Electronic Journal of Probability

Electron. J. Probab. **22** (2017), no. 27, 1–30.
ISSN: 1083-6489 DOI: 10.1214/17-EJP45

Asymptotics of self-similar growth-fragmentation processes

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Abstract

Markovian growth-fragmentation processes introduced in [8, 9] extend the pure-fragmentation model by allowing the fragments to grow larger or smaller between dislocation events. What becomes of the known asymptotic behaviors of self-similar pure fragmentations [6, 11, 12, 14] when growth is added to the fragments is a natural question that we investigate in this paper. Our results involve the terminal value of some additive martingales whose uniform integrability is an essential requirement. Dwelling first on the homogeneous case [8], we exploit the connection with branching random walks and in particular the martingale convergence of Biggins [18, 19] to derive precise asymptotic estimates. The self-similar case [9] is treated in a second part; under the so called Malthusian hypotheses and with the help of several martingale-flavored features recently developed in [10], we obtain limit theorems for empirical measures of the fragments.

Keywords: growth-fragmentation; self-similarity; additive martingale.
AMS MSC 2010: 60J25; 60G18; 60F15.
Submitted to EJP on June 13, 2016, final version accepted on March 6, 2017.
Supersedes arXiv:1606.00829.

1 Introduction

Fragmentation processes are meant to describe the evolution of an object which is subject to random and repeated dislocations over time. The way the mass is spread into smaller fragments during a dislocation event is usually given by a (random) *mass-partition*, that is an element of the space

$$\mathcal{P} := \left\{ \mathbf{p} := (p_i,\, i \in \mathbb{N})\colon p_1 \geq p_2 \geq \cdots \geq 0 \text{ and } \sum_{i=1}^{\infty} p_i \leq 1 \right\}, \tag{1.1}$$

where the total mass need not be conserved, i.e. a positive proportion $1 - \sum_{i\geq 1} p_i$ may disintegrate into dust. The first probabilistic models of fragmentations go back at least to Kolmogorov [28]. Roughly, Kolmogorov imagined a discrete branching system in

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which particles get fragmented according to a conservative distribution ν on $\mathcal{P}$ and in a *homogeneous* manner, that is to say the rate at which a particle splits does not depend on its mass. Under this essential assumption of homogeneity, Kolmogorov showed that a simple rescaling of the empirical measure of the logarithms of the fragments converges with probability one toward the Gaussian distribution. Later, a student of his, Filippov [24] investigated mass-dependent dislocation rates and more precisely the *self-similar* case, in the sense that a particle with size m splits at speed m^α for some fixed constant $\alpha \in \mathbb{R}$ (the homogeneous case then corresponds to $\alpha = 0$). Most notably he discovered a limit theorem for a weighted version of the empirical measure of the fragments when $\alpha > 0$. The special but common *binary* situation, where particles always split into two smaller fragments, has been emphasized by Brennan and Durrett [21, 22], and later reconsidered by Baryshnikov and Gnedin [5] in some variant of the car packing problem. Further extensions and other asymptotic properties in the non-conservative case have also been derived by Bertoin and Gnedin [11] by means of complex analysis and contour integrals.

In the 2000s (see [7, Chapters 1-3] for a comprehensive summary), Bertoin extended and theorized the construction of general fragmentation processes in continuous time. In particular the *dislocation measure* ν need no longer be a probability distribution, as there is only the integrability requirement

$$\int_{\mathcal{P}} (1-p_1)\, \nu(\mathrm{d}\mathbf{p}) < \infty. \tag{1.2}$$

While permitting infinite dislocation rates (so infinitely many dislocation events may occur in a bounded time interval), this condition prevents the total mass from being immediately shattered into dust and leads to a nondegenerate fragmentation process $\mathbf{X}(t) := (X_1(t), X_2(t), \ldots)$, $t \geq 0$, with values in $\mathcal{P}$. When $\alpha = 0$, fragmentation processes can be related (via a simple logarithmic transformation) to *branching random walks*, for which fruitful literature is available, see e.g. the works of Biggins and Uchiyama [18, 42, 19], and [40]. Especially, *additive martingales*, which are processes of the form

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i^q(t)\right]^{-1} \sum_{i=1}^{\infty} X_i^q(t), \qquad t \geq 0, \tag{1.3}$$

for some parameter $q > 0$, play a key role and the question of their uniform integrability inquired by Biggins has successfully led to the asymptotic behavior of homogeneous conservative fragmentations [13, 14]. More generally, in the self-similar case, some specific so called *Malthusian hypotheses* guarantee the existence of an intrinsic martingale associated with the fragmentation and whose convergence again yields many interesting asymptotic results. Among others the results of Kolmogorov and Filippov have been revisited [6], applying known statistics of self-similar Markov processes to the process of a randomly tagged fragment.

More recently, Bertoin [8, 9] introduced a new type of fragmentation processes in which the fragments are allowed to grow during their lifetimes. We expect that most of the aforementioned asymptotic properties extend to these *growth-fragmentation* processes, and it is the main purpose of the present work to derive some of them. We shall first give a bit more description and explain why our task is not completely straightforward. Like in the pure (i.e. without growth) setting, we are interested in the process which describes the (sizes of the) fragments as time passes. For homogeneous growth-fragmentations, namely the *compensated fragmentations* of [8], the basic prototype is simply a dilated homogeneous fragmentation, that is a pure homogeneous fragmentation affected by a deterministic exponential drift. However, there exist much more general compensated fragmentations, where the dislocation measure ν has only to fulfill

$$\int_{\mathcal{P}} (1-p_1)^2 \, \nu(\mathrm{d}\mathbf{p}) < \infty, \tag{1.4}$$

so that the process is nondegenerate and can still be encoded at any time by a non-increasing null sequence. Condition (1.4) is weaker than the necessary and sufficient condition (1.2) for ν to be the dislocation measure of a homogeneous fragmentation, and both are reminiscent of those concerning the jump intensities of Lévy processes, respectively subordinators. Incidentally, it was the main motivation of [8] to establish that, just like the Lévy–Itō construction of Lévy processes in terms of compensated Poisson integrals, compensated fragmentations naturally arise as limits of suitably dilated homogeneous fragmentations [8, Theorem 2]. Though asymptotic properties of pure homogeneous fragmentations immediately transfer to the dilated ones, extending them to general compensated fragmentations would correspond to interchanging two limits, which does not seem obvious at first sight. This is without to mention the self-similar case, that is for the growth-fragmentations in [9], where things look even more complicated.

There, and unlike the compensated fragmentations which are constructed directly as processes in time, the self-similar *cell systems* are rather built from a genealogical point of view: roughly, the (size of the) mother cell evolves like a Markov process on the positive half-line where each negative jump $-y$ is interpreted as a splitting event, giving birth to a daughter cell with initial size y and which then grows independently of the mother particle and according to the same dynamics, i.e. producing in turn great-daughters, and so on. Bertoin focused in particular on the situation where the associated growth-fragmentation process $\mathbf{X} := (\mathbf{X}(t),\, t \geq 0)$, that is the process recording the sizes of all alive cells in the system, fulfills a self-similarity property, namely when there exists $\alpha \in \mathbb{R}$ such that for each $x > 0$, the process $(x\mathbf{X}(x^{\alpha}t),\, t \geq 0)$ has the same law as $\mathbf{X}$ started from a cell whose initial size is x. In the homogeneous case $\alpha = 0$, these growth-fragmentations correspond to the compensated fragmentations of [8] for which the dislocation measure is binary, see [9, Proposition 3]. In the self-similar case $\alpha < 0$, they have been proved to be eventually extinct [9, Corollary 3], an observation which was already made by Filippov [24] in the context of pure fragmentations.

Both for homogeneous and for self-similar fragmentations, the additive martingales (1.3) and more precisely their uniform integrability have turned out to be of greatest importance in the study of asymptotic behaviors. We stress that sufficient conditions to this uniform integrability appear less easily for growth-fragmentations, as they non longer take values in the space of mass-partitions $\mathcal{P}$.

Our work is organized in two independent parts. In Section 2, we deal with the homogeneous case $\alpha = 0$ in the slightly more general setting of compensated fragmentations [8]. With the help of a well-known theorem due to Biggins [19] and by adapting arguments of Bertoin and Rouault [14], we prove the uniform convergence of additive martingales from which, in the realm of branching random walks, we infer precise estimates for the empirical measure of the fragments and the asymptotic behavior of the largest one. This part can be viewed as an application to the study of extremal statistics in certain branching random walks, see e.g. the recent developments by Aïdékon [2], Aïdékon et al. [1], Arguin et al. [3] and Hu et al. [25]. The self-similar case is considered in Section 3 within the framework of [9]. Relying on recent results in [10] and in particular on the uniform integrability of the Malthusian martingale, we establish for $\alpha > 0$ the convergence in probability of the empirical measure of the fragments and that of the largest fragment. In a concluding section we also address the convergence of another empirical measure where fragments are stopped as soon as they become smaller than a vanishing threshold.

2 Compensated fragmentations

2.1 Prerequisites

Recall the space of mass-partitions $\mathcal{P}$ defined in (1.1) and denote by $\mathcal{P}_1$ the subspace of mass-partitions $\mathbf{p} := (p_1, 0, \ldots) \in \mathcal{P}$ having only one single fragment $p_1 \in (0,1]$. A compensated fragmentation process $\mathbf{Z}(t) := (Z_1(t), Z_2(t), \ldots)$, $t \geq 0$, is a stochastic process whose distribution is characterized by a triple (σ^2, c, ν) where $\sigma^2 \geq 0$ is a diffusion coefficient, $c \in \mathbb{R}$ is a growth rate, and ν is a nontrivial measure on $\mathcal{P} \setminus \{(1,0,\ldots)\}$ such that (1.4) holds. It can be seen as the decreasing rearrangement of the exponential of the atoms of a branching process in continuous time. Namely, the process giving the empirical measure of the logarithms of the fragments at time t,

$$\mathcal{Z}^t := \sum_{i=1}^{\infty} \delta_{\log Z_i(t)},$$

is called a *branching Lévy process* in [8], to which we refer for background. In the basic case where $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, i.e. the fragmentation rates are finite, $\mathcal{Z}$ is a generalization of the branching random walk in continuous time introduced by Uchiyama [42]: more precisely, $\mathcal{Z}$ is a branching particle system in which each atom, during its lifetime, is allowed to move in $\mathbb{R}$ independently of the other atoms and according to the dynamics of a spectrally negative Lévy process[1] η with Laplace transform

$$E\big[\exp\big(q\eta(t)\big)\big] = \exp\big(t\psi(q)\big), \qquad t \geq 0,\ q \geq 0,$$

where under (1.4) the Laplace exponent[2]

$$\psi(q) := \frac{1}{2}\sigma^2 q^2 + \left(c + \int_{\mathcal{P}\setminus\mathcal{P}_1} (1-p_1)\,\nu(\mathrm{d}\mathbf{p})\right) q + \int_{\mathcal{P}_1} \big(p_1^q - 1 + q(1-p_1)\big)\,\nu(\mathrm{d}\mathbf{p}) \tag{2.1}$$

is finite for all $q \geq 0$. In words, when $\nu(\mathcal{P}\setminus\mathcal{P}_1) < \infty$, the system can be described as follows. It starts at the origin of space and time with a single particle which evolves like η. Each particle dies after a random exponential time with intensity $\nu(\mathcal{P} \setminus \mathcal{P}_1)$, giving birth to a random family of children $(\eta_1, \eta_2, \ldots)$ whose initial position $(\Delta a_1, \Delta a_2, \ldots)$ relative to the mother particle at its death is such that $(e^{\Delta a_1}, e^{\Delta a_2}, \ldots)$ has the conditional distribution $\nu(\,\cdot \mid \mathcal{P} \setminus \mathcal{P}_1)$.

In the general situation where the dislocation rate $\nu(\mathcal{P} \setminus \mathcal{P}_1)$ may be infinite, the construction is achieved by approximation from compensated fragmentations with finite dislocation rates, using a monotonicity argument (see [8, Lemma 3] recalled in the proof of Proposition 2.8 below).

Let us denote by

$$\mu(\mathrm{d}x) := \mathbb{E}[\mathcal{Z}^1(\mathrm{d}x)] \tag{2.2}$$

the mean intensity of the point process $\mathcal{Z}^1$, so that

$$m(q) := \int e^{qx}\mu(\mathrm{d}x), \qquad q \geq 0,$$

[1] That is a càdlàg stochastic process with stationary and independent increments which has only negative jumps. The results of this section could be quite straightforwardly adapted to also handle positive jumps in the particle motions; we shall however not do so as this would burden the expository and was anyway not considered in [8].

[2] Formula (2.1) is designed in such a way that if $\sigma^2 = 0$, $c = 0$, and $D := \int_{\mathcal{P}} (1-p_1)\,\nu(\mathrm{d}\mathbf{p}) < \infty$, then $\mathbf{Z}$ simply is a pure homogeneous fragmentation $\mathbf{X}$ with dislocation measure ν affected by a dilatation with coefficient D, i.e. $\mathbf{Z}(t) = e^{Dt}\mathbf{X}(t)$, $t \geq 0$. In this case η is a compound Poisson process with jump measure $(\log p_1)\,\nu_{|\mathcal{P}_1}(\mathrm{d}\mathbf{p})$ and drift D, but we stress that $\psi(q) < \infty$ holds in greater generality, namely under (1.4) and $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$. See [8] for details.

is the Laplace transform of μ. An important fact (cf. [8, Theorem 1]) is that, for every $t \geq 0$ and every $q \geq 0$,

$$m(q)^t \;=\; \mathbb{E}\left[\sum_{i=1}^{\infty} Z_i^q(t)\right] \;=\; \exp\bigl(t\kappa(q)\bigr), \tag{2.3}$$

where

$$\kappa(q) \coloneqq \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{P}} \left(\sum_{i=1}^{\infty} p_i^q - 1 + q(1-p_1)\right)\nu(\mathrm{d}\mathbf{p})$$

defines a convex function $\kappa\colon [0,\infty) \to (-\infty,\infty]$. We mention that under $\nu(\mathcal{P}\setminus\mathcal{P}_1) < \infty$, there is the identity

$$\kappa(q) = \psi(q) + \int_{\mathcal{P}\setminus\mathcal{P}_1} \left(\sum_{i=1}^{\infty} p_i^q - 1\right)\nu(\mathrm{d}\mathbf{p}), \qquad q \geq 0. \tag{2.4}$$

As we shall explain in the forthcoming Lemma 2.7, the first summand describes the motion of a particle, while the second outlines the branching mechanism. In better words, κ is merely the log-Laplace transform of the cloud of particles at first generation (i.e. after the first branching event), which is a key feature of branching random walks.

