

## APPROXIMATION OF STABLE LAW IN WASSERSTEIN-1 DISTANCE BY STEIN'S METHOD<sup>1</sup>

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Let  $n \in \mathbb{N}$ , let  $\zeta_{n,1}, \dots, \zeta_{n,n}$  be a sequence of independent random variables with  $\mathbb{E}\zeta_{n,i} = 0$  and  $\mathbb{E}|\zeta_{n,i}| < \infty$  for each  $i$ , and let  $\mu$  be an  $\alpha$ -stable distribution having characteristic function  $e^{-|\lambda|^\alpha}$  with  $\alpha \in (1, 2)$ . Denote  $S_n = \zeta_{n,1} + \dots + \zeta_{n,n}$  and its distribution by  $\mathcal{L}(S_n)$ , we bound the Wasserstein-1 distance of  $\mathcal{L}(S_n)$  and  $\mu$  essentially by an  $L^1$  discrepancy between two kernels. More precisely, we prove the following inequality:

$$d_W(\mathcal{L}(S_n), \mu) \leq C \left[ \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt + \mathcal{R}_{N,n} \right],$$

where  $d_W$  is the Wasserstein-1 distance of probability measures,  $\mathcal{K}_\alpha(t, N)$  is the kernel of a decomposition of the fractional Laplacian  $\Delta^{\frac{\alpha}{2}}$ ,  $K_i(t, N)$  is a  $K$  function (*Normal Approximation by Stein's Method* (2011) Springer) with a truncation and  $\mathcal{R}_{N,n}$  is a small remainder. The integral term

$$\sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt$$

can be interpreted as an  $L^1$  discrepancy.

As an application, we prove a general theorem of stable law convergence rate when  $\zeta_{n,i}$  are i.i.d. and the distribution falls in the normal domain of attraction of  $\mu$ . To test our results, we compare our convergence rates with those known in the literature for four given examples, among which the distribution in the fourth example is not in the normal domain of attraction of  $\mu$ .

**1. Introduction.** Let  $n \in \mathbb{N}$  and let  $\zeta_{n,1}, \dots, \zeta_{n,n}$  be a sequence of independent random variables with  $\mathbb{E}\zeta_{n,i} = 0$  for each  $i$ , denote

$$S_n = \zeta_{n,1} + \dots + \zeta_{n,n}.$$

It is well known that  $S_n$  weakly converges to the standard normal distribution  $\Phi$  if this sequence satisfies the Lindeberg condition and  $\mathbb{E}S_n^2 \rightarrow 1$ . If we further assume

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that  $\mathbb{E}|\zeta_{n,i}|^3 < \infty$  for each  $i$ , then the Berry–Esseen theorem follows

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(S_n \leq x) - \Phi(x)| \leq C \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^3,$$

where  $C > 0$  is some constant.

Stein's method was put forward in the seminal work [45] to study normal approximations such as Berry–Esseen theorem, very soon thereafter Chen applied this method to get the convergence rate of the Poisson approximation [14]. Nowadays, Stein's method has been extended and refined by many authors and become a very important tool for getting bounds of measure approximations; see [1, 5, 6, 9, 11–13, 21, 23–27, 31, 35, 37–41, 43, 44]. For more references, we refer the reader to the webpages: <https://sites.google.com/site/steinsmethod/home> and <https://sites.google.com/site/malliavinstein/home>.

The stable distribution is one of the most important distributions in probability theory and has a lot of applications in economics, finance, physics and so on; see the monographs [30, 47] and the references therein for details. If the above sequence  $\{\zeta_{n,i}\}_{1 \leq i \leq n}$  are assumed to have a suitable heavy tail,  $S_n$  weakly converges to a stable distribution [22], Theorem 3.7.2. However, it seems that there are not many results about the rate of stable law convergence; see [7, 8, 10, 19, 20, 28, 34, 48]. Moreover, all these works are proved by the characteristics function method in Kolmogorov distance.

The goal of this paper is to study the  $\alpha$ -stable law approximation in Wasserstein-1 distance (it is often called  $W_1$  distance or  $L^1$  distance for simplicity) by Stein's method for  $\alpha \in (1, 2)$ . We prove two general theorems, one is a framework which gives a general bound for the  $W_1$  distance between  $S_n$  and  $\mu$ , the other is an application of the framework when  $\{\zeta_{n,i}\}_{1 \leq i \leq n}$  are i.i.d. and their distribution falls in the normal domain of attraction of  $\mu$ . It should be stressed that some known results can give the rate for  $\alpha \in (0, 1]$ , while ours is only for  $\alpha \in (1, 2)$ . The reason is stable distributions do not have 1st moment for  $\alpha \in (0, 1]$ , and the  $W_1$  distance is consequently not well defined in general. Therefore, our assumption  $\alpha \in (1, 2)$  is *essential*.

We apply the two theorems to four examples which have been studied by several authors [18, 20, 28, 34, 48] in Kolmogorov distance, and compare our convergence rates with theirs. A big advantage of our theorems is that one can obtain an explicit bound of convergence rather than only giving the order of rates as in the known literatures.

Our first example is a sequence of i.i.d. random variables having a Pareto distribution density  $p(x) = \frac{\alpha}{2|x|^{\alpha+1}} 1_{\{|x|>1\}}$ , whose sum scaled by  $n^{-1/\alpha}$  weakly converges to a symmetric stable distribution. A convergence rate  $n^{-\frac{2-\alpha}{\alpha}}$  in Kolmogorov distance was obtained in [34], while [20] proved a rate  $n^{-\frac{2-\alpha}{d+\alpha}}$  for  $d$ -dimensional stable law in total variation distance and conjectured that a better rate

should be  $n^{-\frac{2-\alpha}{\alpha}}$  in the  $L^1$  or total variation distance. Our result gives a positive answer to their conjecture for the  $L^1$  distance case when  $d = 1$ .

The second example is from [4, 34]. The distribution of i.i.d. random variables in [34] is a perturbed Pareto distribution with a density  $p(x) = (\frac{A}{|x|^{\alpha+1}} + \frac{B}{|x|^{\beta+1}})1_{\{|x|>a\}}$  for some  $A > 0, B > 0, a > 0$  and  $\beta > \alpha$ . We consider a more general distribution such that the distribution function  $F$  satisfies  $1 - F(x) = \frac{A}{|x|^\alpha} + \frac{B_1(x)}{|x|^\beta}, F(-x) = \frac{A}{|x|^\alpha} + \frac{B_2(x)}{|x|^\beta}$  for large  $x > 0$ , where  $\beta > \alpha, A > 0$  and  $B_1(x), B_2(x)$  are bounded continuous functions. It seems that the technique in [34] is not able to handle this general distribution case. In this paper, we obtain a convergence rate  $n^{-\frac{2-\alpha}{\alpha}}$  for  $\beta \in (2, \infty)$ , while [34] gives the same rate for  $\beta \in (2\alpha, \infty)$ . Note that the example in [4], Appendix B, is covered by this one by taking  $\beta = \alpha + 1$ .

The third example is a special case of [28] by Hall. When the limit distribution is symmetric stable, we can get a rate  $n^{-\frac{2-\alpha}{\alpha}}$  in some situations, while Hall obtained a rate  $n^{-\beta}$  for some  $0 < \beta < \frac{2-\alpha}{\alpha}$ .

The fourth example is from [48], the i.i.d. random variables therein have a density  $p(x) = \frac{C(\log|x|)^\beta}{|x|^{\alpha+1}}1_{\{|x|>c\}}$  with  $\beta \in \mathbb{R}$  and  $c, C > 0$ , which is not in the normal domain of attraction of a stable law. A convergence rate  $(\log n)^{-1}$  in Kolmogorov distance was proved by a very delicate analysis depending on the special form of the distribution. Using our first general theorem, we can obtain a rate  $(\log n)^{-1+\frac{1}{\alpha}}$  in  $W_1$  distance, which is worse than  $(\log n)^{-1}$ . However, our theorem can be used to study more examples which cannot be handled by the characteristics function method in [48] directly. We defer to give the details of this example in the [Appendix](#).

Let us now roughly explain the strategy of our method. In normal approximations, the  $K$  function approach [15] is to write

$$(1.1) \quad \mathbb{E}[S_n f(S_n)] = \sum_{i=1}^n \int_{-\infty}^{\infty} \mathbb{E}[K_i(t) f'(S_n(i) + t)] dt,$$

where  $S_n(i) = S_n - \zeta_{n,i}$  and  $K_i(t) = \mathbb{E}[\zeta_{n,i} 1_{\{0 \leq t \leq \zeta_{n,i}\}} - \zeta_{n,i} 1_{\{\zeta_{n,i} \leq t \leq 0\}}]$ , and bound its difference with  $\mathbb{E}[f'(S_n)]$ .

To prove the convergence rate of stable law, we shall find a solution  $f$  of the Stein equation, (4.9) below, and bound

$$\mathbb{E}\left[\Delta^{\frac{\alpha}{2}} f(S_n) - \frac{1}{\alpha} S_n f'(S_n)\right],$$

where  $\Delta^{\frac{\alpha}{2}}$  is the fractional Laplacian defined by (4.2) below. Inspired by the above observation of  $\mathbb{E}[S_n f(S_n)]$ , we represent

$$(1.2) \quad \mathbb{E}[S_n f'(S_n)] = \sum_{i=1}^n \int_{-N}^N \mathbb{E}[K_i(t, N) f''(S_n(i) + t)] dt + \mathcal{R},$$

where  $N > 0$  is an arbitrary number to be chosen later,  $\mathcal{R}$  is a remainder and

$$(1.3) \quad K_i(t, N) = \mathbb{E}[\zeta_{n,i} 1_{\{0 \leq t \leq \zeta_{n,i} \leq N\}} - \zeta_{n,i} 1_{\{-N \leq \zeta_{n,i} \leq t \leq 0\}}].$$

Due to the heavy tail property of  $\zeta_{n,i}$ , we need to truncate  $\zeta_{n,i}$  and thus get a remainder  $\mathcal{R}$ . On the other hand, we decompose  $\Delta^{\frac{\alpha}{2}} f$  into a linear combination of  $f''$  with a remainder  $\mathcal{R}'$  as the following:

$$(1.4) \quad \Delta^{\frac{\alpha}{2}} f(x) = \int_{-N}^N \mathcal{K}_\alpha(t, N) f''(x+t) dt + \mathcal{R}',$$

where

$$(1.5) \quad \mathcal{K}_\alpha(t, N) = \frac{d_\alpha}{\alpha(\alpha-1)} (|t|^{1-\alpha} - N^{1-\alpha})$$

$$\text{with } d_\alpha = \left( \int_{-\infty}^{\infty} \frac{1 - \cos y}{|y|^{1+\alpha}} dy \right)^{-1}.$$

Using (1.4) and (1.2), we see

$$(1.6) \quad \mathbb{E} \left[ \Delta^{\frac{\alpha}{2}} f(S_n) - \frac{1}{\alpha} S_n f'(S_n) \right]$$

$$= \sum_{i=1}^n \int_{-N}^N \mathbb{E} \left[ \left( \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right) f''(S_n(i) + t) \right] dt + \mathcal{R}'',$$

where  $\mathcal{R}''$  is another remainder. Hence,

$$\left| \mathbb{E} \left[ \Delta^{\frac{\alpha}{2}} f(S_n) - \frac{1}{\alpha} S_n f'(S_n) \right] \right| \leq \left( \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt \right) \|f''\| + |\mathcal{R}''|,$$

where  $\|f''\| = \sup_{x \in \mathbb{R}} |f''(x)|$ . Therefore, in order to obtain the convergence rate, it suffices to bound  $\|f''\|$  and the remainder  $\mathcal{R}''$ .

A recent result about stable convergence by Arras et al. [4], Appendix B, is as the following: for  $\alpha \in (1, 2)$ ,

$$d_{\text{Kol}}(\mathcal{L}(S_n), \mu) \leq C n^{-\frac{1}{2}(1-\frac{\alpha}{2})},$$

$$\sup_{h \in \mathcal{H}_3} \left| \mathbb{E}[h(S_n)] - \int_{\mathbb{R}} h(x) \mu(dx) \right| \leq C n^{\frac{2\alpha}{2\alpha+1}(\frac{1}{2}-\frac{1}{\alpha})},$$

where  $\mathcal{L}(S_n)$  is the distribution of  $S_n$ ,  $d_{\text{Kol}}$  denotes the Kolmogorov distance,  $\mu$  is a stable distribution with characteristic function  $e^{-|\lambda|^\alpha}$ , and  $\mathcal{H}_3$  is the set of all bounded third-order differentiable functions  $h$  such that  $\|h^{(k)}\| \leq 1$  for  $k = 0, 1, 2, 3$ . Their approach is by Stein–Tikhomirov method. Note that [4], Appendix B, is a special case of Example 2 below, in which we show by our general result that a rate  $n^{-\frac{2-\alpha}{\alpha}}$  in  $W_1$  distance can be achieved. By a standard argument, this  $W_1$  rate implies a Kolmogorov rate  $n^{-\frac{2-\alpha}{2\alpha}}$ , which is better than  $n^{-\frac{1}{2}(1-\frac{\alpha}{2})}$ .

More recently, Arras and Houdré found a nice characterization of infinitely divisible law with finite first moment [3], Theorem 3.1, and proved a general upper bound for  $d_{\text{Kol}}(\mu_n, \mu)$  by Fourier analysis as  $\mu_n$  and  $\mu$  are both infinitely divisible. This result was applied to study several examples such as compound Poisson random variables, Pareto type random variables sum and so on, in particular, if  $\mu_n$  is the distribution of a sum of i.i.d. infinitely divisible Pareto-type random variables, it converges to a stable distribution with a rate  $n^{-\frac{2-\alpha}{\alpha}}$  in Kolmogorov distance. They also derived a nice formulation of the related generators for the self-decomposable distribution family [3], Proposition 5.1, which generalized the result in our Lemma 4.6 below. Furthermore, using a methodology very similar to the one developed in our paper, the same authors proved a bound for self-decomposable distribution approximation in a smooth Wasserstein distance  $d_{W_2}$  by Stein’s method; see [3], Section 6. Applying [3], Theorems 6.1, 6.2, to stable approximations, from the discrepancy terms in the bounds therein, we can immediately see that the convergence rate is at most  $n^{-\frac{2-\alpha}{\alpha}}$  in  $d_{W_2}$  distance. Note that  $d_{W_2}$  is smaller than  $W_1$  distance [3], (4.3).

Johnson and Samworth [32] also give a convergence rate for stable approximations in the Mallows distance  $d_r$  with some  $r > 0$ , note that  $d_r$  is the classical Wasserstein- $r$  distance when  $r \geq 1$ . Let  $X_1, \dots, X_n$  be i.i.d. random variables with mean 0 and a distribution function  $F_X$  such that  $F_X(x) = \frac{c_1 + b_X(x)}{|x|^\alpha}$  for  $x < 0$  and  $1 - F_X(x) = \frac{c_2 + b_X(x)}{|x|^\alpha}$  for  $x > 0$ , where  $c_1, c_2 > 0$  and  $b_X(x) = O(\frac{1}{|x|^\gamma})$  with  $\gamma > 0$ , [32], Theorem 1.2, claims that  $S_n = n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i$  converges to a stable distribution  $\mu$  with a rate  $n^{\frac{1}{\beta} - \frac{1}{\alpha}}$  in the distance  $d_\beta$  for some  $\beta \in (\alpha, 2]$ . When  $\alpha \in (1, 2)$ , this rate is worse than the rate  $n^{-\frac{2}{\alpha} + 1}$  in our paper, but  $d_\beta$  is larger than  $W_1$  distance. Moreover, when  $\alpha \geq 1$  and  $\gamma \geq 1$ , one can take  $\beta = 2$  and thus gets a convergence rate  $n^{\frac{1}{2} - \frac{1}{\alpha}}$  in the Wasserstein-2 distance, which is not accessible by our Stein’s method. The theorem was proved by an idea from Lindeberg method and a coupling. More precisely, take a sequence of i.i.d.  $\mu$ -distributed random variables  $Y_1, \dots, Y_n$ , since  $n^{-1/\alpha}(Y_1 + \dots + Y_n)$  has the distribution  $\mu$ , it is easy to see that

$$\begin{aligned} d_\beta^\beta(\mathcal{L}(S_n), \mu) &= n^{-\frac{\beta}{\alpha}} d_\beta^\beta\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right) = n^{-\frac{\beta}{\alpha}} d_\beta^\beta\left(\sum_{i=1}^n X_i^*, \sum_{i=1}^n Y_i^*\right) \\ &\leq n^{-\frac{\beta}{\alpha}} \mathbb{E} \left| \sum_{i=1}^n (X_i^* - Y_i^*) \right|^\beta, \end{aligned}$$

where  $(X_i^*, Y_i^*)$  is a coupling of the distributions of  $X_i$  and  $Y_i$  [32], (3), which enjoys the property  $\mathbb{E}|X_i^* - Y_i^*|^\beta = d_\beta^\beta(X_i, Y_i)$  for each  $i$ , and  $\{(X_i^*, Y_i^*)\}_{1 \leq i \leq n}$  are independent. The previous relation, together with an inequality by von Bahr and Esseen [32], (11), (12), implies that  $d_\beta^\beta(\mathcal{L}(S_n), \mu) \leq 2n^{-\beta/\alpha} \sum_{i=1}^n d_\beta^\beta(X_i, Y_i)$ .

Since  $d_\beta(X_i, Y_i) < \infty$  for some  $\beta > \alpha$  [32], Lemma 5.1, one immediately gets  $d_\beta(\mathcal{L}(S_n), \mu) = O(n^{\frac{1}{\beta} - \frac{1}{\alpha}})$ .

The organization of the paper is as follows. Section 2 introduces notation and gives the two main theorems, while Section 3 applies them to study three examples. The proofs of the two main theorems are given in Sections 4 and 5, respectively, and the regularities of Stein's equation are proved in Section 6. The last section is an Appendix about the fourth example and some details of heat kernel estimates.

