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Disorder relevance without Harris Criterion: the case of pinning model with γ-stable environment*

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Abstract

We investigate disorder relevance for the pinning of a renewal whose inter-arrival law has tail exponent $\alpha > 0$ when the law of the random environment is in the domain of attraction of a stable law with parameter $\gamma \in (1,2)$. We prove that in this case, the effect of disorder is not decided by the sign of the specific heat exponent as predicted by Harris criterion but that a new criterion emerges to decide disorder relevance. More precisely we show that when $\alpha > 1 - \gamma^{-1}$ there is a shift of the critical point at every temperature whereas when $\alpha < 1 - \gamma^{-1}$, at high temperature the quenched and annealed critical points coincide, and the critical exponents are identical.

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1 Introduction

It is a common practice in Theoretical Physics to use simplified lattice models in order to study the qualitative behavior of systems with a large number of interacting components. A prototypical example is the usual Ising model on $\mathbb{Z}^d$, which has been used to understand the phenomenon of ferromagnetic transition. The reason why these lattice models are believed to yield a fair approximation of real world phenomena is that critical phenomena such as phase transitions are not supposed to rely on the detailed structure of interaction and should be preserved after replacing the original system by its simpler lattice version. In their simplest expression these models display homogeneous interactions in the sense that the Hamiltonian is invariant by the lattice symmetries.

Since real world materials always present at least some infinitesimal lack of regularity, an important issue to assert the validity of this approach is thus whether the qualitative

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behavior of a lattice system (such as the presence of a phase transition, its order, etc...) remains the same when a small amount of irregularity is introduced. This natural question leads to consider "disordered" versions of the models where the interactions are given by the realization of an ergodic field, the simplest example being the case of a field of independent identically distributed random variables.

An important step in the understanding of the influence of disorder is due to Harris [25] which introduced a celebrated criterion which allows to predict whether a small amount of disorder may or may not change the critical properties of the system. Harris criterion relies on the analysis of the specific-heat exponent near the critical point and is backed by a heuristic which relies on computing the second moment of the partition function.

The validity of this criterion has been checked in a few cases where the homogeneous model is well understood, in particular for the random field Ising model [1, 27]. A case which has generated a rich literature in the past decade is that of pinning of a one dimensional polymer on a defect line: a model where a renewal process with power-law tails has its law modified due to energetic interactions at renewal points [4, 6, 8, 14, 15, 16, 23, 28, 32]. We refer to the monographs [19, 20] for a detailed introduction to the subject.

For this particular family of models, proofs of disorder relevance (and by this we mean: a radical change in the critical properties of the system) [6, 14, 15, 23] or irrelevance [4, 28, 32] have been given, depending on the value of the exponent associated to the renewal times, all of which confirmed the validity of Harris criterion.

The heuristics behind Harris criterion relies on controlling the asymptotic behavior of the variance of the partition function of the system at the critical point of the system without disorder. As a consequence, a natural question occuring is the following: how could this criterion be altered if one considers systems with heavy tail environments, for which the second moment of the partition function is infinite. We stress that this question is far from being a purely technical one; indeed, heavy tail of the environment is likely to create greater fluctuations of the thermodynamics quantities around their average value, and this could in principle alter the validity of the criterion.

In the particular case of the pinning model detailed heuristic second moment computations were performed in [16] to predict whether disorder is relevant, and in the relevant case in which way the critical point is shifted. These results have since been made rigorous, and in particular the papers which prove lower-bound results on the free energy [4, 28, 32] all rely on controlling the second moment of a partition function.

In the present paper, we study the disordered pinning model in the case where the environment is IID but with a distribution belonging to the domain of attraction of a stable law with parameter $\gamma \in (1, 2)$; this entails in particular that the disorder has a first moment but an infinite second moment. Our aim is to show that in that case, the model falls into a different universality class and that the original formulation of the Harris criterion is not valid anymore. We present and prove an alternative formulation of the criterion in that case, which we believe should hold for a wide class of disordered systems.

2 Model and results

2.1 Setup

Let $\tau = (\tau_n)_{n \geqslant 0}$ be a recurrent integer valued renewal process, that is a random sequence whose increments $(\tau_{n+1} - \tau_n)$ are independent identically distributed (IID) positive integers. We denote by $\mathbf{P}$ the associated probability distribution. We assume that $\tau_0 = 0$, and that the inter-arrival distribution is power-law or more precisely that it satisfies

$$K(n) := \mathbf{P}[\tau_1 = n] \overset{n\to\infty}{\sim} C_K n^{-(1+\alpha)}, \quad \alpha \in (0,\infty), \tag{2.1}$$

where $C_K > 0$ is an arbitrary constant. Note that τ can alternatively be considered as an infinite subset of $\mathbb{N}$ and in our notation, $\{n \in \tau\}$ is equivalent to $\{\exists k \in \mathbb{N},\ \tau_k = n\}$.

We consider a sequence of IID random variables $(\omega_n)_{n\geq 0}$ and denote its law by $\mathbb{P}$. For our main results to hold, we will make some specific assumptions on the distribution of the ω's, namely (2.15), but for the sake of this introduction we will only assume that

$$\mathbb{P}[\omega_1 \geq -1] = 1 \quad \text{and } \mathbb{E}[\omega_1] = 0. \tag{2.2}$$

Given $\beta \in [0,1)$, $h \in \mathbb{R}$, and $N \in \mathbb{N}$, we define a modified renewal measure $\mathbf{P}^{\beta,\omega}_{N,h}$ whose Radon-Nikodym derivative with respect to $\mathbf{P}$ is given by

$$\frac{\mathrm{d}\mathbf{P}^{\beta,\omega}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z^{\beta,\omega}_{N,h}} \left(\prod_{n\in\tau\cap[1,N]} e^h(\beta\omega_n + 1) \right) \delta_N \tag{2.3}$$

where $\delta_n := \mathbf{1}_{\{n\in\tau\}}$ and

$$Z^{\beta,\omega}_{N,h} = \mathbf{E}\left[\left(\prod_{n\in\tau\cap[1,N]} e^h(\beta\omega_n + 1) \right) \delta_N \right] = \mathbf{E}\left[\prod_{n=1}^{N} \left(1 + [e^h(\beta\omega_n+1) - 1]\delta_n\right) \delta_N \right] \tag{2.4}$$

is the partition function. The assumption that ω is bounded from below is present to ensure that the density defined in (2.3)is positive.

In the case $\beta = 0$, we retrieve the homogeneous pinning model where

$$\frac{\mathrm{d}\mathbf{P}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z_{N,h}} e^{h\sum_{n=1}^N \delta_n}\delta_N \quad \text{and} \quad Z_{N,h} := \mathbf{E}\left[e^{h\sum_{n=1}^N \delta_n}\delta_N \right]. \tag{2.5}$$

Note that we assumed that the renewal is recurrent, that is

$$\mathbf{P}[\tau_1 = \infty] := 1 - \sum_{n=1}^{\infty} K(n) = 0.$$

It is a classic observation (see e.g. [19, Remark 1.19]) that this yields no loss of generality: in the case of a terminating renewal process, that is when $\sum_{n=1}^\infty K(n) < 1$, the definition of the partition function $Z^{\beta,\omega}_{N,h}$ is unchanged if the renewal is replaced by a recurrent one with inter-arrival law given by $K(n)/(\sum_{m=1}^\infty K(m))$ and h is replaced by $h + \log(\sum_{m=1}^\infty K(m))$.

Remark 2.1. The expression that we gave for the partition function of the disordered system differs substantially from the one usually found in the literature. However, if we set $\widetilde{\omega}^\beta_n := \log(1+\beta\omega_n) - \mathbb{E}[\log(1+\beta\omega_n)]$ and $\widetilde{h} = h + \mathbb{E}[\log(1+\beta\omega_n)]$ we can rewrite it in a more usual way

$$Z^{\beta,\omega}_{N,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} (\widetilde{\omega}^\beta_n + \widetilde{h})\delta_n \right) \right]. \tag{2.6}$$

Our reason for using a different notation is explained in Section 2.3.

2.2 Free energy and comparison with the annealed model

An important quantity that encodes a lot of information about the asymptotic behavior (by this we mean in the limit when N becomes large) of the renewal τ under the measure

$\mathbf{P}_{N,h}^{\beta,\omega}$ is the free energy per monomer, which is defined as the asymptotic growth rate of the partition function

$$\mathrm{f}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h}^{\beta,\omega} \overset{\mathbb{P}-a.s.}{=} \lim_{N\to\infty} \tfrac{1}{N} \mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] < \infty. \tag{2.7}$$

The existence and the non–randomness of the limit is a well established fact, we refer to [19, Theorem 4.1] for a proof.

The reader can check that $\mathrm{f}(\beta,h)$ is non-negative, and that $h \mapsto \mathrm{f}(\beta,h)$ is non-decreasing and convex (as a limit of non decreasing convex functions). By exchanging limit and derivative, as allowed by convexity, we obtain that the derivative of f w.r.t. h corresponds to the asymptotic contact fraction

$$\partial_h \mathrm{f}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \mathbf{E}_{N,h}^{\beta,\omega}\left[\sum_{n=1}^{N} \delta_n\right], \tag{2.8}$$

as soon as the derivative exists. Note that by convexity we know that $\partial_h \mathrm{f}(\beta,h)$ is defined for all but at most countably many values of h, but more advanced results proved in [24] (for $h \neq h_c(\beta)$)[1] and [14] (for $h = h_c(\beta)$) ascertains that the derivative exists everywhere except when $\beta = h = 0$ and $\alpha > 1$ (see (2.11)).

If one sets

$$h_c(\beta) := \inf\{h \in \mathbb{R} \mid \mathrm{f}(\beta,h) > 0\}, \tag{2.9}$$

then in view of (2.8), $h_c(\beta)$ separates a phase where the contact fraction is vanishing ($h < h_c(\beta)$, the delocalized phase), from another where it is positive ($h > h_c(\beta)$, the localized phase). Very soft arguments exposed below (see eqs. (2.13) and (2.14)) are sufficient to show that this phase transition really occurs, that is that $h_c(\beta) \notin \{-\infty,\infty\}$.

For the homogeneous case, the free energy $\mathrm{f}(h) := \mathrm{f}(0,h)$ can be computed explicitly (see [19])

$$\mathrm{f}(h) = \begin{cases} 0 & \text{if } h \leq 0, \\ g^{-1}(h) & \text{if } h > 0, \end{cases} \tag{2.10}$$

where g is defined on $\mathbb{R}_+$ by

$$g(x) := -\log\left(\sum_{n=1}^{\infty} e^{-nx} K(n)\right).$$

For $\alpha \neq 1$, using some Tauberian theorems, Assumption (2.1) entails that

$$\mathrm{f}(h) \overset{h\to 0+}{\sim} C(K) h^{\max(\alpha^{-1},1)}, \tag{2.11}$$

where $C(K)$ is an explicit (see [19, Theorem 2.1]) function of the renewal function K. When $\alpha = 1$ the same result holds with a slowly varying correction in front of the power of h. We refer to the exponent $\max(\alpha^{-1},1)$ appearing in (2.11) as the critical exponent associated to the free energy.

