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Duality and hypoellipticity: one-dimensional case studies*

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Abstract

To visualize how the randomness of a Markov process X is spreading, one can consider subset-valued dual processes I constructed by intertwining. In the framework of one-dimensional diffusions, we investigate the behavior of such dual processes I in the presence of hypoellipticity for X. The Pitman type property asserting that the measure of I is a time-changed Bessel 3 process is preserved, the effect of hypoellipticity is only found at the level of the time change. It enables to recover the density theorem of Hörmander in this simple degenerate setting, as well as to construct strong stationary times by introducing different dual processes.

Keywords: one-dimensional diffusions; hypoellipticity; duality by intertwining; Bessel 3 process; Hörmander's density theorem; strong stationary times.
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1 Introduction

The technique of *duality by intertwining* associates to a Markov process X a dual Markov process, which here will be taking subsets of the state space of X as values, showing how randomness is spreading. In particular, this approach offers decompositions of the time-marginal laws of X that are useful to deduce that they admit a density with respect to a reference measure, at positive times. In our program to recover Hörmander's theorem by following this probabilistic way, we investigate here the effects of hypoellipticity on duality, by considering the simple one-dimensional framework.

We begin by studying a toy model on $\mathbb{R}$. Consider the hypoelliptic stochastic differential equation (s.d.e.) on $X := (X(t))_{t\in[0,\tau)}$, with $\tau \in (0,+\infty]$ the potential explosion time,

$$\forall\ t \in [0,\tau), \qquad dX(t) \quad = \quad \sqrt{2}X^n(t)\,dB(t) + dt, \tag{1.1}$$

where $n \in \mathbb{N} := \{1,2,3,...\}$ and where $(B(t))_{t\geqslant 0}$ is a standard Brownian motion.

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In the next section, we will check that X is hypoelliptic of order n at 0 and that τ is a.s. infinite.

Let $\mathcal{I}$ stand for the set of nonempty closed intervals from $[-\infty, +\infty]$, which are either included into $[-\infty, 0)$ or into $[0, +\infty]$ and which are different from $\{-\infty\}$ and $\{+\infty\}$. Denote $\mathcal{S}$ the set of singletons from $\mathcal{I}$, i.e. $\mathcal{S} := \{\{x\} : x \in \mathbb{R}\}$. Consider μ_+ an μ_- the speed measures associated to X on $\mathbb{R}_+$ and $(-\infty, 0)$ (whose definition will be recalled in Sections 2 and 3, respectively). We define a Markov kernel Λ from $\mathcal{I}$ to $\mathbb{R}$ by

$$\forall\ \iota \in \mathcal{I},\ \forall\ A \in \mathcal{B}(\mathbb{R}),\ \Lambda(\iota, A) := \begin{cases} \delta_x(A) & \text{, when } \iota \text{ is the singleton } \{x\}, \\ \frac{\mu_-(\iota \cap A)}{\mu_-(\iota)} & \text{, when } \iota \in \mathcal{I} \backslash \mathcal{S} \text{ is included into } [-\infty, 0), \\ \frac{\mu_+(\iota \cap A)}{\mu_+(\iota)} & \text{, when } \iota \in \mathcal{I} \backslash \mathcal{S} \text{ is included into } [0, +\infty], \end{cases} \tag{1.2}$$

where $\mathcal{B}(\mathbb{R})$ stands for the set of Borel subsets from $\mathbb{R}$ and δ_x for the Dirac mass at x. We will check later on that the above expression are well-defined, as the denominators are finite.

Our first goal is the following construction of a dual process I with respect to X, a solution of (1.1):

Theorem 1.1. *There exists a process $I := (I(t))_{t \geqslant 0}$ taking values in $\mathcal{I}$ such that*

$$I(0) = \{X(0)\}, \tag{1.3}$$

$$\forall\ t > 0, \qquad \mathbb{P}[I(t) \in \mathcal{S}] = 0, \tag{1.4}$$

$$\forall\ t \geqslant 0, \qquad \mathcal{L}(X(t)|I[0,t]) = \Lambda(I(t), \cdot), \tag{1.5}$$

where the conditional law in the l.h.s. is with respect to the trajectory $I[0,t] := (I(s))_{s \in [0,t]}$. In particular, we have for any $t \geqslant 0$, the decomposition

$$\mathcal{L}(X(t)) = \int \Lambda(\iota, \cdot)\, \mathcal{L}(I(t))(d\iota),$$

and the r.h.s. is absolutely continuous with respect to the Lebesgue measure for $t > 0$.

As implied by (1.4), I immediately grows into a segment with non-empty interior. But contrary to the elliptic case, where the dual process never return to $\mathcal{S}$, I collapses into the singleton $\{0\}$ at τ_0, the time when X hits 0 (this happens in positive time when $X(0)$ is negative). The process I is continuous (for the Hausdorff topology on the compact subsets of $[-\infty, +\infty]$), except at τ_0, when I may be non left-continuous. Point (1.4) in Theorem 1.1 will be deduced from the fact that the law of τ_0 has no atom outside 0. Note that without this requirement, the trivial dual process defined by $I(t) := \{X(t)\}$, for all $t \geqslant 0$, would be suitable.

Remark 1.2. At first view, the discontinuity of I at τ_0 may be perturbing in the above diffusion context. But it is just a suggestion that the segment-valued process I is not the appropriate object to look at. Indeed, it would be better to consider the probability measure-valued Markov process $(\Lambda(I(t), \cdot))_{t \geqslant 0}$, which is continuous at τ_0, due to the fact that μ_- gives an infinite weight to the left neighborhoods of 0, which implies that

$$\lim_{x \to 0_-} \Lambda([-\infty, x], \cdot) = \delta_0. \tag{1.6}$$

Concerning probability measure-valued process, note that the deterministic flow $(\mathcal{L}(X(t)))_{t \geqslant 0}$ of time-marginal laws can also be seen as a (not very useful) dual, with respect to the kernel $\overline{\Lambda}$ which to a given probability measure associates a random point sampled according to this distribution. In some sense, we are looking for dual processes strictly between the opposite $(\delta_{X(t)})_{t \geqslant 0}$ and $(\mathcal{L}(X(t)))_{t \geqslant 0}$.

After τ_0, the behavior of I depends on n:
• For $n \in \mathbb{N}\backslash\{1\}$, in finite time the process I hits $[0,+\infty]$ and stays there afterward.
• For $n = 1$, the process I converges to $[0,+\infty]$ in large time, but never reaches it (starting from a singleton).

This dichotomy is also valid when $X(0)$ is non-negative and will be reformulated in terms of strong stationary times in the next sections.

But whatever $n \in \mathbb{N}$, Theorem 1.1 recovers, on this example, the density part of Hörmander's theorem, stating that for all $t > 0$, the law of $X(t)$ is absolutely continuous with respect to the Lebesgue measure.

This study can be extended to any hypoelliptic diffusion on $\mathbb{R}$ (or on an interval of $\mathbb{R}$), but we found the circle case more instructive.

Let a and b be two smooth functions on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, such that a is non-negative, $\sqrt{a}$ is smooth and vanishes at most at a finite number of points, write $\mathfrak{N}$ for their set. Assume that for any $x \in \mathfrak{N}$, $b(x) \neq 0$. Consider on $\mathcal{C}^\infty(\mathbb{T})$ the Markov generator

$$L \quad := \quad a\partial^2 + b\partial, \tag{1.7}$$

and let $X := (X(t))_{t\geqslant 0}$ be a corresponding diffusion process. The generator L is hypoelliptic and we are looking for the behavior in law of X for large times.

Let us write $\mathfrak{N} := \{\mathfrak{y}_k \,:\, k \in \mathbb{Z}_N\}$, where the representative points in $[0,1[$ satisfy $0 \leqslant \mathfrak{y}_0 < \mathfrak{y}_1 < \cdots < \mathfrak{y}_{N-1} < 1$ and where $N \in \mathbb{N}$ (what follows is also trivially true in the classical elliptic case where $N = 0$). For $k \in \mathbb{Z}_N$, let $\mathbb{I}_k$ be the projection on $\mathbb{T}$ of the interval $(\mathfrak{y}_k, \mathfrak{y}_{k+1})$ (for $l = N-1$, it is the interval $(\mathfrak{y}_{N-1}, \mathfrak{y}_0 + 1)$), to which is added $\mathfrak{y}_k$ if $b(\mathfrak{y}_k) > 0$ and $\mathfrak{y}_{k+1}$ if $b(\mathfrak{y}_{k+1}) < 0$. Remark that $(\mathbb{I}_k)_{k\in\mathbb{Z}_N}$ forms a partition of $\mathbb{T}$. Denote for $k \in \mathbb{Z}_N$, μ_k the speed measure associated to the restriction of L to $\mathbb{I}_k$. Let $\mathcal{I}$ stand for the set of non-empty closed intervals from $\mathbb{T}$ which are included into one of the $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$ and let $\mathcal{S} := \{\{x\} \,:\, x \in \mathbb{T}\}$. Define a Markov kernel Λ from $\mathcal{I}$ to $\mathbb{T}$ by

$$\forall\ \iota \in \mathcal{I},\ \forall\ A \in \mathcal{B}(\mathbb{T}),\ \Lambda(\iota, A) \quad := \quad \begin{cases} \delta_x(A) & \text{, when } \iota = \{x\} \in \mathcal{S}, \\ \frac{\mu_k(\iota \cap A)}{\mu_k(\iota)} & \text{, when } \iota \in \mathcal{I}\backslash\mathcal{S} \text{ with } \iota \subset \mathbb{I}_k \text{ and } k \in \mathbb{Z}_N. \end{cases} \tag{1.8}$$

In Section 4, it will be checked that the last r.h.s. is well-defined, i.e. $0 < \mu_k(\iota) < +\infty$ for $\iota \in \mathcal{I}\backslash\mathcal{S}$ with $\iota \subset \mathbb{I}_k$ and $k \in \mathbb{Z}_N$.

Theorem 1.1 extends to this context:

Theorem 1.3. *Let X be a diffusion on the circle whose generator is the hypoelliptic elliptic L given in* (1.7)*. There exists a dual process I associated to X satisfying all the statements of Theorem 1.1, where $\mathcal{I}$ and Λ are defined as in* (1.8)*.*

The process I collapses into a singleton when X hits $\mathfrak{N}$. But our definition of the dual process I will not always be optimal, with respect to the construction of strong stationary times. We will see that sometimes it is better to let the dual process I collapses into a pair of points when X exits from the segments $\mathbb{I}_k$ which are open, for $k \in \mathbb{Z}_N$. The description of the evolution of the corresponding dual process is a little more involved and left to Section 4, as well as the definition of another Markov kernel (4.9) replacing (1.8) and Theorem 4.4, the extension of Theorem 1.3 in this situation. Nevertheless and similarly to the toy model case, we deduce from Theorem 4.4 the density part of Hörmander's theorem for the one-dimensional generator (1.7).

Another interest of the dual process I, associated to the Markov kernel (1.8) and constructed in Theorem 1.3, is to quantify the convergence to equilibrium of X, but only when b takes different signs over $\mathfrak{N}$, in which case I converges a.s. for large time. When b has a constant sign over $\mathfrak{N}$, the process I does not converge a.s. for large time. Indeed, writing $I =: [Y, Z]$, one of the two processes Y or Z ends up being Markovian, with a behavior of the same nature as X, after the first time X goes through $\mathfrak{N}$. In this situation, X admits an invariant probability measure π absolutely continuous with respect to the Lebesgue measure. The support of π is $\mathbb{T}$ but its density vanishes on $\mathfrak{N}$. It is then natural to consider $\widetilde{\mathcal{I}}$ the set of non-empty closed intervals from $\mathbb{T}$ and to define a Markov kernel $\widetilde{\Lambda}$ from $\widetilde{\mathcal{I}}$ to $\mathbb{T}$ by

$$\forall\ \iota \in \widetilde{\mathcal{I}},\ \forall\ A \in \mathcal{B}(\mathbb{T}), \qquad \widetilde{\Lambda}(\iota, A) \quad := \quad \begin{cases} \delta_x(A) & \text{, when } \iota = \{x\} \in \mathcal{S}, \\ \frac{\pi(\iota\cap A)}{\pi(\iota)} & \text{, when } \iota \in \widetilde{\mathcal{I}}\backslash\mathcal{S}. \end{cases} \tag{1.9}$$

Theorem 1.3 is still valid when Λ is replaced by $\widetilde{\Lambda}$:

Theorem 1.4. *Let X be a diffusion on the circle whose generator is the hypoelliptic elliptic L given in* (1.7)*, where b has a constant sign over $\mathfrak{N}$. There exists a dual process $\widetilde{I} := (\widetilde{I}(t))_{t\geqslant 0}$ associated to X taking values in $\widetilde{\mathcal{I}}$ and satisfying all the statements of Theorem 1.1, with $\mathcal{I}$ and Λ replaced by $\widetilde{\mathcal{I}}$ and $\widetilde{\Lambda}$ defined in* (1.9).

The dual process $\widetilde{I}$ converges a.s. in finite time to the whole state space $\mathbb{T}$, so we are able to construct a strong stationary time for X and to deduce the weak convergence of $\mathcal{L}(X(t))$ toward π for large times. The dual process $\widetilde{I}$ never collapses to a singleton: in this situation the deduction of the density part of Hörmander's theorem is straightforward, since we have, whatever the initial condition,

$$\mathbb{P}[\forall\ t > 0, \widetilde{I}(t) \in \widetilde{\mathcal{I}}\backslash\mathcal{S}] \quad = \quad 1.$$

The plan of the paper is as follows: the next two sections are respectively devoted to the restriction of the toy model to $\mathbb{R}_+$ and to $\mathbb{R}_-$. In Section 4 we consider the circular hypoelliptic diffusion and its dual process mentioned in Theorem 1.3. Section 5 deals with the situation where b has a constant sign over $\mathfrak{N}$ and in particular Theorem 1.4. Finally Appendix A recalls and adapts some computations from [12] and [3] about the segment-valued dual processes.

2 On $\mathbb{R}_+$

The situation treated here is quite similar to that from [12]. This section serves as a reminder of some notions from the theory of duality by intertwining.

We begin by some general considerations about the diffusion $X := (X(t))_{t\in[0,\tau)}$ whose evolution is described in (1.1). The generator L associated to X is the operator acting on $\mathcal{C}^\infty(\mathbb{R})$ via

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}),\ \forall\ x \in \mathbb{R}, \qquad L[f](x) \quad := \quad x^{2n}\partial^2 f(x) + \partial f(x).$$

The Itô term in (1.1) can be transformed into a Stratanovitch term (see for instance Chapter 4 of Revuz and Yor [13]):

$$\begin{aligned} \sqrt{2}X^n(t)\,dB(t) \quad &= \quad \sqrt{2}X^n(t)\circ dB(t) - \frac{1}{2}d\left\langle \sqrt{2}X^n(t), B\right\rangle \\ &= \quad \sqrt{2}X^n(t)\circ dB(t) - nX^{2n-1}(t)\,dt, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ stands for the bracket of semi-martingales. It follows that the generator can be rewritten under the Hörmander's form $L = V_1^2 + V_0$, where V_0 and V_1 are the vector fields on $\mathbb{R}$, seen as first order differential operators, whose coefficients are given by

$$\forall\ x \in \mathbb{R}, \qquad \begin{cases} V_0(x) &:= \quad 1 - nx^{2n-1}, \\ V_1(x) &= \quad x^n. \end{cases}$$

To see that L satisfies the Hörmander's condition (cf. Hörmander [8] or the pedagogical paper of Hairer [7]), define for all $l \in \mathbb{Z}_+$, the set of vector fields $\mathcal{V}_l$ through the iteration

$$\begin{aligned} \mathcal{V}_0 &\coloneqq \{V_1\}, \\ \forall\, l \in \mathbb{Z}_+, \qquad \mathcal{V}_{l+1} &\coloneqq \mathcal{V}_l \cup \{[U,V] \,:\, U \in \mathcal{V}_l \text{ and } V \in \{V_0, V_1\}\}, \end{aligned}$$

where $[\cdot,\cdot]$ stands for the usual Lie bracket. For any $x \in \mathbb{R}$, let $\mathcal{V}_l(x) \coloneqq \{V(x) \,:\, V \in \mathcal{V}_l\}$. For any $x \in \mathbb{R}\backslash\{0\}$, we have $\mathcal{V}_0(x) \neq \{0\}$, so that L is elliptic on $\mathbb{R}\backslash\{0\}$. At 0, the first $l \in \mathbb{Z}_+$ such that $\mathcal{V}_0(0) \neq \{0\}$ is $l = n$, so that L is hypoelliptic of order n at 0, as announced in the introduction.