**2. Main results.** Recall that  $W_1$  distance between two probability measures  $\mu_1$  and  $\mu_2$  is defined by

$$(2.1) \quad d_W(\mu_1, \mu_2) = \inf_{(X, Y) \in \mathcal{C}(\mu_1, \mu_2)} \mathbb{E}|X - Y|,$$

where  $\mathcal{C}(\mu_1, \mu_2)$  is the set of all the coupling realizations of  $\mu_1, \mu_2$ . By a duality,

$$d_W(\mu_1, \mu_2) = \sup_{h \in \text{Lip}(1)} |\mu_1(h) - \mu_2(h)|,$$

where  $\text{Lip}(1) = \{h : \mathbb{R} \rightarrow \mathbb{R}; |h(y) - h(x)| \leq |y - x|\}$  and

$$\mu_i(h) = \int_{\mathbb{R}} h(x)\mu_i(dx), \quad i = 1, 2.$$

Note that  $d_W$  is also called  $L^1$  distance. The Kolmogorov distance of  $\mu_1$  and  $\mu_2$  is defined by

$$d_{\text{Kol}}(\mu_1, \mu_2) := \sup_{x \in \mathbb{R}} |\mu_1((-\infty, x]) - \mu_2((-\infty, x])|.$$

For a sequence of measures  $\{\nu_n\}_n$ , we say they weakly converge to a measure  $\nu$ , denoted by  $\nu_n \Rightarrow \nu$ , if

$$\lim_{n \rightarrow \infty} \nu_n(f) = \nu(f)$$

for  $f \in \mathcal{C}_b(\mathbb{R})$ , all bounded continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We use  $C_p$  to denote some number which depends on parameter  $p$ , the exact value of  $C_p$  may vary from line to line. We denote  $\mathcal{L}(X)$  the distribution of a given random variable  $X$ .

Recall (1.3) and (1.5) in the Introduction:

$$(2.2) \quad \mathcal{K}_\alpha(t, N) = \frac{d_\alpha}{\alpha(\alpha - 1)} (|t|^{1-\alpha} - N^{1-\alpha}),$$

$$(2.3) \quad \mathcal{K}_i(t, N) = \mathbb{E}[\zeta_{n,i} 1_{\{0 \leq t \leq \zeta_{n,i} \leq N\}} - \zeta_{n,i} 1_{\{-N \leq \zeta_{n,i} \leq t \leq 0\}}],$$

where  $d_\alpha = (\int_{-\infty}^{\infty} \frac{1 - \cos y}{|y|^{1+\alpha}} dy)^{-1}$  and  $1 \leq i \leq n$ . Note  $d_\alpha = \frac{\alpha 2^{\alpha-1} \Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}$  and  $\lim_{\alpha \uparrow 2} \frac{d_\alpha}{2-\alpha} = 1$ , see [17], p. 2800. Recall the Gamma and Beta functions are respectively defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0;$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0, y > 0.$$

Let us now state our first main result, which is a general theorem giving a rate of stable law convergence in  $W_1$  distance.

**THEOREM 2.1.** *Let  $n \in \mathbb{N}$  and let  $\zeta_{n,1}, \dots, \zeta_{n,n}$  be a sequence of independent random variables with  $\mathbb{E}\zeta_{n,i} = 0$  and  $\mathbb{E}|\zeta_{n,i}| < \infty$  for  $1 \leq i \leq n$ .<sup>2</sup> Let  $\mu$  be an  $\alpha$ -stable distribution with characteristic function  $e^{-|\lambda|^\alpha}$  for  $\alpha \in (1, 2)$ . Then we have*

$$d_W(\mathcal{L}(S_n), \mu) \leq D_\alpha \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt + \mathcal{R}_{N,n} \quad \forall N > 0,$$

where  $\mathcal{K}_\alpha(t, N)$  and  $K_i(t, N)$  are defined as above,  $D_\alpha = \frac{4}{\pi} \sqrt{\frac{2\alpha+1}{\alpha}} \mathbf{B}(\frac{\alpha-1}{\alpha}, \frac{2}{\alpha})$ ,

$$\begin{aligned} \mathcal{R}_{N,n} = & 2 \sum_{i=1}^n \mathbb{E}(|\zeta_{n,i}| 1_{\{|\zeta_{n,i}| > N\}}) + \frac{4d_\alpha}{\alpha - 1} \frac{1}{N^{\alpha-1}} \\ & + \frac{D_{\alpha,\gamma}}{n} \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^\gamma \quad \forall \gamma \in (0, 1), \end{aligned}$$

with  $D_{\alpha,\gamma} = \frac{d_\alpha}{\alpha} [\frac{16}{\pi(2-\alpha)} \sqrt{\frac{\alpha+3}{\alpha}} + \frac{16}{\pi(\alpha-1)} \sqrt{\frac{2\alpha+1}{\alpha}}] \mathbf{B}(\frac{1-\gamma}{\alpha}, \frac{\gamma+\alpha}{\alpha})$ .

**REMARK 2.2.** When  $\alpha \leq 1$ , the stable distribution does not have its 1st moment, thus the corresponding  $W_1$  is not well defined; see (2.1). It is expected that  $d_W(S_n, \mu) \rightarrow \infty$  as  $\alpha \downarrow 1$ , this can be seen from

$$\lim_{\alpha \downarrow 1} D_\alpha = \infty, \quad \lim_{\alpha \downarrow 1} D_{\alpha,\gamma} = \infty.$$

Moreover,  $\lim_{\alpha \uparrow 2} D_{\alpha,\gamma} = \frac{2\sqrt{5}}{\pi} \mathbf{B}(\frac{1-\gamma}{2}, \frac{\gamma+2}{2})$  though there is a term  $\frac{1}{2-\alpha}$  in  $D_{\alpha,\gamma}$ . Tables 1 and 2 give the values of  $D_\alpha$  and  $D_{\alpha,\gamma}$ , respectively. Although  $D_{\alpha,\gamma}$  is large, the term  $\frac{D_{\alpha,\gamma}}{n} \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^\gamma$  can be negligible in applications by taking  $\gamma > 2 - \alpha$  and large  $n$ .

**REMARK 2.3.** Due to the lack of concentration phenomena of heavy tailed random variables sum, we can only observe the convergence after sampling a large number of random variables; see [34], Section 5, and Example 1 below. In applications, we take  $\gamma = 0.9$  so that the term  $\frac{D_{\alpha,\gamma}}{n} \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^\gamma$  will be small enough to be negligible as  $n > 10^6$ .

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<sup>2</sup>Elton Hsu pointed out to the author that the condition “ $S_n \Rightarrow \mu$ ” in Theorem 2.1 of the first draft can be removed.

TABLE 1  
The values of  $D_\alpha$

$\alpha$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$D_\alpha$	22.14	11.45	8.04	6.42	5.51	4.94	4.57	4.32	4.15

REMARK 2.4. If  $X$  has a stable distribution  $\mu$  with characteristic function  $e^{-|\lambda|^\alpha}$ , then  $\sigma^{1/\alpha} X$  has a distribution  $\nu$  with characteristic function  $e^{-\sigma|\lambda|^\alpha}$ . By (2.1),

$$(2.4) \quad d_W(\mathcal{L}(\sigma^{1/\alpha} S_n), \nu) = \sigma^{1/\alpha} d_W(\mathcal{L}(S_n), \mu).$$

On the other hand, it is easy to see from the definition of Kolmogorov distance that

$$d_{\text{Kol}}(\mathcal{L}(\sigma^{1/\alpha} S_n), \nu) = d_{\text{Kol}}(\mathcal{L}(S_n), \mu).$$

Theorem 2.1 is a general theorem which bounds the  $W_1$  distance of  $\mathcal{L}(S_n)$  and  $\mu$  by a discrepancy and a small remainder. An application of this theorem is to study the convergence rate of stable law. To this end, we first recall the classical stable law convergence theorem.

THEOREM 2.5 (Theorem 3.7.2 of [22]). *Let  $\xi_1, \dots, \xi_n, \dots$  be i.i.d. with a distribution that satisfies*

$$(i) \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(\xi_1 > x)}{\mathbb{P}(|\xi_1| > x)} = \frac{1}{2}, \quad (ii) \quad \mathbb{P}(|\xi_1| > x) = x^{-\alpha} L(x),$$

where  $\alpha \in (0, 2)$  and  $L : [0, \infty) \rightarrow [0, \infty)$  is a slowly varying function, that is,  $\lim_{x \rightarrow \infty} \frac{L(tx)}{L(x)} = 1$  for all  $t > 0$ . Let  $T_n = \xi_1 + \dots + \xi_n$ ,  $A_n = \inf\{x : \mathbb{P}(|\xi_1| > x) \leq$

TABLE 2  
The values of  $D_{\alpha,\gamma}$

	$\alpha = 1.1$	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\gamma = 0.1$	33.13	19.01	14.40	12.17	10.89	10.09	9.55	9.18	8.91
0.2	35.17	20.33	15.50	13.17	11.83	11.00	10.45	10.06	9.79
0.3	38.59	22.43	17.17	14.64	13.20	12.30	11.70	11.30	11.01
0.4	43.94	25.62	19.66	16.80	15.17	14.16	13.49	13.04	12.71
0.5	52.30	30.53	23.45	20.05	18.12	16.92	16.13	15.59	15.21
0.6	65.91	38.43	29.49	25.20	22.76	21.24	20.24	19.55	19.07
0.7	90.04	52.33	40.06	34.16	30.79	28.69	27.31	26.36	25.68
0.8	140.69	81.33	62.00	52.67	47.34	44.00	41.78	40.25	39.16
0.9	298.18	171.06	129.58	109.52	98.02	90.78	85.95	82.58	80.16

$n^{-1}$ },  $B_n = n\mathbb{E}[\xi_1 1_{(|\xi_1| \leq A_n)}]$ . As  $n \rightarrow \infty$ ,  $(T_n - B_n)/A_n \Rightarrow \nu$ , where  $\nu$  is a symmetric stable distribution with characteristic function  $\exp(-\frac{\alpha|\lambda|^\alpha}{2d_\alpha})$ . In particular, it follows from the property of stable distribution (see Remark 2.4) that as  $n \rightarrow \infty$ ,

$$(2.5) \quad \left(\frac{\alpha}{2d_\alpha}\right)^{-\frac{1}{\alpha}} \frac{T_n - B_n}{A_n} \Rightarrow \mu,$$

where  $\mu$  is a symmetric stable distribution with characteristic function  $e^{-|\lambda|^\alpha}$ .

In [22], Theorem 3.7.2, the limit of (i) is a general  $c \in (0, 1)$  rather than  $\frac{1}{2}$ . When  $c \neq \frac{1}{2}$ , the limiting stable distribution  $\nu$  is not symmetric. From the remark in [22], p. 138, we know that the conditions (i) and (ii) are also necessary for the above weak convergence to stable law. Similar as studying a Berry–Esseen bound for a central limit theorem, we need to strengthen (i) and (ii) to get a rate for the convergence (2.5).

We assume that there exist some  $A > 0$  and two continuous functions  $M_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$  and  $M_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ , with  $\lim_{x \rightarrow \infty} M_1(x) = 0$  and  $\lim_{x \rightarrow \infty} M_2(x) = 0$ , such that for all  $x > A$ ,

$$(i') \quad \frac{\mathbb{P}(\xi_1 > x)}{\mathbb{P}(|\xi_1| > x)} = \frac{1 + M_1(x)}{2}, \quad (ii') \quad \frac{\mathbb{P}(|\xi_1| > x)}{\theta x^{-\alpha}} = 1 + M_2(x),$$

where  $\theta > 0$  is a constant. We note that (i') and (ii') are equivalent to the condition that  $\xi_1$  lies in the normal domain of attraction of  $\mu$ , which is generally stated as for all  $x > A$ ,

$$(2.6) \quad \mathbb{P}(\xi_1 > x) = c_1 x^{-\alpha}(1 + \delta_1(x)), \quad \mathbb{P}(\xi_1 < -x) = c_2 x^{-\alpha}(1 + \delta_2(x)),$$

where  $c_1, c_2 \geq 0$  with  $c_1 + c_2 > 0$  and  $\lim_{x \rightarrow \infty} \delta_1(x) = 0$  and  $\lim_{x \rightarrow \infty} \delta_2(x) = 0$ ; see [29], p. 350, or [32], Definition 5.1. If  $\delta_1$  and  $\delta_2$  both polynomially decay to 0, then we call  $\xi_1$  in the *strong* normal domain of attraction of  $\mu$ ; see [32], Definition 5.2. In our case,  $c_1 = c_2 = \frac{\theta}{2}$ .

Denote  $\ell_n = \frac{\alpha\theta}{2d_\alpha}n$  and  $b_t = \ell_n^{\frac{1}{\alpha}}t + \mathbb{E}\xi_1$  for  $t > 0$  and

$$(2.7) \quad R_t = \frac{1}{2}b_t^{-\alpha}(1 + M_1(b_t))(1 + M_2(b_t))\mathbb{E}\xi_1,$$

$$(2.8) \quad r_t = \frac{1}{2}b_t^{1-\alpha}[M_1 + M_2 + M_1M_2](b_t) + \frac{1}{2} \int_{b_t}^\infty s^{-\alpha}[M_1 + M_2 + M_1M_2](s) ds.$$

Our second main theorem, which is essentially an application of Theorem 2.1, is the following.

**THEOREM 2.6.** *Let  $\alpha \in (1, 2)$ , and let  $\xi_1, \dots, \xi_n, \dots$  be i.i.d. with a distribution satisfying the conditions (i') and (ii'). Write  $\zeta_{n,i} = \ell_n^{-\frac{1}{\alpha}}(\xi_i - \mathbb{E}\xi_i)$  and  $S_n = \zeta_{n,1} + \dots + \zeta_{n,n}$ , then  $\mathcal{L}(S_n) \Rightarrow \mu$  with characteristic function  $e^{-|\lambda|^\alpha}$ . Moreover, we have*

$$d_W(\mathcal{L}(S_n), \mu) \leq \frac{D_\alpha}{\alpha} \int_{-N}^N |\alpha \mathcal{K}_\alpha(t, N) - n K_1(t, N)| dt + \mathcal{R}_{N,n} \quad \forall N > 0,$$

where

$$\begin{aligned} \mathcal{R}_{N,n} &= D_{\alpha,\gamma} \ell_n^{-\frac{\gamma}{\alpha}} \mathbb{E}|\xi_1 - \mathbb{E}\xi_1|^\gamma \\ &+ \frac{4d_\alpha}{\delta_n^{\alpha-1}} \left( \frac{1 + \delta_n^{\alpha-1}}{\alpha - 1} + \frac{1}{\delta_n} + \frac{M_2(\ell_n^{\frac{1}{\alpha}} N \delta_n)}{\delta_n} + \int_{\delta_n}^\infty \frac{M_2(r \ell_n^{\frac{1}{\alpha}} N)}{r^\alpha \delta_n^{1-\alpha}} dr \right) N^{1-\alpha} \\ &\forall \gamma \in (0, 1), \end{aligned}$$

with  $\delta_n = 1 - \ell_n^{-\frac{1}{\alpha}} N^{-1} |\mathbb{E}\xi_1|$ . In particular, if  $\mathbb{E}\xi_1 = 0$ , we have

$$\begin{aligned} \mathcal{R}_{N,n} &= D_{\alpha,\gamma} \mathbb{E}|\xi_1|^\gamma \ell_n^{-\frac{\gamma}{\alpha}} \\ (2.9) \quad &+ 4d_\alpha \left( \frac{\alpha + 1}{\alpha - 1} + M_2(\ell_n^{\frac{1}{\alpha}} N) + \int_1^\infty \frac{M_2(\ell_n^{\frac{1}{\alpha}} N r)}{r^\alpha} dr \right) N^{1-\alpha}. \end{aligned}$$

It is worthy of stating the following corollary of Theorem 2.6, from which we can fast determine the order of convergence rates.

**COROLLARY 2.7.** *Assume that the same conditions as in Theorem 2.6 hold. We have*

$$\begin{aligned} d_W(\mathcal{L}(S_n), \mu) &\leq C_\alpha \left( n^{-\frac{2-\alpha}{\alpha}} + N^{1-\alpha} + n^{1-\frac{1}{\alpha}} N |r_N| \right. \\ &\quad \left. + n^{1-\frac{1}{\alpha}} \int_{4(A+|\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}} \leq |t| \leq N} |r_t| dt \right). \end{aligned}$$

We end this section with the following lemma, which will be used from time to time later.

**LEMMA 2.8.** *Let  $X$  be a random variable, for any  $t > 0$  we have*

$$(2.10) \quad \mathbb{E}[X 1_{\{X > t\}}] = t \mathbb{P}(X > t) + \int_t^\infty \mathbb{P}(X > r) dr.$$

PROOF. Observe by Fubini’s theorem that

$$\begin{aligned}
 \mathbb{E}[X1_{\{X>t\}}] &= \int_0^\infty \mathbb{E}[1_{\{0\leq r < X\}}1_{\{X>t\}}] dr \\
 (2.11) \qquad &= \int_0^t \mathbb{E}[1_{\{X>t\}}] dr + \int_t^\infty \mathbb{E}[1_{\{X>r\}}] dr,
 \end{aligned}$$

from which we immediately obtain the inequality in the lemma, as desired.  $\square$

**3. Three examples.** In this section, we shall use our results to study three examples which have been considered [4, 20, 28, 34], the known literatures only gave the order of convergence rates in Kolmogorov distance. In contrast, using our Theorems 2.1 and 2.6, we can obtain explicit bounds for these examples in  $W_1$  distance, and fast determine the order of the convergence rates by Corollary 2.7. In the regime  $\alpha \in (1, 2)$ , most of our results are as good as or better than the known ones.

In the Appendix, we further consider the fourth example which is out of the scope of normal domain of attraction of stable law. A related example was studied in [48] and the convergence rate in Kolmogorov distance is  $(\log n)^{-1}$ . By our results, we obtain a rate  $(\log n)^{-1+\frac{1}{\alpha}}$ . Because the calculation is very complicated and long, we will not give an explicit bound but only figure out its leading order in the Appendix.

