

Poisson statistics for 1d Schrödinger operators with random decaying potentials

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Abstract

We consider the 1d Schrödinger operators with random decaying potentials in the sub-critical case where the spectrum is pure point. We show that the point process composed of the rescaled eigenvalues in the bulk, together with those zero points of the corresponding eigenfunctions, converges to a Poisson process.

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1 Introduction

The 1d Schrödinger operators with random decaying potentials are known to have rich spectral properties depending on the decay order of the potentials (e.g., [8, 6]). Recently, the level statistics problem of this operators are studied and turned out to be related to the β -ensembles which appear in the random matrix theory[5, 9, 7, 11]. In this paper we consider the following Hamiltonian.

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbf{R})$$

where the function $a \in C^\infty(\mathbf{R})$ is a decay factor satisfying $a(-t) = a(t)$, being non-increasing for $t > 0$, and

$$a(t) = t^{-\alpha}(1 + o(1)), \quad a'(t) = O(t^{-\alpha-1}), \quad t \rightarrow \infty, \quad \alpha > 0.$$

The assumption on a' is technical but we need it to estimate some error terms. $F(X_t)$ is a random factor where $F \in C^\infty(M)$, M is the d-dimensional torus, and

$$\langle F \rangle := \int_M F(x)dx = 0.$$

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$\{X_t\}_{t \in \mathbf{R}}$ is the Brownian motion on M . Since the potential $a(t)F(X_t)$ is compact w.r.t. the free Laplacian $-d^2/dt^2$, the essential spectrum of H is equal to $\sigma_{ess}(H) = [0, \infty)$ which is [8] (1) $\alpha > 1/2$: absolutely continuous, (2) $\alpha < 1/2$: pure point with (sub)exponentially decaying eigenfunctions, and (3) $\alpha = 1/2$: there exists a non-random number $E_c \geq 0$ such that the spectrum is pure point on $[0, E_c]$ and singular continuous on $[E_c, \infty)$.

The purpose of this paper is to study the local fluctuation of the eigenvalues in the positive energy axis. In order for that, let $H_L := H|_{[0,L]}$ be the restriction of H on the interval $[0, L]$ with Dirichlet boundary condition, and let $\{E_j(L)\}_{j \geq j_0}$ ($0 < E_{j_0}(L) < E_{j_0+1}(L) < \dots$) be the set of positive eigenvalues of H_L . Take the reference energy $E_0 > 0$ arbitrary, and consider the point process

$$\xi_L := \sum_{j \geq j_0} \delta_{L(\sqrt{E_j} - \sqrt{E_0})}$$

where we take the square root of each eigenvalues which corresponds to the unfolding with respect to the integrated density of states $N(E) = \pi^{-1}\sqrt{E}$. For a Borel measure μ on \mathbf{R}^d , we denote by $Poisson(\mu)$ the Poisson process on \mathbf{R}^d with intensity measure μ . Similarly, for a constant $c > 0$, we denote by $Poisson(c)$ the Poisson distribution with parameter c . The first theorem of this paper is

Theorem 1.1. *Let $\alpha < 1/2$. Then ξ_L converges in distribution to the Poisson process of intensity $d\lambda/\pi$ ¹*

$$\xi_L \xrightarrow{d} Poisson\left(\frac{d\lambda}{\pi}\right), \quad L \rightarrow \infty.$$

Remark 1.2. When we consider two reference energies $E_1, E_2, E_1 \neq E_2$, then the corresponding point processes ξ_1, ξ_2 jointly converge to the independent Poisson processes of intensity $d\lambda/\pi$.

Remark 1.3. Together with results in [7, 11], we have²

- (1) $\alpha > \frac{1}{2} \implies \xi_L \rightarrow \text{clock process}$
- (2) $\alpha = \frac{1}{2} \implies \xi_L \rightarrow \text{Sine}_\beta \text{ process}$
- (3) $\alpha < \frac{1}{2} \implies \xi_L \rightarrow \text{Poisson process}$

Such kind of results have been known for discrete models: [5] proved (1)-(3) above for CMV matrices, [3] proved “clock behavior” (similar to (1)) for Jacobi matrices, and [9] proved (2) for 1d discrete Schrödinger operators. Hence our result is a continuum analogue of them. The model-independent nature of those results is due to the fact that the Prüfer phases of those models obey the similar equations and thus have similar behavior. The global fluctuation of eigenvalues is studied in [13] which also shows different behavior in above three cases.

Remark 1.4. Let $H'_L := (-\frac{d^2}{dt^2} + L^{-\alpha}F(X_t))|_{[0,L]}$ be the Hamiltonian with decaying coupling constant under the Dirichlet boundary condition. The method of proof of Theorem 1.1 also works for H'_L so that together with results in [11] we have³

- (1) $\alpha > \frac{1}{2} \implies \xi_L \rightarrow \text{clock process}$

¹ We consider the vague topology on the space of point measures on \mathbf{R} . Hence $\xi_L \xrightarrow{d} \xi$ is equivalent to $\lim_{L \rightarrow \infty} \mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E}[e^{-\xi(f)}]$ for any $f \in C_c^+(\mathbf{R})$.

² In (2), $\beta = \beta(E_0) := 8E_0/C(E_0)$ where $C(E) := \langle \nabla g_{\sqrt{E}}, \nabla g_{\sqrt{E}} \rangle, g_{\sqrt{E}} := (L + 2i\sqrt{E})^{-1}F$. $\beta(E)$ is equal to the reciprocal of the Lyapunov exponent of H .

³ In (2), $\tau = \tau(E_0) = C(E_0)/(2E_0) = 4/\beta(E_0)$ [12].

$$(2) \alpha = \frac{1}{2} \implies \xi_L \rightarrow \text{Sch}_\tau \text{ process}$$

$$(3) \alpha < \frac{1}{2} \implies \xi_L \rightarrow \text{Poisson} \left(\frac{d\lambda}{\pi} \right)$$

[9] proved (2) for 1d discrete Schrödinger operators.

Remark 1.5. It would be interesting to study the behavior of eigenvalues near the bottom edge of the essential spectrum (i.e., to study ξ_L for $E_0 = 0$), for which the technique in this paper does not apply. For recent development in this respect, we refer to [2].

To see the outline of proof, we introduce the Prüfer variable as follows. Let x_t be the solution to the Schrödinger equation $Hx_t = \kappa^2 x_t$, $x_0 = 0$, which is represented in the following form.

$$\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = r_t(\kappa) \begin{pmatrix} \sin \theta_t(\kappa) \\ \cos \theta_t(\kappa) \end{pmatrix}, \quad \theta_0(\kappa) = 0.$$

Set

$$\begin{aligned} \Theta_L(\lambda) &:= \theta_L \left(\sqrt{E_0 + \frac{\lambda}{L}} \right) - \theta_L \left(\sqrt{E_0} \right) \\ \phi_L(E_0) &:= \{ \theta_L(\sqrt{E_0}) \}_\pi, \quad \text{where} \quad \{x\}_\pi := x - \left\lfloor \frac{x}{\pi} \right\rfloor \pi. \end{aligned}$$

Since, by Sturm’s oscillation theorem, $E = E_j(L)$ if and only if $\theta_L(\sqrt{E}) = j\pi$, the Laplace transform of ξ_L has the following representation.

$$\mathbf{E} \left[e^{-\xi_L(f)} \right] = \mathbf{E} \left[\exp \left(- \sum_k f \left(\Theta_L^{-1}(k\pi - \phi_L(E_0)) \right) \right) \right] \tag{1.1}$$

$$\text{where} \quad \xi_L(f) = \int_{\mathbf{R}} f(x) \xi_L(dx), \quad f \in C_c^+(\mathbf{R}).$$

Thus our aim is to study the joint limit of $(\Theta_L(\lambda), \phi_L(E_0))$. Here we replace L by n and consider the family H_{nt} ($t \in [0, 1]$) of Hamiltonians. We will show that the following limits exist.

$$\widehat{\Theta}_t(\lambda) \stackrel{d}{=} \lim_{n \rightarrow \infty} \Theta_{nt}(\lambda), \quad \widehat{\phi}_t \stackrel{d}{=} \lim_{n \rightarrow \infty} \phi_{nt}(E_0).$$

In the first equation, both sides are regarded as the non-decreasing function(with the weak topology as a measure)-valued processes in t . Then we have the following theorem.

Theorem 1.6.

(1) For any $t \in (0, 1]$, $\widehat{\phi}_t$ is uniformly distributed on $[0, \pi)$.

(2)

$$\widehat{\Theta}_t(\lambda) = \pi \int_{[0,t] \times [0,\lambda]} \widehat{P}(dsd\lambda')$$

where $\widehat{P} = \text{Poisson}(\pi^{-1}1_{[0,1]}(s)dsd\lambda')$ is the Poisson process on \mathbf{R}^2 whose intensity measure is equal to $\pi^{-1}1_{[0,1]}(s)dsd\lambda'$.

Remark 1.7. The statement in Theorem 1.6(2) is conjectured in [5] for CMV matrices. On the other hand, for the Anderson model $H = -\Delta + V_\omega(x)$ on $l^2(\mathbf{Z}^d)$, the following facts are known [4, 10]. Let $H_L := H|_{\{1, \dots, L\}^d}$ be the restriction of H on the box of size L , with $\{E_j(L)\}_{j \geq 1}$ being its eigenvalues. Let $x_j(L) \in \mathbf{R}^d$ be the localization center corresponding to $E_j(L)$. If E_0 lies in the localized region, we have

$$\sum_j \delta_{(L^d(E_j(L)-E_0), L^{-1}x_j(L))} \xrightarrow{d} \text{Poisson} \left(n(E_0)1_{[0,1]^d}(x)dE \times dx \right) \tag{1.2}$$

where $n(E_0) := \frac{d}{dE}N(E)|_{E=E_0}$ is the density of states at $E = E_0$.

The jump points of the function $t \mapsto \lfloor \Theta_{nt}(\lambda)/\pi \rfloor$ are (modulo some errors) related to the zero points of the eigenfunction such that the corresponding eigenvalue is less than λ . Since the eigenfunction decays sub-exponentially and since the set of jump points of the function $t \mapsto \hat{\Theta}_t(\lambda)/\pi$ has the monotonicity in λ to be described in eq.(1.5), those jump points are close to the localization center of each eigenfunctions. Hence we believe that the statement like eq.(1.2) holds also for our case and that Theorem 1.6 (2) is related to this speculation.

We shall explain the idea of proof. The Prüfer phase satisfies the integral equation (2.1) by which we compute the equation satisfied by $\Theta_{nt}(\lambda)$. By using “Ito’s formula” (2.3) we can show that, up to error terms,

$$d\Theta_{nt}(\lambda) \sim \lambda dt + n^{\frac{1}{2}-\alpha} \operatorname{Re} \left[\left(e^{2i\Theta_{nt}(\lambda)} - 1 \right) t^{-\alpha} dZ_t \right]$$

where $Z_t = X_t + iY_t$ is the complex Brownian motion. At this point, we have a general picture: (1) $\alpha > 1/2$: second term vanishes which implies the convergence to the clock process, (2) $\alpha = 1/2$: $\Theta_{nt}(\lambda)$ converges to the solution to a SDE, and (3) $\alpha < 1/2$: the diffusion term will be dominant so that $\Theta_{nt}(\lambda)$ should be in a vicinity of $\pi\mathbf{Z}$ in order to have $(e^{2i\Theta_{nt}(\lambda)} - 1)$ small. Here we note that $\Theta_{nt}(\lambda) > 0$ for $\lambda > 0$ and $\mathbf{E}[\Theta_{nt}(\lambda)] = \lambda t + o(1)$ (Proposition 2.4). By the change of variables

$$t = s^\gamma, \quad \gamma := \frac{1}{1-2\alpha}, \quad s \in [0, 1],$$

we have

$$d\Theta_{ns^\gamma}(\lambda) \sim \lambda \gamma s^{\gamma-1} ds + n^{\frac{1}{2\gamma}} \operatorname{Re} \left[\left(e^{2i\Theta_{ns^\gamma}(\lambda)} - 1 \right) d\tilde{Z}_s \right].$$

Here we recall the definition of the Sine $_\beta$ -process [14]. Let $\alpha_t(\lambda)$ be the solution to the following SDE.

$$\begin{aligned} d\alpha_t(\lambda) &= \lambda \cdot \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + \operatorname{Re} \left[\left(e^{i\alpha_t(\lambda)} - 1 \right) dZ_t \right] \\ \alpha_0(\lambda) &= 0. \end{aligned} \tag{1.3}$$

Then the function $t \mapsto \lfloor \alpha_t(\lambda)/2\pi \rfloor$ is non-decreasing and the limit $\alpha_\infty(\lambda) := \lim_{t \rightarrow \infty} \alpha_t(\lambda)$ satisfies $\alpha_\infty(\lambda) \in 2\pi\mathbf{Z}$, a.s. Then Sine $_\beta$ -process on the interval $[\lambda_1, \lambda_2]$ is defined by

$$\operatorname{Sine}_\beta[\lambda_1, \lambda_2] \stackrel{d}{=} \frac{\alpha_\infty(\lambda_2) - \alpha_\infty(\lambda_1)}{2\pi}.$$

Allez-Dumaz [1] showed that $\operatorname{Sine}_\beta \xrightarrow{d} \operatorname{Poisson}(d\lambda/2\pi)$ as $\beta \rightarrow 0$. This fact can easily be generalized to other processes where the drift term in the corresponding SDE (1.3) is replaced by functions f with mild conditions[12]. Moreover, by a scaling $t \mapsto \frac{\beta}{4}t$, eq.(1.3) becomes

$$\begin{aligned} d\alpha_t(\lambda) &= \lambda e^{-t} dt + \frac{2}{\sqrt{\beta}} \operatorname{Re} \left[\left(e^{i\alpha_t(\lambda)} - 1 \right) dZ_t \right] \\ \alpha_0(\lambda) &= 0 \end{aligned}$$

so that, by setting $\beta = n^{-\frac{1}{\gamma}}$, we can use the idea of [1]: to study the hitting time of $\Theta_{nt}(\lambda)$ to the set $\pi\mathbf{Z}$, we consider

$$R(nt) := \log \tan \frac{\Theta_{nt}(\lambda)}{2},$$

SDE of which has a diffusion term with constant coefficient so that we may use comparison argument. In fact, modulo error terms, we have (Propositions 3.1, 4.1)

$$dR(nt^\gamma) \sim \left(\lambda\gamma t^{\gamma-1} \cosh R(nt^\gamma) + \frac{C_n^2}{2} \tanh R(nt^\gamma) \right) dt + C_n dM_t \tag{1.4}$$

where $C_n = C(E_0, F)n^{\frac{1}{2\gamma}}$, $d\langle M \rangle_t = (1 + o(1)) dt$,

and $C(E_0, F)$ is a positive constant depending on E_0, F . Here we use assumptions on a, a' to estimate error terms. By a time-change, we can suppose that M_t is a Brownian motion. We divide the interval $[0, 1]$ into small random ones $I_k = [\tau_k/N, \tau_{k+1}/N]$ and consider the stationary processes S_\pm which are the solution to the following SDE's on each I_k 's.

