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Renormalizability of Liouville quantum field theory at the Seiberg bound

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Abstract

Liouville Quantum Field Theory (LQFT) can be seen as a probabilistic theory of 2d Riemannian metrics $e^{\phi(z)}|dz|^2$, conjecturally describing scaling limits of discrete $2d$-random surfaces. The law of the random field ϕ in LQFT depends on weights $\alpha \in \mathbb{R}$ that in classical Riemannian geometry parametrize power law singularities in the metric. A rigorous construction of LQFT has been carried out in [3] in the case when the weights are below the so called Seiberg bound: $\alpha < Q$ where Q parametrizes the random surface model in question. These correspond to studying uniformized surfaces with conical singularities in the classical geometrical setup. An interesting limiting case in classical geometry are the cusp singularities. In the random setup this corresponds to the case when the Seiberg bound is saturated. In this paper, we construct LQFT in the case when the Seiberg bound is saturated which can be seen as the probabilistic version of Riemann surfaces with cusp singularities. The construction involves methods from Gaussian Multiplicative Chaos theory at criticality.

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1 Introduction

Two dimensional statistical physics provides a large class of models of discrete random surfaces (random maps) which are expected to have interesting continuous surfaces as scaling limits. In physics the study of these objects goes under the name “2d gravity” and was pioneered by Polyakov [12] and developed in [10]. That approach

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seeks a description of the geometry of the two dimensional manifold Σ in terms of a probability law in a suitable space of Riemannian metrics defined on Σ. Physics dictates that the law be invariant under the action of the group of diffeomorphisms acting on Σ. In two dimensions the space of smooth metrics modulo diffeomorphisms is rather simple: its elements are (equivalence classes of) $e^{\sigma} g$ where $\sigma : \Sigma \to \mathbb{R}$ and g belongs to a finite dimensional (moduli) space of metrics. Thus, we are basically seeking a law for a random field σ on Σ. The proposal of [10] is that this law is given by

$$\mu_L(dg, dX) = e^{-S_L(X,g)} \mu_0(dg, dX) \tag{1.1}$$

where μ_0 is a "uniform measure" on some space of maps $X : \Sigma \to \mathbb{R}$ and moduli g and S_L is the *Liouville action functional*

$$S_L(X, g) := \frac{1}{4\pi} \int_\Sigma \left(|\nabla_g X|^2 + Q R_g X + 4\pi\mu e^{\gamma X} \right) d\mathrm{v}_g. \tag{1.2}$$

Here we have written $\sigma = \gamma X$ where $\gamma \in (0, 2)$ is a parameter determined by the random surface model and

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{1.3}$$

Furthermore we denoted by ∇_g, R_g and v_g respectively the gradient, Ricci scalar curvature and volume measure in the metric g. Finally the parameter $\mu > 0$ is called " cosmological constant". In [3] we gave a rigorous definition of the measure (1.1) for the case $\Sigma = S^2$ which we recall in Section 2.

The action functional (1.2) has a very natural geometric interpretation in terms of the classical uniformisation theory of Riemann surfaces that goes back to Picard and Poincaré. The Euler-Lagrange equation for the extrema of S_L is given by

$$-2\Delta_g X + Q R_g + 4\pi\mu\gamma e^{\gamma X} = 0 \tag{1.4}$$

where Δ_g is the Laplace-Beltrami operator in the metric g. If we replace Q by its "classical value" $Q_{\mathrm{cl}} = \frac{2}{\gamma}$ and use the relation $R_{e^{\varphi} g} = e^{-\varphi}(R_g - \Delta_g \varphi)$ this equation becomes the *Liouville equation*

$$R_{e^{\gamma X} g} = -2\pi\mu\gamma^2 \tag{1.5}$$

stating that the metric $e^{\gamma X} g$ has *constant negative curvature*. Such metrics are in correspondence to complex structures on the surface Σ through the uniformizing map $\psi : \Sigma \to \mathbb{H}$: pullback under ψ of the Poincaré metric on $\mathbb{H}$ has constant negative curvature. Thus LQFT can be seen as a probabilistic extension of this classical theory.

This correspondence works only if the genus of Σ is at least two. On the sphere S^2 there are no smooth metrics of constant negative curvature since by the Gauss-Bonnet theorem the total curvature is positive. Indeed, the action functional S_L (1.2) is not bounded from below as can be seen by taking $X = c$, a constant. Then by Gauss-Bonnet theorem $\int R_g d\mathrm{v}_g = 8\pi$ and we have

$$S_L(g, c) = 2Qc + 4\pi\mu e^{\gamma c} \tag{1.6}$$

which is not bounded below as $c \to -\infty$. This divergence is also present in the LQFT: the measure (1.1) is not finite and can not be normalized to a probability law [3].

In classical geometry it is known how to obtain a metric with constant negative curvature almost everywhere on the sphere. The idea is to introduce points that are sources of curvature in the Liouville equation. To do this pick n points $z_1, \ldots, z_n$ and weights $\alpha_1, \ldots \alpha_n$ and consider the equation:

$$-2\Delta_g X + Q_{cl} R_g + 4\pi\mu\gamma e^{\gamma X} = 4\pi \sum_i \alpha_i \delta_{z_i} \tag{1.7}$$

This equation is formally the Euler-Lagrange equation of the action functional

$$S_{L,\text{cl}}(X,g)-\sum_i \alpha_i X(z_i). \tag{1.8}$$

For a rigorous treatment one needs to regularize and renormalize this functional, see [15]. Then one finds that the minimizers give rise to the metric $e^{\gamma X(z)}g(z)$ which has singularities at the points z_i. For $\alpha_i < Q_{cl}$ i.e. for $\gamma\alpha_i < 2$ this singularity is *conical*:

$$e^{\gamma X(z)}g(z) \sim |z-z_i|^{-\gamma\alpha_i} \tag{1.9}$$

and for $\alpha_i = Q_{cl}$ the singularity is a *cusp*

$$e^{\gamma X(z)}g(z) \sim (|z-z_i|\ln|z-z_i|)^{-2} \tag{1.10}$$

(see Appendix A for a brief introduction to these concepts). For $\alpha_i > 2/\gamma$ solutions do not exist for integrability reasons. Furthermore for topological reasons (Gauss-Bonnet theorem) one needs also $\sum_i \alpha_i > 2Q_{cl}$ which implies that one needs to introduce at least three singularities on the sphere to have constant negative curvature in their complement.

The probabilistic theory has a complete parallel with the classical one with the important difference being that the parameter $Q_{cl} = 2/\gamma$ is replaced by the quantum value (1.3). Then it was shown in [3] that the measure (1.1) with the action (1.8) (suitably renormalized) has finite mass provided $\sum_i \alpha_i > 2Q$ and the mass is nonzero if and only if $\alpha_i < Q$. This measure can be viewed as a probabilistic theory of metrics with “quantum” conical singularities on the sphere.

In this paper we will extend this theory to the case of “quantum” cusp singularities $\alpha_i = Q$ thus completing the parallel with classical geometry in the setup of random surfaces. This extension requires an extra renormalization of the measure compared to the $\alpha_i < Q$ case. It boils down to an analysis of the Gaussian multiplicative chaos measure in a background measure with density blowing up as $|z-z_i|^{-\gamma Q}$. This in turn leads to an analysis reminiscent to the analysis of the Critical gaussian multiplicative chaos [6, 7].

We conclude this introduction by mentioning that LQFT is interesting per se as it is the first full probabilistic construction of an interacting Conformal Field Theory (CFT for short) and therefore a natural playground to check the whole formalism of CFTs initiated in the celebrated paper [2]. The modification of the action functional (1.8) can be viewed as a correlation function of n random fields:

$$\int \prod_i e^{\alpha_i X(z_i)} d\mu_L \tag{1.11}$$

These correlation functions of LQFT play a prominent role in understanding models of statistical physics models on random planar maps. As an example, the reader can find in appendix A.1 a conjecture on the relationship of these correlation functions of LQFT to random planar maps, in particular a conjecture describing the scaling limit of the correlation functions of the spin field of the Ising model on random planar maps. The case we treat in this paper, i.e. Q-insertions, is especially important for understanding how to embed conformally onto the sphere random planar maps with spherical topology weighted by a $c=1$ conformal field theory (like the Gaussian Free Field). Indeed, in the case $c=1$, one can formulate the conjecture developed in [3, subsection 5.3] with $\gamma=2$ and $Q=2$: the vertex operators with $\gamma=2$ in [3, conjecture 2] are precisely the quantum cusps constructed here. Finally we mention that Riemann surfaces with cusp singularities naturally appear when studying the boundary of the moduli space of higher genus surfaces. Hence the study of Q-insertions plays a prominent role in establishing convergence of the partition function of $2d$-string theory where integrals over the moduli space arise (see [8]).

2 Background and main results

This section contains first a brief summary of the construction and properties of the LQFT carried out in [3] followed by a presentation of our main results and a sketch of proof.

2.1 GFF and multiplicative chaos

We will view the sphere S^2 as the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It can be covered by two copies of $\mathbb{C}$ with coordinates z and z^{-1}. The constant curvature metric is the round metric, $\hat{g}(z)|dz|^2$ with

$$\hat{g}(z) = 4(1 + \bar{z}z)^{-2}.$$

The area is $\int_{\mathbb{C}} \hat{g}(z)dz = 4\pi$ and the scalar curvature $R_g := -4g^{-1}\partial_{\bar{z}}\partial_z \ln g$ is constant for the round metric: $R_{\hat{g}} = 2$. Smooth conformal metrics on $\hat{\mathbb{C}}$ are given by $g = e^{\varphi}\hat{g}$ where $\varphi(z)$ and $\varphi(1/z)$ are smooth and bounded. For such metrics the Gauss-Bonnet theorem holds:

$$\int R_g \, d\mathrm{v}_g = 8\pi.$$

Given a conformal metric on $\hat{\mathbb{C}}$ we can define the Sobolev space $H^1(\hat{C}, g)$ with the norm

$$\|f\|_g^2 := \int (|\partial_z f|^2 + g(z)|f|^2)dz.$$

These norms are equivalent for all continuous conformal metrics and we denote the space simply by $H^1(\hat{\mathbb{C}})$. Finally we define $H^{-1}(\hat{\mathbb{C}})$ as the dual space and denote the dual pairing by $\langle X, f\rangle$.

The LQFT measure will be defined as a measure on $H^{-1}(\hat{\mathbb{C}})$. It will be constructed using the Gaussian Free Field (GFF) on $\hat{\mathbb{C}}$. As is well known the GFF in such a setup is only defined modulo a constant. For LQFT it is important to include this constant as an integration variable. In general the GFF is a Gaussian random field whose covariance is the Green function of the Laplace operator. In our setup the Laplace operator is given by $\Delta_g = 4g(z)^{-1}\partial_{\bar{z}}\partial_z$. Some care is needed here since Δ_g is not invertible. Indeed, $-\Delta_g$ is a non-negative self-adjoint operator on $L^2(\hat{\mathbb{C}}, g)$ (whose inner product we denote by $(f, h)_g = \int \bar{f}hgdz$). It has a point spectrum consisting of eigenvalues λ_n and orthonormal eigenvectors e_n which we take so that $\lambda_n > 0$ except for $\lambda_0 = 0$ with $e_0 = 1/\|1\|_g$. We define the GFF X_g as the random distribution

$$X_g(z) = \sqrt{2\pi} \sum_{n>0} \frac{x_n}{\sqrt{\lambda_n}} e_n(z) \tag{2.1}$$

where x_n are i.i.d. $N(0,1)$. In case of the round metric, we will need later the explicit formula

$$\mathbb{E}[X_{\hat{g}}(z)X_{\hat{g}}(z')] = G_{\hat{g}}(z, z') = \ln \frac{1}{|z - z'|} - \frac{1}{4}(\ln \hat{g}(z) + \ln \hat{g}(z')) + \ln 2 - \tfrac{1}{2}. \tag{2.2}$$

The random field X_g determines probability measure $\mathbb{P}_g$ on $H^{-1}(\hat{\mathbb{C}})$ (supported in the set $\{u \in H^{-1}(\hat{\mathbb{C}}) : \langle u, 1\rangle = 0\}$). The measure (1.1) is intended to contain also the constant fields $X = c$ that are absent from the GFF X_g. Therefore we define the measure μ_{GFF} on $H^{-1}(\hat{\mathbb{C}})$ by

$$\int F(X)\mu_{GFF}(dX) = \int_{\mathbb{R}} \mathbb{E}\, F(X_g + c)dc \tag{2.3}$$

Note that μ_{GFF} is *not* a probability measure: $\int \mu_{GFF}(dX) = \infty$.