Despite the above choice of $\mathbb{R}$ as state space, starting from $\mathbb{R}_+$, the process X lives in $\mathbb{R}_+$. Indeed, to check the status of the point 0 seen from $\mathbb{R}_+$, let us introduce the **scale** and **speed functions** associated to L:

$$\forall\, x > 0, \qquad \sigma_+(x) \;\coloneqq\; \exp\left(-\int_1^x \frac{1}{u^{2n}}\,du\right) \;=\; \exp((x^{1-2n}-1)/(2n-1)), \tag{2.1}$$

$$\mu_+(x) \;\coloneqq\; \frac{1}{x^{2n}\sigma_+(x)} \;=\; v'_+(x), \tag{2.2}$$

where

$$\forall\, x > 0, \qquad v_+(x) \;\coloneqq\; \frac{1}{\sigma_+(x)} \;=\; \exp\left(\int_1^x \frac{1}{u^{2n}}\,du\right) \;=\; \exp((1-x^{1-2n})/(2n-1)).$$

The interest of these functions is that on $(0,+\infty)$, we can write

$$L \;=\; \frac{1}{\mu_+}\partial\left(\frac{1}{\sigma_+}\partial\right). \tag{2.3}$$

The corresponding **scale** and **speed measures**, also written σ_+ and μ_+, are given by

$$\begin{aligned} \forall\, z \geqslant y > 0, \qquad \sigma_+([y,z]) &= \int_y^z \sigma_+(x)\,dx, \\ \mu_+([y,z]) &= \int_y^z \mu_+(x)\,dx \;=\; v_+(z) - v_+(y). \end{aligned}$$

By considering their limits as y goes to 0_+, these expressions can be extended to

$$\begin{aligned} \forall\, z > 0, \qquad \sigma_+([0,z]) &= +\infty, \\ \mu_+([0,z]) &= v_+(z). \end{aligned}$$

We get that

$$\begin{aligned} \int_0^1 \sigma_+([0,x])\,\mu_+(x)dx &= +\infty, \\ \int_0^1 \mu_+([0,x])\,\sigma_+(x)dx &= \int_0^1 v_+(x)\sigma_+(x)\,dx \;=\; \int_0^1 1\,dx \;=\; 1 \;<\; +\infty. \end{aligned}$$

Thus using Chapter 15 from Karlin and Taylor [10], it appears that 0 is an entrance boundary for the restriction of L on $\mathbb{R}_+$: when $X(0)$ is distributed on $\mathbb{R}_+$, the positions of the process X are in $(0,+\infty)$ for any $t \in (0,\tau)$.

The status of $+\infty$ can be investigated similarly. Since $\sigma_+(y)$ converges to $\exp(-1/(2n-1))$ as y goes to $+\infty$, it appears that for any $x > 0$, $\sigma_+([x,+\infty)) = +\infty$ and consequently

$$\int_1^{+\infty} \sigma_+([x,+\infty))\,\mu_+(x)dx \;=\; +\infty.$$

Furthermore, we have as $x > 0$ goes to $+\infty$,

$$\begin{aligned}\mu_+([x,+\infty))\sigma_+(x) &= \frac{v_+(\infty)-v_+(x)}{v_+(x)}\\ &= \left(\exp\left(\int_x^{+\infty}\frac{1}{u^{2n}}\,du\right)-1\right)\\ &\sim \int_x^{+\infty}\frac{1}{u^{2n}}\,du\\ &= x^{1-2n}/(2n-1).\end{aligned}$$

It follows that

$$\int_1^{+\infty}\mu_+([x,+\infty))\,\sigma_+(x)dx \qquad \begin{cases} = +\infty & \text{, if } n=1,\\ < +\infty & \text{, if } n\in\mathbb{N}\backslash\{1\}.\end{cases}$$

Thus when X starts from an initial distribution on $\mathbb{R}_+$, we deduce again from Chapter 15 of Karlin and Taylor [10] that $+\infty$ is a natural boundary if $n = 1$ and an entrance boundary if $n \in \mathbb{N}\backslash\{1\}$. In both cases, $+\infty$ cannot be reached, so that $\tau = +\infty$ a.s.

Following the approach developed in [12], we would like to construct an intertwining dual to X. In this section, we restrict our attention to the case where X starts from $\mathbb{R}_+$.

Consider

$$\begin{aligned}\mathcal{I}_+ &\coloneqq \{(y,z) : y,z\in[0,+\infty],\, y\leqslant z\}\backslash\{(+\infty,+\infty)\},\\ \mathring{\mathcal{I}}_+ &\coloneqq \{(y,z)\in(0,+\infty)^2 : y<z\}\end{aligned}$$

(the interior of $\mathcal{I}_+\cap\mathbb{R}_+^2$) and the diagonal $\mathcal{S}_+ \coloneqq \{(y,y) : y\in\mathbb{R}_+\}\subset\mathcal{I}_+$. As in the introduction, the element $(y,z)\in\mathcal{I}_+$ should be interpreted as the compact interval $[y,z]$ in $\mathbb{R}_+\sqcup\{+\infty\}$ and the elements of $\mathcal{S}_+$, as singletons. This is illustrated by the following definition of the Markov kernel Λ_+ from $\mathcal{I}_+$ to $\mathbb{R}_+$:

$$\forall\ (y,z)\in\mathcal{I}_+,\ \forall\ A\in\mathcal{B}(\mathbb{R}_+),\qquad \Lambda_+((y,z),A) \coloneqq \begin{cases}\delta_y(A) & \text{, if } y=z,\\ \frac{\mu_+([y,z]\cap A)}{\mu_+([y,z])} & \text{, otherwise.}\end{cases}$$

Note that the above expression is well-defined, as we have

$$\mu_+([0,+\infty)) = v_+(+\infty)-v_+(0) = \exp\left(\int_1^{+\infty}\frac{1}{x^{2n}}\,dx\right)-0 < +\infty. \tag{2.4}$$

Let $\mathfrak{L}_+$ be the diffusion generator on $\mathring{\mathcal{I}}_+$ given by

$$\begin{aligned}\mathfrak{L}_+ \coloneqq\ & (z^n\partial_z-y^n\partial_y)^2+(ny^{2n-1}-1)\partial_y+(nz^{2n-1}-1)\partial_z\\ &+2\frac{y^n\mu_+(y)+z^n\mu_+(z)}{\mu_+([y,z])}(z^n\partial_z-y^n\partial_y).\end{aligned} \tag{2.5}$$

Complete this definition on $\{0\}\times(0,+\infty)$ by

$$\mathfrak{L}_+ \coloneqq (z^n\partial_z)^2+(nz^{2n-1}-1)\partial_z+2\frac{z^{2n}\mu_+(z)}{\mu_+([0,z])}\partial_z, \tag{2.6}$$

on $(0,+\infty)\times\{+\infty\}$ by

$$\mathfrak{L}_+ \coloneqq (y^n\partial_y)^2+(ny^{2n-1}-1)\partial_y-2\frac{y^{2n}\mu_+(y)}{\mu_+([y,+\infty))}\partial_y, \tag{2.7}$$

and on $(0,+\infty) \in \mathcal{I}_+$,

$$\mathfrak{L}_+ \quad \coloneqq \quad 0, \tag{2.8}$$

namely $(0,+\infty)$ (alias $[0,+\infty]$) is absorbing for $\mathfrak{L}_+$.

More precisely, $\mathfrak{L}_+$ is defined on $\mathfrak{D}_+$, the set of continuous and bounded functions on $\mathcal{I}_+$ which are smooth on each of the subsets $\mathring{\mathcal{I}}_+$, $\{0\} \times (0,+\infty)$ and $(0,+\infty) \times \{+\infty\}$. Since $\mathfrak{D}_+$ is an algebra, we define the **carré du champs** $\Gamma_{\mathfrak{L}_+}$ associated to $\mathfrak{L}_+$ via

$$\forall\, F,G \in \mathfrak{D}_+, \qquad \Gamma_{\mathfrak{L}_+}[F,G] \quad \coloneqq \quad \frac{1}{2}(\mathfrak{L}_+[FG] - F\mathfrak{L}_+[G] - G\mathfrak{L}_+[F]). \tag{2.9}$$

For instance on $\mathring{\mathcal{I}}_+$, we compute that

$$\forall\, (y,z) \in \mathring{\mathcal{I}}_+, \qquad \Gamma_{\mathfrak{L}_+}[F,G](y,z) \quad = \quad (z^n\partial_z - y^n\partial_y)[F](z^n\partial_z - y^n\partial_y)[G].$$

It is not difficult to check that for any $f \in \mathcal{C}_{\mathrm{b}}^\infty(\mathbb{R}_+)$, the set of bounded smooth functions on $\mathbb{R}_+$, the mapping $\Lambda_+[f]$ is an element of $\mathfrak{D}_+$.

The interest of Λ_+ and $\mathfrak{L}_+$ is the **intertwining relation** $\mathfrak{L}_+\Lambda_+ = \Lambda_+ L$, in the sense that,

$$\forall\, (y,z) \in \mathcal{I}_+ \backslash \mathcal{S}_+,\ \forall\, f \in \mathcal{C}_{\mathrm{b}}^\infty(\mathbb{R}_+), \qquad \mathfrak{L}_+[\Lambda_+[f]](y,z) \quad = \quad \Lambda_+[L[f]](y,z). \tag{2.10}$$

This can be checked by direct computation, as in Lemma 20 from [12]. Alternatively, as in [3], one can resort to an algebra $\mathcal{A}_+$ of convenient observables, containing the mappings $\Lambda_+[f]$ for $f \in \mathcal{C}_{\mathrm{b}}^\infty(\mathbb{R}_+)$, see Appendix A below with $(0,1)$ replaced by $\mathbb{R}_+$.

Following the arguments leading to Proposition 4 from [12], we get that the martingale problems associated to $(\mathfrak{D}_+, \mathfrak{L}_+)$ are well-posed:

Theorem 2.1. *For any probability distribution $\mathfrak{m}_0$ on $\mathcal{I}_+$, there is a unique (in law) continuous Markov process $I \coloneqq (Y(t), Z(t))_{t\geqslant 0}$ whose initial distribution is $\mathfrak{m}_0$ and whose generator is $\mathfrak{L}_+$ in the sense of martingale problems: for any $F \in \mathfrak{D}_+$, the process $M^F \coloneqq (M^F(t))_{t\geqslant 0}$ defined by*

$$\forall\, t \geqslant 0, \qquad M^F(t) \quad \coloneqq \quad F(Y(t),Z(t)) - F(Y(0),Z(0)) - \int_0^t \mathfrak{L}_+[F](Y(s),Z(s))\,ds$$

*is a local martingale. Furthermore the diagonal $\mathcal{S}_+$ is an **entrance boundary** for I: for any $t > 0$, we have $(Y(t),Z(t)) \notin \mathcal{S}_+$.*

Remark 2.2. On $\mathcal{I}_+\backslash\mathcal{S}_+$, the process $(Y(t),Z(t))_{t\geqslant 0}$ is constructed as a solution to the s.d.e.'s associated to the generator $\mathfrak{L}_+$, see Appendix A, with $c = 0$ in (A.1). For instance on $\mathring{\mathcal{I}}_+$, we have, up to the corresponding explosion time,

$$\begin{aligned} dY(t) \quad &= \quad -\sqrt{2}Y^n(t)dW(t) + \left(nY^{2n-1}(t) - 1 - 2\frac{\underline{\mu}_+(\{Y(t),Z(t)\})}{\mu_+([Y(t),Z(t)])}\right)Y^n(t)\,dt, \\ dZ(t) \quad &= \quad \sqrt{2}Z^n(t)dW(t) + \left(nZ^{2n-1}(t) - 1 + 2\frac{\underline{\mu}_+(\{Y(t),Z(t)\})}{\mu_+([Y(t),Z(t)])}\right)Z^n(t)\,dt, \end{aligned}$$

where $(W(t))_{t\geqslant 0}$ is a standard Brownian motion and where

$$\underline{\mu}_+ \quad \coloneqq \quad \sum_{x\in(0,+\infty)} x^n\mu_+(x)\delta_x. \tag{2.11}$$

For any $x_0 \in \mathbb{R}_+$, to get the singleton (x_0,x_0) as a starting point, an approximation by $(x_0, x_0+\epsilon)$, for small $\epsilon > 0$, is performed.

Stone-Weierstrass theorem enables us to see that the algebra $\mathcal{A}_+$ of observables presented in Appendix A is dense in the space of continuous functions on $\mathcal{I}_+\backslash\mathcal{S}_+$, endowed with the uniform convergence on compact subsets (but this is not true on $\mathcal{I}_+$, since the elementary observables vanish on $\mathcal{S}_+$, so that the composed observables from $\mathcal{A}_+$ does not separate the elements of $\mathcal{S}_+$). We strongly believe the martingale problems associated to $(\mathcal{A}_+, \mathfrak{L}_+)$ are equally well-posed (cf. Section 4.4 of Ethier and Kurtz [5] for valuable information in this direction).

As a consequence, we have the following result (this sentence is slightly misleading, since a preliminary version of Corollary 2.3 plays an important role in the proof of Theorem 2.1, to be able to let the process I start from the singletons from $\mathcal{S}_+$, see [12]), for which we need to introduce some notations:

$$\varsigma_+ \quad \coloneqq \quad 2\int_0^{+\infty} \underline{\mu}_+(\partial I(s))^2\, ds \tag{2.12}$$

(where $\underline{\mu}_+(\partial I(s)) = (Y(s))^n\mu_+(Y(s)) + (Z(s))^n\mu_+(Z(s))$, according to (2.11)), with the conventions that $x^n\mu_+(x) = 0$ for $x \in \{0, +\infty\}$, a priori $\varsigma_+ \in (0, +\infty]$, but we will see in Corollary 2.3 below that ς_+ is finite a.s. Let the **time change** $(\theta_+(t))_{t\in[0,\varsigma_+]}$ be defined by

$$\forall\ t \in [0, \varsigma_+), \qquad 2\int_0^{\theta_+(t)} \underline{\mu}_+(\partial I(s))^2\, ds \quad = \quad t, \tag{2.13}$$

and $\theta_+(\varsigma_+) \coloneqq \lim_{t\to(\varsigma_+)_-} \theta_+(t)$.

We are interested in the process $R_+ \coloneqq (R_+(t))_{t\geqslant 0}$ given by

$$\forall\ t \geqslant 0, \qquad R_+(t) \quad \coloneqq \quad \mu_+(I(\theta_+(t\wedge\varsigma_+))). \tag{2.14}$$

Proposition 14 from [12] and its proof lead to the following result.

Corollary 2.3. *The process R_+ is a Bessel process of dimension 3 starting from $\mu_+(I(0))$ and stopped when it hits $\mu_+((0, +\infty))$. In particular, ς_+ is finite a.s. and is the hitting time of $\mu_+((0, +\infty))$ by R_+. More precisely, we have (conditioning by the initial value $I(0)$ for the second point):*
- *for $n \in \mathbb{N}\backslash\{1\}$ or $I(0)$ of the form $(y_0, +\infty)$ for some $y_0 \in [0, +\infty)$, we have $\theta_+(\varsigma_+) < +\infty$ and the process I hits $(0, +\infty)$ in finite time (a.s.)*
- *for $n = 1$ and $I(0)$ not of the form $(y_0, +\infty)$ for some $y_0 \in [0, +\infty)$, we have $\theta_+(\varsigma_+) = +\infty$ and the process I does not hit $(0, +\infty)$ in finite time (a.s.).*

Proof. More precisely, Proposition 14 from [12] shows that $[0, \varsigma_+) \ni t \mapsto R_+(t)$ is a Bessel process of dimension 3 (stopped if $\varsigma_+ < +\infty$). If ς_+ was to be infinite, we would end up with

$$\lim_{t\to+\infty} \mu_+(I(\theta_+(t))) \quad = \quad +\infty,$$

in contradiction with the fact that $\mu_+([0, +\infty)) < +\infty$. So ς_+ must be finite a.s. From (2.12), we deduce that

$$\liminf_{t\to+\infty} \underline{\mu}_+(\partial I(t)) \quad = \quad 0,$$

namely

$$\limsup_{t\to+\infty} I(t) \quad = \quad (0, +\infty),$$

and in particular

$$\liminf_{t\to+\infty} Y(t) = 0,$$
$$\limsup_{t\to+\infty} Z(t) = +\infty.$$

• For $n \in \mathbb{N}\backslash\{1\}$, since 0 and $+\infty$ are entrance boundaries for X, we know from Theorem 1 in [12] that I hits $(0,+\infty)$ in finite time, say τ. So the mapping $\mathbb{R}_+ \ni \theta \mapsto \int_0^\theta \underline{\mu}_+(\partial I(s))^2\,ds$ is increasing on $[0,\tau)$ and constant on $[\tau,+\infty)$. It follows that $\lim_{t\to(\varsigma_+)_-}\theta_+(t) = \tau$.