Since under (1.4),

$$p_1^q - 1 + q(1-p_1) = O\bigl((1-p_1)^2\bigr)$$

is integrable with respect to ν, we easily observe that, if we set

$$\underline{q} \coloneqq \inf\bigl\{q \geq 0\colon \kappa(q) < \infty\bigr\} = \inf\left\{q \geq 0\colon \int_{\mathcal{P}\setminus\mathcal{P}_1} \sum_{i=2}^{\infty} p_i^q\,\nu(\mathrm{d}\mathbf{p}) < \infty\right\},$$

then κ takes finite values and is analytic on the open interval $(\underline{q},\infty)$. Note that (1.4) also implies $\kappa(2) < \infty$, so $\underline{q} \leq 2$. Let us introduce the subspace

$$\ell^{q\downarrow} \coloneqq \left\{\mathbf{z} \coloneqq (z_1, z_2, \ldots)\colon z_1 \geq z_2 \geq \cdots \geq 0 \text{ and } \sum_{i=1}^{\infty} z_i^q < \infty\right\}$$

of the space ℓ^q of q-summable sequences endowed with the distance $\|\mathbf{z}-\mathbf{z}'\|_{\ell^q}^q \coloneqq \sum_{i=1}^{\infty}|z_i - z_i'|^q$. We also denote $\ell^{\infty\downarrow}$ the space of bounded, non-increasing sequences of nonnegative real numbers endowed with the uniform norm $\|\cdot\|_{\ell^\infty}$. We see by (2.3) that the compensated fragmentation $\mathbf{Z} \coloneqq (\mathbf{Z}(t),\, t \geq 0)$ is a $\ell^{q\downarrow}$-valued process for every $q \in (\underline{q},\infty]$, and in particular for $q = 2$. Further if $\mathbf{z} \coloneqq (z_1, z_2, \ldots)$ is in $\ell^{2\downarrow}$ and $\mathbf{Z}^{[1]}, \mathbf{Z}^{[2]}, \ldots$ are independent copies of $\mathbf{Z}$, then the process of the family $(z_j Z_i^{[j]}(t),\, i,j \in \mathbb{N})$, $t \geq 0$, rearranged in the non-increasing order is again in $\ell^{2\downarrow}$, and we denote its distribution by $\mathbb{P}_{\mathbf{z}}$. It has been proved in [8] that $(\mathbf{Z}, (\mathbb{P}_{\mathbf{z}})_{\mathbf{z}\in\ell^{2\downarrow}})$ is a Markov process which fulfills the so called *branching property*: for all $s \geq 0$, the conditional law of $(\mathbf{Z}(t+s))_{t\geq 0}$ given $(\mathbf{Z}(r))_{0\leq r\leq s}$ is $\mathbb{P}_{\mathbf{z}}$, where $\mathbf{z} = \mathbf{Z}(s)$. Without loss of generality we shall assume in the sequel that the fragmentation starts with a single mass with unit size, i.e. $\mathbb{P} \coloneqq \mathbb{P}_{(1,0,\ldots)}$.

Equation (2.3) and the branching property yield an important family of additive martingales. Namely, the $\mathbb{R}$-valued process

$$M(t;q) \coloneqq \exp\bigl(-t\kappa(q)\bigr)\sum_{i=1}^{\infty} Z_i^q(t), \qquad t \geq 0, \tag{2.5}$$

is a martingale for every $q \in (\underline{q},\infty)$. As a first consequence [8, Proposition 2], the compensated fragmentation $\mathbf{Z}$ possesses a càdlàg version in $\ell^{2\downarrow}$, that is a version in the Skorokhod space $D([0,\infty),\ell^{2\downarrow})$ of right continuous with left limits, $\ell^{2\downarrow}$-valued functions. Working with such a version from now on, $\mathbf{Z}$ has actually càdlàg paths in $\ell^{q\downarrow}$ for every $q \in (\underline{q},\infty]$.

Proposition 2.1. *Almost surely, for every $q \in (\underline{q}, \infty]$, $\mathbf{Z}$ has càdlàg paths in $\ell^{q\downarrow}$.*

Proof. Recall that $\|\cdot\|_{\ell^{q'}} \le \|\cdot\|_{\ell^q}$ whenever $q \le q' \le \infty$. Since $\mathbf{Z}$ has càdlàg paths in $\ell^{2\downarrow}$, it has in particular càdlàg paths in $\ell^{\infty\downarrow}$. Let $(q_k,\, k \in \mathbb{N})$ be a sequence decreasing to $\underline{q}$, and define

$$T_m^{(k)} := \inf\left\{t \ge 0\colon M(t; q_k) > m\right\} = \inf\left\{t \ge 0\colon \|\mathbf{Z}(t)\|_{\ell^{q_k}}^{q_k} > m\, e^{t\kappa(q_k)}\right\}$$

for $k, m \in \mathbb{N}$. Applying Doob's maximal inequality to the martingale (2.5) we have that almost surely, for all $k \in \mathbb{N}$, $T_m^{(k)} \uparrow \infty$ as $m \to \infty$. Thus, almost surely, for every $q \in (\underline{q}, \infty]$ and $T \ge 0$ we can find a $k \in \mathbb{N}$ such that $\underline{q} < q_k < q$ and then a $m \in \mathbb{N}$ such that $T < T_m^{(k)}$, whence

$$\begin{aligned}\|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^q}^q &\le \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^\infty}^{q-q_k} \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^{q_k}}^{q_k} \\ &\le m\, 2^{1+q_k}\left(1 + e^{T_m^{(k)}\kappa(q_k)}\right) \|\mathbf{Z}(t) - \mathbf{Z}(s)\|_{\ell^\infty}^{q-q_k}\end{aligned}$$

for all $0 \le s, t < T$. The fact that $\mathbf{Z}$ has càdlàg paths in $\ell^{\infty\downarrow}$ completes the proof. □

We first would like to extend to the compensated fragmentation $\mathbf{Z}$ the asymptotic results obtained by Bertoin and Rouault [13, 14] for pure homogeneous fragmentations. They strongly rely on the work of [18, 19] about the uniform integrability of additive martingales. Essentially, the martingales $(M(t;q))_{t\ge 0}$ will be uniformly integrable if $q\kappa'(q) - \kappa(q) < 0$ and $M(1;q) \in \mathrm{L}^\gamma(\mathbb{P})$ for some $\gamma > 1$. With this in mind, let us introduce

$$\bar{q} := \sup\left\{q > \underline{q}\colon q\kappa'(q) - \kappa(q) < 0\right\}.$$

First note that $\bar{q} < \infty$, because

$$q\kappa'(q) - \kappa(q) = \frac{1}{2}\sigma^2 q^2 + \int_{\mathcal{P}} \left(1 - p_1^q(1 - \log p_1^q)\right) \nu(\mathrm{d}\mathbf{p}) - \int_{\mathcal{P}\setminus\mathcal{P}_1} \sum_{i=2}^\infty p_i^q \left(1 - \log p_i^q\right) \nu(\mathrm{d}\mathbf{p}),$$

which, by Fatou's lemma, is at least $\nu(\mathcal{P})$ as $q \to \infty$. Second, we have $\bar{q} > \underline{q}$ as soon as $q\kappa'(q) - \kappa(q) < 0$ for some $q \ge 0$ such that $\kappa(q) < \infty$ (e.g. for $q = 2$), which is realized when

$$\int_{\mathcal{P}\setminus\mathcal{P}_1} \sum_{i=2}^\infty p_i^q \left(1 - \log p_i^q\right) \nu(\mathrm{d}\mathbf{p}) \ge \left(\frac{1}{2}\sigma^2 + \int_{\mathcal{P}} (1 - p_1)^2\, \nu(\mathrm{d}\mathbf{p})\right) q^2.$$

We distinguish two different regimes for the function $q \mapsto \kappa(q)/q$:

Lemma 2.2. *The function $q \mapsto \kappa(q)/q$ is decreasing on $(\underline{q}, \bar{q})$ and increasing on $(\bar{q}, \infty)$. Further, $\bar{q}\kappa'(\bar{q}) = \kappa(\bar{q})$ when $\bar{q} > \underline{q}$.*

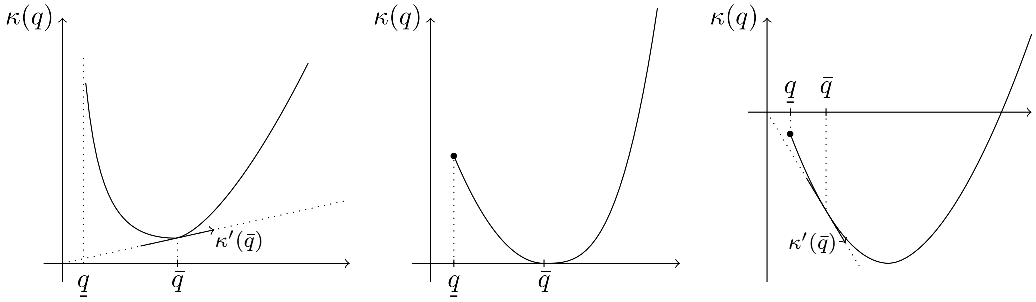
In the context of branching random walks, the value $\kappa'(\bar{q})$ is the asymptotic velocity of the maximal displacement $\log Z_1(t)$; see Figure 1 and Section 2.3.

Proof. On the one hand,

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\frac{\kappa(q)}{q}\right] = \frac{q\kappa'(q) - \kappa(q)}{q^2}.$$

On the other hand, the map $q \mapsto q\kappa'(q) - \kappa(q)$ is increasing on $(\underline{q}, \infty)$ since κ is convex, so it has at most one sign change, occurring at $\bar{q}$ if $\bar{q} > \underline{q}$. □

Our main result provides sufficient conditions for the convergence of the martingales $(M(t;q))_{t\ge 0}$ uniformly in $q \in (\underline{q}, \bar{q})$, both almost surely and in $\mathrm{L}^1(\mathbb{P})$. Most of the coming section is devoted to a precise statement and a proof. As consequences, we ascertain the convergence of a rescaled version of the empirical measure $\mathcal{Z}^t$ and in Section 2.3 we expand on the asymptotic behavior of the largest fragment. One last application is exposed in Section 2.4.



(a) Positive velocity. (b) Null velocity. (c) Negative velocity.

Figure 1: The cumulant function κ, the points $\underline{q}, \bar{q}$, and the velocity $\kappa'(\bar{q}) = \kappa(\bar{q})/\bar{q}$.

2.2 Uniform convergence of the additive martingales

In the remaining of Section 2 we will make, in addition to (1.4), the assumption that the dislocation measure ν fulfills

$$\kappa(0) \in (0, \infty], \tag{2.6}$$

and, for all $\underline{q} < q < 1$,

$$\nu_{|\mathcal{P}\setminus\mathcal{P}_1}\left(\sum_{i=1}^{\infty} p_i^q < 1\right) < \infty. \tag{2.7}$$

Condition (2.6) holds e.g. when $\nu(p_2 > 0) > \nu(p_1 = 0)$ and merely rephrases that the mean number $\mu(\mathbb{R}) = m(0)$ of offspring of particles is greater than 1, i.e. the branching process $\mathcal{Z}$ is supercritical. This implies that the *non-extinction* event $\{\forall t \geq 0,\ Z_1(t) > 0\}$ occurs with positive probability. Condition (2.7) is just a minor technical requirement for the possible values $q < 1$ and is fulfilled in many situations: when $\underline{q} \geq 1$, when $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, or more importantly when the measure ν is *conservative*, i.e. $\sum_{i\geq 1} p_i = 1$ for ν-almost every $\mathbf{p} \in \mathcal{P}$. Observe also that in the conservative case, $\underline{q} < 1$ is possible only if (1.2) holds, i.e. $\mathbf{Z}$ is essentially a dilated pure fragmentation.

We may now state:

Theorem 2.3. *Suppose* (2.6) *and* (2.7)*. Then the following assertions hold almost surely:*

(i) On $(\underline{q}, \bar{q})$, $M(t;\cdot)$ converges locally uniformly as $t \to \infty$. More precisely, there exists a random continuous function $M(\infty;\cdot)\colon (\underline{q}, \bar{q}) \to [0, \infty)$ such that, for any compact subset K of $(\underline{q}, \bar{q})$,

$$\lim_{t\to\infty} \sup_{q\in K} |M(t;q) - M(\infty;q)| = 0,$$

and this convergence also holds in mean. Furthermore for every $q \in (\underline{q}, \bar{q})$, $M(\infty;q) > 0$ conditionally on non-extinction.

(ii) For every $q \in [\bar{q}, \infty)$,

$$\lim_{t\to\infty} M(t;q) = 0.$$

As a first important consequence, we derive uniform estimates for the empirical measure of the fragments, which echo those determined by Bertoin and Rouault [14, Corollary 3]. We shall assume here that the mean intensity measure μ in (2.2) is *non-lattice*, in that it is not supported on $r\mathbb{Z} + s$ for any $r > 0,\ s \in \mathbb{R}$.

Corollary 2.4. *Suppose* (2.6), (2.7), *and* μ *non-lattice. Then for any Riemann integrable function* $f\colon (0,\infty)\to\mathbb{R}$ *with compact support and for all compact subset* K *of* $(\underline{q},\bar{q})$,

$$\lim_{t\to\infty}\sqrt{t}\,e^{-\left(\kappa(q)-q\kappa'(q)\right)t}\sum_{i=1}^{\infty}f\Big(Z_i(t)\,e^{-\kappa'(q)t}\Big) = \frac{M(\infty;q)}{\sqrt{2\pi\kappa''(q)}}\int_0^{\infty}\frac{f(y)}{y^{q+1}}\,\mathrm{d}y$$

uniformly in $q\in K$, *almost surely.*

Remark 2.5. We stress that condition (2.7) is unnecessary if we only deal with $q\geq 1$. In particular it may be removed from the above statements provided that we replace $\underline{q}$ by $\underline{q}\vee 1$.

Before proving these two results, let us give a quick summary on the sizes of particles in a compensated fragmentation. On the one hand, it is easy (see e.g. [7, Corollary 1.4]) to derive from Theorem 2.3.(i) that in the first order, the largest particle $Z_1(t)$ evolves like $e^{\kappa'(\bar{q})t}$ as $t\to\infty$, and we will have a look at the second and third asymptotic orders in Section 2.3. On the other hand, Corollary 2.4 provides the local density of particles at intermediate scales: if $\kappa'(\underline{q})<a<\kappa'(\bar{q})$ and $\kappa^*(a)\coloneqq\kappa(q)-q\kappa'(q)$ for $\kappa'(q)\coloneqq a$, then

$$\lim_{t\to\infty}\frac{1}{t}\log\#\big\{i\in\mathbb{N}\colon e^{at-\varepsilon}\leq Z_i(t)\leq e^{at+\varepsilon}\big\} = \kappa^*(a)$$

for every $\varepsilon>0$, almost surely (just take $f(x)\coloneqq\mathbb{1}_{[-\varepsilon,\varepsilon]}(\log x)$ above). Lastly, we shall observe in Section 2.4 that fragments at untypical levels $a>\kappa'(\bar{q})$ appear with a probability that is roughly of the same order as their expected number (Corollary 2.13.(ii)).