To define the measure (1.1) the exponential $e^{\gamma X}$ needs definition as the GFF X_g is not defined pointwise. To do this regularize X_g by the circle average regularization

$$X_{g,\epsilon}(x) = \frac{1}{2\pi}\int_0^{2\pi} X_g(x+\epsilon e^{i\theta})\, d\theta \tag{2.4}$$

and define the random measure

$$M_{\gamma,\epsilon}(dz) := \epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{g,\epsilon}(z)+Q/2\ln g(z))}\, dz. \tag{2.5}$$

For $\gamma \in [0,2)$, we have the convergence in probability

$$M_\gamma = \lim_{\epsilon\to 0} M_{\gamma,\epsilon} \tag{2.6}$$

in the sense of weak convergence of measures. This limiting measure is non trivial and is an instance of Gaussian multiplicative chaos [9, 13] of the field $X_{\hat{g}}$. In particular for the round metric

$$M_\gamma = e^{\frac{\gamma^2}{2}(\ln 2 - \frac{1}{2})} \lim_{\epsilon\to 0} e^{\gamma X_{\hat{g},\epsilon} - \frac{\gamma^2}{2}\mathbb{E}[X_{\hat{g},\epsilon}^2]}\, d\mathrm{v}_{\hat{g}}. \tag{2.7}$$

and the total mass $M_\gamma(\hat{\mathbb{C}})$ almost surely finite.

2.2 LQFT measure and correlations functions

We may now give the precise definition of the LQFT measure in (1.1). With no loss we work with the round metric $\hat{g}$ from now on. Then

$$\frac{1}{4\pi}\int QR_{\hat{g}}X\, d\mathrm{v}_{\hat{g}} = \frac{1}{4\pi}\int QR_{\hat{g}}(c+X_{\hat{g}})\, d\mathrm{v}_{\hat{g}} = 2Qc$$

where we used the Gauss-Bonnet theorem and $(X_g,1)_g = 0$. Since $\hat{\mathbb{C}}$ has no moduli the LQFT measure μ_L will be a measure only on the conformal factor X. We define

$$\mu_L(dX) = e^{-2Qc} e^{-\mu e^{\gamma c} M_\gamma(\hat{\mathbb{C}})} \mu_{GFF}(dX) \tag{2.8}$$

i.e. concretely

$$\int F(X)\mu_L(dX) = \int e^{-2Qc}\mathbb{E}[F(c+X_{\hat{g}})e^{-\mu e^{\gamma c}M_\gamma(\hat{\mathbb{C}})}]dc := \langle F\rangle_L. \tag{2.9}$$

The rigorous definition of the correlation functions (1.11) proceeds also through regularization. We consider the regularized fields (called vertex operators in the physics literarture)

$$V_{\alpha,\epsilon}(z) = \epsilon^{\frac{\alpha^2}{2}} e^{\alpha(c+X_{\hat{g},\epsilon}(z)+Q/2\ln\hat{g}(z))} \tag{2.10}$$

In [3] it was shown that the limit of their correlation functions

$$\lim_{\epsilon\to 0}\langle\prod_i V_{\alpha_i,\epsilon}(z_i)\rangle_L := \langle\prod_i V_{\alpha_i}(z_i)\rangle_L \tag{2.11}$$

exist if and only if $\sum_i \alpha_i > 2Q$ and the limit is non zero if and only if $\alpha_i < Q$ for all i. These conditions are called called the *Seiberg bounds* [14].

Briefly, the reason of these inequalities is as follows. One can absorb the vertex operators in (2.11) by an application of the Cameron-Martin transform i.e. by a shift of the Gaussian field $X_{\hat{g}} \to X_{\hat{g}} + H$ with

$$H(z) = \sum_i \alpha_i G_{\hat{g}}(z,z_i). \tag{2.12}$$

This leads to

$$\langle\prod_i V_{\alpha_i}(z_i)\rangle_L = K(\mathbf{z})\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\mathbb{E}\Big[e^{-\mu e^{\gamma c}\int e^{\gamma H}dM_\gamma}\Big]dc \tag{2.13}$$

with $K(\mathbf{z})$ an explicit function of the points $(z_i)_i$ and of their weights $(\alpha_i)_i$. The first inequality $\sum_i \alpha_i > 2Q$ is needed for the convergence of the c-integral as $c \to -\infty$. Note the analogy with the classical result (1.6). For the second inequality $\alpha_i < Q$ we note that due to the logarithmic singularity of the Green function $G_{\hat g}$ the integrand $e^{\gamma H(z)}$ blows up as $|z - z_i|^{-\alpha_i\gamma}$ when $z \to z_i$. By analyzing the modulus of continuity of the Gaussian multiplicative chaos measure it was shown in [3] that $\int e^{\gamma H}dM_\gamma$ is a.s. finite if and only if $\alpha_i < Q$. It was further proved in [3] that, provided that the Seiberg bounds hold, the probability measures on $H^{-1}(\hat{\mathbb{C}})$

$$\mathbb{P}_{\alpha,\mathbf{z},\epsilon} := \langle\prod_i V_{\alpha_i,\epsilon}(z_i)\rangle^{-1}\prod_i V_{\alpha_i,\epsilon}(z_i)\,\mu_L(\hat g, dX) \tag{2.14}$$

converge to a probability measure $\mathbb{P}_{\alpha,\mathbf{z}}$ as $\epsilon \to 0$.

The Riemann sphere $\hat{\mathbb{C}}$ has a nontrivial automorphism group $SL(2,\mathbb{C})$ which acts as Möbius transformations $\psi(z) = \frac{az+b}{cz+d}$. By a simple change of variables the classical action functional with $Q = Q_{cl} = 2/\gamma$) satisfies

$$S_L(X\circ\psi^{-1}, \hat g) = S_L(X + \frac{Q}{2}\varphi, \hat g)$$

where $\varphi = |\psi'|^2\hat g\circ\psi/\hat g$. This Möbius covariance is inherited by the Liouville QFT measure: one has

$$\int F(X\circ\psi)d\mu_L = \int F(X - Q\ln|\psi'|)d\mu_L$$

for $F \in L^1(\mu_L)$. One can view this non-compact symmetry group of the measure μ_L as another indication of the fact that it is not normalizable.

The Seiberg bounds $\sum_i \alpha_i > 2Q$ and $\alpha_i < Q$ lead to the conclusion that to have a nontrivial correlation function of vertex operators one needs at least *three* of them. This is in complete analogy with classical geometry as discussed in the Introduction. Note that fixing three points on the sphere removes also the $SL_2(2,\mathbb{C})$ symmetry. In this light it comes as no surprise that the Liouville 2-point correlation functions are not defined: fixing two points on the sphere leaves us the non compact symmetry group of dilations. In [5] two-point quantum spheres are constructed in a quotient space of random measures modulo rotations and dilations. The approach is complementary to ours as it is concerned with a different object, see however [1] for a precise link between the two approaches.

2.3 Main results

Now we describe our main results, which extend the analysis of [3] to the case of vertex operators e^{QX} with weight Q giving rise to quantum cusps. In fact, from now on, we will use a slightly different regularization for the correlation functions than in (2.11). Namely, we will regularize simultaneously the vertex operators (2.10) and the measure μ_L defined by (2.9). Furthermore we will define our objects in the case of a general metric g conformally equivalent to the round metric. So we set

$$\Pi_{\alpha,\mathbf{z},\epsilon} := \int\prod_i V_{\alpha_i,\epsilon}(z_i)\,\mu_L^\epsilon(g, dX) \tag{2.15}$$

where the vertex operators $V_{\alpha_i,\epsilon}(z_i)$ are defined by (2.10) (with g in place of $\hat{g}$) and the measure $\mu_L^\epsilon(g,\cdot)$ is defined by

$$\mu_L^\epsilon(g,dX) := e^{-\frac{Q}{4\pi}\int QR_g(c+X_g)\,dv_g - \mu e^{\gamma c}M_\gamma(D_\epsilon)}\mu_{GFF}(dX) \tag{2.16}$$

where D_ϵ is the complement of the union of the ϵ radius balls centered at those i with $\alpha_i = Q$. In the same spirit as (2.14), we further consider the probability measures

$$\mathbb{P}_{\alpha,\mathbf{z},\epsilon} := \Pi^{-1}_{\alpha,\mathbf{z},\epsilon}\prod_i V_{\alpha_i,\epsilon}(z_i)\,\mu_L^\epsilon(g,dX). \tag{2.17}$$

As explained above, it was proved in [3] that $\Pi_{\alpha,\mathbf{z}} = \lim_{\epsilon\to 0}\Pi_{\alpha,\mathbf{z},\epsilon} = 0$ when one of the α_i is greater or equal to Q. However, an extra renormalization term suffices to obtain a nontrivial limit:

Theorem 2.1. *Let $\sum_i \alpha_i > 2Q$ and $\alpha_i \le Q$ with exactly k of the α_i equal to Q. Then the limit*

$$\lim_{\epsilon\to 0}(-\ln\epsilon)^{\frac{k}{2}}\Pi_{\alpha,\mathbf{z},\epsilon} := \Pi_{\alpha,\mathbf{z}} \tag{2.18}$$

exists and is strictly positive. Moreover, the limit

$$\lim_{\epsilon\to 0}\mathbb{P}_{\alpha,\mathbf{z},\epsilon} := \mathbb{P}_{\alpha,\mathbf{z}} \tag{2.19}$$

exists in the sense of weak convergence of measures on $H^{-1}(\hat{\mathbb{C}})$.