$$dS_+(t) \sim \left(\lambda\gamma \left(\frac{\tau_{k+1}}{N} \right)^{\gamma-1} \cosh S_+(t) + \frac{C_n^2}{2} \tanh S_+(t) \right) dt + C_n dM_t$$

$$dS_-(t) \sim \left(\lambda\gamma \left(\frac{\tau_k}{N} \right)^{\gamma-1} \cosh S_-(t) + \frac{C_n^2}{2} \tanh S_-(t) \right) dt + C_n dM_t.$$

On each I_k , we can bound $R(nt^\gamma)$ by S_\pm from above and below:

$$S_-(t) \leq R(nt^\gamma) \leq S_+(t).$$

We can explicitly compute the explosion times of S_\pm which converge to $Exp(\tilde{\lambda}/\pi)$ as $n \rightarrow \infty$, where $\tilde{\lambda} := \lambda\gamma(\tau_{k+1}/N)^{\gamma-1}$ (Proposition 5.1). By an argument like the convergence of Riemannian sums to the integral, we can show that the jump points of the function $s \mapsto \lfloor \Theta_{ns^\gamma}(\lambda)/\pi \rfloor$ converge to $Poisson(\pi^{-1}\gamma s^{\gamma-1}1_{[0,1]}(s)ds)$ (Proposition 5.7). Hence for an interval $J \subset \mathbf{R}$, $\xi_L(J)$ converges to the Poisson distribution with parameter $\pi^{-1}|J|$. It then suffices to show that the collection of random variables $\xi_L(J_1), \dots, \xi_L(J_n)$ converge jointly to the independent ones for disjoint intervals J_1, J_2, \dots, J_n . For $\lambda_1 < \lambda_2$, let $P_{\lambda_1}, P_{\lambda_2}, P_{\lambda_1, \lambda_2}$ be the limit of those point processes composed by the jump points of functions $s \mapsto \lfloor \Theta_{ns^\gamma}(\lambda_1)/\pi \rfloor, \lfloor \Theta_{ns^\gamma}(\lambda_2)/\pi \rfloor$ and $\lfloor (\Theta_{ns^\gamma}(\lambda_2) - \Theta_{ns^\gamma}(\lambda_1))/\pi \rfloor$ respectively. Then $P_{\lambda_1}, P_{\lambda_2}, P_{\lambda_1, \lambda_2}$ turn out to be the \mathcal{F}_s -Poisson processes under a suitable choice of the filtration \mathcal{F}_s (Lemma 5.10). Letting $\mathcal{P}_{\lambda_1}, \mathcal{P}_{\lambda_2}, \mathcal{P}_{\lambda_1, \lambda_2}$ be the set of atoms, we show (Lemmas 5.11, 5.12)

$$\mathcal{P}_{\lambda_1} \subset \mathcal{P}_{\lambda_2}, \quad \mathcal{P}_{\lambda_1} \cap \mathcal{P}_{\lambda_1, \lambda_2} = \emptyset \tag{1.5}$$

from which the independence of \mathcal{P}_{λ_1} and $\mathcal{P}_{\lambda_1, \lambda_2}$ follows.

Finally we show that $\lim_{n \rightarrow \infty} \Theta_{nt}(\lambda)/\pi \in \mathbf{Z}$, a.s. which proves Theorem 1.6(2). The statement in Theorem 1.6(1) is essentially proved in our previous paper [7] where the condition $\langle F \rangle = 0$ is used. Theorem 1.1 follows from eq.(1.1) and Theorem 1.6.

The rest of this paper is organized as follows. In Section 2, we study the behavior of $\Theta_{nt}(\lambda)$ and derive some properties of the expectation of $\Theta_{nt}(\lambda)$ and the monotonicity of the function $t \mapsto \lfloor \Theta_{nt}(\lambda)/\pi \rfloor$. In Section 3, we derive the Ricatti equation (1.4) satisfied by $R(nt)$. In Section 4, we estimate $R(nt^\gamma)$ from above and below by solutions R_\pm to simple SDE's. In Section 5, following the argument in [1], we consider the stationary approximation S_\pm of R_\pm and compute the explosion time of them. Then we show that the jump points of the function $t \mapsto \lfloor \Theta_{nt}/\pi \rfloor$ converge to a Poisson process and that the processes P_{λ_1} and P_{λ_1, λ_2} mentioned above are independent. In Section 6, we prove Theorems 1.1, 1.6. Sections 7, 8 are appendices. In what follows, C, C' are positive constants which may change from line to line in each argument.

2 Behavior of $\Theta_{nt}(\lambda)$

In this section we introduce notations and derive some basic properties of the relative Prüfer phase $\Theta_{nt}(\lambda)$. Let $\tilde{\theta}_t(\kappa)$ be defined by

$$\theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa)$$

which satisfies the following integral equation.

$$\tilde{\theta}_t(\kappa) = \frac{1}{2\kappa} \operatorname{Re} \int_0^t \left(e^{2i\theta_s(\kappa)} - 1 \right) a(s) F(X_s) ds. \tag{2.1}$$

Set

$$\begin{aligned} \kappa_0 &:= \sqrt{E_0} \\ \kappa_c &:= \kappa_0 + \frac{c}{n}, \quad n > 0, \quad c \in \mathbf{R} \\ r_t^{(n)}(m) &:= e^{2mi\theta_t(\kappa_c)} - e^{2mi\theta_t(\kappa_0)}, \quad m \in \mathbf{Z} \\ A_n(t) &:= -\frac{c}{2\kappa_c \cdot \kappa_0} \operatorname{Re} \left(e^{2i\theta_t(\kappa_c)} - 1 \right) F(X_t) \\ (\Delta f)(m) &:= \frac{1}{2} (f(m+1) + f(m-1)) - f(m). \end{aligned}$$

By (2.1) we have

$$\begin{aligned} \Theta_{nt}(c) &= \theta_{nt}(\kappa_c) - \theta_{nt}(\kappa_0) \\ &= ct + \frac{1}{2\kappa_0} \operatorname{Re} \int_0^{nt} r_s^{(n)}(1) a(s) F(X_s) ds + \frac{1}{n} \int_0^{nt} A_n(s) a(s) ds. \end{aligned} \tag{2.2}$$

Remark 2.1. For large n , we can find $t_0 > 0$ such that for $t \geq t_0$, we have $c > A_n(nt)a(nt)$. Then by eq.(2.2), for $t \geq t_0$, once $\Theta_t^{(n)}(\lambda)$ enters to an interval $((k+1)\pi, (k+2)\pi)$ for some $k \in \mathbf{N}$, it never returns to $(k\pi, (k+1)\pi)$. In other words, the function $t \mapsto \lfloor \Theta_{nt}(\lambda)/\pi \rfloor$ is non-decreasing.

Here we make use of the following identity which is a consequence of Ito's formula [8]: for $f \in C^\infty(M)$ and $\kappa \neq 0$,

$$e^{i\kappa s} f(X_s) ds = d \left(e^{i\kappa s} (R_\kappa f)(X_s) \right) - e^{i\kappa s} (\nabla R_\kappa f)(X_s) dX_s \tag{2.3}$$

$$f(X_s) ds = \langle f \rangle ds + d((R_0 f)(X_s)) - \nabla(R_0 f)(X_s) dX_s \tag{2.4}$$

where $R_\kappa f := (L + i\kappa)^{-1} f$, $R_0 f := L^{-1}(f - \langle f \rangle)$.

L is the generator of X_t . Eq.(2.3) and the integration by parts yields the following equation.

Lemma 2.2. Let $b \in C^\infty([0, \infty))$, $\varphi \in C^\infty(M)$, and let $g_\varphi^{m\kappa_0} := R_{2m\kappa_0} \varphi = (L + 2mi\kappa_0)^{-1} \varphi$. Then we have

$$\begin{aligned} \int_0^t b(s) r_s^{(n)}(m) \varphi(X_s) ds &= (-2mi) \cdot \frac{1}{2\kappa_0} \int_0^t b(s) (\Delta r_s^{(n)})(m) a(s) F(X_s) g_\varphi^{m\kappa_0}(X_s) ds \\ &\quad + \left[b(s) r_s^{(n)}(m) g_\varphi^{m\kappa_0}(X_s) \right]_0^t \\ &\quad - \int_0^t b'(s) r_s^{(n)}(m) g_\varphi^{m\kappa_0}(X_s) ds \\ &\quad - 2mi \cdot \frac{1}{n} \int_0^t b(s) (c + A_n(s) a(s)) e^{2mi\theta_s(\kappa_c)} g_\varphi^{m\kappa_0}(X_s) ds \\ &\quad - \int_0^t b(s) r_s^{(n)}(m) \nabla g_\varphi^{m\kappa_0}(X_s) dX_s. \end{aligned}$$

Putting $m = 1$, $\varphi = F$, and $b(t) = a(t)$ in Lemma 2.2, we have

Lemma 2.3.

$$\Theta_{nt}(c) = ct + M_t^{(n)} + O_t^{(n)} + \delta_t^{(n)}$$

where

$$\begin{aligned} M_t^{(n)} &= -\frac{1}{2\kappa_0} \operatorname{Re} \int_0^{nt} a(s) r_s^{(n)}(1) \nabla g_{F^{\kappa_0}}(X_s) dX_s \\ O_t^{(n)} &= \frac{1}{2\kappa_0} \operatorname{Re} \left(-\frac{2i}{2\kappa_0} \int_0^{nt} a(s)^2 (\Delta r_s^{(n)})(1) F(X_s) g_{F^{\kappa_0}}(X_s) ds \right) \\ \delta_t^{(n)} &= \frac{1}{2\kappa_0} \operatorname{Re} \left\{ \left[a(s) r_s^{(n)}(1) g_{F^{\kappa_0}}(X_s) \right]_0^{nt} - \int_0^{nt} a'(s) r_s^{(n)}(1) g_{F^{\kappa_0}}(X_s) ds \right. \\ &\quad \left. + (-2i) \frac{1}{n} \int_0^{nt} a(s) (c + A_n(s) a(s)) e^{2i\theta_s(\kappa_c)} g_{F^{\kappa_0}}(X_s) ds \right\} \\ &\quad + \frac{1}{n} \int_0^{nt} a(s) A_n(s) ds. \end{aligned}$$

Moreover,

$$\lim_{n \rightarrow \infty} \delta_t^{(n)} = 0.$$

By using Lemmas 2.2, 2.3 we can prove the following Proposition which is necessary to study the behavior of $\mathbf{E}[\Theta_{nt}(\lambda)]$.

Proposition 2.4. *Suppose that*

$$\int_0^\infty a(s)^{j_0} ds < \infty$$

for some $j_0 \geq 1$. Then for $t > 0$, we have

$$\Theta_{nt}(c) = ct + \widetilde{M}_t^{(n)} + o(1), \quad n \rightarrow \infty$$

where $\widetilde{M}_t^{(n)}$ is a martingale.

Proof. Note that $\lim_{n \rightarrow \infty} r_s^{(n)}(m) = 0$. If $j_0 \leq 2$, $O_t^{(n)} = o(1)$ which already proves the statement of Proposition 2.4 with $\widetilde{M}_t^{(n)} = M_t^{(n)}$. If $j_0 \geq 3$, we apply Lemma 2.2 for $O_t^{(n)}$ so that

$$\begin{aligned} O_t^{(n)} &= \frac{1}{2\kappa} \operatorname{Re} \left(-\frac{2i}{2\kappa} \int_0^{nt} a(s)^2 \Delta r_s^{(n)}(1) F(X_s) g_{F^\kappa}(X_s) ds \right) \\ &= \operatorname{Re} \sum_{m=1,2} C_m \int_0^{nt} a(s)^3 r_s^{(n)}(m) G_m^{(n)}(X_s) ds + (\text{martingale}) + o(1) \end{aligned}$$

where $G_m^{(n)}$ is uniformly bounded. Iterating this process until we have $a(s)^{j_0}$ yields

$$O_t^{(n)} = \sum_m c_m \int_0^{nt} a(s)^{j_0} r_s^{(n)}(m) G_m^{(n)}(X_s) ds + (\text{martingale}) + o(1). \quad \square$$

3 Ricatti equation

For a function $\kappa \mapsto f(\kappa)$ we introduce

$$\Delta f := f(\kappa_c) - f(\kappa_d), \quad 0 \leq d < c, \quad \kappa_x := \kappa_0 + \frac{x}{n}.$$

This definition is different from that in Section 2. To study the hitting time of $\Theta_{nt}(\lambda)$ to the set $\pi\mathbf{Z}$, or that of $(\Theta_{nt}(\lambda') - \Theta_{nt}(\lambda))$ in general, we consider

$$R(t) := \log \tan \frac{\Delta\theta_t}{2}.$$

Note that

$$\cosh R(s) = \frac{1}{\sin \Delta\theta_s}, \quad \sinh R(s) = -\frac{\cos \Delta\theta_s}{\sin \Delta\theta_s}. \tag{3.1}$$

Here we recall that, for Sine $_{\beta}$ -process, the corresponding process $\tilde{R}(t) := \log \tan(\alpha_t(\lambda)/4)$ with $\alpha_t(\lambda)$ being the solution to eq.(1.3) satisfies

$$d\tilde{R}(t) = \frac{1}{2} \left(\lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} \cosh \tilde{R}(t) + \tanh \tilde{R}(t) \right) dt + dB_t. \tag{3.2}$$

The following Proposition implies that $R(nt)$ is close to the solution to a SDE which is similar to eq.(3.2).

