \pi_{\mathbf{R}}(B_{\rho,M} \mid A_{\rho,M}).$$

By Lemma 3.2, since $A_{\rho,M}$ implies $|S_M| \le \rho n$, for every $i = 1, \dots, \lfloor q \rfloor$,

$$\mathbb{P}_{\mathrm{col}}\left(\left| |R_i| - \frac{n}{q} \right| \ge 2\rho n \;\middle|\; A_{\rho,M} \right) \le 2e^{-\rho^2 n/(2M)}.$$

If $||R_i| - \frac{n}{q}| < 2\rho n$ for all $i = 1, \dots, \lfloor q \rfloor$, we are left with a remainder set satisfying

$$|R_0| \in \left(\left(1 - \frac{\lfloor q \rfloor}{q} - 2\rho \lfloor q \rfloor\right) n, \left(1 - \frac{\lfloor q \rfloor}{q} + 2\rho \lfloor q \rfloor\right) n \right).$$

Define the event Γ_ρ over colorings of the mean-field FK model as

$$\Gamma_\rho = \left\{ \mathbf{R} \in \mathcal{P} : \left| |R_i| - \frac{n}{q} \right| < 2\rho n \text{ for all } i = 1, \dots, \lfloor q \rfloor \right\},$$

so that the above conclusion can be written as

$$\mathbb{P}_{\mathrm{col}}\left(\Gamma_\rho^c \mid A_{\rho,M}\right) \lesssim \lfloor q \rfloor e^{-\rho^2 n/(2M)}.$$

Combined with the expression for $\pi_{n,p,q}(B_{\rho,M} \mid A_{\rho,M})$, this implies that

$$\begin{aligned} \pi_{n,p,q}(B_{\rho,M} \mid A_{\rho,M}) &\le \max_{\mathbf{R} \in \Gamma_\rho} \frac{\pi_{\mathbf{R}}(B_{\rho,M})}{\pi_{\mathbf{R}}(A_{\rho,M})} + \mathbb{P}_{\mathrm{col}}\left(\Gamma_\rho^c \mid A_{\rho,M}\right) \\ &\lesssim \max_{\mathbf{R} \in \Gamma_\rho} \frac{\pi_{\mathbf{R}}(|S_M| \ge \rho n/2)}{1 - \pi_{\mathbf{R}}(|S_M| \ge \rho n)} + e^{-\rho^2 n/(2M)}. \end{aligned}$$

By a union bound, the first term on the right-hand side is bounded above by

$$\max_{\mathbf{R} \in \Gamma_\rho} \frac{\pi_{|R_0|,p,q-\lfloor q \rfloor}(|S_M| \ge \frac{\rho n}{2\lceil q \rceil}) + \sum_{i=1,\dots,\lfloor q \rfloor} \pi_{|R_i|,p,1}(|S_M| \ge \frac{\rho n}{2\lceil q \rceil})}{1 - \pi_{|R_0|,p,q-\lfloor q \rfloor}(|S_M| \ge \frac{\rho n}{\lceil q \rceil}) - \sum_{i=1,\dots,\lfloor q \rfloor} \pi_{|R_i|,p,1}(|S_M| \ge \frac{\rho n}{\lceil q \rceil})}. \tag{4.4}$$

We lower bound the numerator and upper bound the denominator simultaneously as they entail similar estimates.

Since $\lambda < \lambda_S = q$, there exists $\rho_0(\lambda, q)$ such that for all $\rho < \rho_0$, the random graph $\mathcal{G}(\frac{n}{q} + 2\lfloor q \rfloor \rho n, p)$ is subcritical and the FK model $\mathcal{G}((1 - \frac{\lfloor q \rfloor}{q} + 2\rho \lfloor q \rfloor)n, p, q - \lfloor q \rfloor)$ is also subcritical. In other words, if $\rho < \rho_0(\lambda, q)$, for every $\mathbf{R} \in \Gamma_\rho$, the distributions $\pi_{|R_i|,p}$ for $i = 1, \dots, \lfloor q \rfloor$ and $\pi_{|R_0|,p,q-\lfloor q \rfloor}$ are all subcritical. As such, by Lemma 3.1, there exists $c(\lambda, q) > 0$ such that for every $M \ge M_0(\lambda, \rho)$,

$$\max_{\mathbf{R} \in \Gamma_\rho} \sum_{i=1,\dots,\lfloor q \rfloor} \pi_{|R_i|,p,1}\left(|S_M| \ge \frac{\rho n}{2\lceil q \rceil}\right) \lesssim e^{-c\rho n/2}, \quad \text{and}$$

$$\max_{\mathbf{R} \in \Gamma_\rho} \sum_{i=1,\dots,\lfloor q \rfloor} \pi_{|R_i|,p,1}\left(|S_M| \ge \frac{\rho n}{\lceil q \rceil}\right) \lesssim e^{-c\rho n}.$$