• For $n = 1$ and $Z(0) \neq +\infty$, since $+\infty$ is not an entrance boundary, we know from Theorem 1 in [12] that Z does not hit $+\infty$ in finite time. Thus the mapping $\mathbb{R}_+ \ni \theta \mapsto \int_0^\theta \underline{\mu}_+(\partial I(s))^2\,ds$ is increasing and $\lim_{t\to(\varsigma_+)_-}\theta_+(t) = +\infty$.

When $Z(0) = +\infty$, since 0 is an entrance boundary for X, the proof of Theorem 1 in [12] shows that Y hits 0 in finite time. At this hitting time, I hits $(0,+\infty)$ and we are in the situation where $\theta_+(\varsigma_+) < +\infty$. □

Corollary 2.3 can be seen as an illustration of Theorem 1 from [12] for elliptic diffusions X defined on $\mathbb{R}$ (here $(0,+\infty)$), stating that the dual process hits the whole state space in finite time for all initial distributions if and only if both boundaries are of entrance type. But in the present context, we are not so much concerned with the behavior in large time as with the behavior in small time and with the influence of hypoellipticity. According to Corollary 2.3, the latter does not modify the Pitman-type property that the process of the volumes $(\mu_+(I(t)))_{t\geqslant 0}$ of the dual process is a stopped Bessel 3 process, up to a time change. The impact is to be found in the time change itself:

Proposition 2.4. *Fix $(y,z) \in \mathcal{I}_+$ and consider the process I defined in Theorem 2.1 starting from (y,z). There are several behaviors for the time change θ_+ as t goes to 0_+:*
• If $(y,z) \neq (0,0)$, we have

$$\theta_+(t) \quad\sim\quad \frac{t}{2(y^n\mu_+(y) + z^n\mu_+(z))^2}.$$

• If $(y,z) = (0,0)$, we have

$$\theta_+(t) \quad\sim\quad \frac{1}{((2n-1)\ln(1/t))^{1/(2n-1)}}.$$

Thus in the latter case, the volume $\mu_+[I]$ begins by evolving very slowly (since the inverse function $\theta_+^{-1}(t)$ is negligible with respect to t, for $t \to 0_+$) and the order n of hypoellipticity can be recovered through

$$n \quad=\quad \frac{1}{2}\left(1 + \lim_{t\to 0_+}\frac{\ln(\ln(1/t))}{\ln(1/\theta_+(t))}\right).$$

For multidimensional diffusions X, the hypoellipticity should also impact the *germ* of the shape of the dual process, see [3] for a first approach to the elliptic case.

Proof of Proposition 2.4. When $(y,z) \neq (0,0)$, we have $\underline{\mu}_+(\{y,z\}) = y^n\mu_+(y) + z^n\mu_+(z) > 0$, so by continuity of the diffusion $(I(t))_{t\geqslant 0}$, we get as $\theta \to 0_+$,

$$2\int_0^\theta \underline{\mu}_+(I(s))^2\,ds \quad\sim\quad 2(y^n\mu_+(y) + z^n\mu_+(z))^2\theta,$$

and this leads immediately to the first point.

When $(y,z)=(0,0)$, according to (2.6), the diffusion I is given by

$$\forall\, t\geqslant 0,\qquad I(t)\quad=\quad(0,Z(t)),$$

where $(Z(t))_{t\geqslant 0}$ is solution to the s.d.e.

$$\forall\, t>0,\qquad dZ(t)\quad=\quad\sqrt{2}Z^n(t)dW(t)+\left(nZ^{2n-1}(t)-1+2\frac{\underline{\mu}_+(\{0,Z(t)\})}{\mu_+([0,Z(t)])}\right)dt,$$

where $(W(t))_{t\geqslant 0}$ is a standard Brownian motion. We compute that for all $z>0$,

$$\frac{\underline{\mu}_+(\{0,z\})}{\mu_+([0,z])}\ =\ \frac{\underline{\mu}_+(z)}{\mu_+([0,z])}\ =\ \frac{z^{2n}\mu_+(z)}{v_+(z)}\ =\ \frac{1}{\sigma_+(z)v_+(z)}\ =\ 1,$$

so that the above s.d.e. is

$$\forall\, t>0,\qquad dZ(t)\quad=\quad\sqrt{2}Z^n(t)dW(t)+(nZ^{2n-1}(t)+1)dt,$$

from which we deduce that a.s. $Z(t)\sim t$ for small $t>0$.

Since for any $t>0$, we have $\underline{\mu}_+(z)=v_+(z)$, (2.13) can be rewritten under the form,

$$\int_0^{\theta_+(t)}\exp(-2Z^{1-2n}(s)/(2n-1))\,ds\quad=\quad\frac{\exp(-2/(2n-1))}{2}t.$$

Since for any $\epsilon>0$, we can find (a random) $t>0$ sufficiently small so that for any $s\in(0,t)$, $(1-\epsilon)s\leqslant Z(s)\leqslant(1+\epsilon)s$, we are led to study the behavior for small $\theta>0$ of $\int_0^\theta\exp(-\alpha s^{1-2n})\,ds$, where $\alpha>0$ is a constant (that will take the values $2(1-\epsilon)/(2n-1)$ and $2(1+\epsilon)/(2n-1)$). A usual integration by parts shows that for small $\theta>0$,

$$\int_0^\theta\exp(-\alpha s^{1-2n})\,ds\quad\sim\quad\theta^{2n}\exp(-\alpha\theta^{1-2n}),$$

and by consequence,

$$\ln\left(\int_0^\theta\exp(-\alpha s^{1-2n})\,ds\right)\quad\sim\quad-\frac{\alpha}{\theta^{2n-1}}.$$

These considerations show that for small $t>0$,

$$\frac{2}{(2n-1)\theta^{2n-1}}\quad\sim\quad\ln(1/t),$$

and this leads to the announced result when $(y,z)=(0,0)$. □

Due to (2.10), the arguments of Section 4 of [12] show that the processes X and I can be coupled in the following way:

Theorem 2.5. *Let $\mathfrak{m}_0$ be a probability distribution on $\mathcal{I}_+$ and consider $m_0\coloneqq\mathfrak{m}_0\Lambda_+$. There exists a coupling of X with initial distribution m_0 and of I with initial distribution $\mathfrak{m}_0$ such that for any $t\geqslant 0$,*

$$\mathcal{L}(X(t)|I[0,t])\quad=\quad\Lambda_+(I(t)).$$

Furthermore, the construction of I from X is adapted, in the sense that given the trajectory X, for any $t\geqslant 0$, the conditional law of $I[0,t]$ depends only on $X[0,t]$.

Remark 2.6. Note that conversely, for any probability distribution m_0 on $\mathbb{R}_+$, we can find a law $\mathfrak{m}_0$ on $\mathcal{I}_+$ such that $m_0 = \mathfrak{m}_0\Lambda_+$. It is sufficient for instance to take $\mathfrak{m}_0 \coloneqq \int \delta_{(x,x)}\, m_0(dx)$, as it was done in Theorem 1.1 (at least when $\mathcal{L}(X_0)$ is supported by $\mathbb{R}_+$). But in general it is not the unique possible choice, e.g. when $m_0 = \Lambda_+((y,z),\cdot)$, for some $(y,z) \in \mathcal{I}_+ \backslash \mathcal{S}_+$, just consider $\mathfrak{m}_0 = \delta_{(y,z)}$.

As a classical consequence, going back to Diaconis and Fill [4] in the framework of finite Markov chains (see also [12] for one-dimensional diffusions), we obtain the existence of strong stationary times when $n = 1$. Recall that a **strong stationary time** τ for X is a finite stopping time (with respect to a possibly enlarged filtration for X) such that τ and $X(\tau)$ are independent and such that $X(\tau)$ is distributed according to the **invariant distribution** π, the probability distribution whose density is proportional to μ_+ (π exists due to (2.4)).

Corollary 2.7. *As in Corollary 2.3, there are two situations:*
• *for $n \in \mathbb{N}\backslash\{1\}$, whatever the initial distribution supported by $\mathbb{R}_+$, there exists a strong stationary time for X.*
• *for $n = 1$, for some initial distributions on $\mathbb{R}_+$ (in particular for any initial Dirac measure), a strong stationary time does not exist for X.*

Proof. When $n \in \mathbb{N}\backslash\{1\}$, the first time I hits $(0, +\infty)$ is a strong stationary time for X, see for instance [12] for more details.

When $n = 1$, since $+\infty$ is not an entrance boundary for X, the proof of Theorem 1 in [12] shows that there is no strong stationary time τ for X, if the initial law of X is of the form $\Lambda([0, x_0], \cdot)$, for any $x_0 \in \mathbb{R}_+$ (because τ would be stochastically bounded below by the hitting time of $[0, +\infty]$ by I starting from $(0, x_0)$, which is infinite), see also Fill and Lyzinski [6]. In particular, there is no strong stationary time for X starting with $X(0) = 0$. Let us extend this result to all initial Dirac measure. So let $x_0 \in \mathbb{R}_+$ be given and assume, by contradiction, there is a strong stationary time for X starting from x_0. Then one would be able to construct a strong stationary time for X started from 0, by considering the first time X hits x_0 (which is a.s. finite) and by adding to it a strong stationary time for X starting from x_0. This is in contradiction with our previous observation, so there is no strong stationary time for X starting from x_0.

As at the end of the proof of Corollary 2.3, remark that if the initial distribution of X is of the form $\Lambda_+([x_0, +\infty], \cdot)$, for some $x_0 \in \mathbb{R}_+$, then there exists a strong stationary time for X, consider again the first time I hits $(0, +\infty)$. □

Here we are more interested in the following density result, which is the easy part of the Hörmander's theorem and corresponds to the last statement of Theorem 1.1 when $\mathcal{L}(X_0)$ is supported by $\mathbb{R}_+$.

Corollary 2.8. *Under the assumption of Theorem 2.5, write for any $t \geqslant 0$, $m_t \coloneqq \mathcal{L}(X(t))$ and $\mathfrak{m}_t \coloneqq \mathcal{L}(I(t))$. Then we have*

$$m_t = \int \Lambda_+(\iota, \cdot)\, \mathfrak{m}_t(d\iota).$$

In particular, for any $t > 0$, m_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$.

Proof. The above equality is obtained by taking the expectation in Theorem 2.5. From Theorem 2.1, for any $t > 0$, the set of singletons $\mathcal{S}_+$ is negligible with respect to $\mathfrak{m}_t$. Furthermore for any $\iota \in \mathcal{I}_+ \backslash \mathcal{S}_+$, $\Lambda(\iota, \cdot)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. We can thus conclude to the validity of the last statement of Corollary 2.8. □

3 On $\mathbb{R}_-$

The situation of $\mathbb{R}_-$ follows a pattern similar to the investigation of the previous section. Putting together the results on $\mathbb{R}_-$ and $\mathbb{R}_+$ will lead to Theorem 1.1.

On $\mathbb{R}_-$, it is then more convenient to consider

$$\forall\ x<0, \qquad \sigma_-(x) \ \coloneqq\ \exp\left(-\int_{-1}^{x}\frac{1}{u^{2n}}\,du\right) \ =\ \exp((x^{1-2n}+1)/(2n-1)), \tag{3.1}$$

$$\mu_-(x) \ \coloneqq\ \frac{1}{x^{2n}\sigma_-(x)} \ =\ v_-'(x), \tag{3.2}$$

where

$$\forall\ x<0, \qquad v_-(x) \ \coloneqq\ \frac{1}{\sigma_-(x)} \ =\ \exp\left(\int_{-1}^{x}\frac{1}{u^{2n}}\,du\right) \ =\ \exp(-(x^{1-2n}+1)/(2n-1)).$$

These modified scale and speed functions, where the base point 1 has been replaced by -1, lead to the corresponding scale and speed measures on $\mathbb{R}_-$, still denoted σ_- and μ_-. We compute that

$$\begin{aligned}
\int_{-1}^{0}\sigma_-([x,0])\,\mu_-(x)dx &\ <\ +\infty,\\
\int_{-1}^{0}\mu_-([x,0])\,\sigma_-(x)dx &\ =\ +\infty,\\
\int_{-\infty}^{-1}\sigma_-((-\infty,x])\,\mu_-(x)dx &\ =\ +\infty,\\
\int_{-\infty}^{-1}\mu_-((-\infty,x])\,\sigma_-(x)dx &\ \begin{cases} =\ +\infty & \text{, if } n=1,\\ <\ +\infty & \text{, if } n\in\mathbb{N}\setminus\{1\}.\end{cases}
\end{aligned}$$

Thus when X starts from an initial distribution supported by $\mathbb{R}_-$, 0 is an exit boundary (i.e. it is a.s. attained in finite time). Furthermore, depending on $n=1$ or $n\in\mathbb{N}\setminus\{1\}$, $-\infty$ is an entrance or a natural boundary.

As a summary, conditioning by the initial position, we have the following a.s. behavior for X: starting from $X(0)<0$, the diffusion will reach 0 in finite time and instantaneously pass to $(0,+\infty)$, where X will next live forever. Of course, when $X(0)=0$ or $X(0)>0$, the first stage or the first and second stages of this description has/have to be removed.

We now come to the construction of the dual process I when the initial distribution of X is supported by $(-\infty,0)$.

Consider

$$\begin{aligned}
\mathcal{I}_- &\ \coloneqq\ \{(y,z)\,:\,y,z\in[-\infty,0),\,y\leqslant z\}\setminus\{(-\infty,-\infty)\},\\
\mathring{\mathcal{I}}_- &\ \coloneqq\ \{(y,z)\in(-\infty,0)^2\,:\,y<z\},\\
\mathcal{S}_- &\ \coloneqq\ \{(y,y)\in\mathcal{I}_-\,:\,y\in(-\infty,0)\}.
\end{aligned}$$

Again, the element $(y,z)\in\mathcal{I}_-$ should be interpreted as the compact interval $[y,z]$ in $[-\infty,0)$. Let Λ_- be the Markov kernel from $\mathcal{I}_-$ to $(-\infty,0)$ given by:

$$\forall\ (y,z)\in\mathcal{I}_-,\ \forall\ A\in\mathcal{B}((-\infty,0)), \qquad \Lambda_-((y,z),A) \ \coloneqq\ \begin{cases}\delta_y(A) & \text{, if } y=z,\\ \frac{\mu_-([y,z]\cap A)}{\mu_-([y,z])} & \text{, otherwise.}\end{cases}$$

Note that the above expression is well-defined, as we have for any $x\in(-\infty,0)$,

$$\begin{aligned}
\mu_-((-\infty,x)) &\ =\ v_-(x)-v_-(-\infty)\\
&\ =\ \left(\exp\left(-\frac{x^{1-2n}}{2n-1}\right)-1\right)\exp\left(-\frac{1}{2n-1}\right)\ <\ +\infty.
\end{aligned} \tag{3.3}$$

Let $\mathfrak{L}_-$ be the diffusion generator on $\mathring{\mathcal{I}}_-$ given by

$$\begin{aligned}\mathfrak{L}_- \quad &\coloneqq \quad (z^n\partial_z - y^n\partial_y)^2 + (ny^{2n-1}-1)\partial_y + (nz^{2n-1}-1)\partial_z \\ &\quad +2\frac{y^n\mu_-(y)+z^n\mu_-(z)}{\mu_-([y,z])}(z^n\partial_z - y^n\partial_y),\end{aligned} \tag{3.4}$$

and complete this definition on $\{-\infty\}\times(-\infty,0)$ by

$$\mathfrak{L}_- \quad \coloneqq \quad (z^n\partial_z)^2 + (nz^{2n-1}-1)\partial_z + 2\frac{z^{2n}\mu_-(z)}{\mu_-([0,z])}\partial_z. \tag{3.5}$$

More precisely, $\mathfrak{L}_-$ is defined on $\mathcal{D}(\mathfrak{L}_-)$, the set of continuous functions on $\mathcal{I}_-$ which are smooth on each of the subsets $\mathring{\mathcal{I}}_-$ and $\{-\infty\}\times(-\infty,0)$. It is not difficult to check that for any $f\in\mathcal{C}_{\mathrm{b}}^\infty((-\infty,0))$, the mapping $\Lambda_-[f]$ is an element of $\mathcal{D}(\mathfrak{L}_-)$.