Proposition 3.1.

$$\begin{aligned} R(nt) - R(0) &= \frac{c-d}{n} \int_0^{nt} \cosh R(s) ds \\ &+ \frac{1}{2\kappa_0} Re \left[-\frac{\langle Fg_{\kappa_0} \rangle}{\kappa_0} \right] \int_0^{nt} a(s)^2 \tanh R(s) ds + M_t + E(nt) \end{aligned} \tag{3.3}$$

where M is a martingale with

$$\begin{aligned} d\langle M \rangle_t &= \left(\frac{1}{2\kappa_0} \right)^2 2\langle \psi_{\kappa_0} \rangle na(nt)^2 (1 + o(1)) dt, \quad n \rightarrow \infty \\ \psi_{\kappa_0} &:= [g_{\kappa_0}, \bar{g}_{\kappa_0}], \quad g_{\kappa_0} := R_{2\kappa_0} F = (L + 2i\kappa_0)^{-1} F, \quad [f, g] := \nabla f \cdot \nabla g. \end{aligned} \tag{3.4}$$

The last term $E(nt)$ in eq.(3.3) is an negligible error compared to 1st and 2nd terms of RHS in eq.(3.3), and has the following form.

$$E(nt) = \int_0^{nt} \cosh(R(s))b(s)c_1(s)ds + \int_0^{nt} \tanh(R(s))a(s)^3c_2(s)ds + e^{(n)}(t) + C$$

where C is a non-random constant and

$$\begin{aligned} b(s) &= \frac{1}{n} a(s) + a'(s) + a(s)^{j_0}, \quad j_0 := \min\{j \in \mathbf{N} \mid 1 - j\alpha < 0\} \\ c_1(s), c_2(s) &: \text{ bounded functions} \\ e^{(n)}(t) &\leq C' n^{-\alpha}. \end{aligned}$$

Proof. First of all, we introduce a notation $A \approx B$ meaning that $A - B$ is a sum of an negligible error $E(nt)$ and a martingale N whose quadratic variation is negligible compared to that of M in eq.(3.4):

$$A \approx B \stackrel{def}{\iff} A - B = E(nt) + N_t, \quad d\langle N \rangle_t \leq C \cdot na(nt)^3 dt.$$

By the integral equation (2.1), we have

$$\begin{aligned}
 R(nt) - R(0) &= \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \frac{d}{ds} (\theta_s(\kappa_c) - \theta_s(\kappa_d)) ds \\
 &= \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \left\{ \frac{c-d}{n} + \frac{1}{2\kappa_c} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s)F(X_s) \right. \\
 &\quad \left. - \frac{1}{2\kappa_d} \operatorname{Re} \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s)F(X_s) \right\} ds \\
 &= \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \frac{c-d}{n} ds \\
 &\quad + \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \frac{1}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_d)} \right) a(s)F(X_s) ds \\
 &\quad + \left(\frac{1}{2\kappa_c} - \frac{1}{2\kappa_0} \right) \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s)F(X_s) ds \\
 &\quad - \left(\frac{1}{2\kappa_d} - \frac{1}{2\kappa_0} \right) \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \operatorname{Re} \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s)F(X_s) ds \\
 &=: I + \dots + IV.
 \end{aligned}$$

By (3.1), I is equal to the 1st term of RHS in eq.(3.3). Since $\kappa_c^{-1} - \kappa_0^{-1} = O(n^{-1})$, the integrands of III, IV are equal to $\cosh(R(s)) \cdot a(s)n^{-1}$ multiplied by bounded functions so that $III, IV \approx 0$. Hence it suffices to compute the 2nd term II which has the following form:

$$\begin{aligned}
 II &= \frac{1}{2\kappa_0} \operatorname{Re}[\Delta J] = \frac{1}{2\kappa_0} \operatorname{Re}[J(\kappa_c) - J(\kappa_d)] \\
 \text{where } J(\kappa) &:= \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} e^{2i\theta_s(\kappa)} a(s)F(X_s) ds.
 \end{aligned}$$

In order to compute $J(\kappa)$ we introduce

$$J(k; j; H)(\kappa) := \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} e^{2ik\theta_s(\kappa)} a(s)^j H(X_s) ds$$

for $k \in \mathbf{Z}, j \geq 1$, and $H \in C^\infty(M)$. By Proposition 7.3(1) we have

$$\begin{aligned}
 \Delta J &= \Delta J(1; 1; F) \approx \frac{1}{\kappa_0} \langle F \cdot g_{\kappa_0} \rangle \int_0^{nt} \cos(\Delta\theta_s) a(s)^2 ds \\
 &\quad - \frac{2i}{2\kappa_0} \left\{ \frac{1}{2} \Delta J(2; 2; Fg_{\kappa_0}) - \Delta J(1; 2; Fg_{\kappa_0}) \right\} + N_t \quad (3.5)
 \end{aligned}$$

where we set $g_{\kappa_0} := R_{2\kappa_0} F$. N is a martingale such that

$$\begin{aligned}
 \langle N, N \rangle_t &= o \left(\int_0^{nt} a(s)^2 ds \right) \\
 \langle N, \bar{N} \rangle_t &= 4 \langle \psi \rangle \int_0^{nt} a(s)^2 ds (1 + o(1)), \quad \psi := [g_{\kappa_0}, \bar{g}_{\kappa_0}]
 \end{aligned}$$

as $n \rightarrow \infty$. By (3.1), the 1st term of RHS in eq.(3.5) is equal to the 2nd term of RHS in eq.(3.3). For the 2nd term of RHS in eq.(3.5), we use Theorem 7.3(2). Noting that $J(0; j; H)$ is independent of κ so that $\Delta J(0; j; H) = 0$, we can repeatedly use

Theorem 7.3(2) for $(j_0 - 1)$ - times to obtain the sum of negligible terms of the form: $\Delta J(k; j_0; H) \approx 0$. Therefore

$$\Delta J(2; 2; Fg_{\kappa_0}) \approx 0, \quad \Delta J(1; 2; Fg_{\kappa_0}) \approx 0.$$

Set M to be the sum of $(2\kappa_0)^{-1}ReN$ and all other martingales appeared in the above argument, after taking real part and multiplying $(2\kappa_0)^{-1}$. Then M satisfies eq.(3.4). \square

4 A comparison argument

In this section we consider $\tilde{R} := R - e^{(n)}$, carry out scaling and time-change, and bound from above and below by the diffusions R_{\pm} which obey simple SDE's (4.1), (4.2). We first prepare some notations. Let

$$\tilde{R}(nt) := R(nt) - e^{(n)}(t).$$

$e^{(n)}(t)$ is an error term appeared in Proposition 3.1. Moreover set

$$\begin{aligned} \gamma &:= \frac{1}{1 - 2\alpha} > 1, \\ \delta &= Cn^{-\alpha}, \quad \epsilon = Cn^{-\beta}, \\ \beta &:= \min\{\alpha, j_0\alpha - 1\} = j_0\alpha - 1, \quad C > 0, \\ \cosh_+(r) &:= \sup_{|s-r|<\delta} \cosh s, \quad \cosh_-(r) := \inf_{|s-r|<\delta} \cosh s \\ \tanh_+(r) &:= \sup_{|s-r|<\delta} \tanh s, \quad \tanh_-(r) := \inf_{|s-r|<\delta} \tanh s \\ \tanh_{+,\epsilon}(r) &:= \begin{cases} (1 + \epsilon) \tanh_+(r) & (r > -\delta) \\ (1 - \epsilon) \tanh_+(r) & (r < -\delta) \end{cases} \\ \tanh_{-,\epsilon}(r) &:= \begin{cases} (1 - \epsilon) \tanh_-(r) & (r > \delta) \\ (1 + \epsilon) \tanh_-(r) & (r < \delta) \end{cases} \\ C_n &:= \frac{1}{\kappa_0} \left(\frac{\langle \psi_{\kappa_0} \rangle}{2} \right)^{1/2} \gamma^{\frac{1}{2}} n^{\frac{1}{2\gamma}}. \end{aligned}$$

We consider diffusions R_{\pm} which are the solutions to

$$dR_+ = \left(\lambda(1 + \epsilon) \cosh_+ R_+ \gamma t^{\gamma-1} + \frac{C_n^2}{2} \tanh_{+,\epsilon} R_+ \right) dt + C_n dW_t \tag{4.1}$$

$$dR_- = \left(\lambda(1 - \epsilon) \cosh_- R_- \gamma t^{\gamma-1} + \frac{C_n^2}{2} \tanh_{-,\epsilon} R_- \right) dt + C_n dW_t \tag{4.2}$$

where W_t is a standard Brownian motion starting at 0. Then we have a following bound on \tilde{R} .

Proposition 4.1. *There is a time change $\tau(t)$ with*

$$\tau'(t) = 1 + o(1), \quad n \rightarrow \infty$$

uniformly with respect to $\omega \in \Omega$ such that

$$R_-(t) \leq \tilde{R}(n\tau(t)^\gamma) \leq R_+(t) \tag{4.3}$$

provided the initial values coincide.

Proof. We consider $R(nt^\gamma)$ instead of $R(nt)$ and change variables: $s = nv^\gamma$ in eq.(3.3).

$$R(nt^\gamma) = \lambda \int_0^t \cosh(R(nv^\gamma)) \cdot \gamma v^{\gamma-1} dv$$

$$\begin{aligned}
 & + \frac{1}{2\kappa_0} \operatorname{Re} \left(-\frac{\langle Fg_{\kappa_0} \rangle}{\kappa_0} \right) \int_0^t na(nv^\gamma)^2 \tanh(R(nv^\gamma)) \cdot \gamma v^{\gamma-1} dv \\
 & + M_{nt^\gamma} + E(nt^\gamma) \\
 d\langle M, M \rangle_{nt^\gamma} & = \left(\frac{1}{2\kappa_0} \right)^2 \cdot 2\langle \psi_{\kappa_0} \rangle \cdot na(nt^\gamma)^2 \cdot \gamma t^{\gamma-1} (1 + o(1)) dt, \quad n \rightarrow \infty.
 \end{aligned}$$

We note $\langle \psi_{\kappa_0} \rangle = -2\operatorname{Re}\langle Fg_{\kappa_0} \rangle$ and let

$$D_n := \frac{1}{\kappa_0} \left(\frac{\langle \psi_{\kappa_0} \rangle}{2} \right)^{1/2}, \quad C_n := D_n (\gamma n^{1-2\alpha})^{1/2} = D_n \gamma^{\frac{1}{2}} n^{\frac{1}{2\gamma}}.$$

Then

$$\begin{aligned}
 R(nt^\gamma) & = \lambda \int_0^t \cosh(R(nv^\gamma)) \gamma v^{\gamma-1} dv \\
 & \quad + \frac{D_n^2}{2} \int_0^t \tanh(R(nv^\gamma)) \cdot na(nv^\gamma)^2 \cdot \gamma v^{\gamma-1} dv + M_{nt^\gamma} + E(nt^\gamma) \\
 d\langle M, M \rangle_{nt^\gamma} & = C_n^2 (1 + o(1)) dt, \quad n \rightarrow \infty.
 \end{aligned}$$

Let $N_t := M_{nt^\gamma}/C_n$ and take

$$\tau(t) := \inf \{s \mid \langle N \rangle_s > t\}.$$

Then $W_t := N_{\tau(t)}$ is a Brownian motion, $\tau'(t) \xrightarrow{n \rightarrow \infty} 1 + o(1)$ uniformly with respect to $\omega \in \Omega$, and

$$\begin{aligned}
 R(n\tau(t)^\gamma) & = \lambda \int_0^{\tau(t)} \cosh(R(nv^\gamma)) \gamma v^{\gamma-1} dv \\
 & \quad + \frac{D_n^2}{2} \int_0^{\tau(t)} \tanh(R(nv^\gamma)) \cdot na(nv^\gamma)^2 \cdot \gamma v^{\gamma-1} dv + C_n W_t + E(n\tau(t)^\gamma).
 \end{aligned}$$

Let

$$\tilde{R}(nt) := R(nt) - e^{(n)}(t), \quad \tilde{E}(nt) := E(nt) - e^{(n)}(t).$$

Then

$$\begin{aligned}
 \tilde{R}(n\tau(t)^\gamma) & = \lambda \int_0^{\tau(t)} \cosh \left(\tilde{R}(nv^\gamma) + e^{(n)}(v^\gamma) \right) \gamma v^{\gamma-1} dv \\
 & \quad + \frac{D_n^2}{2} \int_0^{\tau(t)} \tanh \left(\tilde{R}(nv^\gamma) + e^{(n)}(v^\gamma) \right) \cdot na(nv^\gamma)^2 \cdot \gamma v^{\gamma-1} dv \\
 & \quad + C_n W_t + \tilde{E}(n\tau(t)^\gamma) + C. \tag{4.4}
 \end{aligned}$$

Take $t_0 > 0$ small enough. The contribution from $\tilde{E}(nt^\gamma)$ for $t \leq t_0$ is bounded which we ignore. For $t \geq t_0$,

$$\begin{aligned}
 \tilde{E}(nt^\gamma) & = \int_0^t \cosh \left(\tilde{R}(nv^\gamma) + e^{(n)}(v^\gamma) \right) b(nv^\gamma) c_1(nv^\gamma) n \gamma v^{\gamma-1} dv \\
 & \quad + \int_0^t \tanh \left(\tilde{R}(nv^\gamma) + e^{(n)}(nv^\gamma) \right) a(nv^\gamma)^3 c_2(nv^\gamma) n \gamma v^{\gamma-1} dv \\
 d\tilde{E}(nt^\gamma) & \leq \cosh(R(nt^\gamma)) \left\{ \frac{1}{n} a(nt^\gamma) + a'(nt^\gamma) + a(nt^\gamma)^{j_0} \right\} c_1(nt^\gamma) n \gamma t^{\gamma-1} dt \\
 & \quad + |\tanh(R(nt^\gamma))| a(nt^\gamma)^3 c_2(nt^\gamma) n \gamma t^{\gamma-1} dt
 \end{aligned}$$

$$\begin{aligned} &\leq C \cosh(R(nt^\gamma)) \left(\frac{1}{n} \cdot (nt^\gamma)^{-\alpha} + (nt^\gamma)^{-\alpha-1} + (nt^\gamma)^{-\alpha j_0} \right) n\gamma t^{\gamma-1} dt \\ &\quad + C |\tanh(R(nt^\gamma))| (nt^\gamma)^{-3\alpha} n\gamma t^{\gamma-1} dt \\ &\leq C n^{-\beta} \cosh(R(nt^\gamma)) t^{\gamma-1} dt + O(n^{1-3\alpha}) |\tanh(R_{nt^\gamma})| t^{(1-3\alpha)\gamma-1} dt \end{aligned}$$

where $\beta := \min\{\alpha, j_0\alpha - 1\} = j_0\alpha - 1$. Thus in eq.(4.4), $\tilde{E}(n\tau(t)^\gamma)$ is lower order compared to the 1st and the 2nd terms, and then by the comparison theorem, we have

$$R_-(t) \leq \tilde{R}(n\tau(t)^\gamma) \leq R_+(t). \quad \square$$

5 Allez-Dumaz analysis

In this section, we show, along the argument in [1], that (i) the marginal $\xi_L(I)$ ($I = [\lambda_1, \lambda_2]$) of ξ_L converges to Poisson distribution, and (ii) the joint limit of $\xi_L(I_1), \dots, \xi_L(I_N)$ are independent.

Propositions and lemmas in this section can be proved in the same manner as in [1] by putting $\beta = n^{-\frac{1}{\gamma}}$, but we give proofs of them in Appendix II for the sake of completeness.