This theorem means that the vertex operator e^{QX} needs an additional factor $(-\ln\epsilon)^{\frac{1}{2}}$ for its normalization in addition to the $\epsilon^{\frac{\alpha^2}{2}}$ used for $\alpha < Q$. An important ingredient in the proof of convergence (2.18) is to show that the limit agrees (up to a multiplicative constant) with the one constructed with the *derivative vertex operator*

$$\tilde{V}_{Q,\epsilon}(z) = -\frac{d}{d\alpha}V_{\alpha,\epsilon}(z)_{|\alpha=Q} = -(Q\ln\epsilon + c + X_{g,\epsilon} + \tfrac{Q}{2}\ln g)V_{Q,\epsilon}(z). \tag{2.20}$$

Let $\tilde{\Pi}_{\alpha,\mathbf{z},\epsilon}$ be the correlation function where for $\alpha_i = Q$ we use $\tilde{V}_{Q,\epsilon}(z)$ instead of $V_{Q,\epsilon}(z)$. Then

Theorem 2.2.

$$\lim_{\epsilon\to 0}\tilde{\Pi}_{\alpha,\mathbf{z},\epsilon} = (\tfrac{\pi}{2})^{\frac{k}{2}}\Pi_{\alpha,\mathbf{z}}. \tag{2.21}$$

The convergence (2.19) extends to functions of the chaos measure. Let $\mathbb{E}_{\alpha,\mathbf{z},\epsilon}$ denote expectation with respect to $\mathbb{P}_{\alpha,\mathbf{z},\epsilon}$ and let $F = F(X,\nu)$ be a bounded continuous function on $H^{-1}(\hat{\mathbb{C}})\times\mathcal{M}(\hat{\mathbb{C}})$ where $\mathcal{M}(\hat{\mathbb{C}})$ denotes the set of Borel measures on $\hat{\mathbb{C}}$. Define the *Liouville measure*

$$Z := e^{\gamma c}M_\gamma \tag{2.22}$$

and the *Liouville field*

$$\phi := c + X_g + \tfrac{Q}{2}\ln g. \tag{2.23}$$

Then

Theorem 2.3. *With the assumptions of Theorem 2.1, $\mathbb{E}_{\alpha,\mathbf{z},\epsilon}F(\phi,Z)$ converges as $\epsilon\to 0$ to a limit $\mathbb{E}_{\alpha,\mathbf{z}}F(\phi,Z)$ which is conformally covariant, namely*

$$\mathbb{E}_{\alpha,\mathbf{z}}F(\phi,Z) = \mathbb{E}_{\alpha,\psi(\mathbf{z})}F(\phi\circ\psi + Q\ln|\psi'|, Z\circ\psi)$$

for all conformal automorphisms ψ of the sphere, and independent of g in the conformal equivalence class $[g]$. Moreover, the law of $Z(\hat{\mathbb{C}})$ under $\mathbb{P}_{\alpha,\mathbf{z}}$ is given by the Gamma distribution

$$\mathbb{E}_{\alpha,\mathbf{z}}F(Z(\hat{\mathbb{C}})) = \frac{\mu^{\frac{\sigma}{\gamma}}}{\Gamma(\frac{\sigma}{\gamma})}\int_0^\infty F(y)y^{\frac{\sigma}{\gamma}-1}e^{-\mu y}\,dy, \quad \sigma := \sum_i\alpha_i - 2Q \tag{2.24}$$

and the law of the random measure $Z(\cdot)/A$ conditioned on $Z(\hat{\mathbb{C}}) = A$ does not depend on A.

Remark 2.4. The correlation functions $\Pi_{\alpha,\mathbf{z}}$ have the same properties as in the $\alpha_i < Q$ case proven in [3]: **conformal covariance, Weyl covariance and KPZ scaling**. Since the statements are identical we refer the reader to [3] recalling here only the KPZ formula for the μ-dependence:

$$\Pi_{\alpha,\mathbf{z}} = \mu^{\frac{2Q-\sum_i \alpha_i}{\gamma}} \Pi_{\alpha,\mathbf{z}}|_{\mu=1}.$$

Remark 2.5. With some extra work it should be possible to prove that the measures $\mathbb{P}_{\alpha,\mathbf{z}}$ with $\alpha_i < Q$ for all $i = 1, \dots n$ converge as $\alpha_i \uparrow Q$, $i = 1, \dots k$ to the $\mathbb{P}_{\alpha,\mathbf{z}}$ constructed in this paper by proving that

$$\lim_{\alpha_i \uparrow Q} \prod_{i=1}^{k} (Q - \alpha_i)^{-1} \Pi_{\alpha,\mathbf{z}} \tag{2.25}$$

has a limit. We leave that question as an open problem.

Remark 2.6. It is natural to ask about the convergence of the quantum laws $\mathbb{P}_{\alpha,z}$ to the classical solutions of the Liouville equation i.e. the **semiclassical limit** $\gamma \to 0$. For this, let us take, for $i = 1, \dots, k$ $\alpha_i = Q$ and for $i > k$

$$\alpha_i = \frac{\chi_i}{\gamma}$$

with $\chi_i < 2$ and $\mu = \frac{\mu_0}{\gamma^2}$ for some constant $\mu_0 > 0$. Then we conjecture that the law of γX under $\mathbb{P}_{\alpha,z}$ converges towards the minimizer of equation (1.7) which has cusp singularities at z_i, $i \leq k$ and conical ones at the remaining z_i. The case of conical singularities was treated in [11] in the setup where $\hat{\mathbb{C}}$ is replaced by the unit disc.

2.4 Strategy of proof

We will now sketch the main ideas of the proof. We have to control the correlation function (2.15) as $\epsilon \to 0$ when at least one $\alpha_i = Q$. We may assume g is the round metric $\hat{g}$.

First of all, notice that the condition for the convergence of the c-integral remains the same, namely $\sum_i \alpha_i > 2Q$. Second, as explained above a Cameron-Martin transform reduces the analysis of (2.15) to the quantity

$$\Pi_{\alpha,\mathbf{z},\epsilon} = K_\epsilon(\mathbf{z}) \int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c} \mathbb{E}\Big[e^{-\mu e^{\gamma c} \int_{D_\epsilon} e^{\gamma H_\epsilon} dM_\gamma} \Big] dc \tag{2.26}$$

with

$$H_\epsilon(z) = \sum_i \alpha_i G_{g,\epsilon}(z, z_i). \tag{2.27}$$

where $G_{g,\epsilon}$ is a regularization of the covariance of the GFF and $K_\epsilon(\mathbf{z})$ converges as $\epsilon \to 0$ to $K(\mathbf{z})$ of (2.13). Locally around z_i,

$$e^{\gamma H_\epsilon(z)} \asymp \frac{1}{(|z - z_i| \vee \epsilon)^{\gamma \alpha_i}}.$$

The crucial point is thus to determine whether this singularity is integrable in the limit $\epsilon \to 0$ with respect to the measure $M_\gamma(dz)$. Multifractal analysis of the chaos measure shows that this is the case if and only if $\alpha_i < Q$ [3]. Let us see this in more detail to understand how to proceed when $\alpha_i = Q$. Since the problem is local consider the integral for $\alpha \leq Q$

$$I_{\alpha,\mathbf{z},\epsilon}=\int_{\mathbb{R}} e^{(\sum_i \alpha_i-2Q)c}\mathbb{E}\Big[e^{-\mu e^{\gamma c}\int_{C_\epsilon}\frac{1}{|z|^{\gamma\alpha}}\,dM_\gamma}\Big]dc. \tag{2.28}$$

where C_ϵ stands for the annulus $\{z\in\mathbb{C};\epsilon\le|z|\le 1\}$. We use a well known decomposition of the GFF to a "radial" and "angular" part to write the Chaos measure. The radial part of the GFF, defined by

$$X_{\hat{g},r}:=\frac{1}{2\pi}\int X_{\hat{g}}(re^{i\theta})d\theta$$

is a Brownian motion in time $t=\ln r^{-1}$ starting at time zero from $X_{\hat{g},1}$, up to an independent Gaussian random variable of $\mathcal{O}(1)$ variance. Changing to polar coordinated, this leads to the following expression for the chaos integral

$$\int_{C_\epsilon}\frac{1}{|x|^{\gamma\alpha}}\,M_\gamma(dx)=\int_0^{-\ln\epsilon}\int_0^{2\pi} e^{\gamma B_t-\gamma(Q-\alpha)t}\,\mu_Y(dt,d\theta)$$

where $\mu_Y(dt,d\theta)$ is a chaos measure encoding the angular contribution of the angular part of the GFF and independent of the process B_t (see Lemma 4.3). The measure μ_Y requires some care but in order to understand the behavior as $\epsilon\to 0$ it suffices here to consider a simplified problem where we replace it by the Lebesgue measure dt and consider the behaviour of

$$I_\epsilon:=\int_{\mathbb{R}} e^{(\sum_i \alpha_i-2Q)c}\mathbb{E}\Big[e^{-\mu e^{\gamma c}\int_0^{-\ln\epsilon} e^{\gamma B_t-\gamma(Q-\alpha)t}\,dt}\Big]dc. \tag{2.29}$$

as $\epsilon\to 0$. Clearly, when $\alpha<Q$, the drift term in the Brownian motion takes it all making the integral in the exponential converges, hence I_ϵ has a non trivial limit. When $\alpha=Q$, the drift term vanishes so that the integral $\int_0^{-\ln\epsilon} e^{\gamma B_t}\,dt$ diverges to $+\infty$ and I_ϵ goes to 0 as $\epsilon\to 0$. The main idea is that the leading asymptotics for this integral will come from the Brownian paths such that $\int_0^\infty e^{\gamma B_u}\,du<\infty$, which is an event of probability 0 for the Brownian motion. Hence a proper renormalization of this integral will require a conditioning on the event $\{\int_0^\infty e^{\gamma B_u}\,du<\infty\}$, which is the same as conditioning on those paths such that $\{\sup_{u\ge 0} B_u<\infty\}$. Having this picture in mind, it is natural to partition the probability space with the sets

$$A(n,\epsilon)=\{\sup_{u\le-\ln\epsilon} B_u\in]n-1,n]\}$$

for $n\ge 1$. We can then expand $I_\epsilon=\sum_{n\ge 1} I_\epsilon^n$ with

$$I_\epsilon^n:=\int_{\mathbb{R}} e^{(\sum_i \alpha_i-2Q)c}\mathbb{E}\Big[\mathbf{1}_{A(n,\epsilon)}e^{-\mu e^{\gamma c}\int_0^{-\ln\epsilon} e^{\gamma B_u}\,du}\Big]dc.$$

On $A(n,\epsilon)$, the integral $\int_0^{-\ln\epsilon} e^{\gamma B_u}\,du\sim e^{\gamma n}$ and we get

$$I_\epsilon^n\sim\mathbb{P}(A(n,\epsilon))\int_{\mathbb{R}} e^{(\sum_i \alpha_i-2Q)c-\mu e^{\gamma(c+n)}}\,dc\le C\mathbb{P}(A(n,\epsilon))e^{-(\sum_i \alpha_i-2Q)n}.$$

An elementary estimate on Brownian motion gives $\mathbb{P}(A(n,\epsilon))\le\sqrt{2/\pi}\frac{n}{(-\ln\epsilon)^{1/2}}$ so that the series $\sum_n(-\ln\epsilon)^{1/2}I_\epsilon^n$ is dominated by an absolutely convergent series, uniformly with respect to $\epsilon\in]0,1]$. We can thus invert the limits and get

$$\lim_{\epsilon\to 0}(-\ln\epsilon)^{1/2}I_\epsilon=\lim_{n\to\infty}\lim_{\epsilon\to 0}J_\epsilon^n \tag{2.30}$$

with

$$J_\epsilon^n:=(-\ln\epsilon)^{1/2}\int_{\mathbb{R}} e^{(\sum_i \alpha_i-2Q)c}\mathbb{E}\Big[\mathbf{1}_{B(n,\epsilon)}e^{-\mu e^{\gamma c}\int_0^{-\ln\epsilon} e^{\gamma B_u}\,du}\Big]dc$$

where we defined

$$B(n,\epsilon)=\cup_{k=1}^n A(k,\epsilon)=\{\sup_{u\le-\ln\epsilon} B_u\in]0,n]\}.$$

To determine the limit in the right-hand side of (2.30), the first step is to show that one can find a family h_ϵ such that $h_\epsilon \to \infty$, $h_\epsilon/(-\ln\epsilon) \to 0$ and

$$\lim_{\epsilon\to 0} J_\epsilon^n = \lim_{\epsilon\to 0}(-\ln\epsilon)^{1/2}\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\mathbb{E}\Big[\mathbf{1}_{B(n,\epsilon)} e^{-\mu e^{\gamma c}\int_0^{h_\epsilon} e^{\gamma B_u}\,du}\Big]dc. \tag{2.31}$$