As in the previous section, the interest of Λ_- and $\mathfrak{L}_-$ is the intertwining relation $\mathfrak{L}_-\Lambda_-=\Lambda_-L$, in the sense that,

$$\forall\ \iota\in\mathcal{I}_-\backslash\mathcal{S}_-,\ \forall\ f\in\mathcal{C}_{\mathrm{b}}^\infty((-\infty,0)),\qquad \mathfrak{L}_-[\Lambda_-[f]](\iota) \quad = \quad \Lambda_-[L[f]](\iota). \tag{3.6}$$

Again, this can be computed directly as in Lemma 20 of [12] or by introducing, as in [3] (see also Appendix A), an algebra $\mathcal{A}_-\subset\mathcal{D}(\mathfrak{L}_-)$ and a measure $\underline{\mu}_-\coloneqq\sum_{x\in(-\infty,0)}x^{2n}\mu_-(x)$, similarly to what was done in the previous section, replacing $\mathbb{R}_+$ and μ_+ by $(-\infty,0)$ and μ_-.

The martingale problems associated to $(\mathcal{D}(\mathfrak{L}_-),\mathfrak{L}_-)$ are also well-posed:

Theorem 3.1. *For any probability distribution $\mathfrak{m}_0$ on $\mathcal{I}_-$, there is a unique (in law) continuous Markov process $I\coloneqq(Y(t),Z(t))_{t\in[0,\tau_I)}$ whose initial distribution is $\mathfrak{m}_0$ and whose generator is $\mathfrak{L}_-$ in the sense of martingale problems: for any $F\in\mathcal{D}(\mathfrak{L}_-)$, the process $M^F\coloneqq(M^F(t))_{t\geqslant 0}$ defined by*

$$\forall\ t\in[0,\tau_I),\qquad M^F(t) \quad \coloneqq \quad F(Y(t),Z(t))-F(Y(0),Z(0))-\int_0^t \mathfrak{L}_-[F](Y(s),Z(s))\,ds$$

is a local martingale. The diagonal $\mathcal{S}_-$ is an entrance boundary for I: for any $t\in(0,\tau_I)$, we have $(Y(t),Z(t))\notin\mathcal{S}_-$. Furthermore, the explosion time τ_I corresponds to the "hitting" time of 0 by Z, in the sense that

$$\lim_{t\to\tau_I-} Z(t) \quad = \quad 0. \tag{3.7}$$

Proof. The arguments are similar to those of Proposition 4 in [12], except that in this previous paper, the situation of an exit boundary was not considered. So let us sketch the necessary modifications. First consider the case where $\mathfrak{m}_0=\delta_{\iota_0}$, for some $\iota_0\in\mathcal{I}_-$. Consider $\epsilon>0$ such that $\iota_0\subset[-\infty,-2\epsilon)$. Let L_ϵ be the generator acting like L on $(-\infty,-\epsilon)$ and such that $-\epsilon$ is an reflecting boundary (i.e. a Neumann condition is imposed at $-\epsilon$ on the functions entering in the domain of L_ϵ). Use Proposition 4 in [12] to construct the corresponding generator $\mathfrak{L}_\epsilon$ and an associated $\mathcal{I}_{-,\epsilon}$-valued diffusion $I_\epsilon\coloneqq(Y_\epsilon,Z_\epsilon)$, where $\mathcal{I}_{-,\epsilon}$ stands for the elements of $\mathcal{I}_-$ included into $[-\infty,-\epsilon]$. The process Z_ϵ is stopped at the time τ_{I_ϵ} it hits $-\epsilon$. Up to this stopping time τ_{I_ϵ}, I_ϵ is the unique (in law) solution of the martingale problem associated to $\mathfrak{L}_\epsilon$ starting from ι_0. Due to the Dirichlet condition on Z_ϵ, some functions from $\mathcal{D}(\mathfrak{L})$ are missing to conclude that $(I_\epsilon(t\wedge\tau_{I_\epsilon}))_{t\geqslant 0}$ is a stopped solution of the martingale problem associated to $\mathfrak{L}$ and starting from ι_0. To go around this little difficulty, rather stop I_ϵ when Z_ϵ hits -2ϵ. When $\epsilon>0$ varies, all these processes are consistent, so we can apply Kolmogorov's extension theorem to get a process I as in the above theorem. Its uniqueness is shown similarly by stopping. For more general initial distribution $\mathfrak{m}_0$, just condition by $I(0)$, see for instance the book of Ethier and Kurtz [5]. □

Lemma 3.2. *The hitting time τ_I is a.s. finite.*

Proof. This result would be obvious, if we already had Theorem 3.5 below at our disposal, since it provides a coupling such that $Z(t) \geqslant X(t)$ for all $t \in [0, \tau_I)$ and we already know that X hits 0 in finite time.

But the finiteness of τ_I can also be proven directly. According to Appendix A, Z satisfies

$$\forall\ t \in [0, \tau_I), \qquad dZ(t) \quad = \quad \sqrt{2} Z^n dW(t) + \gamma(Y(t), Z(t))dt, \tag{3.8}$$

where

$$\forall\ (y, z) \in \mathcal{I}_- \backslash \mathcal{S}_-, \qquad \gamma(y, z) \quad \coloneqq \quad nz^{2n-1} - 1 + 2\frac{y^n \mu_-(y) + z^n \mu_-(z)}{\mu_-([y, z])} z^n.$$

Define

$$\forall\ z \in (-\infty, 0), \qquad \widetilde{\gamma}(z) \quad \coloneqq \quad \gamma(-\infty, z) \ = \ nz^{2n-1} - 1 + 2\frac{z^{2n}\mu_-(z)}{\mu_-((-\infty, z])}.$$

Since $y^n z^n > 0$, $z^{2n} > 0$ and $\mu_-([y, z]) \leqslant \mu_-((-\infty, z])$ for any $y < z \in (-\infty, 0)$, we get

$$\forall\ (y, z) \in \mathcal{I}_- \backslash \mathcal{S}_-, \qquad \gamma(y, z) \quad \geqslant \quad \widetilde{\gamma}(z). \tag{3.9}$$

Consider the diffusion $\widetilde{Z} \coloneqq (\widetilde{Z}(t))_{t \in [0, \widetilde{\tau})}$ on $(-\infty, 0)$, where $\widetilde{\tau}$ is the explosion time, starting with $\widetilde{Z}(0) = Z(0)$ and solution of the s.d.e.

$$\forall\ t \in [0, \widetilde{\tau}), \qquad d\widetilde{Z}(t) \quad = \quad \sqrt{2}\widetilde{Z}^n dW(t) + \widetilde{\gamma}(\widetilde{Z}(t))dt.$$

Due to (3.9), we have

$$\forall\ t \in [0, \tau_I \wedge \widetilde{\tau}), \qquad \widetilde{Z}(t) \quad \leqslant \quad Z(t), \tag{3.10}$$

so that $\tau_I \leqslant \widetilde{\tau}$. To prove rigorously (3.10), one must come back to the situation of constant diffusion coefficient, namely to consider, when $n \in \mathbb{N} \backslash \{1\}$,

$$\begin{aligned}
dZ^{1-n}(t) \quad &= \quad \sqrt{2}(1-n)dW(t) + \left((1-n)Z^{-n}(t)\gamma(Y(t), Z(t)) + n(n-1)Z^{n-1}(t)\right)dt, \\
d\widetilde{Z}^{1-n}(t) \quad &= \quad \sqrt{2}(1-n)dW(t) + \left((1-n)\widetilde{Z}^{-n}(t)\widetilde{\gamma}(\widetilde{Z}(t)) + n(n-1)\widetilde{Z}^{n-1}(t)\right)dt,
\end{aligned}$$

and when $n = 1$,

$$\begin{aligned}
d\ln(-Z(t)) \quad &= \quad \sqrt{2}dW(t) + \left(-Z^{-1}(t)\gamma(Y(t), Z(t)) - 1\right)dt, \\
d\ln(-\widetilde{Z}(t)) \quad &= \quad \sqrt{2}dW(t) + \left(-\widetilde{Z}^{-1}(t)\widetilde{\gamma}(\widetilde{Z}(t)) - 1\right)dt.
\end{aligned}$$

Classical comparison arguments (see for instance Chapter 6 of Ikeda and Watanabe [9]) are applied on these s.d.e. (be careful of the signs) to get (3.10).

To prove that τ_I is a.s. finite, it remains to show that $\widetilde{\tau}$ is a.s. finite. Since $\widetilde{Z}$ is a diffusion process, it is enough to check that 0 is an exit boundary and that $-\infty$ is not an exit boundary.

We compute that for any $z \in (-\infty, 0)$,

$$\begin{aligned}
\frac{z^{2n}\mu_-(z)}{\mu_-((-\infty, z])} \quad &= \quad \frac{1}{\sigma_-(z)(v_-(z) - v_-(0))} \\
&= \quad \frac{1}{1 - \exp(z^{1-2n}/(2n-1))}.
\end{aligned}$$

The last term converges to 1 as z goes to 0_- and is equivalent to $-(2n-1)z^{2n-1}$ as z goes to $-\infty$. Thus we get

$$\lim_{z\to 0_-} \widetilde{\gamma}(z) \quad = \quad -1,$$

and for z going to $-\infty$

$$\widetilde{\gamma}(z) \quad \sim \quad (-3n+2)z^{2n-1} \quad (\to +\infty).$$

Via the introduction of the corresponding scale and speed functions, Chapter 15 of Karlin and Taylor [10] implies that 0 is an exit boundary and that $-\infty$ is an entrance boundary. □

Transform the definitions given in (2.12), (2.13) and (2.14) into

$$\varsigma_- \quad \coloneqq \quad 2\int_0^{\tau_I} \underline{\mu}_-(\partial I(s))^2\, ds, \tag{3.11}$$

with the convention that $(-\infty)^n\mu_-(-\infty) = 0$, a priori $\varsigma_- \in (0,+\infty]$, but we will see in Corollary 3.3 below that ς_- is infinite a.s. Let the **time change** $(\theta_-(t))_{t\in[0,\varsigma_-]}$ be defined by

$$\forall\ t\in[0,\varsigma_-), \qquad 2\int_0^{\theta_-(t)} \underline{\mu}_-(\partial I(s))^2\, ds \quad = \quad t, \tag{3.12}$$

and $\theta_-(\varsigma_-) \coloneqq \lim_{t\to(\varsigma_-)_-} \theta_-(t)$.

We are interested in the process $R_- \coloneqq (R_-(t))_{t\geqslant 0}$ given by

$$\forall\ t\geqslant 0, \qquad R_-(t) \quad \coloneqq \quad \mu_-(I(\theta_-(t\wedge\varsigma_-))). \tag{3.13}$$

Corollary 3.3. *We have $\varsigma_- = +\infty$, $\theta_-(+\infty) = \tau_I$ and the process R_- is a Bessel process of dimension 3 starting from $\mu_-(I(0))$.*

Proof. Proposition 14 from [12] shows that $[0,\varsigma_-)\ni t\mapsto R_-(t)$ is a Bessel process of dimension 3 (stopped if $\varsigma_- < +\infty$). So to get that R_- is a Bessel process of dimension 3, we must show that the event $\mathcal{E} \coloneqq \{\varsigma_- < +\infty\}$ has probability 0.

Define

$$\widehat{\tau} \quad \coloneqq \quad \inf\{t\geqslant 0\ :\ Z(t)\geqslant Z(0)/2\},$$

which is a.s. finite according to Lemma 3.2. Let us begin by checking that on $\mathcal{E}$, the trajectory $(Z(t))_{t\in[\widehat{\tau},\tau_I]}$ is Hölder of any order $\alpha\in(0,1/2)$. Indeed, taking (3.8) into account, we have for any $s<t\in[\widehat{\tau},\tau_I)$,

$$Z(t)-Z(s) \quad = \quad M(t)-M(s)+\int_s^t nZ^{2n-1}(u)-1+2\frac{\underline{\mu}_-(I(u))}{\mu_-(I(u))}Z^n(u)\, du, \tag{3.14}$$

where

$$\forall\ t\geqslant 0, \qquad M(t) \quad = \quad \sqrt{2}\int_0^{\tau_I\wedge t} Z^n(u)\, dW(u).$$

Since $M \coloneqq (M(t))_{t\geqslant 0}$ is a continuous martingale, up to enlarging the underlying probability space, we can find a standard Brownian motion $\widetilde{W} \coloneqq (\widetilde{W}(t))_{t\geqslant 0}$ so that

$$\forall\ t\geqslant 0, \qquad M(t) \quad = \quad \widetilde{W}\left(2\int_0^{\tau_I\wedge t} Z^{2n}(u)\, du\right).$$

The trajectories of $\widetilde{W}$ are a.s. of order α (see e.g. Chapter 1 of Revuz and Yor [13]), so the same is true for M, since the mapping $\mathbb{R}_+ \ni t \mapsto \int_0^{\tau_I \wedge t} Z^{2n}(u)\, du$ is Lipschitzian (these statements hold a.s., i.e. the corresponding "constants" are random). The mapping $\mathbb{R}_+ \ni t \mapsto \int_0^{\tau_I \wedge t} nZ^{2n-1}(u) - 1\, du$ is also Lipschitzian, so according to (3.14), it remains to bound the term $\int_s^t \frac{\underline{\mu}_-(I(u))}{\mu_-(I(u))} Z^n(u)\, du$. This is done via Cauchy-Schwartz' inequality, for $s, t \in [\widehat{\tau}, \tau_I]$:

$$\begin{aligned} \left| \int_s^t \frac{\underline{\mu}_-(I(u))}{\mu_-(I(u))}\, du \right| &\leqslant \sqrt{\int_s^t \underline{\mu}_-(I(u))^2\, du} \sqrt{\int_s^t \frac{Z^{2n}(u)}{\mu_-^2(I(u))}\, du} \\ &\leqslant \max_{u \in [\widehat{\tau}, \tau_I]} \frac{|Z^n(u)|}{\mu_-(I(u))} \sqrt{\int_0^{\tau_I} \underline{\mu}_-(I(u))^2\, du} \sqrt{t-s}. \end{aligned}$$

The quantity $\max_{u \in [\widehat{\tau}, \tau_I]} |Z^n(u)|$ is finite by continuity of Z and $\max_{u \in [\widehat{\tau}, \tau_I]} 1/\mu_-(I(u))$ is finite due to the fact that the Bessel process of dimension 3 R_- does not hit zero once it has left 0 (this the reason for the introduction of $\widehat{\tau}$). Since on $\mathcal{E}$, $\sqrt{\int_0^{\tau_I} \underline{\mu}_-(I(u))^2\, du}$ is also finite, we deduce the trajectory $(Z(t))_{t \in [\widehat{\tau}, \tau_I]}$ is Hölder of order α. In particular, there exists a (random) constant $C > 0$ such that for all

$$\forall\, s \in [\widehat{\tau}, \tau_I], \qquad |Z_s| \;=\; |Z_s - Z_{\tau_I}| \;\leqslant\; C\, |\tau_I - s|^{1/4}.$$

We deduce that on $\mathcal{E}$,

$$\begin{aligned} \varsigma_- &\geqslant 2 \int_{\widehat{\tau}}^{\tau_I} \underline{\mu}_-(Z(s))^2\, ds \\ &= 2 \int_{\widehat{\tau}}^{\tau_I} v_-(Z(s))^2\, ds \\ &= 2 \int_{\widehat{\tau}}^{\tau_I} \exp(-2(Z^{1-2n}(s) + 1)/(2n-1))\, ds \\ &\geqslant 2 \int_{\widehat{\tau}}^{\tau_I} \exp(-2Z^{1-2n}(s)/(2n-1))\, ds \\ &\geqslant 2 \int_0^{\tau_I - \widehat{\tau}} \exp(2C^{1-2n} s^{(1-2n)/4}/(2n-1))\, ds \\ &= +\infty, \end{aligned}$$

in contradiction with the definition of $\mathcal{E}$. Since all the above assertions are a.s., we get that $\mathcal{E}$ is negligible.