5.1 Preliminary: explosion time of stationary approximation

In this subsection we study the explosion time of the stationary approximation S_\pm of R_\pm which are the solution to another SDE's (5.1) where the coefficient $\gamma t^{\gamma-1}$ in the drift term in eq.(4.1), (4.2) are replaced by 1:

$$dS_\pm = \left(\lambda(1 \pm \epsilon) \cosh_\pm(S_\pm) + \frac{C_n^2}{2} \tanh_{\pm, \epsilon}(S_\pm) \right) dt + C_n dW_t. \quad (5.1)$$

If $|S_\pm| > \delta$, the drift term of these SDE's are just the constant multiples of the shift of \cosh, \tanh , so that the analysis in [1] also works. Because the potential corresponding to the drift term in SDE (5.1) has a barrier between the local minimum in the well and the local maximum, we have a "memory-loss effect" so that the explosion time converges to the exponential distribution. More precisely, let ζ_\pm be the explosion time of S_\pm and let

$$\begin{aligned} t_n^{(\pm)}(r) &:= \mathbf{E}[\zeta_\pm | S_\pm(0) = r] \\ g_n^{(\pm)}(r) &:= \mathbf{E}[e^{-\xi \cdot \frac{\lambda}{\pi} \zeta_\pm} | S_\pm(0) = r] \end{aligned}$$

be the expectation value and the Laplace transform of ζ_\pm conditioned $S_\pm(0) = r$ respectively. We then have

Proposition 5.1.

$$\begin{aligned} \lim_{r \downarrow -\infty} \lim_{n \rightarrow \infty} t_n^{(\pm)}(r) &= \frac{\pi}{\lambda} \\ \lim_{r \downarrow -\infty} \lim_{n \rightarrow \infty} g_n^{(\pm)}(r) &= \frac{1}{1 + \xi}. \end{aligned}$$

5.2 Poisson convergence for marginals

In this subsection, we prove that the marginal $\xi_L(I)$ of ξ_L on an interval I converges to a Poisson distribution by showing that the jump points of the function $t \mapsto \lfloor \Theta_{n\tau(t)^\gamma} \rfloor$ converges to a Poisson process. This will be done by dividing the time interval $[0, 1]$ into small random ones I_k and approximating R_\pm by S_\pm on each I_k 's. In order that such approximation work, we need to show that $\{\Theta_{n\tau(t)^\gamma}(\lambda)\}_\pi$ is sufficiently small on sufficiently large portion of the time interval, which is guaranteed by Lemma 5.4. In order to prove Lemma 5.4, we need some estimates on the explosion time for

$$R^{(n)}(t) := \tilde{R}(n\tau(t)^\gamma)$$

which are done in Lemmas 5.2, 5.3. Lemmas 5.5, 5.6 are rephrase of Lemmas 5.2, 5.4 respectively. Since $\tau'(t) = 1 + o(1)$ uniformly in $\omega \in \Omega$, all statements in this subsection are also valid for $\tilde{R}(nt^\gamma)$. Let

$$T_r := \inf \left\{ s \mid R^{(n)}(s) = r \right\}$$

be the hitting time of $R^{(n)}$ to $r \in \mathbf{R} \cup \{+\infty\}$. We denote by \mathbf{P}_{r_0, t_0} the law of $R^{(n)}$ conditioned $R^{(n)}(t_0) = r_0$. If $t_0 = 0$, we simply write $\mathbf{P}_{r_0, t_0} = \mathbf{P}_{r_0}$.

Lemma 5.2. *Let $0 < \epsilon < 1, c > \gamma + \frac{1}{2}$. Then we can find a constant $c' > 0$ such that*

$$\mathbf{P}_{\epsilon \log n^{\frac{1}{\gamma}}} \left(T_{+\infty} < \frac{5c}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \geq 1 - n^{-\frac{c'}{\gamma}}.$$

Idea of proof: (i) we derive the probability of the event that $R^{(n)}$ reaches $c \log n^{\frac{1}{\gamma}}$ before hitting $\frac{\epsilon}{2} \log n^{\frac{1}{\gamma}}$, by the time $\frac{4c}{C_n^2} \log n^{\frac{1}{\gamma}}$. Since the drift term is bounded from below by $\frac{1}{4} C_n^2 dt$, this is possible provided the Brownian motion term satisfies $C_n \inf\{W_t \mid 0 \leq t \leq \frac{4c}{C_n^2} \log n^{\frac{1}{\gamma}}\} \geq -\frac{\epsilon}{2} \log n^{\frac{1}{\gamma}}$ which happens with probability $\geq 1 - n^{-\frac{c'}{\gamma}}$. (ii) Once $R^{(n)}$ reaches $c \log n^{\frac{1}{\gamma}}$, it explodes by the time $\frac{c}{C_n^2} \log n^{\frac{1}{\gamma}}$ which can be proved by studying the explosion time of an ODE explicitly.

Lemma 5.3.

$$\mathbf{P}_{-\frac{1}{4} \log n^{\frac{1}{\gamma}}} \left(T_{+\infty} < \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \geq n^{-\frac{1}{2\gamma}}.$$

Idea of Proof: on account of Lemma 5.2 with $\epsilon = 1/4$, it is sufficient to estimate the probability $\mathbf{P}_{-\frac{1}{4} \log n^{\frac{1}{\gamma}}} \left(T_{\frac{1}{4} \log n^{\frac{1}{\gamma}}} < \frac{1}{C_n^2} \log n^{\frac{1}{\gamma}} \right)$ which can be done similarly by the idea (i) for Lemma 5.2.

Lemma 5.4. *Let*

$$\Xi_n(t) := \mathbf{E}_{-\infty} \left[\int_0^t 1 \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}} \right) du \right].$$

Then we can find a constant C such that

$$\Xi_n(t) \leq C n^{-\frac{1}{2\gamma}} \log n^{\frac{1}{\gamma}}.$$

Idea of Proof: by Lemma 5.3, if $R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}}$, we have $T_{+\infty} < \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}}$, that is, it will explode by the time $\frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}}$, with a good probability. Hence the quantity inside the expectation in the definition of $\Xi_n(t)$ is bounded from above by the number of explosions multiplied by $\frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}}$. On the other hand, the expectation value of the number of explosions is bounded from above.

We shall study the distribution of the jump points of the function $t \mapsto \lfloor \Theta_{n\tau(t)\gamma}(\lambda)/\pi \rfloor$. The corresponding point process is defined by

$$\tilde{\mu}_\lambda^{(n)} := \sum_k \delta_{\tilde{\zeta}_k^\lambda}$$

where $\tilde{\zeta}_k^\lambda := \inf \{ t \in [0, 1] \mid \Theta_{n\tau(t)\gamma}(\lambda) \geq k\pi \}$.

Then the statements of Lemma 5.2, 5.4 have the following form.

Lemma 5.5. *Let $0 < \epsilon < 1, c > \gamma + \frac{1}{2}$. Then conditioned on $\{\Theta_0(\lambda)\}_\pi = \pi - 2 \arctan n^{-\frac{\epsilon}{\gamma}}$, we have*

$$\mathbf{P} \left(\tilde{\zeta}_1^\lambda < \frac{5c}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \geq 1 - n^{-\frac{c'}{\gamma}}.$$

Lemma 5.6. *Let*

$$\Xi_n(t) := \mathbf{E} \left[\int_0^t 1 \left(\Theta_{n\tau(u)\gamma}(\lambda) \geq 2 \arctan n^{-\frac{1}{4\gamma}} \right) du \right].$$

Then we can find a constant C such that

$$\Xi_n(t) \leq C n^{-\frac{1}{2\gamma}} \log n^{\frac{1}{\gamma}}.$$

We can now prove that the jump points of the function $t \mapsto \lfloor \Theta_{n\tau(t)\gamma}(\lambda)/\pi \rfloor$ converges to a Poisson process.

Proposition 5.7.

$$\tilde{\mu}_\lambda^{(n)} \xrightarrow{d} \text{Poisson} \left(\frac{\lambda}{\pi} \gamma t^{\gamma-1} 1_{[0,1]}(t) dt \right)$$

and the same statement also holds for the point process $\mu_\lambda^{(n)}$ whose atoms consist of

$$\zeta_k^\lambda := \inf \{ t \in [0, 1] \mid \Theta_{nt^\gamma}(\lambda) \geq k\pi \}.$$

Idea of Proof: Let

$$I_k := \left[\frac{T_k}{N}, \frac{T_{k+1}}{N} \right] \quad \text{where} \quad T_k := \sum_{i=1}^k \tau_i, \quad \tau_i = \text{unif} \left(\frac{1}{2}, \frac{3}{2} \right).$$

Let $S_\pm^{(n)}$ be the solution to the following SDE's where the constant λ in SDE (5.1) is replaced by $\gamma \left(\frac{T_{k+1}}{N} \right)^{\gamma-1}$, $\gamma \left(\frac{T_k}{N} \right)^{\gamma-1}$ respectively:

$$dS_\pm^{(n)} = \left(\lambda_k^\pm (1 \pm \epsilon) \cosh_\pm(S_\pm^{(n)}) + \frac{C_n^2}{2} \tanh_{\pm, \epsilon}(S_\pm^{(n)}) \right) dt + C_n dB_t, \quad t \in I_k$$

$$\text{where} \quad \lambda_k^+ = \gamma \left(\frac{T_{k+1}}{N} \right)^{\gamma-1}, \quad \lambda_k^- = \gamma \left(\frac{T_k}{N} \right)^{\gamma-1}$$

with initial values $S_\pm^{(n)} \left(\frac{T_k}{N} \right) := R^{(n)} \left(\frac{T_k}{N} \right)$ on each I_k . We remark that, once $S_\pm^{(n)}$ explode to $+\infty$, it starts at $-\infty$ again and so on. Let $\Theta_\pm^{(n)}$ defined by

$$S_\pm^{(n)} = \log \tan \frac{\Theta_\pm^{(n)}}{2},$$

in other words, $\Theta_\pm^{(n)} := 2 \arctan e^{S_\pm^{(n)}}$. Then by eq.(4.3) and using comparison theorem between S_\pm and R_\pm ,

$$\Theta_{-,t}^{(n)}(\lambda) \leq \Theta_{n\tau(t)\gamma}(\lambda) \leq \Theta_{+,t}^{(n)}(\lambda).$$

Thus we can estimate the number of jump points of $\lfloor \Theta_{n\tau(t)\gamma}(\lambda)/\pi \rfloor$ from above and below by those of $\lfloor \Theta_{\pm,t}^{(n)}(\lambda)/\pi \rfloor$. By Lemma 5.6 and by the definition of T_k , on each starting point of the interval I_k , we can suppose $\Theta_{n\tau(t)\gamma}(\lambda) \leq 2 \arctan n^{-\frac{1}{4\gamma}}$ with a good probability, so that by Proposition 5.1, the explosion time of $\Theta_\pm^{(n)}$ converges to the exponential distribution on each intervals, which proves the statement of Proposition 5.7 for $\Theta_{n\tau(t)\gamma}(\lambda)$. Since $\tau'(t) = 1 + o(1)$ uniformly in $\omega \in \Omega$, the same statement also holds for $\mu_\lambda^{(n)}$.

Remark 5.8. Let $\lambda < \lambda'$ and let

$$\mu_{\lambda, \lambda'}^{(n)} := \sum_k \delta_{\zeta_k^{\lambda, \lambda'}}$$

where $\zeta_k^{\lambda, \lambda'} := \inf \{t \in [0, 1] \mid \Theta_{nt^\gamma}(\lambda') - \Theta_{nt^\gamma}(\lambda) \geq k\pi\}$.

We can apply all the arguments in previous sections for $\Theta_{nt^\gamma}(\lambda') - \Theta_{nt^\gamma}(\lambda)$ yielding

$$\mu_{\lambda, \lambda'}^{(n)} \xrightarrow{d} \text{Poisson} \left(\frac{\lambda' - \lambda}{\pi} \gamma t^{\gamma-1} 1_{[0,1]} dt \right).$$

5.3 Limiting coupled Poisson process

For $0 < \lambda < \lambda'$, let $P_\lambda := \lim_{n \rightarrow \infty} \mu_\lambda^{(n)}$, $P_{\lambda'} := \lim_{n \rightarrow \infty} \mu_{\lambda'}^{(n)}$, $P_{\lambda, \lambda'} := \lim_{n \rightarrow \infty} \mu_{\lambda, \lambda'}^{(n)}$ be the limiting Poisson processes described in Proposition 5.7 and Remark 5.8. In this subsection, we show that (i) they are realized jointly as \mathcal{F}_t -Poisson processes under suitable filtration (Lemma 5.10), (ii) the sets $\mathcal{P}_\lambda, \mathcal{P}_{\lambda'}, \mathcal{P}_{\lambda, \lambda'}$ of corresponding atoms satisfy $\mathcal{P}_\lambda \subset \mathcal{P}_{\lambda'}$ (Lemma 5.11), and (iii) $\mathcal{P}_\lambda \cap \mathcal{P}_{\lambda, \lambda'} = \emptyset$ (Lemma 5.12). The independence of $P_\lambda, P_{\lambda, \lambda'}$ (and thus independence of finite number of marginals of ξ_L on intervals) then follows from those observations. But first of all we need to show that the “fractional part” of $\Theta(\lambda), \Theta(\lambda')$ also obey the same ordering as λ, λ' for sufficiently large portions in time (Lemma 5.9). We recall $\{x\}_\pi := x - \lfloor x/\pi \rfloor \pi$.

Lemma 5.9. Let $0 < \lambda < \lambda'$ and

$$\Upsilon_n(t) := \mathbf{E} \left[\int_0^t 1(\{\Theta_{nu^\gamma}(\lambda')\}_\pi \leq \{\Theta_{nu^\gamma}(\lambda)\}_\pi) du \right]$$

then we can find a constant C such that

$$\Upsilon_n(t) \leq C n^{-\frac{c'}{\gamma}}.$$

Idea of Proof: let

$$\begin{aligned} \mathcal{E}_u &:= \{ \{\Theta_{nu^\gamma}(\lambda')\}_\pi \leq \{\Theta_{nu^\gamma}(\lambda)\}_\pi \}, \quad u \in [0, 1] \\ \zeta_u &:= \sup \left\{ \zeta_k^{\lambda'} \mid \zeta_k^{\lambda'} \leq u \right\}, \quad \zeta_k^{\lambda'} := \inf \{t \in [0, 1] \mid \Theta_{nt^\gamma}(\lambda') \geq k\pi\} \\ u_0 &:= u - \frac{5c}{C_n^2} \log n^{\frac{1}{\gamma}}, \quad c > \gamma + \frac{1}{2}. \end{aligned}$$

On the event \mathcal{E}_u , we consider the following three possibilities.

- (i) the latest jump of the function $t \mapsto \lfloor \Theta_{nt^\gamma}(\lambda')/\pi \rfloor$ before u occurs after u_0
- (ii) the latest jump of $\lfloor \Theta_{nt^\gamma}(\lambda')/\pi \rfloor$ before u occurs before u_0 , and

$$\{\Theta_{nu_0^\gamma}(\lambda)\}_\pi \leq 2 \arctan n^{-\frac{1}{4\gamma}},$$

- (iii) the latest jump of $\lfloor \Theta_{nt^\gamma}(\lambda')/\pi \rfloor$ before u occurs before u_0 , and

$$\{\Theta_{nu_0^\gamma}(\lambda)\}_\pi > 2 \arctan n^{-\frac{1}{4\gamma}}.$$

Then

- (i) the probability of the event (i) is bounded from above by $n^{-\frac{1}{2\gamma}} \log n^{\frac{1}{\gamma}} \cdot \mathbf{E}[\mu_\lambda^n[0, t]]$.
- (ii) Let $\tilde{\zeta}_{2\pi}$ be the explosion time of $\Theta_{nt^\gamma}(\lambda, \lambda') := \Theta_{nt^\gamma}(\lambda') - \Theta_{nt^\gamma}(\lambda)$ for which we can carry out the arguments in previous sections. Then in Case (ii) we must have $\tilde{\zeta}_{2\pi} \geq \frac{5c}{C_n^2} \log n^{\frac{1}{\gamma}}$ of which the probability is bounded from above by $n^{-\frac{c'}{\gamma}}$
- (iii) Lemma 5.6 gives the bound on the probability of Case (iii).