To try to understand the behavior of the disordered pinning model, it is tempting to use comparison with the homogeneous one. Making use of Jensen's inequality and (2.2) we obtain

$$\mathbb{E}[\log Z_{N,h}^{\beta,\omega}] \leqslant \log \mathbb{E}[Z_{N,h}^{\beta,\omega}] = \log(Z_{N,h}), \tag{2.12}$$

[1] In [24], it is assumed that the environment satisfies a concentration inequality which our choice for ω clearly violates. The interested reader can check however that this condition is only needed to prove differentiability in β. The proof for the infinite differentiability in h relies on [24, Equation (3.2)] which can be shown to be valid for general IID environement by a simple comparison argument.

and hence

$$\mathrm{f}(\beta,h) \leqslant \mathrm{f}(h) \quad \text{and } h_c(\beta) \geq 0. \tag{2.13}$$

On the other hand, some other convexity considerations (see [19, Proposition 5.1] and recall Remark 2.1 concerning the difference in notational convention) yield

$$\mathrm{f}(\beta,h) \geqslant \mathrm{f}(h+\mathbb{E}[\log(1+\beta\omega_1)]) \quad \text{and } h_c(\beta) \leq -\mathbb{E}[\log(1+\beta\omega_1)]. \tag{2.14}$$

2.3 Harris criterion, second moment and stable laws

While (2.14) is never sharp (see [5]) the question whether $h_c(\beta)$ is equal to zero is a much more subtle one and is very much related to the question of disorder relevance:

"Does the introduction of a disorder of small amplitude (small β)
implies a change of the critical behavior of the system?"

More precisely the question can be decomposed in two points:

(A) Does the critical point of the disordered system coincide with the one obtained after averaging (2.12)? (With our conventions: is $h_c(\beta) = 0$?)

(B) If at the vicinity of the critical point we have

$$\mathrm{f}(\beta, h_c(\beta)+u) \approx u^\nu,$$

does ν coincide with the exponent of the pure system $\max(\alpha^{-1},1)$?

These questions received a lot of attention from the mathematical community since the publication of heuristic predictions made by Derrida *et al.* [16] based on an interpretation of the Harris criterion [25].

In substance, the argument in [16] is based on the following observation: if one considers the disordered system at the pure critical point $h=0$, then the variance of the partition function diverges (exponentially with the size of the system) for any $\beta>0$ if the return exponent is larger or equal to $1/2$ and remains bounded for small values of β if α is strictly smaller than $1/2$. From these observations and some heuristic computations, they conclude that

(1) When β is small and $\alpha<1/2$, the critical point $h_c(\beta)$ is equal to 0 which is that of the pure system, and furthermore, the critical exponent associated to the free energy is equal to α^{-1}.

(2) When $\alpha \geq 1/2$, there is a shift of the critical point ($h_c(\beta)>0$) for all values of β, which for $\alpha>1/2$ is of order $\beta^{\frac{2\alpha}{2\alpha-1}}$ for small β, ad exponentially small for $\alpha=1/2$.

These predictions were confirmed when $\alpha<1/2$ [4, 32, 28], $\alpha>1/2$ [15, 6] and $\alpha=1/2$ [22, 8], in the case where ω has a finite second moment.

From the proof heuristics, one is led to believe that the assumption $\mathbb{E}[\omega_n^2]<\infty$ is not only a technical detail, and that considering disorders with heavier tail might considerably change the picture of disorder relevance.

In this paper we decide to consider the case where the ω_n's are heavy tailed, and more precisely are in the domain of attraction of a γ-stable law with $\gamma\in(1,2)$. To keep things simple we assume that there exists a constant $C_{\mathbb{P}}$ such that

$$\mathbb{P}[\omega_n \geq x] \overset{x\to\infty}{\sim} C_{\mathbb{P}} x^{-\gamma}, \quad \gamma\in(1,2). \tag{2.15}$$

This justifies our unorthodox choice for the writing of the partition function (underlined in Remark 2.1): writing things in the usual way, we would end up with an exponent γ which depends on β which would be unpractical.

Remark 2.2. All the results presented in this paper, would also extend to the case where one allows the presence of a slowly varying function $L(\cdot)$ instead of the constants C_K and $C_{\mathbb{P}}$ in the tail distribution of the renewal process (2.1) or of the environment (2.15) (keeping the conditions (2.2) on the distribution of ω). We made the choice of a more restrictive assumption to simplify the notation.

2.4 Results

Our main achievement is to show that when an environment with heavier tails is considered, Harris criterion in its standard formulation is not valid anymore. The alternative criterion in our context is the following: the value of α that separates disorder relevance and irrelevance is no longer equal to $1/2$ but to $1-\gamma^{-1}$. In all the results stated below we assume that (2.1) and (2.15) holds.

Our first result states that when $\alpha < [1-\gamma^{-1}]$, the critical point and critical exponent coincide with the one of the pure system.

Theorem 2.3. *If $\alpha < [1-\gamma^{-1}]$, there exists $\beta_0 > 0$ (depending on the renewal function K and on the distribution of ω) such that for all $\beta \in (0, \beta_0)$, we have $h_c(\beta) = 0$ and*

$$\lim_{h \to 0+} \frac{\log \mathrm{f}(\beta, h)}{\log h} = \frac{1}{\alpha}. \tag{2.16}$$

Our second result deals with the relevant disorder case: we show that when $\alpha > [1-\gamma^{-1}]$ the critical point is shifted for every value of β.

Theorem 2.4. *If $\alpha > [1-\gamma^{-1}]$, then for any $\beta \in (0,1)$ we have $h_c(\beta) > 0$. Furthermore we have the following estimate on the critical point shift*

$$\lim_{\beta \to 0+} \frac{\log h_c(\beta)}{(\log \beta)} = \begin{cases} \frac{\alpha\gamma}{(\alpha-1)\gamma+1}, & \text{if } \alpha \in (1-\gamma^{-1}, 1) \\ \gamma & \text{if } \alpha \geq 1. \end{cases} \tag{2.17}$$

Note that in the case where ω has finite second moment both (2.16) and (2.17) hold (with proofs given respectively in [4, 32, 28] and [15, 6]) with γ replaced by 2.

Remark 2.5. We prove in fact more quantitative upper and lower bounds for (2.16) and (2.17), but our upper and lower bounds do not match. We believe that for $\alpha < [1-\gamma^{-1}]$, $\beta \in (0, \beta_0)$, the annealed bound (2.13) is asymptotically sharp, that is

$$\mathrm{f}(\beta, h) \overset{h \to 0+}{\sim} \mathrm{f}(0, h).$$

When $\alpha \in (1-\gamma^{-1}, 1)$ we believe, that, similarly to what occurs in the L_2 case [13] we have

$$h_c(\beta) \overset{\beta \to 0+}{\sim} c\beta^{\frac{\alpha\gamma}{(\alpha-1)\gamma+1}},$$

(See also [7] for a result covering the case $\alpha > 1$).

2.5 Open questions and conjectures

2.5.1 The marginal case

An important observation about our results is that they do not solve the marginal case $\alpha = [1-\gamma^{-1}]$. In a companion paper [29], we show that when $\alpha = [1-\gamma^{-1}]$ and (2.1) is satisfied, disorder is also relevant, a result that bears some similarity with that proved in [22].

However this does not completely solve the problem of disorder relevance: for instance when (2.15) holds, we would like to find a necessary and sufficient condition on K (assuming only regular variations) for the occurrence of a critical point shift at every temperature similar to the one proved in [8] under the assumption of second moment.

Heuristic computations suggest the following picture:

Conjecture 2.6. *Assuming that Assumption* (2.15) *holds and that the inter-arrival law* $K(\cdot)$ *is regularly varying, the following equivalence holds*

$$\{\forall \beta > 0,\ h_c(\beta) > 0\} \Leftrightarrow \left\{ \sum_{n\geq 1} \mathbf{P}[n \in \tau]^\gamma = \infty \right\}. \tag{2.18}$$

While it seems plausible that, with a consequent amount of work, the techniques developed in [8] could be adapted to prove one side of the implication (that is, that the proposition on the r.h.s. implies that on the l.h.s.), the other direction seems to be much more challenging with the techniques we have at hand.

2.6 Smoothing of the phase transition

It is a general paradigm that the presence of disorder tends to make the free energy curve smoother at the vicinity of the critical point (see e.g. [1] for a celebrated result of this kind for the random field Ising model). For disordered pinning models in particular the first result of this type was proved in [23] and generalized in [14]. This last generalization applies without restriction to our setup, indeed the reader can check that that the first line of [14, Assumption (1.3)] is equivalent in our context to

$$\mathbb{E}[(1+\beta\omega_n)^t] < \infty, \text{ for } |t| < t_0,$$

which is readily implied by our assumptions (2.15) and $\beta < 1$. We obtain from [14, Theorem 1.9] that there exists a constant $C_\beta > 0$ such that for all $u \in [0,1]$,

$$\mathrm{f}(\beta, h_c(\beta) + u) \leq C_\beta u^2. \tag{2.19}$$

There are various reasons to believe that an heavy-tailed environment should make the free energy curve even smoother than quadratic at criticality. More precisely

Conjecture 2.7. *Assuming that* (2.1) *and* (2.15) *holds, for every* $\beta > 0$ *there exists* $C_\beta > 0$ *such that for all* $u \in [0,1]$*,*

$$\mathrm{f}(\beta, h_c(\beta) + u) \leq C_\beta u^{\frac{\gamma}{\gamma-1}}. \tag{2.20}$$

A first justification for this conjecture is that for relevant disorder, the free energy of the disordered system should be smoother than that of the pure system. Thus the critical exponent for the disordered system should be larger than α^{-1} for every $\alpha > 1 - \gamma^{-1}$. Perhaps a more convincing one is that the proofs in [14, 23] are based on localization strategies which take advantage of rare fluctuations of the environment. With heavy tailed ω's, this strategy could in principle be improved using the presence of larger fluctuations. However, there are serious technical obstacles to transform these heuristics into a proof.

2.6.1 Scaling limits

Important efforts have been recently performed in the community to understand in which way systems with relevant disorder scale to continuous limits by tuning the intensity of the disorder to zero while the system grows [2, 11, 12]. So far, to our knowledge only the case of disorder with finite second moment (that is, such that the partition function possesses a finite second moment) has been considered, and limits have been found to be related to Gaussian multiplicative chaos.

Here, due to the different nature of the noise, the scaling limit should no longer be Gaussian but should involve some Levy noise, and the results we obtain also suggest that the appropriate scaling should be different (note that the scaling considered in [11, Equation (1.11)] would correspond to $\gamma = 2$).

Conjecture 2.8. *When $\alpha \in (1-\gamma^{-1}, 1)$, given $\widehat{\beta} > 0$ and $\widehat{h} \in \mathbb{R}$, if one sets*

$$\begin{cases} \beta_N & := \widehat{\beta} N^{1-\alpha-\gamma^{-1}}, \\ h_N & := \widehat{h} N^{-\alpha}, \end{cases} \tag{2.21}$$

then the sequence $(N^{1-\alpha} Z^{\beta_N,\omega}_{N,h_N})_N$ converges in law to a non-degenerate random variable.

The factor $N^{1-\alpha}$ is present to compensate for the cost of the conditioning $N \in \tau$ and would not be present if one considered another type of boundary condition. We believe that the limit could be expressed as a multiplicative chaos over a Levy Noise similar to the Levy multiplicative chaos defined in [30] but defined on a state of paths (see [31] for a general definition of chaos in the Gaussian setup).

When $\alpha > 1$, $h_N = \widehat{h} N^{-1}$ and $\beta_N := \widehat{\beta} N^{-\gamma^{-1}}$, the sequence $Z^{\beta_N,\omega_N}_{N,h_N}$ should also converge to some non-degenerate distribution. While the rigorous proof in the case $\alpha > 1$ does not seem to present a big challenge, Conjecture 2.8 on the other hand could prove to be quite daring as most of the tools used in [11] do not seem to adapt to the Levy case, and some new ideas should be developed.

2.7 About the proofs

2.7.1 Disorder irrelevance and upper bound on the critical point shift

The proof of Theorem 2.3 requires a new method as all the proof of results of this type in the literature rely on controlling the second moment which is infinite in our case. What we do instead is to try to control the moment of order p for some p in the interval $(1, \gamma)$.