EXAMPLE 1 (Pareto distribution case [20, 34]). Assume that  $\xi_1, \dots, \xi_n, \dots$  be i.i.d. with a Pareto distribution with  $\alpha \in (1, 2)$ , that is,

$$\mathbb{P}(\xi_1 \geq x) = \frac{1}{2|x|^\alpha}, \quad x \geq 1, \quad \mathbb{P}(\xi_1 \leq x) = \frac{1}{2|x|^\alpha}, \quad x \leq -1,$$

that is,  $\xi_1$  has a density function  $p(x)$ :

$$p(x) = 0, \quad |x| \leq 1; \quad p(x) = \frac{\alpha}{2|x|^{\alpha+1}}, \quad |x| > 1.$$

By Theorem 2.5, we have  $B_n = 0$  and  $A_n = n^{1/\alpha}$ . Denote  $\ell_n = \frac{\alpha}{2d_\alpha}n$  and

$$\zeta_{n,i} = \ell_n^{-\frac{1}{\alpha}} \xi_i, \quad i = 1, \dots, n,$$

$S_n$  weakly converges to a stable distribution  $\mu$  with characteristic function  $e^{-|\lambda|^\alpha}$ . We can directly apply Theorem 2.6 to get a convergence rate  $n^{-\frac{2-\alpha}{\alpha}}$ , but it is very instructive to prove this rate by applying Theorem 2.1 directly.

It is straightforward to check that the terms in  $\mathcal{R}_{N,n}$  are

$$\begin{aligned}
 2 \sum_{i=1}^n \mathbb{E}(|\zeta_{n,i}| 1_{\{|\zeta_{n,i}|>N\}}) &= \frac{4d_\alpha}{\alpha - 1} N^{1-\alpha}, \\
 \frac{D_{\alpha,\gamma}}{n} \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^\gamma &= \frac{\alpha D_{\alpha,\gamma}}{\alpha - \gamma} \left(\frac{2d_\alpha}{\alpha}\right)^\frac{\gamma}{\alpha} n^{-\frac{\gamma}{\alpha}},
 \end{aligned}$$

thus

$$\mathcal{R}_{N,n} = \frac{8d_\alpha}{\alpha - 1} N^{1-\alpha} + \frac{\alpha D_{\alpha,\gamma}}{\alpha - \gamma} \left(\frac{2d_\alpha}{\alpha}\right)^{\frac{\gamma}{\alpha}} n^{-\frac{\gamma}{\alpha}}.$$

It remains to compute the integral term in the bound of Theorem 2.1. Recall (2.3), when  $t \geq 0$ ,

$$\begin{aligned} K_1(t, N) &= \mathbb{E}[\zeta_{n,1} 1_{\{0 \leq t \leq \zeta_{n,1} \leq N\}}] \\ &= \ell_n^{-\frac{1}{\alpha}} \int_{\ell_n^{1/\alpha} t}^{\ell_n^{1/\alpha} N} x p(x) dx \\ (3.1) \quad &= \frac{\alpha}{2\ell_n(\alpha - 1)} \left[ \left(t \vee \frac{1}{\ell_n^{1/\alpha}}\right)^{-\alpha+1} - \frac{1}{N^{\alpha-1}} \right]. \end{aligned}$$

By the symmetry property of  $p(x)$ , we have

$$(3.2) \quad K_1(t, N) = \frac{\alpha}{2\ell_n(\alpha - 1)} \left[ \left(|t| \vee \frac{1}{\ell_n^{1/\alpha}}\right)^{-\alpha+1} - \frac{1}{N^{\alpha-1}} \right], \quad t \leq 0.$$

Hence,

$$\begin{aligned} &\sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt \\ (3.3) \quad &= \int_{-N}^N \left| \frac{d_\alpha}{\alpha(\alpha - 1)} \left( \frac{1}{|t|^{\alpha-1}} - \frac{1}{N^{\alpha-1}} \right) - \frac{nK_1(t, N)}{\alpha} \right| dt \\ &= \frac{d_\alpha}{\alpha(\alpha - 1)} \int_{-N}^N \left| \frac{1}{|t|^{\alpha-1}} - \left( |t| \vee \frac{1}{\ell_n^{1/\alpha}} \right)^{-\alpha+1} \right| dt \\ &= \frac{1}{2 - \alpha} \left( \frac{2d_\alpha}{\alpha} \right)^{\frac{2}{\alpha}} n^{-\frac{2-\alpha}{\alpha}}. \end{aligned}$$

So, we have

$$\begin{aligned} (3.4) \quad d_W(\mathcal{L}(S_n), \mu) &\leq \frac{8d_\alpha}{\alpha - 1} N^{1-\alpha} + \frac{D_\alpha}{2 - \alpha} \left(\frac{2d_\alpha}{\alpha}\right)^{\frac{2}{\alpha}} n^{-\frac{2-\alpha}{\alpha}} \\ &\quad + \frac{\alpha D_{\alpha,\gamma}}{\alpha - \gamma} \left(\frac{2d_\alpha}{\alpha}\right)^{\frac{\gamma}{\alpha}} n^{-\frac{\gamma}{\alpha}}. \end{aligned}$$

Since  $N$  is arbitrary, let  $N \rightarrow \infty$ , we get

$$\begin{aligned} (3.5) \quad d_W(\mathcal{L}(S_n), \mu) &\leq \frac{D_\alpha}{2 - \alpha} \left(\frac{2d_\alpha}{\alpha}\right)^{\frac{2}{\alpha}} n^{-\frac{2-\alpha}{\alpha}} \\ &\quad + \frac{\alpha D_{\alpha,\gamma}}{\alpha - \gamma} \left(\frac{2d_\alpha}{\alpha}\right)^{\frac{\gamma}{\alpha}} n^{-\frac{\gamma}{\alpha}} \quad \forall \gamma \in (0, 1). \end{aligned}$$

TABLE 3  
Exact bounds of  $d_W(\mathcal{L}(S_n), \mu)$  with  $n = 10^6$

	$\alpha = 1.1$	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\gamma = 0.1$	9.906	6.245	5.121	4.636	4.424	4.399	4.588	5.114	6.174
0.2	3.176	2.213	1.975	1.925	1.970	2.112	2.418	3.030	4.154
0.3	1.066	0.818	0.792	0.833	0.921	1.087	1.407	2.032	3.177
0.4	0.377	0.317	0.333	0.380	0.462	0.617	0.926	1.544	2.694
0.5	0.142	0.131	0.149	0.186	0.255	0.396	0.692	1.300	2.451
0.6	0.059	0.058	0.073	0.101	0.160	0.289	0.576	1.177	2.327
0.7	0.027	0.029	0.040	0.063	0.115	0.238	0.518	1.114	2.263
0.8	0.016	0.018	0.026	0.046	0.095	0.214	0.490	1.084	2.232
0.9	0.014	0.015	0.023	0.042	0.091	0.210	0.487	1.081	2.230

Let us compare our result with the known results in literatures. The reference [34] gave a convergence rate:

$$d_{\text{Kol}}(\mathcal{L}(S_n), \mu) \leq C_\alpha \begin{cases} n^{-\frac{2-\alpha}{\alpha}}, & \alpha \in (1, 2), \\ n^{-1}, & \alpha \in (0, 1], \end{cases}$$

where an exact value of  $C_\alpha$  was not given.

When  $\alpha \in (1, 2)$ , the authors of [20] obtained a rate  $n^{-\frac{2-\alpha}{d+\alpha}}$  for  $d$  dimensional stable law in total variation distance and conjectured that the rate can be improved to  $n^{-\frac{2-\alpha}{\alpha}}$  in  $L^1$  or total variation distance. Our result gives a positive answer to their conjecture for the  $L^1$  distance case when  $d = 1$ .

Table 3 gives exact bounds of  $d_W(\mathcal{L}(S_n), \mu)$  as  $n = 10^6$  according to (3.5), which vary according to  $\alpha$  and  $\gamma$ . Due to the lack of concentration phenomena in heavy detailed random variables sum, in simulations one has to take large samples (often more than  $10^6$ ) to observe the convergence. [34], Section 5, only simulated the limiting behavior of  $\mathbb{E}|S_n|$  for  $\alpha = 1.5$ , the convergence can be well observed only after the sample size reaches  $10^6$ .

EXAMPLE 2 (Convergence rate of Pareto densities with modified tails [34], Section 3, [4], Appendix B). In [34], Section 3, a sequence of i.i.d. random variables  $(\xi_n)_{n \geq 1}$  with the following density were considered:

$$(3.6) \quad p(x) = \frac{A}{|x|^{1+\alpha}} + \frac{B}{|x|^{1+\beta}} \quad \text{for } |x| > a, \quad p(x) = 0 \quad \text{for } |x| \leq a,$$

where  $0 < \alpha < 2$ ,  $\alpha < \beta$ ,  $A > 0$  and  $B > 0$ . When  $\beta > 2\alpha$ , it was proved that  $n^{-1/\alpha} \sum_{i=1}^n \xi_i$  converges to a stable distribution  $\nu$  in Kolmogorov distance with a rate  $n^{-\frac{2-\alpha}{\alpha}}$  for  $\alpha \in (1, 2)$  and a rate  $n^{-1}$  for  $\alpha \in (0, 1]$ . When  $\beta \in (\alpha, 2\alpha)$ , the rate is  $n^{-\min(\frac{\beta}{\alpha}-1, \frac{2-\alpha}{\alpha})}$  for  $\beta \neq 2$  and  $n^{-\frac{2-\alpha}{\alpha}} \log n$  for  $\beta = 2$ ; see [34], (3.6), Table 1.

We now determine an explicit bound for  $d_W(\mathcal{L}(S_n), \mu)$  by Theorem 2.6. Without loss of generality, we assume  $a = 1$  (otherwise take  $\tilde{\xi}_i = a^{-1}\xi_i$ ), and thus have

$$(3.7) \quad p(x) = \frac{A}{2|x|^{1+\alpha}} + \frac{B}{2|x|^{1+\beta}} \quad \text{for } |x| > 1, \quad p(x) = 0 \quad \text{for } |x| \leq 1,$$

with  $\frac{A}{\alpha} + \frac{B}{\beta} = 1$  and  $\beta \in (\alpha, \infty)$ . We can easily determine  $\theta = \frac{A}{\alpha}$ ,  $\ell_n = \frac{A}{2d_\alpha}n$  and  $S_n = \ell_n^{-\frac{1}{\alpha}} \sum_{i=1}^n \xi_i$  and

$$M_1(x) = 0, \quad M_2(x) = (B\alpha)(A\beta)^{-1}x^{-(\beta-\alpha)}, \quad x \geq 1.$$

To use Theorem 2.6, we need to compute  $n\mathcal{K}_\alpha(t, N)$  and  $\mathcal{R}_{N,n}$  therein. By a straightforward calculation, we have

$$\begin{aligned} n\mathcal{K}_1(t, N) &= \frac{d_\alpha}{\alpha - 1} \left( (|t| \vee \ell_n^{-\frac{1}{\alpha}})^{1-\alpha} - N^{1-\alpha} \right) \\ &\quad + \frac{Bn\ell_n^{-\frac{\beta}{\alpha}}}{2(\beta - 1)} \left( (|t| \vee \ell_n^{-\frac{1}{\alpha}})^{1-\beta} - N^{1-\beta} \right). \end{aligned}$$

By a similar computation as in Example 1, when  $\beta \neq 2$ ,

$$\int_{-N}^N |\alpha\mathcal{K}_\alpha(t, N) - n\mathcal{K}_1(t, N)| dt = \frac{2d_\alpha\ell_n^{\frac{\alpha-2}{\alpha}}}{2-\alpha} + \frac{2Ad_\alpha}{B(\beta-2)} (\ell_n^{\frac{\alpha-2}{\alpha}} - \ell_n^{\frac{\alpha-\beta}{\alpha}} N^{2-\beta});$$

when  $\beta = 2$ ,

$$\int_{-N}^N |\alpha\mathcal{K}_\alpha(t, N) - n\mathcal{K}_1(t, N)| dt = \frac{2d_\alpha\ell_n^{\frac{\alpha-2}{\alpha}}}{2-\alpha} + \frac{2Ad_\alpha}{B} \ell_n^{\frac{\alpha-2}{\alpha}} \left[ \log N + \frac{1}{\alpha} \log \ell_n \right].$$

By (2.9), we immediately obtain

$$(3.8) \quad \begin{aligned} \mathcal{R}_{N,n} &= D_{\alpha,\gamma} \left( \frac{A}{\alpha-\gamma} + \frac{B}{\beta-\gamma} \right) \ell_n^{-\frac{\gamma}{\alpha}} \\ &\quad + 4d_\alpha \left[ \frac{\alpha+1}{\alpha-1} + \frac{B\alpha}{A(\beta-1)} \ell_n^{\frac{\alpha-\beta}{\alpha}} N^{\alpha-\beta} \right] N^{1-\alpha}. \end{aligned}$$

Combining the previous relations, we immediately obtain an explicit bound for  $d_W(\mathcal{L}(S_n), \mu)$ , which has a leading term and a remainder  $\mathcal{R}(n)$ , both having explicit values. More precisely, (note  $\ell_n = \frac{A}{2d_\alpha}n$ ), we have

(1) When  $\beta > 2$ , take  $N \rightarrow \infty$ ,

$$d_W(\mathcal{L}(S_n), \mu) \leq \frac{2d_\alpha D_\alpha}{\alpha} \left( \frac{1}{2-\alpha} + \frac{A}{B(\beta-2)} \right) \ell_n^{-\frac{2-\alpha}{\alpha}} + \mathcal{R}(n)$$

with

$$\mathcal{R}(n) = D_{\alpha,\gamma} \left( \frac{A}{\alpha-\gamma} + \frac{B}{\beta-\gamma} \right) \ell_n^{-\frac{\gamma}{\alpha}}.$$

(2) When  $\beta = 2$ , take  $N = \ell_n^q$  with  $q \geq \frac{2-\alpha}{\alpha(\alpha-1)}$  being arbitrary,

$$d_W(\mathcal{L}(S_n), \mu) \leq \frac{2d_\alpha D_\alpha A(\alpha q + 1)}{\alpha^2 B} \ell_n^{-\frac{2-\alpha}{\alpha}} \log \ell_n + \mathcal{R}(n)$$

with

$$\begin{aligned} \mathcal{R}(n) &= \frac{2d_\alpha D_\alpha}{(2-\alpha)\alpha} \ell_n^{-\frac{2-\alpha}{\alpha}} + D_{\alpha,\gamma} \left( \frac{A}{\alpha-\gamma} + \frac{B}{2-\gamma} \right) \ell_n^{-\frac{\gamma}{\alpha}} \\ &\quad + 4d_\alpha \left( \frac{\alpha+1}{\alpha-1} \ell_n^{-(\alpha-1)q} + \frac{\alpha B}{A} \ell_n^{-\frac{2-\alpha}{\alpha}-q} \right). \end{aligned}$$

(3) When  $\alpha < \beta < 2$ , take  $N = \ell_n^q$  with  $q = \frac{\beta-\alpha}{\alpha(\alpha+1-\beta)}$  being arbitrary,

$$d_W(\mathcal{L}(S_n), \mu) \leq 2d_\alpha \left[ \frac{D_\alpha}{(2-\beta)\alpha} + \frac{2(\alpha+1)}{\alpha-1} \right] \ell_n^{-\frac{(\beta-\alpha)(\alpha-1)}{\alpha(\alpha+1-\beta)}} + \mathcal{R}(n)$$

with

$$\begin{aligned} \mathcal{R}(n) &= 2d_\alpha \left( \frac{1}{2-\alpha} - \frac{A}{(2-\beta)B} \right) \ell_n^{-\frac{2-\alpha}{\alpha}} + D_{\alpha,\gamma} \left( \frac{A}{\alpha-\gamma} + \frac{B}{2-\gamma} \right) \ell_n^{-\frac{\gamma}{\alpha}} \\ &\quad + \frac{4A\alpha d_\alpha}{B(\beta-1)} \ell_n^{-\frac{2\alpha+2+\alpha\beta-3\beta-\alpha^2}{\alpha(\alpha+1-\beta)}}. \end{aligned}$$

Note that the case (1) covers the example considered in [4], Appendix B, in which  $\beta = 1 + \alpha$ . By a standard argument, the bound in (1) implies

$$d_{\text{Kol}}(\mathcal{L}(S_n), \mu) \leq Cn^{-\frac{2-\alpha}{2\alpha}},$$

which is better than the rate  $n^{-\frac{1}{2}(1-\frac{\alpha}{2})}$  in [4].