In what follows, we set $\lambda < \lambda' < \lambda''$. Since the set of triples $\{(\mu_\lambda^{(n)}, \mu_{\lambda'}^{(n)}, \lambda_{\lambda', \lambda''}^{(n)}), n \geq 0\}$ is tight as a set of Radon measures on \mathbf{R}_+ , we can find a subsequence (n_k) such that

$$(\mu_\lambda^{(n_k)}, \mu_{\lambda'}^{(n_k)}, \lambda_{\lambda', \lambda''}^{(n_k)}) \rightarrow (P_\lambda, P_{\lambda'}, P_{\lambda', \lambda''})$$

where $P_\lambda, P_{\lambda'}, P_{\lambda', \lambda''}$ are Poisson processes which turn out to be independent of the choice of convergent subsequences.

Lemma 5.10. *Let*

$$\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$$

$$\mathcal{F}_t := \sigma(P_\lambda(s), P_{\lambda'}(s), P_{\lambda', \lambda''}(s); 0 \leq s \leq t).$$

Then $P_\lambda, P_{\lambda'}, P_{\lambda', \lambda''}$ are the (\mathcal{F}_t) -Poisson processes whose intensity measures are equal to $\pi^{-1} \lambda \gamma t^{\gamma-1} 1_{[0,1]}(t) dt$, $\pi^{-1} \lambda' \gamma t^{\gamma-1} 1_{[0,1]}(t) dt$, and $\pi^{-1} (\lambda'' - \lambda') \gamma t^{\gamma-1} 1_{[0,1]}(t) dt$ respectively.

Let $\mathcal{P}_\lambda, \mathcal{P}_{\lambda'}, \mathcal{P}_{\lambda', \lambda''}$ be the set of atoms of $P_\lambda, P_{\lambda'}, P_{\lambda', \lambda''}$ respectively.

Lemma 5.11. *If $\lambda < \lambda'$, $\mathcal{P}_\lambda \subset \mathcal{P}_{\lambda'}$ a.s.*

Idea of Proof: suppose that there are no atoms of $\mu_{\lambda'}^{(n)}$ near the atom ξ of $\mu_\lambda^{(n)}$ for large n . Then we should have $\{\Theta_{nt^\gamma}(\lambda')\}_\pi < \{\Theta_{nt^\gamma}(\lambda)\}_\pi$ near ξ of which the probability is estimated from above by Lemma 5.9.

Lemma 5.12. *We have $\mathcal{P}_\lambda \cap \mathcal{P}_{\lambda', \lambda''} = \emptyset$. Hence by Lemma 5.10, \mathcal{P}_λ and $\mathcal{P}_{\lambda', \lambda''}$ are independent.*

Idea of Proof: we shall show $\mathcal{P}_\lambda \cap \mathcal{P}_{\lambda, \lambda'} = \emptyset$. Otherwise, we can find an atom ξ of $\mu_{\lambda, \lambda'}^{(n)}$ near those of $\mu_\lambda^{(n)}$ for large n . If we have $\{\Theta_{nt^\gamma}(\lambda')\}_\pi < \{\Theta_{nt^\gamma}(\lambda)\}_\pi$ near ξ , this probability is estimated from above by Lemma 5.9. If, on the contrary, we have $\{\Theta_{nt^\gamma}(\lambda')\}_\pi \geq \{\Theta_{nt^\gamma}(\lambda)\}_\pi$, then $\lfloor \Theta_{nt^\gamma}(\lambda')/\pi \rfloor$ jumps twice in a neighborhood of ξ . Since the jump points of $\lfloor \Theta_{nt^\gamma}(\lambda')/\pi \rfloor$ converges to a Poisson process, the probability of such events are relatively small.

By using these lemmas, we can show

Proposition 5.13. *Let $\nu^{(n)}$ be a point process on \mathbf{R} defined by*

$$\nu^{(n)}[\lambda_1, \lambda_2] = \left\lfloor \frac{\Theta_n(\lambda_2) - \Theta_n(\lambda_1)}{\pi} \right\rfloor$$

then

$$\nu^{(n)} \xrightarrow{d} \text{Poisson} \left(\frac{d\lambda}{\pi} \right).$$

6 Proof of Theorems

6.1 Proof of Theorem 2

The first statement (1) of Theorem 1.6 can be proved in the same manner as [7] Proposition 7.1: the only major difference is to show

$$\lim_{t \rightarrow \infty} \int_1^t s^{-3\alpha} \exp \left(- \int_s^t u^{-2\alpha} du \right) ds = 0$$

which is straightforward. For the second statement (2) of Theorem 1.6, we summarize the facts obtained in previous sections.

(1) Let

$$\zeta^{(n)}(\lambda) := \sum_j \delta_{\tau_j^{(n)}(\lambda)}$$

where $\tau_j^{(n)}(\lambda) := \inf \{t \in [0, 1] \mid \Theta_{nt}(\lambda) = j\pi\}$.

Then by Proposition 5.7

$$\zeta^{(n)}(\lambda) \rightarrow Q_\lambda := \text{Poisson} \left(\frac{\lambda}{\pi} 1_{[0,1]} dt \right).$$

In other words, the function $t \mapsto \lfloor \Theta_{nt}(\lambda)/\pi \rfloor$ converges to a Poisson jump process.

(2) By Proposition 2.4, $\mathbf{E}[\Theta_{nt}(\lambda)] \rightarrow \lambda t$.

(3) For $0 < \lambda < \lambda'$, let

$$\zeta^{(n)}(\lambda, \lambda') = \sum_j \delta_{\tau_j^{(n)}(\lambda, \lambda')}$$

$$\text{where } \tau_j^{(n)}(\lambda, \lambda') := \inf \{t \in [0, 1] \mid \Theta_{nt}(\lambda') - \Theta_{nt}(\lambda) = j\pi\}.$$

Then

$$\zeta^{(n)}(\lambda, \lambda') \rightarrow Q_{\lambda, \lambda'} := \text{Poisson} \left(\frac{\lambda' - \lambda}{\pi} 1_{[0,1]} dt \right)$$

and Q_λ and $Q_{\lambda, \lambda'}$ are independent.

By (1), (2), we have

$$\mathbf{E} \left[\left\lfloor \frac{\Theta_{nt}(\lambda)}{\pi} \right\rfloor \right] \rightarrow \frac{\lambda}{\pi} t, \quad \mathbf{E} \left[\frac{\Theta_{nt}(\lambda)}{\pi} \right] \rightarrow \frac{\lambda}{\pi} t$$

so that, writing

$$\frac{\Theta_\lambda(t)}{\pi} = \left\lfloor \frac{\Theta_\lambda(t)}{\pi} \right\rfloor + \epsilon_t^{(n)}, \quad \epsilon_t^{(n)} \geq 0$$

we have $\mathbf{E}[\epsilon_t^{(n)}] \rightarrow 0$ which implies $\epsilon_t^{(n)} \rightarrow 0$ in probability⁴. It follows that $t \mapsto \Theta_{nt}(\lambda)/\pi$ also converges to a Poisson jump process, and in particular,

$$\widehat{\Theta}_t(\lambda) := \lim_{n \rightarrow \infty} \Theta_{nt}(\lambda)$$

takes values in $\pi\mathbf{Z}$ for a.e. t . Moreover, by Remark 2.1 and Lemma 5.11, $\widehat{\Theta}_t(\lambda)$ is non-decreasing with respect to (t, λ) , so that it is a distribution function of a point process η on \mathbf{R}^2 whose marginals on rectangles have Poisson distribution. Let

$$N(t_1, t_2; \lambda_1, \lambda_2) = \left(\widehat{\Theta}_{t_2}(\lambda_2) - \widehat{\Theta}_{t_1}(\lambda_2) \right) - \left(\widehat{\Theta}_{t_2}(\lambda_1) - \widehat{\Theta}_{t_1}(\lambda_1) \right)$$

be the number of atoms of η in $[t_1, t_2] \times [\lambda_1, \lambda_2]$. By Lemma 5.12,

$$N(t_1, t'_1; \lambda_1, \lambda'_1), \dots, N(t_n, t'_n; \lambda_n, \lambda'_n)$$

are independent obeying $\text{Poisson}(\pi^{-1}(\lambda'_j - \lambda_j)(t'_j - t_j))$, $j = 1, 2, \dots, n$ which proves the statement (2) of Theorem 1.6.

6.2 Proof of Theorem 1.1

By Proposition 5.13, we have

$$(\Theta_n(c_i) - \Theta_n(d_i), i = 1, \dots, k) \xrightarrow{d} (\widehat{\Theta}_1(c_i) - \widehat{\Theta}_1(d_i), i = 1, \dots, k)$$

for any $k \in \mathbf{N}$, $c_i, d_i \in \mathbf{R}$ and $\widehat{\Theta}_1(\cdot)$ is a Poisson jump process. By [7] Lemma 9.1,

$$\Theta_n(\cdot) \xrightarrow{d} \widehat{\Theta}_1(\cdot)$$

⁴ In [14], they showed that, for $\beta \leq 2$, $\alpha_t(\lambda)$ converges to $\alpha_\infty(\lambda)$ from above which is consistent with this argument.

as a non-decreasing function valued process. By Skorohod's theorem, we may suppose that

$$\Theta_n(c) \rightarrow \widehat{\Theta}_1(c), \quad a.s.$$

at any continuity point of $\widehat{\Theta}_1(c)$. Fix a.s. $\omega \in \Omega$, $K \in \mathbf{N}$, $\epsilon > 0$ and let τ_1, τ_2, \dots be the jump points of $\widehat{\Theta}_1(\cdot)$. Then for large n ,

$$\begin{aligned} |\Theta_n(\tau_k - \epsilon) - (k - 1)\pi| &< \epsilon \\ |\Theta_n(\tau_k + \epsilon) - k\pi| &< \epsilon, \quad k = 1, 2, \dots, K. \end{aligned}$$

By the monotonicity of $\Theta_n(\cdot)$, if $\Theta_n(\tau_k - \epsilon) < y < \Theta_n(\tau_k + \epsilon)$, we have

$$|(\Theta_n)^{-1}(y) - \tau_k| < \epsilon$$

so that, if $(k - 1)\pi + \epsilon < y < k\pi - \epsilon$, we have

$$|(\Theta_n)^{-1}(y) - \tau_k| < \epsilon.$$

Let $\Xi(y)$ be the inverse of the Poisson jump process $\widehat{\Theta}_1(\cdot)$ (it may be set to take arbitrary values at the discontinuity points). Since $\widehat{\phi}_t$ is uniformly distributed on $[0, \pi)$, its distribution never have a atom at 0 so that, taking $n \rightarrow \infty$ in (1.1), we have

$$\mathbf{E}[e^{-\xi_L(f)}] \rightarrow \mathbf{E} \left[\exp \left(- \sum_{n \in \mathbf{Z}} f(\Xi(n\pi + \theta)) \right) \right] = \mathbf{E}[e^{-\zeta_P(f)}]$$

where $\zeta_P = Poisson(\pi^{-1}d\lambda)$.

7 Appendix I

In this section we prepare some estimates necessary to prove Proposition 3.1. The basic strategy of our computation is that, for the terms whose integrand contains a factor of the form $e^{i\kappa s} H(X_s) ds$ ($\kappa \neq 0$), we use eq.(2.3) and perform the integration by parts to obtain the terms whose integrands are multiplied by $a(s)$ or $a'(s)$ so that they have better decay. We may continue this process as many times we need to finally obtain the negligible terms. On the other hand, for the terms with $H(X_s) ds$ (that is, $\kappa = 0$), we use eq.(2.4) instead to obtain the 2nd term of RHS in eq.(3.3).

We first consider the following quantity which often appears in the computation of $J(k; j; H)$.

$$\begin{aligned} K(k, l; j; H) &:= \int_0^{nt} \sin(\Delta\theta_s) e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} a(s)^j H(X_s) ds, \\ k, l \in \mathbf{Z}, \quad j \in \mathbf{N}, \quad H \in C^\infty(M). \end{aligned}$$

Lemma 7.1. *Suppose $(k, l) \neq (0, 0)$ and $j \geq 2$. Then*

$$\begin{aligned} &K(k, l; j; H) \\ &\approx -\frac{2ik}{2\kappa_0} \left\{ \frac{1}{2} \cdot K(k + 2, l; j + 1; FR_{2(k+l)\kappa_0} H) + \frac{1}{2} \cdot K(k - 2, l; j + 1; FR_{2(k+l)\kappa_0} H) \right. \\ &\quad \left. - K(k, l; j + 1; FR_{2(k+l)\kappa_0} H) \right\} \\ &\quad -\frac{2il}{2\kappa_0} \left\{ \frac{1}{2} \cdot K(k, l + 2; j + 1; FR_{2(k+l)\kappa_0} H) + \frac{1}{2} \cdot K(k, l - 2; j + 1; FR_{2(k+l)\kappa_0} H) \right. \\ &\quad \left. - K(k, l; j + 1; FR_{2(k+l)\kappa_0} H) \right\}. \end{aligned}$$

Proof. We note

$$2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d) = 2i(k+l)\kappa_0s + \frac{2i(ck+dl)}{n}s + 2ik\tilde{\theta}_s(\kappa_c) + 2il\tilde{\theta}_s(\kappa_d).$$

Using (2.3) with $\kappa = 2(k+l)\kappa_0$ and $f = H$, we have

$$\begin{aligned} & K(k, l; j; H) \\ &= \int_0^{nt} \sin(\Delta\theta_s) \exp \left[\frac{2i(ck+dl)}{n}s + 2ik\tilde{\theta}_s(\kappa_c) + 2il\tilde{\theta}_s(\kappa_d) \right] a(s)^j e^{2i(k+l)\kappa_0s} H(X_s) ds \\ &= \int_0^{nt} \sin(\Delta\theta_s) \exp \left[\frac{2i(ck+dl)}{n}s + 2ik\tilde{\theta}_s(\kappa_c) + 2il\tilde{\theta}_s(\kappa_d) \right] a(s)^j \\ &\quad \times \left\{ d \left(e^{2i(k+l)\kappa_0s} R_{2(k+l)\kappa_0} H(X_s) \right) - e^{2i(k+l)\kappa_0s} \nabla R_{2(k+l)\kappa_0} H(X_s) dX_s \right\} \end{aligned}$$

For simplicity, we set

$$\tilde{H} := R_{2(k+l)\kappa_0} H.$$

Integration by parts yields

$$\begin{aligned} & K(k, l; j; H) \\ &= \left[\sin(\Delta\theta_s) e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} a(s)^j \tilde{H}(X_s) \right]_0^{nt} \\ &\quad - \int_0^{nt} \cos(\Delta\theta_s) \\ &\quad \times \left\{ \frac{c-d}{n} + \frac{1}{2\kappa_c} \operatorname{Re} \left[e^{2i\theta_s(\kappa_c)} - 1 \right] a(s) F(X_s) - \frac{1}{2\kappa_d} \operatorname{Re} \left[e^{2i\theta_s(\kappa_d)} - 1 \right] a(s) F(X_s) \right\} \\ &\quad \times e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} a(s)^j \tilde{H}(X_s) ds \\ &\quad - \int_0^{nt} \sin(\Delta\theta_s) \\ &\quad \times \left\{ \frac{2i(ck+dl)}{n} + \frac{2ik}{2\kappa_c} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s) F(X_s) + \frac{2il}{2\kappa_d} \operatorname{Re} \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s) F(X_s) \right\} \\ &\quad \times e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} a(s)^j \tilde{H}(X_s) ds \\ &\quad - \int_0^{nt} \sin(\Delta\theta_s) e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} (a(s)^j)' \tilde{H}(X_s) ds \\ &\quad - \int_0^{nt} \sin(\Delta\theta_s) e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} a(s)^j \nabla \tilde{H}(X_s) dX_s \\ &=: K_1 + \dots + K_5. \end{aligned}$$