Finally, the equality $\theta_-(+\infty) = \tau_I$ is a consequence of the (strict) monotonicity of the mapping $[0, \tau_I) \ni \theta \mapsto \int_0^\theta \underline{\mu}_-(\partial I(s))^2\, ds$. □

Remark 3.4. As a consequence of Corollary 3.3, we have

$$\lim_{t \to \tau_I -} \mu_-(I(t)) \;=\; +\infty. \tag{3.15}$$

It suggests the following behavior for approximations: for $\epsilon > 0$, consider the elliptic generator $L_\epsilon \coloneqq (x^{2n} + \epsilon)\partial^2 + \partial$ (not to be mistaken with the reflecting generator introduced in the proof of Theorem 3.1). The associated speed function μ_ϵ is defined by

$$\forall\, x \in \mathbb{R}, \qquad \mu_\epsilon(x) \;\coloneqq\; \frac{1}{x^{2n} + \epsilon} \exp\left(- \int_{-1}^x \frac{1}{u^{2n} + \epsilon}\, du \right).$$

It is also possible to define dual processes $(I_\epsilon(t))_{t \geqslant 0}$ with values in the set of closed intervals in the extended line $[-\infty, +\infty]$ (except the singletons $\{-\infty\}$ and $\{+\infty\}$). Assume that $I_\epsilon(0)$ is a fixed element of $\mathcal{I}_-$. Then we guess that

$$\lim_{\epsilon\to 0_+} \mu_\epsilon(I_\epsilon(\tau_\epsilon)) \quad = \quad +\infty,$$

where $\tau_\epsilon \coloneqq \inf\{t > 0 \,:\, 0 \in I_\epsilon(t)\}$ (or at least with $\tau_\epsilon \coloneqq \inf\{t > 0 \,:\, \eta \in I_\epsilon(t)\}$, for all fixed $\eta > 0$).

Due to (3.6), the processes X and I can be coupled in the following way:

Theorem 3.5. *Let $\mathfrak{m}_0$ be a probability distribution on $\mathcal{I}_-$ and consider $m_0 \coloneqq \mathfrak{m}_0\Lambda_-$. There exists a coupling of X with initial distribution m_0 and of I with initial distribution $\mathfrak{m}_0$ such that for any $t \geqslant 0$, we have on $\{\tau_I > t\}$,*

$$\mathcal{L}(X(t)|I[0,t]) \quad = \quad \Lambda_-(I(t), \cdot). \tag{3.16}$$

Furthermore, the construction of I from X is adapted.

With the above coupling, we get that $\tau_I = \tau_0$, the hitting time of 0 by X seen in the introduction:

Proposition 3.6. *In addition to* (3.7)*, we have*

$$\lim_{t\to \tau_I -} X(t) \quad = \quad 0.$$

Proof. Since a.s., for all $t \in [0, \tau_I)$, we have $X(t) \leqslant Z(t)$, it follows that $\tau_I \leqslant \tau_0$. To see the converse inequality, define for any $\epsilon > 0$,

$$\tau_\epsilon \quad \coloneqq \quad \inf\{t \geqslant 0 \,:\, Z(t) \geqslant -\epsilon\}.$$

We have

$$\lim_{\epsilon\to 0_+} \tau_\epsilon \quad = \quad \tau_0,$$

thus by continuity of the the trajectories of X, a.s.

$$\lim_{\epsilon\to 0_+} X(\tau_\epsilon) \quad = \quad X(\tau_I).$$

To get $X(\tau_I) = 0$, it is sufficient to check that $X(\tau_\epsilon)$ converges in probability toward 0 as ϵ goes to 0_+. The relation (3.16) is also true when t is replaced by a stopping time for I (see Diaconis and Fill [4]), so we have

$$\mathcal{L}(X(\tau_\epsilon)|I[0,\tau_\epsilon]) \quad = \quad \Lambda_-(I(\tau_\epsilon), \cdot).$$

It follows that for any given $\eta > 0$,

$$\mathbb{P}[X(\tau_\epsilon) \in [-\eta, 0]|I[0,\tau_\epsilon]) \quad = \quad \Lambda_-(I(\tau_\epsilon), [-\eta, 0]).$$

Taking expectation, we deduce that

$$\mathbb{P}[X(\tau_\epsilon) \in [-\eta, 0]] \quad = \quad \mathbb{E}[\Lambda_-(I(\tau_\epsilon), [-\eta, 0])].$$

Note that we have $\Lambda_-(I(\tau_\epsilon), [-\eta, 0]) \leqslant \Lambda_-([-\infty, Z(\tau_\epsilon)], [-\eta, 0])$, so by dominated convergence, (1.6) implies that

$$\lim_{\epsilon\to 0_+} \mathbb{P}[X(\tau_\epsilon) \in [-\eta, 0]] \;=\; \mathbb{E}[\Lambda_-(I(\tau_\epsilon), [-\eta, 0])] \;=\; 0,$$

as desired. □

In general, we cannot conclude that $\lim_{t\to(\tau_0)_-} Y(t) = 0$ (convergence which should be sufficiently slow to be compatible with (3.15)), e.g. if we started with $Y(0) = -\infty$, then $Y(t) = -\infty$ for all $t \in [0, \tau_0)$. Anyway, Proposition 3.6 enables to set $(Y(\tau_0), Z(\tau_0)) \coloneqq (0, 0)$ while preserving the validity of (3.16). See also Remark 1.2, where Λ is just Λ_- in (1.6).

Next we extend the process I after time τ_0 as in Theorem 2.1, starting from $(0, 0)$. Note that the Markov kernel Λ from $\mathcal{I}_- \sqcup \mathcal{I}_+$ to $\mathbb{R}$ defined in (1.2), is obtained by imposing that $\Lambda = \Lambda_-$ on $\mathcal{I}_- \times \mathcal{B}(\mathbb{R})$ and $\Lambda = \Lambda_+$ on $\mathcal{I}_+ \times \mathcal{B}(\mathbb{R})$. Taking into account this observation, we can merge Theorems 2.5 and 3.5 and Corollary 2.8 into Theorem 1.1.

Remark 3.7. Corollary 2.7 is still valid, replacing $\mathbb{R}_+$ by $\mathbb{R}$. Indeed, the unique invariant measure remains π, the probability measure defined before Corollary 2.7. The first time I hits $(0, +\infty)$ is a strong stationary time, as soon as it is finite.

To deduce the density part of Hörmander's theorem, stating that for any $t > 0$, $\mathcal{L}(X(t))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, it remains to show the next result:

Lemma 3.8. *For any $t > 0$, we have $\mathbb{P}[I(t) = (0, 0)] = 0$.*

Proof. According to the previous considerations, we have for $t > 0$, a.s.

$$\{I(t) = (0, 0)\} \ = \ \{X(t) = 0\} \ = \ \{\tau_0 = t\}.$$

To prove that $\mathbb{P}[\tau_0 = t] = 0$, up to conditioning with respect to $X(0)$, we can assume that $X(0) = x_0$ for some $x_0 \in \mathbb{R}$. When $x_0 \geqslant 0$, the previous section shows that $\mathbb{P}[X(t) = 0] = 0$ for all $t > 0$. So assume that $x_0 < 0$ and decompose $\tau_0 = \widetilde{\tau} + \widehat{\tau}$, with

$$\begin{aligned} \widetilde{\tau} &\coloneqq \inf\{t \geqslant 0 \,:\, X(t) = x_0/2\}, \\ \widehat{\tau} &\coloneqq \inf\{t \geqslant 0 \,:\, X(\widetilde{\tau} + t) = 0\}. \end{aligned}$$

Due to the strong Markov property of X, $\widetilde{\tau}$ and $\widehat{\tau}$ are independent. Thus to get that the law of τ_0 has no atom, it is sufficient to see that $\mathcal{L}(\widetilde{\tau})$ has no atom. By contradiction, assume there exists $s > 0$ such that $\mathbb{P}[\widetilde{\tau} = s] > 0$. We would have $\mathbb{P}[X(s) = x_0/2] > 0$. Couple X with $I = (Y, Z)$ starting from (x_0, x_0) as in Theorem 3.5. Taking into account the equality $\tau_0 = \tau_I$ and (3.16), we have

$$\begin{aligned} \mathbb{P}[X(s) = x_0/2] &= \mathbb{P}[X(s) = x_0/2,\, \tau_0 > s] \\ &= \mathbb{P}[X(s) = x_0/2,\, \tau_I > s] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{1}_{X(s)=x_0/2} | I[0, s]] \mathbb{1}_{\tau_I > s}] \\ &= \mathbb{E}[\Lambda_-(I(s), x_0/2) \mathbb{1}_{\tau_I > s}] \\ &= 0, \end{aligned}$$

because for $s \in (0, \tau_I)$, $\Lambda_-(I(s), \cdot)$ is absolutely continuous with respect to the Lebesgue measure. This is the wanted contradiction. □

4 On the circle

In the circle framework presented in the introduction, we begin by studying X and its dual I on each of the segments $\mathbb{I}_k$, with $k \in \mathbb{Z}_N$. The global behavior of (X, I) is deduced by putting together the obtained informations, similarly to what was done in the previous section.

Let $\mathbb{I}$ be one of the segments $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$. To simplify the notation, we see $\mathbb{I}$ as a subset of $\mathbb{R}$ and up to an affine transformation, we assume that the interior of $\mathbb{I}$ is $(0, 1)$ (where the boundaries 0 and 1, may or not be the same in $\mathbb{T}$). There are four possibilities for the status of the boundaries of $\mathbb{I}$, that we investigate below. First we recall some classical definitions, valid in the four cases. To the restriction on $\mathbb{I}$ of the generator L defined in (1.7), we associate its scale and speed functions:

$$\forall\, x \in (0,1), \qquad \sigma(x) \;\coloneqq\; \exp\left(-\int_{1/2}^{x} \frac{b(u)}{a(u)}\,du\right), \tag{4.1}$$

$$\mu(x) \;\coloneqq\; \frac{1}{a(x)\sigma(x)}. \tag{4.2}$$

The interest of these functions is that on $(0,1)$, we can write

$$L \;=\; \frac{1}{\mu}\partial\left(\frac{1}{\sigma}\partial\right). \tag{4.3}$$

The corresponding scale and speed measures, also written σ and μ, are given by

$$\forall\, z \geqslant y \in (0,1), \qquad \sigma([y,z]) \;=\; \int_y^z \sigma(x)\,dx,$$
$$\mu([y,z]) \;=\; \int_y^z \mu(x)\,dx.$$

With the notation of Chapter 15 from Karlin and Taylor [10], define

$$\Sigma(0) \;\coloneqq\; \int_0^{1/2} \sigma((0,u))\,\mu(u)du, \qquad N(0) \;\coloneqq\; \int_0^{1/2} \mu((0,u))\,\sigma(u)du,$$
$$\Sigma(1) \;\coloneqq\; \int_{1/2}^{1} \sigma((u,1))\,\mu(u)du, \qquad N(1) \;\coloneqq\; \int_{1/2}^{1} \mu((u,1))\,\sigma(u)du.$$

The finiteness or not of $\Sigma(0)$ and $N(0)$ determine the status of the boundary 0 with respect to the diffusion X associated to L, seen from $\mathbb{I}$, and similarly for 1. To get these status of the boundaries, as well as their orders of ellipticity, we only need the asymptotic behavior of a and b near the boundaries. That is why we assumed $\sqrt{a}$ to be smooth, so that by considering expansions of $\sqrt{a}$ near the boundaries, we can come back to the computations made in Sections 2 and 3. Probably these computations can be extended to more general positive exponents n, in particular with $n = 1/2$ we would only need to assume that a is smooth. We refrained from this generality, just to avoid the emergence of singularities in the formulation of Hörmander's condition.

Define $\mathcal{I}$ the set of compact subsegments included in $\mathbb{I}$ and $\mathcal{S}$ the set of singletons from $\mathcal{I}$. Consider the Markov kernel Λ from $\mathcal{I}$ to $[0,1]$:

$$\forall\, [y,z] \in \mathcal{I}, \qquad \Lambda([y,z],\cdot) \;\coloneqq\; \begin{cases} \delta_y & \text{, if } y = z, \\ \frac{\mu([y,z]\cap\cdot)}{\mu([y,z])} & \text{, otherwise.} \end{cases}$$

• Case (C1): $\mathbb{I} = [0,1]$, namely $b(0) > 0$ and $b(1) < 0$, by considering the behavior of μ and σ near 0 and 1, we compute that $\Sigma(0) = +\infty$, $N(0) < +\infty$, $\Sigma(1) = +\infty$ and $N(1) < +\infty$, so that 0 and 1 are entrance boundaries for X. It follows that under the initial condition $X(0) = x_0$, where x_0 is fixed in $[0,1]$, the process X stays forever in $[0,1]$ and, more precisely, in $(0,1)$ for positive times. Since $\lim_{x\to 0_+} \mu(x) = 0 = \lim_{x\to 1_-} \mu(x)$, the measure μ has a finite weight over $\mathbb{I}$. It is also clear that μ is positive on $(0,1)$. It justifies the above definition of Λ and enables to define π as the normalization of μ into a probability measure, which is just $\Lambda([0,1],\cdot)$.

As in Section 2 and in [12], it is possible to construct a $\mathcal{I}$-valued dual process $I \coloneqq (I(t))_{t\geqslant 0}$, so that Theorem 1.1 is valid. It follows that for any $t > 0$, $\mathcal{L}(X(t))$

is absolutely continuous with respect to μ (or equivalently to the Lebesgue measure restricted to $[0,1]$), because $\mathcal{S}$ is an entrance boundary for I. More precisely, note that μ satisfies (A.1) with $c = 0$, so according to Appendix A, I can be described in the following way. Writing $I \coloneqq (Y,Z) \coloneqq ((Y(t),Z(t))_{t\geqslant 0}$, the processes Y and Z are solutions, up to the time (finite a.s.) when either Y hits 0 or Z hits 1, of the s.d.e.

$$\left\{\begin{array}{rcl} dY(t) & = & \left(a'(Y(t)) - b(Y(t)) - 2\frac{\sqrt{a(Y(t))}\mu(Y(t))+\sqrt{a(Z(t))}\mu(Z(t))}{\mu([Y(t),Z(t)])}\sqrt{a(Y(t))}\right)dt \\ & & -\sqrt{2a(Y(t))}\,dW(t), \\ dZ(t) & = & \left(a'(Z(t)) - b(Z(t)) + 2\frac{\sqrt{a(Y(t))}\mu(Y(t))+\sqrt{a(Z(t))}\mu(Z(t))}{\mu([Y(t),Z(t)])}\sqrt{a(Z(t))}\right)dt \\ & & +\sqrt{2a(Z(t))}\,dW(t). \end{array}\right. \tag{4.4}$$

where $(W(t))_{t\geqslant 0}$ is a standard Brownian motion. Assume for instance that Y hits 0 before Z hits 1, after the corresponding hitting time and up to the time Z hits 1, Z is solution of the s.d.e.

$$dZ(t) \quad = \quad \left(a'(Z(t)) - b(Z(t)) + 2\frac{\mu(Z(t))}{\mu([0,Z(t)])}a(Z(t))\right)dt + \sqrt{2a(Z(t))}\,dW(t). \tag{4.5}$$

Once Z hits 1, I remains at $[0,1]$. Furthermore, the covering time

$$\boldsymbol{\tau} \quad \coloneqq \quad \inf\{t\geqslant 0\,:\, I(t)=[0,1]\}$$

is finite a.s. and is a strong stationary time for X. Recall that the separation discrepancy between two probability measures m and π is defined in general via

$$\mathfrak{s}(m,\pi) \quad \coloneqq \quad \operatorname*{ess\,inf}_{\pi}\left(1-\frac{dm}{d\pi}\right),$$

where $dm/d\pi$ is the Radon-Nikodym derivative of the absolutely continuous part of m with respect to π. We have the following bound, due to Diaconis and Fill [4] (in the case of finite Markov chains, but valid in general):

$$\forall\, t\geqslant 0, \qquad \|\mathcal{L}(X(t))-\pi\|_{\mathrm{tv}} \;\leqslant\; \mathfrak{s}(\mathcal{L}(X(t)),\pi) \;\leqslant\; \mathbb{P}[\boldsymbol{\tau}\geqslant t], \tag{4.6}$$

where the norm in the l.h.s. is the total variation. In particular, $X(t)$ converges in law toward π for large $t\geqslant 0$.

• Case (C2): $\mathbb{I} = [0,1)$, namely $b(0) > 0$ and $b(1) > 0$, we get that $\Sigma(0) = +\infty$, $N(0) < +\infty$, $\Sigma(1) < +\infty$ and $N(1) = +\infty$, so that 0 is an entrance boundary and 1 an exit boundary for X. It follows that under the initial condition $X(0) = x_0$, where x_0 is fixed in $[0,1)$, the process X ends up exiting $[0,1)$ by hitting 1 in finite time, say at $\tau \coloneqq \inf\{t\geqslant 0\,:\, X(t)=1\}$. We have $\lim_{x\to 0_+}\mu(x) = 0$ (but $\lim_{x\to 1_-}\mu(x) = +\infty$), so any compact segment included into $\mathbb{I}$ has a finite weight, which is positive if it is not reduced to a singleton. Thus the Markov kernel Λ is well-defined.