Similar bounds under $\pi_{|R_0|,p,q-\lfloor q \rfloor}$ follow immediately for a different $c(\lambda, q) > 0$ from Lemma 4.3. Altogether, this implies that for every $\rho < \rho_0$ and every $M \ge M_0(\lambda, \rho)$, there exists $c(\rho, M, \lambda, q) > 0$ such that

$$\pi_{n,p,q}(B_{\rho,M} \mid A_{\rho,M}) \lesssim e^{-cn}. \qquad \square$$

Proof of Theorem 2: the case $np = \lambda \in [\lambda_c, \lambda_S)$. For $\rho, M > 0$, let $A_{\rho,M}$ and $B_{\rho,M}$ be as in (4.3). By Proposition 2.1, for $\lambda \in [\lambda_c, \lambda_S)$, for sufficiently small $\rho > 0$ and large M, there exists $c(\lambda, q) > 0$ such that $\pi_{n,p,q}(A^c_{\rho,M}) \ge c$. Then by (4.1) it suffices to prove an exponentially decaying upper bound on

$$\frac{Q(A_{\rho,M}, A^c_{\rho,M})}{\pi_{n,p,q}(A_{\rho,M})} \lesssim \max_{X_0 \in A_{\rho,M} - B_{\rho,M}} P\big(X_0, A^c_{\rho,M}\big) + \pi_{n,p,q}(B_{\rho,M} \mid A_{\rho,M}), \tag{4.5}$$

where P, Q are the transition matrix and edge measure, respectively, of the Chayes–Machta dynamics. We first bound the first term in the right-hand side of (4.5).

Consider some $X_0 \in A_{\rho,M} - B_{\rho,M}$. In the activation stage of the Chayes–Machta dynamics, clusters are activated with probability $\frac{1}{q}$; denote by $\mathcal{A}_1$ the set of activated vertices in this stage of the dynamics. Since $X_0 \in A_{\rho,M} - B_{\rho,M}$, by Lemma 3.2 with the choice of $\varepsilon = \delta = \rho/2$,

$$\mathbb{P}_{X_0}\left(\left||\mathcal{A}_1| - \frac{n}{q}\right| \ge \rho n\right) \le 2e^{-\rho^2 n/(8M)}.$$

Since $\lambda < \lambda_S = q$, for $\rho < \lambda^{-1} - q^{-1}$, the random graph $\mathcal{G}((\frac{1}{q} + \rho)n, p)$ is subcritical. In that case, by Lemma 3.1, there exists $c(\lambda, \rho) > 0$ such that for every $M \ge M_0(\lambda, \rho)$,

$$\begin{aligned}\mathbb{P}_{X_0}\left(X_1 \notin A_{\rho,M} \;\middle|\; \left||\mathcal{A}_1| - \frac{n}{q}\right| < \rho n\right) &\le \mathbb{P}_{X_0}\left(\left|S_M(X_1\restriction_{\mathcal{A}_1})\right| \ge \frac{\rho n}{2} \;\middle|\; \left||\mathcal{A}_1| - \frac{n}{q}\right| < \rho n\right) \\ &\lesssim e^{-c\rho n/2},\end{aligned}$$

Union bounding over the event $||\mathcal{A}_1| - n/q| \ge \rho n$ and its complement, there exists $c(\rho, M, \lambda, q) > 0$ such that for every $\rho < \lambda^{-1} - q^{-1}$, for every $M \ge M_0(\lambda, \rho)$,

$$\max_{X_0 \in A_{\rho,M} - B_{\rho,M}} P\big(X_0, A^c_{\rho,M}\big) \lesssim e^{-cn}.$$

Lemma 4.5 yields a similar exponentially decaying upper bound on the second term on the right-hand side of (4.5), concluding the proof. □

4.2. *The subcritical/critical regime*, $np = \lambda \in (\lambda_s, \lambda_c]$

Recall the definitions of $\Theta^*(\lambda, q)$ and $\Theta_r(\lambda, q)$ corresponding to the drift function g from Proposition 2.6. When $\lambda \in (\lambda_s, \lambda_c]$, we will need the following intermediate lemma, before proceeding to the analogue of Lemma 4.5. This is a straightforward adaptation of an argument of [1].