The reason why one can find such a family h_ϵ is that conditioning the Brownian motion on not exceeding n will force it on going to $-\infty$ with a speed making the integral $\int_0^\infty e^{\gamma B_u}\,du$ finite. To compute the integral in the right-hand side in (2.31), we use the Markov property of the Brownian motion. Let $\mathcal{F}_t$ be the sigma algebra generated by the Brownian motion up to time t. Then this integral can be estimated by

$$(-\ln\epsilon)^{1/2}\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\mathbb{E}\Big[\mathbf{1}_{B(n,h_\epsilon)}\mathbb{E}[\mathbf{1}_{B(n,\epsilon)}|\mathcal{F}_{h_\epsilon}] e^{-\mu e^{\gamma c}\int_0^{h_\epsilon} e^{\gamma B_u}\,du}\Big]dc.$$

Once again, a standard computation related to the supremum of the Brownian motion shows that

$$\mathbb{E}[\mathbf{1}_{B(n,\epsilon)}|\mathcal{F}_{h_\epsilon}] \sim \sqrt{2/\pi}\,\frac{n - b_{h_\epsilon}}{(-\ln\epsilon - h_\epsilon)^{1/2}}.$$

Plugging this relation into the expression of J_ϵ^n, we deduce that

$$\lim_{\epsilon\to 0} J_\epsilon^n = \lim_{\epsilon\to 0}\sqrt{2/\pi}\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\mathbb{E}\Big[(n - B_{h_\epsilon})\mathbf{1}_{B(n,h_\epsilon)} e^{-\mu e^{\gamma c}\int_0^{h_\epsilon} e^{\gamma B_u}\,du}\Big]dc.$$

It turns out that, under the probability measure $d\tilde{\mathbb{P}} = \frac{1}{n}(n - B_{h_\epsilon})\mathbf{1}_{B(n,h_\epsilon)}$ (with expectation $\tilde{\mathbb{E}}$), the process $\beta_t = n - B_t$ is a $3d$-Bessel process. Rewriting the above integral, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0} J_\epsilon^n &= \lim_{\epsilon\to 0}\sqrt{2/\pi}\,n\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\tilde{\mathbb{E}}\Big[e^{-\mu e^{\gamma(c+n)}\int_0^{h_\epsilon} e^{-\gamma\beta_u}\,du}\Big]dc\\ &= \sqrt{2/\pi}\,n\int_{\mathbb{R}} e^{(\sum_i \alpha_i - 2Q)c}\tilde{\mathbb{E}}\Big[e^{-\mu e^{\gamma(c+n)}\int_0^{h_\epsilon} e^{-\gamma\beta_u}\,du}\Big]dc.\end{aligned}$$

As a Bessel process β_t goes to ∞ as $t \to \infty$ roughly at speed $\sqrt{t}$, the integral $\int_0^{h_\epsilon} e^{-\gamma\beta_u}\,du$ converges $\tilde{\mathbb{P}}$-almost surely towards $\int_0^\infty e^{-\gamma\beta_u}\,du$. This explains the convergence of $(-\ln\epsilon)^{1/2} I_\epsilon$ towards a non trivial limit as $\epsilon \to 0$. The main lines of our proof follows the thread of this heuristic.

3 Partition of the probability space

The singularity at the Q-insertions will be studied by partitioning the probability space according to the maximum of the circle average fields around them. As we will see this is a local operation and it will suffice to consider the case with only one Q-insertion, say $\alpha_1 = Q$, $\alpha_i < Q$, $i > 1$. We may also assume that the Q-insertion is located at $z_1 = 0$ and, for notational convenience, we further assume that the other z_i are in the complement of the disc $B(0,1)$. Also, we will work from now on with the round metric $\hat{g}$; the general case $g = e^{\phi}\hat{g}$ is treated as in [3].

Recalling the definitions (2.16) and (2.15) we need to study

$$\Pi_{\alpha,\mathbf{z},\epsilon}(F) = \int_{\mathbb{R}} e^{\sigma c}\mathbb{E}\Big[F(c + X_{\hat{g}})\prod_i V_{z_i,\alpha_i,\epsilon}(z_i) e^{-\mu e^{\gamma c} M_\gamma(D_\epsilon)}\Big]\,dc \tag{3.1}$$

where we use throughout the paper the notation

$$\sigma := \sum_i \alpha_i - 2Q \tag{3.2}$$

as in (2.24) and $D_\epsilon := \mathbb{C} \setminus B(z_1, \epsilon)$. We have then

$$\mathbb{E}_{z,\alpha,\epsilon} F = \Pi_{\alpha,\mathbf{z},\epsilon}(F)/\Pi_{\alpha,\mathbf{z},\epsilon}(1). \tag{3.3}$$

It will be convenient to replace the GFF $X_{\hat{g}}$ with the GFF X_0 with vanishing mean on the circle

$$X_0 := X_{\hat{g}} - m_{\mathcal{C}}(X_{\hat{g}}), \quad \text{with} \quad m_{\mathcal{C}}(X_{\hat{g}}) := \frac{1}{2\pi}\int_0^{2\pi} X_{\hat{g}}(e^{i\theta})\, d\theta,$$

which is more adapted to the local analysis around 0, as its covariance kernel

$$G_0(x,y) = \ln \frac{1}{|x-y|} + \ln|x|\mathbf{1}_{\{|x|\geq 1\}} + \ln|y|\mathbf{1}_{\{|y|\geq 1\}}, \tag{3.4}$$

is of exact log type in the ball $B(0,1)$, hence facilitates the analysis around 0. The replacement can be performed by making the change of variables $c \to c - m_{\mathcal{C}}(X_{\hat{g}})$ in the expression (3.1) to get

$$\Pi_{\alpha,\mathbf{z},\epsilon}(F) = \int_{\mathbb{R}} e^{\sigma c}\mathbb{E}\Big[e^{-\sigma m_{\mathcal{C}}(X_{\hat{g}})} F(c+X_0)\prod_i V_{z_i,\alpha_i,\epsilon}(z_i) e^{-\mu e^{\gamma c} M_\gamma^0(D_\epsilon)}\Big]\, dc \tag{3.5}$$

where

$$M_\gamma^0(dz) := \lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{0,\epsilon}(z)+\frac{Q}{2}\ln\hat{g}(z))}\, dz = e^{\gamma X_0(z) - \frac{\gamma^2}{2}\mathbb{E}[X_0^2(z)]}(|z|\vee 1)^{\gamma^2}\hat{g}(z)^{\frac{\gamma Q}{2}}\, dz,$$

and the vertex operators $V_{z_i,\alpha_i,\epsilon}(z_i)$ are defined as in (2.10) with $X_{\hat{g}}$ replaced by X_0. The Cameron-Martin argument then gives $\Pi_{\alpha,\mathbf{z},\epsilon}(F) = K_\epsilon(\mathbf{z})A_\epsilon(F)$ with

$$A_\epsilon(F) = \int_{\mathbb{R}} e^{\sigma c}\mathbb{E}\Big[F(c+X_0+H_\epsilon^0) e^{-\mu e^{\gamma c}\int_{D_\epsilon} e^{\gamma H_\epsilon^0} dM_\gamma^0}\Big]\, dc. \tag{3.6}$$

where

$$H_\epsilon^0(z) = \sum_i \alpha_i \int_0^{2\pi} G_0(z_i+\epsilon e^{i\theta}, z)\tfrac{d\theta}{2\pi} - \sigma(\tfrac{1}{2}\ln(1+|z|^2) - \ln|z|1_{\{|z|\geq 1\}}) + \sigma\tfrac{1}{2}(\ln 2 - 1) \tag{3.7}$$

and $K_\epsilon(\mathbf{z})$ (the variance of the Cameron-Martin transform) converges to some explicit K as $\epsilon \to 0$; we do not write the explicit expression for K as we do not need it in the following. The sum over i comes from the shift of the vertex operators $V_{z_i,\alpha_i,\epsilon}(z_i)$ in (3.5) and the remaining part from the shift induced by $e^{-\sigma m_{\mathcal{C}}}$.

Similarly for the derivative vertex operator (2.20) we get

$$\begin{aligned}\tilde{\Pi}_{\alpha,\mathbf{z},\epsilon}(F) = -K_\epsilon(\mathbf{z})\int_{\mathbb{R}} e^{\sigma c}\mathbb{E}\Big[&F(c+X_0+H_\epsilon^0)(Q\ln\epsilon + \tilde{H}_\epsilon^0 + c + X_{0,\epsilon}(z_1) \\ &+ \tfrac{Q}{2}\ln\hat{g}(z_1)) e^{-\mu e^{\gamma c}\int_{D_\epsilon} e^{\gamma H_\epsilon^0} dM_\gamma^0}\Big]\, dc.\end{aligned} \tag{3.8}$$

where

$$\tilde{H}_\epsilon^0 = \sum_i \alpha_i \int_0^{2\pi}\int_0^{2\pi} G_0(z_1+\epsilon e^{i\theta_2}, z_i+\epsilon e^{i\theta_2})\tfrac{d\theta_1}{2\pi}\tfrac{d\theta_2}{2\pi}. \tag{3.9}$$

Using (3.4) we see that the $Q\ln\epsilon$ singularity in (3.8) is cancelled by the one in the $i=1$ term in (3.9) so that $Q\ln\epsilon + \tilde{H}_\epsilon + \frac{Q}{2}\ln g(z_1)$ is bounded uniformly in ϵ. Since $\Pi_{z,\alpha,\epsilon}(F) \to 0$ as $\epsilon \to 0$ ([3]) we conclude that the limit, if it exits, of $\tilde{\Pi}_{z,\alpha,\epsilon}(F)$ equals the limit of $K_\epsilon(\mathbf{z})\tilde{A}_\epsilon(F)$ where

$$\tilde{A}_\epsilon(F) = \int_{\mathbb{R}} e^{\sigma c}\mathbb{E}\Big[F(c+X_{0,\epsilon}+H_\epsilon^0)(-c-X_{0,\epsilon}(z_1)) e^{-\mu e^{\gamma c}\int_{D_\epsilon} e^{\gamma H_\epsilon^0} dM_\gamma^0}\Big]\, dc. \tag{3.10}$$

Hence Theorems 2.1 and 2.2 follow if we prove

Proposition 3.1. *Let F be bounded and continuous on $H^{-1}(\hat{\mathbb{C}})$. Then the following limits*

$$A(F)=\lim_{\epsilon\to 0}(-\ln\epsilon)^{\frac{1}{2}}A_\epsilon(F)=\sqrt{2/\pi}\lim_{\epsilon\to 0}\tilde{A}_\epsilon(F) \tag{3.11}$$

exist and $A(1)>0$.