As in Section 3, it is possible to construct a $\mathcal{I}$-valued dual process $I \coloneqq ([Y(t), Z(t)])_{t\in[0,\tau)}$, so that Theorem 3.5 is valid, see also Appendix A with $c=0$. Up to the time τ, the processes Y and Z are solutions to (4.4) (or (4.5), after Y has hit 0, this may happen or not before Z hits 1). We have a.s.

$$\lim_{t\to\tau_-} Z(t) \quad = \quad 1,$$

and the natural way to extend I after time τ is to define $I(\tau) = \{1\}$ and to let I start from there into the corresponding segment. Note that for any time $t\geqslant 0$, we can write

$$\mathcal{L}(X(t\wedge\tau)) \quad = \quad \mathbb{P}[\tau<t]\mathcal{L}(X(t)|\tau<t) + \mathbb{P}[\tau\geqslant t]\delta_1,$$

with

$$\mathcal{L}(X(t)|\tau < t) \quad = \quad \int \Lambda(\iota, \cdot)\, \mathbb{P}[I(t) \in d\iota | \tau < t],$$

so that the conditional law in the l.h.s. is absolutely continuous with respect to the Lebesgue measure for $t > 0$. As in Lemma 3.8, we show that for any given $t \geqslant 0$, $\mathbb{P}[\tau = t] = 0$.

• Case (C3): $\mathbb{I} = (0, 1]$, namely $b(0) < 0$ and $b(1) < 0$, we get that $\Sigma(0) < +\infty$, $N(0) = +\infty$, $\Sigma(1) = +\infty$ and $N(1) < +\infty$, so that 0 is an exit boundary and 1 an entrance boundary for X. This situation can be described as in the above case $\mathbb{I} = [1, 0)$, by symmetry.

• Case (C4): $\mathbb{I} = (0, 1)$, namely $b(0) < 0$ and $b(1) > 0$, we get that $\Sigma(0) < +\infty$, $N(0) = +\infty$, $\Sigma(1) < +\infty$ and $N(1) = +\infty$, so that 0 and 1 are exit boundaries for X. It follows that under the initial condition $X(0) = x_0$, where x_0 is fixed in $(0, 1)$, the process X ends up exiting $(0, 1)$ by hitting 0 or 1 in finite time, say $\tau_X \coloneqq \inf\{t \geqslant 0 : X(t) \in \{0, 1\}\}$. Since μ as function is continuous and positive, any compact segment included into $\mathbb{I}$ has a finite weight, which is positive if it is not reduced to a singleton. Again the Markov kernel Λ is well-defined.

As in Section 3, it is possible to construct a $\mathcal{I}$-valued dual process $I \coloneqq ([Y(t), Z(t)])_{t \in [0,\tau_I)}$, where $\tau_I > 0$ is the explosion time, so that Theorem 3.5 is valid, see also Appendix A with $c = 0$. Up to the time τ_I, the processes Y and Z are still solutions to (4.4). A priori the explosion time τ_I is such that $\tau_I \leqslant \tau_X$, but the arguments of Proposition 3.6 show that

$$\lim_{t \to \tau_I -} Y(t) \;=\; 0 \quad \text{or} \quad \lim_{t \to \tau_I -} Z(t) \;=\; 1,$$

and $\tau_I = \tau_X$.

When $\lim_{t \to \tau_I -} Y(t) = 0$ and $\lim_{t \to \tau_I -} Z(t) < 1$, it is safe to set $I(\tau_I) = \{0\}$. In this situation we have $X(\tau_I) = 0$, according to the proof of Proposition 3.6. We can thus let (X, I) start from $(0, \{0\})$ into the segment containing 0. Symmetrically when $\lim_{t \to \tau_I -} Y(t) > 0$ and $\lim_{t \to \tau_I -} Z(t) = 1$, we set $I(\tau_I) = \{1\}$ and we have $X(\tau_I) = 1$, so we can let (X, I) start from $(1, \{1\})$ into the segment containing 1.

Consider now the case where $\lim_{t \to \tau_I -} Y(t) = 0$ and $\lim_{t \to \tau_I -} Z(t) = 1$. When furthermore we have

$$\lim_{t \to \tau_I -} \Lambda(I(t), \cdot) \;=\; \delta_0 \quad \text{or} \quad \lim_{t \to \tau_I -} \Lambda(I(t), \cdot) \;=\; \delta_1, \tag{4.7}$$

again we can respectively define $I(\tau_I) = \{0\}$ and $I(\tau_I) = \{1\}$.

But what should we do when the limit of $\Lambda(I(t), \cdot)$, as t goes to τ_I-, charges both 0 and 1, or worse, if this limit does not exist? In fact, we believe the former alternative is always true (killing even the possibility of (4.7)):

Conjecture 4.1. *In the Case (C4), we have a.s.*

$$\lim_{t \to \tau_I -} \Lambda(I(t), \cdot) \quad = \quad \mathcal{L}(X(\tau_I)).$$

Whether this assertion is true or wrong, it is always possible to look at $X(\tau_I)$, which is either 0 or 1, and to set $I(\tau_I) = \{X(\tau_I)\}$. This idea was also used in Copros [2], in the context of denumerable Markov processes. Immediately after τ_I, X and I will evolve in the segment containing $\{X(\tau_I)\}$. This choice leads to a dual process I satisfying Theorem 1.1.

Remark 4.2. One does not need to wait that X pass through 0 or 1 for making an observation of X and subsequently concentrate I to a singleton: at any stopping time ζ for X, one can decide to change the value of I and impose that $I(\zeta) = \{X_\zeta\}$. This quantum physics sounding property does not impact condition (1.5), but of course it may destroy condition (1.4), for instance if ζ is the minimum of a positive deterministic time with τ_X. Note that the observation may also be imperfect: assume that $(0, 1)$ is decomposed into a measurable partition $\sqcup_{s\in S} A_s$, where S is a denumerable index set, and that we observe that $X(\zeta) \in A_s$, then we can replace $I(\zeta)$ by $I(\zeta) \cap A_s$. In general we are looking for the largest dual processes, so the above observation/concentration procedure should be avoided, see Example 4.3 below.

More precisely, let us come back to the circle setting described before Theorem 1.3. Consider the segments $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$, as the vertices of an oriented graph whose edges are as follows: there is an edge from $\mathbb{I}_k$ to $\mathbb{I}_{k+1}$, if $\mathfrak{y}_{k+1} \in \mathbb{I}_{k+1}$ and an edge from $\mathbb{I}_{k+1}$ to $\mathbb{I}_k$, if $\mathfrak{y}_{k+1} \in \mathbb{I}_k$. Except when the segments are all of type (C2), or all of type (C3), following the oriented edges, one goes from segments of type (C4) or **springs** to segments of type (C1) or **sinks**, after possibly visiting a successive sequence of segments of type (C2), turning anti-clockwise, or a successive sequence of segments of type (C3), turning clockwise. In particular, it appears that the number of springs is the number of sinks. Inside each segment, the dual process is constructed according to its type. From the above considerations, we get all the requirements on the dual process I presented in Theorem 1.1.

Note that the segments are all of type (C2) (respectively (C3)) if and only if b is positive (resp. negative) on $\mathfrak{N}$. Thus assuming the drift b does not take a fixed sign on $\mathfrak{N}$, whatever the starting point, X ends up into a sink in finite time, since the exit times from segments of type (C2), (C3) and (C4) are all a.s. finite. In this situation, for large times, the process X converges in law, the process I converges a.s. and the limit law of X is $\mathbb{E}[\Lambda(I(+\infty), \cdot)]$, where $I(+\infty) \coloneqq \lim_{t\to+\infty} I(t)$ (convergence taking place in finite time). Except when there is only one sink (in which case it is possible to construct a strong stationary time, since there is a unique invariant probability measure, namely the normalizations of the speed measure on the sink), the limit law of X depends on its initial condition. E.g. starting from a spring, the process X have positive probabilities (depending on the exact initial position in the spring) to exit it from the right or from the left and with the same probabilities, I collapse on the right or on the left boundary. After that, I will converge toward the closest sink following the above edges. The limit law of X is then a convex combinaison (with the previous probabilities) of the normalizations of the corresponding speed measures.

When b has a fixed sign on $\mathfrak{N}$, the process I does not converge a.s. since it appears that $I(t_n) = \{X(t_n)\}$ for all $n \in \mathbb{N}$, where $(t_n)_{n\in\mathbb{N}}$ is the unbounded increasing sequence of times $t \geqslant 0$ such that $X(t) \in \mathfrak{N}$. More precisely, assume for instance that b is positive on $\mathfrak{N}$, after the first time X hits $\mathfrak{N}$, according to (4.5), we have, according to Appendix A with $c = 0$,

$$dZ(t) \quad = \quad \left(a'(Z(t)) - b(Z(t)) + 2\frac{\mu_{K(t)}(Z(t))}{\mu_{K(t)}([y_{K(t)}, Z(t)])}a(Z(t))\right) dt + \sqrt{2a(Z(t))}\, dW(t),$$

where $K(t)$ is the unique index $k \in \mathbb{Z}_N$ such that $Z(t) \in \mathbb{I}_k$ (furthermore, we have $Y(t) = y_{K(t)}$). Thus it appears that Z becomes a Markov processes, whose behavior is quite similar to that of X (they even coincide at each time X pass through $\mathfrak{N}$). The dual process I is not very helpful to understand the convergence in law of X. Indeed, as announced in the introduction, another dual process $\widetilde{I}$ should be considered to go in this direction. It will be done in the following section.

Let us now present an example showing the above dual is not optimal with respect to the construction of a strong stationary time.

Example 4.3. Consider on $\mathbb{T} \coloneqq \mathbb{R}/(2\pi\mathbb{Z})$, the operator $L \coloneqq a\partial^2 + b\partial$, with

$$\forall\, x \in \mathbb{T}, \qquad \begin{cases} a(x) & \coloneqq & \cos^2(x), \\ b(x) & = & \sin(x). \end{cases}$$

We have $N = 2$, $\mathfrak{y}_0 = \pi/2$ and $\mathfrak{y}_1 = 3\pi/2$, so that $\mathbb{I}_0 = (-\pi/2, \pi/2)$ is of type (C4) and $\mathbb{I}_1 = [\pi/2, 3\pi/2]$ is of type (C1). Consider the initial condition $X(0) = 0$. Due to the symmetry of $\mathbb{I}_0$ and of the coefficients a and b (anti-symmetric) with respect to 0, we deduce from (4.4) that we have $Z = -Y$ until X hits $\{-\pi/2, \pi/2\}$, say at time τ. In this situation, it appears that

$$\lim_{t \to \tau_-} \Lambda([-Z(t), Z(t)], \cdot) \quad = \quad \frac{1}{2}\left(\delta_{-\pi/2} + \delta_{\pi/2}\right).$$

Thus the natural extension seems to be $I(\tau) \coloneqq \{-\pi/2, \pi/2\}$, instead of $I(\tau) \coloneqq \{X(\tau)\}$. Indeed, in the former case, for $t \geqslant \tau$, we can construct a dual process of the form

$$I(t) \quad = \quad [-Z(t), -\pi/2] \sqcup [\pi/2, Z(t)],$$

where Z takes values in $[\pi/2, \pi]$ and solves the s.d.e.

$$\begin{aligned} dZ(t) \quad &= \quad \left(a'(Z(t)) - b(Z(t)) + 2\frac{\sqrt{a(Z(t))}\mu_1(Z(t)) + \sqrt{a(-Z(t))}\mu_1(-Z(t))}{\mu_1([-Z(t), -\pi/2] \sqcup [\pi/2, Z(t)])}\sqrt{a(Z(t))} \right) dt \\ &\quad + \sqrt{2a(Z(t))}\, dW(t) \qquad\qquad (4.8) \\ &= \quad \left(a'(Z(t)) - b(Z(t)) + 2\frac{\sqrt{a(Z(t))}\mu_1(Z(t))}{\mu_1([\pi/2, Z(t)])}\sqrt{a(Z(t))} \right) dt + \sqrt{2a(Z(t))}\, dW(t), \end{aligned}$$

where μ_1 is the speed measure associated to $\mathbb{I}_1$ and $(W(t))_{t \geqslant 0}$ is a standard Brownian motion. For the second equality, we used the symmetry of $\mathbb{I}_1$ and of L (with respect to the real axis, when $\mathbb{T}$ is seen as the unit circle in $\mathbb{C}$). When Z hits π, I hits $[\pi/2, 3\pi/2]$ and the corresponding hitting time is a strong stationary time for X.

Consider now the case where we set $I(\tau) = \{X(\tau)\}$ and assume for instance that $X(\tau) = \pi/2$. For $t \geqslant \tau$, our construction for Theorem 1.3 leads to a dual of the form $I(t) = [\pi/2, Z(t)]$, where Z takes values in $[\pi/2, 3\pi/2]$ and solves the s.d.e. (4.8). The dual process will be absorbed at $\mathbb{I}_1$ when Z hits $3\pi/2$ and this provides a strong stationary larger than the previous one, since Z must go through π before hitting $3\pi/2$.

As just seen, starting from 0, this example can be brought back to the case of a diffusion on a segment starting from its boundary. This situation is well-understood (see Fill and Lyzinski [6]) and the strong stationary time constructed in the former case is in fact sharp, namely stochastically smaller than any other strong stationary time.

For the remaining part of this section, let us assume that Conjecture 4.1 is true. To construct a dual process $J \coloneqq (J(t))_{t \geqslant 0}$ able to collapse on pairs of points, we modify the definitions given in the introduction in the following way. Let $\mathcal{I}_1$ stand for the set of non-empty closed intervals from $\mathbb{T}$ which are included into one of the $\mathbb{I}_k$, for $k \in \mathbb{Z}_N$ and $\mathcal{I}_2$ the set of pairs (ι_1, ι_2), where $\iota_1, \iota_2 \in \mathcal{I}_1$ are disjoints. Now set $\mathcal{I} \coloneqq \mathcal{I}_1 \sqcup \mathcal{I}_2$ and $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2$, with $\mathcal{S}_1 \coloneqq \{\{x\} \,:\, x \in \mathbb{T}\}$ and $\mathcal{S}_2 \coloneqq \{(\mathfrak{y}_k, \mathfrak{y}_l) \,:\, k \neq l \in \mathbb{Z}_N\}$. For any $\alpha \in [0, 1]$, define a Markov kernel Λ_α from $\mathcal{I}$ to $\mathbb{T}$ by

$$\forall\ \iota\in\mathcal{I},\qquad \Lambda_\alpha(\iota,\cdot)\ \coloneqq\ \begin{cases}\delta_x(\cdot) & \text{, when } \iota=\{x\}\in\mathcal{S}_1,\\ \alpha\delta_{\mathfrak{y}_k}(\cdot)+(1-\alpha)\delta_{\mathfrak{y}_l}(\cdot) & \text{, when } \iota=(\mathfrak{y}_k,\mathfrak{y}_l)\in\mathcal{S}_2,\\ \frac{\mu_k(\iota\cap\cdot)}{\mu_k(\iota)} & \text{, when } \iota\in\mathcal{I}_1\backslash\mathcal{S}_1 \text{ and } \iota\subset\mathbb{I}_k,\\ \frac{\alpha\mu_k(\iota_1\cap\cdot)+(1-\alpha)\mu_l(\iota_2\cap\cdot)}{\alpha\mu_k(\iota_1)+(1-\alpha)\mu_l(\iota_2)} & \text{, when } \iota=(\iota_1,\iota_2)\in\mathcal{I}_2\backslash\mathcal{S}_2,\ \iota_1\subset\mathbb{I}_k\\ & \quad\text{and } \iota_2\subset\mathbb{I}_l.\end{cases}\tag{4.9}$$

Then Theorem 1.3 can be extended into:

Theorem 4.4. *There exists a process $J\coloneqq(J(t))_{t\geqslant 0}$ taking values in $\mathcal{I}$, whose construction is adapted with respect to X, such that*

$$\begin{aligned} J(0) &= \{X(0)\},\\ \forall\ t>0,\qquad \mathbb{P}[J(t)\in\mathcal{S}] &= 0,\\ \forall\ t\geqslant 0,\qquad \mathcal{L}(X(t)|J[0,t]) &= \Lambda_{\alpha(I(0))}(J(t),\cdot),\end{aligned}$$

where $\alpha(I(0))\in[0,1]$ only depends on $I(0)$ (or equivalently on $X(0)$). In particular, when $\mathcal{L}(X(0))=\delta_{x_0}$ for some $x_0\in\mathbb{T}$, we have for any $t\geqslant 0$, the decomposition

$$\mathcal{L}(X(t))\ =\ \int\Lambda_{\alpha(\{x_0\})}(\iota,\cdot)\,\mathcal{L}(J(t))(d\iota).$$

Proof. When $X(0)$ does not belong to a spring, the dual process J is the same as I in Theorem 1.3 and the introduction of $\mathcal{I}_2$ and $\mathcal{S}_2$ are not necessary. When $X(0)=x_0$ belongs to a spring, say $\mathbb{I}_k$, let τ its exit time from $\mathbb{I}_k$ and $\alpha(\{x_0\})\coloneqq\mathbb{P}_{x_0}[X(\tau)\in\mathbb{I}_{k-1}]$. Before τ, J is constructed as I in Theorem 1.3, but at τ, we impose $J(\tau)\coloneqq(\{\mathfrak{y}_k\},\{\mathfrak{y}_{k+1}\})$. Conjecture 4.1 enables us to see that

$$\mathcal{L}(X(\tau)|J[0,\tau])\ =\ \Lambda_{\alpha(I(0))}(J(\tau),\cdot),$$

from which we can keep up constructing J after the time τ, by setting

$$\begin{aligned}\forall\ t\in[0,\widetilde{\tau}),\qquad J(\tau+t) &\coloneqq (I_1(t),I_2(t)),\\ \widetilde{\tau} &\coloneqq \inf\{t\geqslant 0\ :\ I_1(t)\cap I_2(t)\neq\varnothing\},\end{aligned}$$

where I_1 and I_2 are the same as in Theorem 1.3, starting with $I_1(0)=\{\mathfrak{y}_k\}$ and $I_2(0)=\{\mathfrak{y}_{k+1}\}$, and directed by the same Brownian motion $(W(t))_{t\geqslant 0}$ in (4.4) and (4.5). When $\widetilde{\tau}<+\infty$ (as in Example 4.3, where it corresponds to the time Z hits π), we set $J(\widetilde{\tau}+t)\coloneqq I_1(\widetilde{\tau})\cup I_2(\widetilde{\tau})$ for all $t\geqslant 0$. □

As in Remark 1.2, the probability measure-valued Markov process $(\Lambda_{\alpha(\{x_0\})}(J(t),\cdot)$ is continuous and seems the right object to consider as a dual.