Lemma 4.6. *Consider the mean-field FK model on n vertices with parameters (p, q) with $np = \lambda \in (\lambda_s, \lambda_S)$; let $\omega_0 \in A_{\rho,\varepsilon,M}$ where*

$$A_{\rho,\varepsilon,M} = \big\{\omega : \mathscr{L}_1 \ge \big(\Theta^* + \varepsilon\big)n, |S_M - \mathcal{C}_1| < \rho n\big\}.$$

Color $\mathcal{C}_1$ red and independently color each cluster in $\omega_0 - \mathcal{C}_1$ red with probability $\frac{1}{q}$; let R be the set of all red vertices. Resample $\omega_0\restriction_R \sim \pi_{|R|,p,1}$ and let ω_1 be the resulting configuration on n vertices. Then there exists $c(\rho, \varepsilon, M, \lambda) > 0$ so that for sufficiently small $\rho, \varepsilon > 0$, for every $M \ge M_0(\lambda, \rho)$, uniformly in $\omega_0 \in A_{\rho,\varepsilon,M}$,

$$\mathbb{P}\big(\mathscr{L}_1(\omega_1) \le \big(\Theta^* + \varepsilon\big)n\big) \lesssim e^{-cn}.$$

Proof. Fix any $\omega_0 \in A_{\rho,\varepsilon,M}$ and let $n\theta_0 = \mathscr{L}_1(\omega_0)$ for $\theta_0 \ge \Theta^* + \varepsilon$. Then

$$\mathbb{E}\big[|R|\big] = \theta_0 n + \frac{1}{q}(1 - \theta_0)n =: \mu_0,$$

so that by Lemma 3.2, for all $\rho > 0$,

$$\mathbb{P}\big(\big||R| - \mu_0\big| \ge \rho n\big) \le 2e^{-\rho^2 n/(8M)}.$$

Therefore, we can write for every $\delta > 0$,

$$\mathbb{P}\big(\big|\mathscr{L}_1(\omega_1) - nf(\theta_0)\big| \geq \delta n\big) \\ \leq \max_{a:|a-\mu_0|\leq \rho n} \pi_{a,p}\big(\big|\mathscr{L}_1 - nf(\theta_0)\big| \geq \delta n\big) + 2e^{-\rho^2 n/(8M)}.$$

For all $\theta_0 \geq \Theta^* + \varepsilon$, for sufficiently small $\rho > 0$, using $\theta_0 > \Theta^* > \Theta_{\min}$, since $\lambda < \lambda_S$,the random graph $\mathcal{G}(\mu_0 - \rho n, p)$ is supercritical. By continuity of f, for any $\delta > 0$, there exists $\rho > 0$ sufficiently small such that $\max_{a:|a-\mu_0|\leq\rho n} |f(\theta_0) - \theta_{\lambda a/n}| < \delta$; moreover, by Proposition 2.4, for every $\delta > 0$

$$\max_{a:|a-\mu_0|\leq \rho n} \pi_{a,p}\big(|\mathscr{L}_1 - \theta_{\lambda a/n} n| \geq \delta n\big) \lesssim e^{-c\delta^2 n},$$

for some $c(\lambda, \rho) > 0$. Thus, for sufficiently small $\rho > 0$, we have, for some $c(\rho, M, \lambda) > 0$,

$$\mathbb{P}\big(\big|\mathscr{L}_1(\omega_1) - nf(\theta_0)\big| \geq 2\delta n\big) \lesssim e^{-c\delta^2 n} + e^{-\rho^2 n/(8M)}.$$