Now we partition the probability space according to the values of the maximum of the mapping $u\mapsto X_{0,u}(z_1)$ over $u\in[\epsilon,1]$. So we set

$$M_{n,\epsilon}=\big\{\max_{u\in[\epsilon,1]}X_{0,u}(z_1)\in[n-1,n]\big\},\quad n\geq 1, \tag{3.12}$$

$$M_{0,\epsilon}=\big\{\max_{u\in[\epsilon,1]}X_{0,u}(z_1)\leq 0\big\}., \tag{3.13}$$

and we expand the integral $A_\epsilon(F)$ along the partition made up of these sets $(M_{n,\epsilon})_n$:

$$A_\epsilon(F)=\sum_{n\geq 0}\int_{\mathbb{R}}e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}F(c+X_0+H^0_\epsilon)e^{-\mu e^{\gamma c}\int_{D_\epsilon}e^{\gamma H^0_\epsilon}\,dM^0_\gamma}\Big]\,dc:=\sum_{n\geq 0}A_\epsilon(F,n). \tag{3.14}$$

For $\tilde{A}_\epsilon(F)$ we write

$$\tilde{A}_\epsilon(F)=\sum_{n\geq 0}(\tilde{A}_\epsilon(F,n)+B_\epsilon(F,n))$$

with

$$\tilde{A}_\epsilon(F,n)=\int_{\mathbb{R}}e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}\big(n-X_{0,\epsilon}(z_1)\big)F(c+X_0+H^0_\epsilon)e^{-\mu e^{\gamma c}\int_{D_\epsilon}e^{\gamma H^0_\epsilon}\,dM^0_\gamma}\Big]\,dc \tag{3.15}$$

and

$$B_\epsilon(F,n)=-\int_{\mathbb{R}}e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}(n+c)F(c+X_0+H^0_\epsilon)e^{-\mu e^{\gamma c}\int_{D_\epsilon}e^{\gamma H^0_\epsilon}\,dM^0_\gamma}\Big]\,dc. \tag{3.16}$$

Note that $\tilde{A}_\epsilon(F,n)\geq 0$ for $F\geq 0$. We prove

Lemma 3.2. *Let F be bounded and continuous on $H^{-1}(\hat{\mathbb{C}})$. Then for all $n\geq 0$ the limits*

$$A(F,n)=\lim_{\epsilon\to 0}(-\ln\epsilon)^{\frac{1}{2}}A_\epsilon(F,n)=\sqrt{2/\pi}\lim_{\epsilon\to 0}\tilde{A}_\epsilon(F,n). \tag{3.17}$$

exist and $A(1,n)>0$. Moreover

$$\sum_{n\geq 0}\sup_{\epsilon\in]0,1]}(-\ln\epsilon)^{\frac{1}{2}}A_\epsilon(1,n)<\infty \tag{3.18}$$

$$\sum_{n\geq 0}\sup_{\epsilon\in]0,1]}\tilde{A}_\epsilon(1,n)<\infty \tag{3.19}$$

$$\sum_{n\geq 0}B_\epsilon(F,n)\to 0,\quad \textit{as } \epsilon\to 0. \tag{3.20}$$

Proposition 3.1 then follows from Lemma 3.2 since $\lim_{\epsilon\to 0}A_\epsilon(F,\epsilon)=\sum A(F,n)$ follows from (3.17) and (3.18) by the dominated convergence theorem, similarly for $\tilde{A}$. The remaining part of this paper is devoted to proving this lemma.

4 Decomposition of the GFF and chaos measure

We denote by $\mathcal{F}_\delta$ ($\delta>0$) the sigma-algebra generated by the field X_0 "away from the disc $B(0,\delta)$", namely

$$\mathcal{F}_\delta=\sigma\{X_0(f);\mathrm{supp}\,f\in B(0,\delta)^c\}. \tag{4.1}$$

$\mathcal{F}_\infty$ stands for the sigma algebra generated by $\bigcup_{\delta>0}\mathcal{F}_\delta$.

First we collect a few old and classical observations (see [3, 5, 13] for more on this)

Lemma 4.1. *For all $\delta > 0$, the process*

$$t \mapsto X_{0,\delta e^{-t}}(0) - X_{0,\delta}(0)$$

evolves as a Brownian motion independent of the sigma algebra $\mathcal{F}_\delta$.

The following decomposition of the field X_0 will be useful for the analysis (this observation was already made in [5])

Lemma 4.2. *The field X_0 may be decomposed (recall that the fields we consider are understood in terms of distributions in the sense of Schwartz)*

$$X_0(z) = X_{0,|z|}(0) + Y(z) \tag{4.2}$$

where the process $r \in \mathbb{R}_+^* \mapsto X_{0,r}(0)$ is independent of the field $Y(z)$. The latter has the following covariance

$$\mathbb{E}[Y(z)Y(z')] = \ln \frac{|z| \vee |z'|}{|z-z'|}$$

Proof. From (3.4) we get using rotational invariance $\mathbb{E}[X_0(z)X_{0,|z'|}(0)] = \mathbb{E}[X_{0,|z|}(0)X_{0,|z'|}(0)]$, which in turn leads to independence:

$$\mathbb{E}X_0(z)X_0(z') = \mathbb{E}X_{0,|z|}(0)X_{0,|z'|}(0) + \mathbb{E}Y(z)Y(z').$$

Furthermore we calculate

$$\mathbb{E}[Y(z)Y(z')] = G_0(z,z') - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G_0(|z|e^{iu}, |z'|'e^{iv})dudv.$$

The claim follows from $\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G_0(|z|e^{iu}, |z'|'e^{iv})dudv = \ln\frac{1}{|z|\vee|z'|} + \ln|z|\mathbf{1}_{\{|z|\geq 1\}} + \ln|z'|\mathbf{1}_{\{|z'|\geq 1\}}$. ☐

Now, we get the decomposition

$$M_\gamma^0(dz) = \hat{g}(z)^{\frac{\gamma Q}{2}}|z|^{\frac{\gamma^2}{2}}e^{\gamma X_{0,|z|}(0)}\, M_\gamma^0(dz, Y)$$

where $M_\gamma(dz,Y)$ is the multiplicative chaos measure of the field Y with respect to the Lebesgue measure λ (i.e. $\mathbb{E}\, M_\gamma(dz,Y) = \lambda(dz)$).

We will now make change of variables $z = e^{-s+i\theta}$, $s \in \mathbb{R}_+$, $\theta \in [0,2\pi)$ and let $\mu_Y(ds,d\theta)$ be the multiplicative chaos measure of the field $Y(e^{-s+i\theta})$ with respect to the measure $dsd\theta$. We will denote by x_s the process

$$s \in \mathbb{R}_+ \to x_s := X_{0,e^{-s}}(0).$$

We have arrived at the following useful decomposition of the chaos measure around $z_1 = 0$:

Lemma 4.3. *On the ball $B(0,1)$ we have the following decomposition of the measure M_γ:*

$$\int_A \frac{1}{|x|^{\gamma Q}}\, M_\gamma^0(dx) = \int_0^\infty \int_0^{2\pi} \mathbf{1}_A(e^{-s}e^{i\theta})e^{\gamma x_s}\hat{g}(e^{-s})^{\frac{\gamma Q}{2}}\, \mu_Y(ds,d\theta)$$

for all $A \subset B(0,1)$ where $\mu_Y(ds,d\theta)$ is a measure independent of the whole process $(x_s)_{s\geq 0}$. Furthermore, for all $q \in]-\infty;\frac{4}{\gamma^2}[$, we have

$$\sup_{a>0}\mathbb{E}\Big[\Big(\int_a^{a+1}\int_0^{2\pi} e^{\gamma(x_s-x_a)}\mu_Y(ds,d\theta)\Big)^q\Big] < +\infty. \tag{4.3}$$

Proof. We have for $0 \leq q < \frac{4}{\gamma^2}$

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_a^{a+1}\int_0^{2\pi} e^{\gamma(x_s-x_a)}\mu_Y(ds,d\theta)\Big)^q\Big] \\
&\leq (2\pi)^q \mathbb{E}\Big[e^{q\gamma \sup_{\sigma\in[0,1]}(x_{a+\sigma}-x_a)}\Big]\mathbb{E}\Big[\mu_Y([a,a+1]\times[0,2\pi])^q\Big] \\
&= (2\pi)^q \mathbb{E}\Big[e^{q\gamma \sup_{\sigma\in[0,1]}(x_{a+\sigma}-x_a)}\Big]\mathbb{E}\Big[\mu_Y([0,1]\times[0,2\pi])^q\Big],
\end{aligned}$$

by stationarity of $(s,\theta)\in\mathbb{R}_+^*\times[0,2\pi]\mapsto Y(e^{-s}e^{i\theta})$. By Lemma 4.1 the first exponent is Brownian motion and hence the expectation is bounded uniformly in a. From Gaussian multiplicative chaos theory [13, Theorem 2.11], we have finiteness of the quantity $\mathbb{E}\Big[\mu_Y([0,1]\times[0,2\pi])^q\Big]<\infty$, hence we get (4.3). For $q<0$, this is the same argument by replacing $\sup_{\sigma\in[0,1]}(x_{a+\sigma}-x_a)$ by $\min_{\sigma\in[0,1]}(x_{a+\sigma}-x_a)$ and using [13, Theorem 2.12]. □

It will be useful in the proofs to introduce for all $a\geq 1$ the stopping times T_a defined by

$$T_a = \inf\{s;\ x_s \geq a-1\}, \tag{4.4}$$

and we denote by $\mathcal{G}_{T_a}$ the associated filtration. We have the following analog of (4.3) with stopping times

Lemma 4.4. *For all $q\leq 0$, $n\geq 1$,*

$$\mathbb{E}\Big[\Big(\int_{T_{n-1}}^{T_n}\int_0^{2\pi} e^{\gamma(x_s-x_{T_{n-1}})}\mu_Y(ds,d\theta)\Big)^q\Big]<\infty. \tag{4.5}$$

Proof. Using the independence of the processes x_r and Y, Lemma 4.1 and stationarity of $Y(s,\theta)$ in s we see that (4.5) is equivalent to proving

$$\mathbb{E}\Big[\Big(\int_0^{\tau}\int_0^{2\pi} e^{\gamma\beta_s}\mu_Y(ds,d\theta)\Big)^q\Big]<\infty. \tag{4.6}$$

where β is a Brownian motion independent of Y and $\tau=\inf\{s;\beta_s\geq 1\}$. We have (recall that $q\leq 0$)

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^{\tau}\int_0^{2\pi} e^{\gamma\beta_s}\mu_Y(ds,d\theta)\Big)^q\Big] &\leq \mathbb{E}\Big[\mathbf{1}_{\tau\leq 1}\Big(\int_0^{\tau}\int_0^{2\pi} e^{\gamma\beta_s}\mu_Y(ds,d\theta)\Big)^q\Big] \\
&\quad + \mathbb{E}\Big[\Big(\int_0^{1}\int_0^{2\pi} e^{\gamma\beta_s}\mu_Y(ds,d\theta)\Big)^q\Big]
\end{aligned}$$

The second term is bounded by Lemma 4.3. The first one equals

$$\begin{aligned}
&\sum_{k\geq 1}\mathbb{E}\Big[\mathbf{1}_{1/2^{k+1}<\tau\leq 1/2^k}\Big(\int_0^{\tau}\int_0^{2\pi} e^{\gamma\beta_s}\mu_Y(ds,d\theta)\Big)^q\Big] \\
&\leq \sum_{k\geq 1}\mathbb{E}\Big[\mathbf{1}_{1/2^{k+1}<\tau\leq 1/2^k}\Big(\int_0^{1/2^{k+1}}\int_0^{2\pi} e^{\gamma\beta_r}\mu_Y(dr,d\theta)\Big)^q\Big] \\
&\leq \sum_{k\geq 1}\mathbb{P}(1/2^{k+1}<\tau\leq 1/2^k)^{1/2}\mathbb{E}\Big[\Big(\int_0^{1/2^{k+1}}\int_0^{2\pi} e^{\gamma\beta_r}\mu_Y(dr,d\theta)\Big)^{2q}\Big]^{1/2} \\
&\leq \sum_{k\geq 1}\mathbb{P}(1/2^{k+1}<\tau\leq 1/2^k)^{1/2}\mathbb{E}\Big[e^{2q\gamma\sup_{\sigma\in[0,2^{-k-1}]}\beta(\sigma)}\Big]^{\frac12}\mathbb{E}\Big[\mu_Y([0,2^{-k-1}]\times[0,2\pi])^{2q}\Big]^{\frac12} \\
&\leq C\sum_{n\geq 1} e^{-c2^n}\mathbb{E}\Big[\mu_Y([0,2^{-k-1}]\times[0,2\pi])^{2q}\Big]^{\frac12}.
\end{aligned}$$

$$\leq C \sum_{n\geq 1} e^{-c2^n} \mathbb{E}\Big[\mu_Y([0,2^{-k-1}]\times[0,2^{-k-1}])^{2q}\Big]^{\frac{1}{2}}.$$