We can consider a more general distribution:

$$(3.9) \quad \begin{aligned} \mathbb{P}(\xi_1 > x) &= \frac{A}{2|x|^\alpha} + \frac{B_1(x)}{2|x|^\beta}, & x > a, \\ \mathbb{P}(\xi_1 < x) &= \frac{A}{2|x|^\alpha} + \frac{B_2(x)}{2|x|^\beta}, & x < -a, \end{aligned}$$

where  $\alpha \in (1, 2)$ ,  $\alpha < \beta$ ,  $a > 0$ ,  $B_1(x)$  and  $B_2(x)$  are both continuous functions such that  $-L \leq B_1(x), B_2(x) \leq L$  for all  $x \in \mathbb{R}$  and some constant  $L > 0$ . Take  $\theta = A$  and  $\ell_n = \frac{\theta\alpha}{2d_\alpha}n$ , we have

$$S_n \Rightarrow \mu \quad \text{with } S_n = \ell_n^{-\frac{1}{\alpha}} \sum_{i=1}^n (\xi_i - \mathbb{E}\xi_i).$$

By Theorem 2.6, we can obtain an explicit bound for  $d_W(\mathcal{L}(S_n), \mu)$  by a similar but much more complicated calculation. Here, we would like to omit the detailed

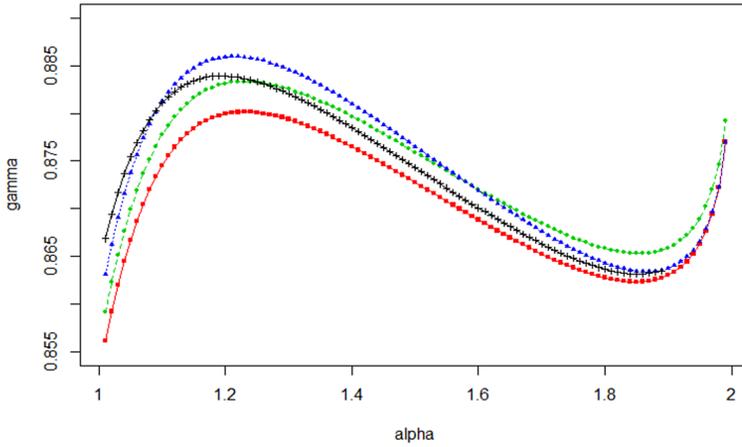


FIG. 1. The optimal choice of  $\gamma$  with  $n = 10^6$ .

calculation but get the order of the rate. More precisely, by Corollary 2.7 we have

$$d_W(\mathcal{L}(S_n), \mu) \leq C(n^{\frac{\alpha-2}{\alpha}} + N^{1-\alpha} + N^{2-\beta}n^{1-\frac{\beta}{\alpha}}), \quad \beta \neq 2,$$

$$d_W(\mathcal{L}(S_n), \mu) \leq C(n^{\frac{\alpha-2}{\alpha}} + N^{1-\alpha} + n^{1-\frac{2}{\alpha}} \log n + n^{1-\frac{2}{\alpha}} \log N), \quad \beta = 2,$$

where  $C$  depends on  $\alpha, \beta, a, L, A, B$ . Hence,

(i) When  $\beta > 2$ , let  $N \rightarrow \infty$ , we get

$$d_W(\mathcal{L}(S_n), \mu) \leq Cn^{-\frac{2-\alpha}{\alpha}}.$$

(ii) When  $\beta = 2$ , taking  $N = n^{\frac{2-\alpha}{\alpha(\alpha-1)}}$ , we get

$$d_W(\mathcal{L}(S_n), \mu) \leq Cn^{-\frac{2-\alpha}{\alpha}} \log n.$$

(iii) When  $\alpha < \beta < 2$ , taking  $N = n^{\frac{\beta-\alpha}{\alpha(1+\alpha-\beta)}}$ , we have

$$d_W(\mathcal{L}(S_n), \mu) \leq Cn^{-\frac{(\alpha-1)(\beta-\alpha)}{\alpha(1+\alpha-\beta)}}.$$

As seen from the results in the previous two examples, the  $\gamma$  in the bounds of  $d_W(\mathcal{L}(S_n), \mu)$  may vary from 0 to 1; its optimal choice depends on  $\alpha$  and  $n$ . When  $n = 10^6$ , we plot Figure 1 to demonstrate the optimal  $\gamma$  as a function of  $\alpha$  for  $\alpha = 1 + j/100$  with  $j \in \{1, \dots, 99\}$  for the following four cases: (1) Example 1 (red line); (2) Example 2 with  $\beta = 4$  and  $A = B = \frac{\alpha\beta}{\alpha+\beta}$  (green line); (3) Example 2 with  $A = B$  and  $\beta = 2$  (blue line); (4) Example 2 with  $A = B$  and  $\beta = \alpha + 0.1$  (black line).

EXAMPLE 3 (An example of Hall [28]). Let  $Z_1, \dots, Z_n, \dots$  be a sequence of i.i.d. random variables such that  $Z_1$  has a density function  $f(x)$  in the interval  $-\varepsilon < x < \varepsilon$  for some  $\varepsilon > 0$ . For further use, we denote  $f(x) = a_0 + h_1(x)$  for  $x \in [0, \varepsilon)$  and  $f(x) = b_0 + h_2(|x|)$  for  $x \in (-\varepsilon, 0]$  where  $a_0 > 0, b_0 > 0, h_1, h_2$  are both continuous positive functions from  $[0, \varepsilon)$  to  $\mathbb{R}^+$ . Let  $X_i = \text{sgn}(Z_i)|Z_i|^{-\frac{1}{\alpha}}$ , Hall studied the convergence rate of the following sum:

$$(3.10) \quad n^{-\frac{1}{\alpha}} \left[ \sum_{i=1}^n X_i - k_n(\alpha) \right],$$

where  $k_n(\alpha)$  is some number and  $0 < \alpha < 2$ , he proved the following.

THEOREM 3.1 (Theorem 3 of [28]). Suppose  $1 < \alpha < 2, f(x) = a_0 + h_1(x)$  for  $x \in [0, \varepsilon)$  and  $f(x) = b_0 + h_2(|x|)$  for  $x \in (-\varepsilon, 0]$  where  $|h_1(x)| + |h_2(x)| \leq b|x|^c$  for some  $c > 0$  with  $\alpha(c + 1) < 2$ , that is,  $0 < c < \frac{2-\alpha}{\alpha}$ , then  $n^{-\frac{1}{\alpha}}(\sum_{i=1}^n X_i - n\mathbb{E}X_1)$  weakly converges to a stable distribution with the distribution function  $A(x)$ . Moreover,

$$(3.11) \quad \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left[ n^{-\frac{1}{\alpha}} \left( \sum_{i=1}^n X_i - n\mathbb{E}X_1 \right) \leq x \right] - A(x) \right| = O(n^{-c}).$$

As an application of the case (3.7) above, we can study a special case of (3.10) and give an explicit bound of the convergence in  $W_1$  distance. More precisely, let  $\alpha \in (1, 2)$ , we assume  $a_0 = b_0 = a$  and  $h_1(x) = h_2(x) = bx^c$  for  $[0, 1]$ , by a straightforward calculation,

$$\begin{aligned} \mathbb{P}(X_1 > x) &= \frac{2a}{2x^\alpha} + \frac{\frac{2b}{c+1}}{2x^{\alpha(c+1)}}, & x > 1; \\ \mathbb{P}(X_1 < x) &= \frac{2a}{2|x|^\alpha} + \frac{\frac{2b}{c+1}}{2|x|^{\alpha(c+1)}}, & x < -1. \end{aligned}$$

We have the following theorem about explicit bound of convergence rate in  $W_1$  distance.

THEOREM 3.2. Let the above assumptions hold. Take  $A = 2a, B = \frac{2b}{c+1}, \beta = \alpha(c + 1), \ell_n = \frac{A\alpha}{2d_\alpha}n$  and  $S_n = \ell_n^{-\frac{1}{\alpha}} \sum_{i=1}^n \xi_i$ , we have  $S_n \Rightarrow \mu$  with  $\mu$  a stable distribution having characteristic function  $e^{-|\lambda|^\alpha}$ . Moreover,  $d_W(\mathcal{L}(S_n), \mu)$  has an explicit bound the same as the cases (1)–(3) in Example 2. More precisely,

- If  $\beta > 2$ , that is,  $c > \frac{2-\alpha}{\alpha}$ , the bound of the case (1) holds with leading order  $n^{-\frac{2-\alpha}{\alpha}}$ .
- If  $\beta = 2$ , that is,  $c = \frac{2-\alpha}{\alpha}$ , the bound of the case (2) holds with leading order  $n^{-\frac{2-\alpha}{\alpha}} \log n$ .

- If  $\alpha < \beta < 2$ , that is,  $c \in (0, \frac{2-\alpha}{\alpha})$ , the bound of the case (3) holds with leading order  $n^{-\frac{(\alpha-1)c}{1-\alpha c}}$ .

We can give the exact values of coefficients in Theorem 3.2 and those of  $d_W(\mathcal{L}(S_n), \mu)$ , the results are the same as those in Example 2, thus we omit them. Similar as the case (3.9) in the previous example, we can apply Theorem 2.6 to more general distributions and get an explicit bound for the corresponding  $d_W(\mathcal{L}(S_n), \mu)$  by much more complicated calculations. Here, we only use Corollary 2.7 to give the order of the convergence rate as the following.

**THEOREM 3.3.** *Suppose  $1 < \alpha < 2$ ,  $a_0 = b_0 = a$  and  $|h_1(x)| + |h_2(x)| \leq b|x|^c$  for  $x \in [0, \varepsilon)$  with  $b > 0, \varepsilon > 0, c > 0$ . Let  $\ell_n = \frac{a\alpha}{d_\alpha}n$ , then  $S_n := \ell_n^{-\frac{1}{\alpha}}(\sum_{i=1}^n X_i - n\mathbb{E}X_1)$  weakly converges to a stable distribution  $\mu$  with characteristic function  $e^{-|\lambda|^\alpha}$ . Moreover,*

$$(3.12) \quad d_W(\mathcal{L}(S_n), \mu) \leq C \begin{cases} n^{-\frac{2-\alpha}{\alpha}}, & c > \frac{2-\alpha}{\alpha}, \\ n^{-\frac{2-\alpha}{\alpha}} \log n, & c = \frac{2-\alpha}{\alpha}, \\ n^{-\frac{(\alpha-1)c}{1-\alpha c}}, & 0 < c < \frac{2-\alpha}{\alpha}. \end{cases}$$

**4. Proof of Theorem 2.1: Stein's method.**

4.1. *Stein's equation and its regularity estimates.* Let us first recall the definition of fractional Laplacian  $\Delta^{\frac{\alpha}{2}}$  with  $\alpha \in (0, 2)$ ; see, for instance, [36]. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function, for any  $x \in \mathbb{R}$ ,  $\Delta^{\frac{\alpha}{2}} f(x)$  is defined by

$$(4.1) \quad \Delta^{\frac{\alpha}{2}} f(x) = d_\alpha \left( \text{p.v.} \int_{\mathbb{R}} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy \right),$$

provided the principal value  $\text{p.v.} \int_{\mathbb{R}} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy$  exists, where  $d_\alpha = (\int_{-\infty}^{\infty} \frac{1 - \cos y}{|y|^{1+\alpha}} dy)^{-1}$  and

$$\text{p.v.} \int_{\mathbb{R}} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy = \lim_{r \downarrow 0} \int_{\mathbb{R} \setminus (-r, r)} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy.$$

The definition (4.1) with a principle value is not convenient for use, if some suitable regularity of  $f$  is further assumed,  $\Delta^{\alpha/2} f(x)$  can be rewritten in a form without limit.

When  $\alpha > 1$ , if  $f'$  and  $f''$  are both bounded, then  $\Delta^{\alpha/2} f(x)$  is well defined for all  $x \in \mathbb{R}$  and can be rewritten as

$$(4.2) \quad \Delta^{\frac{\alpha}{2}} f(x) = d_\alpha \int_{\mathbb{R}} \frac{f(x+y) - f(x) - yf'(x)}{|y|^{1+\alpha}} dy.$$

Indeed, using Taylor’s expansions, we easily see that

$$\begin{aligned}
 & \left| \int_{\mathbb{R}} \frac{f(x+y) - f(x) - yf'(x)}{|y|^{1+\alpha}} dy \right| \\
 (4.3) \quad & \leq \left| \int_{|y| \leq 1} \frac{\frac{1}{2}f''(x + \theta_1 y)y^2}{|y|^{1+\alpha}} dy \right| \\
 & \quad + \left| \int_{|y| > 1} \frac{f'(x + \theta_2 y)y - f'(x)y}{|y|^{1+\alpha}} dy \right| < \infty,
 \end{aligned}$$

where  $\theta_1, \theta_2 \in (0, 1)$ . On the other hand,

$$\begin{aligned}
 & \int_{\mathbb{R}} \frac{f(x+y) - f(x) - yf'(x)}{|y|^{1+\alpha}} dy \\
 (4.4) \quad & = \lim_{r \downarrow 0} \int_{\mathbb{R} \setminus (-r, r)} \frac{f(x+y) - f(x) - yf'(x)}{|y|^{1+\alpha}} dy \\
 & = \lim_{r \downarrow 0} \int_{\mathbb{R} \setminus (-r, r)} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy.
 \end{aligned}$$

In our paper, thanks to that the solution  $f$  of Stein’s equation has bounded first- and second- order derivatives, we will use the form (4.2) to avoid the limit in (4.1). Moreover,  $\Delta^{\frac{\alpha}{2}} f(x)$  can be rewritten as (4.20) and (4.21) below, these two new formulations will play an important role in our analysis.

It is well known that  $\Delta^{\alpha/2}$  is the infinitesimal generator of the standard 1d symmetric  $\alpha$ -stable process  $(Z_t)_{t \geq 0}$  [2] with  $Z_0 = 0$ , the distribution of  $Z_t$  has a density  $p(t, x)$  satisfying

$$(4.5) \quad \int_{-\infty}^{\infty} e^{i\lambda x} p(t, x) dx = e^{-t|\lambda|^\alpha},$$

it is well known that  $p(t, x)$  is uniquely determined by its characteristic function [22], Section 3.3.1. Note that  $p(t, x)$  is called Green’s function of symmetric process and satisfies the differential equation:

$$(4.6) \quad \partial_t p(t, x) = \Delta^{\alpha/2} p(t, x), \quad p(0, x) = \delta_0(x),$$

where  $\delta_0(x)$  is Dirac function at 0, that is,  $\delta_0(x) = 0$  for all  $x \neq 0$  and  $\int_{-\infty}^{\infty} \delta_0(x) dx = 1$ ; see [33], (1.8), with  $A \equiv 0$  and  $d = 1$  therein.

Let us now consider the Orenstein–Uhlenbeck  $\alpha$ -stable process as the following:

$$(4.7) \quad dX_t = -\frac{1}{\alpha} X_t dt + dZ_t, \quad X_0 = x,$$

we denote by  $X_t(x)$  the solution to the SDE (4.7). Its infinitesimal generator is

$$\mathcal{A}f(x) = \Delta^{\frac{\alpha}{2}} f(x) - \frac{1}{\alpha} x f'(x) \quad \forall f \in \mathcal{S}(\mathbb{R}, \mathbb{R}),$$

where  $\mathcal{S}(\mathbb{R}, \mathbb{R})$  is the Schwartz function space, the set of all smooth functions whose derivatives are rapidly decreasing [46]. The domain  $\mathcal{D}(\mathcal{A})$  of the operator  $\mathcal{A}$  is the closure of  $\mathcal{S}(\mathbb{R}, \mathbb{R})$  by a standard procedure depending on the underlying function space that we consider [42], Chapter 2.

The following characterization theorem of stable distribution is well known; see [2], Proposition 3.2, for instance.

**THEOREM 4.1.** *Let  $Y$  be a random variable. If the following equation holds:*

$$(4.8) \quad \mathbb{E}[\Delta^{\frac{\alpha}{2}} f(Y)] - \frac{1}{\alpha} \mathbb{E}[Yf'(Y)] = 0 \quad \forall f \in \mathcal{S}(\mathbb{R}, \mathbb{R}),$$

where  $\alpha \in (0, 2)$ , then  $Y$  has a symmetric  $\alpha$ -stable distribution  $\mu$  with the characteristic function  $e^{-|\lambda|^\alpha}$ . Moreover, the distribution of  $Y$  is uniquely determined by (4.8).

**PROOF.** By [2], Proposition 3.2, with  $a_1 = 0$  and  $a_2 = \alpha d_\alpha$  in (3.6) therein (note that the linear operator in [2] is  $L = \alpha \mathcal{A}$ ), we get  $\mu$  is the unique invariant measure of  $\mathcal{A}$  in the sense that

$$\int_{\mathbb{R}} \mathcal{A}f(x) \mu(dx) = 0.$$

See [2], Definition 3.1. This means that  $Y$  has a distribution  $\mu$  and this distribution is uniquely determined.  $\square$

For any Lipschitz function  $h : \mathbb{R} \rightarrow \mathbb{R}$ , Stein's equation is

$$(4.9) \quad \Delta^{\frac{\alpha}{2}} f(x) - \frac{1}{\alpha} x f'(x) = h(x) - \mu(h),$$

that is,

$$(4.10) \quad \mathcal{A}f(x) = h(x) - \mu(h).$$

It is also known that equation (4.10) is called a Poisson equation; we can represent its solution by the stochastic process generated by  $\mathcal{A}$ . More precisely,

**LEMMA 4.2.** *Equation (4.9) has a solution*

$$(4.11) \quad f(x) = - \int_0^\infty \int_{-\infty}^\infty p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}} x) (h(y) - \mu(h)) dy dt,$$

where  $p(\cdot, \cdot)$  is determined by its characteristic function (4.5).

We will leave the proof of Lemma 4.2 later. With the help of this lemma, we shall prove the following regularity results of  $f$ , which plays a crucial role in the proof of Theorem 2.1.

PROPOSITION 4.3. *Let  $f$  be the solution to equation (4.9) defined by (4.11). We have the following estimates:*

$$(4.12) \quad \|f'\| \leq \alpha \|h'\|,$$

$$(4.13) \quad \|f''\| \leq \frac{4}{\pi} \sqrt{\frac{2\alpha + 1}{\alpha}} \mathbf{B}\left(\frac{\alpha - 1}{\alpha}, \frac{2}{\alpha}\right) \|h'\|,$$

where  $\|\cdot\|$  is the uniform norm, that is,  $\|g\| = \sup_{x \in \mathbb{R}} |g(x)|$  for any bounded measurable function  $g$ .