It remains to argue that for $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ such that for all $\theta_0 \geq \Theta^* + \varepsilon$, we have $nf(\theta_0) - 2\delta n \geq (\Theta^* + \varepsilon)n$. If $\theta_0 > \Theta_r - \varepsilon$, then by [1, Lemma 2.14], $f(\theta_0) \geq \Theta_r - \varepsilon > \Theta^* + \varepsilon$ and for small enough ε letting $\delta = \frac{1}{2}(\Theta_r - \Theta^* - 2\varepsilon) > 0$ yields the desired. If $\theta_0 \leq \Theta_r - \varepsilon$, since g is positive on (Θ^*, Θ_r), for ε small, $f(\theta_0) > \theta_0 \geq \Theta^* + \varepsilon$. By continuity of f, for $\varepsilon < \frac{1}{2}(\Theta_r - \Theta^*)$, letting $\delta = \frac{1}{2}\min_{[\Theta^*+\varepsilon,\Theta_r-\varepsilon]} g$, we obtain

$$f(\theta_0) - 2\delta \geq \theta_0 + g(\theta_0) - \min_{[\Theta^*+\varepsilon,\Theta_r-\varepsilon]} g \geq \theta_0 \geq \Theta^* + \varepsilon.$$

Together, for $\varepsilon > 0$ sufficiently small, there exists $c(\rho, \varepsilon, M, \lambda) > 0$ such that

$$\mathbb{P}\big(\mathscr{L}_1(\omega_1) \leq \big(\Theta^* + \varepsilon\big)n\big) \lesssim e^{-cn}. \qquad \square$$

The following is the analogue of Lemma 4.5 in the presence of a giant component.

Lemma 4.7. *Consider the mean-field FK model on n vertices with parameters (p, q) with $q > 2$ and $np = \lambda \in (\lambda_s, \lambda_S)$; for every $\rho, \varepsilon, M > 0$ let*

$$E_{\rho,\varepsilon,M} = \left\{\mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n, \frac{\rho n}{2} < |S_M - \mathcal{C}_1| < \rho n\right\}.$$

There exists $c(\rho, M, \lambda, q) > 0$ such that for sufficiently small $\rho, \varepsilon > 0$, for $M \geq M_0(\lambda, \rho)$,

$$\pi_{n,p,q}\big(E_{\rho,\varepsilon,M} \mid \mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n, |S_M - \mathcal{C}_1| < \rho n\big) \lesssim e^{-cn}.$$

Proof. Fix $np = \lambda > \lambda_s$ and for $\rho, \varepsilon, M > 0$, define the sets

$$A_{\rho,\varepsilon,M} = \big\{\mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n, |S_M - \mathcal{C}_1| < \rho n\big\},$$
$$B_{\rho,M} = \left\{\frac{\rho n}{2} < |S_M - \mathcal{C}_1| < \rho n\right\}.$$

We prove the lemma similarly to Lemma 4.5, after treating the giant component separately. Using the coloring scheme of Corollary 4.2, with $\mathbb{P}_{\mathrm{col}}$ and $\pi_{\mathbf{R}}$ defined as before, by considering the color class to which $\mathcal{C}_1$ belongs, and using symmetry, we obtain

$$\pi_{n,p,q}(E_{\rho,\varepsilon,M} \mid A_{\rho,\varepsilon,M}) = \frac{q}{\lfloor q \rfloor} \sum_{\mathbf{R}\in\mathcal{P}} \mathbb{P}_{\mathrm{col}}(\mathbf{R} \mid \mathcal{C}_1 \subset R_1, A_{\rho,\varepsilon,M}) \pi_{\mathbf{R}}(E_{\rho,\varepsilon,M} \mid \mathcal{C}_1 \subset R_1, A_{\rho,\varepsilon,M}) \\ + \frac{q}{q - \lfloor q \rfloor} \sum_{\mathbf{R}\in\mathcal{P}} \mathbb{P}_{\mathrm{col}}(\mathbf{R} \mid \mathcal{C}_1 \subset R_0, A_{\rho,\varepsilon,M}) \pi_{\mathbf{R}}(E_{\rho,\varepsilon,M} \mid \mathcal{C}_1 \subset R_0, A_{\rho,\varepsilon,M}).$$

Call the two sums on the right hand side **I** and **II** respectively and consider them separately. Conditional on $A_{\rho,\varepsilon,M}$ and $\mathcal{C}_1 \subset R_1$, if $\mu_{\mathbf{I}} = (\Theta^* + \varepsilon)n + \frac{1}{q}(1 - \Theta^* - \varepsilon)n$,

$$\mathbb{P}_{\mathrm{col}}\big(|R_1| \geq \mu_{\mathbf{I}} - 2\rho n \mid \mathcal{C}_1 \subset R_1, A_{\rho,\varepsilon,M}\big) \leq e^{-\rho^2 n/(2M)},$$

where we used Lemma 3.2 with $\varepsilon = \delta = \rho$. Following the proof of Lemma 4.5, let

$$\Gamma_\rho^{\mathbf{I}} = \left\{ \mathbf{R} : |R_1| \geq \mu_{\mathbf{I}} - 2\rho n, \left| |R_i| - \frac{n - |R_1|}{q-1} \right| < 2\rho n \text{ for all } i = 2, \ldots, \lfloor q \rfloor \right\}.$$

By Lemma 3.2 and a union bound, $\mathbb{P}_{\mathrm{col}}((\Gamma_\rho^{\mathbf{I}})^c \mid \mathcal{C}_1 \subset R_1, A_{\rho,\varepsilon,M}) \leq 2\lceil q \rceil e^{-\rho^2 n/(2M)}$.