The problem that arises then is that while integer moments of partition functions have generally a nice expression involving several replicas of the system, this is not the case for fractional moments. To obtain suitable upper bounds on the fractional moments, we first rewrite the problem as the estimation of the $(p-1)$-th moment of a partition of the system where the environment has been tilted along a quenched renewal trajectory.

To obtain an upper bound on this modified partition function, we then perform an adequate partial annealing and a decomposition of the partition function which takes into account the high cost of making long jumps. These computations require a fine intuition of the mechanism that yields self-averaging in the partition function.

To control the value of $h_c(\beta)$, like in [6, 32], we try to control the rate of explosion of the partition function for small values of β. Depending on the value of α, we perform this by controlling either the second moment ($\alpha \geq 1/2$) or a fractional moment $\alpha < 1/2$ of a partition function with truncated environment.

2.7.2 Disorder relevance

In the case of relevant disorder, we estimate the q-fractional moments of the partition function for some $q \in (0, 1)$ with the help of a coarse graining procedure combined with a change of measure argument. This method has been introduced in [21] and has been improved several times [22, 8]. The underlying idea is to introduce a penalization for atypical environments, that is environments which have small probability but give an important contribution to the annealed partition function $\mathbb{E}[Z^{\beta,\omega}_{N,h}]$.

The important novelty added in this paper is that we do not penalize the environment for which the empirical mean of the ω is too large like in the L^2 case, but we choose to penalize environments for which the extremal values of ω are large, as heavier-tailed distributions tend to make these values meaningful.

2.8 Organization of the paper

In Section 3 we introduce a couple of technical results which are required for the proofs. Sections 4 and 5 are dedicated to the proof of Theorem 2.3, this is the most novel part of the paper where an original method is introduced to treat disorder irrelevance in the absence of second moment. In Section 6 the upper bound on $h_c(\beta)$ present in Equation (2.17) is proved, partly using the method used to prove disorder irrelevance. Finally in Section 7 we prove the lower bounds from Equation (2.17) which completes the proof of Theorem 2.4.

3 A few technical tools

3.1 Estimate on probability of visiting a given point

Set $u(n) = \mathbf{P}[n \in \tau]$. Under our power-law tail assumption the asymptotic behavior of $u(n)$ is well identified.

Lemma 3.1. *If the assumption* (2.1) *holds, then there exists C'_K (depending on the renewal function K) such that*

$$u(n) \sim \begin{cases} C'_K n^{\alpha-1}, & \text{if } \alpha \in (0,1), \\ C'_K (\log n)^{-1}, & \text{if } \alpha = 1, \\ C'_K & \text{if } \alpha > 1. \end{cases} \tag{3.1}$$

The case $\alpha \in (0,1)$ has been established in [17], and the case $\alpha > 1$ is a consequence of the renewal theorem (we refer to [19, Theorems 2.2(3), A.6 and A.7] for full references, including the case $\alpha = 1$).

3.2 Finite volume criteria

To obtain a lower bound on the free energy, it is sufficient to obtain a bound on the partition function of finite size. Indeed we can observe that $\mathbb{E}[\log Z_{N,h}^{\beta,\omega}]$ is a super-additive sequence (see e.g. [19, Proposition 4.2]) and thus, for every N:

$$\mathrm{f}(\beta, h) \geq \frac{1}{N} \mathbb{E}[\log Z_{N,h}^{\beta,\omega}]. \tag{3.2}$$

However, in our lower bound computations, it will be much more convenient for us to work with the free-boundary partition function where the constraint $\{N \in \tau\}$ is dropped, that is

$$Z_{N,h}^{\beta,\omega,\mathrm{f}} := \mathbf{E}\left[\prod_{n \in \tau \cap [1,N]} e^h(\beta\omega_n + 1) \right].$$

Note that $Z_{N,h}^{\beta,\omega,\mathrm{f}}$ compares well with $Z_{N,h}^{\beta,\omega}$ (cf. [19, Equation (4.25)]); indeed, this quantity also provides a lower bound on the free energy with the loss of a $\log$ factor (for a proof see [3, Proposition 2.6]).

Lemma 3.2. *There exists a constant $C(\beta)$ such that for any value of N and h,*

$$\mathrm{f}(\beta, h) \geq \frac{1}{N} \mathbb{E}[\log Z_{N,h}^{\beta,\omega,\mathrm{f}}] - C(\beta) \frac{\log N}{N}. \tag{3.3}$$

4 Disorder irrelevance: the proof of Theorem 2.3

The idea used in the proof is similar to the one introduced in [28]: if at the pure critical point the partition function behaves like its average, it implies that the measure $\mathbf{P}_{N,0}^{\beta,\omega}$ is in a sense close to the original one $\mathbf{P}$. We want to use this information to prove that the expected number of contacts at criticality is large, from which we get a lower bound on the partition function $Z_{N,h}^{\beta,\omega,\mathrm{f}}$ at positive times, and finally conclude using (3.3). Using this procedure, we can prove the following result:

Proposition 4.1. *Assume that* (2.15) *holds and that* $\alpha < [1-\gamma^{-1}]$. *There exists* β_0 *such that for all* $\beta \in [0, \beta_0]$*, there exists* C_β *(that may depend also on the inter-arrival distribution and the distribution of* ω*) such that*

$$\forall h \in [0, 1/2], \quad \mathrm{f}(\beta, h) \geq C_\beta \left(\frac{h}{|\log h|^{\alpha-1}} \right)^{\alpha^{-1}}. \tag{4.1}$$

Note that combining (4.1) and (2.11) with (2.13) immediately implies both $h_c(\beta) = 0$ and (2.16).

Contrarily to what we do in [28], we do not use the convergence of the partition function (as a martingale), but find a more efficient way to use uniform integrability in order to extract quantitative statements. Then we use the same technique to prove upper-bounds on $h_c(\beta)$ in the disorder relevant case.

4.1 Decomposition of the proof

In this short section, we show how Proposition 4.1 follows from two key statements. The most important one, whose proof is detailed in Section 5, is that some non-integer moments of order $p > 1$ of the partition function are uniformly bounded in the size of the system.

Proposition 4.2. *For any* $p \in (1, \gamma)$ *and for* $\beta \leq \beta_0(p)$*, we have*

$$\sup_{N \geq 0} \mathbb{E}\left[(Z_{N,0}^{\beta,\omega,\mathrm{f}})^p \right] < \infty. \tag{4.2}$$

The second result, whose short proof is detailed in Section 4.2 uses this bound to show that typical events for $\mathbf{P}$ cannot be atypical for $\mathbf{P}_{N,0}^{\beta,\omega,\mathrm{f}}$.

Lemma 4.3. *Given* $p > 1$*, if* $\mathbb{E}\left[(Z_{N,0}^{\beta,\omega,\mathrm{f}})^p \right] =: M < \infty$*, there exists* $\delta = \delta(M, p) > 0$ *such that for any event* A

$$\{ \mathbf{P}[A] \geq 1 - \delta \} \quad \Rightarrow \quad \left\{ \mathbb{E}\left[\mathbf{P}_{N,0}^{\beta,\omega,\mathrm{f}}[A] \right] \geq \delta \right\}. \tag{4.3}$$

Let $\mathcal{N}_N(\tau) := \#([1, N] \cap \tau)$ be the number of renewal points in the interval $[1, N]$. Given $\varepsilon > 0$, we apply Lemma 4.3 to the event

$$A_{N,\varepsilon} := \{ \tau \, : \, \mathcal{N}_N(\tau) \geq \varepsilon N^\alpha \}.$$

As a consequence of the convergence of $(n^{-1/\alpha} \tau_{\lceil nt \rceil})_{t \geq 0}$ to an α-stable subordinator (see [18, Chapter XVII]), $N^{-\alpha} \mathcal{N}_N(\tau)$ converges in law to a random variable (the first hitting time of $[1, \infty)$ for this subordinator) whose distribution has no atom at zero. In particular

$$\lim_{\varepsilon \to 0} \liminf_{N \to \infty} \mathbf{P}(A_{N,\varepsilon}) = 1. \tag{4.4}$$

Proof of Proposition 4.1. We have now all the ingredients to prove (4.1). We fix $p \in (1, \gamma)$ arbitrarily, consider $\beta \leq \beta_0(p)$, set $M = M(\beta) := \sup_{N \geq 0} \mathbb{E}\left[(Z_{N,0}^{\beta,\omega,\mathrm{f}})^p \right]$ and choose ε_0 and N_0 such that $\mathbf{P}(A_{N,\varepsilon_0}) \geq 1 - \delta(M, p)$, for all $N \geq N_0$. From Lemma 4.3, we have

$$\mathbb{E}\left[\mathbf{P}_{N,0}^{\beta,\omega,\mathrm{f}}[A_{N,\varepsilon_0}] \right] > \delta. \tag{4.5}$$

Using the convexity of the function $h \mapsto \log Z_{N,h}^{\beta,\omega,\mathrm{f}}$, we observe that for some constant C

$$\begin{aligned}\log Z_{N,h}^{\beta,\omega,\mathrm{f}} &\geq \log Z_{N,0}^{\beta,\omega,\mathrm{f}} + h\partial_u \log Z_{N,u}^{\beta,\omega,\mathrm{f}}|_{u=0} \\ &\geq \log \mathbf{P}[\tau_1 > N] + h\mathbf{E}_{N,0}^{\beta,\omega,\mathrm{f}}[\mathcal{N}_N(\tau)] \geq h\varepsilon_0 N^\alpha \mathbf{P}_{N,0}^{\beta,\omega,\mathrm{f}}[A_{N,\varepsilon_0}] - \alpha[\log N] - C. \end{aligned} \tag{4.6}$$

Taking the expectation and using (4.5) and (3.3) we obtain that for some N_0 sufficiently large, for any $h > 0$ and $N \geq N_0$ we have

$$\mathrm{f}(\beta, h) \geq hN^{\alpha-1}\varepsilon_0\delta - C'\frac{\log N}{N}, \tag{4.7}$$

and we conclude by taking $N = C''\left(h^{-1}|\log h|\right)^{\alpha^{-1}}$, for C'' sufficiently large. □

4.2 Proof of Lemma 4.3

We have

$$\begin{aligned}\mathbb{E}\mathbf{P}_{N,0}^{\beta,\omega,\mathrm{f}}\left[A^\complement\right] &= \mathbb{E}\left[\frac{1}{Z_{N,0}^{\beta,\omega,\mathrm{f}}}\mathbf{E}\left[\mathbf{1}_{A^\complement}\prod_{i=1}^N(1+\beta\omega_n\delta_n)\right]\right] \\ &\leq 2\mathbb{E}\mathbf{E}\left[\mathbf{1}_{A^\complement}\prod_{i=1}^N(1+\beta\omega_n\delta_n)\right] + \mathbb{P}\left[Z_{N,0}^{\beta,\omega,\mathrm{f}} < 1/2\right] \\ &= 2\mathbf{P}\left[A^\complement\right] + \left(1 - \mathbb{P}\left[Z_{N,0}^{\beta,\omega,\mathrm{f}} \geq 1/2\right]\right). \end{aligned} \tag{4.8}$$

To conclude we use the following estimate for $\theta = 1/2$.

Lemma 4.4. *If X is a positive random variable and $p > 1$, we have*

$$\mathbb{P}\left[X \geq \theta\mathbb{E}[X]\right] \geq (1-\theta)^{\frac{p}{p-1}}\frac{\mathbb{E}[X]^{\frac{p}{p-1}}}{\mathbb{E}[X^p]^{\frac{1}{p-1}}}. \tag{4.9}$$

Using the above result for $X = Z_{N,0}^{\beta,\omega,\mathrm{f}}$ and $\theta = 1/2$ together with the assumption on A this yields

$$\mathbb{E}\mathbf{P}_{N,0}^{\beta,\omega,\mathrm{f}}\left[A^\complement\right] \leq 2\delta + 1 - 2^{-\frac{p}{p-1}}M^{-\frac{1}{p-1}} \leq 1-\delta, \tag{4.10}$$

provided δ is chosen sufficiently small.