One can find some constant $C>0$ such that the covariance $\mathbb{E}[Y(e^{-s}e^{i\theta})Y(e^{-s'}e^{i\theta'})]$ is bounded by $\ln\frac{1}{|se^{i\theta}-s'e^{i\theta'}|}+C$ hence by Kahane's convexity inequality [13, Theorem 2.1] one gets the existence of some constant $C>0$ such that

$$\mathbb{E}\Big[\mu_Y([0,2^{-n-1}]\times[0,2^{-n-1}])^{q}\Big] \leq C\frac{1}{2^{n\xi(-q)}},$$

with $\xi(-q)=-(2+\frac{\gamma^2}{2})q-\gamma^2\frac{q^2}{2}$. Hence $\sum_{n\geq 1} e^{-c2^n}\mathbb{E}\Big[\mu_Y([0,2^{-n-1}]\times[0,2^{-n-1}])^{2q}\Big]^{\frac{1}{2}}<\infty$, which concludes the proof. □

Now let us consider the martingale $(f_\epsilon^n)_{\epsilon\in]0,1]}$ defined by

$$f_\epsilon^n = \mathbf{1}_{\left\{\min_{u\in[\epsilon,1]} n-x_{\ln\frac{1}{u}}\geq 0\right\}}(n-x_{\ln\frac{1}{\epsilon}}). \tag{4.7}$$

The martingale property of $(f_\epsilon^n)_{\epsilon\in]0,1]}$ is classical: it results from Lemma 4.1 as well as the optional stopping theorem. We can define for each $\epsilon\in]0,1]$ a probability measure on $\mathcal{F}_\epsilon$ by

$$\Theta_\epsilon^n = \frac{1}{\mathbb{E}[f_\epsilon^n]} f_\epsilon^n\, d\mathbb{P},$$

where one has the following bound $\mathbb{E}[f_\epsilon^n]=\mathbb{E}[f_1^n]\leq n+C$ for some constant C. Because of Lemma 4.1 and the martingale property of the family $(f_\epsilon^n)_{\epsilon\in]0,1]}$, it is plain to check that these probability measures are compatible in the sense that, for $\epsilon'<\epsilon$

$$\Theta_{\epsilon'}^n|_{\mathcal{F}_\epsilon} = \Theta_\epsilon^n. \tag{4.8}$$

By Caratheodory's extension theorem we can find a probability measure Θ^n on $\mathcal{F}_\infty$ such that for all $\epsilon\in]0,1]$

$$\Theta^n|_{\mathcal{F}_\epsilon} = \Theta_\epsilon^n. \tag{4.9}$$

We denote by $\mathbb{E}^{\Theta^n}$ the corresponding expectation.

Recall the following explicit law of the Brownian motion conditioned to stay positive

Lemma 4.5. *Under the probability measure Θ^n, the process*

$$t\mapsto n-x_t$$

evolves as a $3d$-Bessel process starting from $n-x_0$ where x_0 is distributed like $X_{0,1}$ (under $\mathbb{P}$) conditioned to be less or equal to n.

We will sometimes use the following classical representation: under Θ^n, the process $t\mapsto n-x_t$ is distributed like $|n-x_0+B_t|$ where B_t is a standard 3d Brownian motion starting from 0 (here, we identify $n-x_0$ with $(n-x_0)(1,0,0)$).

5 Construction of the derivative Q-vertex

In this section, we prove the claims in Lemma 3.2 concerning $\tilde{A}_\epsilon$. We register here a simple Lemma on Brownian motion that is used repeatedly and whose proof is elementary and left to the reader:

Lemma 5.1. *Let B be a standard real valued Brownian motion. We have for $\beta>0$*

$$\mathbb{P}(\sup_{u\leq t} B_u\leq\beta) = \sqrt{\frac{2}{\pi}}\int_0^{\frac{\beta}{\sqrt{t}}} e^{-\frac{u^2}{2}}\,du \leq \sqrt{\frac{2}{\pi}}\frac{\beta}{\sqrt{t}}.$$

5.0.1 Proof of (3.20)

From (3.16) we get

$$|B_\epsilon(F,n)| \leq C\int_{\mathbb{R}} e^{\sigma c}\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}|n+c|e^{-\mu e^{\gamma c}\int_{D_\epsilon} e^{\gamma H_\epsilon^0}\,dM_\gamma^0}\Big]\,dc. \tag{5.1}$$

Recall that $|z_i|>1$ for $i\geq 2$. Then, recalling (3.7) and (3.4) we get for $\epsilon\leq |z|\leq 1$

$$e^{\gamma H_\epsilon^0(z)} \geq C|z|^{-\gamma Q}. \tag{5.2}$$

By Lemma 4.3, we get

$$|B_\epsilon(F,n)| \leq C\int_{\mathbb{R}} e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}|n+c|e^{-C\mu e^{\gamma c}\int_0^{\ln\frac{1}{\epsilon}} e^{\gamma x_r}\,\mu_Y(dr)}\Big]\,dc, \tag{5.3}$$

where $\mu_Y(dr)$ is the measure defined by $\mu_Y(dr)=\int_0^{2\pi}\mu_Y(dr,d\theta)$.

Below, we want to show that the integral in the exponential term above carries a big amount of mass, and we will look for this mass at some place where the process $r\mapsto x_r$ takes on values close to its maximum, which is between $n-1$ and n on the set $M_{n,\epsilon}$. To locate this place, we use the stopping times T_{n-1} and T_n defined by (4.4) which are finite and belong to $[0,\ln\frac{1}{\epsilon}]$ on $M_{n,\epsilon}$. We deduce

$$|B_\epsilon(F,n)| \leq C\int_{\mathbb{R}} e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}|n+c|e^{-\mu e^{\gamma c}Ce^{\gamma(n-1)}I_n}\Big]\,dc$$

where we have set

$$I_n=\int_{T_{n-1}}^{T_n} e^{\gamma(x_r-x_{T_{n-1}})}\,\mu_Y(dr). \tag{5.4}$$

By making the change of variables $y=e^{\gamma(c+n)}I_n$, we get

$$B_\epsilon(F,n) \leq Ce^{-n\sigma}\int_0^\infty y^{\frac{\sigma}{\gamma}-1}(1+|\ln y|)e^{-\mu Ce^{-\gamma}y}\,dy\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}(1+|\ln I_n|)I_n^{-\frac{\sigma}{\gamma}}\Big].$$

Then we bound

$$\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}(1+|\ln I_n|)I_n^{-\frac{\sigma}{\gamma}}\Big] \leq \mathbb{P}(M_{n,\epsilon})^{1/2}\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}(1+|\ln I_n|)^2 I_n^{-\frac{2\sigma}{\gamma}}\Big]^{\frac{1}{2}}.$$

Hence, by Lemma 4.4 we conclude

$$B(F;\epsilon,n) \leq Ce^{-n\sigma}\mathbb{P}(M_{n,\epsilon})^{1/2}.$$

The claim (3.20) then follows by the dominated convergence theorem since for each fixed n, the probability $\mathbb{P}(M_{n,\epsilon})$ goes to 0 as ϵ goes to 0 (see Lemma 5.1). □

5.0.2 Proof of (3.19)

Proceeding as in the proof of (3.20) we get

$$\tilde{A}_\epsilon(1,n) \leq Ce^{-n\sigma}\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}(n-x_{\ln\frac{1}{\epsilon}})I_n^{-\frac{\sigma}{\gamma}}\Big],$$

where I_n is as in (5.4). Now, we have

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{M_{n,\epsilon}}(n-x_{\ln\frac{1}{\epsilon}})|\mathcal{F}_{T_n}\vee\sigma(Y)] &= \mathbb{E}[\mathbf{1}_{\min_{s\in[0,\ln\frac{1}{\epsilon}]}(n-x_s)\geq 0}(n-x_{\ln\frac{1}{\epsilon}})|\mathcal{F}_{T_n}\vee\sigma(Y)]\mathbf{1}_{T_n\leq\ln\frac{1}{\epsilon}}\\
&= \mathbf{1}_{\min_{s\in[0,T_n]}(n-x_s)\geq 0}(n-x_{T_n})\mathbf{1}_{T_n\leq\ln\frac{1}{\epsilon}}\\
&= \mathbf{1}_{\min_{s\in[0,T_n]}(n-x_s)\geq 0}\mathbf{1}_{T_n\leq\ln\frac{1}{\epsilon}}
\end{aligned}$$

so that

$$\tilde{A}_\epsilon(1,n) \leq Ce^{-n\sigma}\mathbb{E}\,I_n^{-\frac{\sigma}{\gamma}} \leq Ce^{-n\sigma}.$$

from which the estimate (3.19) follows.

5.0.3 Proof of the first part of (3.17), i.e. the existence of $\lim_{\epsilon\to 0}\tilde{A}_\epsilon(F,n)$

Now, we need to establish the existence and non triviality of the limit of $\tilde{A}_\epsilon(F,n)$, i.e. one part of (3.17). Since H^0_ϵ converges in $H^{-1}(\hat{\mathbb{C}})$ towards H^0, it suffices to study the convergence and non triviality of the limit for $F=1$ and fixed n. We claim that this will result from the convergence in probability of the quantity $\int_{D_\epsilon} e^{\gamma H^0_\epsilon}\,dM^0_\gamma$ under the probability measure Θ^n towards a non trivial limit. To see this, make the change of variables $y=e^{\gamma c}\int_{D_\epsilon} e^{\gamma H^0_\epsilon}\,dM^0_\gamma$ to get

$$\begin{aligned}\int e^{\sigma c}\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}(n-x_{\ln\frac{1}{\epsilon}})e^{-\mu e^{\gamma c}\int_{D_\epsilon} e^{\gamma H^0_\epsilon}\,dM^0_\gamma}\Big]\,dc\\ =\gamma^{-1}\mathbb{E}[f_1^n]\int_0^\infty y^{\frac{\sigma}{\gamma}-1}e^{-\mu y}\,dy\times\mathbb{E}^{\Theta^n}\Big[\mathbf{1}_{M_{n,\epsilon}}\Big(\int_{D_\epsilon} e^{\gamma H^0_\epsilon}\,dM^0_\gamma\Big)^{-\frac{\sigma}{\gamma}}\Big].\end{aligned}$$

Under the probability measure Θ^n the process $t\mapsto(n-x_t)$ is a 3d Bessel process hence $\min_{s\in[0,\ln\frac{1}{\epsilon}]}(n-x_s)$ converges almost surely to a finite random variable as ϵ goes to 0 and therefore $\mathbf{1}_{M_{n,\epsilon}}$ converges to $\mathbf{1}_{\max_{s\in[0,\infty]}(x_s)\in[n-1,n]}$.