Theorem 2.3 is essentially a version of a theorem of Biggins [19] in the context of compensated fragmentations. In this respect, one important requirement to derive part (i) is that $\mathbb{E}[M(1;q)^{\gamma}]<\infty$ for some $\gamma>1$. We start with a lemma controlling the finiteness of

$$W^{\gamma}_{\nu,q}\coloneqq\int_{\mathcal{P}\setminus\mathcal{P}_1}\left|1-\sum_{i=1}^{\infty}p_i^q\right|^{\gamma}\nu(\mathrm{d}\mathbf{p}).$$

Lemma 2.6. *Let* $q>\underline{q}$ *and suppose either* (2.7) *or* $q\geq 1$. *Then* $W^{\gamma}_{\nu,q}<\infty$ *for some* $\gamma\in(1,2]$.

Proof. Suppose first $q\geq 1$. Then for $\gamma\coloneqq 2$ and for all $\mathbf{p}\in\mathcal{P}$,

$$0\leq\left(1-\sum_{i=1}^{\infty}p_i^q\right)^2\leq(1-p_1^q)^2\leq q^2\,(1-p_1)^2$$

(the last inequality resulting from the convexity of $x\mapsto x^q$), so $W^2_{\nu,q}<\infty$ by (1.4). Suppose now $q<1$. Then

$$W^{\gamma}_{\nu,q}\leq\nu_{|\mathcal{P}\setminus\mathcal{P}_1}\left(\sum_{i=1}^{\infty}p_i^q<1\right)+\int_{\mathcal{P}\setminus\mathcal{P}_1}\mathbb{1}_{\left\{\sum\limits_{i=1}^{\infty}p_i^q\geq 1\right\}}\left(\sum_{i=1}^{\infty}p_i^q-1\right)^{\gamma}\nu(\mathrm{d}\mathbf{p}).$$

Under (2.7), $W^{\gamma}_{\nu,q}$ is finite as soon as the latter integral is finite. But by Jensen's inequality, the integrand is bounded from above by

$$\left(\sum_{i=2}^{\infty}p_ip_i^{q-1}\right)^{\gamma}\leq\sum_{i=2}^{\infty}p_i^{1+\gamma(q-1)},$$

which is ν-integrable if $1+\gamma(q-1)=q-(\gamma-1)(1-q)>\underline{q}$, i.e. provided $\gamma\in(1,2]$ is close enough to 1. $\square$

We then derive an upper bound for $\mathbb{E}[M(t;q)^\gamma]$ in terms of $W^\gamma_{\nu,q}$:

Lemma 2.7. *Suppose that $\nu(\mathcal{P}\setminus\mathcal{P}_1)<\infty$. Then for every $q\in(\underline{q},\infty)$, $\gamma\in(1,2]$ and $t\geq 0$,*

$$\mathbb{E}[M(t;q)^\gamma]\leq\ \mathfrak{c}_\gamma\,W^\gamma_{\nu,q}\,f\big(t,\psi(\gamma q)-\gamma\psi(q),\kappa(\gamma q)-\gamma\kappa(q)\big), \tag{2.8}$$

where ψ is given by (2.1)*, $f(t,x,y):=(e^{tx}-e^{ty})/(x-y)$, and $\mathfrak{c}_\gamma$ is a finite constant depending only on γ.*

Proof. Lemma 2 in [8] states that the branching Lévy process $\mathcal{Z}$ can be obtained by superposing independent spatial Lévy motions to a "steady" branching random walk. Specifically, for each $t\geq 0$,

$$\mathbf{Z}(t)\overset{d}{=}\big(e^{\beta_1}X_1(t),e^{\beta_2}X_2(t),\ldots\big),$$

where $\mathbf{X}(t):=(X_1(t),X_2(t),\ldots)$ are the atoms at time t of a homogeneous fragmentation $\mathbf{X}$ with dislocation measure $\nu_{|\mathcal{P}\setminus\mathcal{P}_1}$ and $(\beta_i)_{i\in\mathbb{N}}$ is an independent sequence of i.i.d. random variables with Laplace transform $\mathbb{E}[\exp(q\beta_i)]=\exp(t\psi(q))$, $q\geq 0$. Applying Jensen's inequality and conditioning on $\mathbf{X}(t)$ produce

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{i=1}^\infty Z_i^q(t)\right)^\gamma\right] &= \mathbb{E}\left[\left(\sum_{j=1}^\infty X_j^q(t)\right)^\gamma\left(\sum_{i=1}^\infty e^{q\beta_i}\frac{X_i^q(t)}{\sum_j X_j^q(t)}\right)^\gamma\right]\\ &\leq \mathbb{E}\left[\left(\sum_{j=1}^\infty X_j^q(t)\right)^{\gamma-1}\sum_{i=1}^\infty e^{\gamma q\beta_i}X_i^q(t)\right]\\ &= \exp\big(t\psi(\gamma q)\big)\,\mathbb{E}\left[\left(\sum_{i=1}^\infty X_i^q(t)\right)^\gamma\right].\end{aligned} \tag{2.9}$$

We now recall from [6] (see the proof of its Theorem 2) how to estimate the latter expectation. Denoting

$$\phi(q):=\int_{\mathcal{P}\setminus\mathcal{P}_1}\left(\sum_{i=1}^\infty p_i^q-1\right)\nu(\mathrm{d}\mathbf{p})<\infty,$$

the process

$$N(t;q):=\exp\big(-t\phi(q)\big)\sum_{i=1}^\infty X_i^q(t),\qquad t\geq 0, \tag{2.10}$$

is a purely discontinuous martingale. It is then deduced from an inequality of Burkholder–Davis–Gundy that

$$\mathbb{E}[N(t;q)^\gamma]\leq\mathfrak{c}_\gamma\,\mathbb{E}[V^\gamma(t;q)],$$

where $\mathfrak{c}_\gamma<\infty$ is some constant, and V^γ is the γ-variation process of N:

$$V^\gamma(t;q):=\sum_{0<s\leq t}|N(s;q)-N(s-;q)|^\gamma.$$

Since in this setting

$$|N(s;q)-N(s-;q)|^\gamma=\exp\big(-s\gamma\phi(q)\big)X_k^{\gamma q}(s-)\left|1-\sum_{i=1}^\infty p_i^q\right|^\gamma,$$

$(s, \mathbf{p}, k) \in (0,t] \times \mathcal{P} \times \mathbb{N}$ being the atoms of a Poisson random measure with intensity $\mathrm{d}t \otimes \nu_{|\mathcal{P}\setminus\mathcal{P}_1} \otimes \sharp$, it follows that V^γ has predictable compensator

$$\left(\int_{\mathcal{P}\setminus\mathcal{P}_1} \left|\sum_{i=1}^\infty 1 - p_i^q\right|^\gamma \nu(\mathrm{d}\mathbf{p})\right) \int_0^t \exp\bigl(-s\gamma\phi(q)\bigr) \sum_{i=1}^\infty X_i^{\gamma q}(s)\,\mathrm{d}s,$$

and therefore

$$\mathbb{E}[N(t;q)^\gamma] \le \mathfrak{c}_\gamma\, W_{\nu,q}^\gamma\, f\bigl(t, 0, \phi(\gamma q) - \gamma\phi(q)\bigr). \tag{2.11}$$

Now recall (2.5), (2.10) and the identity $\phi(q) + \psi(q) = \kappa(q)$ already observed in (2.4). Multiplying (2.9) by $e^{-t\gamma\kappa(q)}$ and then reporting the bound (2.11), we end up with

$$\mathbb{E}[M(t;q)^\gamma] \le \mathfrak{c}_\gamma\, W_{\nu,q}^\gamma\, f\bigl(t, \psi(\gamma q) - \gamma\psi(q), \kappa(\gamma q) - \gamma\kappa(q)\bigr),$$

as desired. □

Putting the previous results together now yields:

Proposition 2.8. *Let $q > \underline{q}$ and suppose either (2.7) or $q \ge 1$. Then there exists $\gamma \in (1,2]$ such that $M(t;q) \in \mathrm{L}^\gamma(\mathbb{P})$ for all $t \ge 0$.*

Proof. By Lemma 2.6 we can choose $\gamma \in (1,2]$ such that $W_{\nu,q}^\gamma < \infty$. Let us first assume $\nu(\mathcal{P} \setminus \mathcal{P}_1) < \infty$, so that we may apply Proposition 2.7. Note that

$$\psi(\gamma q) - \gamma\psi(q) = \frac{1}{2}\sigma^2(\gamma - 1)\gamma q^2 + \int_{\mathcal{P}_1} (p_1^{\gamma q} - \gamma p_1^q + \gamma - 1)\,\nu(\mathrm{d}\mathbf{p}),$$

with

$$0 \le p_1^{\gamma q} - \gamma p_1^q + \gamma - 1 = O\bigl((1-p_1)^2\bigr).$$

Then the inequality (2.8) is

$$\mathbb{E}[M(t;q)^\gamma] \le\; \mathfrak{c}_\gamma\, W_{\nu,q}^\gamma\, f\left(t, \frac{1}{2}\sigma^2(\gamma-1)\gamma q^2 + \int_{\mathcal{P}_1} (p_1^{\gamma q} - \gamma p_1^q + \gamma - 1)\,\nu(\mathrm{d}\mathbf{p}), \kappa(\gamma q) - \gamma\kappa(q)\right),$$

where f is a continuous function. This bound is finite for each $t \ge 0$, and we shall show by approximation that this also holds when $\nu(\mathcal{P} \setminus \mathcal{P}_1) = \infty$. The measures $\nu^{(b)}$, images of ν by the maps

$$\mathbf{p} \longmapsto \bigl(p_1, p_2 \mathbb{1}_{\{p_2 > e^{-b}\}}, p_3 \mathbb{1}_{\{p_3 > e^{-b}\}}, \ldots\bigr), \qquad b > 0,$$

define a consistent family of dislocation measures such that $\nu^{(b)}(\mathcal{P} \setminus \mathcal{P}_1) < \infty$. Thanks to [8, Lemma 3] we can consider that $\mathbf{Z}$ arises from the inductive limit $\mathbf{Z} := \lim\!\uparrow \mathbf{Z}^{(b)}$ as $b \uparrow \infty$, where the $\mathbf{Z}^{(b)}$, $b > 0$, are suitably embedded compensated fragmentations with characteristics $(\sigma^2, c, \nu^{(b)})$. With obvious notations, we deduce from the monotone convergence theorem that $\mathbb{E}[\|\mathbf{Z}^{(b)}(t)\|_{\ell^q}^{\gamma q}] \to \mathbb{E}[\|\mathbf{Z}(t)\|_{\ell^q}^{\gamma q}]$ as $b \to \infty$, and from the dominated convergence theorem that $\kappa^{(b)}(q) \to \kappa(q)$ and $W_{\nu^{(b)},q}^\gamma \to W_{\nu,q}^\gamma$ (working like in the proof of Lemma 2.6). The proof is then completed by Fatou's lemma. □

We are finally ready to tackle the proof of the convergence of the martingale $(M(t;q))_{t\ge0}$.

Proof of Theorem 2.3. Since $(M(t;q))_{t\ge0}$ is a nonnegative càdlàg martingale, its limit $M(\infty;q)$ as $t \to \infty$ exists almost surely. If $q \in [\bar{q}, \infty)$, then $q\kappa'(q) - \kappa(q) \ge 0$ so that condition (3.3) in [18] fails, and therefore $M(\infty;q) = 0$. This proves (ii). For (i), we follow the lines of [14]. From Proposition 2.1 we know that almost surely, for every $t \geq 0$ and $(t_n,\, n \in \mathbb{N})$ such that $t_n \downarrow t$ as $n \to \infty$, the sequence of random functions on K

$$q \mapsto \big(1 + Z_1(t_n)\big)^{-q} \sum_{i=1}^{\infty} Z_i^q(t_n), \qquad n \in \mathbb{N},$$

which all are non-increasing because of the leading factors $(1 + Z_1(t_n))^{-q}$, converges pointwise to the random continuous function

$$q \mapsto \big(1 + Z_1(t)\big)^{-q} \sum_{i=1}^{\infty} Z_i^q(t).$$

By a classical counterpart of Dini's theorem (see e.g. [39, Problem II.3.127]), the convergence is actually uniform in $q \in K$. Multiplying by the continuous function $q \mapsto (1 + Z_1(t))^q \exp(-t\kappa(q))$ and dealing similarly with the left limits of $\mathbf{Z}$, we can therefore view $(M(t;\cdot))_{t\geq 0}$ as a martingale with càdlàg paths in the Banach space $\mathcal{C}(K, \mathbb{R})$ of continuous functions on K.

We now observe that the process

$$\mathcal{Z}^n = \sum_{i=1}^{\infty} \delta_{\log Z_i(n)}, \qquad n \in \mathbb{N},$$

is a branching random walk (in discrete time), and check the two conditions to apply the results of Biggins [19, Theorems 1 & 2]: first, if $q \in (\underline{q}, \bar{q})$ then by Proposition 2.8 we have $\mathbb{E}[M(1;q)^\gamma] < \infty$ for some $\gamma \in (1, 2]$; second, using Lemma 2.2 we can find $\alpha \in (1, \gamma]$ such that $\alpha q \in (\underline{q}, \bar{q})$, hence

$$\frac{m(\alpha q)}{m(q)^\alpha} = \exp\left\{\alpha q \left(\frac{\kappa(\alpha q)}{\alpha q} - \frac{\kappa(q)}{q}\right)\right\} < 1. \tag{2.12}$$

It thus follows that the $\mathcal{C}(K, \mathbb{R})$-valued discrete-time martingale

$$M(n;\cdot)\colon q \mapsto \exp\big(-n\kappa(q)\big) \int e^{qx}\, \mathcal{Z}^n(\mathrm{d}x), \qquad n \in \mathbb{N},$$

converges as $n \to \infty$ to a random function $M(\infty;\cdot) \in \mathcal{C}(K, \mathbb{R})$, almost surely and in mean. Now, the uniform norm $\|\cdot\|$ of $\mathcal{C}(K, \mathbb{R})$ is a convex map and thus for any integer $n \geq 0$ the process $(\|M(t;\cdot) - M(n;\cdot)\|)_{t\geq n}$ is a nonnegative submartingale with càdlàg paths. If $t \geq 0$ and n is chosen so that $n \leq t < n+1$, we have in particular

$$\mathbb{E}[\|M(t;\cdot) - M(n;\cdot)\|] \leq \mathbb{E}[\|M(n+1;\cdot) - M(n;\cdot)\|],$$

and consequently

$$\mathbb{E}[\|M(t;\cdot) - M(\infty;\cdot)\|] \leq \mathbb{E}[\|M(n+1;\cdot) - M(n;\cdot)\|] + \mathbb{E}[\|M(n;\cdot) - M(\infty;\cdot)\|].$$

The convergence in $\mathrm{L}^1(\mathbb{P})$ of the continuous-time martingale $(M(t;\cdot))_{t\geq 0}$ then follows from the one in discrete time. The almost sure convergence is established by applying Doob's maximal inequality and the Borel–Cantelli lemma, like in the proof of [14]: indeed for every $\varepsilon > 0$,

$$\mathbb{P}(\exists t \geq n\colon \|M(t;\cdot) - M(n;\cdot)\| > \varepsilon) \;\leq\; \varepsilon^{-1}\, \mathbb{E}[\|M(\infty;\cdot) - M(n;\cdot)\|] \xrightarrow[n\to\infty]{} 0.$$

We finally deal with the almost sure positivity of the terminal value $M(\infty;q)$ conditionally on non-extinction. We derive from the branching property at time n that, for every $q \in (\underline{q}, \bar{q})$,

$$\mathbb{P}\big(M(\infty;q) = 0 \bigm| \mathcal{Z}^n\big) = \prod_{z\in\mathcal{Z}^n} \mathbb{P}_z(M(\infty;q) = 0),$$

where by scaling, the probability $\mathbb{P}_z(M(\infty;q)=0)$ does actually not depend on the initial size z. Hence $\rho := \mathbb{P}(M(\infty;q)=0) = \mathbb{E}[\rho^{\#(n)}]$, where $\#(n) := \mathcal{Z}^n(\mathbb{R})$, the number of particles at time $n \in \mathbb{N}$, defines a supercritical Galton–Watson process. Since $\rho < 1$ (because $\mathbb{E}[M(\infty;q)] = \mathbb{E}[M(0;q)] = 1$), ρ is its probability of extinction. The two events $\{\text{extinction}\} \subseteq \{M(\infty;q)=0\}$ having thus the same probability we conclude that they coincide up to a negligible event. $\square$

Remark 2.9. Under the conditions of Theorem 2.3 we have also from [19, Theorem 5] that for each $q \in (\underline{q}, \bar{q})$ and $\alpha \in (1,\gamma]$ as in (2.12), the martingale $(M(t;q))_{t\geq 0}$ converges in $\mathrm{L}^\alpha(\mathbb{P})$.