PROPOSITION 4.4. *For any  $\gamma \in (0, 1)$ , we have*

$$(4.14) \quad \sup_{x \neq y} \frac{|\Delta^{\frac{\alpha}{2}} f(x) - \Delta^{\frac{\alpha}{2}} f(y)|}{|x - y|^\gamma} \leq \frac{d_\alpha}{\alpha} \left[ \frac{16}{\pi(2 - \alpha)} \sqrt{\frac{\alpha + 3}{\alpha}} + \frac{16}{\pi(\alpha - 1)} \sqrt{\frac{2\alpha + 1}{\alpha}} \right] \mathbf{B}\left(\frac{1 - \gamma}{\alpha}, \frac{\gamma + \alpha}{\alpha}\right) \|h'\|.$$

4.2. *Proof of Theorem 2.1.* Recall that  $\zeta_{n,1}, \dots, \zeta_{n,n}$  are a sequence of independent random variables with  $\mathbb{E}\zeta_{n,i} = 0$  and  $\mathbb{E}|\zeta_{n,i}| < \infty$  for  $1 \leq i \leq n$ . Recall the notation

$$S_n = \zeta_{n,1} + \dots + \zeta_{n,n};$$

$$S_n(i) = S_n - \zeta_{n,i}, \quad 1 \leq i \leq n.$$

LEMMA 4.5. *We have*

$$(4.15) \quad \mathbb{E}[S_n f'(S_n)] = \sum_{i=1}^n \int_{-N}^N \mathbb{E}[K_i(t, N) f''(S_n(i) + t)] dt + \mathcal{R}_1 \quad \forall N > 0,$$

where  $K_i(t, N) = \mathbb{E}[\zeta_{n,i} 1_{\{0 \leq t \leq \zeta_{n,i} \leq N\}} - \zeta_{n,i} 1_{\{-N \leq \zeta_{n,i} \leq t \leq 0\}}]$ , and

$$(4.16) \quad \mathcal{R}_1 = \sum_{i=1}^n \mathbb{E}\{\zeta_{n,i} [f'(S_n) - f'(S_n(i))] 1_{\{|\zeta_{n,i}| > N\}}\} \quad \forall N > 0.$$

PROOF. By the independence and  $\mathbb{E}\zeta_{n,i} = 0$  for each  $i$ , we have

$$(4.17) \quad \begin{aligned} \mathbb{E}[S_n f'(S_n)] &= \sum_{i=1}^n \mathbb{E}[\zeta_{n,i} f'(S_n)] \\ &= \sum_{i=1}^n \mathbb{E}\{\zeta_{n,i} [f'(S_n) - f'(S_n(i))]\} = \sum_{i=1}^n I(i) + \mathcal{R}_1, \end{aligned}$$

where

$$(4.18) \quad \begin{aligned} I(i) &= \mathbb{E}\{\zeta_{n,i}[f'(S_n) - f'(S_n(i))]1_{\{|\zeta_{n,i}| \leq N\}}\}, \\ \mathcal{R}_1 &= \sum_{i=1}^n \mathbb{E}\{\zeta_{n,i}[f'(S_n) - f'(S_n(i))]1_{\{|\zeta_{n,i}| > N\}}\}. \end{aligned}$$

For  $I(i)$ , we have

$$(4.19) \quad \begin{aligned} I(i) &= \mathbb{E}\{\zeta_{n,i}[f'(S_n) - f'(S_n(i))]1_{\{|\zeta_{n,i}| \leq N\}}\} \\ &= \mathbb{E}\left\{\zeta_{n,i} \left[ \int_0^{\zeta_{n,i}} f''(S_n(i) + t) dt \right] 1_{\{|\zeta_{n,i}| \leq N\}}\right\} \\ &= \mathbb{E}\left\{\zeta_{n,i} \left[ \int_{-\infty}^{\infty} f''(S_n(i) + t) (1_{\{0 \leq t \leq \zeta_{n,i}\}} - 1_{\{\zeta_{n,i} \leq t \leq 0\}}) dt \right] 1_{\{|\zeta_{n,i}| \leq N\}}\right\} \\ &= \int_{-\infty}^{\infty} \mathbb{E}[f''(S_n(i) + t) (1_{\{0 \leq t \leq \zeta_{n,i}\}} - 1_{\{\zeta_{n,i} \leq t \leq 0\}}) \zeta_{n,i} 1_{\{|\zeta_{n,i}| \leq N\}}] dt \\ &= \int_{-\infty}^{\infty} \mathbb{E}[f''(S_n(i) + t)] \mathbb{E}[(1_{\{0 \leq t \leq \zeta_{n,i}\}} - 1_{\{\zeta_{n,i} \leq t \leq 0\}}) \zeta_{n,i} 1_{\{|\zeta_{n,i}| \leq N\}}] dt \\ &= \int_{-N}^N K_i(t, N) \mathbb{E}[f''(S_n(i) + t)] dt, \end{aligned}$$

where the last second inequality is by the independence of  $S_n(i)$  and  $\zeta_{n,i}$ .

Combining all the relations above, we immediately get the equality in the lemma, as desired.  $\square$

LEMMA 4.6. *For all  $x \in \mathbb{R}$ , we have*

$$(4.20) \quad \Delta^{\frac{\alpha}{2}} f(x) = \frac{d_\alpha}{\alpha} \int_{-\infty}^{\infty} \frac{f'(x+z) - f'(x)}{\text{sgn}(z)|z|^\alpha} dz.$$

Moreover, for all  $x \in \mathbb{R}$ ,

$$(4.21) \quad \Delta^{\frac{\alpha}{2}} f(x) = \int_{-N}^N \mathcal{K}_\alpha(t, N) f''(x+t) dt + \mathcal{R}_2(x),$$

where  $N > 0$  is an arbitrary number and

$$\begin{aligned} \mathcal{K}_\alpha(t, N) &= \frac{d_\alpha}{\alpha(\alpha - 1)} (|t|^{1-\alpha} - N^{1-\alpha}), \\ \mathcal{R}_2(x) &= \frac{d_\alpha}{\alpha} \int_{|z| > N} \frac{f'(x+z) - f'(x)}{\text{sgn}(z)|z|^\alpha} dz. \end{aligned}$$

PROOF. We observe

$$\begin{aligned} \Delta^{\frac{\alpha}{2}} f(x) &= d_{\alpha} \int_{\mathbb{R}} \frac{f(x+y) - f(x) - yf'(x)}{|y|^{1+\alpha}} dy \\ &= \int_{\mathbb{R}} \frac{d_{\alpha}}{|y|^{1+\alpha}} \int_0^y (f'(x+z) - f'(x)) dz dy \\ &= \int_0^{\infty} \frac{d_{\alpha}}{y^{1+\alpha}} \int_0^y (f'(x+z) - f'(x)) dz dy \\ &\quad + \int_{-\infty}^0 \frac{d_{\alpha}}{(-y)^{1+\alpha}} \int_0^y (f'(x+z) - f'(x)) dz dy. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\int_0^{\infty} \frac{d_{\alpha}}{y^{1+\alpha}} \int_0^y (f'(x+z) - f'(x)) dz dy \\ &= \int_0^{\infty} (f'(x+z) - f'(x)) \int_z^{\infty} \frac{d_{\alpha}}{y^{1+\alpha}} dy dz \\ &= \frac{d_{\alpha}}{\alpha} \int_0^{\infty} \frac{f'(x+z) - f'(x)}{z^{\alpha}} dz. \end{aligned}$$

Similarly,

$$\int_{-\infty}^0 \frac{d_{\alpha}}{(-y)^{1+\alpha}} \int_0^y (f'(x+z) - f'(x)) dz dy = -\frac{d_{\alpha}}{\alpha} \int_{-\infty}^0 \frac{f'(x+z) - f'(x)}{(-z)^{\alpha}} dz.$$

Combining the previous two relations, we immediately obtain (4.20).

Now we write (4.20) as

$$\Delta^{\frac{\alpha}{2}} f(x) = \mathcal{J}_1(x) - \mathcal{J}_2(x) + \mathcal{R}_2(x),$$

with

$$\begin{aligned} \mathcal{J}_1(x) &= \frac{d_{\alpha}}{\alpha} \int_0^N \frac{f'(x+z) - f'(x)}{z^{\alpha}} dz, \\ \mathcal{J}_2(x) &= \frac{d_{\alpha}}{\alpha} \int_{-N}^0 \frac{f'(x+z) - f'(x)}{(-z)^{\alpha}} dz, \\ \mathcal{R}_2(x) &= \frac{d_{\alpha}}{\alpha} \int_N^{\infty} \frac{f'(x+z) - f'(x)}{z^{\alpha}} dz \\ &\quad - \frac{d_{\alpha}}{\alpha} \int_{-\infty}^{-N} \frac{f'(x+z) - f'(x)}{(-z)^{\alpha}} dz. \end{aligned} \tag{4.22}$$

Moreover,

$$\begin{aligned}
 \mathcal{J}_1(x) &= \int_0^N \frac{d_\alpha}{\alpha z^\alpha} \int_0^z f''(x+t) dt dz \\
 &= \int_0^\infty \int_0^z \frac{d_\alpha}{\alpha z^\alpha} f''(x+t) 1_{\{0 \leq t \leq z \leq N\}} dt dz \\
 (4.23) \quad &= \int_0^\infty \int_t^\infty \frac{d_\alpha}{\alpha z^\alpha} 1_{\{0 \leq t \leq z \leq N\}} dz f''(x+t) dt \\
 &= \frac{d_\alpha}{\alpha(\alpha-1)} \int_0^N (t^{-\alpha+1} - N^{-\alpha+1}) f''(x+t) dt.
 \end{aligned}$$

Similarly, we have

$$(4.24) \quad \mathcal{J}_2(x) = \frac{-d_\alpha}{\alpha(\alpha-1)} \int_{-N}^0 [(-t)^{-\alpha+1} - N^{-\alpha+1}] f''(x+t) dt.$$

Combining the above relations of  $\mathcal{J}_1(x)$ ,  $\mathcal{J}_2(x)$  and  $\mathcal{R}_2(x)$ , we immediately conclude the proof.  $\square$

Combining Lemmas 4.5 and 4.6, we prove the following.

LEMMA 4.7. *The following equality holds:*

$$\begin{aligned}
 &\mathbb{E} \left[ \Delta^{\frac{\alpha}{2}} f(S_n) - \frac{1}{\alpha} S_n f'(S_n) \right] \\
 (4.25) \quad &= \sum_{i=1}^n \int_{-N}^N \mathbb{E} \left[ \left( \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right) f''(S_n(i) + t) \right] dt \\
 &\quad - \frac{1}{\alpha} \mathcal{R}_1 + \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathcal{R}_2(S_n(i))] + \mathcal{R}_3,
 \end{aligned}$$

where  $\mathcal{R}_1$  and  $\mathcal{R}_2(x)$  are defined in Lemmas 4.5 and 4.6, respectively, and

$$\mathcal{R}_3 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\Delta^{\frac{\alpha}{2}} f(S_n) - \Delta^{\frac{\alpha}{2}} f(S_n(i))].$$

PROOF. Observe

$$\mathbb{E}[\Delta^{\frac{\alpha}{2}} f(S_n)] - \frac{1}{\alpha} \mathbb{E}[S_n f'(S_n)] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\Delta^{\frac{\alpha}{2}} f(S_n(i))] - \frac{1}{\alpha} \mathbb{E}[S_n f'(S_n)] + \mathcal{R}_3.$$

By Lemmas 4.5 and 4.6, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\Delta^{\frac{\alpha}{2}} f(S_n(i))] - \frac{1}{\alpha} \mathbb{E}[S_n f'(S_n)] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left\{ \int_{-N}^N \mathcal{K}_\alpha(t, N) f''(S_n(i) + t) dt + \mathcal{R}_2(S_n(i)) \right\} \\ & \quad - \frac{1}{\alpha} \sum_{i=1}^n \int_{-N}^N \mathbb{E}[K_i(t, N) f''(S_n(i) + t)] dt - \frac{1}{\alpha} \mathcal{R}_1 \\ &= \sum_{i=1}^n \int_{-N}^N \mathbb{E} \left[ \left( \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right) f''(S_n(i) + t) \right] dt \\ & \quad + \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathcal{R}_2(S_n(i))] - \frac{1}{\alpha} \mathcal{R}_1. \end{aligned}$$

Hence, the lemma is proved.  $\square$

PROOF OF THEOREM 2.1. By equation (4.9), we have

$$\mathbb{E}[h(S_n)] - \mu(h) = \mathbb{E} \left[ \Delta^{\frac{\alpha}{2}} f(S_n) - \frac{1}{\alpha} S_n f'(S_n) \right].$$

To bound  $|\mathbb{E}[h(S_n)] - \mu(h)|$ , by Lemma 4.7, it suffices to bound the four terms on the right-hand side of (4.25). By (4.12), we have

$$\begin{aligned} \mathcal{R}_1 &\leq 2\alpha \|h'\| \sum_{i=1}^n \mathbb{E}[|\zeta_{n,i}| 1_{\{|\zeta_{n,i}| > N\}}], \\ \frac{1}{n} \sum_{i=1}^n \mathbb{E}|\mathcal{R}_2(S_n(i))| &\leq 2d_\alpha \|h'\| \int_{|z| > N} \frac{1}{|z|^\alpha} dz \leq \frac{4d_\alpha}{\alpha - 1} \|h'\| N^{1-\alpha}. \end{aligned}$$

For the integral term, by (4.13) we have

$$\begin{aligned} & \left| \sum_{i=1}^n \int_{-N}^N \mathbb{E} \left[ \left( \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right) f''(S_n(i) + t) \right] dt \right| \\ & \leq \|f''\| \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt \\ & \leq D_\alpha \|h'\| \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt. \end{aligned}$$

Finally, for  $\mathcal{R}_3$ , by Proposition 4.4, for all  $\gamma \in (0, 1)$  we have

$$\begin{aligned} |\mathcal{R}_3| &\leq \frac{1}{n} \sum_{i=1}^n |\mathbb{E}[\Delta^{\frac{\alpha}{2}} f(S_n) - \Delta^{\frac{\alpha}{2}} f(S_n(i))]| \\ &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \frac{|\Delta^{\frac{\alpha}{2}} f(S_n) - \Delta^{\frac{\alpha}{2}} f(S_n(i))|}{|\zeta_{n,i}|^\gamma} |\zeta_{n,i}|^\gamma \right] \leq \frac{D_{\alpha,\gamma}}{n} \sum_{i=1}^n \mathbb{E} |\zeta_{n,i}|^\gamma \|h'\|. \end{aligned}$$

Combining the above estimates, we immediately obtain the inequality in the theorem, as desired.  $\square$

**5. Proofs of Theorem 2.6 and Corollary 2.7.** Let us first prove Theorem 2.6 and then Corollary 2.7, and as stressed before, Corollary 2.7 can help us to fast determine the leading order of convergence rates, while Theorem 2.6 can give us an explicit bounds for  $d_W(\mathcal{L}(S_n), \mu)$ .

**PROOF OF THEOREM 2.6.** It suffices to prove the inequality in the theorem by bounding the integral and the remainder  $\mathcal{R}_{N,n}$  in Theorem 2.1. For the integral term, we have

$$(5.1) \quad \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| = \frac{1}{\alpha} \int_{-N}^N |\alpha \mathcal{K}_\alpha(t, N) - n K_1(t, N)| dt.$$

Let us first estimate  $\mathcal{R}_{N,n}$ , in which we need to bound the two sums. Recall  $\zeta_{n,1} = \ell_n^{-\frac{1}{\alpha}} (\xi_1 - \mathbb{E}\xi_1)$ , for the first sum, by Lemma 2.8,

$$\begin{aligned} &\sum_{i=1}^n \mathbb{E} (|\zeta_{n,i}| \mathbb{1}_{\{|\zeta_{n,i}| > N\}}) \\ &= n \ell_n^{-\frac{1}{\alpha}} \left[ \ell_n^{\frac{1}{\alpha}} N \mathbb{P}(|\xi_1 - \mathbb{E}\xi_1| > \ell_n^{\frac{1}{\alpha}} N) + \int_{\ell_n^{\frac{1}{\alpha}} N}^\infty \mathbb{P}(|\xi_1 - \mathbb{E}\xi_1| > r) dr \right] \\ &\leq n N \mathbb{P}(|\xi_1| > \ell_n^{\frac{1}{\alpha}} N - |\mathbb{E}\xi_1|) + n \ell_n^{-\frac{1}{\alpha}} \int_{\ell_n^{\frac{1}{\alpha}} N - |\mathbb{E}\xi_1|}^\infty \mathbb{P}(|\xi_1| > r) dr \\ &= n N \mathbb{P}(|\xi_1| > \ell_n^{\frac{1}{\alpha}} N \delta_n) + n \ell_n^{-\frac{1}{\alpha}} \int_{\ell_n^{\frac{1}{\alpha}} N \delta_n}^\infty \mathbb{P}(|\xi_1| > r) dr, \end{aligned}$$

where  $\delta_n = 1 - \ell_n^{-\frac{1}{\alpha}} N^{-1} |\mathbb{E}\xi_1|$ . This and the assumption (ii') yields

$$n N \mathbb{P}(|\xi_1| > \ell_n^{\frac{1}{\alpha}} N \delta_n) = n N \frac{\theta(1 + M_2(\ell_n^{\frac{1}{\alpha}} N \delta_n))}{(\ell_n^{\frac{1}{\alpha}} N \delta_n)^\alpha} = \frac{2d_\alpha(1 + M_2(\ell_n^{\frac{1}{\alpha}} N \delta_n))}{\alpha \delta_n^\alpha N^{\alpha-1}}$$

and

$$\begin{aligned} n\ell_n^{-\frac{1}{\alpha}} \int_{\ell_n^{\frac{1}{\alpha}} N\delta_n}^{\infty} \mathbb{P}(|\xi_1| > r) \, dr &= n\ell_n^{-\frac{1}{\alpha}} \frac{\theta(\ell_n^{\frac{1}{\alpha}} N\delta_n)^{-\alpha+1}}{(\alpha-1)} + n\ell_n^{-\frac{1}{\alpha}} \int_{\ell_n^{\frac{1}{\alpha}} N\delta_n}^{\infty} \frac{M_2(r)}{r^\alpha} \, dr \\ &= \frac{2d_\alpha(\delta_n N)^{1-\alpha}}{(\alpha-1)\alpha} + \frac{2d_\alpha}{\alpha N^{\alpha-1}} \int_{\delta_n}^{\infty} \frac{M_2(r\ell_n^{\frac{1}{\alpha}} N)}{r^\alpha} \, dr. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^n \mathbb{E}(|\zeta_{n,i}| 1_{\{|\zeta_{n,i}| > N\}}) \\ &\leq \frac{2d_\alpha}{\alpha\delta_n^{\alpha-1}} \left( \frac{1}{\alpha-1} + \frac{1}{\delta_n} + \frac{M_2(\ell_n^{\frac{1}{\alpha}} N\delta_n)}{\delta_n} + \int_{\delta_n}^{\infty} \frac{M_2(r\ell_n^{\frac{1}{\alpha}} N)}{r^\alpha \delta_n^{1-\alpha}} \, dr \right) N^{1-\alpha}. \end{aligned}$$

Moreover, the other sum can be bounded as follows: we immediately obtain

$$\frac{D_{\alpha,\gamma}}{n} \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^\gamma = D_{\alpha,\gamma} \ell_n^{-\frac{\gamma}{\alpha}} \mathbb{E}|\xi_1 - \mathbb{E}\xi_1|^\gamma.$$

Combining all the estimates with the inequality in Theorem (2.1), we immediately obtain the estimate in the theorem, as desired.  $\square$