Take any non empty closed ball B of $\mathbb{R}^2$ containing no insertions z_i. Then $\sup_\epsilon H^0_\epsilon$ is bounded in B and thus

$$\Big(\int_{D_\epsilon} e^{\gamma H^0_\epsilon}\,dM^0_\gamma\Big)^{-\frac{\sigma}{\gamma}}\leq CM^0_\gamma(B)^{-\frac{\sigma}{\gamma}}.$$

Let $\delta>0$ be such that $B\subset B(0,\delta)^c$. Then

$$\mathbb{E}^{\Theta^n}\Big[M^0_\gamma(B)^{-\frac{\sigma}{\gamma}}\Big]\leq C(n+1)^{-1}\mathbb{E}[f^n_\delta M^0_\gamma(B)^{-\frac{\sigma}{\gamma}}]\leq C(n+1)^{-1}\mathbb{E}[(f^n_\delta)^2]^{\frac{1}{2}}\,\mathbb{E}\Big[M_\gamma(B)^{-\frac{2\sigma}{\gamma}}\Big]^{\frac{1}{2}}.$$

Because GMC admits moments of negative order [13, theorem 2.12], the last expectation is finite. Hence the dominated convergence theorem entails that to prove our claim it is enough to establish the convergence in probability of the quantity $\int_{D_\epsilon} e^{\gamma H^0_\epsilon}\,dM^0_\gamma$ under the probability measure Θ^n towards a non trivial limit. Because M^0_γ is a positive measure and because of the bound (5.2), this is clearly equivalent to the finiteness under Θ^n of the quantity $\int_{\mathbb{R}^2} e^{\gamma H^0}\,dM^0_\gamma$. Outside of the ball $B(0,1)$, the finiteness results from the fact that $\int_{D_1} e^{\gamma H^0}\,dM^0_\gamma<\infty$ under $\mathbb{P}$ (see see [3, proof of Th. 3.2]), and the absolute continuity of Θ^n with respect to $\mathbb{P}$ when restricted to $\mathcal{F}_1$. The main point is thus to analyze the integrability inside the ball $B(0,1)$. It is clearly enough to show

$$\int_{B(0,1)}\frac{1}{|x|^{\gamma Q}}\,dM^0_\gamma<\infty,\quad\text{a.s. under }\Theta^n. \tag{5.5}$$

This follows from the following Lemma 5.2: □

Lemma 5.2. *The measure M^0_γ satisfies*

$$\mathbb{E}^{\Theta^n}\Big[\int_{B(0,1)}\frac{1}{|x|^{\gamma Q}}M^0_\gamma(dx)\Big]<\infty. \tag{5.6}$$

Proof. Under the measure Θ^n, the process $t\mapsto n-x_t$ is distributed like $|n-x_0+B_t|$ where B_t is a standard 3 dimensional Brownian motion (here, we identify $n-x_0$ with $(n-x_0)(1,0,0)$). We suppose the Brownian motion lives on the same probability space. Then, if N denotes a standard 3d Gaussian variable (under some expectation we will also denote $\mathbb{E}$), we have

$$\begin{aligned}
\mathbb{E}^{\Theta^n}\Big[\int_{B(0,1)}\frac{1}{|x|^{\gamma Q}}M_\gamma^0(dx)\Big] &\leq C\mathbb{E}^{\Theta^n}\Big[\int_0^\infty\int_0^{2\pi} e^{\gamma x_r}\,\mu_Y(dr,d\theta)\Big]\\
&= Ce^{\gamma n}\mathbb{E}^{\Theta^n}\Big[\int_0^\infty\int_0^{2\pi} e^{-\gamma|n-x_0+B_r|}\,\mu_Y(dr,d\theta)\Big]\\
&= Ce^{\gamma n}\mathbb{E}^{\Theta^n}\Big[\int_0^\infty e^{-\gamma|n-x_0+B_r|}\,dr\Big]\\
&\leq Ce^{2\gamma n}\mathbb{E}\,[e^{\gamma|x_0|}]\mathbb{E}^{\Theta^n}\Big[\int_0^\infty e^{-\gamma|B_r|}\,dr\Big]\\
&= Ce^{2\gamma n}\mathbb{E}\Big[\int_0^\infty e^{-\gamma\sqrt{r}|N|}\,dr\Big],\\
&= Ce^{2\gamma n}(\int_0^\infty e^{-\gamma\sqrt{r}}\,dr)\mathbb{E}\Big[\frac{1}{|N|^2}\Big]<\infty. \qquad \square
\end{aligned}$$

6 Renormalization of the Q-vertex operators

6.1 Proof of (3.18)

Using (5.2) and proceeding as for (5.3) we get

$$A_\epsilon(1,n)\leq C\int_{\mathbb{R}} e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}}\exp\Big(-\mu e^{\gamma c}C\int_0^{\ln\frac{1}{\epsilon}} e^{\gamma x_r}\,\mu_Y(dr)\Big)\Big]\,dc$$

The stopping time $T_n=\inf\{s;\,x_s\geq n-1\}$ is finite and belongs to $[0,\ln\frac{1}{\epsilon}]$ on $M_{n,\epsilon}$. We deduce that

$$\begin{aligned}
A_\epsilon(1,n)\leq& C\int_{\mathbb{R}} e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}\cap\{T_n<\ln\frac{1}{\epsilon}-1\}}\exp\Big(-\mu e^{\gamma c}Ce^{\gamma(n-1)}I(T_n)\Big)\Big]\,dc && (6.1)\\
&+C\int_{\mathbb{R}} e^{\sigma c}\,\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}\cap\{T_n\geq\ln\frac{1}{\epsilon}-1\}}\exp\Big(-\mu e^{\gamma c}Ce^{\gamma x_{\ln\frac{1}{\epsilon}-1}}I(\ln\frac{1}{\epsilon}-1)\Big)\Big]\,dc\\
&=:a_\epsilon(n)+b_\epsilon(n) && (6.2)
\end{aligned}$$

where we have set

$$I(z)=\int_z^{z+1} e^{\gamma(x_r-x_z)}\,\mu_Y(dr).$$

We will show that there exists a constant $C>0$ such that for all n

$$(\ln\tfrac{1}{\epsilon})^{\frac12}a_\epsilon(n),\ (\ln\tfrac{1}{\epsilon})^{\frac12}b_\epsilon(n)\leq Cne^{-\sigma n}, \tag{6.3}$$

which is enough to complete the proof of (3.18).

We begin with $a_\epsilon(n)$. By making the change of variables $y=e^{\gamma(c+n)}I(T_n)$, we get

$$a_\epsilon(n)\leq C\,e^{-n\sigma}\int_0^\infty y^{\frac{\sigma}{\gamma}-1}e^{-\mu Ce^{-\gamma}y}\,dy\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}\cap\{T_n<\ln\frac{1}{\epsilon}-1\}}I(T_n)^{-\frac{\sigma}{\gamma}}\Big].$$

It suffices to estimate the last expectation. Obviously, we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}\cap\{T_n<\ln\frac{1}{\epsilon}-1\}}I(T_n)^{-\frac{\sigma}{\gamma}}\Big] && (6.4)\\
\leq&\mathbb{E}\Big[\mathbf{1}_{\{\min_{u\in[0,T_n]}n-x_u\geq0\}}\mathbf{1}_{\{\min_{u\in[T_n+1,\ln\frac{1}{\epsilon}]}(n-x_{T_n+1})-(x_u-x_{T_n+1})\geq0\}}\frac{\mathbf{1}_{\{T_n+1<\ln\frac{1}{\epsilon}\}}}{I(T_n)^{\frac{\sigma}{\gamma}}}\Big].
\end{aligned}$$

By conditioning on the the sigma algebra $\mathcal{H}_{T_n}$ generated by $\{x_r,r\leq T_n\}$, $\{x_r-x_{T_n+1},r\geq T_n+1\}$ and $\{x_{T_n+1}-n\}$, we see that we have to estimate the quantity

$$\mathbb{E}[I(a)^{-\frac{\sigma}{\gamma}}|x_{a+1}-x_a].$$

We claim

Lemma 6.1. *There exists a constant C (independent of any relevant quantity) such that for all $a > 0$*

$$\mathbb{E}[I(a)^{-\frac{\sigma}{\gamma}}|x_{a+1}-x_a] \leq C\big(e^{-\sigma(x_{a+1}-x_a)}+1\big).$$

The proof of this lemma is given just below. Admitting it for a while and given the fact that the random variable $x_{T_n+1}-n$ is a standard Gaussian random variable, the conditioning on $\mathcal{H}_{T_n}$ of the expectation (6.4) thus gives

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}\cap\{T_n<\ln\frac{1}{\epsilon}-1\}}\frac{1}{I(T_n)^{\frac{\sigma}{\gamma}}}\Big]\\
\leq & C\int_{\mathbb{R}}\mathbb{E}\Big[\mathbf{1}_{\{\min_{u\in[0,T_n]}n-x_u\geq 0\}}\mathbf{1}_{\{\min_{u\in[T_n+1,\ln\frac{1}{\epsilon}]}-y-(x_u-x_{T_n+1})\geq 0\}}\Big]\big(e^{-\sigma(y+1)}+1\big)e^{-y^2/2}\,dy.
\end{aligned}$$

To estimate the expectation in the integral, use the strong Markov property of the Brownian motion to write

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{\min_{u\in[0,T_n]}n-x_u\geq 0\}}\mathbf{1}_{\{\min_{u\in[T_n+1,\ln\frac{1}{\epsilon}]}-y-(x_u-x_{T_n+1})\geq 0\}}\Big]\\
=&\mathbb{E}\Big[\mathbf{1}_{\{\min_{u\in[0,T_n]}n-x_u\geq 0\}}\mathbf{1}_{\{\min_{u\in[T_n,\ln\frac{1}{\epsilon}-1]}-y-(x_u-x_{T_n})\geq 0\}}\Big]\\
\leq&\mathbb{E}\Big[\mathbf{1}_{\{\min_{u\in[0,\ln\frac{1}{\epsilon}-1]}n+\max(0,-y)-x_u\geq 0\}}\Big]\leq(\tfrac{2}{\pi})^{\frac{1}{2}}\frac{n+\max(0,-y)}{(\ln\frac{1}{\epsilon}-1)^{\frac{1}{2}}},
\end{aligned}$$

where in the last inequality we have used Lemma 5.1. We deduce

$$\mathbb{E}\Big[\mathbf{1}_{M_{n,\epsilon}\cap\{T_n<\ln\frac{1}{\epsilon}-1\}}\frac{1}{I(T_n)^{\frac{\sigma}{\gamma}}}\Big]\leq C(\ln\tfrac{1}{\epsilon})^{-\frac{1}{2}}ne^{-n\sigma}.$$

All in all, we have obtained

$$\sup_{\epsilon\in]0,1]}(\ln\tfrac{1}{\epsilon})^{\frac{1}{2}}a_\epsilon(n)\leq Cne^{-n\sigma},$$

which proves the claim. The same argument holds for $b_\epsilon(n)$. □

Proof of Lemma 6.1. Notice that the joint law of $\Big((x_r-x_a)_{r\in[a,a+1]},x_{a+1}-x_a\Big)$ is that of $\big((B_u-B_a)_{u\in[a,a+1]},B_{a+1}-B_a\big)$ where B is a standard Brownian motion starting from 0 (independent of Y). Hence the law of $I(a)$ conditionally on $x_{a+1}-x_a=x$ is given by

$$\int_0^1 e^{\gamma\mathrm{Bridge}_r^{0,x}}\,\mu_Y(dr)$$

where $(\mathrm{Bridge}_r^{0,x})_{r\leq 1}$ is a Brownian bridge between 0 et x with lifetime 1. Hence it has the law of $r\mapsto B_r-rB_1+ux$. By convexity of the mapping $x\mapsto x^{-q}$ for $q>0$ and the fact that the covariance kernel of the Brownian Bridge and the Brownian motion are comparable up to fixed constant, we can apply Kahane's inequality [9] to get that

$$\mathbb{E}[I(a)^{-\frac{\sigma}{\gamma}}|x_{a+1}-x_a=x]\leq C\mathbb{E}\Big[\Big(\int_a^{a+1}e^{\gamma(B_r-B_a)+(r-a)x}\,\mu_Y(dr)\Big)^{-\frac{\sigma}{\gamma}}\Big].$$