We close this section with the proof of Corollary 2.4.

Proof of Corollary 2.4. Let us define the tilted measures

$$\mathcal{Z}_q^t(\mathrm{d}x) := \frac{e^{qx}}{m(q)^t}\,\mathcal{Z}^t(\mathrm{d}x), \quad t \geq 0, \qquad \text{and} \qquad \mu_q(\mathrm{d}x) := \frac{e^{qx}}{m(q)}\,\mu(\mathrm{d}x).$$

Using (2.3), μ_q is a probability measure with mean

$$c_q := m(q)^{-1}\,\mathbb{E}\left[\sum_{i=1}^\infty Z_i^q(1)\log Z_i(1)\right] = \kappa'(q),$$

and variance

$$\sigma_q^2 := m(q)^{-1}\,\mathbb{E}\left[\sum_{i=1}^\infty Z_i^q(1)\log^2 Z_i(1)\right] - c_q^2 = \kappa''(q).$$

On the one hand, we observe that for every $n \in \mathbb{N}$,

$$e^{-\left(\kappa(q)-q\kappa'(q)\right)n}\sum_{i=1}^\infty f\left(Z_i(n)\,e^{-\kappa'(q)n}\right) = \int_{\mathbb{R}} f(e^x)\,e^{-qx}\,\mathcal{Z}_q^n(nc_q+\mathrm{d}x).$$

On the other hand, by a local limit theorem due to Stone [41, Theorem 2],

$$\sqrt{n}\,\mu_q^{(\star n)}(nc_q+\mathrm{d}x) \;\approx\; p_q\left(\frac{x}{\sqrt{n}}\right)\mathrm{d}x, \qquad n\to\infty,$$

uniformly for $x \in \mathbb{R}$ and q in compact subsets of $(\underline{q},\infty)$, where $\mu_q^{(\star n)}$, $n\in\mathbb{N}$, is the n^{th} convolution of μ_q with itself and $p_q(x)$ denotes the Gaussian density with mean c_q and variance σ_q^2. Thanks to the uniform convergence in Theorem 2.3, this in terms of the branching random walk translates into

$$\sqrt{n}\,\mathcal{Z}_q^n(nc_q+\mathrm{d}x) \;\approx\; M(\infty;q)\,p_q\left(\frac{x}{\sqrt{n}}\right)\mathrm{d}x, \qquad n\to\infty,$$

uniformly for $x\in\mathbb{R}$ and q in compact subsets of $(\underline{q},\bar{q})$, almost surely. The corollary then results from a Riemann sum argument. We leave details and refer the interested reader to Corollary 4 in [19] and its "continuous-time" extension discussed on page 150 there. $\square$

2.3 On the largest fragment

Alike the observation made by Bertoin [6, Equation (9)] for pure homogeneous fragmentations, Theorem 2.3 readily reveals the asymptotic velocity of the largest fragment Z_1: if $\mathbb{P}^*$ denotes the probability $\mathbb{P}$ conditionally on non-extinction, then

$$\lim_{t\to\infty}\frac{1}{t}\log Z_1(t) = \kappa'(\bar{q}), \qquad \mathbb{P}^*\text{-almost surely,}$$

where $\kappa'(\bar{q}) = \kappa(\bar{q})/\bar{q}$ and provided that $\bar{q} > \underline{q}$. We shall now delve deeper into the analogy with branching random walks and tell a bit more about the asymptotic expansion of $Z_1(t)$. To this end, we proceed to a renormalization of the branching process $\mathcal{Z}^t$: specifically, for

$$\widetilde{\mathcal{Z}}^t := \sum_{i=1}^{\infty} \delta_{\kappa(\bar{q})t - \bar{q} \log Z_i(t)},$$

which has the log-Laplace transform

$$\widetilde{\kappa}(q) \;:=\; \frac{1}{t} \log \mathbb{E}\left[\int_{\mathbb{R}} e^{-qy}\, \widetilde{\mathcal{Z}}^t(\mathrm{d}y)\right] \;=\; \kappa(q\bar{q}) - q\kappa(\bar{q}), \qquad q \geq 0,$$

we are now in the so called *boundary case*, namely $\widetilde{\kappa}(0) > 0$ and $\widetilde{\kappa}(1) = \widetilde{\kappa}'(1) = 0$. Let us also introduce the process

$$D(t) := \int_{\mathbb{R}} y e^{-y}\, \widetilde{\mathcal{Z}}^t(\mathrm{d}y) = -\bar{q}\, \frac{\mathrm{d}}{\mathrm{d}q} M(t;q)\bigg|_{q=\bar{q}}, \qquad t \geq 0,$$

which is easily seen from the branching property to be a martingale (rightly called the *derivative martingale*) and will serve our purpose.

Corollary 2.10. *Suppose* (2.6), (2.7), *and* $\bar{q} > \underline{q}$.

(a) Then

$$\lim_{t\to\infty} \frac{\log Z_1(t) - \kappa'(\bar{q})t}{\log t} = -\frac{3}{2\bar{q}}, \qquad \textit{in } \mathbb{P}^*\textit{-probability}. \tag{2.13}$$

(b) If further μ is non-lattice, then there exist a constant $C^* > 0$ and a nonnegative random variable D_∞ such that, for every $x > 0$,

$$\lim_{t\to\infty} \mathbb{P}\Big(t^{3/2\bar{q}}\, e^{-\kappa'(\bar{q})t}\, Z_1(t) \leq x\Big) \;=\; \mathbb{E}\Big[e^{-C^* D_\infty / x}\Big]. \tag{2.14}$$

Moreover $D_\infty > 0$, $\mathbb{P}^*$-almost surely.

Remark 2.11. (i) Kyprianou et al. [30] recently derived an analogue of (a) for pure homogeneous fragmentations. However the method we employ here (for both statements) is different: basically, we directly transfer the known results on branching random walks to discrete skeletons of the growth-fragmentation, and then infer the behavior of the whole process with the help of Lemma 2.12 below.
(ii) The logarithmic fluctuations [40, Theorem 5.23] also show that

$$\limsup_{t\to\infty} \frac{\log Z_1(t) - \kappa'(\bar{q})t}{\log t} \geq -\frac{1}{2\bar{q}}, \qquad \mathbb{P}^*\text{-almost surely}$$

(we conjecture that there is in fact equality), so the convergence (2.13) cannot be strengthened.
(iii) Other interesting facts from the literature of branching random walks could be inherited. For instance, by specializing a recent result due to Aïdékon and Shi [40, Theorem 5.29] one infers a so called Seneta–Heyde renormalization for the convergence of $M(t;\bar{q})$ in Theorem 2.3.(ii): namely

$$\lim_{t\to\infty} \sqrt{t}\, M(t;\bar{q}) = \sqrt{\frac{2}{\pi \bar{q}^2 \kappa''(\bar{q})}}\, D_\infty, \qquad \text{in } \mathbb{P}^*\text{-probability}$$

(with D_∞ as above), and again this convergence cannot be strengthened (the $\limsup$ is infinite $\mathbb{P}^*$-almost surely.).

Lemma 2.12 (Croft–Kingman, [27, Theorem 2]). *Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuous function such that for every $h > 0$, the sequence $f(nh)$, $n \in \mathbb{N}$, converges. Then $f(x)$ has a limit as $x \to \infty$.*

Proof of Corollary 2.10. Let $h > 0$ be any fixed time mesh. It is plain from the branching property that $\widetilde{\mathcal{Z}}^{nh}$ corresponds to the individuals at generation $n \in \mathbb{N}$ of a branching random walk on $\mathbb{R}$ whose offspring point process is distributed like $\widetilde{\mathcal{Z}}^h$. On the one hand, there is

$$\mathbb{E}\left[\int_{\mathbb{R}} y^2\, e^{-y}\, \widetilde{\mathcal{Z}}^h(\mathrm{d}y)\right] \;=\; h\widetilde{\kappa}''(1) \;=\; h\bar{q}^2\kappa''(\bar{q}) \;<\; \infty.$$

On the other hand, with the notation $u_+ \coloneqq \max(u,0)$ for any $u \in \mathbb{R}$ and

$$X \coloneqq \int_{\mathbb{R}} e^{-y}\, \widetilde{\mathcal{Z}}^h(\mathrm{d}y) = M(h;\bar{q}), \qquad \widetilde{X} \coloneqq \int_{\mathbb{R}} y_+\, e^{-y}\, \widetilde{\mathcal{Z}}^h(\mathrm{d}y),$$

Proposition 2.8 readily entails that

$$\mathbb{E}\big[X(\log X)_+^2\big] < \infty \qquad \text{and} \qquad \mathbb{E}\Big[\widetilde{X}(\log \widetilde{X})_+\Big] < \infty.$$

(For the second, we use that $|\log(x)| \le (x^{\varepsilon} + x^{-\varepsilon})/\varepsilon$ for every $x > 0$ and any $0 < \varepsilon < \bar{q} - \underline{q}$.) As a result, Assumption (H) of [40, § 5.1] is fulfilled[3]. From Theorem 5.12 there, we obtain that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{\log Z_1(nh) - \kappa'(\bar{q})nh}{\log nh} + \frac{3}{2\bar{q}}\right| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0, \qquad \text{for each } h > 0.$$

As the left-hand side is a continuous function of $t \coloneqq nh$, the proof of (a) follows from Lemma 2.12. Similarly, when μ is non-lattice, then $\widetilde{\mathcal{Z}}^h$ is non-lattice as well and Theorem 5.15 of [40] (likewise, Theorem 1.1 of [2]) applies: for every $y \in \mathbb{R}$ and every $h > 0$, the quantity

$$\mathbb{P}\left(\log Z_1(nh) - \kappa'(\bar{q})nh + \frac{3}{2\bar{q}}\log nh \le y\right)$$

has a limit as $n \to \infty$. Applying Croft–Kingman's lemma once more gives the convergence (2.14), where the limit is e.g. that for $h = 1$. In (b), the random variable D_∞ can be taken as the terminal value of the derivative martingale $(D(n),\, n \in \mathbb{N})$: Theorem 5.2 of [40] shows that it exists almost surely and is positive on non-extinction. That D_∞ is at least nonnegative holds simply because the smallest atom of $\widetilde{\mathcal{Z}}^n$,

$$\kappa(\bar{q})n - \bar{q}\log Z_1(n),$$

is bounded from below by $-\log M(n;\bar{q})$, which tends to ∞ a.s. due to Theorem 2.3.(ii). $\square$

2.4 On abnormally large fragments

In this last section we give an estimation for the probability of presence of fragments at scale greater than $\kappa'(\bar{q})$ in a compensated fragmentation. We simply perform the very same analysis as done in [14] for homogeneous pure fragmentations. Let us fix two real numbers $\alpha < \beta$ and introduce

$$U(t,x) \coloneqq \mathbb{P}\Big(\mathcal{Z}^t([x+\alpha, x+\beta]) > 0\Big),$$

$$V(t,x) \coloneqq \mathbb{E}\Big[\mathcal{Z}^t([x+\alpha, x+\beta])\Big],$$

for every $t \ge 0$ and $x \in \mathbb{R}$.

[3] Strictly speaking, [40] also requires a *finite* branching ($\widetilde{\mathcal{Z}}^t(\mathbb{R}) < \infty$ a.s.), but this condition turns out to be unnecessary (see e.g. [34]; besides, it is not needed in the latest version of [2] that we invoke to prove (b), and the conclusion of (b) obviously implies (a)).

Corollary 2.13. *Let $q > \underline{q}$. Suppose* (2.6)*, μ non-lattice, and either* (2.7) *or $q \geq 1$.*

(i) Then

$$\lim_{t\to\infty} \sqrt{t}\, e^{-\left(\kappa(q)-q\kappa'(q)\right)t}\, V\big(t, t\kappa'(q)\big) = \frac{e^{-q\alpha} - e^{-q\beta}}{q\sqrt{2\pi\kappa''(q)}}.$$

(ii) If further $q > \bar{q}$ (so that $\kappa(q) - q\kappa'(q) < 0$), then

$$\lim_{t\to\infty} \frac{U\big(t, t\kappa'(q)\big)}{V\big(t, t\kappa'(q)\big)} = K_q,$$

where K_q is some positive finite constant.

Remark 2.14. In the range $q \in (\underline{q}, \bar{q})$, (i) is the counterpart in mean of the convergence stated in Corollary 2.4 for $f := \mathbb{1}_{[\alpha,\beta]}$. This convergence thus holds in $\mathrm{L}^1(\mathbb{P})$ thanks to the Riesz–Scheffé lemma.