It is easy to verify that (i') and (ii') imply

$$\begin{aligned} (5.2) \quad \mathbb{P}(\xi_1 > x) &= \frac{1 + M_1(x)}{2} (1 + M_2(x)) \theta x^{-\alpha}, \quad x > A; \\ \mathbb{P}(\xi_1 < x) &= \frac{1 - M_1(|x|)}{2} (1 + M_2(|x|)) \theta |x|^{-\alpha}, \quad x < -A. \end{aligned}$$

PROOF OF COROLLARY 2.7. By Theorem 2.6 and noticing  $\ell_n = \frac{\alpha\theta}{2d_\alpha} n$ , we have

$$\mathcal{R}_{N,n} \leq C_\alpha [n^{-\frac{2-\alpha}{\alpha}} + N^{1-\alpha}].$$

It remains to bound the integral

$$\int_{-N}^N |\alpha \mathcal{K}_\alpha(t, N) - n K_1(t, N)| \, dt.$$

Recall the definitions of  $\mathcal{K}_\alpha(t, N)$ ,  $K_1(t, N)$  and  $\zeta_{n,1}$ , we have

$$\begin{aligned} &\int_{|t| \leq 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} |\alpha \mathcal{K}_\alpha(t, N) - n K_1(t, N)| \, dt \\ &\leq \frac{d_\alpha}{\alpha-1} \int_{|t| \leq 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} \frac{1}{|t|^{\alpha-1}} \, dt + n \int_{|t| \leq 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} \mathbb{E}|\zeta_{n,1}| \, dt \\ &\leq C_\alpha n^{-\frac{2-\alpha}{\alpha}}. \end{aligned}$$

Now let us estimate

$$\int_{|t| \geq 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} |\alpha\mathcal{K}_\alpha(t, N) - nK_1(t, N)| dt.$$

For  $t > 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}$ , we have

$$\alpha\mathcal{K}_\alpha(t, N) - nK_1(t, N) = I_1 - I_2,$$

where  $b_t = \ell_n^{\frac{1}{\alpha}}t + \mathbb{E}\xi_1$  and

$$\begin{aligned} I_1 &= \frac{d_\alpha}{\alpha - 1} t^{1-\alpha} - n\ell_n^{-\frac{1}{\alpha}} [\mathbb{E}(\xi_1 1_{\{\xi_1 \geq b_t\}}) - \mathbb{P}(\xi_1 \geq b_t)\mathbb{E}\xi_1], \\ I_2 &= \frac{d_\alpha}{\alpha - 1} N^{1-\alpha} - n\ell_n^{-\frac{1}{\alpha}} [\mathbb{E}(\xi_1 1_{\{\xi_1 \geq b_N\}}) - \mathbb{P}(\xi_1 \geq b_N)\mathbb{E}\xi_1]. \end{aligned}$$

By Lemma 2.8 and (5.2), we have

$$\mathbb{E}[\xi_1 1_{\{\xi_1 > b_t\}}] = b_t \mathbb{P}(\xi_1 > b_t) + \int_{b_t}^\infty \mathbb{P}(\xi_1 > r) dr = \frac{\alpha\theta}{2(\alpha - 1)} b_t^{1-\alpha} + r_t,$$

where  $r_t$  is defined by (2.8). Therefore,

$$\begin{aligned} I_1 &= \frac{d_\alpha}{\alpha - 1} t^{1-\alpha} - \frac{\alpha\theta n\ell_n^{-\frac{1}{\alpha}}}{2(\alpha - 1)} b_t^{1-\alpha} - n\ell_n^{-\frac{1}{\alpha}} r_t + n\ell_n^{-\frac{1}{\alpha}} \mathbb{P}(\xi_1 > b_t)\mathbb{E}\xi_1 \\ &= \frac{d_\alpha}{\alpha - 1} t^{1-\alpha} - \frac{d_\alpha}{\alpha - 1} t^{1-\alpha} (1 + t^{-1}\ell_n^{-\frac{1}{\alpha}}\mathbb{E}\xi_1)^{1-\alpha} - n\ell_n^{-\frac{1}{\alpha}} r_t + n\ell_n^{-\frac{1}{\alpha}} R_t, \end{aligned}$$

where  $R_t$  is defined by (2.7).

As  $t > 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}$ , we have  $|t^{-1}\ell_n^{-\frac{1}{\alpha}}\mathbb{E}\xi_1| \leq \frac{1}{4}$ . By Taylor expansion  $|(1 + x)^{1-\alpha} - 1| \leq 4|x|$  with  $|x| \leq \frac{1}{4}$  and the easy fact  $|R_t| \leq Ct^{-\alpha}n^{-1}$ , we get

$$|I_1| \leq C_\alpha(t^{-\alpha}n^{-\frac{1}{\alpha}} + n^{1-\frac{1}{\alpha}}|r_t|).$$

Similarly, we have

$$|I_2| \leq C_\alpha(N^{-\alpha}n^{-\frac{1}{\alpha}} + n^{1-\frac{1}{\alpha}}|r_N|).$$

Hence,

$$\begin{aligned} &\int_{t > 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} \left| \frac{d_\alpha}{\alpha - 1} \left( \frac{1}{|t|^{\alpha-1}} - \frac{1}{N^{\alpha-1}} \right) - nK_1(t, N) \right| dt \\ &\leq C_\alpha \left( n^{-\frac{2-\alpha}{\alpha}} + N^{1-\alpha}n^{-\frac{1}{\alpha}} + n^{1-\frac{1}{\alpha}} \int_{t > 4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} |r_t| dt + n^{1-\frac{1}{\alpha}}N|r_N| \right). \end{aligned}$$

By the same argument, we get

$$\int_{t < -4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} \left| \frac{d_\alpha}{\alpha - 1} \left( \frac{1}{|t|^{\alpha-1}} - \frac{1}{N^{\alpha-1}} \right) - nK_1(t, N) \right| dt \leq C_\alpha \left( n^{-\frac{2-\alpha}{\alpha}} + N^{1-\alpha} n^{-\frac{1}{\alpha}} + n^{1-\frac{1}{\alpha}} \int_{t < -4(A + |\mathbb{E}\xi_1|)\ell_n^{-\frac{1}{\alpha}}} |r_t| dt + n^{1-\frac{1}{\alpha}} N|r_N| \right).$$

Combining the previous two inequalities, we get the inequality in the corollary. □

**6. Proofs of Lemma 4.2 and Propositions 4.3 and 4.4.** Before proving the lemma and propositions, we first list some well-known results about symmetric  $\alpha$ -stable process and  $\Delta^{\alpha/2}$  that we shall use. It is easy to verify by the definition of  $\Delta^{\alpha/2}$  that if  $z = x - y$ , then

$$(6.1) \quad \Delta_x^{\alpha/2} f(x - y) = \Delta_y^{\alpha/2} f(x - y) = \Delta_z^{\alpha/2} f(z),$$

where  $\Delta_x^{\alpha/2}$  means that the operator  $\Delta^{\alpha/2}$  acts on the variable  $x$ . Similarly, for  $z = cx$  for some constant  $c \in \mathbb{R}$ , we have

$$(6.2) \quad \Delta_x^{\alpha/2} f(cx) = |c|^\alpha \Delta_z^{\alpha/2} f(z).$$

Recall that  $p(t, x)$  is the transition probability density of standard symmetric  $\alpha$ -stable process  $Z_t$ , it is well known that

$$(6.3) \quad p(t, x) = t^{-1/\alpha} p(1, t^{-1/\alpha}x), \quad t > 0, x \in \mathbb{R}.$$

We have the following estimate.

LEMMA 6.1. *Let  $p(1, x)$  be the transition probability density of  $Z_1$ , we have*

$$\begin{aligned} |\partial_x p(1, x)| &\leq \frac{1}{\alpha\pi}, & |\partial_x p(1, x)| &\leq \frac{2\alpha + 1}{\pi} \frac{1}{x^2}; \\ |\partial_x^2 p(1, x)| &\leq \frac{2}{\alpha\pi}, & |\partial_x^2 p(1, x)| &\leq \frac{2\alpha + 6}{\pi} \frac{1}{x^2}. \end{aligned}$$

PROOF. The proof is based on the inverse Fourier transform and will be given in the [Appendix](#). □

REMARK 6.2. A sharp heat kernel estimate of  $p(1, x)$  is as the following ([18], (1.3) and [18], (2.11)):

$$(6.4) \quad \partial_x^k p(1, x) \leq \frac{C_{k,\alpha}}{(1 + |x|)^{\alpha+1+k}}, \quad k \in \mathbb{N} \cup \{0\},$$

but exact values of the above constants  $C_{k,\alpha}$  are often difficult to be found. See [16] for more details about heat kernel estimates of stable-type processes.

6.1. Proof of Lemma 4.2.

PROOF OF LEMMA 4.2. Note that  $\mu$  has a density  $p(1, x)$ , by the property  $p(t, x) = t^{-1/\alpha} p(1, t^{-1/\alpha}x)$  and a change of variable, we have

$$\begin{aligned}
 & \int_{-\infty}^{\infty} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) dy \\
 (6.5) \quad &= \int_{\mathbb{R}} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)h(y) dy - \int_{\mathbb{R}} p(1, y)h(y) dy \\
 &= \int_{\mathbb{R}} p(1 - e^{-t}, y)h(y + e^{-\frac{t}{\alpha}}x) dy - \int_{\mathbb{R}} p(1, y)h(y) dy \\
 &= \int_{\mathbb{R}} p(1, y)h((1 - e^{-t})^{\frac{1}{\alpha}}y + e^{-\frac{t}{\alpha}}x) dy - \int_{\mathbb{R}} p(1, y)h(y) dy.
 \end{aligned}$$

This implies

$$\begin{aligned}
 & \left| \int_{-\infty}^{\infty} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) dy \right| \\
 & \leq C_{\alpha} \|h'\| e^{-\frac{t}{\alpha}} \left( |x| + \int_{\mathbb{R}} |y|p(1, y) dy \right) \\
 & \leq C_{\alpha} \|h'\| e^{-\frac{t}{\alpha}} (|x| + 1),
 \end{aligned}$$

and hence  $f(x)$  is well defined for all  $x \in \mathbb{R}$ .

By the Fubini theorem, we have

$$(6.6) \quad \Delta^{\frac{\alpha}{2}} f(x) = - \int_0^{\infty} \int_{-\infty}^{\infty} \Delta_x^{\frac{\alpha}{2}} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) dy dt.$$

On the other hand, denote  $s = 1 - e^{-t}$  and  $z = y - e^{-\frac{t}{\alpha}}x$ , we have

$$\begin{aligned}
 & \frac{d}{dt} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) \\
 &= e^{-t} \partial_s p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) + \frac{1}{\alpha} e^{-\frac{t}{\alpha}}x \partial_z p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) \\
 &= e^{-t} \Delta_z^{\frac{\alpha}{2}} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) + \frac{1}{\alpha} e^{-\frac{t}{\alpha}}x \partial_z p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) \\
 &= \Delta_x^{\frac{\alpha}{2}} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) - \frac{x}{\alpha} \partial_x p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x),
 \end{aligned}$$

where the second equality is by (4.6) and the third one is by (6.1) and (6.2). Substituting the previous relation into (6.6), we get

$$\Delta^{\frac{\alpha}{2}} f(x) = - \int_0^{\infty} \int_{-\infty}^{\infty} \frac{d}{dt} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) dy dt$$

$$\begin{aligned}
 (6.7) \quad & - \int_0^\infty \int_{-\infty}^\infty \frac{x}{\alpha} \partial_x p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) \, dy \, dt \\
 & = - \int_0^\infty \int_{-\infty}^\infty \frac{d}{dt} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) \, dy \, dt + \frac{x}{\alpha} f'(x).
 \end{aligned}$$

By (4.6), the Fubini theorem and a straightforward calculation, we get

$$(6.8) \quad - \int_0^\infty \int_{-\infty}^\infty \frac{d}{dt} p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)(h(y) - \mu(h)) \, dy \, dt = h(x) - \mu(h).$$

Hence,  $f(x)$  solves equation (4.9).  $\square$

6.2. *Proof of Proposition 4.3.* In this and the next subsections, we shall often exchange differential operators and integrals without detailed proofs, since the exchangeability can be justified by a standard argument thanks to Lemma 6.1.

LEMMA 6.3. *The density of  $X_t(x)$  is  $p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x)$  where  $p(t, x)$  is the probability density function determined by equation (4.5).*

PROOF. For  $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ , define

$$Q_t f(x) = \int_{-\infty}^\infty p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) f(y) \, dy, \quad t > 0.$$

We shall show that

$$(6.9) \quad \partial_t Q_t f(x) = \Delta^{\alpha/2} Q_t f(x) - \frac{1}{\alpha} x (Q_t f)'(x), \quad Q_0 f(x) = f(x).$$

Note that equation (6.9) is the Kolmogorov backward equation associated to SDE (4.7), which admits a unique solution with the form

$$Q_t f(x) = \mathbb{E}[f(X_t(x))].$$

Since  $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$  is arbitrary, the probability of  $X_t(x)$  has a density function as in the lemma.

It remains to prove equation (6.9).  $Q_0 f(x) = f(x)$  is obvious. Let us now show the equation. Denote  $s = 1 - e^{-t}$  and  $z = y - e^{-\frac{t}{\alpha}}x$ , we have

$$\begin{aligned}
 (6.10) \quad \partial_t Q_t f(x) &= \partial_t \int_{-\infty}^\infty p(s, z) f(y) \, dy \\
 &= \int_{-\infty}^\infty \partial_t p(s, z) f(y) \, dy \\
 &= \int_{-\infty}^\infty e^{-t} \partial_s p(s, z) f(y) \, dy + \frac{1}{\alpha} x e^{-\frac{t}{\alpha}} \int_{-\infty}^\infty \partial_z p(s, z) f(y) \, dy.
 \end{aligned}$$

On the one hand, by (4.6), we have

$$\begin{aligned}
 \partial_s p(s, z) &= \Delta_z^{\alpha/2} p(s, z) \\
 &= d_\alpha \int_{\mathbb{R}} \frac{p(s, z + u) - p(s, z)}{|u|^{1+\alpha}} du \\
 &= d_\alpha \int_{\mathbb{R}} \frac{p(s, y - e^{-\frac{t}{\alpha}}x + u) - p(s, y - e^{-\frac{t}{\alpha}}x)}{|u|^{1+\alpha}} du \\
 (6.11) \quad &= e^t d_\alpha \int_{\mathbb{R}} \frac{p(s, y - e^{-\frac{t}{\alpha}}(x + \tilde{u})) - p(s, y - e^{-\frac{t}{\alpha}}x)}{|\tilde{u}|^{1+\alpha}} d\tilde{u} \\
 &= e^t \Delta_x^{\alpha/2} p(s, y - e^{-\frac{t}{\alpha}}x) \\
 &= e^t \Delta_x^{\alpha/2} p(s, z),
 \end{aligned}$$

where the fourth equality is by taking  $\tilde{u} = -e^{\frac{t}{\alpha}}u$ .

On the other hand, it is easy to check

$$e^{-\frac{t}{\alpha}} \partial_z p(s, z) = -\partial_x p(s, z).$$

Combing the previous three relations, we immediately obtain

$$\begin{aligned}
 \partial_t Q_t f(x) &= \int_{-\infty}^{\infty} \Delta_x^{\alpha/2} p(s, z) f(y) dy - \frac{1}{\alpha} x \int_{-\infty}^{\infty} \partial_x p(s, z) f(y) dy \\
 (6.12) \quad &= \Delta_x^{\alpha/2} \int_{-\infty}^{\infty} p(s, z) f(y) dy - \frac{1}{\alpha} x \partial_x \int_{-\infty}^{\infty} p(s, z) f(y) dy \\
 &= \Delta^{\alpha/2} Q_t f(x) - \frac{1}{\alpha} x \partial_x Q_t f(x). \quad \square
 \end{aligned}$$

PROOF OF PROPOSITION 4.3. By Lemma 4.2, we have

$$(6.13) \quad f(x) = - \int_0^\infty \int_{-\infty}^\infty p(1 - e^{-t}, y - e^{-\frac{t}{\alpha}}x) (h(y) - \mu(h)) dy dt.$$

Denote  $s = 1 - e^{-t}$  and  $z = y - e^{-\frac{t}{\alpha}}x$ , it is easy to check

$$\partial_x p(s, z) = -e^{-\frac{t}{\alpha}} \partial_z p(s, z), \quad \partial_y p(s, z) = \partial_z p(s, z).$$

We have

$$\begin{aligned}
 f'(x) &= \int_0^\infty \int_{-\infty}^\infty \partial_x p(s, z) (h(y) - \mu(h)) dy dt \\
 (6.14) \quad &= - \int_0^\infty \int_{-\infty}^\infty e^{-\frac{t}{\alpha}} \partial_y p(s, z) (h(y) - \mu(h)) dy dt \\
 &= \int_0^\infty \int_{-\infty}^\infty e^{-\frac{t}{\alpha}} p(s, z) h'(y) dy dt.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \|f'\| &\leq \|h'\| \int_0^\infty e^{-\frac{t}{\alpha}} \int_{-\infty}^\infty p(s, z) \, dy \, dt \\
 (6.15) \qquad &= \|h'\| \int_0^\infty e^{-\frac{t}{\alpha}} \int_{-\infty}^\infty p(s, z) \, dz \, dt = \alpha \|h'\|.
 \end{aligned}$$

We further have

$$(6.16) \qquad f''(x) = - \int_0^\infty \int_{-\infty}^\infty e^{-\frac{2t}{\alpha}} \partial_z p(s, z) h'(y) \, dy \, dt.$$

Thanks to the property  $p(s, z) = s^{-1/\alpha} p(s^{-1/\alpha} z)$  with  $p(x) = p(1, x)$  for  $x \in \mathbb{R}$ , we have

$$(6.17) \qquad \|f''\| \leq \|h'\| \int_0^\infty s^{-\frac{1}{\alpha}} e^{-\frac{2t}{\alpha}} \int_{-\infty}^\infty s^{-\frac{1}{\alpha}} |p'(s^{-\frac{1}{\alpha}} z)| \, dy \, dt.$$