Then $K_1 = O(n^{-\alpha}) \approx 0$. Since $j \geq 2$, K_2, K_4 is included in $E(nt)$ and thus negligible: $K_2, K_4 \approx 0$. K_5 is a martingale with negligible quadratic variation: $\langle K_5 \rangle = O\left(\int_0^{nt} a(s)^{2j} ds\right)$ so that $K_5 \approx 0$. Therefore

$$K(j; k, l; H) \approx K_3.$$

In the integrand of K_3 , the 1st term has $O(n^{-1})$ factor and thus negligible. In the 2nd and 3rd terms, we can replace $2ik/2\kappa_c, 2il/2\kappa_d$ by $2ik/2\kappa_0, 2il/2\kappa_0$ respectively which produces negligible $O(n^{-1})$ error. Hence

$$\begin{aligned}
 & K_3 \\
 \approx & - \int_0^{nt} \sin(\Delta\theta_s) \\
 & \times \left\{ \frac{2ik}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s)F(X_s) + \frac{2il}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s)F(X_s) \right\} \\
 & \times e^{2ik\theta_s(\kappa_c) + 2il\theta_s(\kappa_d)} a(s)^j \tilde{H}(X_s) ds \\
 = & - \frac{2ik}{2\kappa_0} \left\{ \frac{1}{2} \cdot K(k+2, l; j+1; F \cdot \tilde{H}) + \frac{1}{2} \cdot K(k-2, l; j+1; F \cdot \tilde{H}) - K(k, l; j+1; F \cdot \tilde{H}) \right\} \\
 & - \frac{2il}{2\kappa_0} \left\{ \frac{1}{2} \cdot K(k, l+2; j+1; F \cdot \tilde{H}) + \frac{1}{2} \cdot K(k, l-2; j+1; F \cdot \tilde{H}) - K(k, l; j+1; F \cdot \tilde{H}) \right\}.
 \end{aligned}$$

□

Lemma 7.2. Suppose $(k, l) \neq (0, 0)$ and $j \geq 2$. Then

$$K(k, l; j; H) - K(l, k; j; H) \approx 0.$$

Proof. We compute each terms by Lemma 7.1. If we have terms of the form $K(0, 0; j+1; H')$, it equally comes from the 1st and 2nd terms and cancels each other. Therefore the terms of the form $K(k', l'; j+1; H')$ with $(k', l') \neq (0, 0)$ only remain so that we can continue to use Lemma 7.1 at least for $(j_0 - j)$ - times so that the quantity in question is equal to the sum of the terms of the form $K(k', l'; j_0; H')$ which are negligible. □

Here we recall the definition of $J(k; j; H)$:

$$J(k; j; H)(\kappa) := \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} e^{2ik\theta_s(\kappa)} a(s)^j H(X_s) ds$$

where $k \in \mathbf{Z}$, $j \geq 1$, and $H \in C^\infty(M)$. We compute $J(k; j; H)$ by using Lemmas 7.1, 7.2.

Proposition 7.3.

(1) $j = 1, k = 1$:

$$\begin{aligned}
 \Delta J(k; j; H) \approx & \frac{1}{\kappa_0} \langle F \cdot R_{2k\kappa_0} H \rangle \int_0^{nt} \cos(\Delta\theta_s) a(s)^{j+1} ds \\
 & - \frac{2ik}{2\kappa_0} \left\{ \frac{1}{2} \Delta J(k+1; j+1; FR_{2k\kappa_0} H) - \Delta J(k; j+1; FR_{2k\kappa_0} H) \right\} \\
 & + M_t
 \end{aligned} \tag{7.1}$$

where M is a martingale whose quadratic variation satisfies

$$\begin{aligned}
 \langle M, M \rangle_t &= o \left(\int_0^{nt} a(s)^{2j} ds \right) \\
 \langle M, \overline{M} \rangle_t &= 4 \langle \psi \rangle \int_0^{nt} a(s)^{2j} ds (1 + o(1)), \quad \psi := [R_{2k\kappa_0}(H), \overline{R_{2k\kappa_0}(H)}].
 \end{aligned}$$

(2) $j \geq 2, k \neq 0$:

$$\begin{aligned}
 \Delta J(k; j; H) \approx & - \frac{2ik}{2\kappa_0} \left\{ \frac{1}{2} \Delta J(k+1; j+1; FR_{2k\kappa_0} H) + \frac{1}{2} \Delta J(k-1; j+1; FR_{2k\kappa_0} H) \right. \\
 & \left. - \Delta J(k; j+1; FR_{2k\kappa_0} H) \right\}.
 \end{aligned} \tag{7.2}$$

Proof. We use (2.3) with $k = 2k\kappa_0$. Setting $\tilde{H} := R_{2k\kappa_0}H$ for simplicity, we have

$$\begin{aligned}
 J(k; j; H)(\kappa_x) &= \left[\frac{1}{\sin(\Delta\theta_s)} e^{2ik\theta_s(\kappa_x)} a(s)^j \tilde{H}(X_s) \right]_0^{nt} \\
 &+ \int_0^{nt} \frac{\cos(\Delta\theta_s)}{\sin^2(\Delta\theta_s)} \left\{ \frac{c-d}{n} + \frac{1}{2\kappa_c} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s) F(X_s) \right. \\
 &\quad \left. - \frac{1}{2\kappa_d} \operatorname{Re} \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s) F(X_s) \right\} e^{2ik\theta_s(\kappa_x)} a(s)^j \tilde{H}(X_s) ds \\
 &- \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \left\{ 2ik \cdot \frac{x}{n} + \frac{2ik}{2\kappa_x} \operatorname{Re} \left(e^{2i\theta_s(\kappa_x)} - 1 \right) a(s) F(X_s) \right\} \\
 &\quad \times e^{2ik\theta_s(\kappa_x)} a(s)^j \tilde{H}(X_s) ds \\
 &- \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} e^{2ik\theta_s(\kappa_x)} (a(s)^j)' \tilde{H}(X_s) ds \\
 &- \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} e^{2ik\theta_s(\kappa_x)} a(s)^j \nabla \tilde{H}(X_s) dX_s \\
 &=: J_1 + \dots + J_5.
 \end{aligned}$$

We estimate $\Delta J_1, \dots, \Delta J_5$ separately. It will turn out that $\Delta J_1, \Delta J_4$ are negligible, ΔJ_2 is equal to the 1st term of RHS in (7.1) modulo error, ΔJ_3 is equal to the 2nd term of RHS in (7.1) or is equal to RHS in (7.2).

(1) J_1 : By an elementary equality

$$e^{2i\theta_1} - e^{2i\theta_2} = 2i \sin(\theta_1 - \theta_2) e^{i\theta_1 + i\theta_2} \tag{7.3}$$

we have

$$\Delta J_1 = \left[\frac{1}{\sin(\Delta\theta_s)} \cdot 2i \sin(k\Delta\theta_s) e^{ik(\theta_s(\kappa_c) + \theta_s(\kappa_d))} a(s)^j \tilde{H}(X_s) \right]_0^{nt}.$$

Therefore $\Delta J_1 = O(n^{-j\alpha}) + C \approx 0$.

(2) J_2 : we separate the discussion into the following two cases.

(i) $j \geq 2$: as in the proof of Lemma 7.1, we may ignore the term with $(c-d)/n$ factor and replace $1/2\kappa_c, 1/2\kappa_d$ by $1/2\kappa_0$:

$$J_2 \approx \int_0^{nt} \frac{\cos(\Delta\theta_s)}{\sin^2(\Delta\theta_s)} \frac{1}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_d)} \right) e^{2ik\theta_s(\kappa_x)} a(s)^{j+1} (F \cdot \tilde{H})(X_s) ds.$$

And we compute ΔJ_2 using (7.3):

$$\begin{aligned}
 \Delta J_2 &\approx \int_0^{nt} \frac{\cos(\Delta\theta_s)}{\sin^2(\Delta\theta_s)} \frac{1}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_d)} \right) \left(e^{2ik\theta_s(\kappa_c)} - e^{2ik\theta_s(\kappa_x)} \right) \\
 &\quad \times a(s)^{j+1} (F \cdot \tilde{H})(X_s) ds \\
 &= \int_0^{nt} \frac{\cos(\Delta\theta_s)}{\sin(\Delta\theta_s)} \cdot \sin(k\Delta\theta_s) \frac{1}{2\kappa_0} \operatorname{Re} \left[2ie^{i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} \right] \left(2ie^{ik(\theta_s(\kappa_c) + \theta_s(\kappa_d))} \right) \\
 &\quad \times a(s)^{j+1} (F \cdot \tilde{H})(X_s) ds
 \end{aligned}$$

which is negligible if $j \geq 2$: $\Delta J_2 \approx 0$.

(ii) $j = 1, k = 1$: we further decompose as follows.

$$\begin{aligned} \Delta J_2 &\approx \frac{1}{\kappa_0} \int_0^{nt} \cos(\Delta\theta_s) \left(1 - e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))}\right) a(s)^{j+1} (F \cdot \tilde{H})(X_s) ds \\ &=: \Delta J_{2-1} + \Delta J_{2-2}. \end{aligned}$$

For ΔJ_{2-1} , we use (2.4):

$$\begin{aligned} \Delta J_{2-1} &= \frac{1}{\kappa_0} \int_0^{nt} \cos(\Delta\theta_s) a(s)^{j+1} \left\{ \langle F \cdot \tilde{H} \rangle - d \left(R_0(F \cdot \tilde{H}) \right) - \nabla R_0(F \cdot \tilde{H}) dX_s \right\} \\ &=: \Delta J_{2-1-1} + \dots + \Delta J_{2-1-3}. \end{aligned}$$

ΔJ_{2-1-1} already has the desired form. For ΔJ_{2-1-2} , integration by parts yields

$$\begin{aligned} \Delta J_{2-1-2} &= \frac{1}{\kappa_0} \left\{ \left[\cos(\Delta\theta_s) a(s)^{j+1} R_0(F \cdot \tilde{H})(X_s) \right]_0^{nt} \right. \\ &\quad + \int_0^{nt} \sin(\Delta\theta_s) \left\{ \frac{c-d}{n} + \frac{1}{2\kappa_c} Re \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s) F(X_s) \right. \\ &\quad \left. \left. - \frac{1}{2\kappa_d} Re \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s) F(X_s) \right\} a(s)^{j+1} R_0(F \cdot \tilde{H})(X_s) ds \right. \\ &\quad \left. - \int_0^{nt} \cos(\Delta\theta_s) (a(s)^{j+1})' R_0(F \cdot \tilde{H})(X_s) ds. \right. \end{aligned}$$

As in the proof of Lemma 7.1, 1st and 3rd terms are negligible ; in the 2nd term, the term with $(c-d)/n$ factor is also negligible and $1/2\kappa_c, 1/2\kappa_d$ may be replaced by $1/2\kappa_0$ up to negligible error:

$$\begin{aligned} \Delta J_{2-1-2} &\approx \frac{1}{\kappa_0} \int_0^{nt} \sin(\Delta\theta_s) \frac{1}{2\kappa_0} Re \left(e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_d)} \right) a(s)^{j+2} F \cdot R_0(F \cdot \tilde{H})(X_s) ds \\ &= \frac{1}{\kappa_0} \cdot \frac{1}{2\kappa_0} Re \left\{ K(2, 0; j+2; F \cdot R_0(F \cdot \tilde{H})) - K(0, 2; j+2; F \cdot R_0(F \cdot \tilde{H})) \right\} \\ &\approx 0. \end{aligned}$$

In the last step, we used Lemma 7.2. For ΔJ_{2-1-3} ,

$$\langle \Delta J_{2-1-3}, \Delta J_{2-1-3} \rangle = O \left(\int_0^{nt} a(s)^{2j+2} \right) = o \left(\int_0^{nt} a(s)^{2j} \right)$$

so that $\Delta J_{2-1-3} \approx 0$. Therefore, we have

$$\Delta J_{2-1} \approx \Delta J_{2-1-1} = \frac{1}{\kappa_0} \langle F \cdot \tilde{H} \rangle \int_0^{nt} \cos(\Delta\theta_s) a(s)^{j+1} ds.$$

For ΔJ_{2-2} , we use (2.3) with $\kappa = 4\kappa_0$, perform the integration by parts, estimate as before, and use Lemma 7.2:

$$\begin{aligned} \Delta J_{2-2} &= -\frac{1}{\kappa_0} \left\{ \left[\cos(\Delta\theta_s) e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} a(s)^{j+1} R_{4\kappa_0}(F \cdot \tilde{H})(X_s) \right]_0^{nt} \right. \\ &\quad + \int_0^{nt} \sin(\Delta\theta_s) \left\{ \frac{c-d}{n} + \frac{1}{2\kappa_c} Re \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s) F(X_s) \right. \\ &\quad \left. \left. - \frac{1}{2\kappa_d} Re \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s) F(X_s) \right\} e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} \right. \end{aligned}$$

$$\begin{aligned}
 & \times a(s)^{j+1} R_{4\kappa_0}(F \cdot \tilde{H})(X_s) ds \\
 & - \int_0^{nt} \cos(\Delta\theta_s) \left\{ 2i \cdot \frac{c+d}{n} + \frac{1}{2\kappa_c} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - 1 \right) a(s) F(X_s) \right. \\
 & \quad \left. + \frac{1}{2\kappa_d} \operatorname{Re} \left(e^{2i\theta_s(\kappa_d)} - 1 \right) a(s) F(X_s) \right\} e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} \\
 & \times a(s)^{j+1} R_{4\kappa_0}(F \cdot \tilde{H})(X_s) ds \\
 & - \int_0^{nt} \cos(\Delta\theta_s) e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} (a(s)^{j+1})' R_{4\kappa_0}(F \cdot \tilde{H})(X_s) ds \\
 & - \int_0^{nt} \cos(\Delta\theta_s) e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} a(s)^{j+1} \nabla R_{4\kappa_0}(F \cdot \tilde{H})(X_s) dX_s \left. \right\} \\
 & \approx -\frac{1}{\kappa_0} \int_0^{nt} \sin(\Delta\theta_s) \frac{1}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_d)} \right) e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} \\
 & \quad \times a(s)^{j+2} F \cdot R_{4\kappa_0}(F \cdot \tilde{H})(X_s) ds \\
 & = -\frac{1}{\kappa_0} \cdot \frac{1}{2\kappa_0} \cdot \frac{1}{2} \\
 & \quad \times \left\{ K(4, 2; j+2; F \cdot R_{4\kappa_0}(F \cdot \tilde{H})) + K(0, 2; j+2; F \cdot R_{4\kappa_0}(F \cdot \tilde{H})) \right. \\
 & \quad \left. - K(2, 4; j+2; F \cdot R_{4\kappa_0}(F \cdot \tilde{H})) - K(2, 0; j+2; F \cdot R_{4\kappa_0}(F \cdot \tilde{H})) \right\} \\
 & \approx 0.
 \end{aligned}$$

To summarize:

$$\Delta J_2 \approx \frac{1}{\kappa_0} \langle F \cdot \tilde{H} \rangle \int_0^{nt} \cos(\Delta\theta_s) a(s)^{j+1} ds.$$

(3) J_3 : after cutting out negligible terms we have

$$\begin{aligned}
 J_3 & \approx - \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \frac{2ik}{2\kappa_0} \operatorname{Re} \left(e^{2i\theta_s(\kappa_x)} - 1 \right) e^{2ik\theta_s(\kappa_x)} a(s)^{j+1} (F \cdot \tilde{H})(X_s) ds \\
 & = -\frac{2ik}{2\kappa_0} \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \left(\frac{e^{2i(k+1)\theta_s(\kappa_x)} + e^{2i(k-1)\theta_s(\kappa_x)}}{2} - e^{2ik\theta_s(\kappa_x)} \right) \\
 & \quad \times a(s)^{j+1} (F \cdot \tilde{H})(X_s) ds \\
 & = -\frac{2ik}{2\kappa_0} \left\{ \frac{1}{2} J(k+1; j+1; F\tilde{H})(\kappa_x) + \frac{1}{2} J(k-1; j+1; F\tilde{H})(\kappa_x) \right. \\
 & \quad \left. - J(k; j+1; F\tilde{H})(\kappa_x) \right\}.
 \end{aligned}$$

(4) J_4 : this is clearly negligible:

$$\Delta J_4 = - \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} \cdot 2i \sin(k\Delta\theta_s) e^{ik(\theta_s(\kappa_c) + \theta_s(\kappa_d))} (a(s)^j)' \tilde{H}(X_s) ds \approx 0.$$

(5) J_5 : using (7.3) we have

$$\Delta J_5 = - \int_0^{nt} \frac{1}{\sin(\Delta\theta_s)} 2i \sin(k\Delta\theta_s) e^{ik(\theta_s(\kappa_c) + \theta_s(\kappa_d))} a(s)^j \nabla \tilde{H}(X_s) dX_s$$

We consider the following two cases.

