

# Location of the path supremum for self-similar processes with stationary increments

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**Abstract.** In this paper we consider the distribution of the location of the path supremum in a fixed interval for self-similar processes with stationary increments. A point process is constructed and its relation to the distribution of the location of the path supremum is studied. Using this framework, we show that the distribution has a spectral-type representation, in the sense that it is always a mixture of a special group of absolutely continuous distributions, plus point masses on the two boundaries. An upper bound for the value of the density function is established. We further discuss self-similar Lévy processes as an example. Most of the results in this paper can be generalized to a group of random locations, including the location of the largest jump, etc.

**Résumé.** Dans cet article, nous considérons la distribution de la position du supremum de la trajectoire d'un processus auto-similaire à accroissements stationnaires dans un intervalle fixé. Un processus ponctuel est construit et sa relation avec la distribution de la position du supremum est étudiée. Dans ce cadre, nous montrons que cette distribution a une représentation de type spectral, dans le sens où il s'agit toujours d'un mélange d'un groupe particulier de distributions absolument continues et de masses ponctuelles aux bords de l'intervalle. Une borne supérieure pour la valeur de la fonction de densité est obtenue. De plus, à titre d'exemple, nous discutons des processus de Lévy auto-similaires. La plupart des résultats de cet article peuvent être généralisés à un groupe de positions aléatoires, y compris la position du plus grand saut, etc.

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## 1. Introduction

Self-similar processes are stochastic processes whose distributions do not change under proper rescaling in time and space. The study of self-similar processes as a unified concept dates back to [6], and this class of processes have attracted attention of researchers from various fields since then, due to their theoretical tractability and broad applications. The book [5], the lecture note [1], and the review papers [4] and [8] are all excellent sources for general introduction and existing results. A special subclass of self-similar processes, self-similar processes with stationary increments, or *ss,si* processes in short, are of particular interest. They combine the two probabilistic symmetries given by the self-similarity and the stationary increments, and include famous examples such as fractional Brownian motions and self-similar Lévy processes.

In this paper, we consider the distributional properties of the location of the path supremum over a fixed interval for self-similar processes with stationary increments. Compared to the values of the extremes, their locations received relatively less attention. On one hand, there exist results for some special cases. For instance, the distribution of the location of path supremum for a Brownian motion is well-known as the (third) arcsin law. More generally, the result for self-similar Lévy processes was given in [2]. While the exact result for fractional Brownian motions remains unclear, approximate distributions were studied in [3] using perturbation theory. On the other hand, there has not been

any structural study of the distribution of the location of path supremum for general self-similar processes. Our goal in this paper is to establish a framework which works for general self-similar processes, and to derive properties for the distribution of the location of path supremum.

## 2. Basic settings

Let  $\mathbf{X} = \{X(t)\}_{t \in \mathbb{R}}$  be a stochastic process defined on some probability space  $(\Omega, \mathcal{F}, P)$  and taking values in  $\mathbb{R}$ , whose path is almost surely càdlàg.  $\{X(t)\}_{t \in \mathbb{R}}$  is said to be self-similar, if  $\{X(at)\}$  and  $\{a^H X(t)\}$  have the same distribution, for some  $H \geq 0$ . The constant  $H$  is called the exponent of the self-similar process. In this work,  $\{X(t)\}_{t \in \mathbb{R}}$  is always assumed self-similar. We further assume that  $\{X(t)\}_{t \in \mathbb{R}}$  has stationary increments. Such a self-similar process with stationary increments is often referred as a  $H$ -ss,si process, or a ss,si process when it is not necessary to specify  $H$ .

We are interested in the distribution of the location of the path supremum for the ss,si process  $\{X(t)\}_{t \in \mathbb{R}}$  over an interval with a fixed length, say,  $T > 0$ . By the stationarity of the increments, the distribution will be the same for any interval with length  $T$ , and consequently, the interval can be chosen as  $[0, T]$ . In this case, we denote the location of the path supremum of  $\{X(t)\}_{t \in \mathbb{R}}$  over  $[0, T]$  by

$$\tau_{\mathbf{X}, T} := \inf \left\{ t \in [0, T] : \limsup_{s \rightarrow t} X(s) = \sup_{s \in [0, T]} X(s) \right\},$$

where the infimum means that in case where the supremum is achieved at multiple locations, the leftmost point among them is taken.

Alternatively, one can first define

$$\tau_{\mathbf{X}, T} := \inf \left\{ t \in [0, T] : X(t) = \sup_{s \in [0, T]} X(s) \right\}$$

for stochastic processes with upper semicontinuous paths, in which case the supremum can be indeed achieved at some point. Then for a càdlàg process  $\mathbf{X}$ , define its location of path supremum over  $[0, T]$  as that of the modified process

$$X'(t) = X(t-) \vee X(t),$$

which is now upper semicontinuous.

**Proposition 2.1.**  $\mathbf{X}'$  is a modification of  $\mathbf{X}$ .

**Proof.** Since  $\mathbf{X}$  has stationary increments, for any  $t \in \mathbb{R}$  and  $\varepsilon > 0$ ,  $X(t) - X(t - \varepsilon) \stackrel{d}{=} X(\varepsilon) - X(0)$ . As  $\mathbf{X}$  has càdlàg sample paths, we have  $X(\varepsilon) \rightarrow X(0)$  almost surely, hence  $X(t - \varepsilon) \rightarrow X(t)$  in probability, as  $\varepsilon \downarrow 0$ . On the other hand, by definition, we have  $X(t - \varepsilon) \rightarrow X(t-)$  almost surely as  $\varepsilon \downarrow 0$ . Hence  $X(t) = X(t-)$  almost surely for any  $t \in \mathbb{R}$ . Since  $\mathbf{X}'$  and  $\mathbf{X}$  only differ when  $X(t-) \neq X(t)$ , we conclude that  $X(t) = X'(t)$  almost surely for any  $t \in \mathbb{R}$ .  $\square$

The sample path of  $\mathbf{X}'$  has the property that the left limit and right limit  $X'(t-)$  and  $X'(t+)$  exists for all  $t \in \mathbb{R}$ , and  $X'(t) = X'(t-) \vee X'(t+)$ . Denote by  $D'$  the collection of all such functions on  $\mathbb{R}$ . Note that if  $\mathbf{X}$  is ss,si, so is  $\mathbf{X}'$ . Therefore, in the rest of the paper, we can always assume that the process  $\mathbf{X}$  has paths in  $D'$ , thus in particular, is upper semicontinuous.