Proof of Lemma 4.4. Using Hölder's inequality (with $p' = \frac{p}{p-1}$), we get

$$\mathbb{E}[X] \leq \theta\mathbb{E}[X] + \mathbb{E}\left[X\mathbf{1}_{\{X\geq\theta\mathbb{E}[X]\}}\right] \leq \theta\mathbb{E}[X] + (\mathbb{E}[X^p])^{1/p}(\mathbb{P}\left[X \geq \theta\mathbb{E}[X]\right])^{1/p'}. \tag{4.11}$$

□

5 Bounding the fractional moments: Proof of Proposition 4.2

5.1 Decomposing the proof

By monotonicity of $p \mapsto \mathbb{E}[(Z_N)^p]^{1/p}$, it is sufficient to treat the case $p \in \left(\frac{1}{1-\alpha}, \gamma\right)$. Thus, setting $q = p - 1$, we have

$$\frac{\alpha}{1-\alpha} < q < \gamma - 1. \tag{5.1}$$

While it is clear from the assumption (2.15) that for $p \geq \gamma$ the moment of order p is equal to infinity, it is not obvious at this stage why we also need a lower bound on q. This will appear in the course of the proof when using coarse graining arguments.

Our proof goes as follows; first (Section 5.2) we rewrite the p moment of the partition function as the q moment of a different partition function involving an extra quenched copy of the renewal and a tilted environment. We also perform some partial annealing to simplify the expression which is obtained.

In a second step (Section 5.3), we perform a decomposition of this new partition function based on the classical inequality $(\sum a_i)^q \le \sum a_i^q$. This helps us to reduce our proof to the estimate of (the q moment of) the partition function of a system of finite size.

Finally we use a change of measure technique (Section 5.4) to show that this last partition function is small.

5.2 Rewriting the partition function using size-biasing

To simplify the quantity we have to bound, we decide to rewrite it is as $\widetilde{\mathbb{E}}_N\left[\left(Z_{N,0}^{\beta,\omega,\mathrm{f}}\right)^q\right]$, where $\widetilde{\mathbb{P}}_N$ is the probability defined by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}}(\omega) = Z_{N,0}^{\beta,\omega,\mathrm{f}}. \tag{5.2}$$

The partition function having expectation one under $\mathbb{P}$, it defines indeed a probability density. The reason for which this consideration might be useful is that more techniques are available to control p moments for p in the interval $(0,1)$ than for p in $(1,2)$ (we refer for example to the key role of the inequality (5.16) in our proof). A first important thing to do however, is to re-express $\widetilde{\mathbb{P}}_N$ in a form which will be more adequate to perform computations. This representation of the sized-biased measure which we present below, sometimes called “spine representation” in the case of branching structures (see the recent monograph [34] and references therein) is classical, and has been used several times in the framework of polymer measures (see e.g. [10]).

Let τ' be an independent copy of the renewal τ (we denote its law by $\mathbf{P}'$). We notice that, for a fixed realization of τ', the quantity $\prod_{n=1}^N (1+\beta\omega_n \mathbf{1}_{\{n\in\tau'\}})$ averages to one under $\mathbb{P}$, and thus can be considered as a probability density. Given a realization of τ', we introduce the probability measure $\mathbb{P}_{\tau',N}$ whose density with respect to $\mathbb{P}$ is given by

$$\mathbb{P}_{\tau',N}(d\omega) = \prod_{n=1}^N (1+\beta\omega_n \mathbf{1}_{\{n\in\tau'\}})\mathbb{P}(d\omega). \tag{5.3}$$

With these notations, we can write

$$\mathbb{E}\left[\left(Z_{N,0}^{\beta,\omega,\mathrm{f}}\right)^p\right] = \mathbb{E}\mathbf{E}'\left[\prod_{n=1}^N (1+\beta\omega_n \mathbf{1}_{\{n\in\tau'\}})\left(Z_{N,0}^{\beta,\omega,\mathrm{f}}\right)^q\right] = \mathbf{E}'\left[\mathbb{E}_{\tau',N}\left[\left(Z_{N,0}^{\beta,\omega,\mathrm{f}}\right)^q\right]\right]. \tag{5.4}$$

Note that under $\mathbb{P}_{\tau',N}$, the random variables ω_n are still independent but that they are no longer identically distributed, since the law of $(\omega_n)_{n\in[1,N]\cap\tau'}$ has been tilted by the quantity $(1+\beta\omega_n)$. However we can construct an environment $\widehat{\omega}$ of law $\mathbb{P}_{\tau',N}$ using two IID environment as follows:

(1) First we set $(\widetilde{\omega}_n)_{n\geqslant 1}$ to be an IID tilted environment; namely, all $\widetilde{\omega}_n$'s are IID, and

$$\widetilde{\mathbb{P}}[\widetilde{\omega}_1 \in \mathrm{d}x] := (1+\beta x)\mathbb{P}[\omega_1 \in \mathrm{d}x]. \tag{5.5}$$

(2) Given a realization of $\tau', \omega, \widetilde{\omega}$, we define the sequence $\widehat{\omega}$ in the following way

$$\widehat{\omega}_n := \widehat{\omega}_n(\omega,\widetilde{\omega},\tau',N) = \omega_n \mathbf{1}_{\{n\notin\tau'\cap[1,N]\}} + \widetilde{\omega}_n \mathbf{1}_{\{n\in\tau'\cap[1,N]\}}. \tag{5.6}$$

With this notation we have for every realization of τ'

$$\mathbb{P}_{\tau',N}\left[\omega \in \cdot\right] = \widetilde{\mathbb{P}} \otimes \mathbb{P}\left[\widehat{\omega} \in \cdot\right],$$

and Fubini's identity yields

$$\mathbf{E}'\left[\mathbb{E}_{\tau',N}\left[\left(Z_{N,0}^{\beta,\omega,\mathrm{f}}\right)^q\right]\right] = \widetilde{\mathbb{E}} \otimes \mathbb{E} \otimes \mathbf{E}'\left[\left(Z_{N,0}^{\beta,\widehat{\omega},\mathrm{f}}\right)^q\right].$$

Using Jensen's inequality with respect to the measure $\mathbb{E}$ (recall that we have chosen $q \in (0,1)$), we obtain:

$$\widetilde{\mathbb{E}} \otimes \mathbb{E} \otimes \mathbf{E}'\left[\left(Z_{N,0}^{\beta,\widehat{\omega},\mathrm{f}}\right)^q\right] \le \widetilde{\mathbb{E}}\mathbf{E}'\left[\left(\mathbb{E}\left[Z_{N,0}^{\beta,\widetilde{\omega},\mathrm{f}}\right]\right)^q\right] = \widetilde{\mathbb{E}}\mathbf{E}'\left[(Z_N^{\beta,\widetilde{\omega}}[\tau'])^q\right] \tag{5.7}$$

where, for a given realization of τ', we defined

$$Z_N^{\beta,\widetilde{\omega}}[\tau'] := \mathbf{E}\left[\prod_{n=1}^N (1+\beta\widetilde{\omega}_n \mathbf{1}_{\{n\in\tau\cap\tau'\}})\right].$$

To conclude this section, let us thus recall that we reduced the proof of Proposition 4.2 to the proof of the following statement.

Lemma 5.1. *For any $q \in \left(\frac{\alpha}{1-\alpha}, \gamma-1\right)$, and $\beta \le \beta_0(q)$ we have*

$$\sup_{N\ge 0} \widetilde{\mathbb{E}}\mathbf{E}'\left[(Z_N^{\beta,\widetilde{\omega}}[\tau'])^q\right] < \infty. \tag{5.8}$$

Remark 5.2. We used Jensen's inequality here to obtain a more tractable expression to estimate (in particular we are back to only one IID environment). A way to justify that this step does not make us lose much is that when disorder is irrelevant (which is what we aim to prove) the disorder is self-averaging. On the other hand, we cannot simply use Jensen's inequality for $\widetilde{\mathbb{E}}$ as $\widetilde{\omega}$ has infinite mean. Applying Jensen's inequality to $\mathbf{P}'$ is also not optimal as in some cases, one could prove that under our assumptions, $\widetilde{\mathbb{E}}\left[(\mathbf{E}'[Z_N^{\beta,\widetilde{\omega}}[\tau']])^q\right]$ diverges.

5.3 Coarse graining

Our idea to estimate the q-th moment of the partition function $Z_N^{\beta,\widetilde{\omega}}[\tau']$ is to use a change of measure argument; namely, given a positive function G of $\widetilde{\omega}$, by Hölder's inequality we have

$$\widetilde{\mathbb{E}}\left[(Z_N^{\beta,\widetilde{\omega}}[\tau'])^q\right] \le \widetilde{\mathbb{E}}\left[(G(\widetilde{\omega})Z_N^{\beta,\widetilde{\omega}}[\tau'])\right]^q \widetilde{\mathbb{E}}\left[(G(\widetilde{\omega}))^{-\frac{q}{1-q}}\right]^{1-q}. \tag{5.9}$$

If we apply this inequality to $G(\widetilde{\omega}) = \prod_{i=1}^N g(\widetilde{\omega}_i)$ with $\widetilde{\mathbb{E}}[g(\widetilde{\omega}_i)] = 1$, the first term corresponds to the expectation of $Z_N^{\beta,\widetilde{\omega}}[\tau']$ under a new measure for which the law of $\widetilde{\omega}$ has been changed and the second one can be interpreted as a cost for this change of measure.

The problem with this approach is that the cost grows exponentially in the size of the system N and the first term, which is always larger than $\mathbf{P}[\tau_1 > N] \approx N^{-\alpha}$ cannot compensate for it. We want to apply this idea in a more subtle way by coupling it with a coarse graining argument. To perform this we decompose the partition function in order to make sure that we apply our change of measure in regions that have a positive density of contacts involving the environment $\widetilde{\omega}$ (i.e contacts for $\tau\cap\tau'$). Hence we decompose $\widetilde{Z}$ according to the location of the large jumps. Let us introduce a few notations in order to apply this decomposition.

Given a realization of τ', for $b > a$ we set

$$u(a,b,\tau') = \mathbf{1}_{\{a,b\in\tau'\}}\mathbf{P}[b \in \tau \text{ and } \tau\cap\tau'\cap(a,b)=\emptyset \mid a\in\tau], \tag{5.10}$$

which is the probability that b is the next point in $\tau\cap\tau'$ after a. Decomposing the partition function according to the realization of $(\tau\cap\tau')$ and setting by convention $t_0 := 0$, we obtain

$$\begin{aligned} Z_N^{\beta,\widetilde{\omega}}[\tau'] &= \sum_{k\geq 1}\sum_{0<t_1<\cdots<t_k\leq N}\prod_{i=1}^{k} u(t_{i-1},t_i,\tau')(1+\beta\widetilde{\omega}_{t_i}) \\ &\quad\times \mathbf{P}[(t_k,N]\cap\tau\cap\tau'=\emptyset] + \mathbf{P}[(\tau\cap\tau')_1 > N] \\ &\leq \sum_{k\geq 1}\sum_{0<t_1<\cdots<t_k}\prod_{i=1}^{k} u(t_{i-1},t_i,\tau')(1+\beta\widetilde{\omega}_{t_i}) + 1 =: Z^{\beta,\widetilde{\omega}}[\tau'] + 1. \end{aligned} \tag{5.11}$$

In the remaining part of this proof, we will be interested only in bounding the q-th moment of $Z^{\beta,\widetilde{\omega}}[\tau']$.