(i) $k = 1$: setting

$$\varphi := [\widetilde{H}, \widetilde{H}], \quad \psi := [\widetilde{H}, \overline{\widetilde{H}}],$$

we have

$$\begin{aligned} \langle \Delta J_5, \Delta J_5 \rangle &= (-4) \int_0^{nt} e^{2i(\theta_s(\kappa_c) + \theta_s(\kappa_d))} a(s)^{2j} \varphi(X_s) ds \\ \langle \Delta J_5, \overline{\Delta J_5} \rangle &= 4 \int_0^{nt} a(s)^{2j} \psi(X_s) ds. \end{aligned}$$

to which we apply (2.3), (2.4) respectively. By the same argument as in the estimate of $\Delta J_2, \Delta J_3$ we have

$$\begin{aligned} \langle \Delta J_5, \Delta J_5 \rangle &= o\left(\int_0^{nt} a(s)^{2j} ds\right) \\ \langle \Delta J_5, \overline{\Delta J_5} \rangle &= 4\langle \psi \rangle \int_0^{nt} a(s)^{2j} ds (1 + o(1)), \quad n \rightarrow \infty. \end{aligned}$$

(ii) $k \geq 2$: by a direct computation, it is easy to see

$$\langle \Delta J_5, \Delta J_5 \rangle, \langle \Delta J_5, \overline{\Delta J_5} \rangle = O\left(\int_0^{nt} a(s)^{2j} ds\right)$$

so that $\Delta J_5 \approx 0$ for $j \geq 2$. □

8 Appendix II

In Appendix II, we provide the proofs of Proposition 5.1 and statements in Section 5 for the sake of completeness, all of which are done by tracing those in [1].

Proof of Proposition 5.1. We discuss the computation of $t_n^{(+)}(r)$ only, for $t_n^{(-)}(r)$ can be treated similarly. We write eq.(5.1) as in the following manner:

$$\begin{aligned} dS_+ &= -W_+(S_+)dt + C_n dW_t \\ \text{where } -W_+(r) &:= \lambda(1 + \epsilon) \cosh_+ r + \frac{C_n^2}{2} \tanh_{+, \epsilon} r. \end{aligned}$$

Then

$$\begin{aligned} -V_+(r) &:= \lambda(1 + \epsilon) \{ \sinh(r \pm \delta) \mp \sinh \delta \} 1(\pm r > 0) \\ &\quad + \frac{C_n^2}{2} (1 \pm \epsilon) \log \frac{\cosh(r + \delta)}{\cosh \delta} 1(\pm r > -\delta) \end{aligned}$$

satisfies $V_+'(r) = W_+(r)$ for $r \neq 0, -\delta$. We first derive the critical points $r = a_n, b_n$ such that $W_+(r) = 0$:

$$\begin{aligned} a_n &= \delta + \log \frac{\tilde{\lambda}}{C_n^2} + O(C_n^{-2}) \\ b_n &= -\frac{2\tilde{\lambda}}{C_n^2} \cosh(2\delta)(1 + O(C_n^{-2})) - \delta \end{aligned}$$

where $\tilde{\lambda} := (1 + \epsilon)\lambda/(1 - \epsilon)$. Moreover we have

$$V_+(a_n + y) = -\lambda(1 + \epsilon) \left\{ \frac{\tilde{\lambda}}{2C_n^2} e^{y + \delta \pm \delta + O(C_n^{-2})} - \frac{C_n^2}{2\tilde{\lambda}} e^{-y + \delta \pm \delta + O(C_n^{-2})} \mp \sinh \delta \right\}$$

$$\begin{aligned}
 & \times 1(\pm(a_n + x) > 0) \\
 & - \frac{C_n^2}{2}(1 \pm \epsilon) \left\{ \log \left(\frac{\tilde{\lambda}}{2C_n^2} e^{y+2\delta+O(C_n^{-2})} + \frac{C_n^2}{2\tilde{\lambda}} e^{-y-2\delta+O(C_n^{-2})} \right) - \log \cosh \delta \right\} \\
 & \times 1(\pm(a_n + x) > -\delta) \\
 V_+(b_n + x) = & -\lambda(1 + \epsilon) \left\{ \sinh \left(x - \delta \pm \delta - \frac{2\tilde{\lambda}}{C_n^2} \cosh(2\delta)(1 + O(C_n^{-2})) \right) \mp \sinh \delta \right\} \\
 & \times 1(\pm(b_n + x) > 0) \\
 & - \frac{C_n^2}{2}(1 \pm \epsilon) \left\{ \log \cosh \left(x - \frac{2\tilde{\lambda}}{C_n^2} \cosh(2\delta)(1 + O(C_n^{-2})) \right) - \log \cosh \delta \right\} \\
 & \times 1(\pm(b_n + x) > -\delta).
 \end{aligned}$$

Since $t_n^{(+)}(r)$ satisfies

$$\frac{C_n^2}{2} f'' - W_+(r) f' = -1, \quad f(\infty) = 0,$$

we have

$$t_n^{(+)}(r) = \frac{2}{C_n^2} \int_r^\infty dx \int_{-\infty}^x dy \exp \left\{ \frac{2}{C_n^2} (V_+(x) - V_+(y)) \right\}.$$

Substituting above equations, we have

$$\begin{aligned}
 t_n^+(r) = & \frac{2}{C_n^2} \int_{r-b}^\infty dx \\
 & \exp \left[-\frac{2}{C_n^2} \lambda(1 + \epsilon) \left\{ \sinh \left(x - \delta \pm \delta - \frac{2\tilde{\lambda}}{C_n^2} \cosh(2\delta)(1 + O(C_n^{-2})) \right) \mp \sinh \delta \right\} \right. \\
 & \left. \times 1(\pm(b_n + x) > 0) \right] \\
 & \cdot \left\{ \frac{\cosh \delta}{\cosh \left(x - \frac{2\tilde{\lambda}}{C_n^2} \cosh(2\delta)(1 + O(C_n^{-2})) \right)} \right\}^{1 \pm \epsilon} 1(\pm(b_n + x) > -\delta) \\
 & \times \int_{-\infty}^{(b-a)+x} dy \\
 & \exp \left[\left\{ \frac{\lambda \cdot \tilde{\lambda}(1 + \epsilon)}{C_n^4} e^{y+\delta \pm \delta + O(C_n^{-2})} - \frac{\lambda(1 + \epsilon)}{\tilde{\lambda}} e^{-y+\delta \pm \delta + O(C_n^{-2})} \mp \frac{2}{C_n^2} \cdot \lambda(1 + \epsilon) \cdot \sinh \delta \right\} \right. \\
 & \left. \times 1(\pm(a_n + x) > 0) \right] \\
 & \cdot \left(\frac{\tilde{\lambda}}{2C_n^2} e^{y+2\delta+O(C_n^{-2})} + \frac{C_n^2}{2\tilde{\lambda}} e^{-y-2\delta+O(C_n^{-2})} \right)^{1 \pm \epsilon} \cdot \frac{1}{(\cosh \delta)^{1 \pm \epsilon}} \cdot 1(\pm(a_n + x) > -\delta)
 \end{aligned}$$

Noting that $\epsilon \rightarrow 0$, $\tilde{\lambda} \rightarrow \lambda$, $a_n \rightarrow -\infty$, $b_n \rightarrow 0$ as $n \rightarrow \infty$, we have

$$t_n^{(+)}(r) \xrightarrow{n \rightarrow \infty} \frac{1}{\lambda} \int_r^\infty \frac{dx}{\cosh x} \int_{-\infty}^\infty dy e^{-y-e^{-y}} = \frac{1}{\lambda} \int_r^\infty \frac{dx}{\cosh x}.$$

Thus

$$\lim_{r \downarrow -\infty} \lim_{n \rightarrow \infty} t_n^{(+)}(r) = \frac{\pi}{\lambda}.$$

The statement for the Laplace transform is derived by the same way as in the proof of Proposition 2.2 [1]. \square

Proof of Lemma 5.2. LHS of the inequality in question is bounded from below by

$$\begin{aligned} LHS &\geq \mathbf{P}_{\epsilon \log n^{\frac{1}{\gamma}}} \left(T_{c \log n^{\frac{1}{\gamma}}} < \frac{4c}{C_n^2} \log n^{\frac{1}{\gamma}} \wedge T_{\frac{\epsilon}{2} \log n^{\frac{1}{\gamma}}} \right) \\ &\quad \times \mathbf{P}_{c \log n^{\frac{1}{\gamma}}, \frac{4c}{C_n^2} \log n^{\frac{1}{\gamma}}} \left(T_{+\infty} < \frac{c}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \\ &=: (1) \times (2). \end{aligned}$$

which we estimate separately.

(1) If $\frac{\epsilon}{2} \log n^{\frac{1}{\gamma}} < r < c \log n^{\frac{1}{\gamma}}$, the drift term of the SDE for R_- satisfies (drift term) $\geq \frac{1}{2} C_n^2 \tanh r \geq \frac{1}{4} C_n^2$ so that the first factor (1) is bounded from below by the probability of the following event.

$$\mathcal{E} := \left\{ \inf_{0 < t < 4 \frac{c}{C_n^2} \log n^{\frac{1}{\gamma}}} C_n B_t \geq -\frac{\epsilon}{2} \log n^{\frac{1}{\gamma}} \right\}$$

where B_t is a Brownian motion with $B_0 = 0$. By the reflection principle, we have

$$\mathbf{P}(\mathcal{E}) = \mathbf{P} \left(C_n \left| B \left(\frac{4c}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \right| \leq \frac{\epsilon}{2} \log n^{\frac{1}{\gamma}} \right) \geq 1 - \left(n^{-\frac{1}{\gamma}} \right)^{c'}.$$

(2) Let

$$\tilde{\mathcal{E}} := \left\{ \sup_{0 \leq t \leq \frac{c}{C_n^2} \log n^{\frac{1}{\gamma}}} C_n |B(t)| < \frac{\epsilon}{2} \log n^{\frac{1}{\gamma}} \right\}.$$

Then $\mathbf{P}(\tilde{\mathcal{E}}) \geq 1 - \left(n^{-\frac{1}{\gamma}} \right)^{c''}$ for some $c'' > 0$, and under the event $\tilde{\mathcal{E}}$, $G(t) := R(t) - C_n B(t)$ satisfies

$$\begin{aligned} G'(t) &\geq \frac{\lambda}{2} e^{G(t)} \cdot e^{-\frac{\epsilon}{2} \log n^{\frac{1}{\gamma}}} \cdot \gamma \cdot \left(\frac{4c}{C_n^2} \log n^{\frac{1}{\gamma}} \right)^{\gamma-1} + \frac{C_n^2}{2} \tanh(G(t) + C_n B) \\ &\geq C \cdot \left(n^{-\frac{1}{\gamma}} \right)^{\gamma-1+\frac{\epsilon}{2}} e^{G(t)} - \frac{C_n^2}{2}. \end{aligned}$$

Therefore the explosion time of G satisfies $T_{+\infty} \sim \left(n^{-\frac{1}{\gamma}} \right)^{c-(\gamma-1+\frac{\epsilon}{2})}$. \square

Proof of Lemma 5.3. LHS of the inequality in question is bounded from below by

$$\begin{aligned} LHS &\geq \mathbf{P}_{-\frac{1}{4} \log n^{\frac{1}{\gamma}}} \left(T_{\frac{1}{4} \log n^{\frac{1}{\gamma}}} < \frac{1}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \\ &\quad \times \mathbf{P}_{\frac{1}{4} \log n^{\frac{1}{\gamma}}, \frac{1}{C_n^2} \log n^{\frac{1}{\gamma}}} \left(T_{+\infty} < 5 \frac{c}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \\ &=: (1) \times (2). \end{aligned}$$

The second factor (2) has been estimated in Lemma 5.2. For the first factor (1), since $R^{(n)}(t) \geq -\frac{C_n^2}{2}t + C_n B_t$ we have

$$(1) \geq \mathbf{P} \left(-\frac{C_n^2}{2} \cdot \frac{1}{C_n^2} \log n^{\frac{1}{\gamma}} + C_n B_{\frac{1}{C_n^2} \log n^{\frac{1}{\gamma}}} \geq \frac{1}{2} \log n^{\frac{1}{\gamma}} \right) \geq C \left(n^{-\frac{1}{\gamma}} \right)^{1/2}. \quad \square$$

Proof of Lemma 5.4. Conditioning at time u and using the Markov property, we have

$$\begin{aligned} & \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}} \right) \\ & \leq \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}}, T_{+\infty} < \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \\ & \quad + \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}}, T_{+\infty} \geq \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \\ & \leq \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}}, T_{+\infty} < \frac{5c+1}{C^2} \log n^{\frac{1}{\gamma}} \right) \\ & \quad + \left(1 - \left(\frac{1}{n^{\frac{1}{\gamma}}} \right)^{1/2} \right) \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}} \right). \end{aligned}$$