It is not difficult to check that  $\tau_{\mathbf{X}, T}$  is a well-defined random variable. Moreover, since the process is self-similar, the distributions of  $\tau_{\mathbf{X}, T}$  and  $\tau_{\mathbf{X}, 1}$  are the same up to a scaling. Therefore we focus on the case where  $T = 1$ , and use the simplified notation  $\tau_{\mathbf{X}} = \tau_{\mathbf{X}, 1}$ .

Additionally, define set

$$M = \left\{ \omega \in \Omega : X(t_i) = \sup_{s \in [0, 1]} X(s) \text{ for at least two different } t_1, t_2 \in [0, 1] \right\},$$

and assume that the process  $\{X(t)\}_{t \in \mathbb{R}}$  satisfies

**Assumption U.**  $P(M) = 0$ .

Most processes that we encounter do satisfy Assumption U. A necessary and sufficient condition for this assumption for processes with continuous paths can be found in [9]. Under Assumption U, the supremum in  $[0, 1]$  is attained at a unique point, and the infimum in the definition of  $\tau_{\mathbf{X}}$  can be removed. Notice that Assumption U also excludes the case where  $H = 0$  provided that the process is stochastically continuous at  $t = 0$ , since in that case  $X(t)$  must be a constant, which is trivial anyway. Hence we always assume  $H > 0$  for the rest of the paper.

In [12], the author showed that for a stochastic process  $\{X(t)\}_{t \in \mathbb{R}}$  with stationary increments, the distribution of  $\tau_{\mathbf{X}}$  can have point masses on the two boundaries 0 and 1, but must be absolutely continuous in the interior of the interval  $[0, 1]$ , and its density function, denoted as  $f(t)$ , can be taken as the right derivative of the cumulated distribution function of  $\tau_{\mathbf{X}}$ . Moreover, this version of the density function is càdlàg everywhere.

In the presence of self-similarity, it turns out that the distribution of  $\tau_{\mathbf{X}}$  is closely related to a point process, constructed as below.

For  $t \in \mathbb{R}$ , define  $l(t) = \inf\{s > 0 : X(t - s) \geq X(t)\}$  and  $r(t) = \inf\{s > 0 : X(t + s) \geq X(t)\}$ , with the tradition that  $\inf(\emptyset) = +\infty$ . Intuitively,  $l(t)$  and  $r(t)$  are the distances by which the process will return to the level  $X(t)$  or higher at the left and the right of the point  $t$ . It is clear that  $t$  is a (strict) local maximum if and only if both  $l(t)$  and  $r(t)$  are strictly positive.

Let  $S = \{s \in \mathbb{R} : l(s) > 0, r(s) > 0\}$  be the set of all the local maxima of  $\mathbf{X}$ . Notice that since  $|s_1 - s_2| \geq \min\{l(s_1), r(s_1), l(s_2), r(s_2)\}$  for any  $s_1, s_2 \in S$ ,  $S$  is at most countable. For each  $s_i \in S \cap [0, 1]$ , define point  $\epsilon_i = (l(s_i), r(s_i))$ , then  $\epsilon_i$  is a point in the first quadrant of  $\overline{\mathbb{R}}^2$ , where  $\overline{\mathbb{R}} = [-\infty, \infty]$ . The collection of these points, denoted as  $\mathcal{E}$ , or more precisely, the (random) counting measure determined by it, denoted as  $\xi := \sum_{\epsilon_i \in \mathcal{E}} \delta_{\epsilon_i}$ , forms a point process in  $(\overline{\mathbb{R}}^+)^2$ , where  $\overline{\mathbb{R}}^+ = (0, \infty]$ .

Let  $\nu$  be the mean measure of the point process  $\xi$ :

$$\nu(A) = E(\xi(A)) \quad \text{for every } A \in \mathcal{B}((\overline{\mathbb{R}}^+)^2), \tag{1}$$

where  $\mathcal{B}((\overline{\mathbb{R}}^+)^2)$  is the Borel  $\sigma$ -algebra on  $(\overline{\mathbb{R}}^+)^2$ , with  $+\infty$  treated as a separate point in  $\overline{\mathbb{R}}^+$ . Again, since the points in  $\mathcal{E}$  have the property that  $|s_1 - s_2| \geq \min\{l(s_1), r(s_1), l(s_2), r(s_2)\}$ ,  $\nu(A)$  is finite whenever the set  $A$  is bounded away from the axes.

### 3. Location of the supremum for ss,si processes

We start by exploring the structure of the measure  $\nu$ . Firstly, the following result shows that  $\nu$  has mass 0 on the boundaries at  $+\infty$ . As a result, we can effectively remove infinity from the definition of  $\xi$  when only  $\nu$  is considered.

**Proposition 3.1.** *Let  $\mathbf{X}$  be a ss,si process satisfying Assumption U, and  $\nu$  be defined as at the end of Section 2. Then*

$$\nu(\mathbb{R}^+ \times \{+\infty\}) = \nu(\{+\infty\} \times \mathbb{R}^+) = \nu(\{+\infty\} \times \{+\infty\}) = 0.$$

**Proof.** First, notice that the set  $S_\infty := \{t \in \mathbb{R} : l(t) = r(t) = +\infty\}$  contains at most one single point, which is the location of the strict global maximum of the process  $\mathbf{X}$  over the whole real line. Denote by  $\tau_{\mathbf{X}}^*$  this location if it exists. Since it is compatible with horizontal translation and invariant under vertical shift of the path, we have for any  $s \in \mathbb{R}$ ,  $\tau_{\mathbf{Y}_s}^* = \tau_{\mathbf{X}}^* + s$ , where the process  $\mathbf{Y}_s$  is defined by  $Y_s(t) = X(t - s) - X(-s)$ , and  $\tau_{\mathbf{Y}_s}^*$  is the location of the global maximum of  $\mathbf{Y}_s$ . On the other hand, by the stationarity of the increments,  $\{Y_s(t)\}_{t \in \mathbb{R}} \stackrel{d}{=} \{X(t) - X(0)\}_{t \in \mathbb{R}}$ , hence we have

$$P(\tau_{\mathbf{X}}^* \in [0, 1]) = P(\tau_{\mathbf{Y}_s}^* \in [s, s + 1]) = P(\tau_{\mathbf{X}}^* \in [s, s + 1])$$

for all  $s \in \mathbb{R}$ . However,

$$\sum_{s \in \mathbb{Z}} P(\tau_{\mathbf{X}}^* \in [s, s + 1]) = P(\tau_{\mathbf{X}}^* \text{ exists}) \leq 1,$$

implying that  $P(\tau_{\mathbf{X}}^* \in [0, 1])$  must be 0. Consequently, we conclude that

$$\nu(\{+\infty\} \times \{+\infty\}) = P(S_\infty \cap [0, 1] \neq \emptyset) = P(\tau_{\mathbf{X}}^* \in [0, 1]) = 0.$$

Note that the above reasoning actually shows that for a process with stationary increments, the global (strict) maximum almost surely does not exist.