We want to express $Z^{\beta,\widetilde{\omega}}[\tau']$ in terms of partition functions where all the gaps in $\tau\cap\tau'$ are smaller than L. For $L>0$ (meant to be large) and $b>a>0$ all integers, we define $Z^L_{[a,b]}[\tau']$ as the partition function on the interval $[a,b]$ restricted to trajectories for which $\tau\cap\tau'$ has no gap larger than L, that is

$$Z^L_{[a,b]}[\tau'] := (1+\beta\widetilde{\omega}_a)\sum_{k\geq 1}\sum_{\substack{a=t_0<t_1<\cdots<t_k=b \\ t_i-t_{i-1}\leq L}}\prod_{i=1}^{k} u(t_{i-1},t_i,\tau')(1+\beta\widetilde{\omega}_{t_i}). \tag{5.12}$$

By convention we set

$$Z^L_{[a,a]}[\tau'] = (1+\beta\widetilde{\omega}_a)\mathbf{1}_{\{a\in\tau'\}}. \tag{5.13}$$

Note that $Z^L_{[a,b]}[\tau'] = 0$ if either $a\notin\tau'$, $b\notin\tau'$ or $\tau'_{i+1}-\tau'_i > L$ for some i with $\tau'_i\in[a,b)$ and that $Z^L_{[a,b]}[\tau'] > 0$ in all other cases.

Decomposing according to the cardinality, the locations and the lengths of the excursions which are longer than L, we get the expression

$$(1+\beta\widetilde{\omega}_0)Z^{\beta,\widetilde{\omega}}[\tau'] = \sum_{k\geq 1}\sum_{(\mathbf{t},\mathbf{t}')\in\mathcal{T}_L^k}\left(\prod_{i=1}^{k-1} Z^L_{[t_i,t'_i]}[\tau']u(t'_i,t_{i+1},\tau')\right) Z^L_{[t_k,t'_k]}[\tau'], \tag{5.14}$$

where

$$\mathcal{T}_L^k := \{(\mathbf{t},\mathbf{t}')\in\mathbb{N}^{2k} \ : \ t_0=0,\ \forall i\in[\![1,k]\!], t'_i\geq t_i \text{ and } t_{i+1}-t'_i > L\}. \tag{5.15}$$

Hence using the inequality

$$\left(\sum_{i\in I} a_i\right)^q \leqslant \sum_{i\in I} a_i^q \tag{5.16}$$

valid for any collection of positive numbers, and combining it with the IID nature of the environment, we obtain

$$\begin{aligned} &\widetilde{\mathbb{E}}\left[(1+\beta\widetilde{\omega}_0)^q (Z^{\beta,\widetilde{\omega}}[\tau'])^q\right] \\ &\quad\leq \sum_{k\geq 1}\sum_{(\mathbf{t},\mathbf{t}')\in\mathcal{T}_L^k}\left(\prod_{i=1}^{k-1}\widetilde{\mathbb{E}}\left[\left(Z^L_{[t_i,t'_i]}[\tau']\right)^q\right] u(t'_i,t_{i+1},\tau')^q\right)\widetilde{\mathbb{E}}\left[\left(Z^L_{[t_k,t'_k]}[\tau']\right)^q\right]. \end{aligned} \tag{5.17}$$

The terms in the sum are zero unless $t_i \in \tau'$ and $t'_i \in \tau'$ for all i. We use the spatial Markov property for τ' and obtain that

$$\mathbf{E}'\widetilde{\mathbb{E}}\left[(1+\beta\widetilde{\omega}_0)^q(Z_N^{\beta,\widetilde{\omega}}[\tau'])^q\right] \leq \sum_{k\geq 1}\sum_{(\mathbf{t},\mathbf{t}')\in\mathcal{T}_L^k} \mathbf{P}'\left[\forall i\in[\![1,k]\!],\ t_i,t'_i\in\tau'\right]$$
$$\times\left(\prod_{i=1}^{k-1}\mathbf{E}'\widetilde{\mathbb{E}}\left[\left(Z^L_{[t_i,t'_i]}[\tau']\right)^q \mid t_i,t'_i\in\tau'\right]\mathbf{E}'[u(t'_i,t_{i+1},\tau')^q \mid t'_i,t_{i+1}\in\tau']\right)$$
$$\times\widetilde{\mathbb{E}}\left[\mathbf{E}'\left[\left(Z^L_{[t_k,t'_k]}[\tau']\right)^q \mid t_k,t'_k\in\tau'\right]\right]. \quad (5.18)$$

Let us now observe that most of the terms in the above expression are translation invariant which helps for factorization. We have

$$\mathbf{E}'\widetilde{\mathbb{E}}\left[\left(Z^L_{[t_i,t'_i]}[\tau']\right)^q \mid t_i,t'_i\in\tau'\right] = \mathbf{E}'\widetilde{\mathbb{E}}\left[\left(Z^L_{[0,t'_i-t_i]}[\tau']\right)^q \mid t'_i-t_i\in\tau'\right]$$
$$\leq \widetilde{\mathbb{E}}\left[\left(\mathbf{E}'\left[Z^L_{[0,t'_i-t_i]}[\tau'] \mid t'_i-t_i\in\tau'\right]\right)^q\right], \quad (5.19)$$

where we used Jensen's inequality in the last line. We consider the partition function $\widetilde{Z}_n^{L,\widetilde{\omega}}$ associated to the (terminating) renewal $\widetilde{\tau}$ (with probability denoted by $\widetilde{\mathbf{P}}^L$) whose inter-arrival distribution is given by

$$\widetilde{\mathbf{P}}^L[\widetilde{\tau}_1=n]=\widetilde{K}^L(n)$$
$$:=\mathbf{P}\otimes\mathbf{P}'\left[n\in\tau\cap\tau'\ ;\ (0,n)\cap\tau\cap\tau'=\emptyset\right]\mathbf{1}_{\{n\leq L\}} =: \widetilde{K}(n)\mathbf{1}_{\{n\leq L\}}. \quad (5.20)$$

Note that as $\alpha<1-\gamma<1/2$, the renewal $\tau\cap\tau'$ is terminating and thus we have

$$\sum_{n\geq 1}\widetilde{K}(n)<1. \quad (5.21)$$

We define accordingly

$$\widetilde{Z}_n^{L,\widetilde{\omega}} := \widetilde{\mathbf{E}}^L\left[\prod_{i=0}^n(1+\beta\widetilde{\omega}_i\mathbf{1}_{\{i\in\widetilde{\tau}\}})\mathbf{1}_{\{n\in\widetilde{\tau}\}}\right]. \quad (5.22)$$

Now we observe that for $b>a$,

$$\mathbf{E}'[Z^L_{[a,b]}[\tau']|b-a\in\tau'] = u(b-a)\widetilde{Z}^{L,\theta_a\widetilde{\omega}}_{b-a}, \quad (5.23)$$

where θ_a is the shift operator on the environment. Hence the last term appearing in the right hand side of (5.19) can be rewritten as

$$u(t'_i-t_i)^{-q}\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}^{L,\widetilde{\omega}}_{t'_i-t_i}\right)^q\right]. \quad (5.24)$$

Regarding the contribution of the long jumps, we can ignore the constraint $\tau\cap\tau'\cap(a,b)=\emptyset$ in (5.10) and obtain that

$$\mathbf{E}'[u(t'_i,t_{i+1},\tau')^q \mid t'_i,t_{i+1}\in\tau']$$
$$=\mathbf{E}'[u(0,t_{i+1}-t'_i,\tau')^q \mid t_{i+1}-t'_i\in\tau'] \leq u(t_{i+1}-t'_i)^q. \quad (5.25)$$

Factorizing $\mathbf{P}'\left[\forall i\in[\![1,k]\!],\ t_i,t'_i\in\tau'\right]$ and reorganizing the sum in the r.h.s. of (5.18), we have

$$\mathbf{E}'\widetilde{\mathbb{E}}\left[(1+\beta\widetilde{\omega}_0)^q\left(Z_N^{\beta,\widetilde{\omega}}[\tau']\right)^q\right]$$
$$\leq\sum_{k\geq 1}\left(\sum_{m=L}^{\infty}u(m)^{q+1}\right)^{k-1}\left(\sum_{n=0}^{\infty}u(n)^{1-q}\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right]\right)^k. \quad (5.26)$$

Hence we can conclude the proof of Lemma 5.1 as soon as we can show that for some arbitrary L

$$\left(\sum_{m=L}^{\infty} u(m)^{q+1}\right)\left(\sum_{n=0}^{\infty} u(n)^{1-q}\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right]\right) < 1. \tag{5.27}$$

The first sum is easy to estimate considering Lemma 3.1 and (5.1). We have

$$\sum_{m=L}^{\infty} u(m)^{q+1} \leq CL^{1-(1-\alpha)(q+1)}, \tag{5.28}$$

which can be made as small as we wish by choosing L large.

We need thus a uniform bound on the second sum. The idea is that for β small, and ignoring the constraint of having no long jumps, $\widetilde{Z}_n^{L,\widetilde{\omega}}$ looks like the partition function of a pinning model in the delocalized phase, and thus it should be of order $\widetilde{K}(n)$ (see [19, Theorem 2.2]), which, as the associated renewal is transient, is of the same order as $\widetilde{\mathbf{P}}[n \in \widetilde{\tau}] = u(n)^2$. Hence we should try to prove that $\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right]$ is of order $u(n)^{2q}$. We prove the following result in the next section.

Lemma 5.3. *Given $q \leq \gamma - 1$ and $L \geq 1$, there exist a constant $C = C(q)$ and $\beta = \beta_0(L, q)$ such that for all $\beta \in (0, \beta_0)$ and for all $n \geq 0$*

$$\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right] \leq Cu(n)^{2q}. \tag{5.29}$$

Recalling (5.1) again, this result implies that

$$\sum_{n=0}^{\infty} u(n)^{1-q}\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right] \leq C \sum_{n \geqslant 0} u(n)^{1+q} \leq C'. \tag{5.30}$$

Since the last constant C' does not depend on L, we can combine this with (5.28) to prove (5.27) for an adequate choice of L. This concludes our proof of Lemma 5.1.

5.4 Change of measure: the proof of Lemma 5.3

The proof relies on the idea exposed at the beginning of the previous section. The partition function $\widetilde{Z}_n^{L,\widetilde{\omega}}$ being a decreasing function of L, we can assume that L is sufficiently large. We use (5.9) for the partition function $\widetilde{Z}_n^{L,\widetilde{\omega}}$ with

$$G(\widetilde{\omega}) := \prod_{i=0}^{n} g(\widetilde{\omega}_i) \quad \text{where} \quad g(\widetilde{\omega}_i) := (1+\beta\widetilde{\omega}_i)^{q-1}. \tag{5.31}$$

We want the distribution of $\widetilde{\omega}_0$ affected by G because $\widetilde{Z}_n^{L,\widetilde{\omega}}$ also takes into account the environment at 0 (recall (5.12)). Using Hölder's inequality, we have

$$\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right] \leqslant \left(\widetilde{\mathbb{E}}\left[\prod_{0=1}^{n} g(\widetilde{\omega}_i)^{-\frac{q}{1-q}}\right]\right)^{1-q}\left(\widetilde{\mathbb{E}}\left[\prod_{i=0}^{n} g(\widetilde{\omega}_i)\widetilde{Z}_n^{L,\widetilde{\omega}}\right]\right)^{q}. \tag{5.32}$$

Since the environment is IID, the first term is equal to

$$\left(\widetilde{\mathbb{E}}\left[g(\widetilde{\omega}_1)^{-\frac{q}{1-q}}\right]\right)^{(1-q)(n+1)}, \tag{5.33}$$

while the expectation in the second term is equal to (recall (5.20))

$$\sum_{k\geq 1}\sum_{\substack{0=t_0<\cdots<t_k=n\\ t_i-t_{i-1}\leqslant L}} \widetilde{\mathbb{E}}[g(\widetilde{\omega}_1)(1+\beta\widetilde{\omega}_1)]^{k+1}\prod_{i=1}^{k}\widetilde{K}(t_i-t_{i-1}). \tag{5.34}$$

Note that with our choice for g and the definition (5.5), the terms integrated w.r.t. $\widetilde{\mathbb{P}}$ appearing in (5.33) and (5.34) are equal. Indeed

$$\widetilde{\mathbb{E}}[g(\widetilde{\omega}_1)^{-\frac{q}{1-q}}] = \widetilde{\mathbb{E}}\left[(1+\beta\widetilde{\omega}_1)g(\widetilde{\omega}_1)\right] = \mathbb{E}[(1+\beta\omega_1)^{1+q}].$$

Thanks to our assumption (5.1), this expectation is finite. We also need it to be small and for this we will make use of the following immediate consequence of the Dominated Convergence Theorem.