Hence

$$\begin{aligned} & \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}} \right) \\ & \leq \left(n^{\frac{1}{\gamma}} \right)^{1/2} \mathbf{P}_{-\infty} \left(R^{(n)}(u) \geq -\frac{1}{4} \log n^{\frac{1}{\gamma}}, T_{+\infty} < \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}} \right) \\ & \leq \left(n^{\frac{1}{\gamma}} \right)^{1/2} \mathbf{P}_{-\infty} \left(\left[u, u + \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}} \right] \text{ contains at least one explosion} \right). \end{aligned}$$

Let $k := \#\{ \text{explosions of } R^{(n)} \text{ in } [0, t] \}$ with $\{\zeta_j\}_{j=1}^k$ being the explosion points, we have

$$\int_0^t 1 \left(\exists i : \zeta_i \in \left[u, u + \frac{5c+1}{C_n^2} \log n^{\frac{1}{\gamma}} \right] \right) du \leq \frac{5c+1}{C_n^2} \cdot \log n^{\frac{1}{\gamma}} \cdot (k+1).$$

It thus suffices to take the expectation of both sides and use the following inequality:
 $\mathbf{E}[\#\{ \text{explosions of } R^{(n)} \text{ in } [0, t] \}] \leq \mathbf{E}[\Theta_{nt}(\lambda)/\pi] \leq C\lambda t/\pi.$ □

From now on, for the sake of simplicity, we use the following notation.

$$\Theta_{\lambda}^{(n)}(u) := \Theta_{nu^{\gamma}}(\lambda), \quad \Theta_{\lambda, \lambda'}^{(n)}(u) := \Theta_{\lambda'}^{(n)}(u) - \Theta_{\lambda}^{(n)}(u).$$

Proof of Proposition 5.7. It suffices to show,

$$\begin{aligned} (1) \quad & \mathbf{E}[\mu_{\lambda}^{(n)}(I)] \rightarrow \frac{\lambda}{\pi} \int_I \gamma t^{\gamma-1} 1_{[0, 1]} dt \\ (2) \quad & \mathbf{P} \left(\mu_{\lambda}^{(n)}(I) = 0 \right) \rightarrow \exp \left(-\frac{\lambda}{\pi} \int_I \gamma t^{\gamma-1} 1_{[0, 1]} dt \right) \end{aligned}$$

for the finite union $I \subset [0, 1]$ of disjoint intervals. Let

$$\mathcal{C}_k := \left\{ \left\{ \Theta_{\lambda}^{(n)} \left(\frac{T_k}{N} \right) \right\} \leq 2 \arctan \left(n^{-\frac{1}{\gamma}} \right)^{1/4} \right\}, \quad \mathcal{C} := \bigcap_{k=1}^{2N+1} \mathcal{C}_k.$$

Then by Lemma 5.6,

$$\begin{aligned} \mathbf{P}(\mathcal{C}^c) & \leq \sum_{k=0}^{2N+1} \mathbf{P} \left(\left\{ \Theta_{\lambda}^{(n)} \left(\frac{T_k}{N} \right) \right\} > 2 \arctan \left(n^{-\frac{1}{\gamma}} \right)^{1/4} \right) \\ & \leq 2\mathbf{E} \left[\int_0^{3N+3} 1 \left(\Theta_{\lambda}^{(n)} \left(\frac{u}{N} \right) > 2 \arctan \left(n^{-\frac{1}{\gamma}} \right)^{1/4} \right) du \right] \end{aligned}$$

$$\leq \left(n^{-\frac{1}{\gamma}}\right)^{1/2} N \log n^{\frac{1}{\gamma}} \rightarrow 0, \quad n \rightarrow \infty.$$

(1) We may take $I = [0, t]$. Upper bound simply follows from

$$\mathbf{E}[\mu_\lambda^{(n)}[0, t]] = \mathbf{E}\left[\frac{\Theta_\lambda^{(n)}(t)}{\pi}\right] \leq \frac{1}{\pi} \lambda \int_0^t \gamma s^{\gamma-1} ds = \frac{\lambda t^\gamma}{\pi}.$$

For the lower bound, we consider

$$N_k^\pm := \# \left\{ \text{jumps of } \Theta_{\lambda, \pm}^{(n)} \text{ in } I_k \right\}$$

$$N_k := \# \left\{ \text{jumps of } \Theta_\lambda^{(n)} \text{ in } I_k \right\}.$$

Then

$$\mathbf{E}\left[\mu_\lambda^{(n)}[0, t]\right] \geq \sum_{k=0}^{2Nt+1} \mathbf{E}\left[N_k^- \mathbf{1}\left(\frac{T_k}{N} < t\right) \middle| \mathcal{C}_k\right] - \sum_k \mathbf{E}\left[N_k^- \mathbf{1}\left(\frac{T_k}{N} < t\right) \middle| \mathcal{C}_k\right] \mathbf{P}(\mathcal{C}_k^c),$$

the 2nd term of which vanishes as $n \rightarrow \infty$:

$$\begin{aligned} \text{2nd term} &\leq \mathbf{E}\left[\# \left\{ \text{jumps of } \Theta_\lambda^{(n)} \text{ in } [0, 3t] \right\}\right] \times \sup_k \frac{\mathbf{P}(\mathcal{C}_k^c)}{\mathbf{P}(\mathcal{C}_k)} \\ &\leq \mathbf{E}\left[\# \left\{ \text{jumps of } \Theta_\lambda^{(n)} \text{ in } [0, 3t] \right\}\right] \times \frac{\mathbf{P}(\mathcal{C}^c)}{1 - \mathbf{P}(\mathcal{C}^c)} \rightarrow 0. \end{aligned}$$

For the 1st term, we note that $\mathbf{E}[N_k^- | \mathcal{C}_k] = \pi^{-1} \lambda \gamma \left(\frac{T_k}{N}\right)^{\gamma-1} \cdot \frac{\tau_{k+1}}{N}$ by Proposition 5.1. Hence by the convergence of the Riemannian sum to the integral,

$$\sum_k \mathbf{E}[N_k^- | \mathcal{C}_k] = \frac{\lambda}{\pi} \sum_k \gamma \left(\frac{T_k}{N}\right)^{\gamma-1} \cdot \frac{\tau_{k+1}}{N} \mathbf{1}\left(\frac{T_k}{N} < t\right) \rightarrow \frac{\lambda}{\pi} \int_0^t \gamma s^{\gamma-1} ds$$

as $N \rightarrow \infty$.

(2) We first suppose $I = [t_1, t_2]$. Since

$$\mathbf{P}\left(\mu_\lambda^{(n)}[t_1, t_2] = 0\right) \leq \mathbf{E}\left[\prod_{k \geq 0} \mathbf{P}\left[N_k^- = 0 \middle| \mathcal{C}_k, (\tau_i)_i\right] \mathbf{1}\left(\frac{T_{k+1}}{N} \geq t_1, \frac{T_k}{N} \leq t_2\right)\right] + \mathbf{P}(\mathcal{C}^c)$$

and since

$$\mathbf{P}\left[N_k^- = 0 \middle| \mathcal{C}_k\right] \rightarrow \mathbf{E}\left[\exp\left(-\frac{\tau_{k+1}}{N} \cdot \frac{\lambda}{\pi} \cdot \gamma \left(\frac{T_{k+1}}{N}\right)^{\gamma-1}\right)\right]$$

we have

$$\begin{aligned} &\limsup \mathbf{P}\left(\mu_\lambda^{(n)}[t_1, t_2] = 0\right) \\ &\leq \mathbf{E}\left[\prod_{k \geq 0} \exp\left(-\frac{\lambda}{\pi} \gamma \left(\frac{T_{k+1}}{N}\right)^{\gamma-1} \cdot \frac{\tau_{k+1}}{N}\right) \mathbf{1}\left(\frac{T_{k+1}}{N} \geq t_1, \frac{T_k}{N} \leq t_2\right)\right]. \end{aligned}$$

Taking $N \rightarrow \infty$ proves (2) for $I = [t_1, t_2]$. General case easily follows from the Markov property. \square

Proof of Lemma 5.9. We decompose $\mathbf{P}(\mathcal{E}_u)$ as follows.

$$\begin{aligned} \mathbf{P}(\mathcal{E}_u) &\leq \mathbf{P}(\mathcal{E}_u \cap \{\zeta_u \in [u_0, u]\}) + \mathbf{P}(\mathcal{E}_u \cap \{\zeta_u < u_0\}) \\ &\leq \mathbf{P}(\mathcal{E}_u \cap \{\zeta_u \in [u_0, u]\}) + \mathbf{P}\left(\bigcap_{s \in [u_0, u]} \mathcal{E}_s\right) \\ &\leq \mathbf{P}(\{\zeta_u \in [u_0, u]\}) + \mathbf{P}\left(\bigcap_{s \in [u_0, u]} \mathcal{E}_s \cap \left\{\Theta_\lambda^{(n)}(u_0)\right\} \leq 2 \arctan n^{-\frac{1}{4\gamma}}\right) \\ &\quad + \mathbf{P}\left(\left\{\Theta_\lambda^{(n)}(u_0)\right\} \geq 2 \arctan n^{-\frac{1}{4\gamma}}\right) \\ &\leq \mathbf{P}(\{\zeta_u \in [u_0, u]\}) + n^{-\frac{c'}{\gamma}} + \mathbf{P}\left(\left\{\Theta_\lambda^{(n)}(u_0)\right\} \geq 2 \arctan n^{-\frac{1}{4\gamma}}\right) \end{aligned} \tag{8.1}$$

where we used the monotonicity of $[\Theta_{\lambda, \lambda'}^{(n)}]$ in the 2nd inequality. In the last inequality, we used the fact that, when $\{\Theta_\lambda^{(n)}(u_0)\}_\pi \leq 2 \arctan n^{-\frac{1}{4\gamma}}$, we necessarily have $\{\Theta_{\lambda, \lambda'}^{(n)}(u_0)\}_\pi \geq \pi - 2 \arctan n^{-\frac{1}{4\gamma}}$. Hence by Lemma 5.5 we have

$$\mathbf{P}\left(\bigcap_{s \in [u_0, u]} \mathcal{E}_s \cap \left\{\Theta_\lambda^{(n)}(u_0)\right\} \leq 2 \arctan n^{-\frac{1}{4\gamma}}\right) \leq n^{-\frac{c'}{\gamma}},$$

proving the last inequality in (8.1). Now we integrate both sides of (8.1) and use Lemma 5.6 for the 1st and 3rd terms of RHS. \square

For the proof of Lemmas 5.11, 5.12, let $(\xi_i^\lambda), (\xi_i^{\lambda'}), (\xi_i^{\lambda', \lambda''})$ be the atoms of $P_\lambda, P_{\lambda'}, P_{\lambda', \lambda''}$ respectively. Also, let $(\zeta_i^\lambda), (\zeta_i^{\lambda'}), (\zeta_i^{\lambda', \lambda''})$ be the atoms of $\mu_\lambda^{(n)}, \mu_{\lambda'}^{(n)}, \mu_{\lambda', \lambda''}^{(n)}$ respectively.

Proof of Lemma 5.11. For $N \in \mathbf{N}$, let

$$p_N^n := \mathbf{P}\left(\exists i : \zeta_i^\lambda < t, \forall j \geq i, |\zeta_i^\lambda - \zeta_j^{\lambda'}| > \frac{1}{2N}\right).$$

It is then sufficient to show $\limsup_{n \rightarrow \infty} p_N^n = 0$. Let $(T_k)_k$ be the random division of intervals used in the proof of Proposition 5.7. Then we have

$$\begin{aligned} p_N^n &\leq \mathbf{P}\left(\exists k \leq [2Nt] + 1 : \left[\frac{\Theta_\lambda^{(n)}}{\pi}\right] \text{ jumps on } \left[\frac{T_k}{N}, \frac{T_k + 2}{N}\right] \text{ but not } \left[\frac{\Theta_{\lambda'}^{(n)}}{\pi}\right]\right) \\ &\leq \sum_{k=1}^{[2Nt]+1} \mathbf{P}\left(\left\{\Theta_{\lambda'}^{(n)}\left(\frac{T_k}{N}\right)\right\}_\pi \leq \left\{\Theta_\lambda^{(n)}\left(\frac{T_k}{N}\right)\right\}_\pi\right) \end{aligned}$$

where we used the monotonicity of $[\Theta_{\lambda, \lambda'}^{(n)}/\pi]$. It thus suffices to use Lemma 5.9. \square

Proof of Lemma 5.12. As in the proof of Lemma 5.11, it is sufficient to show

$$p_N^n := \mathbf{P}\left(\exists i, j \in \mathbf{N} : \zeta_i^\lambda < t, \zeta_j^{\lambda, \lambda'} < t, |\zeta_i^\lambda - \zeta_j^{\lambda, \lambda'}| < \frac{1}{2N}\right)$$

satisfies $\limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} p_N^n = 0$.

$$p_N^n \leq \mathbf{P}\left(\exists i, j \in \mathbf{N}, \exists k \leq [2Nt] + 1, \frac{T_k}{N} \leq \zeta_i^\lambda, \zeta_j^{\lambda, \lambda'} \leq \frac{T_k + 2}{N}\right)$$

$$\begin{aligned}
 &= \mathbf{P} \left(\exists k \leq [2Nt] + 1 : \left[\frac{\Theta_{\lambda}^{(n)}}{\pi} \right], \left[\frac{\Theta_{\lambda'}^{(n)} - \Theta_{\lambda}^{(n)}}{\pi} \right] \text{ both jump on } \left[\frac{T_k}{N}, \frac{T_k + 2}{N} \right] \right) \\
 &\leq \sum_{k=1}^{[2Nt]+1} \mathbf{P} \left(\left\{ \Theta_{\lambda'}^{(n)} \left(\frac{T_k}{N} \right) \right\}_{\pi} \leq \left\{ \Theta_{\lambda}^{(n)} \left(\frac{T_k}{N} \right) \right\}_{\pi} \right) \\
 &\quad + \sum \mathbf{P} \left(\left[\frac{\Theta_{\lambda'}^{(n)}}{\pi} \right] \text{ jumps more than 2-times on } \left[\frac{T_k}{N}, \frac{T_k + 2}{N} \right] \right) \tag{8.2}
 \end{aligned}$$

where we used the monotonicity of $\lfloor \Theta_{\lambda, \lambda'}^{(n)} / \pi \rfloor$ in the last inequality. The 1st term in RHS of (8.2) has been estimated in the proof of Lemma 5.11. For the 2nd term, we use Proposition 5.7.

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \sum_{k=1}^{[2Nt]+1} \mathbf{P} \left(\mu_{\lambda'}^n \left[\frac{T_k}{N}, \frac{T_k + 2}{N} \right] \geq 2 \right) \\
 &\leq C \sum_{k=1}^{[2Nt]+1} \mathbf{E} \left[\exp \left[-\frac{\lambda}{\pi} \int_{\frac{T_k}{N}}^{\frac{T_k+2}{N}} \gamma u^{\gamma-1} du \right] \cdot \left(-\frac{\lambda}{\pi} \int_{\frac{T_k}{N}}^{\frac{T_k+2}{N}} \gamma u^{\gamma-1} du \right)^2 \right] \\
 &= O(N^{-1}). \quad \square
 \end{aligned}$$

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