Setting  $u = s^{-1/\alpha} z$  and applying the two estimates of  $p'(x)$  in Lemma 6.1 to the two integrals  $\int_{|u| \leq \sqrt{\alpha(2\alpha+1)}}$  and  $\int_{|u| > \sqrt{\alpha(2\alpha+1)}}$  below, we have

$$\begin{aligned}
 \int_{-\infty}^\infty s^{-\frac{1}{\alpha}} |p'(s^{-\frac{1}{\alpha}} z)| \, dy &= \int_{-\infty}^\infty |p'(u)| \, du \\
 (6.18) \qquad &= \int_{|u| \leq \sqrt{\alpha(2\alpha+1)}} \frac{1}{\alpha\pi} \, du + \int_{|u| > \sqrt{\alpha(2\alpha+1)}} \frac{2\alpha+1}{\pi u^2} \, du \\
 &\leq \frac{4}{\pi} \sqrt{\frac{2\alpha+1}{\alpha}}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \|f''\| &\leq \frac{4}{\pi} \sqrt{\frac{2\alpha+1}{\alpha}} \int_0^\infty s^{-\frac{1}{\alpha}} e^{-\frac{2t}{\alpha}} \, dt \|h'\| \\
 (6.19) \qquad &= \frac{4}{\pi} \sqrt{\frac{2\alpha+1}{\alpha}} \mathbf{B}\left(\frac{\alpha-1}{\alpha}, \frac{2}{\alpha}\right) \|h'\|,
 \end{aligned}$$

where the last equality is by the change of variable  $u = e^{-t}$ .  $\square$

### 6.3. Proof of Proposition 4.4.

LEMMA 6.4. *Let  $f \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R})$ , the space of all second-order differentiable functions with bounded zero, first-, second-order derivatives. For any differentiable  $h$  such that  $\lim_{x \rightarrow \pm\infty} f(x)h(x) = 0$ , we have*

$$(6.20) \qquad \int_{-\infty}^\infty \Delta^{\frac{\alpha}{2}} f(x)h(x) \, dx = \int_{-\infty}^\infty \mathcal{I}(f)(x)h'(x) \, dx,$$

where

$$\mathcal{I}(f)(x) = -\frac{d_\alpha}{\alpha} \int_{-\infty}^\infty \frac{f(x+w) - f(x)}{\operatorname{sgn}(w)|w|^\alpha} dw.$$

PROOF. Recalling (4.20),

$$\Delta^{\frac{\alpha}{2}} f(x) = \frac{d_\alpha}{\alpha} \int_{-\infty}^\infty \frac{f'(x+z) - f'(x)}{\operatorname{sgn}(z)|z|^\alpha} dz,$$

and using Fubini's theorem two times and an integration by parts, we get

$$\begin{aligned} (6.21) \quad \int_{-\infty}^\infty \Delta^{\frac{\alpha}{2}} f(x)h(x) dx &= \frac{d_\alpha}{\alpha} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f'(x+z) - f'(x)}{\operatorname{sgn}(z)|z|^\alpha} h(x) dx dz \\ &= -\frac{d_\alpha}{\alpha} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x+z) - f(x)}{\operatorname{sgn}(z)|z|^\alpha} dz h'(x) dx. \end{aligned}$$

The proof is complete.  $\square$

PROOF OF PROPOSITION 4.4. Observe

$$\begin{aligned} (6.22) \quad &\Delta^{\frac{\alpha}{2}} f(x) - \Delta^{\frac{\alpha}{2}} f(y) \\ &= \int_0^\infty \int_{-\infty}^\infty \Delta_x^{\frac{\alpha}{2}} p(1 - e^{-t}, z - e^{-\frac{t}{\alpha}}x)(h(z) - \mu(h)) dz dt \\ &\quad - \int_0^\infty \int_{-\infty}^\infty \Delta_y^{\frac{\alpha}{2}} p(1 - e^{-t}, z - e^{-\frac{t}{\alpha}}y)(h(z) - \mu(h)) dz dt. \end{aligned}$$

Denote  $s = 1 - e^{-t}$  and  $p(x) = p(1, x)$ , we have

$$(6.23) \quad p(s, z - e^{-\frac{t}{\alpha}}x) = s^{-\frac{1}{\alpha}} p(s^{-\frac{1}{\alpha}}(z - e^{-\frac{t}{\alpha}}x)).$$

By (6.1) and (6.2), we have

$$\begin{aligned} \Delta_x^{\frac{\alpha}{2}} p(s, z - e^{-\frac{t}{\alpha}}x) &= s^{-\frac{1}{\alpha}} \Delta_x^{\frac{\alpha}{2}} p(s^{-\frac{1}{\alpha}}(z - e^{-\frac{t}{\alpha}}x)) \\ &= s^{-\frac{1}{\alpha}} e^{-t} \Delta_z^{\frac{\alpha}{2}} p(s^{-\frac{1}{\alpha}}(z - e^{-\frac{t}{\alpha}}x)). \end{aligned}$$

Hence, by Lemma 6.4,

$$\begin{aligned} (6.24) \quad &\int_0^\infty \int_{-\infty}^\infty \Delta_x^{\frac{\alpha}{2}} p(s, z - e^{-\frac{t}{\alpha}}x)(h(z) - \mu(h)) dz dt \\ &= \int_0^\infty s^{-\frac{1}{\alpha}} e^{-t} \int_{-\infty}^\infty \Delta_z^{\frac{\alpha}{2}} p(s^{-\frac{1}{\alpha}}(z - e^{-\frac{t}{\alpha}}x))(h(z) - \mu(h)) dz dt \\ &= \int_0^\infty s^{-\frac{1}{\alpha}} e^{-t} \int_{-\infty}^\infty \mathcal{I}(p(s^{-\frac{1}{\alpha}}(\cdot - e^{-\frac{t}{\alpha}}x)))(z)h'(z) dz dt \\ &= \int_0^\infty s^{-1+\frac{1}{\alpha}} e^{-t} \int_{-\infty}^\infty \mathcal{I}(p(\cdot - s^{-\frac{1}{\alpha}}e^{-\frac{t}{\alpha}}x))(z)h'(\frac{1}{s^\frac{1}{\alpha}}z) dz dt, \end{aligned}$$

where the last equality is by a change of variables on  $z$  and the  $w$  in  $\mathcal{I}$ . Similarly,

$$\begin{aligned}
 (6.25) \quad & \int_0^\infty \int_{-\infty}^\infty \Delta_y^{\frac{\alpha}{2}} p(s, z - e^{-\frac{t}{\alpha}} y)(h(z) - \mu(h)) \, dz \, dt \\
 & = \int_0^\infty s^{-1+\frac{1}{\alpha}} e^{-t} \int_{-\infty}^\infty \mathcal{I}(p(\cdot - s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} y))(z) h'(s^{\frac{1}{\alpha}} z) \, dz \, dt.
 \end{aligned}$$

Observe

$$\begin{aligned}
 & \frac{1}{|x - y|^\gamma} |\mathcal{I}(p(\cdot - s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} x))(z) - \mathcal{I}(p(\cdot - s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} y))(z)| \\
 & = \frac{d_\alpha}{\alpha} \left| \int_{-\infty}^\infty \frac{\delta_w p(z - s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} x) - \delta_w p(z - s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} y)}{\operatorname{sgn}(w) |w|^\alpha |x - y|^\gamma} \, dw \right| \\
 & = \frac{d_\alpha}{\alpha} s^{-\frac{\gamma}{\alpha}} e^{-\frac{\gamma t}{\alpha}} \left| \int_{-\infty}^\infty \frac{\delta_w p(z - \tilde{x}) - \delta_w p(z - \tilde{y})}{\operatorname{sgn}(w) |w|^\alpha |\tilde{x} - \tilde{y}|^\gamma} \, dw \right|,
 \end{aligned}$$

where  $\delta_w p(z) = p(z + w) - p(z)$ ,  $\tilde{x} = s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} x$  and  $\tilde{y} = s^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}} y$ . Therefore, for any  $x \neq y$ ,

$$\begin{aligned}
 (6.26) \quad & \frac{|\Delta_y^{\frac{\alpha}{2}} f(x) - \Delta_y^{\frac{\alpha}{2}} f(y)|}{|x - y|^\gamma} \\
 & \leq \frac{d_\alpha \|h'\|}{\alpha} \int_0^\infty s^{-\frac{\gamma+\alpha-1}{\alpha}} e^{-\frac{(\gamma+\alpha)t}{\alpha}} \\
 & \quad \times \left| \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\delta_w p(z - \tilde{x}) - \delta_w p(z - \tilde{y})}{\operatorname{sgn}(w) |w|^\alpha |\tilde{x} - \tilde{y}|^\gamma} \, dw \right| \, dz \, dt.
 \end{aligned}$$

Let us bound the integral above. When  $|\tilde{x} - \tilde{y}| \leq 1$ , observe

$$\begin{aligned}
 \delta_w p(z - \tilde{x}) - \delta_w p(z - \tilde{y}) & = \int_0^w \int_{\tilde{x}}^{\tilde{y}} p''(z + r - a) \, da \, dr, \\
 \delta_w p(z - \tilde{x}) - \delta_w p(z - \tilde{y}) & = \int_{\tilde{x}}^{\tilde{y}} (p'(z + w - a) - p'(z - a)) \, da,
 \end{aligned}$$

we have

$$\begin{aligned}
 (6.27) \quad & \int_{-\infty}^\infty \left| \int_{-\infty}^\infty \frac{\delta_w p(z - \tilde{x}) - \delta_w p(z - \tilde{y})}{\operatorname{sgn}(w) |w|^\alpha |\tilde{x} - \tilde{y}|^\gamma} \, dw \right| \, dz \\
 & \leq \int_{-\infty}^\infty \int_{|w| \leq 1} \frac{1}{|w|^\alpha |\tilde{x} - \tilde{y}|^\gamma} \left| \int_0^w \int_{\tilde{x}}^{\tilde{y}} |p''(z + r - a)| \, da \, dr \right| \, dw \, dz \\
 & \quad + \int_{-\infty}^\infty \int_{|w| > 1} \frac{1}{|w|^\alpha |\tilde{x} - \tilde{y}|^\gamma} \left| \int_{\tilde{x}}^{\tilde{y}} |p'(z + w - a) - p'(z - a)| \, da \right| \, dw \, dz.
 \end{aligned}$$

Applying the two estimates of  $\partial_x^2 p(1, x)$  in Lemma 6.1 to the two integrals  $\int_{|z| \leq \sqrt{\alpha(\alpha+3)}}$  and  $\int_{|z| > \sqrt{\alpha(\alpha+3)}}$  below respectively, we obtain

$$\begin{aligned}
 \int_{-\infty}^{\infty} |p''(z+r-a)| dz &= \int_{-\infty}^{\infty} |p''(z)| dz \\
 (6.28) \qquad &\leq \int_{|z| \leq \sqrt{\alpha(\alpha+3)}} \frac{2}{\alpha\pi} dz + \int_{|z| > \sqrt{\alpha(\alpha+3)}} \frac{2\alpha+6}{\pi z^2} dz \\
 &\leq \frac{8}{\pi} \sqrt{\frac{\alpha+3}{\alpha}}.
 \end{aligned}$$

Applying the two estimates of  $\partial_x p(1, x)$  in Lemma 6.1 similarly, we obtain

$$\begin{aligned}
 \int_{-\infty}^{\infty} |p'(z+w-a) - p'(z-a)| dz &\leq 2 \int_{-\infty}^{\infty} |p'(z)| dz \\
 &= 2 \left( \int_{|z| \leq \sqrt{\alpha(2\alpha+1)}} + \int_{|z| > \sqrt{\alpha(2\alpha+1)}} \right) |p'(z)| dz \\
 &\leq \frac{8}{\pi} \sqrt{\frac{2\alpha+1}{\alpha}}.
 \end{aligned}$$

Hence, these two inequalities and (6.27), together with Fubini's theorem, imply

$$\begin{aligned}
 \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{\delta_w p(z-\tilde{x}) - \delta_w p(z-\tilde{y})}{\text{sgn}(w)|w|^\alpha |\tilde{x}-\tilde{y}|^\gamma} dw \right| dz \\
 (6.29) \qquad &\leq \frac{8}{\pi} \sqrt{\frac{\alpha+3}{\alpha}} \int_{|w| \leq 1} \frac{1}{|w|^{\alpha-1}} dw + \frac{8}{\pi} \sqrt{\frac{2\alpha+1}{\alpha}} \int_{|w| > 1} \frac{1}{|w|^\alpha} dw \\
 &\leq \frac{16}{\pi(2-\alpha)} \sqrt{\frac{\alpha+3}{\alpha}} + \frac{16}{\pi(\alpha-1)} \sqrt{\frac{2\alpha+1}{\alpha}}, \quad |\tilde{x}-\tilde{y}| \leq 1.
 \end{aligned}$$

When  $|\tilde{x}-\tilde{y}| > 1$ , observe

$$(6.30) \quad \delta_w p(z-\tilde{x}) - \delta_w p(z-\tilde{y}) = \int_0^w (p'(z+r-\tilde{x}) - p'(z+r-\tilde{y})) dr,$$

we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{\delta_w p(z-\tilde{x}) - \delta_w p(z-\tilde{y})}{\text{sgn}(w)|w|^\alpha |\tilde{x}-\tilde{y}|^\gamma} dw \right| dz \\
 \leq \int_{-\infty}^{\infty} \int_{|w| \leq 1} \frac{1}{|w|^\alpha} \left| \int_0^w |p'(z+r-\tilde{x}) - p'(z+r-\tilde{y})| dr \right| dw dz \\
 + \int_{-\infty}^{\infty} \int_{|w| > 1} \frac{1}{|w|^\alpha} [|\delta_w p(z-\tilde{x})| + |\delta_w p(z-\tilde{y})|] dw dz.
 \end{aligned}$$

By a similar argument as above and  $\int_{\mathbb{R}} |\delta_w p(z - c)| dz \leq 2$  for any  $c \in \mathbb{R}$ , we have

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{\delta_w p(z - \tilde{x}) - \delta_w p(z - \tilde{y})}{\operatorname{sgn}(w)|w|^\alpha |\tilde{x} - \tilde{y}|^\gamma} dw \right| dz \\
 (6.31) \quad & \leq \frac{8}{\pi} \sqrt{\frac{2\alpha + 1}{\alpha}} \int_{|w| \leq 1} \frac{1}{|w|^{\alpha-1}} dw + 4 \int_{|w| > 1} \frac{1}{|w|^\alpha} dw \\
 & \leq \frac{16}{\pi(2 - \alpha)} \sqrt{\frac{2\alpha + 1}{\alpha}} + \frac{8}{\alpha - 1}, \quad |\tilde{x} - \tilde{y}| > 1.
 \end{aligned}$$

Combining (6.26), (6.29) and (6.31), we immediately obtain

$$\begin{aligned}
 & \frac{|\Delta^{\frac{\alpha}{2}} f(x) - \Delta^{\frac{\alpha}{2}} f(y)|}{|x - y|^\gamma} \\
 & \leq \frac{d_\alpha}{\alpha} \left[ \frac{16}{\pi(2 - \alpha)} \sqrt{\frac{\alpha + 3}{\alpha}} + \frac{16}{\pi(\alpha - 1)} \sqrt{\frac{2\alpha + 1}{\alpha}} \right] \|h'\| \int_0^\infty s^{-\frac{\gamma + \alpha - 1}{\alpha}} e^{-\frac{(\gamma + \alpha)t}{\alpha}} dt \\
 & = \frac{d_\alpha}{\alpha} \left[ \frac{16}{\pi(2 - \alpha)} \sqrt{\frac{\alpha + 3}{\alpha}} + \frac{16}{\pi(\alpha - 1)} \sqrt{\frac{2\alpha + 1}{\alpha}} \right] \mathbf{B}\left(\frac{1 - \gamma}{\alpha}, \frac{\gamma + \alpha}{\alpha}\right) \|h'\|. \quad \square
 \end{aligned}$$

APPENDIX

**A.1. Example 4: An example in [48].** Let us assume that  $\xi_1, \dots, \xi_n, \dots$  be a sequence of i.i.d. random variables. The authors of [48] considered the following case:  $\xi_1$  has a density function as

$$(A.1) \quad p(x) = K_0 \frac{(\log |x|)^\beta}{|x|^{1+\alpha}} \quad \text{for } |x| > x_0, \quad p(x) = 0 \quad \text{for } |x| \leq x_0,$$

where  $K_0 > 0, x_0 > 0, \alpha \in (0, 2)$  and  $\beta \in \mathbb{R}$ . It is easy to check that this example is out of the scope of Theorem 2.6 because the conditions (i') and (ii') are not satisfied.

By [48], Proposition 1, we have  $B_n = 0$  and  $A_n = n^{1/\alpha} h(n)$  with  $h(n) = C \log^\frac{\gamma}{\alpha} (C n^\frac{1}{\alpha} \log^\frac{\gamma}{\alpha} (n^\frac{1}{\alpha}))$  and as  $n \rightarrow \infty$ ,

$$T_n/A_n \Rightarrow \nu,$$

where  $T_n = \xi_1 + \dots + \xi_n$  and  $\nu$  is a symmetric stable distribution with characteristic function  $\exp(-\frac{\alpha|\lambda|^\alpha}{2d_\alpha})$ . The following bound was proved in [48], Proposition 1:

$$d_{\text{Kol}}(\mathcal{L}(T_n/A_n), \nu) = O((\log n)^{-1}),$$

whose proof heavily depends on the special form of (A.1). Recall  $S_n = (\frac{\alpha}{2d_\alpha})^{-\frac{1}{\alpha}} \frac{T_n}{A_n}$ , by Remark 2.4, we have

$$(A.2) \quad d_{\text{Kol}}(\mathcal{L}(S_n), \mu) = O((\log n)^{-1}),$$

where  $\mu$  is a symmetric stable distribution with characteristic function  $e^{-|\lambda|^\alpha}$ . Applying Theorem 2.1, we can prove that if (A.1) is satisfied with  $\alpha \in (1, 2)$ , a convergence rate  $O((\log n)^{-1+\frac{1}{\alpha}})$  can be obtained in  $W_1$  distance.