Lemma 5.4. *For any given $\varepsilon > 0$, we can choose $\beta_0(\varepsilon) > 0$ such that for every $\beta \in (0, \beta_0)$, we have*

$$\mathbb{E}[(1+\beta\omega_1)^{1+q}] \leqslant (1+\varepsilon). \tag{5.35}$$

In the following computations we choose $\varepsilon = L^{-2}$. As a consequence of the above Lemma 5.4, (5.32) implies that for $\beta < \beta_0$ we have

$$\begin{aligned}\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right] &\leq (1+\varepsilon)^{(1-q)(n+1)}\left(\sum_{k\geq 1}\sum_{\substack{0=t_0<\cdots<t_k=n\\ t_i-t_{i-1}<L}}(1+\varepsilon)^{k+1}\prod_{i=1}^k \widetilde{K}(t_i-t_{i-1})\right)^q\\ &\leq (1+\varepsilon)^{(n+1)}\left(\sum_{k\geq 1}\sum_{\substack{0=t_0<\cdots<t_k=n\\ t_i-t_{i-1}\leqslant L}}\prod_{i=1}^k \widetilde{K}(t_i-t_{i-1})\right)^q.\end{aligned} \tag{5.36}$$

Recalling (5.21) we set

$$c := -\log\left(\sum_{j=1}^{\infty}\widetilde{K}(j)\right) > 0.$$

Then we note that

$$\begin{aligned}&\sum_{k\geq 1}\sum_{\substack{0=t_0<\cdots<t_k=n\\ t_i-t_{i-1}<L}}\prod_{i=1}^k \widetilde{K}(t_i-t_{i-1})\\ &\qquad\leq \min\left(\sum_{k\geq n/L}\left(\sum_{j=1}^{\infty}\widetilde{K}(j)\right)^k, \sum_{k\geq 1}\sum_{0=t_0<\cdots<t_k=n}\prod_{i=1}^k \widetilde{K}(t_i-t_{i-1})\right).\end{aligned} \tag{5.37}$$

The second term in the $\min$ is equal to $\mathbf{P}\otimes\mathbf{P}'(n\in\tau\cap\tau') = u(n)^2$, and is a sufficient bound in the case where $n \leq L^2$ as the pre-factor $(1+\varepsilon)^{n+1}$ in (5.36) is bounded by e in that case. The first term is smaller than $(1-e^{-c})^{-1}e^{-cnL^{-1}}$ which together with (5.36) gives (cf. (3.1))

$$\widetilde{\mathbb{E}}\left[\left(\widetilde{Z}_n^{L,\widetilde{\omega}}\right)^q\right] \leq (1-e^{-c})^{-q}(1+\varepsilon)^{n+1}e^{-cnqL^{-1}} \leq C(q)u(n)^{2q} \tag{5.38}$$

for all $n \geq L^2$ provided that L is chosen large enough. □

6 Upper bound on the critical point shift

In this section, we adapt the tools used in the proof of Theorem 2.3 in order to obtain a lower bound on the free energy when $\alpha > 1-\gamma^{-1}$, which yields an upper bound on the critical point shift. More precisely we prove

Proposition 6.1. *There exists a constant C such that for every $\beta \in [0,1)$*

$$h_c(\beta) \le \begin{cases} C\beta^{\gamma} \text{ if } \alpha \ge 1, \\ C|\log\beta|\beta^{\frac{\alpha\gamma}{1-\gamma(1-\alpha)}} & \text{if } \alpha \in (1/2,1), \\ C\beta^{\frac{\gamma}{2-\gamma}}|\log\beta|^{\frac{4-\gamma}{2(2-\gamma)}}, & \text{if } \alpha = 1/2. \end{cases} \tag{6.1}$$

If $\alpha \in (1-\gamma^{-1}, 1/2)$, given $\delta > 0$ there exists a constant $C_\delta > 0$ such that for every $\beta \in [0,1)$

$$h_c(\beta) \le C_\delta \beta^{\frac{\alpha\gamma}{1-\gamma(1-\alpha)}-\delta}. \tag{6.2}$$

Remark 6.2. While we tried to optimize the factor in front of the power of β for $\alpha \ge 1/2$, we did not perform such an operation in the case $\alpha < 1/2$, in order to keep the computation simple. In any case the correction needed would be worse than a logarithmic power. We believe that these corrections are artifacts of the proof, and that the asymptotic behavior of $h_c(\beta)$ should be given by a pure power of β in most cases.

The case $\alpha \ge 1$ is the easiest one: in that case the result follows directly from (2.14). Indeed it is a simple exercise (observing that $\mathbb{E}[\omega_n] = 0$ and differentiating $\log(1+\beta\omega) - \beta\omega$ to compute the integral) to check that (2.15) implies

$$\mathbb{E}\left[\log(1+\beta\omega_n)\right] \overset{\beta\searrow 0}{=\sim} -C_{\mathbb{P}}\beta^{\gamma}\left(\int_0^\infty \frac{x^{1-\gamma}}{1+x}\,\mathrm{d}x\right). \tag{6.3}$$

For the rest, we treat separately the two cases $\alpha \in (1-\gamma^{-1}, 1/2]$ and $\alpha \in (1/2, 1)$, and use different methods for each. In both cases we replace the environment ω by a truncated version which ignores high values of ω. In order to know where to perform the truncation we need to fix a referential size for the system. For the rest of the computation we set

$$N = N_\beta := \begin{cases} c_1\beta^{-\frac{\gamma}{1-\gamma(1-\alpha)}}, & \text{if } \alpha \in (1/2,1), \\ c_1\left(\beta^2(1+|\log\beta|)\right)^{-\frac{\gamma}{2-\gamma}}, & \text{if } \alpha = 1/2, \\ c_1\beta^{-\frac{\gamma(1-\delta)}{1-\gamma(1-\alpha)}}, & \text{if } \alpha \in (1-\gamma^{-1}, 1/2), \end{cases} \tag{6.4}$$

where c_1 is a small positive constant, and $\delta > 0$ is chosen arbitrarily small. Of course we have to consider the integer part of the above but we chose to omit it to simplify the notation. Now we introduce $\widehat{\omega}^\beta$ given by

$$\widehat{\omega}_n^\beta := \min(\omega_n, N_\beta^{\gamma^{-1}}). \tag{6.5}$$

The truncation level is chosen so that $\widehat{\omega}_n^\beta$ is a reasonable approximation of ω when restricted to the segment $[\![1, N_\beta]\!]$; indeed the reader can check that ω coincides with $\widehat{\omega}^\beta$ on that segment with positive probability in the sense that

$$\inf_{\beta\in[0,1)} \mathbb{P}[\forall n \in [\![1,N_\beta]\!], \widehat{\omega}_n^\beta \neq \omega_n] > 0.$$

We also set

$$h_\beta := -\log \mathbb{E}[1+\beta\widehat{\omega}_1^\beta]. \tag{6.6}$$

The quantity h_β is chosen in such a way that the partition function associated to the truncated environment has expected value one (for all values of N);

$$\mathbb{E}[Z_{N,h_\beta}^{\widehat{\omega}^\beta,\beta,\mathrm{f}}] = 1. \tag{6.7}$$

As we have $h_\beta \sim \beta\mathbb{E}[\widehat{\omega}_1^\beta]$, and

$$\mathbb{E}[\widehat{\omega}_1^\beta] = \mathbb{E}[\omega_1 - \widehat{\omega}_1^\beta] = \mathbb{E}[(\omega_1 - N_\beta^{\gamma-1})_+],$$

using the expression for N_β and (2.15) we have

$$h_\beta \sim \begin{cases} c'_\alpha \beta^{\frac{\alpha\gamma}{1-\gamma(1-\alpha)}}, & \text{if } \alpha \in (1/2, 1), \\ c'_\alpha \beta^{\frac{\gamma}{2-\gamma}} |\log \beta|^{\frac{\gamma-1}{2-\gamma}}, & \text{if } \alpha = 1/2, \\ c'_\alpha \beta^{\frac{\alpha\gamma}{1-\gamma(1-\alpha)} - \frac{\delta(\gamma-1)}{1-\gamma(1-\alpha)}}, & \text{if } \alpha \in (1-\gamma^{-1}, 1/2), \end{cases} \tag{6.8}$$

where c'_α is a constant that depends on c_1 (in (6.4)) and γ.

The idea of our proof is to control either the second moment (if $\alpha \geq 1/2$) or the p-moment for some $p \in (1,2)$ (if $\alpha < 1/2$) of $Z^{\widehat{\omega}^\beta,\beta,\mathrm{f}}_{N,h_\beta}$ in order to be able to apply the results of Section 4.

6.1 The case $\alpha \geq 1/2$

The main thing we have to prove is that the second moment of the truncated partition function for the system of size N_β is bounded. With this result at hand, we perform the same computation as in Section 4.1 to conclude.

Lemma 6.3. *If c_1 is chosen sufficiently small, we have*

$$\sup_{\beta\in[0,1]} \mathbb{E}\left[\left(Z^{\widehat{\omega}^\beta,\beta,\mathrm{f}}_{N_\beta,h_\beta}\right)^2\right] \leq 2. \tag{6.9}$$

With our choice for h_β we have $\mathbb{E}[e^{h_\beta}(1+\beta\widehat{\omega}^\beta_n)] = 1$, so that h_β corresponds to the annealed critical point associated to the environment ω^β_n, Using the result above we can apply Lemma 4.3 for $p = 2$ and prove exactly as we proved Proposition 4.1, that there exist constants C and δ such that for every $u \geq 0$

$$\mathrm{f}(\beta, h_\beta + u) \geq \frac{1}{N_\beta}\mathbb{E}\left[\log Z^{\widehat{\omega}^\beta,\beta,\mathrm{f}}_{N_\beta,h_\beta+u}\right] - C\frac{\log N_\beta}{N_\beta} \geq \delta N_\beta^{\alpha-1} u - C'\frac{\log N_\beta}{N_\beta}. \tag{6.10}$$

Hence setting

$$u_\beta := C'' N_\beta^{-\alpha} \log N_\beta,$$

for a sufficiently large constant C'' we conclude that

$$\mathrm{f}(\beta, h_\beta + u_\beta) > 0 \text{ and } h_c(\beta) \leq h_\beta + u_\beta.$$

As for any value of $\alpha \in (1/2, 1)$, u_β is of a larger order magnitude than h_β, we conclude that (6.1) holds by replacing N_β by its value.

Proof of Lemma 6.3. Using the definition of h_β we readily obtain that

$$\mathbb{E}\left[\left(Z^{\widehat{\omega}^\beta,\beta,\mathrm{f}}_{N_\beta,h_\beta}\right)^2\right] = \mathbf{E}^{\otimes 2}\left[e^{\chi \sum_{n=1}^{N_\beta} \mathbf{1}_{\{n\in\tau^{(1)}\cap\tau^{(2)}\}}}\right], \tag{6.11}$$

where

$$\chi = \chi(\beta) := 2h_\beta + \log(\mathbb{E}[(1+\beta\widehat{\omega}^\beta_n)^2]) = \log(1 + e^{2h_\beta}\beta^2 \mathrm{V}ar(\widehat{\omega}_n)).$$

and $\tau^{(i)}$, $i = 1, 2$ are two IID renewal processes with distribution $\mathbf{P}$. Hence the quantity appearing in (6.11) is simply the partition function of the homogeneous pinning model associated to $\widetilde{\tau} := \tau^{(1)} \cap \tau^{(2)}$, which is a recurrent renewal process since $\alpha \geq 1/2$.