Using the fact that for every $\varepsilon > 0$, $E_{\rho,\varepsilon,M} \subset B_{\rho,M}$, we can write

$$\begin{aligned}
\mathbf{I} &\leq \mathbb{P}_{\mathrm{col}}\big((\Gamma_\rho^{\mathbf{I}})^c \mid \mathcal{C}_1 \subset R_1, A_{\rho,\varepsilon,M}\big) + \max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \pi_{\mathbf{R}}(E_{\rho,\varepsilon,M} \mid A_{\rho,\varepsilon,M}, \mathcal{C}_1 \subset R_1) \\
&\lesssim \frac{q\lceil q \rceil}{\lfloor q \rfloor} e^{-\rho^2 n/(2M)} + \frac{q}{\lfloor q \rfloor} \max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \frac{\pi_{\mathbf{R}}(B_{\rho,M} \mid \mathscr{L}_1 \geq (\Theta^* + \varepsilon)n, \mathcal{C}_1 \subset R_1)}{\pi_{\mathbf{R}}(A_{\rho,\varepsilon,M} \mid \mathscr{L}_1 \geq (\Theta^* + \varepsilon)n, \mathcal{C}_1 \subset R_1)}.
\end{aligned}$$

If $\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}$, for sufficiently small $\rho > 0$, the definition of Θ^* and $\lambda > \lambda_s$ implies $\mathcal{G}(|R_1|, p)$ is supercritical, and both $\mathcal{G}(\frac{n-|R_1|}{q-1} + 2\rho n, p)$ and $\mathcal{G}(|R_0|, p, q - \lfloor q \rfloor)$ are subcritical. By a union bound we can expand the numerator above as at most

$$\begin{aligned}
&\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \Bigg(\pi_{|R_1|,p}\bigg(|S_M - \mathcal{C}_1| \geq \frac{\rho n}{2\lceil q \rceil} \mid \mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n \bigg) + \sum_{i=2,\ldots,\lfloor q \rfloor} \pi_{|R_i|,p}\bigg(|S_M| \geq \frac{\rho n}{2\lceil q \rceil} \bigg) \\
&\quad + \pi_{|R_0|,p,q-\lfloor q \rfloor}\bigg(|S_M| \geq \frac{\rho n}{2\lceil q \rceil} \bigg) \Bigg) + e^{-c\Theta^* n},
\end{aligned}$$

and analogously, the denominator as at least

$$\begin{aligned}
&\min_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \Bigg(1 - \pi_{|R_1|,p}\bigg(|S_M - \mathcal{C}_1| \geq \frac{\rho n}{\lceil q \rceil} \mid \mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n \bigg) - \pi_{|R_0|,p,q-\lfloor q \rfloor}\bigg(|S_M| \geq \frac{\rho n}{\lceil q \rceil} \bigg) \\
&\quad - \sum_{i=2,\ldots,\lfloor q \rfloor} \pi_{|R_i|,p}\bigg(|S_M| \geq \frac{\rho n}{\lceil q \rceil} \bigg) \Bigg) - e^{-c\Theta^* n}.
\end{aligned}$$

(In both of the above, we paid a cost of $e^{-c\Theta^* n}$ for the assumption $\mathscr{L}_1(\omega) = \mathscr{L}_1(\omega \restriction_{R_1})$.) By Lemma 3.4, (for every $\mathscr{L}_1 \geq (\Theta^* + \varepsilon)n$ and $\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}$, $\mathcal{G}(|R_1| - \mathscr{L}_1, p)$ is subcritical) there exists $c(\lambda, q) > 0$ such that for sufficiently small $\rho, \varepsilon > 0$ and every $M \geq M_0(\lambda, \rho)$,

$$\begin{aligned}
&\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \pi_{|R_1|,p,1}\bigg(|S_M - \mathcal{C}_1| \geq \frac{\rho n}{2\lceil q \rceil} \mid \mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n \bigg) \lesssim e^{-c\rho n/2}, \quad \text{and} \\
&\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \pi_{|R_1|,p,1}\bigg(|S_M - \mathcal{C}_1| \geq \frac{\rho n}{\lceil q \rceil} \mid \mathscr{L}_1 \geq \big(\Theta^* + \varepsilon\big)n \bigg) \lesssim e^{-c\rho n}.
\end{aligned}$$