For the proof of the rest part of the proposition and future use, we introduce the notion of “compatible set”. Denote by  $(D', \mathcal{C})$  the collection of all modified càdlàg paths equipped with the cylindrical  $\sigma$ -field, and  $(M_P, \mathcal{M}_P)$  the standard measurable space for point processes on the real line.

**Definition 3.2.** A compatible set  $I$  is a measurable mapping from  $(D', \mathcal{C})$  to  $(M_P, \mathcal{M}_P)$ , satisfying

1. (Shift compatibility)  $I(\theta_c \circ (g + d)) = \theta_c \circ I(g)$  for all  $g \in D'$  and  $c, d \in \mathbb{R}$ , where  $\theta_c$  is the shift operator:  $\theta_c \circ g(t) := g(t + c)$ ,  $t \in \mathbb{R}$ ;  $\theta_c \circ \Gamma := \Gamma - c$ ,  $\Gamma \in M_P$ , and  $(g + d)(t) := g(t) + d$ .
2. (Scaling compatibility)  $I(d \cdot g(c \cdot)) = c^{-1}I(g(\cdot))$  for all  $g \in D'$  and  $c, d \in \mathbb{R}^+$ .

By the above definition and the corresponding probabilistic symmetries possessed by  $ss, si$  processes, it is clear that the distribution of the point process  $I(\mathbf{X})$  will be both stationary and scaling invariant if the underlying process  $\mathbf{X}$  is  $ss, si$  and  $I$  is a compatible set. Then we have

**Lemma 3.3.** Let  $\mathbf{X}$  be a  $ss, si$  process with paths in  $D'$ , and  $I$  be a compatible set. Then  $P(I(\mathbf{X}) \text{ is dense in } \mathbb{R}) + P(I(\mathbf{X}) = \emptyset) = 1$ .

That is, a compatible set of a  $ss, si$  process is either empty or dense. The proof of this lemma is very simple.

**Proof of Lemma 3.3.** Assume that  $P(I(\mathbf{X}) \text{ is neither dense nor empty}) > 0$ . By stationarity of  $I(\mathbf{X})$ , this implies that

$$P(I(\mathbf{X}) \neq \emptyset, d(0, I(\mathbf{X})) > 0) > 0,$$

where  $d$  denotes the Euclidean distance between points or sets. Indeed, assume that  $P(I(\mathbf{X}) \neq \emptyset, d(0, I(\mathbf{X})) > 0) = 0$ , then since  $I(\mathbf{X})$  is stationary,  $P(I(\mathbf{X}) \neq \emptyset, d(a, I(\mathbf{X})) > 0) = 0$  for all  $a \in \mathbb{Q}$ , hence given  $I(\mathbf{X}) \neq \emptyset$ ,  $I(\mathbf{X})$  is dense almost surely, contradicting our assumption.

However, since the distribution of  $I(\mathbf{X})$  is invariant under rescaling, so is the distribution of  $d(0, I(\mathbf{X}))$ , which implies that  $d(0, I(\mathbf{X})) \in \{0, +\infty\}$  almost surely, contradicting the above result. Thus we conclude that

$$P(I(\mathbf{X}) \text{ is dense in } \mathbb{R}) + P(I(\mathbf{X}) = \emptyset) = 1. \quad \square$$

Now consider the set  $S_r := \{s \in S : r(s) = +\infty\}$ . Note that  $S_r$  is a compatible set. Therefore, by Lemma 3.3,  $P(S_r \text{ is dense in } \mathbb{R}) + P(S_r = \emptyset) = 1$ . Assume  $S_r$  is dense for some path of  $\mathbf{X}$ . Then for any fixed  $t_2 \in \mathbb{R}$  and  $s \in S_r$  such that  $s < t_2$ ,  $X(s) > X(t_2)$ . Recall that the paths of  $\mathbf{X}$  are in  $D'$  and therefore, upper semicontinuous. Consequently, for any  $t_1 < t_2$ , taking a sequence of such  $s$  converging to  $t_1$  leads to the result that  $X(t_1) \geq X(t_2)$ . Moreover, since  $S_r$  is dense, there exists  $s_0 \in S_r \cap (t_1, t_2)$ . For such  $s_0$  we must have  $X(t_1) \geq X(s_0) > X(t_2)$  based on the above argument. Consequently, the whole path is strictly decreasing. However, the definition of  $S_r$  requires that both  $l(s)$  and  $r(s)$  be strictly positive for  $s \in S_r$ , thus  $S_r$  should be empty if the path is strictly monotonic, contradicting the assumption that  $S_r$  is dense. Therefore we conclude that the set  $S_r$  can never be dense for any realization, in other words,  $S_r$  is empty almost surely.

Symmetrically,  $S_l := \{s \in S : l(s) = +\infty\}$  is empty almost surely. As a result,

$$\nu(\mathbb{R}^+ \times \{+\infty\}) = \nu(\{+\infty\} \times \mathbb{R}^+) = 0. \quad \square$$

Note that if  $s$  is the location of a local maximum of  $\mathbf{X}$ , then for any  $a > 0$ ,  $s' = as$  is the location of a local maximum of  $\mathbf{Y}$  defined by  $Y(at) = X(t)$ ,  $t \in \mathbb{R}$ . Moreover,  $l_{\mathbf{Y}}(s') = al(s)$  and  $r_{\mathbf{Y}}(s') = ar(s)$ , where  $l_{\mathbf{Y}}$  and  $r_{\mathbf{Y}}$  are defined for  $\mathbf{Y}$  in the same way as  $l$  and  $r$  for  $\mathbf{X}$ . Therefore by self-similarity, for any  $A \in \mathcal{B}$ , we have

$$\nu(aA) = a^{-1}\nu(A), \tag{2}$$

where  $aA := \{(al, ar) : (l, r) \in A\}$ , and the factor  $a^{-1}$  on the right hand side comes from the fact that the measure  $\nu$  still counts the expected number of qualified points in an interval with length 1 rather than length  $a$ .