Repeating the computations performed in [8, Equations (6.24)–(6.27)], we obtain

$$\mathbb{E}\left[\left(Z^{\widehat{\omega}^\beta,\beta,\mathrm{f}}_{N_\beta,h_\beta}\right)^2\right] \leq 1 + \chi \sum_{k=1}^{N_\beta} \exp\left(k\left[\chi + \log \mathbf{P}^{\otimes 2}[\widetilde{\tau}_1 \leq N_\beta]\right]\right). \tag{6.12}$$

This is uniformly bounded by 2 if we can show that

$$\log \mathbf{P}^{\otimes 2}[\widetilde{\tau}_1 \leq N_\beta] \leq -2\chi. \tag{6.13}$$

We observe that the function

$$D(N) := \sum_{n=1}^{N} \mathbf{P}^{\otimes 2}[n \in \widetilde{\tau}] = \sum_{n=1}^{N} u(n)^2 \tag{6.14}$$

is regularly varying: according to (3.1) it is asymptotically equivalent to either $\log N$ (if $\alpha = 1/2$) or $N^{2\alpha-1}$ times a constant (if $\alpha \in (1/2, 1)$). Thus we can use [9, Theorem 8.7.3] which implies that

$$\mathbf{P}^{\otimes 2}[\widetilde{\tau}_1 > N] \overset{n\to\infty}{\sim} \frac{1}{\Gamma(2\alpha)\Gamma(2(1-\alpha))D(N)}. \tag{6.15}$$

This yields that for an appropriate constant, whose precise value is irrelevant to us, that

$$-\log \mathbf{P}^{\otimes 2}[\widetilde{\tau}_1 \leq N_\beta] \overset{\beta\to 0}{\sim} \begin{cases} C(K)(\log N_\beta)^{-1}, \alpha = 1/2 \\ C(K)N_\beta^{1-2\alpha}, \alpha \in (1/2, 1). \end{cases} \tag{6.16}$$

On the other hand from (2.15) we have

$$\chi(\beta) \overset{\beta\to 0}{\sim} \beta^2 \mathbb{V}ar(\widehat{\omega}_n) \overset{\beta\to 0}{\sim} \frac{2C_{\mathbb{P}}}{2-\gamma}\beta^2 N_\beta^{\frac{2-\gamma}{\gamma}}. \tag{6.17}$$

Replacing N_β by its value, we can check that (6.13) is satisfied for β sufficiently small if c_1 in (6.4) is chosen small enough. $\square$

Remark 6.4. When $\alpha < 1/2$, the method exposed above would also provide an upper bound on $h_c(\beta)$. As in that case the intersection of two independent renewals is a terminating renewal, one would have to choose N_β in a way such that $\chi(\beta)$ remains bounded by a small constant. However, this would not yield the right exponent for $h_c(\beta)$.

6.2 The case $\alpha \in (1-\gamma^{-1}, 1/2)$

In this other case we do not require the size of the system to be N_β. The important statement to prove is the following.

Lemma 6.5. *If $\beta \leq \beta_0$ and c_1 is chosen sufficiently small, there exists $p > 1$ such that*

$$\sup_{N\geq 0} \mathbb{E}\left[\left(Z_{N,h_\beta}^{\widehat{\omega}^\beta,\beta,\mathrm{f}}\right)^p\right] < \infty. \tag{6.18}$$

Following the steps of Section 4.1 we can deduce from Lemma 6.5 that for any $\beta \leqslant \beta_0$, any $u > 0$ and $N \geq 1$, we have

$$\mathrm{f}(\beta, h_\beta + u) \geq \frac{1}{N}\mathbb{E}\left[\log Z_{N,h_\beta+u}^{\widehat{\omega}^\beta,\beta,\mathrm{f}}\right] - C\frac{\log N}{N} \geq \delta N^{\alpha-1}u - C'\frac{\log N}{N}.$$

Choosing N sufficient large (depending on u), the r.h.s. becomes positive, which implies $h_c(\beta) \leqslant h_\beta$. This completes the proof of Proposition 6.1.

Proof of Lemma 6.5. We choose

$$p = \frac{1+\delta'}{1-\alpha} < 2,$$

where $\delta' > 0$ is to depend on δ (which enters in the definition of h_β) in a way which we determine later. In view of (5.26), it is sufficient to show that for some $L \in \mathbb{N}$

$$\left(\sum_{m\geq L} u(m)^p\right)\left(\sum_{n=0}^{\infty} u(n)^{2-p}\widetilde{\mathbb{E}}^\beta\left[\left(\widetilde{Z}_{n,h_\beta}^{L,\widetilde{\omega}^\beta}\right)^{p-1}\right]\right) < 1, \tag{6.19}$$

where $u(m)$ is defined in (3.1) and the tilted measure $\widetilde{\mathbb{P}}^\beta$ for the environment $\widetilde{\omega}^\beta$ is defined as a product measure on $\mathbb{R}^{\mathbb{Z}}$ whose marginal law is

$$\widetilde{\mathbb{P}}^\beta[\widetilde{\omega}^\beta \in \mathrm{d}x] = e^{h_\beta}(1+\beta x)\mathbb{P}[\widehat{\omega}^\beta \in \mathrm{d}x], \tag{6.20}$$

and the partition function $\widetilde{Z}^{L,\widetilde{\omega}^\beta}_{n,h_\beta}$ is similar to the one defined as in (5.22), but including h_β. More explicitly, recalling the definition of $\widetilde{\tau}$ (5.20), its definition is the following

$$\widetilde{Z}^{L,\widetilde{\omega}^\beta}_{n,h_\beta} = \widetilde{\mathbf{E}}^L\left[\prod_{i=0}^{n}\left(e^{h_\beta}(1+\beta\widetilde{\omega}_i^\beta\mathbf{1}_{\{i\in\widetilde{\tau}\}})+\mathbf{1}_{\{i\notin\widetilde{\tau}\}}\right)\right]. \tag{6.21}$$

In that case, one deduces from Lemma 3.1 that the first sum in (6.19) is smaller than

$$\sum_{m\geq L} u(m)^p \leq CL^{-\delta'} \tag{6.22}$$

which tends to zero when L tends to infinity. To control the second term we must prove that

$$\widetilde{\mathbb{E}}^\beta\left[\left(\widetilde{Z}^{L,\widetilde{\omega}^\beta}_{n,h_\beta}\right)^{p-1}\right] \leq Cu(n)^{2(p-1)}. \tag{6.23}$$

The key point is to show that given L, p and $\varepsilon > 0$, for β sufficiently small, we have

$$\mathbb{E}\left[\left[e^{h_\beta}(1+\beta\widehat{\omega}_1^\beta)\right]^p\right] \leq 1+\varepsilon, \tag{6.24}$$

as the rest follows exactly as in the proof of Lemma 5.4. First we notice that the e^{h_β} term can be neglected as it tends to one. As for the rest, since $p/2 < 1$ we have

$$(1+\beta\widehat{\omega}_1^\beta)^p = (1+2\beta\widehat{\omega}_1^\beta+(\beta\widehat{\omega}_1^\beta)^2)^{p/2} \leq 1+(2\beta|\widehat{\omega}_1^\beta|)^{p/2}+|\beta\widehat{\omega}_1^\beta|^p \tag{6.25}$$

The expectation of the second term is smaller than $(2\beta\mathbb{E}[|\widehat{\omega}_1|])^{p/2}$ and thus tends to zero when β tends to 0. As for the last term, using (2.15) and replacing N_β by its value, we obtain

$$\mathbb{E}\left[|\beta\widehat{\omega}_1^\beta|^p\right] \leq C\beta^p N_\beta^{\frac{p-\gamma}{\gamma}} = C\beta^{\frac{\delta}{1-\alpha}-\frac{\delta'\gamma}{1-\gamma(1-\alpha)}+\frac{\delta\delta'}{(1-\alpha)(1-\gamma(1-\alpha))}}. \tag{6.26}$$

The above exponent is positive if $\delta'(1-\alpha)\gamma \leq [1-\gamma(1-\alpha)]\delta$, which holds if δ' is sufficiently small. This allows to conclude that (6.24) holds if β is sufficiently small. ☐

7 Disorder relevance

7.1 Reduction to a fractional moment bound

The aim of this section is to prove that when $\alpha > 1-\gamma^{-1}$, there is a critical point shift whose magnitude is of order $\beta^{\frac{\alpha\gamma}{(\alpha-1)\gamma+1}}$ up to logarithmic corrections. To prove such a statement we adapt the technique used in [15] which combines the use of a finite volume criterion (Proposition 7.2) and a change of measure argument.

Although some refinements of this method involving a more delicate coarse graining procedure have been developed in the literature (introduced in [33] and developed in e.g. [22, 8]) and should yield a better estimate on $h_c(\beta)$, a lighter proof seemed more appropriate here. Let us set

$$h_\beta^{(2)} = \begin{cases} c_2\left(\frac{\beta}{|\log\beta|+1}\right)^{\frac{\alpha\gamma}{1-\gamma(1-\alpha)}}, & \text{if } \alpha \in (1-\gamma^{-1},1),\\ c_2\frac{\beta^\gamma}{(|\log\beta|+1)^{3\gamma-2}}, & \text{if } \alpha = 1,\\ c_2\left(\frac{\beta}{(|\log\beta|+1)}\right)^{\gamma}, & \text{if } \alpha > 1, \end{cases} \tag{7.1}$$

where c_2 is a positive constant whose value is to be specified later.

The conclusion we obtain is the following.

Proposition 7.1. *If $\alpha > 1 - \gamma^{-1}$, then $h_c(\beta) > 0$ for every $\beta \in (0,1)$. Moreover there exists a choice for the constant c_2 in* (7.1) *such that for all β*

$$h_c(\beta) \geqslant h_\beta^{(2)}. \tag{7.2}$$

A first important observation is that in order to control the free energy, it is sufficient to control the growth of fractional moments of the partition function. Indeed, we have, for any $\theta \in (0,1)$

$$\mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] = \frac{1}{\theta}\mathbb{E}\left[\log(Z_{N,h}^{\beta,\omega})^\theta\right] \leq \frac{1}{\theta}\log \mathbb{E}\left[(Z_{N,h}^{\beta,\omega})^\theta\right]. \tag{7.3}$$

Hence to show that $\mathrm{f}(\beta, h_\beta^{(2)}) = 0$, we only need to prove that

$$A_N := \mathbb{E}\left[\left(Z^{\beta,\omega}_{N,h_\beta^{(2)}}\right)^\theta\right] \tag{7.4}$$

is uniformly bounded (we set by convention $A_0 = 1$). We will do so by using a bootstrapping argument from [15] which shows that if the first k values $(A_1, \dots, A_k)$ are small enough then A_N is uniformly bounded. More precisely, given $\beta \in (0,1)$, $h \in \mathbb{R}$, $k \in \mathbb{N}$ and $\theta \in (0,1)$, we set

$$\rho(\beta,h,k,\theta) := \mathbb{E}\left[e^{\theta h}(1+\beta\omega_1)^\theta\right]\sum_{n=k}^{\infty}\sum_{j=0}^{k-1} K(n-j)^\theta A_j. \tag{7.5}$$

We have the following:

Proposition 7.2 (Proposition 2.5 in [15])**.** *Given h and $\beta \in (0,1)$, if we can find $k \in \mathbb{N}$ and $\theta \in (0,1)$ such that*

$$\rho(\beta,h,k,\theta) \leq 1,$$

then $\mathrm{f}(\beta,h) = 0$.