Moreover, as in the proof of Lemma 4.5, by Lemmas 3.1 and 4.3, we also have that for $i = 2, \ldots, \lfloor q \rfloor$ that there exists $c(\lambda, q) > 0$ such that for every $M \geq M_0(\lambda, \rho)$,

$$\begin{aligned}
&\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \pi_{|R_i|,p,1}\bigg(|S_M| \geq \frac{\rho n}{\lceil q \rceil} \bigg) \lesssim e^{-c\rho n}, \quad \text{and} \\
&\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \pi_{|R_0|,p,q-\lfloor q \rfloor}\bigg(|S_M| \geq \frac{\rho n}{\lceil q \rceil} \bigg) \lesssim \sqrt{n} e^{-c\rho n}.
\end{aligned}$$

Clearly, analogous bounds hold for the above when replacing $\frac{\rho n}{\lceil q \rceil}$ with $\frac{\rho n}{2\lceil q \rceil}$. Combining all of the above bounds and plugging them in to the right-hand side of

$$\mathbf{I} \lesssim \max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{I}}} \frac{\pi_{\mathbf{R}}(B_{\rho,M} \mid \mathscr{L}_1 \geq (\Theta^* + \varepsilon)n, \mathcal{C}_1 \subset R_1)}{\pi_{\mathbf{R}}(A_{\rho,\varepsilon,M} \mid \mathscr{L}_1 \geq (\Theta^* + \varepsilon)n, \mathcal{C}_1 \subset R_1)} + e^{-\rho^2 n/(2M)},$$

yields an exponentially decaying upper bound on the sum **I**. The bound on the sum **II** is very similar. Letting $\mu_{\mathbf{II}} = (\Theta^* + \varepsilon)n + \frac{q - \lfloor q \rfloor}{q}(1 - \Theta^* - \varepsilon)n$, we define

$$\Gamma_\rho^{\mathbf{II}} = \left\{ |R_0| \ge \mu_{\mathbf{II}} - 2\rho n, \left| |R_i| - \frac{n - |R_0|}{\lfloor q \rfloor} \right| < 2\rho n \text{ for all } i = 1, \dots, \lfloor q \rfloor \right\}.$$

As before, by Lemma 3.2, we can write

$$\mathbf{II} \lesssim e^{-\rho^2 n/(2M)} + \max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{II}}} \frac{\pi_{\mathbf{R}}(B_{\rho,M} \mid \mathscr{L}_1 \ge (\Theta^* + \varepsilon)n, \mathcal{C}_1 \subset R_0)}{\pi_{\mathbf{R}}(A_{\rho,\varepsilon,M} \mid \mathscr{L}_1 \ge (\Theta^* + \varepsilon)n, \mathcal{C}_1 \subset R_0)},$$

and observe that for every $\mathbf{R} \in \Gamma_\rho^{\mathbf{II}}$, since $\lambda \in (\lambda_s, \lambda_S)$, for sufficiently small $\rho > 0$, the FK model $\pi_{|R_0|, p, q - \lfloor q \rfloor}$ is supercritical and the random graphs $\mathcal{G}(|R_i|, \lambda)$ are subcritical for all $i = 1, \dots, \lfloor q \rfloor$. By Lemmas 3.1 and 4.4, there exists $c(\lambda, q) > 0$ such that for every $\rho > 0$ sufficiently small and every $M \ge M_0(\lambda, \rho)$,

$$\begin{aligned} &\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{II}}} \pi_{|R_0|, p, q - \lfloor q \rfloor}\left(|S_M - \mathcal{C}_1| \ge \frac{\rho n}{\lceil q \rceil} \mid \mathscr{L}_1 \ge \left(\Theta^* + \varepsilon\right)n \right) \\ &\quad \le \max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{II}}} \pi_{|R_0|, p, q - \lfloor q \rfloor}\left(|S_M - \mathcal{C}_1| \ge \frac{\rho n}{\lceil q \rceil} \right) \lesssim e^{-c\rho n}, \quad \text{and} \\ &\max_{\mathbf{R} \in \Gamma_\rho^{\mathbf{II}}} \pi_{|R_i|, p, 1}\left(|S_M| \ge \frac{\rho n}{\lceil q \rceil} \right) \lesssim e^{-c\rho n} \quad \text{for all } i = 1, \dots, \lfloor q \rfloor, \end{aligned}$$