Define bijection  $\Psi : (l, r) \rightarrow (u, v)$  by

$$u := l, \quad v := \frac{l}{l+r},$$

and  $\nu' = \nu \circ \Psi^{-1}$ . We have the following factorization result.

**Lemma 3.4.**

$$\nu' = \eta \times \mu, \tag{3}$$

where  $\mu$  is a measure on  $((0, 1), \mathcal{B}(0, 1))$ ,  $\eta$  is an absolutely continuous measure with density

$$g(u) = cu^{-2}, \quad u > 0$$

for some positive constant  $c$ .

**Proof.** After the change of variable  $\Psi$ , the relation (2) becomes

$$\nu'(\varphi_a^u(A')) = a^{-1}\nu'(A'), \quad A' \in \mathcal{B},$$

where  $\varphi_a^u(A') = \{(au, v) : (u, v) \in A'\}$ .

Let  $\mu$  be a measure determined by  $\mu([v_1, v_2]) = \nu'([c, \infty) \times [v_1, v_2])$ ,  $0 < v_1 < v_2 < 1$ . Note that since  $\Psi^{-1}([c, \infty) \times [v_1, v_2])$  is bounded away from the axes,  $\mu([v_1, v_2])$  is always finite. For any  $0 < u_1 < u_2$ ,

$$\begin{aligned} \nu'([u_1, u_2] \times [v_1, v_2]) &= \nu'([u_1, \infty) \times [v_1, v_2]) - \nu'([u_2, \infty) \times [v_1, v_2]) \\ &= c(u_1^{-1} - u_2^{-1})\nu'([c, \infty) \times [v_1, v_2]) \\ &= c(u_1^{-1} - u_2^{-1})\mu([v_1, v_2]) \\ &= \eta([u_1, u_2])\mu([v_1, v_2]), \end{aligned}$$

hence the desired factorization. □

The following theorem reveals a key relation between  $f(t)$ , the density of the location of the path supremum, and the mean measure  $\nu$  of the point process that we introduced.

**Theorem 3.5.** *Let  $\{X(t)\}_{t \in \mathbb{R}}$  be H-ss,si for  $H > 0$ , satisfying Assumption U.  $f(t)$  is the density in  $(0, 1)$  of  $\tau_{\mathbf{X}}$ , and  $\nu$  is defined as in (1). Then*

$$f(t) = \nu([t, \infty) \times [1-t, \infty)), \quad t \in (0, 1). \tag{4}$$

**Proof.** We first show

$$\nu((t, \infty) \times (1-t, \infty)) \leq f(t) \leq \nu([t, \infty) \times [1-t, \infty)), \tag{5}$$

and then

$$\nu(\{t\} \times [1-t, \infty) \cup [t, \infty) \times \{1-t\}) = 0. \tag{6}$$

Once we have these two results, their combination clearly gives (4).

In order to show (5), let  $0 < \varepsilon < \min\{t, \frac{1-t}{2}\}$ . Note that if there exists a point  $s \in S$  located in  $[t, t + \varepsilon]$ , such that  $l(s) > t + \varepsilon, r(s) > 1 - t$ , then  $X(s)$  is the supremum of the process over the interval  $[s - (t + \varepsilon), s + 1 - t]$ . Since  $[0, 1] \subseteq [s - (t + \varepsilon), s + (1 - t)]$ ,  $\tau_X = s \in [t, t + \varepsilon]$ . On the other hand, if  $\tau_X \in [t, t + \varepsilon]$ , then by definition,  $l(\tau_X) \geq \tau_X \geq t$ , and  $r(\tau_X) \geq 1 - \tau_X \geq 1 - t - \varepsilon$ , hence there exists  $s \in S$  located in  $[t, t + \varepsilon]$ , such that  $l(s) \geq t, r(s) \geq 1 - t - \varepsilon$ . Therefore,

$$\begin{aligned} & \{ \text{there exists } s \in S \cap [t, t + \varepsilon] \text{ satisfying } l(s) > t + \varepsilon, r(s) > 1 - t \} \\ & \subseteq \{ \tau_X \in [t, t + \varepsilon] \} \\ & \subseteq \{ \text{there exists } s \in S \cap [t, t + \varepsilon] \text{ satisfying } l(s) \geq t, r(s) \geq 1 - t - \varepsilon \}. \end{aligned}$$

Since for any point  $s$  in the set in the first or the third line,  $\min\{l(s), r(s)\} \geq \min\{t, 1 - t - \varepsilon\} > \varepsilon$ , there can only be at most one such point in the interval  $[t, t + \varepsilon]$ . Thus by the stationarity of the increments,

$$\begin{aligned} & P(\{ \text{there exists } s \in S \cap [t, t + \varepsilon] \text{ satisfying } l(s) > t + \varepsilon, r(s) > 1 - t \}) \\ & = E(|s_i \in S \cap [t, t + \varepsilon] : \epsilon_i \in (t + \varepsilon, \infty) \times (1 - t, \infty)|) \\ & = \varepsilon E(|s_i \in S \cap [0, 1] : \epsilon_i \in (t + \varepsilon, \infty) \times (1 - t, \infty)|) \\ & = \varepsilon E\left(\sum_{\epsilon_i \in \mathcal{E}} \delta_{\epsilon_i}((t + \varepsilon, \infty) \times (1 - t, \infty))\right) \\ & = \varepsilon v((t + \varepsilon, \infty) \times (1 - t, \infty)), \end{aligned}$$

where  $|\cdot|$  for a set gives the number of the elements in the set. Similarly,

$$\begin{aligned} & P(\{ \text{there exists } s \in S \cap [t, t + \varepsilon] \text{ satisfying } l(s) \geq t, r(s) \geq 1 - t - \varepsilon \}) \\ & = E(|s_i \in S \cap [t, t + \varepsilon] : \epsilon_i \in [t, \infty) \times [1 - t - \varepsilon, \infty)|) \\ & = \varepsilon E(|s_i \in S \cap [0, 1] : \epsilon_i \in [t, \infty) \times [1 - t - \varepsilon, \infty)|) \\ & = \varepsilon E\left(\sum_{\epsilon_i \in \mathcal{E}} \delta_{\epsilon_i}([t, \infty) \times [1 - t - \varepsilon, \infty))\right) \\ & = \varepsilon v([t, \infty) \times [1 - t - \varepsilon, \infty)). \end{aligned}$$