Proof. The proof is a straightforward adaptation of that of Theorem 5 in [14]. In our setting, we have

$$a := \kappa'(q), \qquad \Lambda^*(a) := q\kappa'(q) - \kappa(q),$$

and for any time mesh $h > 0$,

$$\Lambda_h(q) \;:=\; \log \mathbb{E}\left[\sum_{i=1}^{\infty} Z_i^q(h)\right] \;=\; h\kappa(q),$$

by (2.3). From Equation (12) in [13] we readily get

$$\sqrt{nh}\, e^{nh\Lambda^*(a)} V(nh, anh) \xrightarrow[n\to\infty]{} \frac{e^{-q\alpha} - e^{-q\beta}}{q\sqrt{2\pi\kappa''(q)}}. \tag{2.15}$$

If furthermore $\Lambda^*(a) > 0$ (i.e. $q > \bar{q}$), then Proposition 2.8 ensures that the conditions of Theorem 2 in [13] are fulfilled and therefore

$$\frac{U(nh, anh)}{V(nh, anh)} \xrightarrow[n\to\infty]{} K_q^{(h)}, \tag{2.16}$$

where $K_q^{(h)}$ is a positive constant. Besides, the time mesh $h > 0$ in (2.15) and (2.16) is arbitrary and the left-hand sides are both continuous functions of the variable $t := nh$. The existence of limits as $t \to \infty$ then comes again from Lemma 2.12. (In particular, the constant $K_q^{(h)}$ in (2.16) does actually not depend on h.) □

3 Self-similar growth-fragmentations

As opposed to the previous part, a *self-similar* growth-fragmentation will now allow inhomogeneous fragmentation rates. Loosely speaking, one can picture it as a homogeneous fragmentation where each fragment is “sped up” all along its history by a fixed power $\alpha \in \mathbb{R}$ of its current size. If as before the Laplace transform of the fragment sizes at genealogical births may be related through a cumulant function κ, self-similarity induces significant changes when we look at processes over time. Mainly, in the case $\alpha > 0$ we shall mostly focus on, where positive growth in the fragments is thus compensated by higher dislocation rates, the typical sizes will no longer be of exponential order (given through the derivative κ'), but will instead encounter a polynomial decay of the type $t^{-1/\alpha}$. Another side effect is that additive martingales appear less nicely, so specific assumptions will be needed.

3.1 Prerequisites

We begin with a quick summary of the construction and important properties of self-similar growth-fragmentations processes. These were introduced in [9]; greater details as well as some applications to random planar maps can be found in [10].

Let $\xi := (\xi(t),\, t \geq 0)$ be a possibly killed Lévy process and $(\sigma^2, b, \Lambda, \mathsf{k})$ denote its characteristic quadruple in the following sense. The Gaussian coefficient $\sigma^2 \geq 0$, the drift coefficient $b \in \mathbb{R}$, the Lévy measure Λ (that is, a measure on $\mathbb{R}$ with $\int (1 \wedge y^2)\, \Lambda(\mathrm{d}y) < \infty$), and the killing rate $\mathsf{k} \in [0, \infty)$ may be recovered from this slight variation of the Lévy–Khinchin formula:

$$E\big[\exp\big(q\xi(t)\big)\big] = \exp\big(t\Psi(q)\big), \qquad t, q \geq 0,$$

the Laplace exponent Ψ being written in the form

$$\Psi(q) := -\mathsf{k} + \frac{1}{2}\sigma^2 q^2 + bq + \int_{\mathbb{R}} \big(e^{qy} - 1 + q(1 - e^y)\big)\, \Lambda(\mathrm{d}y), \qquad q \geq 0.$$

The case $\Lambda((-\infty, 0)) = 0$ will be uninteresting and is therefore excluded. We shall also assume that $\int_{(1,\infty)} e^y\, \Lambda(\mathrm{d}y) < \infty$ (which always holds when the support of Λ is bounded from above), and that

$$\mathsf{k} > 0 \quad \text{or} \quad \big(\mathsf{k} = 0 \quad \text{and} \quad \Psi'(0+) \in [-\infty, 0)\big) \tag{3.1}$$

(in other words, that $\Psi(q) < 0$ for some $q > 0$). This latter condition means that $\xi(t)$ either has a finite lifetime or tends to $-\infty$ as $t \to \infty$, almost surely. Let now $\alpha \in \mathbb{R}$ and, for each $x > 0$, P_x be the law of the process

$$X(t) := x \exp\big\{\xi(\tau_{x^\alpha t})\big\}, \qquad t \geq 0,$$

where

$$\tau_t := \inf\left\{u \geq 0 \colon \int_0^u \exp\big(-\alpha\xi(s)\big)\, \mathrm{d}s \geq t\right\},$$

and with the convention that $X(t) := \partial$ for $t \geq \zeta := x^{-\alpha}\int_0^\infty \exp(-\alpha\xi(s))\, \mathrm{d}s$. This *Lamperti transform* ([32]; see also [31, Theorem 13.1]) makes $(X, (P_x)_{x>0})$ be a positive self-similar Markov process (for short, *pssMp*), in the sense that:

$$\text{For all } x > 0, \qquad \text{the law of } \big(xX(x^\alpha t), t \geq 0\big) \text{ under } P_1 \text{ is } P_x. \tag{3.2}$$

(Following the terminology in [16], we say that X is a pssMp with index $1/(-\alpha)$.) Moreover, this transformation is reversible, and since the law of the Lévy process ξ is uniquely determined by its Laplace exponent Ψ, the pair (Ψ, α) characterizes the law of X; we call it the characteristics of the pssMp X. Note that under (3.1), X either is eventually absorbed to the cemetery point ∂ added to the positive half-line $(0, \infty)$, or it converges to 0 as $t \to \infty$.

The process X above will portray the typical size of a cell in the system and is thus referred as the *cell process*. Specifically, a cell system is a process $((\mathcal{X}_u, b_u),\, u \in \mathbb{U})$ indexed on the Ulam–Harris tree

$$\mathbb{U} := \bigcup_{i=0}^{\infty} \mathbb{N}^i,$$

with the following classical notations: $\mathbb{N}^0$ is reduced to the *root* of $\mathbb{U}$, labeled $\varnothing$, and for any *node* $u := u_1 u_2 \cdots u_i \in \mathbb{U}$ in this tree, $|u| := i \in \{0, 1, 2, \ldots\}$ refers to its *generation* (or *height*), and $u1, u2, \ldots$ to its children. For each $u \in \mathbb{U}$, $(\mathcal{X}_u(t),\, t \geq 0)$ is a càdlàg process on $(0,\infty)\cup\{\partial\}$ driven by X and recording the size of the cell labeled by u since its birth time b_u, which shall be implicitly encoded in the notation $\mathcal{X}_u$. In this system $\mathcal{X}_\varnothing$ refers to Eve cell, born at time $b_\varnothing := 0$, and each negative[4] jump of a cell is interpreted as the birth of a daughter cell. More precisely for every $u\in\mathbb{U}$ and $j\in\mathbb{U}$, $\mathcal{X}_{uj}$ is the process of the j^{th} daughter cell of u, born at the absolute time $b_{uj} := b_u+\beta_{uj}$, where β_{uj} is the instant of the j^{th} biggest positive jump[5] of $-\mathcal{X}_u$. The law $\mathcal{P}_x$ of $\mathcal{X}$ is then defined recursively as the unique probability distribution such that $\mathcal{X}_\varnothing$ has the law P_x and, conditionally on $\mathcal{X}_\varnothing$, the processes $(\mathcal{X}_{iu},\, u\in\mathbb{U})$, $i\in\mathbb{N}$, are independent with respective laws $\mathcal{P}_{x_i}$, $i\in\mathbb{N}$, where $(x_1,\beta_1),(x_2,\beta_2),\dots$ is the sequence of positive jump sizes and times of $-\mathcal{X}_\varnothing$, sorted by decreasing sizes (with $\beta_i<\beta_{i+1}$ if $x_i=x_{i+1}$). Here, we agree that $x_i:=\partial$ and $\beta_i:=\infty$ if $\mathcal{X}_\varnothing$ has less than i negative jumps, and we let $\mathcal{P}_\partial$ denote the law of the degenerate cell system where $\mathcal{X}_u :\equiv \partial$ for every $u\in\mathbb{U}$, $b_\varnothing := 0$ and $b_u:=\infty$ for $u\neq\varnothing$.

The associated growth-fragmentation process is the process of the family of (the sizes of) all alive cells in the system:

$$\mathbf{X}(t) := \left\{\!\!\left\{\mathcal{X}_u(t-b_u)\colon u\in\mathbb{U},\, b_u\le t<d_u\right\}\!\!\right\}, \qquad t\ge 0$$

(with b_u and d_u denoting respectively the birth time and the death time of the cell labeled by u). Additionally to the scaling parameter α, one other specific quantity is

$$\kappa(q) := \Psi(q) + \int_{(-\infty,0)} (1-e^y)^q\,\Lambda(\mathrm{d}y), \qquad q\ge 0.$$

If $\alpha=0$ and Λ has support in $[-\log 2,0]$, then [9, Proposition 3] $\mathbf{X}$ is merely a compensated fragmentation of the type considered in Section 2, and the notation κ there is compatible with the one we use here: more precisely, $\mathbf{X}$ has diffusion coefficient σ^2, growth rate b and dislocation measure $\nu := \mathsf{k}\,\delta_{\mathbf{0}}+\nu_2$, where $\mathbf{0}:=(0,0,\dots)\in\mathcal{P}$ is the null mass-partition and ν_2 is the image of Λ by the map $x\in[-\log 2,0]\mapsto(e^x,1-e^x,0,\dots)\in\mathcal{P}$ (the fragmentation is binary).

We shall work under the assumption

$$\exists q\ge 0,\ \kappa(q)\le 0, \tag{3.3}$$

see [9, Theorem 2]. Then for each time t, the family $\mathbf{X}(t)$ may be ranked in the non-increasing order, i.e. $\mathbf{X}(t) := (X_1(t),X_2(t),\dots)$ with $X_1(t)\ge X_2(t)\ge\cdots\ge 0$. Further, the self-similarity property (3.2) extends to the process $\mathbf{X} := (\mathbf{X}(t),\, t\ge 0)$, and there is the branching property. Formally, if $\mathbb{P}_x$ denotes the law of $\mathbf{X}$ under $\mathcal{P}_x$, then firstly, for every $x>0$, the distribution of $(x\mathbf{X}(x^\alpha t),\, t\ge 0)$ under $\mathbb{P}_1$ is $\mathbb{P}_x$, and secondly, for each $s\ge 0$, conditionally on $\mathbf{X}(s)=(x_1,x_2,\dots)$, the process $(\mathbf{X}(t+s),\, t\ge 0)$ is independent of $(\mathbf{X}(r),\, 0\le r\le s)$ and has the same law as the non-increasing rearrangement of the family $(X_j^{(i)},\, i,j\in\mathbb{N})$, where the $\mathbf{X}^{(i)}$ are independent self-similar growth-fragmentations with respective laws $\mathbb{P}_{x_i}$.

In the sequel we mainly focus on large time asymptotics for the growth-fragmentation process $\mathbf{X}$. Since we can refer to Section 2 when $\alpha=0$, and because the growth-fragmentation is eventually extinct when the scaling parameter α is negative [9, Corollary 3], we will mostly suppose $\alpha>0$. Note in this case that (3.3) is a necessary and

[4] In [9], the author only considered spectrally negative Lévy processes so jumps were always of negative sign. However, allowing the cells to have sudden positive growths during their lifetimes is relevant in some applications such as those exposed in [10]. Their slightly more general setting, which we have also chosen to adopt, does not invalidate the results of [9] — the significant point being that only the negative jumps correspond to division events while the possible positive jumps just remain part of the trajectories of the cells.

[5] Recall that the cell process is either absorbed in finite time or converges to 0, so the positive jumps of $-\mathcal{X}_u$ may indeed be ranked in the decreasing order.

sufficient condition preventing local explosion of the fragmentation [15], that is a phenomenon causing infinitely many particles of arbitrary large sizes to be produced in almost surely finite time (which in particular would impede us to list the elements of $\mathbf{X}(t)$ in the non-increasing order). Like in Section 2, the function $\kappa\colon [0,\infty) \to (-\infty,\infty]$ will be of greatest importance in the study. It is clearly convex; therefore the equation $\kappa(q) = 0$ has at most two solutions. We assume from here on that these two solutions exist — more precisely that the *Malthusian hypotheses* hold:

$$\text{there exist } 0 < \omega_- < \omega_+ \text{ such that} \quad \kappa(\omega_-) = \kappa(\omega_+) = 0 \ \text{ and } \ \kappa'(\omega_-) > -\infty \tag{3.4}$$

(note then that $\kappa'(\omega_-) < 0$, by convexity). Condition (3.4) implies that $\kappa(q) < 0$ for some $q > 0$, which in turn implies (3.3), and (3.1) (because $\Psi \le \kappa$).

As before, limit theorems for the growth-fragmentation process $\mathbf{X}$ will involve the terminal value of some additive martingale, namely the *Malthusian martingale*

$$M^-(t) \coloneqq \sum_{i=1}^{\infty} X_i^{\omega_-}(t), \qquad t \ge 0.$$

In this direction, results of [10] will be of fundamental use; we restate some of them here for sake of reference.

Proposition 3.1 (from [10, Theorem 3.10.(ii), Corollaries 3.7.(ii) and 3.9])**.** *Suppose $\alpha > 0$.*

(i) The process $(M^-(t),\, t \ge 0)$ under $\mathbb{P}_x$ is a uniformly integrable martingale; more precisely it is bounded in $\mathrm{L}^p(\mathbb{P}_x)$ for every $1 < p < \omega_+/\omega_-$.

(ii) For every $0 < q < (\omega_+ - \omega_-)/\alpha$, the process

$$\sum_{i=1}^{\infty} X_i^{q\alpha+\omega_-}(t), \qquad t \ge 0,$$

is a supermartingale converging to 0 in $\mathrm{L}^1(\mathbb{P}_x)$: more precisely,

$$\mathbb{E}_x\left[\sum_{i=1}^{\infty} X_i^{q\alpha+\omega_-}(t)\right] \sim\ c(q)\, x^{\omega_-}\, t^{-q}$$

as $t \to \infty$, for some constant $c(q) > 0$.

Remark 3.2. We find relevant to mention that [10] also introduced the genealogical martingale

$$\mathcal{M}^-(n) \coloneqq \sum_{|u|=n+1} \mathcal{X}_u^{\omega_-}(0), \qquad\qquad n \ge 0,$$

called the *intrinsic martingale*, which under $\mathcal{P}_x$ is always uniformly integrable. When $\alpha \ge 0$, there is the remarkable fact

$$M^-(t) \ = \ \mathcal{E}_x\big[\mathcal{M}^-(\infty) \bigm| \mathcal{F}_t\big], \qquad\quad t \ge 0,$$

with $(\mathcal{F}_t)_{t\ge0}$ the canonical filtration of $\mathbf{X}$. In particular, $M^-(\infty) = \mathcal{M}^-(\infty)$ almost surely.