The main advantage of Theorem 4.4 over Theorem 1.3, i.e. of the Markov kernel given in (4.9) over the Markov kernel (1.8), is that it enables to extend the construction of strong time stationary times $\mathfrak{r}$, in the sense that the position $X_{\mathfrak{r}}$ is distributed according to an invariant probability measure (maybe non longer the unique invariant probability measure as before). This is possible when $X(0)$ starts from a fixed position $x_0\in\mathbb{T}$ and when b does not take a fixed sign on $\mathfrak{N}$. Indeed, in this case the dual process ends up being absorbed in a state $J(\infty)$ depending only on x_0, which is either a closed segment from $\{\mathbb{I}_k\ :\ k\in\mathbb{Z}_N\}$ or a disjoint union of two such segments. Since $\Lambda_{\alpha(\{x_0\})}(J(\infty),\cdot)$ is an invariant probability measure for X depending only on x_0, classical arguments from Diaconis and Fill [4] then show that the absorbing time for J is a strong stationary time.

5 The turning diffusion

Here we consider more precisely the circle situation where b has a fixed sign on $\mathcal{N}$, to show Theorem 1.4 and to deduce the convergence of X in law for large time.

Up to conjugacy with respect to $\mathbb{T} \ni x \mapsto -x \in \mathbb{T}$, it is sufficient to study the case where $b > 0$ on $\mathcal{N}$. We begin by investigating the invariant measure for the generator L given in (1.7). For $k \in \mathbb{Z}_N$, recall that μ_k is the speed measure of the restriction of L on $\mathbb{I}_k$. It is defined up to a positive factor by

$$\forall\, x \in \mathbb{I}_k, \qquad \mu_k(x) \quad \coloneqq \quad \frac{1}{a(x)} \exp\left(\int_{[\mathfrak{z}_k, x]} \frac{b(u)}{a(u)}\, du\right),$$

where $\mathfrak{z}_k$ is a chosen point belonging to $\mathbb{I}_k$ and where a segment $[u, v] \subset \mathbb{T}$ will always be understood as the path going from u to v anti-clockwise. For any family of non-negative numbers $(p_k)_{k\in\mathbb{Z}_N}$, the measure $\mu \coloneqq \sum_{k\in\mathbb{Z}_N} p_k\mu_k$ satisfies $\mu[L[f]] = 0$ for any smooth f with compact support in $\mathbb{T}\backslash\mathfrak{N}$. But this is not sufficient for μ to be a invariant measure. Furthermore we are here looking for an invariant probability measure and it can be easily check that $\mu(\mathbb{T}) = +\infty$, except in the trivial case where all the p_k, for $k \in \mathbb{Z}_N$, are equal to zero. In fact, for fixed $k \in \mathbb{Z}_N$, the restriction of L to smooth functions with compact support in $\mathbb{I}_k$ is symmetric in $\mathbb{L}^2(\mu_k)$ but the problem at hand is really non reversible since the diffusion X has a strong tendency to turn anti-clockwise around $\mathbb{T}$. Lemma A.1 in Appendix A suggests to rather look for the solutions η_k of the equation given on the interior of $\mathbb{I}_k$ by

$$(a\eta_k)' \quad = \quad b\eta_k - c_k,$$

where c_k is a constant. When a did not vanish on $\{\mathfrak{y}_k, \mathfrak{y}_{k+1}\}$, it is not difficult to check that the general solution of this equation is, for any $x \in \mathring{\mathbb{I}}_k$,

$$\eta_k(x) \quad = \quad \frac{1}{a(x)}\left(p_k \int_{[\mathfrak{y}_k,x]} \exp\left(\int_{[u,x]} \frac{b}{a}(v)\, dv\right) du + q_k \int_{[x,\mathfrak{y}_{k+1}]} \exp\left(-\int_{[x,v]} \frac{b}{a}(v)\, dv\right)\right),$$

where p_k and q_k are two constants such that $p_k + q_k = c_k$. If we want this expression to converge when a does vanish on $\{\mathfrak{y}_k, \mathfrak{y}_{k+1}\}$ and b is positive on $\{\mathfrak{y}_k, \mathfrak{y}_{k+1}\}$, we must take $p_k = 0$. It leads us to consider

$$\forall\, x \in \mathring{\mathbb{I}}_k, \quad \eta_k(x) \quad = \quad \int_{[x,\mathfrak{y}_{k+1}]} \exp\left(-\int_{[x,v]} \frac{b}{a}(v)\, dv\right).$$

We compute that

$$\lim_{x\to\mathfrak{y}_{k+}} \eta_k(x) \;=\; 0 \;=\; \lim_{x\to\mathfrak{y}_{k+1-}} \eta_k(x),$$

so define $\eta_k(\mathfrak{y}_k) \coloneqq 0 \eqqcolon \eta_k(\mathfrak{y}_{k+1})$. Since we have $(a\eta_k)' = b\eta_k - 1$, we deduce from the decomposition (A.12) with $c = 1$ that for any $f \in \mathcal{C}^\infty([\mathfrak{y}_k, \mathfrak{y}_{k+1}])$, we have

$$\begin{aligned} \eta_k[L[f]] \quad &= \quad [a\eta_k f']_{\mathfrak{y}_k}^{\mathfrak{y}_{k+1}} - [f]_{\mathfrak{y}_k}^{\mathfrak{y}_{k+1}} \\ &= \quad -(f(\mathfrak{y}_{k+1}) - f(\mathfrak{y}_k)). \end{aligned}$$

Define the function η on $\mathbb{T}$ by imposing that η coincide with η_k on $\mathbb{I}_k$ for all $k \in \mathbb{Z}_N$. Also denote η the measure admitting η as density with respect to the Lebesgue measure and remark that this density is continuous (and vanish on $\mathcal{N}$), so that $\eta(\mathbb{T}) < +\infty$. Furthermore we have for any $f \in \mathcal{C}^\infty(\mathbb{T})$,

$$\eta[L[f]] \quad = \quad -\sum_{k\in\mathbb{Z}_N} f(\mathfrak{y}_{k+1}) - f(\mathfrak{y}_k) \;=\; 0,$$

namely η is invariant for L. The probability π appearing in (1.9) is just the normalization of η into a probability measure.

Let us now describe the evolution of the dual process $\widetilde{I} \coloneqq (\widetilde{Y}, \widetilde{Z})$. Assume that $X(0) = x_0 \in \mathbb{I}_k = [\mathfrak{y}_k, \mathfrak{y}_{k+1})$, for some $k \in \mathbb{Z}_N$. Following (A.2) and (A.6), we begin by defining $(\widetilde{Y}(t), \widetilde{Z}(t))_{t\in[0,\tau_1)}$ as the solution of the s.d.e.

$$\left\{\begin{array}{rcl} \lim_{t\to 0_+} \widetilde{Y}(t) &=& x_0, \\ \lim_{t\to 0_+} \widetilde{Z}(t) &=& x_0, \\ d\widetilde{Y}(t) &=& \left(a'(\widetilde{Y}(t)) - b(\widetilde{Y}(t)) + \frac{2}{\eta(\widetilde{Y}(t))}\right) dt - \sqrt{2a(\widetilde{Y}(t))}\, dW(t) \\ && -2\frac{\sqrt{a(\widetilde{Y}(t))}\eta(\widetilde{Y}(t))+\sqrt{a(\widetilde{Z}(t))}\eta(\widetilde{Z}(t))}{\eta([\widetilde{Y}(t),\widetilde{Z}(t)])}\sqrt{a(\widetilde{Y}(t))}dt, \\ d\widetilde{Z}(t) &=& \left(a'(\widetilde{Z}(t)) - b(\widetilde{Z}(t)) + \frac{2}{\eta(\widetilde{Z}(t))}\right) dt + \sqrt{2a(\widetilde{Z}(t))}\, dW(t) \\ && +2\frac{\sqrt{a(\widetilde{Y}(t))}\eta(\widetilde{Y}(t))+\sqrt{a(\widetilde{Z}(t))}\eta(\widetilde{Z}(t))}{\eta([\widetilde{Y}(t),\widetilde{Z}(t)])}\sqrt{a(\widetilde{Z}(t))}, \end{array}\right.$$

for $t \in (0, \tau_1)$, where τ_1 is the first time either $\widetilde{Y}$ hits $\mathfrak{y}_k$ or $\widetilde{Z}$ hits $\mathfrak{y}_{k+1}$.

First, assume that $\widetilde{Y}(\tau_1) = \mathfrak{y}_k$. We extend the process $(\widetilde{Y}, \widetilde{Z})$ after time τ_1 by letting $\widetilde{Y}(t) = \mathfrak{y}_k$, for all $t \geqslant \tau_1$, and by solving for $\widetilde{Z}$ the s.d.e., for $t \in [\tau_1, \tau_2)$,

$$\begin{array}{rcl} d\widetilde{Z}(t) &=& \left(a'(\widetilde{Z}(t)) - b(\widetilde{Z}(t)) + \dfrac{2}{\eta(\widetilde{Z}(t))} + 2\dfrac{\eta(\widetilde{Z}(t))}{\eta([\mathfrak{y}_k, \widetilde{Z}(t)])}a(\widetilde{Z}(t))\right) dt \\ && +\sqrt{2a(\widetilde{Z}(t))}\, dW(t), \end{array} \tag{5.1}$$

where τ_2 is the first time after τ_1 that $\widetilde{Z}$ hits $\mathfrak{y}_{k+1}$. This time is a.s. finite, because $\mathfrak{y}_{k+1}$ is an exit boundary for $\widetilde{Z}$ (as well as for X) on $[\mathfrak{y}_k, \mathfrak{y}_{k+1})$. Next for $t \in [\tau_2, \tau_3)$, we ask that $\widetilde{Z}$ solves again the s.d.e. (5.1), where τ_3 is the first time after τ_2 that $\widetilde{Z}$ hits $\mathfrak{y}_{k+2}$. This time is a.s. finite, because $\mathfrak{y}_{k+2}$ is an exit boundary for $\widetilde{Z}$ on $[\mathfrak{y}_{k+1}, \mathfrak{y}_{k+2})$. We keep solving this equation until $\widetilde{Z}$ ends up hitting $\mathfrak{y}_k$, say at time $\mathfrak{\tau}$, which is also a.s. finite. After $\mathfrak{\tau}$, we take $\widetilde{I}$ to be equal to $\mathbb{T}$.

Since the generator of $\widetilde{I} \coloneqq (\widetilde{Y}, \widetilde{Z})$ is intertwined with L through $\widetilde{\Lambda}$, we construct a coupling of $\widetilde{I}$ with the diffusion X, associated with the generator L, so that

$$\begin{array}{rrcl} & \widetilde{I}(0) &=& \{X(0)\}, \\ \forall\ t \geqslant 0, & \mathcal{L}(X(t)|\widetilde{I}[0,t]) &=& \widetilde{\Lambda}(\widetilde{I}(t), \cdot). \end{array}$$

Then we get that $\mathfrak{\tau}$ is a strong stationary time for X.

It follows that X converges toward π in separation and in total variation in large time, due to the general bounds (4.6). As pointed out by the referee, since by compactness the above hitting times can be bounded uniformly with respect to the starting point, these convergences are uniform with respect to the starting point, implying an exponential convergence.

A About segment-valued dual processes

Putting together considerations from [12] and [3], we present here some computations that were used throughout the paper.

On $(0, 1)$, consider a generator $L \coloneqq a\partial^2 + b\partial$, where $a > 0$ and b are smooth functions on $(0, 1)$. Let $\eta > 0$ be a smooth function on $(0, 1)$ satisfying

$$(a\eta)' \quad = \quad b\eta - c, \tag{A.1}$$

where c is a constant. Then the measure (still denoted η) admitting η with respect to the Lebesgue measure λ on $(0,1)$ is invariant for L in the following sense:

Lemma A.1. *For any $f \in \mathcal{C}_c^\infty(0,1)$, the space of smooth functions with compact support inside $(0,1)$, we have $\eta[L[f]] = 0$. Furthermore, η is reversible with respect to L, in the sense that for all $f, g \in \mathcal{C}_c^\infty(0,1)$, $\eta[gL[f]] = \eta[fL[g]]$, if and only if $c = 0$.*

Proof. These results are immediate consequences of the following integration by parts: for all $f, g \in \mathcal{C}_c^\infty(0,1)$,

$$\begin{aligned}
\eta[gL[f]] &= \int_0^1 a\eta g f'' + b\eta g f' \, d\lambda \\
&= \int_0^1 -(a\eta g)' f' + b\eta g f' \, d\lambda \\
&= \int_0^1 -(a\eta)' g f' - a\eta g' f' + b\eta g f' \, d\lambda \\
&= -\int_0^1 a g' f' \, d\eta + c \int g f' \, d\lambda.
\end{aligned}$$

When $g = \mathbb{1}$ (the mapping always taking the value 1, here on $(0,1)$), the r.h.s. is equal to

$$c\int_0^1 f' \, d\lambda = 0,$$

showing the first assertion of the above lemma. Concerning the second one, the reversibility is equivalent to

$$\forall\ f, g \in \mathcal{C}_c^\infty(0,1), \qquad c\int g f' \, d\lambda = c\int f g' \, d\lambda.$$

By another integration by parts, the r.h.s. is equal to $-c\int g f' \, d\lambda$, so we must have $c\int g f' \, d\lambda = 0$, fo all $f, g \in \mathcal{C}_c^\infty(0,1)$ and this is true if and only if $c = 0$. □

Let be given $y_0 < z_0 \in (0,1)$ and β a smooth function on $(0,1)$ that will be specified later, in (A.6). Consider a solution $(Y, Z) \coloneqq (Y(t), Z(t))_{t \in [0,\tau)}$ of the s.d.e.