Thus

$$\begin{aligned} & \varepsilon v((t + \varepsilon, \infty) \times (1 - t, \infty)) \\ & \leq P(\tau_X \in [t, t + \varepsilon]) \\ & \leq \varepsilon v([t, \infty) \times [1 - t - \varepsilon, \infty)). \end{aligned}$$

Recall that  $f(t)$  is right continuous with left limits, and equals to the right derivative of the cumulative distribution function of  $\tau_X$  everywhere in  $(0, 1)$ . Therefore by dividing all the expressions by  $\varepsilon$  and taking limit  $\varepsilon \downarrow 0$ , we have

$$\begin{aligned} v((t, \infty) \times (1 - t, \infty)) & = \lim_{\varepsilon \downarrow 0} v((t + \varepsilon, \infty) \times (1 - t, \infty)) \leq f(t) \\ & \leq \lim_{\varepsilon \downarrow 0} v([t, \infty) \times [1 - t - \varepsilon, \infty)) = v([t, \infty) \times [1 - t, \infty)). \end{aligned}$$

The second step is to establish (6). By symmetry, it suffices to prove that  $v(\{t\} \times [1 - t, \infty)) = 0$ . To this end, we use the change of variables  $\Psi$  and apply Lemma 3.4. More precisely, note that

$$\Psi(\{t\} \times [1 - t, \infty)) = \{(u, v) : u = t, 0 < v \leq t\}.$$

Hence by Lemma 3.4,

$$\begin{aligned} & \nu(\{t\} \times [1 - t, \infty)) \\ &= \nu'(\{(u, v) : u = t, 0 < v \leq t\}) \\ &= \sum_{i=1}^{\infty} \nu'(\{(u, v) : u = t, v \in (2^{-i}t, 2^{-i+1}t]\}) \\ &= \sum_{i=1}^{\infty} \eta(\{t\})\mu((2^{-i}t, 2^{-i+1}t]). \end{aligned}$$

Since  $\eta$  is absolutely continuous,  $\eta(\{t\}) = 0$ . Moreover, as discussed in the proof of Lemma 3.4,  $\mu([v_1, v_2]) < \infty$  for any  $0 < v_1 < v_2 < 1$ . Hence  $\mu((2^{-i}t, 2^{-i+1}t]) < \infty$  for all  $i \in \mathbb{N}$ . As a result,  $\eta(\{t\})\mu((2^{-i}t, 2^{-i+1}t]) = 0$ ,  $i = 1, 2, \dots$ , so is  $\nu(\{t\} \times [1 - t, \infty))$ . Thus, we have shown (6), which completes the proof.  $\square$

It is now straightforward to derive the following spectral-type result.

**Theorem 3.6.** *Let  $\{X(t)\}_{t \in \mathbb{R}}$  and  $f(t)$  be defined as in Theorem 3.5. Then*

$$f(t) = \int_0^1 f_v(t)\mu_1(dv), \quad 0 < t < 1, \tag{7}$$

where

$$f_v(t) = \begin{cases} \frac{1-v}{-v \ln(v) - (1-v) \ln(1-v)} (1-t)^{-1}, & t \leq v, \\ \frac{v}{-v \ln(v) - (1-v) \ln(1-v)} t^{-1}, & t > v, \end{cases}$$

and  $\mu_1$  is a sub-probability measure on  $(0, 1)$  (i.e.,  $\mu_1(0, 1) \leq 1$ ).

**Proof.** It is not difficult to see that

$$\Psi([t, \infty) \times [1 - t, \infty)) = \{(u, v) : u \geq h(v, t)\},$$

where

$$h(v, t) = \begin{cases} t, & 0 < v < t, \\ \frac{v}{1-v}(1-t), & t \leq v < 1. \end{cases}$$

Hence by Lemma 3.4 and Theorem 3.5,

$$\begin{aligned} f(t) &= \nu([t, \infty) \times [1 - t, \infty)) \\ &= \int_{v=0}^1 \left( \int_{u=h(v,t)}^{\infty} cu^{-2} du \right) \mu(dv) \\ &= \int_{v=0}^1 ch(v, t)^{-1} \mu(dv) \\ &= \int_{v=0}^1 f_v(t)c(v)\mu(dv), \end{aligned}$$

where

$$c(v) = \frac{c(-v \ln(v) - (1 - v) \ln(1 - v))}{v}.$$

Defining measure  $\mu_1$  by  $\frac{d\mu_1}{d\mu}(v) = c(v)$  leads to the desired expression. Finally, since  $\int_0^1 f_v(t) dt = 1$  for any  $v \in (0, 1)$  and  $\int_0^1 f(t)dt \leq 1$ ,  $\mu_1$  is a sub-probability measure. Note that  $\mu_1$  is not necessarily a probability measure due to the potential mass of the distribution on the boundaries 0 and 1.  $\square$

The next result gives a universal entropy-type upper bound for the density function  $f(t)$ . It can be obtained using only the basic properties of self-similarity, but here we are going to prove it using the result of Theorem 3.6.

**Corollary 3.7.** *For any given  $t \in (0, 1)$ ,*

$$f(t) \leq (-t \ln(t) - (1 - t) \ln(1 - t))^{-1}. \tag{8}$$

**Proof.** By Theorem 3.6, it suffices to check (8) for  $f_v(t)$  for all  $0 < v < 1$ , which can be done using fundamental calculus.  $\square$

Corollary 3.7 is a significant improvement of the corresponding result derived in [12] for general processes with stationary increments but not necessarily with self-similarity. More precisely, the upper bound of  $f(t)$  is improved from  $\max\{t^{-1}, (1 - t)^{-1}\}$  to the current form. The factor of improvement varies from  $-\ln(t)$  when  $t \rightarrow 0$  and  $-\ln(1 - t)$  when  $t \rightarrow 1$  to  $\frac{2}{\ln(2)}$  when  $t = \frac{1}{2}$ .