Additive martingales — in the present context, the Malthusian martingale $(M^-(t))_{t\ge0}$, are of important interest since the celebrated work of Lyons et al. [33]. Roughly speaking, one can perform a change of probability measure in terms of the terminal value $M^-(\infty)$ so that the genealogical system may be observed from the point of view of a randomly tagged branch. Specifically, write $\partial\mathbb{U}$ for the set of *leaves* of $\mathbb{U}$, each of which determines a unique branch from the root. For every leaf $\ell \in \partial\mathbb{U}$, let $\ell(n)$ denote its unique ancestor at generation $n \geq 0$, and $\mathcal{X}_\ell := (\mathcal{X}_\ell(t),\, t \geq 0)$ be the process of the cell on the branch from $\varnothing$ to ℓ:

$$\mathcal{X}_\ell(t) := \mathcal{X}_{\ell[t]}(t - b_{\ell[t]}), \qquad t \geq 0,$$

where $\ell[t]$ labels the cell in this branch which is alive at time t (i.e. $\ell[t]$ is the unique ancestor u of ℓ such that $b_u \leq t < b_{\ell(|u|+1)}$), with the convention that $\mathcal{X}_\ell(t) := \partial$ for $t > \lim\uparrow_{n\to\infty} b_{\ell(n)} =: b_\ell$. Next we consider a random leaf $\mathcal{L} \in \partial\mathbb{U}$ and we define for every $x > 0$ the joint distribution $\widehat{\mathcal{P}}_x^-$ of $(\mathcal{X}, \mathcal{L})$ as follows. Under $\widehat{\mathcal{P}}_x^-$, the law of $\mathcal{X} := (\mathcal{X}_u,\, u \in \mathbb{U})$ is absolutely continuous with respect to $\mathcal{P}_x$ with density $x^{-\omega_-} M^-(\infty)$, and the law of $\mathcal{L}$ conditionally on $\mathcal{X}$ is

$$\widehat{\mathcal{P}}_x^-(u \text{ ancestor of } \mathcal{L} \mid \mathcal{X}) := \lim_{n\to\infty} \frac{1}{\mathcal{M}^-(\infty)} \sum_{|v|=n} \mathcal{X}_{uv}^{\omega_-}(0). \tag{3.5}$$

Denoting $\widehat{\mathcal{X}} := \mathcal{X}_{\mathcal{L}}$ the *randomly tagged cell*, Bertoin et al. [10] derived:

Proposition 3.3 (from [10, Theorem 4.7 and Proposition 4.6])**.**

(i) *The process* $(\widehat{\mathcal{X}}, (\widehat{\mathcal{P}}_x^-)_{x>0})$ *is a pssMp with characteristics* (Φ^-, α)*, where*

$$\Phi^-(q) := \kappa(q + \omega_-), \qquad q \geq 0. \tag{3.6}$$

(ii) Many-to-one formula. *For every* $x > 0$*, every* $t \geq 0$*, and every measurable function* $f\colon (0,\infty) \to (0,\infty)$*, we have*

$$\mathbb{E}_x\left[\sum_{i=1}^{\infty} X_i^{\omega_-}(t)\, f\big(X_i(t)\big)\right] = x^{\omega_-}\, \widehat{\mathcal{E}}_x^-\Big[f\big(\widehat{\mathcal{X}}(t)\big)\Big], \tag{3.7}$$

with the convention $f(\partial) := 0$*.*

Formula (3.7) will be a key ingredient for our purpose. Roughly speaking, it says that the intensity of the weighted point measure $\sum_{z\in\mathbf{X}} z^{\omega_-}\delta_z$ is captured by the law of the randomly tagged cell $\widehat{\mathcal{X}}$ (hence the denomination "many-to-one").

When $\alpha > 0$, unlike in the homogeneous case, a *polynomial* decrease in the size of the fragments is expected. Large-time asymptotics for their empirical measure will be retrieved in the next section. In Section 3.3 we center our attention on the largest fragment. Lastly, in Section 3.4, we discuss the convergence of the empirical measure of the fragments taken at the instant when they become smaller than a vanishing threshold.

3.2 Convergence of the empirical measure

We are here especially interested in the convergence of the empirical measure $\rho_t^{(\alpha)}$ given by

$$\big\langle \rho_t^{(\alpha)}, f\big\rangle := \sum_{i=1}^{\infty} X_i^{\omega_-}(t)\, f\big(t^{1/\alpha} X_i(t)\big),$$

for $\alpha > 0$. From now on, we shall suppose that the Lévy process ξ^- associated with the tagged cell $\widehat{\mathcal{X}}$ via Lamperti's transformation is not *arithmetic*, in the sense that there is no $r > 0$ such that $\mathbb{P}(\xi^-(t) \in r\mathbb{Z}) = 1$ for all $t \geq 0$. To state our result, let us define the probability distribution ρ on $(0,\infty)$ by

$$\int_0^\infty f(y)\, \rho(\mathrm{d}y) := \frac{-1}{\alpha\kappa'(\omega_-)}\, E\Big[I^{-1}\, f\big(I^{1/\alpha}\big)\Big],$$

where

$$I := \int_0^\infty \exp\bigl(\alpha\xi^-(s)\bigr)\,\mathrm{d}s \tag{3.8}$$

is the so called *exponential functional* of $\alpha\xi^-$. The following completes the results of Bertoin [6] and Bertoin and Gnedin [11] relative to self-similar pure fragmentations, and differs substantially from the homogeneous case (Corollary 2.4).

Theorem 3.4. *For every $1 < p < \omega_+/\omega_-$ and for every bounded continuous function $f\colon (0,\infty) \to \mathbb{R}$,*

$$\lim_{t\to\infty} \langle \rho_t^{(\alpha)}, f\rangle = M^-(\infty)\int_0^\infty f(y)\,\rho(\mathrm{d}y), \qquad \text{in } \mathrm{L}^p(\mathbb{P}_1).$$

Consequently, the random measure $\rho_t^{(\alpha)}$ converges in $\mathbb{P}_1$-probability to $M^-(\infty)\,\rho$ as $t \to \infty$, in the space of finite measures on $(0,\infty)$ endowed with the topology of weak convergence.

Remark 3.5. Note the presence of the random factor $M^-(\infty)$, which does not appear in [6] because the Malthusian martingale is trivial for *conservative* pure fragmentations[6]. It does nonetheless appear in the non-conservative case [11]; however the method used there leads to a L^2-convergence that we cannot hope for growth-fragmentations when $\omega_+/\omega_- < 2$, and it seems anyway difficult to generalize.

Exponential functionals of Lévy processes such as (3.8) arise in a variety of contexts and their laws have been widely studied, see the survey [17] and the recent works [36, 38, 37, 4]. In particular, Pardo et al. [36] showed that under mild assumptions, they can be factorized into the product of two independent exponential functionals associated with companion Lévy processes, and the distributions of both these functionals are uniquely determined by either their positive or their negative moments. To name just one example, in the common situation where ξ^- is spectrally negative ($\Lambda((0,\infty)) = 0$), we have [36, Corollary 2.1] that $I \overset{d}{=} J/\Gamma$, with J the exponential functional of the descending ladder height process of $\alpha\xi^-$ and Γ an independent Gamma random variable with parameter $(\omega_+ - \omega_-)/\alpha$. Further, the density of I has a polynomial tail of order $1 + (\omega_+ - \omega_-)/\alpha$ and admits a semi-explicit series expansion.

The distribution of I (likewise, ρ) naturally takes part in asymptotics of the tagged cell $\widehat{\mathcal{X}}$:

Lemma 3.6. *As $t \to \infty$, the random variable $t^{1/\alpha}\widehat{\mathcal{X}}(t)$ under $\widehat{\mathcal{P}}_1^-$ converges in distribution to ρ. Moreover, $\int_0^\infty y^{q\alpha}\,\rho(\mathrm{d}y) < \infty$ for every $0 \le q < 1 + (\omega_+ - \omega_-)/\alpha$.*

Proof. Clearly, $(1/\widehat{\mathcal{X}}(t),\, t \geq 0)$ is a pssMp with self-similarity index $1/\alpha$ associated with $-\xi^-$, where ξ^- is a Lévy process with the Laplace exponent Φ^- in (3.6). According to [16, Theorem 1], all we need to check to prove the first part of the statement is that $-\xi^-(1)$ admits a finite and positive first moment, which is implied by the Malthusian hypotheses (3.4): indeed,

$$E[-\xi^-(1)] = -(\Phi^-)'(0+) = -\kappa'(\omega_-) \in (0,\infty).$$

The existence of moments is quite straightforwardly adapted from the proof of [17, Theorem 3]. □

We now turn to the proof of Theorem 3.4, arguing along the lines of [7, Theorem 1.3]. The main idea is that, by the branching and scaling properties, the empirical measure of

[6] Since then the cumulant function vanishes at 1 and the total mass of the fragments at any generation is constant.

the fragments can be rewritten as the sum of identically distributed pieces arising from an intermediate (arbitrary large) time, which are all independent conditionally on the past. With the help of a (conditional) law of large numbers, we are then reduced to a first moment estimate for some additive functional of the growth-fragmentation, which we can work out thanks to the many-to-one formula and the asymptotic behavior of the tagged fragment above.

Proof of Theorem 3.4. Using the branching property at time t and the self-similarity of $\mathbf{X}$ we can write, on the event $\{\mathbf{X}(t) = (x_1, x_2, \ldots)\}$,

$$\langle \rho^{(\alpha)}_{t+t^2}, f \rangle \coloneqq \sum_{i=1}^{\infty} X_i^{\omega_-}\left(t+t^2\right) f\Big(\left(t+t^2\right)^{1/\alpha} X_i\big(t+t^2\big)\Big) = \sum_{i=1}^{\infty} \lambda_i(t) Y_i(t),$$

where $\lambda_i(t) \coloneqq X_i^{\omega_-}(t) = x_i^{\omega_-}$ and

$$Y_i(t) \coloneqq \sum_{j=1}^{\infty} X_{i,j}^{\omega_-}\left(x_i^\alpha\, t^2\right) f\Big(\left(t+t^2\right)^{1/\alpha} x_i X_{i,j}\big(x_i^\alpha\, t^2\big)\Big), \tag{3.9}$$

the families $(X_{i,1}, X_{i,2}, \ldots)$, $i \geq 1$, being i.i.d. copies independent of $\mathbf{X}$, having all the same law $\mathbb{P}_1$. Clearly, the Y_i are independent conditionally on $\boldsymbol{\lambda}(t) \coloneqq (\lambda_1(t), \lambda_2(t), \ldots)$ and, if we introduce

$$\bar{Y}_i \coloneqq \|f\|_\infty \sup_{t\geq 0} \sum_{j=1}^{\infty} X_{i,j}^{\omega_-}(t),$$

then thanks to Proposition 3.1.(i) and Doob's maximal inequality, the $\bar{Y}_i$ are i.i.d. random variables in $\mathrm{L}^p(\mathbb{P}_1)$ such that $|Y_i(t)| \leq \bar{Y}_i$ for all $t \geq 0$. For the same reason,

$$\sup_{t\geq 0} \mathbb{E}_1\left[\left(\sum_{i=1}^{\infty} \lambda_i(t)\right)^p\right] < \infty$$

and further, using Proposition 3.1.(ii),

$$\lim_{t\to\infty} \mathbb{E}_1\left[\sum_{i=1}^{\infty} \lambda_i^p(t)\right] = 0.$$

By a variation of the law of large numbers ([35]; see also [7, Lemma 1.5]) we then have

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \lambda_i(t)\Big(Y_i(t) - \mathbb{E}_1[Y_i(t) \mid \boldsymbol{\lambda}(t)]\Big) = 0, \qquad \text{in } \mathrm{L}^p(\mathbb{P}_1).$$

Consequently, the proof boils down to showing that

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \lambda_i(t)\, \mathbb{E}_1[Y_i(t) \mid \boldsymbol{\lambda}(t)] = M^-(\infty) \int_0^\infty f(y)\, \rho(\mathrm{d}y), \qquad \text{in } \mathrm{L}^p(\mathbb{P}_1), \tag{3.10}$$

where, applying the many-to-one formula (3.7),

$$\mathbb{E}_1[Y_i(t) \mid \boldsymbol{\lambda}(t)] = \widehat{\mathcal{E}}_1^-\Big[f\Big((1+t^{-1})^{1/\alpha}\, x_i\, t^{2/\alpha}\, \widehat{\mathcal{X}}(x_i^\alpha t^2)\Big)\Big].$$

But we know from Lemma 3.6 that as $s \to \infty$, the law of $s^{1/\alpha}\widehat{\mathcal{X}}(s)$ under $\widehat{\mathcal{P}}_1^-$ converges weakly to ρ. On the one hand, it thus follows that

$$\widehat{\mathcal{E}}_1^-\Big[f\Big((1+t^{-1})^{1/\alpha}\, x_i\, t^{2/\alpha}\, \widehat{\mathcal{X}}(x_i^\alpha t^2)\Big)\Big] \xrightarrow[t\to\infty]{} \int_0^\infty f(y)\, \rho(\mathrm{d}y)$$

uniformly in i such that, say, $x_i^\alpha t^2 > \sqrt{t}$, i.e. $x_i > t^{-3/2\alpha}$. On the other hand, applying again (3.7), the quantity

$$\sum_{i=1}^{\infty} X_i^{\omega_-}(t)\, \mathbb{1}_{\{X_i(t)\le t^{-3/2\alpha}\}} \tag{3.11}$$

has, under $\mathbb{P}_1$, mean

$$\widehat{\mathcal{P}}_1^-\left(t^{1/\alpha}\widehat{\mathcal{X}}(t) < t^{-1/2\alpha}\right),$$

which tends to 0 as $t\to\infty$. Since (3.11) is bounded in $\mathrm{L}^q(\mathbb{P}_1)$ for every $p<q<\omega_+/\omega_-$, it also converges to 0 in $\mathrm{L}^p(\mathbb{P}_1)$ (by Hölder's inequality). Putting everything together yields (3.10), and thus the first part of the statement.