and by the same reasoning, analogous bounds hold when replacing $\frac{\rho n}{\lceil q \rceil}$ with $\frac{\rho n}{2\lceil q \rceil}$. Then expanding the fraction in the upper bound on **II** as done in the bound on **I** implies there exists $c(\lambda, q) > 0$ such that for sufficiently small $\rho, \varepsilon > 0$ and every $M \ge M_0(\lambda, \rho)$,

$$\pi_{n,p,q}(E_{\rho,\varepsilon,M} \mid A_{\rho,\varepsilon,M}) \lesssim \mathbf{I} + \mathbf{II} \lesssim e^{-c\rho n} + e^{-c\Theta^* n} + e^{-\rho^2 n/(2M)}. \qquad \square$$

We are now in position to complete the proof of Theorem 2.

Proof of Theorem 2: the case $np = \lambda \in (\lambda_s, \lambda_c]$**.** The proof when $\lambda \in (\lambda_s, \lambda_c]$ is similar to the extension of slow mixing for the Swendsen–Wang dynamics when $\lambda \in [\lambda_c, \lambda_S)$ to $\lambda \in (\lambda_s, \lambda_c]$. Recall that for fixed $\lambda > \lambda_s$, the two zeros of $g(\theta) = f(\theta) - \theta$ were denoted $\Theta^* < \Theta_r$ so that g is positive on (Θ^*, Θ_r). We again use a conductance estimate to lower bound the inverse gap of the Chayes–Machta dynamics. Define for every $\rho, \varepsilon, M > 0$,

$$\begin{aligned} A_{\rho,\varepsilon,M} &= \left\{ \mathscr{L}_1 \ge \left(\Theta^* + \varepsilon\right)n, |S_M - \mathcal{C}_1| < \rho n \right\}, \\ E_{\rho,\varepsilon,M} &= \left\{ \mathscr{L}_1 \ge \left(\Theta^* + \varepsilon\right)n, \frac{\rho n}{2} < |S_M - \mathcal{C}_1| < \rho n \right\}. \end{aligned}$$

As in (4.5), by (4.1) it suffices to show an exponentially decaying upper bound on

$$\frac{Q(A_{\rho,\varepsilon,M}, A_{\rho,\varepsilon,M}^c)}{\pi_{n,p,q}(A_{\rho,\varepsilon,M})} \lesssim \max_{X_0 \in A_{\rho,\varepsilon,M} - E_{\rho,\varepsilon,M}} P\left(X_0, A_{\rho,\varepsilon,M}^c\right) + \pi_{n,p,q}(E_{\rho,\varepsilon,M} \mid A_{\rho,\varepsilon,M}),$$

for sufficiently small $\rho, \varepsilon > 0$ and large M; this is because by Proposition 2.1, for all small enough ε, ρ, we have $\pi_{n,p,q}(A_{\rho,\varepsilon,M}^c) \ge c > 0$. We bound the two terms above separately as in the proof for $\lambda \in [\lambda_c, \lambda_S)$. First of all, note by Lemma 4.7 that the second term on the right-hand side is bounded above by e^{-cn} for some $c(\rho, M, \lambda, q) > 0$ for every sufficiently small $\varepsilon, \rho > 0$ and every $M \ge M_0(\lambda, \rho)$.

Now consider any $X_0 \in A_{\rho,\varepsilon,M} - E_{\rho,\varepsilon,M}$ and bound $P(X_0, A_{\rho,\varepsilon,M}^c)$ under the Chayes–Machta dynamics. We split the transition probability of the Chayes–Machta dynamics into the case when $\mathcal{C}_1(X_0)$ is activated and $\mathcal{C}_1(X_0)$ is not activated; let $\mathcal{A}_1$ denote the set of activated vertices. If $\mathcal{C}_1(X_0) \not\subset \mathcal{A}_1$, we have $\mathbb{E}[|\mathcal{A}_1| \mid \mathcal{C}_1 \not\subset \mathcal{A}_1] \le \frac{1}{q}(1 - \Theta^* - \varepsilon)n$ and since $X_0 \in A_{\rho,\varepsilon,M}$, by Lemma 3.2, if $\varepsilon > \rho$, then

$$\mathbb{P}_{X_0}\left(|\mathcal{A}_1| \ge \frac{1}{q}\left(1 - \Theta^* - \varepsilon\right)n + \varepsilon n \mid \mathcal{C}_1(X_0) \not\subset \mathcal{A}_1 \right) \le 2e^{-\varepsilon^2 n/(2M)}.$$