**Remark 3.8.** In the excellent work of [7] the authors proved that

$$f(t) \leq f(s) \max\left(\frac{s}{t}, \frac{1-s}{1-t}\right)$$

for any  $s, t \in (0, 1)$ . In particular,  $f$  is always continuous. Moreover, assuming the existence of the left and the right derivatives, denoted as  $f'(t-)$  and  $f'(t+)$  respectively, the above result easily leads to the following bounds:

$$f'(t-) \leq \frac{f(t)}{1-t}, \tag{9}$$

$$f'(t+) \geq -\frac{f(t)}{t}. \tag{10}$$

Our framework provides an alternative way to derive these bounds: similar as in Corollary 3.7, one can directly check that the above bounds are satisfied by all the basis functions  $f_v(t), 0 < v < 1$ , hence they must hold for all density functions  $f$ . This method also guarantees the existence of the left and the right derivatives.

The following immediate corollary of Theorem 3.6 gives bounds for the expectation of any function of the location of the path supremum. The proof is omitted.

**Corollary 3.9.** *Let  $\{X(t)\}_{t \in \mathbb{R}}$  be  $H$ -ss,si for  $H > 0$ , satisfying Assumption U, and  $\tau_{\mathbf{X}}$  be the location of its path supremum over  $[0, 1]$ . Let  $g$  be a bounded, or non-negative, measurable function on  $[0, 1]$ . Then*

$$\begin{aligned} & \min \left\{ g(0), g(1), \inf_{v \in (0,1)} \int_0^1 g(t) f_v(t) dt \right\} \\ & \leq \mathbb{E}(g(\tau_{\mathbf{X}})) \\ & \leq \max \left\{ g(0), g(1), \sup_{v \in (0,1)} \int_0^1 g(t) f_v(t) dt \right\}. \end{aligned}$$

Corollary 3.9 can be used to derive, for example, the upper bound for the probability that the path supremum falls into an interval  $[c, d]$ :  $P(\tau_{\mathbf{X}} \in [c, d])$ .

In many cases the process  $\mathbf{X}$  is time-reversible, i.e.,  $\{X_t\}_{t \in \mathbb{R}} \stackrel{d}{=} \{X(-t)\}_{t \in \mathbb{R}}$ . For instance, all fractional Brownian motions are time-reversible. This property further improves the spectral-type representation result and the related bound.

**Proposition 3.10.** *Let  $\{X(t)\}_{t \in \mathbb{R}}$  be a time-reversible,  $ss,si$  process, and  $f(t)$  be the density in  $(0, 1)$  of the location of the path supremum for  $\{X(t)\}_{t \in \mathbb{R}}$ . Then*

$$f(t) = \int_0^{1/2} \tilde{f}_v(t) \tilde{\mu}_1(dv),$$

where

$$\tilde{f}_v(t) = \begin{cases} \frac{1}{2(-v \ln(v) - (1-v) \ln(1-v))} (1-t)^{-1}, & 0 < t < v, \\ \frac{v}{2(-v \ln(v) - (1-v) \ln(1-v))} (t^{-1} + (1-t)^{-1}), & v \leq t < 1-v, \\ \frac{1}{2(-v \ln(v) - (1-v) \ln(1-v))} t^{-1}, & 1-v \leq t < 1, \end{cases}$$

and  $\tilde{\mu}_1$  is a sub-probability measure on  $(0, \frac{1}{2}]$ .

To see this result, simply use the fact that  $\mu_1$  in Theorem 3.6 now needs to be symmetric due to the time-reversibility, then define  $\tilde{f}_v(t) = \frac{1}{2}(f_v(t) + f_{1-v}(t))$ . We omit the details. The corresponding upper bound for  $f(t)$  becomes

$$f(t) \leq \begin{cases} \frac{1}{2(1-t)} (-t \ln(t) - (1-t) \ln(1-t))^{-1}, & 0 < t < \frac{1}{2}, \\ \frac{1}{2t} (-t \ln(t) - (1-t) \ln(1-t))^{-1}, & \frac{1}{2} \leq t < 1. \end{cases}$$

We end this section by generalizing the results to other random locations such as the location of the largest jump in a fixed interval.

In [11] the authors introduced the notion of *intrinsic location functional*, which is a large family of random locations including the location of the path supremum, the first hitting time to a fixed level, among many others. It was later shown in [12] that there exists an equivalent characterization of the intrinsic location functionals using partially ordered random sets, which we take here as the definition.

Let  $H$  be a space of real valued functions on  $\mathbb{R}$ , closed under translation. That is, for any  $f \in H$  and  $c \in \mathbb{R}$ ,  $\theta_c f \in H$ . Let  $\mathcal{I}$  be the set of all compact, non-degenerate intervals in  $\mathbb{R}$ .

**Definition 3.11 ([12]).** A mapping  $L = L(f, I)$  from  $H \times \mathcal{I}$  to  $\mathbb{R} \cup \{\infty\}$  is called an intrinsic location functional, if

1.  $L(\cdot, I)$  is measurable for  $I \in \mathcal{I}$ .
2. For each function  $f \in H$ , there exists a subset  $S(f)$  of  $\mathbb{R}$ , equipped with a partial order  $\preceq$ , satisfying:
  - (a) For any  $c \in \mathbb{R}$ ,  $S(f) = S(\theta_c f) + c$ .
  - (b) For any  $c \in \mathbb{R}$  and any  $t_1, t_2 \in S(f)$ ,  $t_1 \preceq t_2$  implies  $t_1 - c \preceq t_2 - c$  in  $S(\theta_c f)$ , such that for any  $I \in \mathcal{I}$ , either  $S(f) \cap I = \emptyset$ , in which case  $L(f, I) = \infty$ , or  $L(f, I)$  is the maximal element in  $S(f) \cap I$  according to  $\preceq$ .

Briefly, an intrinsic location functional always takes the maximal element in a random set in the interval of interest, according to some partial order. Infinity was added as a possible value to deal with the case where some random location may not be well-defined for certain path and interval.

For the case of the location of the path supremum over an interval, the set  $S(f)$  is the set of all the points  $t \in \mathbb{R}$  such that  $f(t)$  is the supremum of  $f$  in either  $[t - s, t]$  or  $[t, t + s]$  (or both) for some  $s > 0$ , and the order  $\preceq$  is the natural order of the value  $f(t)$ . A review of the proofs in this section shows that they did not use any specific property of the location of the path supremum, but rather two general properties that this location possesses, in terms of its partially ordered random set representation:

1. The set  $S$  is a compatible set, as defined in Definition 3.2, with space  $D'$  replaced by  $H$ ;
2. The partial order  $\leq$  is also compatible with rescaling. That is,  $t_1 \leq t_2$  in  $S(f(\cdot))$  implies  $c^{-1}t_1 \leq c^{-1}t_2$  in  $S(d \cdot f(c \cdot))$  for  $c, d \in \mathbb{R}^+$ .