The second part is derived from standard arguments: the space $\mathcal{C}_c((0,\infty))$ of continuous functions on $(0,\infty)$ with compact support being separable, a diagonal extraction procedure easily entails, for every sequence $t_n\to\infty$, that there exists an extraction $\sigma\colon\mathbb{N}\to\mathbb{N}$ such that, almost surely,

$$\forall f\in\mathcal{C}_c((0,\infty)),\quad \left\langle \rho^{(\alpha)}_{t_{\sigma(n)}}, f\right\rangle \xrightarrow[n\to\infty]{} M^-(\infty)\int_0^\infty f(y)\,\rho(\mathrm{d}y),$$

i.e. $\rho^{(\alpha)}_{t_{\sigma(n)}}$ converges vaguely to $M^-(\infty)\,\rho$, a.s. Since the total mass is conserved, that is

$$\left\langle \rho^{(\alpha)}_t, 1\right\rangle = \sum_{i=1}^{\infty} X_i^{\omega_-}(t) \xrightarrow[t\to\infty]{} M^-(\infty) = \left\langle M^-(\infty)\,\rho, 1\right\rangle \qquad \text{a.s.},$$

the convergence of $\rho^{(\alpha)}_{t_{\sigma(n)}}$ toward $M^-(\infty)\,\rho$ is actually weak. The conclusion follows easily. ☐

The existence of moments for ρ (Lemma 3.6) allows us to strengthen Theorem 3.4:

Corollary 3.7. *For every $0<q<(\omega_+-\omega_-)/\alpha$, every measurable function $f\colon(0,\infty)\to\mathbb{R}$ such that $f(y)=O(y^{q\alpha})$, and every $1<p<\omega_+/(q\alpha+\omega_-)$,*

$$\lim_{t\to\infty}\left\langle \rho^{(\alpha)}_t, f\right\rangle = M^-(\infty)\int_0^\infty f(y)\,\rho(\mathrm{d}y), \qquad \textit{in } \mathrm{L}^p(\mathbb{P}_1).$$

Proof. Approximating $y\mapsto y^{-q\alpha}f(y)$ by simple functions, it is enough to do the proof for $f=f_q\colon y\mapsto y^{q\alpha}$, that is to prove:

$$\lim_{t\to\infty} t^q\sum_{i=1}^{\infty} X_i^{q\alpha+\omega_-}(t) = M^-(\infty)\int_0^\infty y^{q\alpha}\,\rho(\mathrm{d}y), \qquad \text{in } \mathrm{L}^p(\mathbb{P}_1).$$

This is of course not a direct consequence to Theorem 3.4 because f_q is not a bounded continuous function; nevertheless we can repeat the argument used in the previous proof. Observing that $q\alpha+\omega_-\in(\omega_-,\omega_+)$ and defining $Y_i(t)$ as in (3.9) but with f_q in place of f, we easily check with the help of Proposition 3.1 and Hölder's inequality that conditionally on $\mathbf{X}$, the Y_i are independent supermartingales bounded in $\mathrm{L}^p(\mathbb{P}_1)$ for every $1<p<\omega_+/(q\alpha+\omega_-)$. Therefore, all that remains to show is the convergence

$$t^q\,\mathbb{E}_1\left[\sum_{i=1}^{\infty} X_i^{q\alpha+\omega_-}(t)\right] = \widehat{\mathcal{E}}_1^-\left[\left(t^{1/\alpha}\widehat{\mathcal{X}}(t)\right)^{q\alpha}\right] \xrightarrow[t\to\infty]{} \int_0^\infty y^{q\alpha}\,\rho(\mathrm{d}y),$$

where the equality is just an application of the many-to-one formula (3.7). Since we already know that $t^{1/\alpha}\widehat{\mathcal{X}}(t)$ converges in distribution toward ρ (Lemma 3.6), it suffices to show that $((t^{1/\alpha}\widehat{\mathcal{X}}(t))^{q\alpha})_{t\ge 0}$ is bounded in $\mathrm{L}^r(\widehat{\mathcal{P}}_1^-)$ for some $r>1$, which is immediate using again the many-to-one formula and the convergence rate in Proposition 3.1.(ii) (we can take $1<r<(\omega_+-\omega_-)/q\alpha$). ☐

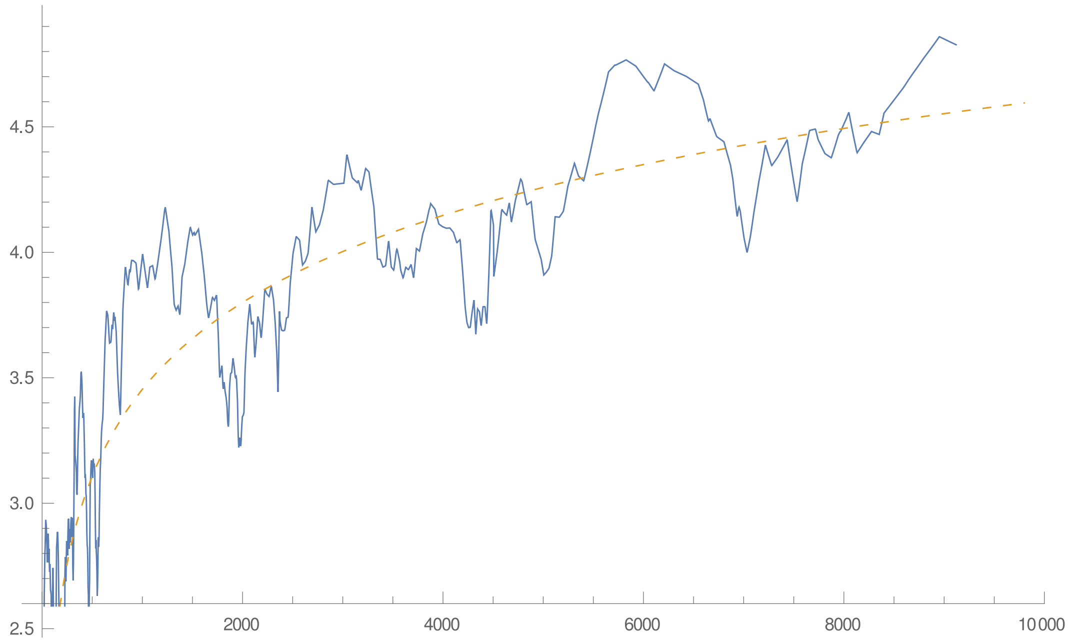


Figure 2: Simulation of $-\log X_1$ in a self-similar growth-fragmentation process with scaling $\alpha = 2$. (The dashed line represents the map $t \mapsto \frac{1}{\alpha}\log t$.)

3.3 Asymptotic behavior of the largest fragment

For pure self-similar fragmentations with scaling parameter $\alpha > 0$, it is known [6] that the size of the largest fragment decreases like $t^{-1/\alpha}$ as $t \to \infty$. The same holds for growth-fragmentations[7]:

Theorem 3.8. *Assume again* (3.4), $\alpha > 0$, *and that* ξ^- *is not arithmetic, and suppose further that* $\Lambda((0,\infty)) = 0$. *Let* $S := \{\forall t \geq 0,\ \mathbf{X}(t) \neq \varnothing\}$ *be the non-extinction event, and* $\mathcal{P}^* := \mathcal{P}_1(\,\cdot \mid S)$. *Then*

$$\lim_{t\to\infty} \frac{\log X_1(t)}{\log t} = -\frac{1}{\alpha}, \qquad \text{in } \mathcal{P}^*\text{-probability.}$$

Proof of the lower bound. The fact that the $\mathcal{P}^*$-$\liminf$ of $\log X_1(t)/\log t$ as $t \to \infty$ is at least $-1/\alpha$ follows by comparison with the randomly tagged cell $\widehat{\mathcal{X}}$. Indeed, we know by Lemma 3.6 that $\log \widehat{\mathcal{X}}(t)/\log t$ converges to $-1/\alpha$ in $\widehat{\mathcal{P}}_1^-$-probability. Because $X_1(t)$ is the size of the largest fragment and $\widehat{\mathcal{X}}(t)$ is that of some other fragment in the system, we deduce that for every $\eta > 0$,

$$\widehat{\mathcal{P}}_1^-\left(\frac{\log X_1(t)}{\log t} + \frac{1}{\alpha} < -\eta\right) \leq \widehat{\mathcal{P}}_1^-\left(\frac{\log \widehat{\mathcal{X}}(t)}{\log t} + \frac{1}{\alpha} < -\eta\right) \xrightarrow[t\to\infty]{} 0.$$

Since $\mathrm{d}\widehat{\mathcal{P}}_1^-/\mathrm{d}\mathcal{P}_1 = M^-(\infty)$, which by the branching property is positive $\mathcal{P}_1$-a.s. on S, the latter convergence also holds with $\mathcal{P}^*$ in place of $\widehat{\mathcal{P}}_1^-$. $\square$

For the other direction, we need to make sure that asymptotically, if the largest fragment ever exceeds the level $t^{-1/\alpha}$, it is unlikely that one of its parents has gone far below this level before t. To this end, we write $X_1(t) := \mathcal{X}_{u^*(t)}(t - b_{u^*(t)})$ with

[7] For simplicity, we suppose that the cell process has no positive jumps, though this restriction is probably superfluous.

$u^*(t) \coloneqq \arg\max_{u\in\mathbb{U}, b_u \le t < d_u} \mathcal{X}_u(t-b_u)$ (in case of *ex aequo*, we choose $u^*(t)$ to be minimal in lexicographic order), and introduce the event

$$\mathcal{H}_t(\varepsilon)\colon \quad \mathcal{X}_v(s-b_v) < \varepsilon, \quad \text{for some time } s \text{ and ancestor } v \preccurlyeq u^*(t) \text{ with } b_v \le s < d_v \wedge t.$$

The following statement is tailored for our purpose.

Proposition 3.9. *There exists $\theta \ge \omega_+$ such that*

$$\left\{ \sup_{x>1} \, , \, \lim_{x\to\infty} \right\} \frac{1}{\log x} \log \mathcal{P}_1\left(\sup_{s\ge 0} X_1(s) > x \right) = -\theta. \tag{3.12}$$

Furthermore, for every $\gamma, \varepsilon \in (0,1)$ and every $t > 0$,

$$\mathcal{P}_1\Big(\mathcal{H}_t(\varepsilon),\, X_1(t) > \varepsilon^\gamma\Big) \le \varepsilon^{(1-\gamma)\theta}.$$

Proof. We may assume $\alpha = 0$ as the statement does not depend on α. The first assertion is a large deviation estimate for the probability $F(x)$ that $T^+(x) < \infty$, where $T^+(x) \coloneqq \inf\{s \ge 0\colon X_1(s) > x\}$. To eventually obtain a fragment larger than xy in the growth-fragmentation, for $x, y > 1$, it is enough that the largest particle X_1 first reaches some level $z > x$, and that the subsequent fragmentation of this particle produces a fragment with size larger than xy. But by scaling we have, for any $z > x$,

$$\mathcal{P}_z\big(T^+(xy) < \infty\big) = \mathcal{P}_1\big(T^+(xy/z) < \infty\big) = F\big(xy/z\big) \ge F(y) = \mathcal{P}_1\big(T^+(y) < \infty\big),$$

so that the branching property at $T^+(x)$ yields, since $z \coloneqq X_1(T^+(x)) > x$ on the event $\{T^+(x) < \infty\}$,

$$F(xy) = \mathcal{P}_1(T^+(xy) < \infty) \ge \mathcal{E}_1\big[\mathbb{1}_{\{T^+(x)<\infty\}} F\big(xy/X_1\big(T^+(x)\big)\big)\big] \ge F(x)F(y).$$

Eq. (3.12) then arises from the subadditive lemma (see e.g. [29, Theorem 16.2.9]). The lower bound $\theta \ge \omega_+$ is just a consequence of Doob's maximal inequality applied to the process

$$M^+(s) \coloneqq \sum_{i=1}^\infty X_i^{\omega_+}(s), \qquad s \ge 0,$$

which [10, Corollary 3.7.(i)] is a martingale (for $\alpha = 0$): namely

$$F(x) = \mathcal{P}_1\left(\sup_{s\ge 0} X_1(s) > x\right) \le \mathcal{P}_1\left(\sup_{s\ge 0} M^+(s) > x^{\omega_+}\right) \le x^{-\omega_+}, \qquad x > 1.$$

Next, we take $x \coloneqq \varepsilon^{\gamma-1}$ and apply again the scaling property: we deduce that, for every $0 < y < \varepsilon$,

$$F(\varepsilon^\gamma/y) = \mathcal{P}_y\left(\sup_{s\ge 0} X_1(s) > \varepsilon^\gamma\right) \le \varepsilon^{(1-\gamma)\theta}.$$

But the event $\mathcal{H}_t(\varepsilon)$ holds precisely when the cell process following the ancestral lineage of $u^*(t)$ has reached a value $0 < y < \varepsilon$ before t. Using the branching property at the first time this happens, the second assertion is then easily proved. ☐

We can now derive the upper bound and complete the proof of Theorem 3.8.

Proof of the upper bound. Let $0 < \eta < 1$ and observe that $\delta \coloneqq \eta - (1-\eta)(1-\gamma)/\gamma$ lies in $(0, \eta)$ for any $\gamma \in (1-\eta, 1)$ arbitrarily fixed. Define $\varepsilon \coloneqq t^{-(1-\delta)/\alpha}$ for $t > 1$, so that $\varepsilon^\gamma = t^{-(1-\eta)/\alpha}$, and

$$\begin{aligned}\mathcal{P}_1\Big(X_1(t) > t^{-(1-\eta)/\alpha}\Big) &= \mathcal{P}_1(X_1(t) > \varepsilon^\gamma) \\ &= \mathcal{P}_1\Big(\mathcal{H}_t(\varepsilon),\, X_1(t) > \varepsilon^\gamma\Big) + \mathcal{P}_1\Big(\mathcal{H}_t(\varepsilon)^\complement,\, X_1(t) > \varepsilon^\gamma\Big).\end{aligned}$$

By Proposition 3.9,

$$\mathcal{P}_1\Big(\mathcal{H}_t(\varepsilon),\, X_1(t) > \varepsilon^{\gamma}\Big) \;\leq\; t^{-(1-\delta)(1-\gamma)\theta/\alpha} \;\xrightarrow[t\to\infty]{}\; 0.$$

To estimate the second term, we shall exploit the fact that a self-similar growth-fragmentation can be constructed from a homogeneous one by performing an appropriate Lamperti time-substitution on each cell in the system (see [9, Corollary 2] or [15, Section 2.1]). Specifically, there exists a cell system $\mathcal{Z} := ((\mathcal{Z}_u, \beta_u) \colon u \in \mathbb{U})$, with same cumulant function κ, such that every element in $\mathbf{X}(t)$ with label $v \in \mathbb{U}$ equals $\mathcal{Z}_u(\tau - \beta_u)$ for some $u \in \mathbb{U}$ and $\tau \geq 0$ fulfilling

$$\tau = \int_0^t \Big(\mathcal{X}_{\bar{v}(s)}\big(s - b_{\bar{v}(s)}\big)\Big)^{\alpha} \,\mathrm{d}s, \tag{3.13}$$

where $\bar{v}(s)$ labels the cell in $\mathcal{X}$ corresponding to the unique ancestor of v that is alive at time s. Further, the connection with compensated fragmentations [9, Proposition 3] entails that for every $q \geq 0$ with $\kappa(q) < \infty$,

$$\mathcal{E}_1\left[\sum_{u\in\mathbb{U}} \Big(\mathcal{Z}_u(\tau - \beta_u)\Big)^q\right] = \exp\big(\tau\kappa(q)\big).$$

On the one hand, if $Z_1(\tau)$ denotes the size of the largest cell at time τ in $\mathcal{Z}$, then Markov's inequality yields

$$\mathcal{P}_1(Z_1(\tau) > \varepsilon^{\gamma}) \leq \varepsilon^{-\gamma q} \exp\big(\tau\kappa(q)\big).$$

On the other hand, if we purposely take $v := u^*(t)$ then, on the complementary event of $\mathcal{H}_t(\varepsilon)$, we have $\mathcal{X}_{\bar{v}(s)}(s - b_{\bar{v}(s)}) \geq \varepsilon$ for all $s \in [0,t)$ and thus, by (3.13), $X_1(t) = \mathcal{Z}_u(\tau - b_u)$ with $\tau \geq t\,\varepsilon^{\alpha} = t^{\delta}$. Hence, fixing $q \in (\omega_-, \omega_+)$ (so that $\kappa(q) < 0$),

$$\mathcal{P}_1\Big(\mathcal{H}_t(\varepsilon)^{\complement},\, X_1(t) > \varepsilon^{\gamma}\Big) \;\leq\; \mathcal{P}_1\big(Z_1(\tau) > \varepsilon^{\gamma}\big) \;\leq\; t^{q(1-\eta)/\alpha} \exp\big(t^{\delta}\kappa(q)\big) \;\xrightarrow[t\to\infty]{}\; 0.$$