Here, we consider a new example which is more complicated than (A.1), more precisely,

$$(A.3) \quad \mathbb{P}(|\xi_1| > x) = \frac{K_0(\log x)^\beta}{x^\alpha}, \quad x > x_0.$$

Note that  $K_0$  and  $x_0$  here may be different from those in (A.1). The corresponding density function is

$$p(x) = \frac{K_0[\alpha(\log |x|)^\beta - \beta(\log |x|)^{\beta-1}]}{2|x|^{\alpha+1}}, \quad |x| > x_0;$$

$$p(x) = 0, \quad |x| \leq x_0.$$

It seems that the method in [48] cannot deal with this example directly. However, by our first main result Theorem 2.1, we can prove

$$(A.4) \quad d_W(\mathcal{L}(S_n), \mu) \leq C_{\alpha,\beta}(\log n)^{-1+\frac{1}{\alpha}}.$$

It can be seen from the proof that (A.4) also holds under the condition (A.1) by a similar but simpler argument. Because the proof of (A.4) under the condition (A.3) is long, we only give the leading order of the convergence.

Let  $L, A$  be two quantities with  $A > 0$ , if there exist some  $C > 0$  (which may depend on some parameters) such that

$$|L| \leq CA,$$

we denote  $L = O(A)$ .

By Theorem 2.5,  $A_n = \inf\{x > 0 : \mathbb{P}(|\xi_1| > x) \leq \frac{1}{n}\}$  can be determined by  $\frac{K_0(\log A_n)^\beta}{A_n^\alpha} = \frac{1}{n}$ , which gives

$$(A.5) \quad \frac{n}{A_n^\alpha} = \frac{1}{K_0(\log A_n)^\beta}.$$

It is easy to see  $C_{\alpha,\beta}n^{\frac{1}{\alpha}} \leq A_n \leq C_{\alpha,\beta}n^{\frac{1}{\alpha}}(\log n)^{\frac{\beta}{\alpha}}$ . By the symmetry property,  $B_n = n\mathbb{E}[\xi_1 1_{\{|\xi_1| \leq A_n\}}] = 0$ .

Now we apply Theorem 2.1 with  $N = (\log A_n)^{\frac{1}{\alpha}}$  and

$$\zeta_{n,i} = \frac{1}{A_n} \xi_i \quad \text{with } \tilde{A}_n = \left(\frac{\alpha}{2d_\alpha}\right)^{1/\alpha} A_n.$$

Let us first estimate the remainder term  $\mathcal{R}_{N,n}$ . Let  $\gamma = 2 - \alpha$ , we get

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}|\zeta_{n,i}|^{2-\alpha} \leq C_\alpha A_n^{-2+\alpha}.$$

By Lemma 2.8, we get

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}(|\zeta_{n,i}| \mathbf{1}_{\{|\zeta_{n,i}| > N\}}) &= \frac{n}{\tilde{A}_n} \mathbb{E}(|\xi_1| \mathbf{1}_{\{|\xi_1| > \tilde{A}_n N\}}) \\ &= nN\mathbb{P}(|\xi_1| > \tilde{A}_n N) + \frac{n}{\tilde{A}_n} \int_{\tilde{A}_n N}^\infty \mathbb{P}(|\xi_1| > r) \, dr. \end{aligned}$$

By (A.5),  $N = (\log A_n)^{\frac{1}{\alpha}}$  and  $A_n \geq C_{\alpha,\beta} n^{\frac{1}{\alpha}}$ ,

$$nN\mathbb{P}(|\xi_1| > \tilde{A}_n N) = \frac{2d_\alpha N^{1-\alpha}}{\alpha} \left( \frac{\log(\tilde{A}_n N)}{\log A_n} \right)^\beta \leq C_{\alpha,\beta} N^{1-\alpha}.$$

Moreover, by (A.5) and a change of variable  $s = \frac{r}{\tilde{A}_n N}$ ,

$$\begin{aligned} \frac{n}{\tilde{A}_n} \int_{\tilde{A}_n N}^\infty \mathbb{P}(|\xi_1| > r) \, dr &= \frac{N^{1-\alpha}}{K_0(\log \tilde{A}_n)^\beta} \int_1^\infty \frac{(\log s + \log(\tilde{A}_n N))^\beta}{s^\alpha} \, ds \\ &= \frac{2d_\alpha N^{1-\alpha}}{\alpha K_0} \left( \frac{\log(\tilde{A}_n N)}{\log A_n} \right)^\beta \int_1^\infty \frac{(1 + \frac{\log s}{\log(\tilde{A}_n N)})^\beta}{s^\alpha} \, ds \\ &\leq C_{\alpha,\beta} N^{1-\alpha}, \end{aligned}$$

where the inequality is by  $A_n \geq C_{\alpha,\beta} n^{\frac{1}{\alpha}}$  and an easy observation that the above integral is bounded. Hence,

$$\sum_{i=1}^n \mathbb{E}(|\zeta_{n,i}| \mathbf{1}_{\{|\zeta_{n,i}| > N\}}) \leq C_{\alpha,\beta} N^{1-\alpha}.$$

Collecting all the above estimates, we immediately obtain

$$(A.6) \quad \mathcal{R}_{N,n} \leq C_{\alpha,\beta} \left( A_n^{-2+\alpha} + \frac{1}{N^{\alpha-1}} \right) \leq C_{\alpha,\beta} (\log n)^{-1+\frac{1}{\alpha}}.$$

Now let us estimate the integral term in the theorem, observe

$$\begin{aligned} \sum_{i=1}^n \int_{-N}^N \left| \frac{\mathcal{K}_\alpha(t, N)}{n} - \frac{K_i(t, N)}{\alpha} \right| dt &= \int_{-N}^N \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt \\ (A.7) \quad &= \left( \int_{|t| \leq \frac{x_0}{A_n}} + \int_{\frac{x_0}{A_n} < |t| < N} \right) \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt. \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 (A.8) \quad & \int_{|t| \leq \frac{x_0}{\tilde{A}_n}} \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt \\
 & \leq C_\alpha \left( \int_{|t| \leq \frac{x_0}{\tilde{A}_n}} |t|^{1-\alpha} dt + \int_{|t| \leq \frac{x_0}{\tilde{A}_n}} n \tilde{A}_n^{-1} \mathbb{E}|\xi_1| dt \right) \\
 & \leq C_{\alpha, \beta} \frac{(\log n)^\beta}{n^{\frac{2}{\alpha}-1}}.
 \end{aligned}$$

We shall show below that

$$(A.9) \quad \int_{\frac{x_0}{\tilde{A}_n} < |t| < N} \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt \leq C_{\alpha, \beta} \frac{N}{\log A_n}.$$

By  $N = (\log n)^{\frac{1}{\alpha}}$  and  $A_n \geq C_{\alpha, \beta} n^{\frac{1}{\alpha}}$ , we have

$$\int_{-N}^N \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt \leq C_{\alpha, \beta} (\log n)^{-1 + \frac{1}{\alpha}}.$$

Combining this with that of  $\mathcal{R}_{N, n}$ , we immediately obtain the estimate (A.4), as desired.

It remains to prove (A.9). For  $t > \frac{x_0}{\tilde{A}_n}$ , we have

$$\begin{aligned}
 (A.10) \quad nK_1(t, N) &= n \mathbb{E} \left[ \frac{1}{\tilde{A}_n} \xi_1 1_{\{\tilde{A}_n t \leq \xi_1 \leq \tilde{A}_n N\}} \right] \\
 &= \frac{n}{\tilde{A}_n} \left[ \mathbb{E}(\xi_1 1_{\{\xi_1 > \tilde{A}_n t\}}) - \mathbb{E}(\xi_1 1_{\{\xi_1 > \tilde{A}_n N\}}) \right] \\
 &= nt \mathbb{P}(\xi_1 > \tilde{A}_n t) - nN \mathbb{P}(\xi_1 > \tilde{A}_n N) \\
 &\quad + \frac{n}{\tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \mathbb{P}(\xi_1 > r) dr,
 \end{aligned}$$

where the last equality is by Lemma 2.8. For the first term in the last line above, by (A.3), (A.5) and a straightforward computation, we get

$$\begin{aligned}
 & nt \mathbb{P}(\xi_1 > \tilde{A}_n t) - nN \mathbb{P}(\xi_1 > \tilde{A}_n N) \\
 &= \frac{d_\alpha}{\alpha} \left[ t^{1-\alpha} \left( \frac{\log(\tilde{A}_n t)}{\log A_n} \right)^\beta - N^{1-\alpha} \left( \frac{\log(\tilde{A}_n N)}{\log A_n} \right)^\beta \right] \\
 &= (\alpha - 1) \mathcal{K}_\alpha(t, N) + O \left( \frac{\log N}{\log A_n} \right) t^{1-\alpha},
 \end{aligned}$$

where  $\mathcal{K}_\alpha(t, N) = \frac{d_\alpha}{\alpha(\alpha-1)}(t^{1-\alpha} - N^{1-\alpha})$ . For the integral term in (A.10), as  $t > \frac{x_0}{\tilde{A}_n}$ , by (A.5), we have

$$\begin{aligned} & \frac{n}{\tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \mathbb{P}(\xi_1 > r) \, dr \\ &= \frac{nK_0}{2\tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \frac{(\log r)^\beta}{r^\alpha} \, dr \\ &= \frac{nK_0}{2\tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \frac{(\log \tilde{A}_n)^\beta}{r^\alpha} \, dr + \frac{nK_0}{2\tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \frac{(\log r)^\beta - (\log \tilde{A}_n)^\beta}{r^\alpha} \, dr \\ &= \mathcal{K}_\alpha(t, N) + O\left(\frac{\log N}{\log A_n}\right)t^{1-\alpha} + \frac{nK_0}{2\tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \frac{(\log r)^\beta - (\log \tilde{A}_n)^\beta}{r^\alpha} \, dr. \end{aligned}$$

From the previous estimate, it is easy to check

$$\begin{aligned} \text{(A.11)} \quad & \mathcal{K}_\alpha(t, N) - \frac{nK_{N,1}(t, N)}{\alpha} \\ &= \frac{K_0 n}{2\alpha \tilde{A}_n} \int_{\tilde{A}_n t}^{\tilde{A}_n N} \frac{(\log r)^\beta - (\log \tilde{A}_n)^\beta}{r^\alpha} \, dr + O\left(\frac{\log N}{\log A_n}\right)t^{1-\alpha}. \end{aligned}$$

When  $t > \frac{x_0}{\tilde{A}_n}$ , we first observe

$$\begin{aligned} \text{(A.12)} \quad & |(\log r)^\beta - (\log \tilde{A}_n)^\beta| \\ & \leq \left| \left( \log \tilde{A}_n + \log \frac{r}{\tilde{A}_n} \right)^\beta - (\log \tilde{A}_n)^\beta \right| \\ & \leq (\log \tilde{A}_n)^\beta \left| \left( 1 + \frac{\log \frac{r}{\tilde{A}_n}}{\log \tilde{A}_n} \right)^\beta - 1 \right| \\ & \leq C_{\alpha,\beta} (\log A_n)^{\beta-1} \left| \log \frac{r}{\tilde{A}_n} \right|, \end{aligned}$$

where the last inequality is by Taylor's expansion and the easy fact  $\left| \frac{\log \frac{r}{\tilde{A}_n}}{\log \tilde{A}_n} \right| < 1$  when  $1 \leq r \leq N\tilde{A}_n$ . The previous two relations, (A.5) and a change of variable  $s = r/\tilde{A}_n$  yield

$$\begin{aligned} \text{(A.13)} \quad & \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| \\ & \leq C_{\alpha,\beta} \left[ \frac{n(\log A_n)^{\beta-1} K_0}{2A_n^\alpha} \int_t^N \frac{|\log s|}{s^\alpha} \, ds + \frac{\log N}{\log A_n} t^{1-\alpha} \right] \\ & = C_{\alpha,\beta} \left[ (\log A_n)^{-1} \int_t^N \frac{|\log s|}{s^\alpha} \, ds + \frac{\log N}{\log A_n} t^{1-\alpha} \right]. \end{aligned}$$

It is easy to check when  $t \geq 1$ ,

$$(A.14) \quad \int_t^N \frac{|\log s|}{s^\alpha} ds \leq \frac{C_\alpha}{(\alpha - 1)^2},$$

when  $0 < t \leq 1$ , we have

$$(A.15) \quad \begin{aligned} \int_t^N \frac{|\log s|}{s^\alpha} ds &= \int_t^1 \frac{|\log s|}{s^\alpha} ds + \int_1^N \frac{|\log s|}{s^\alpha} ds \\ &\leq \frac{|\log t|t^{1-\alpha}}{\alpha - 1} + \frac{2t^{1-\alpha}}{(\alpha - 1)^2}. \end{aligned}$$

Collecting the above estimates, we get

$$(A.16) \quad \begin{aligned} &\left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| \\ &\leq C_{\alpha, \beta} \begin{cases} (\log A_n)^{-1} (1 + t^{1-\alpha} \log N), & 1 \leq t \leq N; \\ (\log A_n)^{-1} t^{1-\alpha} (1 + |\log t| + \log N), & \frac{x_0}{\tilde{A}_n} \leq t < 1. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \int_{x_0/\tilde{A}_n}^N \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt &\leq C_{\alpha, \beta} \left( \frac{N}{\log A_n} + \frac{N^{2-\alpha} \log N}{\log A_n} + \frac{1}{\log A_n} \right) \\ &\leq C_{\alpha, \beta} \frac{N}{\log A_n}. \end{aligned}$$

By the same argument, we get

$$\int_{-N}^{-x_0/\tilde{A}_n} \left| \mathcal{K}_\alpha(t, N) - \frac{n}{\alpha} K_1(t, N) \right| dt \leq C_{\alpha, \beta} \frac{N}{\log A_n}.$$

Hence, (A.9) is proved.

**A.2. Proof of Lemma 6.1.** For notational simplicity, we write  $p(x) = p(1, x)$ . Due to the symmetry property  $p(x) = p(-x)$  for all  $x \in \mathbb{R}$ , it suffices to consider  $p(x)$  for  $x \geq 0$ . We shall frequently use the easy relations

$$\Gamma(z + 1) = z\Gamma(z) \quad \forall z \in \mathbb{R}; \quad \Gamma(z) \leq 1 \quad \forall z \in (1, 2).$$

For  $\theta \in (-1, \infty)$ , we denote

$$I_\theta(x) = \int_0^\infty \lambda^\theta e^{-\lambda^\alpha} \cos(\lambda x) d\lambda, \quad J_\theta(x) = \int_0^\infty \lambda^\theta e^{-\lambda^\alpha} \sin(\lambda x) d\lambda.$$

It is easy to verify by the easy estimate  $|\cos(\lambda x)|, |\sin(\lambda x)| \leq 1$  and a change of variable  $t = \lambda^\alpha$  that

$$|I_\theta(x)| \leq \frac{\Gamma(\frac{\theta+1}{\alpha})}{\alpha}, \quad |J_\theta(x)| \leq \frac{\Gamma(\frac{\theta+1}{\alpha})}{\alpha}.$$

By the inverse of Fourier transform, we have

$$(A.17) \quad p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\lambda|^\alpha} e^{-ix\lambda} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\lambda|^\alpha} \cos(\lambda x) d\lambda = \frac{I_0(x)}{\pi}.$$

Hence,

$$(A.18) \quad p(x) \leq \frac{\Gamma(\frac{1}{\alpha})}{\pi\alpha} = \frac{\Gamma(1 + \frac{1}{\alpha})}{\pi} \leq \frac{1}{\pi}.$$

For  $x > 0$ , using integration by parts two times, we get

$$(A.19) \quad I_0(x) = \frac{\alpha(\alpha - 1)I_{\alpha-2}(x) - \alpha^2 I_{2\alpha-2}(x)}{x^2}.$$

Moreover,

$$(A.20) \quad |I_{\alpha-2}(x)| \leq \frac{\Gamma(1 - \frac{1}{\alpha})}{\alpha} = \frac{\Gamma(2 - \frac{1}{\alpha})}{\alpha - 1} \leq \frac{1}{\alpha - 1},$$

$$(A.21) \quad |I_{2\alpha-2}(x)| \leq \frac{\Gamma(2 - \frac{1}{\alpha})}{\alpha} \leq \frac{1}{\alpha}.$$

Hence,

$$(A.22) \quad p(x) = \frac{I_0(x)}{\pi} \leq \frac{2\alpha}{\pi x^2}.$$

Now we estimate  $p'(x)$ . It is obvious that

$$I_0'(x) = -J_1(x),$$

and thus

$$|I_0'(x)| \leq \frac{\Gamma(\frac{2}{\alpha})}{\alpha} \leq \frac{1}{\alpha}.$$

Hence,

$$|p'(x)| = \frac{|I_0'(x)|}{\pi} \leq \frac{1}{\pi\alpha}.$$

For  $x > 0$ , using integration by parts, we have

$$(A.23) \quad I_0(x) = \frac{\alpha J_{\alpha-1}(x)}{x},$$

which implies

$$(A.24) \quad I_0'(x) = -\frac{\alpha J_{\alpha-1}(x)}{x^2} + \frac{\alpha I_\alpha(x)}{x}.$$

It is easy to check

$$(A.25) \quad \left| \frac{\alpha J_{\alpha-1}(x)}{x^2} \right| \leq \frac{1}{x^2}.$$

Using integration by parts, we have

$$(A.26) \quad I_\alpha(x) = \frac{\alpha J_{2\alpha-1}(x) - \alpha J_{\alpha-1}(x)}{x},$$

which gives

$$(A.27) \quad \left| \frac{\alpha I_\alpha(x)}{x} \right| \leq \frac{\alpha^2}{x^2} \left( \frac{\Gamma(2)}{\alpha} + \frac{\Gamma(1)}{\alpha} \right) = \frac{2\alpha}{x^2}.$$

Hence,

$$|p'(x)| \leq \frac{|I'_0(x)|}{\pi} \leq \frac{(2\alpha + 1)}{\pi x^2}.$$

For  $p''(x)$ , we have

$$(A.28) \quad p''(x) = -\frac{I_2(x)}{\pi},$$

which immediately implies

$$|p''(x)| \leq \frac{\Gamma(\frac{3}{\alpha})}{\alpha\pi} \leq \frac{2}{\alpha\pi}.$$

Using integration by parts two times,

$$(A.29) \quad I_2(x) = -\frac{2}{x^2} I_0(x) + \frac{\alpha^2 + 3\alpha}{x^2} I_\alpha(x) - \frac{\alpha^2}{x^2} I_{2\alpha}(x).$$

By a similar computation, we get the second estimate of  $p''(x)$ , as desired.

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