Consequently, all the results in Section 3 can be generalized to any intrinsic location functional satisfying the two properties above. In particular, the spectral-type result, Theorem 3.6, and its two corollaries, also apply to the location of the largest jump in  $[0, 1]$ , defined as

$$\delta_{\mathbf{X}} := \inf \left\{ t \in [0, 1] : |X(t) - X(t-)| = \sup_{s \in [0, 1]} |X(s) - X(s-)| \right\}, \quad (11)$$

provided that it is unique, or the location of the largest drawdown by considering only the downward jumps.

#### 4. Supremum location of self-similar Lévy processes

In this section we consider the special case of self-similar Lévy processes, which are ss,si processes with independent increments. Various results are already known for this family of processes. Most importantly, a non-constant Lévy process is self-similar with exponent  $H$  if and only if it is  $\alpha$ -stable for index  $\alpha = 1/H$ , and the exponent  $H$  must be greater than or equal to  $1/2$ . Intuitively, this means that a non-constant self-similar Lévy process has no Brownian motion part unless  $H = 1/2$ , and that its Lévy measure must have power densities on  $(0, \infty)$  and on  $(-\infty, 0)$  with the same power  $-1 - \alpha$ . Moreover, when  $H = 1/2$  the process is necessarily a Brownian motion (with no drift); when  $H \in (1/2, 1)$  the process has zero mean; when  $H \in (1, \infty)$  the process has zero drift; when  $H = 1$ , the process either has a symmetric Lévy measure or is a deterministic linear function with non-zero slope. See Chapter 3 of [10] for details.

Recall that in order to make the location of the path supremum well-defined, we are using the upper semicontinuous modification of the Lévy process in this section.

Applying the general results derived in the previous section would require Assumption U. To this end, we first show that a non-constant, self-similar Lévy process satisfies Assumption U automatically. Note that such a result is a consequence of self-similarity and does not hold for Lévy processes in general, for which one should consider the compound Poisson processes separately, as in [2].

**Lemma 4.1.** *Let  $\mathbf{X} = \{X(t)\}_{t \in \mathbb{R}}$  be a non-constant, self-similar Lévy process, then  $\mathbf{X}$  satisfies Assumption U.*

**Proof.** Define  $M_1 := \sup_{s \in [0, 1]} X(s)$ ,  $M_2 := \sup_{s \in [0, 1]} (X(s) - X(1))$ , and  $d := \inf\{s > 0 : X(s) > X(0) = 0\}$ . By Blumenthal's 0–1 law (see, for example, Proposition 40.4 in [10]),  $P(d > 0) = 1$  or  $P(d = 0) = 1$ . We discuss these two cases separately.

Case 1:  $P(d > 0) = 1$ . Since the distribution of  $d$  is invariant under rescaling due to self-similarity, in this case we must have  $P(d = \infty) = 1$ , which means that the set  $\{s > 0 : X(s) > X(0) = 0\}$  is empty. As a result,  $\mathbf{X}$  is (non-strictly) decreasing due to the stationarity of the increments. Since  $\mathbf{X}$  is non-constant, Lévy-Itô decomposition implies that  $\mathbf{X}$  is spectrally negative, with no diffusion component. Moreover, as discussed at the beginning of this section, the Lévy measure of  $\mathbf{X}$  must have power densities, hence the intensity of jumps with size  $a$  or larger tends to infinity as  $a$  decreases to 0, making  $\mathbf{X}$  strictly decreasing. Thus in this case, Assumption U is satisfied, with the unique path supremum on  $[0, 1]$  achieved at  $t = 0$ .

Case 2:  $P(d = 0) = 1$ . In this case  $P(M_1 > 0) = 1$ . Let  $A_1 = \{x : P(M_1 = x) > 0\}$  and  $A_2 = \{x : P(M_2 = x) > 0\}$  be the set of atoms of  $M_1$  and  $M_2$ , respectively, then  $0 \notin M_1$ . Since  $\mathbf{X}$  is self-similar and has independent and stationary increments, for  $t \in (0, 1)$ , we have

$$\begin{aligned} P\left(\sup_{s \in [0, t]} X(s) = \sup_{s \in [t, 1]} X(s)\right) &= P\left(\sup_{s \in [0, t]} (X(s) - X(t)) = \sup_{s \in [t, 1]} (X(s) - X(t))\right) \\ &= P(t^H M_2' = (1-t)^H M_1') \\ &= E(P((t/1-t)^H M_2' = M_1' | M_1')), \end{aligned}$$

where  $M'_1$  and  $M'_2$  are independent random variables having the same distributions as  $M_1$  and  $M_2$ , respectively. For any  $x \notin (t/1 - t)^H A_2$ ,  $P((t/1 - t)^H M'_2 = M'_1 | M'_1 = x) = 0$ . Thus,  $P(\sup_{s \in [0, t]} X(s) = \sup_{s \in [t, 1]} X(s)) > 0$  only if  $P(M'_1 \in (t/1 - t)^H A_2) > 0$ , or in other words,  $A_1 \cap (t/1 - t)^H A_2 \neq \emptyset$ . Since  $A_1, A_2$  are at most countable, and  $0 \notin A_1$ , such a condition can only hold for at most countably many  $t$ . On the other hand, suppose Assumption U is not satisfied, then with positive probability there exist at least two locations in  $[0, 1]$  at which  $\sup_{s \in [0, 1]} X(s)$  is achieved. Let  $\tau_1 < \tau_2$  be two such points (e.g., the infimum and the supremum of these points, which also achieve the path supremum by the upper semicontinuity). Then there exists  $\epsilon > 0$ , such that  $P(\tau_2 - \tau_1 \geq \epsilon) > 0$ . For such  $\epsilon$ , there exists  $n \in \mathbb{N}$  satisfying

$$P\left(\tau_1 \in \left[\frac{(n-1)\epsilon}{2}, \frac{n\epsilon}{2}\right], \tau_2 - \tau_1 \geq \epsilon\right) > 0.$$