Putting the two pieces together we have just showed that, for every $0 < \eta < 1$,

$$\mathcal{P}_1\left(\frac{\log X_1(t)}{\log t} + \frac{1}{\alpha} > \frac{\eta}{\alpha}\right) \xrightarrow[t\to\infty]{} 0,$$

which is the upper bound we wanted. □

3.4 Freezing the fragmentation

Suppose now that we "freeze" every cell as soon as its size falls under a fixed diameter $\varepsilon > 0$ (which may occur at birth), in the sense that frozen cells no longer grow or split. To put things more formally we need a more chronological point of view in the cells genealogy. For this reason we suppose that the growth-fragmentation has been constructed as in [15, Section 2.1], where cells are now labeled on the infinite binary tree

$$\mathbb{B} := \bigcup_{n=0}^{\infty} \{1,2\}^n \subset \mathbb{U}.$$

Roughly speaking, any jump from a size $x > 0$ to some smaller size $x - y \in (0, x)$ of a cell with label, say, $u \in \mathbb{B}$, causes the death of that cell while at the same time two independent cells labeled by $u1$ and $u2$ are born with initial sizes $x - y$ and y respectively. We implicitly reuse the notations of Section 3.1 within this new description, e.g. $\mathcal{P}_x$ is the distribution of the cell system $\mathcal{X} := (\mathcal{X}_u \colon u \in \mathbb{B})$ when the mother cell starts at size $x > 0$ (i.e. has the law P_x). Analogously, $\ell \in \partial\mathbb{B}$ refers to a leaf of $\mathbb{B}$, and $\ell[t]$ and $\mathcal{X}_\ell(t)$

respectively denote the label and the process of the unique cell in the branch from $\varnothing$ to ℓ that is alive at time t. Let us then introduce the first passage times

$$\mathcal{T}_v(\varepsilon) := \inf\big\{t \geq 0 \colon \mathcal{X}_v(t) < \varepsilon\big\}, \qquad v \in \mathbb{B} \cup \partial\mathbb{B},$$

so that the family of frozen cells can be defined as

$$\big\{\chi_{i,\varepsilon}\big\}_{i=1}^{\infty} := \Big(\mathcal{X}_u\big(\mathcal{T}_u(\varepsilon)\big) \colon u \in \mathbb{B}(\varepsilon)\Big),$$

with $\mathbb{B}(\varepsilon) := \{u \in \mathbb{B} \colon u = \ell[\mathcal{T}_\ell(\varepsilon)] \text{ for some } \ell \in \partial\mathbb{B}\}$. Note that this procedure of freezing cells does not depend on the scaling parameter α of the growth-fragmentation (changing α just affects the speed at which particles get frozen). It is proved [10, Proposition 2.5] that for each $x > 0$, the process of the sum of the sizes of frozen cells raised to the power ω_-,

$$\mathcal{M}^-(\varepsilon) := \sum_{i=1}^{\infty} \chi_{i,\varepsilon}^{\omega_-}, \qquad 0 < \varepsilon \leq x,$$

is a backward martingale converging to $M^-(\infty)$ as $\varepsilon \to 0^+$, almost surely and in $\mathrm{L}^1(\mathcal{P}_x)$. In the same vein as in [12], we investigate the empirical measure $\varphi(\varepsilon)$ defined by

$$\big\langle \varphi(\varepsilon), f \big\rangle := \sum_{i=1}^{\infty} \chi_{i,\varepsilon}^{\omega_-} f\Big(\frac{\chi_{i,\varepsilon}}{\varepsilon}\Big).$$

Again, we let ξ^- denote the Lévy process with Laplace exponent Φ^- associated with the pssMp $\widehat{\mathcal{X}}$ via Lamperti's transformation; see (3.6). We can check that its Lévy measure Λ^- is given by

$$\int_{\mathbb{R}} g(y)\,\Lambda^-(\mathrm{d}y) = \int_{\mathbb{R}} \big[e^{y\omega_-} g(y) + \mathbb{1}_{\{y<0\}}(1-e^y)^{\omega_-} g\big(\log(1-e^y)\big)\big]\,\Lambda(\mathrm{d}y),$$

see [31, Theorem 3.9].

Theorem 3.10. *Suppose* (3.4), $\Lambda((0,\infty)) = 0$, *and that* ξ^- *is not arithmetic. Then as* $\varepsilon \to 0^+$, *the random measure* $\varphi(\varepsilon)$ *converges in* $\mathcal{P}_1$*-probability to* $M^-(\infty)\,\varphi$, *where* φ *is a deterministic probability measure on* $(0,1)$ *specified by*

$$\big\langle \varphi, f \big\rangle := \frac{\omega_+ - \omega_-}{-\kappa'(\omega_-)} \iint_{(-\infty,0)^2} f(e^x)\, e^{(\omega_+-\omega_-)y}\,\Lambda^-\big((-\infty, x+y)\big)\,\mathrm{d}x\mathrm{d}y. \tag{3.14}$$

Proof. As said previously we may suppose $\alpha = 0$, so that $\widehat{\mathcal{X}}$ is just the exponential of ξ^-. After Jagers [26], we can see that the random set $\mathbb{B}(\varepsilon) \subset \mathbb{B}$ is a so called *optional line* for which the strong branching property holds — intuitively, freezing the cells below ε is equivalent to freezing those which would descend from a family of cells that have first been frozen below $\varepsilon + \delta$, with $\delta > 0$ fixed. Specifically, by choosing $\delta := \sqrt{\varepsilon} - \varepsilon$ for $0 < \varepsilon < 1$ and scaling, we can write

$$\big\langle \varphi(\varepsilon), f \big\rangle = \sum_{i=1}^{\infty} \underbrace{\chi_{i,\sqrt{\varepsilon}}^{\omega_-}}_{\lambda_i(\varepsilon)} \underbrace{\sum_{j=1}^{\infty} \chi_{i,j,\varepsilon_i}^{\omega_-} f\Big(\frac{\chi_{i,\sqrt{\varepsilon}}\,\chi_{i,j,\varepsilon_i}}{\varepsilon}\Big)}_{Y_i(\varepsilon)},$$

where conditionally on $\boldsymbol{\lambda}(\varepsilon) := (\lambda_i(\varepsilon))_{i\geq 1}$, the $\{\chi_{i,j,\varepsilon_i}\}_{j=1}^{\infty}$, $i = 1, 2, \ldots$, are independent cell families respectively frozen below $\varepsilon_i := \varepsilon/\chi_{i,\sqrt{\varepsilon}}$. For every $1 < p < \omega_+/\omega_-$, (conditional) Jensen's inequality easily shows that the closed martingale $\mathcal{M}^-(\varepsilon)$ is bounded (by $\mathcal{E}_1[M^-(\infty)^p]$) in $\mathrm{L}^p(\mathcal{P}_1)$. Hence

$$\sup_{0<\varepsilon<1} \mathcal{E}_1\left[\left(\sum_{i=1}^{\infty} \lambda_i(\varepsilon)\right)^p\right] < \infty,$$

and, because $\mathcal{E}_1[\mathcal{M}^-(\varepsilon)] = \mathcal{E}_1[M^-(\infty)] = 1$,

$$\mathcal{E}_1\left[\sum_{i=1}^{\infty} \lambda_i^p(\varepsilon)\right] \le \varepsilon^{(p-1)\omega_-} \xrightarrow[\varepsilon\to 0^+]{} 0.$$

The proof then continues like that of Theorem 3.4. Similarly to the many-to-one formula, Lemma 3.11 below gives

$$\mathcal{E}_1\big[\langle\varphi(\varepsilon), f\rangle\big] = \widehat{\mathcal{E}}_1^-\Big[f\Big(\widehat{\mathcal{X}}\big(\mathcal{T}_{\mathcal{L}}(\varepsilon)\big)/\varepsilon\Big)\Big],$$

where $\mathcal{T}_{\mathcal{L}}(\varepsilon) = \inf\{t \ge 0\colon \widehat{\mathcal{X}}(t) < \varepsilon\}$. It thus remains to find the distributional limit of $\widehat{\mathcal{X}}(\mathcal{T}_{\mathcal{L}}(\varepsilon))/\varepsilon$ as $\varepsilon \to 0^+$. Observe that up to taking the inverse exponential, this random variable corresponds to the overshoot above $-\log\varepsilon$ of the spectrally positive Lévy process $-\xi^-$, which drifts to ∞ a.s. (since $E[-\xi^-(1)] = -(\Phi^-)'(0+) = -\kappa'(\omega_-) \in (0,\infty)$). By a classical result of renewal theory (see e.g. [20] or [31, Theorem 5.7]) we have, for every continuous function $g\colon (0,\infty) \to \mathbb{R}$ with compact support,

$$E\big[g\big(-\xi^- - (-\log\varepsilon)\big)\big] \xrightarrow[\varepsilon\to 0^+]{} \frac{1}{\mu}\int_{(0,\infty)^2} g(x)\,\Pi(y+\mathrm{d}x)\mathrm{d}y, \tag{3.15}$$

with Π and μ respectively the jump measure and the expectation at time 1 of the ascending ladder height process associated with $-\xi^-$. On the one hand, from [23, Corollary 4.4.4.(iv)] we get

$$\mu = \frac{E[-\xi^-(1)]}{\mathsf{k}^*},$$

where k^* is the killing rate of the ascending ladder height process associated with ξ^-, and equals the right inverse at 0 of the Laplace exponent Φ^- (see for instance [31, Example 6.11]):

$$\mathsf{k}^* = \sup\big\{t \ge 0\colon \Phi^-(t) = 0\big\} = \omega_+ - \omega_-.$$

On the other hand, we know since the work of Vigon [43] (see also [31, Corollary 7.9]) that Π fulfills

$$\Pi\big((y,\infty)\big) = \int_0^{\infty} e^{-\mathsf{k}^* x}\Lambda^-\big((-\infty, -x-y)\big)\,\mathrm{d}x, \qquad y > 0.$$

An easy computation then enables us to identify the right-hand sides of (3.14) and (3.15) (with $g(x) \coloneqq f(e^{-x})$). $\square$

Lemma 3.11. *For every $x > 0$ and every bounded measurable function $f\colon (0,\infty) \to \mathbb{R}$,*

$$\mathcal{E}_x\left[\sum_{u\in\mathbb{B}(\varepsilon)} \mathcal{X}_u^{\omega_-}\big(\mathcal{T}_u(\varepsilon)\big)\, f\Big(\mathcal{X}_u\big(\mathcal{T}_u(\varepsilon)\big)\Big)\right] = x^{\omega_-}\,\widehat{\mathcal{E}}_x^-\Big[f\Big(\widehat{\mathcal{X}}\big(\mathcal{T}_{\mathcal{L}}(\varepsilon)\big)\Big)\Big],$$

with the usual convention $f(\partial) \coloneqq 0$.

Proof. We slightly adapt the proof of [10, Proposition 4.1]. To this end, recall the intrinsic martingale $\mathcal{M}^-$ evoked in Remark 3.2 and, in the paragraph following that remark, the definition of the randomly tagged branch $\mathcal{L}$. It is here convenient to write $u \succ \mathbb{B}(\varepsilon)$ if $u \in \mathbb{B}$ stems from a (unique) node in $\mathbb{B}(\varepsilon)$ that we then call $\bar{u}$ (i.e. $\bar{u} \in \mathbb{B}(\varepsilon)$ is a prefix of u). The (conditional) distribution of $\mathcal{L}$ in (3.5) gives

$$\begin{aligned}\widehat{\mathcal{E}}_x^-\Big[f\Big(\widehat{\mathcal{X}}\big(\mathcal{T}_{\mathcal{L}}(\varepsilon)\big)\Big)\mathbb{1}_{\{\mathcal{L}(k+1)\succ\mathbb{B}(\varepsilon)\}}\Big] &= \widehat{\mathcal{E}}_x^-\left[\sum_{|u|=k+1}\mathbb{1}_{\{u\succ\mathbb{B}(\varepsilon)\}}\mathbb{1}_{\{u\text{ ancestor of }\mathcal{L}\}}\, f\Big(\mathcal{X}_{\bar{u}}\big(\mathcal{T}_{\bar{u}}(\varepsilon)\big)\Big)\right]\\ &= \widehat{\mathcal{E}}_x^-\left[\frac{1}{\mathcal{M}^-(\infty)}\lim_{n\to\infty}\sum_{\substack{|u|=k+1\\|v|=n}}\mathbb{1}_{\{u\succ\mathbb{B}(\varepsilon)\}}\mathcal{X}_{uv}^{\omega_-}(0)\, f\Big(\mathcal{X}_{\bar{u}}\big(\mathcal{T}_{\bar{u}}(\varepsilon)\big)\Big)\right].\end{aligned}$$

Rewriting the latter in terms of $\mathcal{P}_x$ simplifies out $\mathcal{M}^-(\infty)$. The branching property at u and the martingale property of $\mathcal{M}^-$ then entail

$$\widehat{\mathcal{E}}_x^-\left[f\Big(\widehat{\mathcal{X}}\big(\mathcal{T}_{\mathcal{L}}(\varepsilon)\big)\Big)\mathbb{1}_{\{\mathcal{L}(k+1)\succ\mathbb{B}(\varepsilon)\}}\right] = x^{-\omega_-}\mathcal{E}_x\left[\sum_{|u|=k+1}\mathbb{1}_{\{u\succ\mathbb{B}(\varepsilon)\}}\mathcal{X}_u^{\omega_-}(0)\,f\Big(\mathcal{X}_{\bar{u}}\big(\mathcal{T}_{\bar{u}}(\varepsilon)\big)\Big)\right].$$

If we now gather the nodes u which have the same ancestor $v := \bar{u} \in \mathbb{B}(\varepsilon)$ and repeat the previous argument, we obtain

$$\widehat{\mathcal{E}}_x^-\left[f\Big(\widehat{\mathcal{X}}\big(\mathcal{T}_{\mathcal{L}}(\varepsilon)\big)\Big)\mathbb{1}_{\{\mathcal{L}(k+1)\succ\mathbb{B}(\varepsilon)\}}\right] = x^{-\omega_-}\mathcal{E}_x\left[\sum_{|v|\le k}\mathbb{1}_{\{v\in\mathbb{B}(\varepsilon)\}}\mathcal{X}_v^{\omega_-}\big(\mathcal{T}_v(\varepsilon)\big)\,f\Big(\mathcal{X}_v\big(\mathcal{T}_v(\varepsilon)\big)\Big)\right].$$

Since the event $\{\mathcal{L}(k+1)\succ\mathbb{B}(\varepsilon)\}$ must occur for k large enough when $\lim_{t\to\infty}\widehat{\mathcal{X}}(t)=0$ and $f(\partial)=0$ anyway, letting $k\to\infty$ yields the result by dominated convergence. □

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Acknowledgments. The author is grateful to Jean Bertoin for his kind assistance and fruitful discussions throughout this work, to Alexander R. Watson for pointing out several mistakes in an earlier version of Proposition 2.8, and to Bastien Mallein for his help to prove Theorem 3.8. Thanks are also addressed to Gabriel Berzunza, Robin Stephenson, and an anonymous referee for interesting comments.