Proof. The proof relies on (5.16) and on a decomposition of the partition function $Z_{N,h}^{\beta,\omega}$ according to the position of the first contact point in the interval $[N-k+2, N]$ and the last one in the interval $[0,N]$. For full details we refer to [15]. ☐

For the rest of this section, we fix $h = h_\beta^{(2)}$,

$$k = \begin{cases} h^{-\frac{1}{\alpha}}, & \text{if } \alpha \in (1-\gamma^{-1}, 1), \\ h^{-1}|\log h|^{-2} & \text{if } \alpha = 1, \\ h^{-1} & \text{if } \alpha > 1. \end{cases} \tag{7.6}$$

and choose $\theta = 1 - (\log k)^{-1}$.

In this setup, to prove that ρ is smaller than one, it is enough to show that for all $j < k$, A_j is significantly smaller (that is by a large constant factor) than $u(j) := \mathbf{P}[j \in \tau]$. More precisely we need the following estimate

Lemma 7.3. *There exists $\eta(c_2) > 0$ which can be made arbitrarily small by choosing c_2 adequately such that for all $j \geq \eta k$:*

$$A_j \leq \eta u(j). \tag{7.7}$$

Moreover for c_2 sufficiently small we also have for all $j \leq \eta k$

$$A_j \leq 2e^2 u(j). \tag{7.8}$$

Proof of Proposition 7.1. Let us focus first on the case $\alpha \in (1-\gamma^{-1}, 1)$ for simplicity. By Jensen's inequality

$$\mathbb{E}\left[(e^h(\beta\omega_1+1))^\theta\right] \le e^{\theta h} \le 2, \tag{7.9}$$

provided c_2 is chosen small enough. Now using assumption (2.1), the definition of θ and usual comparisons between sums and integrals, we get

$$\begin{aligned}\sum_{n=k}^{\infty} K(n-j)^\theta \le C \int_{k-j}^{\infty} x^{-(1+\alpha)\theta}\,\mathrm{d}x &= \frac{C}{(1+\alpha)\theta-1}(k-j)^{1-(1+\alpha)\theta} \\ &\le C' \exp((1+\alpha)(1-\theta)\log(k-j))(k-j)^{-\alpha} \le C''(k-j)^{-\alpha}, \end{aligned} \tag{7.10}$$

where we used $\theta > \frac{1}{1+\alpha}$, which holds for k large enough, that is c_2 small enough.

Thus, using Lemma 7.3 and (3.1), we have

$$\rho \le C\sum_{j=0}^{k-1}(k-j)^{-\alpha}A_j \le C\eta \sum_{j=\eta k}^{k-1}(k-j)^{-\alpha} j^{\alpha-1} + C'\sum_{j=0}^{\eta k} j^{\alpha-1}(k-j)^{-\alpha}. \tag{7.11}$$

The reader can check via a Riemann sum approximation that the sum can be made arbitrarily small by choosing η small.

The proof in the case $\alpha \ge 1$ follows exactly the same line, we leave the details of the computation to the reader. □

7.2 Proof of Lemma 7.3

The idea is to use a change of measure similar to the one used in the proof of Lemma 5.4. A change of measure that penalizes environments which make the partition function anomalously large, since these environments might give a significant contribution to $\mathbb{E}[Z_{i,h}^{\beta,\omega}]$, and not to A_i. Given G such a penalization function, we have by Hölder's inequality

$$A_j \le \mathbb{E}[G(\omega)^{-\frac{\theta}{1-\theta}}]^{1-\theta}\left(\mathbb{E}[G(\omega)Z_{j,h}^{\beta,\omega}]\right)^\theta. \tag{7.12}$$

We choose to use a change of measure that penalizes high values of ω. As the exponent $\frac{\theta}{1-\theta}$ is large, we cannot choose a very high penalization. Thus we set

$$G(\omega) := \exp\left(-(\log k)^{-1}\sum_{i=1}^{k}\mathbf{1}_{\{\omega_i \ge k^{\gamma^{-1}}\}}\right) =: \prod_{i=1}^{k} g(\omega_i). \tag{7.13}$$

We are going to prove the two following results

Lemma 7.4. *There exists a constant C such that for any value of k,*

$$\mathbb{E}[G(\omega)^{-\frac{\theta}{1-\theta}}] \le C. \tag{7.14}$$

Lemma 7.5. *There exists $\eta(c_2)$, which can be chosen arbitrarily small if c_2 is chosen adequately, such that*

$$\mathbb{E}[G(\omega)Z_{j,h}^{\beta,\omega}] \le \left(\mathbf{1}_{\{j<\eta k\}} + \eta\mathbf{1}_{\{j\ge\eta k\}}\right)u(j). \tag{7.15}$$

Lemma 7.3 can be immediately deduced from (7.12), Lemma 7.4 and Lemma 7.5. Indeed from Lemma 7.4 the first term in (7.12) can be bounded by 2 if k is sufficiently large (that is c_2 sufficiently small), while the second term is smaller than

$$\left(\mathbf{1}_{\{j<\eta k\}} + \eta^\theta\mathbf{1}_{\{j\ge\eta k\}}\right)u(j)^\theta,$$

and we can conclude using the fact that if k is sufficiently large, from (3.1) and our choice for θ, for all $j \le k$

$$u(j)^\theta \leqslant e^2 u(j).$$

Proof of Lemma 7.4. Because of the assumption on the tail distribution of ω we have

$$\mathbb{P}\left[\omega_1 \geq k^{\gamma^{-1}}\right] \leq Ck^{-1}.$$

Hence

$$\mathbb{E}[G(\omega)^{-\frac{\theta}{1-\theta}}] \leq \mathbb{E}\left[e^{\sum_{j=1}^k \mathbf{1}_{\{\omega_1 \geq k^{\gamma^{-1}}\}}}\right] = \left(1+(e-1)\mathbb{P}\left[\omega_j \geq k^{\gamma^{-1}}\right]\right)^k \leq C'. \tag{7.16}$$

□

Proof of Lemma 7.5. Using the product structure of G, we have

$$\mathbb{E}[G(\omega)Z_{j,h}^{\beta,\omega}] := \mathbf{E}\left[\exp\left(-\sum_{i=1}^k \left(\eta_1 \mathbf{1}_{\{i\notin\tau\cap[1,j]\}}\right) + \eta_2 \mathbf{1}_{\{i\in\tau\cap[1,j]\}}\right)\right)\right], \tag{7.17}$$

where

$$\eta_1 := -\log \mathbb{E}\left[g(\omega_1)\right] \text{ and } \eta_2 := -\log \mathbb{E}\left[g(\omega_1)(1+\beta\omega_1)\right] - h. \tag{7.18}$$

Both η_1 and η_2 are positive for our choices of parameter (and this fact is sufficient to treat the case $j \leq \eta k$). For η_1 this is obvious as $g(\omega_1) \leq 1$. Concerning η_2 we note that

$$\mathbb{E}\left[g(\omega_1)(1+\beta\omega_1)\right] \leq 1 + \beta\mathbb{E}[g(\omega_1)\omega_1] \tag{7.19}$$

and as $\mathbb{E}[\omega_1] = 0$, we have

$$\begin{aligned}\mathbb{E}[g(\omega_1)\omega_1] &= \mathbb{E}[(g(\omega_1)-1)\omega_1] \\ &= \left(e^{-(\log k)^{-1}} - 1\right)\mathbb{E}\left[\omega_1 \mathbf{1}_{\{\omega_1 \geq k^{\gamma^{-1}}\}}\right] \leq -c(\log k)^{-1} k^{\frac{1-\gamma}{\gamma}},\end{aligned} \tag{7.20}$$

where c is a positive constant that only depends on the distribution of ω_1. Thus we have

$$\eta_2 \geq c\beta(\log k)^{-1} k^{\frac{1-\gamma}{\gamma}} - h. \tag{7.21}$$

Using our choice of parameters (recall (7.6)), we see that when c_2 is chosen sufficiently small, the r.h.s. in (7.21) is negative and that $\eta_2 \leq -\widetilde{h}$

$$\widetilde{h}(M,k) = \begin{cases} Mk^{-\min(\alpha,1)} & \text{if } \alpha \neq 1, \\ Mk^{-1}(\log k)^2 & \text{if } \alpha = 1, \end{cases} \tag{7.22}$$

where M can be chosen arbitrarily large if c_2 is chosen sufficiently small. Hence recalling (7.17), we have

$$\mathbb{E}[G(\omega)Z_{j,h}^{\beta,\omega}] \leq \mathbf{E}\left[e^{-\widetilde{h}\sum_{i=1}^j \delta_i}\delta_j\right] = u(j)\mathbf{E}\left[e^{-\widetilde{h}\sum_{i=1}^j \delta_i} \mid j \in \tau\right]. \tag{7.23}$$

To conclude, we need to show that the r.h.s. can be made much smaller than $u(j)$ for all $j \geq \eta k$ if M is chosen sufficiently large. To this purpose it is sufficient to use the following result (proved below).

Lemma 7.6. *We set*

$$\mathcal{A}(N,\varepsilon) = \begin{cases} \left\{\mathcal{N}_N(\tau) \leq \varepsilon N^{\min(\alpha,1)}\right\} & \text{if } \alpha \neq 1, \\ \left\{\mathcal{N}_N(\tau) \leq \varepsilon N(\log N)^{-2}\right\} & \text{if } \alpha = 1. \end{cases} \tag{7.24}$$

We have

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbf{P}\left[\mathcal{A}(N,\varepsilon) \mid N \in \tau\right] = 0. \tag{7.25}$$

For $\alpha \neq 1$ and $j \geq \eta k$, using (7.22), we obtain

$$\mathbf{E}\left[e^{-\widetilde{h}\sum_{i=1}^{j}\delta_i} \mid j \in \tau\right] \leq \mathbf{P}\left[\mathcal{A}(j,\varepsilon) \mid j \in \tau\right] + e^{-\varepsilon j^{\min(\alpha,1)}\widetilde{h}}. \tag{7.26}$$

If we choose ε sufficiently small and j sufficiently large, both terms can be made arbitrarily small (and the same argument works for $\alpha = 1$). □

Proof of Lemma 7.6. In the case $\alpha > 1$, the result is a consequence of the law of large numbers, since the renewal theorem (recall (3.1)) yields that $N \in \tau$ has a probability bounded away from zero. The case $\alpha < 1$ is a direct consequence of [15, Proposition A.3]. For the case $\alpha = 1$, we note that

$$\begin{aligned}\mathbf{P}\left[\mathcal{A}(N,\varepsilon)\ ;\ N \in \tau\right] &= \mathbf{P}\left[\sum_{i=1}^{\varepsilon N(\log N)^{-2}} (\tau_i - \tau_{i-1}) = N\right] \\ &= \mathbf{P}\left[\sum_{i=1}^{\varepsilon N(\log N)^{-2}} (\tau_i - \tau_{i-1})\mathbf{1}_{\{\tau_i - \tau_{i-1}\} \leq N\}} = N\right].\end{aligned} \tag{7.27}$$

Now we notice that

$$\mathbf{E}\left[\sum_{i=1}^{\varepsilon N(\log N)^{-2}} (\tau_i - \tau_{i-1})\mathbf{1}_{\{\tau_i - \tau_{i-1}\} \leq N\}}\right] = \varepsilon N(\log N)^{-2}\mathbf{E}\left[\tau_1 \mathbf{1}_{\{\tau_1 \leq N\}}\right] \leq C\varepsilon N \log N^{-1}. \tag{7.28}$$

Hence the probability in (7.27) is smaller than $C\varepsilon \log N^{-1}$ from Markov's inequality, and we can conclude using (3.1). □

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