$$\left\{\begin{array}{rcl}
Y(0) &=& y_0, \\
Z(0) &=& z_0, \\
dY(t) &=& \left(a'(Y(t)) - \beta(Y(t)) - 2\frac{\sqrt{a(Y(t))}\eta(Y(t)) + \sqrt{a(Z(t))}\eta(Z(t))}{\eta([Y(t), Z(t)])}\sqrt{a(Y(t))}\right) dt \\
&& -\sqrt{2a(Y(t))}\, dW(t), \\
dZ(t) &=& \left(a'(Z(t)) - \beta(Z(t)) + 2\frac{\sqrt{a(Y(t))}\eta(Y(t)) + \sqrt{a(Z(t))}\eta(Z(t))}{\eta([Y(t), Z(t)])}\sqrt{a(Z(t))}\right) dt \\
&& +\sqrt{2a(Z(t))}\, dW(t),
\end{array}\right. \tag{A.2}$$

where the explosion time τ is such that either $\lim_{t\to\tau_-} Z(t) - Y(t) = 0$, or $\lim_{t\to\tau_-} Y(t) = 0$ or $\lim_{t\to\tau_-} Z(t) = 1$. Denote $\triangle \coloneqq \{(y,z) \,:\, 0 < y < z < 1\}$. For any $f \in \mathcal{C}_c^\infty(0,1)$, define the **elementary observable**

$$\forall\ (y,z) \in \triangle, \qquad F_f(y,z) \coloneqq \int_y^z f(x)\, \eta(dx). \tag{A.3}$$

It will be also convenient to consider for $(y,z) \in \triangle$,

$$\begin{aligned}
G_f(y,z) &\coloneqq f(z)\sqrt{a(z)}\eta(z) + f(y)\sqrt{a(y)}\eta(y), \\
H(y,z) &\coloneqq \frac{G_{\mathbb{1}}(y,z)}{F_{\mathbb{1}}(y,z)}.
\end{aligned}$$

We compute that for any $f \in \mathcal{C}_c^\infty(0,1)$ and $(y,z) \in \triangle$,

$$\begin{array}{rcl} \partial_y F_f(y,z) & = & -f(y)\eta(y), \\ \partial_z F_f(y,z) & = & f(z)\eta(z), \\ \partial_y^2 F_f(y,z) & = & -(f\eta)'(y), \\ \partial_z^2 F_f(y,z) & = & (f\eta)'(z), \\ \partial_z\partial_y F_f(y,z) & = & 0. \end{array}$$

It follows from Itô's formula that

$$\begin{array}{rl} & dF_f(Y(t),Z(t)) \\ = & \partial_z F_f(Y(t),Z(t))dZ(t) + \partial_y F_f(Y(t),Z(t))dY(t) + \frac{1}{2}\partial_z^2 F_f(Y(t),Z(t))d\langle Z\rangle(t) \\ & +\frac{1}{2}\partial_y^2 F_f(Y(t),Z(t))d\langle Y\rangle(t) + \partial_z\partial_y F_f(Y(t),Z(t))d\langle Y,Z\rangle(t) \\ = & \partial_z F_f(Y(t),Z(t))dZ(t) + \partial_y F_f(Y(t),Z(t))dY(t) + \partial_z^2 F_f(Y(t),Z(t))a(Z(t))dt \\ & +\partial_y^2 F_f(Y(t),Z(t))a(Y(t))dt \\ = & (f\eta)(Z(t))dZ(t) - (f\eta)(Y(t))dY(t) + (f\eta)'(Z(t))a(Z(t))dt - (f\eta)'(Y(t))a(Y(t))dt \\ = & dM^f(t) + A(Y(t),Z(t))dt, \end{array} \tag{A.4}$$

where $M^f \coloneqq (M_t^f)_{t\in[0,\tau)}$ is the local martingale defined by

$$\begin{array}{rrcl} \forall\ t \in [0,\tau), & M^f(t) & \coloneqq & \int_0^t (f\eta)(Z(s))\sqrt{2a(Z(s))} + (f\eta)(Y(s))\sqrt{2a(Y(s))}\,dW(s) \\ & & = & \sqrt{2}\int_0^t G_f(Y(s),Z(s))\,dW(s), \end{array} \tag{A.5}$$

and where

$$\begin{array}{rl} & A(y,z) \\ \coloneqq & (f\eta)(z)\left(a'(z) - \beta(z) + 2H(y,z)\sqrt{a(z)}\right) - (f\eta)(y)\left(a'(y) - \beta(y) - 2H(y,z)\sqrt{a(y)}\right) \\ & +(f\eta)'(z)a(z) - (f\eta)'(y)a(y) \\ = & (fa\eta)'(z) - (fa\eta)'(y) - (\beta f\eta)(z) + (\beta f\eta)(y) + 2H(y,z)G_f(y,z) \\ = & (f'a\eta)(z) - (f'a\eta)(y) + (f((a\eta)' - \beta\eta))(z) - (f((a\eta)' - \beta\eta))(y) + 2H(y,z)G_f(y,z). \end{array}$$

The first term of the r.h.s. can be transformed into

$$\begin{array}{rcl} (f'a\eta)(z) - (f'a\eta)(y) & = & \int_{[y,z]} (f'a\eta)'(x)\,dx \\ & = & \int_{[y,z]} (a\eta)(x)f''(x) + (a\eta)'f'(x)\,dx \\ & = & \int_{[y,z]} (a\eta)(x)f''(x) + (b\eta)(x)f'(x)\,dx - \int_{[y,z]} cf'(x)\,dx \\ & = & \int_{[y,z]} L[f](x)\,\eta(dx) - c(f(z) - f(y)) \\ & = & F_{L[f]}(y,z) - c(f(z) - f(y)), \end{array}$$

where we took into account (A.1). We deduce that

$$\begin{array}{rcl} A(y,z) & = & F_{L[f]}(y,z) + (f((a\eta)' - \beta\eta - c))(z) - (f((a\eta)' - \beta\eta - c))(y) + 2H(y,z)G_f(y,z) \\ & = & F_{L[f]}(y,z) + (f((b-\beta)\eta - 2c))(z) - (f((b-\beta)\eta - 2c))(y) + 2H(y,z)G_f(y,z). \end{array}$$

It leads us to consider

$$\beta \quad := \quad b - 2\frac{c}{\eta}, \tag{A.6}$$

so that

$$A(y,z) \quad = \quad F_{L[f]}(y,z) + 2H(y,z)G_f(y,z). \tag{A.7}$$

Remark A.2. Let us make the link with the formulation adopted in [3] in the context of Riemannian geometry in dimension strictly larger than 1. Endow $(0,1)$ with the Riemannian metric given by $1/a$ (so that the norm of the usual unit vector 1 above $x \in (0,1)$ is $1/\sqrt{a(x)}$, or equivalently, $\pm\sqrt{a(x)}$ are the unit vectors above x in the new Riemannian metric). Let d be the corresponding distance and for any $A \subset (0,1)$ and $\epsilon > 0$, let $A_\epsilon := \{x \in (0,1) : d(x,A) \leqslant \epsilon\}$, the ϵ-enlargement of A. Then we have for any $f \in \mathcal{C}_c^\infty(0,1)$ and $(y,z) \in \triangle$,

$$\begin{aligned} \partial_\epsilon \int_{[y,z]_\epsilon} f\, d\eta \bigg|_{\epsilon=0} \quad &= \quad G_f(y,z) \\ &= \quad \int_{\partial[y,z]} f\, d\underline{\eta}, \end{aligned} \tag{A.8}$$

where $\underline{\eta}$ is the (non-σ-finite) measure given by

$$\underline{\eta} \quad := \quad \sum_{x\in(0,+\infty)} \sqrt{a(x)}\eta(x)\delta_x \tag{A.9}$$

($\underline{\eta}$ will only serve to measure the boundaries $\partial[y,z] = \{y,z\}$ of segments $[y,z]$, with $(y,z) \in \triangle$, we used the symbol $\int$ in (A.8) instead of a sum over the two elements of $\partial[y,z]$ to adopt the same notation as in higher dimensional Riemannian geometry). It appears that

$$\partial_\epsilon^2 \int_{[y,z]_\epsilon} f\, d\eta \bigg|_{\epsilon=0} \quad = \quad \int_{\partial[y,z]} \langle \nabla f, \nu\rangle + \langle \nabla U, \nu\rangle \, d\underline{\eta}, \tag{A.10}$$

where $\langle\cdot,\cdot\rangle$ and ∇ are relative to the considered Riemannian metric and where ν is the "unit exterior normal vector" on $\partial[y,z]$. The function $U := \ln(d\eta/d\gamma)$ is the logarithm of the Radon-Nikodym density of η with respect to the Riemannian measure γ, which admits itself the density $1/\sqrt{a}$ with respect to the usual Lebesgue measure. Thus we have $d\eta/d\gamma = \underline{\eta}$, where by a traditional abuse of notation, we also interpret $\underline{\eta}$ as the function $\sqrt{a}\eta$. In the usual definitions in higher dimensional Riemannian geometry (see e.g. Proposition 1.2.1 of Mantegazza [11]), the r.h.s. of (A.10) should contain a supplementary term $\int f\rho\, d\underline{\eta}$ where ρ would be the "mean" curvature on the boundary $\partial[y,z]$ with respect to the unit exterior normal vectors. Thus we recover that in dimension 1, the mean curvature of a boundary of dimension 0 vanishes: $\rho \equiv 0$. To see the coherence of (A.2) with the formulation of [3] in the context of diffusions in Riemannian manifolds of dimension larger or equal to 2, we should check that

$$a' - \beta \quad = \quad -\langle \nabla U - b_{\mathrm{H}}, \nu\rangle \nu - \rho\nu, \tag{A.11}$$

where b_{H} is such that the Helmoltz-Hodge decomposition $b = \nabla U + b_{\mathrm{H}}$ holds (note the change of sign with respect to (A.11)), i.e.

$$\begin{aligned} b_{\mathrm{H}} \quad &= \quad b - \nabla U \\ &= \quad b - aU' \\ &= \quad b - a\frac{(\sqrt{a}\eta)'}{\sqrt{a}\eta} \\ &= \quad b - a\frac{\eta'}{\eta} - \frac{a'}{2}. \end{aligned}$$

It follows that the r.h.s. of (A.11) is equal to

$$\begin{aligned} -\langle \nabla U - b_{\mathrm{H}}, \nu\rangle \nu &= b_{\mathrm{H}} - \nabla U \\ &= 2b_{\mathrm{H}} - b \\ &= b - 2a\frac{\eta'}{\eta} - a' \\ &= b - 2\left(\frac{(a\eta)'}{\eta} - a'\right) - a' \\ &= a' - \beta, \end{aligned}$$

as wanted, where we used (A.1) and (A.6).

Remark that in general the Helmoltz-Hodge decomposition $b = \nabla U + b_{\mathrm{H}}$ is different from the decomposition $b = (a\eta)'/\eta + c/\eta$, which enables to write

$$L = \frac{1}{\eta}\partial(a\eta\partial) + \frac{c}{\eta}\partial, \tag{A.12}$$

where $\frac{1}{\eta}\partial(a\eta\partial)$ is symmetric in $\mathbb{L}^2(\mu)$ and $\frac{c}{\eta}\partial$ is skew-symmetric in $\mathbb{L}^2(\mu)$.

Let $(\mathfrak{D}, \mathfrak{L})$ be the generator $\mathfrak{L}$ of (Y, Z) in the sense of (local) martingale problems. It follows from (A.4) and (A.7) that $\mathfrak{L}$ acts on the elementary observable F_f, with $f \in \mathcal{C}_{\mathrm{c}}^{\infty}(0,1)$, by

$$\mathfrak{L}[F_f](y,z) = F_{L[f]}(y,z) + 2H(y,z)G_f(y,z).$$

Furthermore, the carré du champs $\Gamma_{\mathfrak{L}}$ associated to $\mathfrak{L}$ is such that the bracket of the martingale M^f defined in (A.5) satisfies

$$\forall\, t \in [0,\tau), \qquad \left\langle M^f\right\rangle_t = 2\int_0^t \Gamma_{\mathfrak{L}}[f,f](Y(s),Z(s))\,ds.$$

It follows by polarization that all $f, g \in \mathcal{C}_{\mathrm{c}}^{\infty}(0,1)$,

$$\forall\, (y,z) \in \triangle, \qquad \Gamma_{\mathfrak{L}}[F_f, F_g](y,z) = G_f(y,z)G_g(y,z).$$

Since (Y,Z) is a **diffusion** (namely a Markov process with continuous trajectories), the generator $\mathfrak{L}$ and the carré du champs $\Gamma_{\mathfrak{L}}$ extend in the following way (see e.g. the book of Bakry, Gentil and Ledoux [1]). Consider the algebra $\mathcal{A}$ consisting of the composed observables of the form $\mathfrak{F} \coloneqq \mathfrak{f}(F_{f_1}, \ldots, F_{f_n})$, where $n \in \mathbb{Z}_+$, $f_1, \ldots, f_n \in \mathcal{C}_{\mathrm{c}}^{\infty}(0,1)$ and $\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^{\infty}$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\triangle$ by $(F_{f_1}, \ldots, F_{f_n})$. Then $\mathcal{A}$ is included into $\mathfrak{D}$ and since $\mathfrak{L}$ is a differential operator of order 2 without terms of order 0, we have for any $\mathfrak{F} \coloneqq \mathfrak{f}(F_{f_1}, \ldots, F_{f_n})$ and $\mathfrak{G} \coloneqq \mathfrak{g}(F_{g_1}, \ldots, F_{g_m})$ belonging to $\mathcal{A}$,

$$\begin{aligned} \mathfrak{L}[\mathfrak{F}] &= \sum_{j \in [\![n]\!]} \partial_j \mathfrak{f}(F_{f_1}, \ldots, F_{f_n})\mathfrak{L}[F_{f_j}] + \sum_{k,l \in [\![n]\!]} \partial_{k,l}\mathfrak{f}(F_{f_1}, \ldots, F_{f_n})\Gamma_{\mathfrak{L}}[F_{f_k}, F_{f_l}], \\ \Gamma_{\mathfrak{L}}[\mathfrak{F}, \mathfrak{G}] &= \sum_{l \in [\![n]\!],\, k \in [\![m]\!]} \partial_l \mathfrak{f}(F_{f_1}, \ldots, F_{f_n})\partial_k \mathfrak{g}(F_{g_1}, \ldots, F_{g_m})\Gamma_{\mathfrak{L}}[F_{f_l}, F_{g_k}] \end{aligned}$$

(where $[\![n]\!] \coloneqq \{1, 2, \ldots, n\}$).

Define a Markov kernel Λ from $\triangle$ to $(0,1)$ by

$$\forall\, (y,z) \in \triangle,\ \forall\, A \in \mathcal{B}(0,1), \qquad \Lambda([y,z], A) \coloneqq \frac{\eta([y,z] \cap A)}{\eta([y,z])}.$$

Note that for any $f \in \mathcal{C}_c^\infty(0,1)$, we have $\Lambda[f] = F_f/F_{\mathbb{1}}$, so $\Lambda[f] \in \mathcal{A}$ and the above formulas lead without difficulty to the intertwining relation

$$\forall\ (y,z) \in \triangle,\ \forall\ f \in \mathcal{C}_b^\infty(0,1), \qquad \mathfrak{L}[\Lambda[f]](y,z) \quad = \quad \Lambda[L[f]](y,z). \tag{A.13}$$

Furthermore, by considering observables of the form $\mathfrak{f}(F_{\mathbb{1}})$, where $\mathfrak{f} \in \mathcal{C}^\infty(\mathbb{R})$, it appears that $(\eta([Y(t),Z(t)]))_{t\in[0,\tau)}$ is a (possibly stopped) time-changed Bessel process of dimension 3. This property enables us to let the process (Y,Z) start from the singleton (y_0,y_0), by passing to the limit as z_0 goes to y_{0+} and to see that the set of the singletons is an entering boundary for (Y,Z), see Section 2 from [12]. Under the assumption that $\mathcal{L}(Y_0,Z_0)\Lambda = \mathcal{L}(X_0)$, proceeding as in Section 4 from [12], we construct a coupling of the diffusion X associated to the generator L with the process (Y,Z), so that

$$\forall\ T \geqslant 0, \qquad \mathcal{L}(X_t|(Y_t,Z_t)_{t\in[0,T]}) \quad = \quad \Lambda((Y_T,Z_T),\cdot). \tag{A.14}$$

Alternatively, conditioning furthermore by the initial condition X_0, we can also couple X and (Y,Z) so that $Y_0 = X_0 = Y_0$, in addition to (A.14).

When $\eta([0,x]) < +\infty$ for (one or all) $x \in (0,1)$, in the above considerations Y can be fixed equal to 0 (and symmetrically, Z can be fixed equal to 1, when $\eta([x,1]) < +\infty$ for $x \in (0,1)$). In particular, we can impose this restriction once Y has hit 0 (or Z has hit 1). Then the natural extensions of the previous results hold.

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