Note that  $\tau_1 \in [\frac{(n-1)\epsilon}{2}, \frac{n\epsilon}{2}]$  and  $\tau_2 - \tau_1 \geq \epsilon$  imply that  $[\frac{n\epsilon}{2}, \frac{(n+1)\epsilon}{2}] \subseteq [\tau_1, \tau_2]$ . Hence for any  $t \in [\frac{n\epsilon}{2}, \frac{(n+1)\epsilon}{2}]$ ,

$$P\left(\sup_{s \in [0, t]} X(s) = \sup_{s \in [t, 1]} X(s)\right) > P\left(\left[\frac{n\epsilon}{2}, \frac{(n+1)\epsilon}{2}\right] \subseteq [\tau_1, \tau_2]\right) > 0.$$

However, we just shown that  $P(\sup_{s \in [0, t]} X(s) = \sup_{s \in [t, 1]} X(s)) > 0$  can only hold for at most countably many  $t$ , which can not cover any interval. Thus we conclude that Assumption U is satisfied in the case where  $P(d = 0) = 1$ . Combining the two cases completes the proof.  $\square$

**Proposition 4.2.** *Let  $\{X(t)\}_{t \in \mathbb{R}}$  be a non-constant, self-similar Lévy process with exponent  $H > 0$ , and  $\nu$  be the same as previously defined. Then*

$$\nu = \nu_1 \times \nu_2,$$

where  $\nu_1$  and  $\nu_2$  are measures on  $(0, \infty)$  with survival functions  $\bar{F}_1(l) := \nu_1(l, \infty) = l^{-c_1}$  and  $\bar{F}_2(r) := \nu_2(r, \infty) = c_0 r^{-c_2}$ , respectively. The constants  $c_0, c_1, c_2 > 0$  and  $c_1 + c_2 = 1$ .

**Proof.** Consider  $\nu((l, \infty) \times (r, \infty))$  for  $l, r$  satisfying  $l > 1, r > 1$ . Notice that by the construction of the set of points  $\mathcal{E}$ , there is at most one point in  $(1, \infty) \times (1, \infty)$ . Thus

$$\begin{aligned} \nu((l, \infty) \times (r, \infty)) &= P(\text{there exists } s \in [0, 1], \text{ such that } l(s) > l, r(s) > r) \\ &= P(l(s) > l, r(s) > r | E)P(E), \end{aligned}$$

where the event  $E := \{\text{there exists a unique } s \in [0, 1], \text{ such that } l(s) > 1, r(s) > 1\}$ . By the independence of increments, we further have

$$\begin{aligned} P(l(s) > l, r(s) > r | E)P(E) &= P(l(s) > l | l(s) > 1)P(r(s) > r | r(s) > 1)P(E) \\ &=: \bar{F}'_1(l)\bar{F}'_2(r)P(E), \quad l > 1, r > 1. \end{aligned}$$

The condition  $l > 1$  and  $r > 1$  is not essential due to the self-similarity. Thus

$$\nu((l, \infty) \times (r, \infty)) \propto \bar{F}_1(l)\bar{F}_2(r)$$

for some functions  $\bar{F}_1$  and  $\bar{F}_2$ . Taking  $A = (l, \infty) \times (r, \infty)$  in (2), we have

$$\bar{F}_1(al)\bar{F}_2(ar) = a^{-1}\bar{F}_1(l)\bar{F}_2(r)$$

for any  $a > 0$ .

Standard procedure leads to the conclusion that the only solutions of this functional equation, which make both  $\overline{F}_1$  and  $\overline{F}_2$  non-increasing, are of the form

$$\begin{aligned}\overline{F}_1(l) &= c'l^{-c_1}, \\ \overline{F}_2(r) &= c''r^{-c_2},\end{aligned}$$

where  $c_1, c_2 > 0$  and  $c_1 + c_2 = 1$ . Finally  $c'$  and  $c''$  can obviously be combined as  $c_0$ , and put only in front of  $\overline{F}_2$ .  $\square$

With the help of Lemma 4.1, Proposition 4.2 immediately leads to a new way to prove the following result regarding the distribution of  $\tau_{\mathbf{X}}$ , the supremum location for the self-similar Lévy process  $\mathbf{X}$  over  $[0, 1]$ . This result was first established in [2] by considering the joint distribution of the location and the value of the path supremum for stable Lévy processes. The special case of Brownian motion is well known and can be found in, for instance, [13].

**Theorem 4.3.** *Let  $\mathbf{X} = \{X(t)\}_{t \in \mathbb{R}}$  be a non-constant, self-similar Lévy process, then one of the three following scenarios is true:*

1.  $\mathbf{X}$  is almost surely strictly decreasing, hence  $P(\tau_{\mathbf{X}} = 0) = 1$ .
2.  $\mathbf{X}$  is almost surely strictly increasing, hence  $P(\tau_{\mathbf{X}} = 1) = 1$ .
3.  $\mathbf{X}$  is not monotone,  $\tau_{\mathbf{X}} \sim \text{Beta}(1 - c_1, 1 - c_2)$ , where  $c_1, c_2 > 0$  and  $c_1 + c_2 = 1$ .

**Proof.** The first two cases are trivial. Now let us consider the case where  $\mathbf{X}$  is not strictly monotone. It is clear by Proposition 4.2 that the density function of  $\tau_{\mathbf{X}}$  in  $(0, 1)$ ,  $f(t)$ , satisfies

$$f(t) \propto t^{-c_1}(1-t)^{-c_2}, \quad 0 < t < 1.$$

Therefore it suffices to prove that when  $\mathbf{X}$  is not strictly monotone,  $P(\tau_{\mathbf{X}} = 0) = P(\tau_{\mathbf{X}} = 1) = 0$ . Indeed, suppose  $P(\tau_{\mathbf{X}} = 0) > 0$ . Let  $d = \inf\{s > 0 : X(s) > X(0) = 0\}$  as defined in the proof of Lemma 4.1. Since  $\tau_{\mathbf{X}} = 0$  implies that  $d > 1$ ,  $P(d > 0) > 0$ . A same reasoning as in the proof of Lemma 4.1 then leads to the result that  $P(d = \infty) = 1$ , and that  $\mathbf{X}$  is strictly decreasing, which contradicts the assumption in this case that  $\mathbf{X}$  is not strictly monotone. Therefore we conclude that  $P(\tau_{\mathbf{X}} = 0) = 0$ . Symmetrically,  $P(\tau_{\mathbf{X}} = 1) = 0$ .  $\square$

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