If $|\mathcal{A}_1| \le \frac{1}{q}(1-\Theta^* - \varepsilon)n + \varepsilon n$, for sufficiently small $\varepsilon > 0$, since $\lambda < \lambda_S = q$, the random graph $\mathcal{G}(|\mathcal{A}_1|, p)$ is subcritical, in which case with probability at least $1 - e^{-c\Theta^* n}$, $\mathcal{C}_1(X_1) = \mathcal{C}_1(X_0)$. By Lemma 3.1, there exists $c(\rho, M, \lambda, q) > 0$ such that for $0 < \rho < \varepsilon$ sufficiently small and every $M \ge M_0(\lambda, \rho)$,

$$\begin{aligned}
&\mathbb{P}_{X_0}\big(\big|S_M(X_1) - \mathcal{C}_1(X_1)\big| \ge \rho n \mid \mathcal{C}_1(X_0) \not\subset \mathcal{A}_1\big) \\
&\quad\lesssim \pi_{\frac{1}{q}(1-\Theta^*+(q-1)\varepsilon)n, p, 1}\left(|S_M| \ge \frac{\rho n}{2}\right) + e^{-\varepsilon^2 n/(2M)} + e^{-c\Theta^* n} \\
&\quad\lesssim e^{-c\rho n} + e^{-\varepsilon^2 n/(2M)} + e^{-c\Theta^* n}.
\end{aligned}$$

Thus, for some $c(\rho, \varepsilon, M, \lambda, q) > 0$, for small enough $0 < \rho < \varepsilon$, and every $M \ge M_0(\lambda, \rho)$,

$$\max_{X_0 \in A_{\rho,\varepsilon,M} - E_{\rho,\varepsilon,M}} \mathbb{P}_{X_0}\big(X_1 \notin A_{\rho,\varepsilon,M} \mid \mathcal{C}_1(X_0) \not\subset \mathcal{A}_1\big) \lesssim e^{-cn}.$$

Now suppose that $\mathcal{C}_1(X_0) \subset \mathcal{A}_1$; then one step of Chayes–Machta dynamics is described precisely by the set up of Lemma 4.6, with ρ replaced by $\rho/2$, yielding

$$\max_{X_0 \in A_{\rho,\varepsilon,M} - E_{\rho,\varepsilon,M}} \mathbb{P}_{X_0}\big(\mathscr{L}_1 \le \big(\Theta^* + \varepsilon\big)n \mid \mathcal{C}_1(X_0) \subset \mathcal{A}_1\big) \lesssim e^{-c'n}$$

for some $c'(\varepsilon, \rho, M, \lambda, q) > 0$ for all sufficiently small $\varepsilon, \rho > 0$ and $M \ge M_0(\lambda, \rho)$. On the complement of that event, deterministically $\mathcal{C}_1(X_1) = \mathcal{C}_1(X_1 \restriction_{\mathcal{A}_1})$. By Lemma 3.4, for some $c(\lambda, q) > 0$, for small $\varepsilon, \rho > 0$ and large $M \ge M_0(\lambda, \rho)$,

$$\mathbb{P}\big(\big|S_M(X_1 \restriction_{\mathcal{A}_1}) - \mathcal{C}_1(X_1 \restriction_{\mathcal{A}_1})\big| \ge \rho n/2 \mid \mathcal{C}_1(X_0) \subset \mathcal{A}_1\big) \lesssim e^{-c\rho n/2}.$$

Combining the above, we deduce that there exists $c(\rho, \varepsilon, M, \lambda, q) > 0$ such that for all sufficiently small $0 < \rho < \varepsilon$, for every $M \ge M_0(\lambda, \rho)$, we have $P(X_0, A^c_{\rho,\varepsilon,M}) \lesssim e^{-cn}$, concluding the proof of Theorem 2 when $\lambda \in (\lambda_s, \lambda_c]$. $\square$

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