From Lemma 4.3 and the fact that $e^{(r-a)x}\geq e^x\wedge 1$ for $r\in[a,a+1]$, this quantity is less than

$$\mathbb{E}[I(a)^{-\frac{\sigma}{\gamma}}|x_{a+1}-x_a=x]\leq C\big(e^{-\sigma x}\vee 1\big).$$

This proves the claim. □

6.2 Proof of (3.17)

First notice that

$$\sum_{n=0}^{N} A_\epsilon(1,n) = \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} \exp\Big(-\mu e^{\gamma c} \int_{D_\epsilon} e^{\gamma H_\epsilon^0}\, dM_\gamma^0 \Big)\Big]\, dc$$

where

$$B_{N,\epsilon} = \{ \min_{u\in[\epsilon,1]} N - x_u \geq 0 \}.$$

Let us denote by Z_ϵ the measure $e^{\gamma H_\epsilon^0}\, dM_\gamma^0$ and define

$$s_\epsilon := (\ln \tfrac{1}{\epsilon})^{\frac{1}{6}}, \quad h_\epsilon := e^{-s_\epsilon}.$$

Now we prove the upper bound. We have

$$\begin{aligned}\sum_{n=0}^{N} A_\epsilon(1,n) &\leq \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} \exp\Big(-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon}) \Big)\Big]\, dc \\ &= \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{B_{N,\epsilon}} | \mathcal{F}_{s_\epsilon}\big] \exp\Big(-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon}) \Big)\Big]\, dc.\end{aligned}$$

Using the standard estimate $\mathbb{E}\big[\mathbf{1}_{B_{N,\epsilon}} | \mathcal{F}_{s_\epsilon}\big] \leq \sqrt{2/\pi}\frac{N-x_{s_\epsilon}}{\sqrt{\ln\frac{1}{\epsilon}-s_\epsilon}}$ (see Lemma 5.1) we deduce

$$\limsup_{\epsilon\to 0} \sum_{n=0}^{N} (\ln \tfrac{1}{\epsilon})^{\frac{1}{2}} A_\epsilon(1,n) \leq \sqrt{2/\pi} \sum_{n=0}^{N} \tilde{A}(1,n).$$

which completes the upper bound.

Let us now investigate the lower bound. We denote by $C(\epsilon)$ the annulus $\{x : \epsilon \leq |x| \leq h_\epsilon\}$ and by I_ϵ the set

$$I_\epsilon = \{ \min_{u\in[s_\epsilon, -\ln\epsilon]} (N - x_u) \geq s_\epsilon^\theta \}$$

where $\theta \in]0,1/2[$. We have

$$\sum_{n=0}^{N} A_\epsilon(1,n) \geq \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} \mathbf{1}_{I_\epsilon} \exp\Big(-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon}) - \mu e^{\gamma c} Z_\epsilon(C(\epsilon)) \Big)\Big]\, dc.$$

Using $e^{-u} \geq 1 - u^{\frac{1}{2}}$, we deduce

$$\begin{aligned}\sum_{n=0}^{N} A_\epsilon(1,n) &\geq \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} \mathbf{1}_{I_\epsilon} e^{-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon})} \Big(1 - \mu^{\frac{1}{2}} e^{\frac{1}{2}\gamma c} Z_\epsilon(C(\epsilon))^{\frac{1}{2}} \Big)\Big]\, dc \\ &= \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} e^{-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon})}\Big]\, dc - \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} \mathbf{1}_{I_\epsilon^c} e^{-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon})}\Big]\, dc \\ &\quad - \mu^{\frac{1}{2}} \int_{\mathbb{R}} e^{c\left(\frac{1}{2}\gamma+\sigma\right)}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} \mathbf{1}_{I_\epsilon} e^{-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon})} Z_\epsilon(C(\epsilon))^{\frac{1}{2}}\Big]\, dc \\ &=: B_1(N,\epsilon) - B_2(N,\epsilon) - B_3(N,\epsilon). \end{aligned} \tag{6.5}$$

We now estimate the above three terms.

We start with $B_1(N,\epsilon)$. We have

$$\mathbb{E}\big[\mathbf{1}_{B_{N,\epsilon}} | \mathcal{F}_{h_\epsilon}\big] = \mathbf{1}_{B_{N,h_\epsilon}} (\tfrac{2}{\pi})^{\frac{1}{2}} \int_0^{\frac{N-x_{s_\epsilon}}{\sqrt{\ln\frac{1}{\epsilon}-s_\epsilon}}} e^{-\frac{u^2}{2}}\, du$$

$$
\begin{aligned}
&\geq \mathbf{1}_{B_{N,h_\epsilon}} \mathbf{1}_{\{N-x_{s_\epsilon} \leq (\ln \frac{1}{\epsilon}-s_\epsilon)^{\frac14}\}} (\tfrac{2}{\pi})^{\frac12} \int_0^{\frac{N-x_{s_\epsilon}}{\sqrt{\ln \frac{1}{\epsilon}-s_\epsilon}}} e^{-\frac{u^2}{2}}\, du \\
&\geq \mathbf{1}_{B_{N,h_\epsilon}} \frac{N-x_{s_\epsilon}}{\sqrt{\ln \frac{1}{\epsilon}-s_\epsilon}} \mathbf{1}_{\{N-x_{s_\epsilon} \leq (\ln \frac{1}{\epsilon}-s_\epsilon)^{\frac14}\}} (\tfrac{2}{\pi})^{\frac12} e^{-\frac12 (\ln \frac{1}{\epsilon}-s_\epsilon)^{-\frac12}}.
\end{aligned}
$$

Plugging this relation into $B_1(N,\epsilon)$ we deduce

$$
\begin{aligned}
&B_1(N,\epsilon) \\
&\geq (\tfrac{2}{\pi})^{\frac12} e^{-\frac12 (\ln \frac{1}{\epsilon}-s_\epsilon)^{-\frac12}} (\ln \tfrac{1}{\epsilon}-s_\epsilon)^{-\frac12} \Big(\int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}} (N-x_{s_\epsilon}) \exp\Big(-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon})\Big)\Big]\, dc \\
&- \int_{\mathbb{R}} e^{\sigma c}\, \mathbb{E}\Big[\mathbf{1}_{\{N-x_{s_\epsilon} > (\ln \frac{1}{\epsilon}-s_\epsilon)^{\frac14}\}} \mathbf{1}_{B_{N,h_\epsilon}} (N-x_{s_\epsilon}) e^{-\mu e^{\gamma c} Z_\epsilon(D_{h_\epsilon})}\Big]\, dc \\
&=: \Delta_1(\epsilon)+\Delta_2(\epsilon).
\end{aligned}
$$

It is clear that

$$
\lim_{\epsilon\to 0} (\ln \tfrac{1}{\epsilon})^{\frac12} \Delta_1(\epsilon) = \lim_{\epsilon\to 0} \sum_{n=0}^N \tilde{A}_\epsilon(1,n) = \sum_{n=0}^N \tilde{A}(1,n).
$$

It remains to treat $\Delta_2(\epsilon)$. By making the change of variables $y = e^{\gamma c} Z_\epsilon(D_{h_\epsilon})$, we get

$$
(\ln \tfrac{1}{\epsilon})^{\frac12} \Delta_2(\epsilon) \leq C \mathbb{E}^{\Theta^N}\Big[\mathbf{1}_{\{N-x_{s_\epsilon} > (\ln \frac{1}{\epsilon}-s_\epsilon)^{\frac14}\}} Z_\epsilon(D_{h_\epsilon})^{-\frac{\sigma}{\gamma}}\Big].
$$

Now we will use the fact that under Θ^N the event in the above expectation is very unlikely. Using the elementary inequality $ab \leq a^2/2 + b^2/2$ we get

$$
\begin{aligned}
&\mathbb{E}^{\Theta^N}\Big[\mathbf{1}_{\{N-x_{s_\epsilon} > (\ln \frac{1}{\epsilon}-s_\epsilon)^{\frac14}\}} Z_\epsilon(D_{h_\epsilon})^{-\frac{\sigma}{\gamma}}\Big] \\
&\quad \leq (\ln \tfrac{1}{\epsilon})^{\kappa} \mathbb{E}^{\Theta^N}\Big[\mathbf{1}_{\{N-x_{s_\epsilon} > (\ln \frac{1}{\epsilon}-s_\epsilon)^{\frac14}\}}\Big] + (\ln \tfrac{1}{\epsilon})^{-\kappa} \mathbb{E}^{\Theta^N}\Big[Z_\epsilon(D_1)^{-2\frac{\sigma}{\gamma}}\Big].
\end{aligned}
$$

Using the fact that a Gaussian Multiplicative Chaos has negative moments of all orders on all open balls, the expectation in the second term in the above expression is easily seen to be bounded uniformly in ϵ. Hence, the second term tends to 0 as $\epsilon \to 0$. Concerning the first term, recall Lemma 4.5 and the estimate, for a $3d$-Bessel process β_t and $u > x$

$$
\mathbb{P}_x(\beta_t > u) = \mathbb{P}_{x/t^{\frac12}}(\beta_1 > u/t^{\frac12}) \leq C \frac{t^{\frac12}}{u-x} \wedge 1.
$$

Therefore

$$
\begin{aligned}
\Theta^N\Big(N - x_{s_\epsilon} > (\ln \tfrac{1}{\epsilon}-s_\epsilon)^{\frac14}\Big) &\leq C\mathbb{E}\Big[\frac{s_\epsilon^{1/2}}{|\ln \frac{1}{\epsilon}-s_\epsilon - x_0|} \wedge 1\Big] \\
&\leq 2C\mathbb{E}\Big[\frac{s_\epsilon^{1/2}}{|\ln \frac{1}{\epsilon}-s_\epsilon - x_0|}\Big] + \mathbb{P}(x_0 > \tfrac12 (\ln \tfrac{1}{\epsilon}-s_\epsilon)^{\frac14}) \\
&\leq 2C \ln(\tfrac{1}{\epsilon})^{-\frac16} + C(\ln \tfrac{1}{\epsilon})^{\frac14} e^{-\frac12 (\ln \frac{1}{\epsilon})^{\frac12}}.
\end{aligned}
$$

Hence, choosing $\kappa < 1/6$ leads to $\lim_{\epsilon\to 0} (\ln \frac{1}{\epsilon})^{\frac12} \Delta_2(\epsilon) = 0$. Thus

$$
\liminf_{\epsilon\to 0} (\ln \tfrac{1}{\epsilon})^{\frac12} B_1(N,\epsilon) \geq \sum_{n=0}^N \tilde{A}(1,n). \tag{6.6}
$$

Now we treat $B_3(N,\epsilon)$. To this purpose, we use first the change of variables $y = e^{\gamma c} Z_\epsilon(D_{h_\epsilon})$ to get

$$\begin{aligned}
B_3(N,\epsilon) \leq & C\mathbb{E}[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon} Z_\epsilon(D_{h_\epsilon})^{-\frac{\frac{1}{2}\gamma+\sigma}{\gamma}} Z_\epsilon(C(\epsilon))^{\frac{1}{2}}] \\
= & C\mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon}\mathbb{E}[Z_\epsilon(D_{h_\epsilon})^{-\frac{\frac{1}{2}\gamma+\sigma}{\gamma}} Z_\epsilon(C(\epsilon))^{\frac{1}{2}}|(x_s)_{s<\infty}]\Big] \\
\leq & C\mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon}\mathbb{E}[Z_\epsilon(D_{h_\epsilon})^{-\frac{\gamma+2\sigma}{\gamma}}|(x_s)_{s<\infty}]^{\frac{1}{2}}\mathbb{E}[Z_\epsilon(C(\epsilon))|(x_s)_{s<\infty}]^{\frac{1}{2}}\Big] \\
= & C\mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon}\mathbb{E}[Z_\epsilon(D_{h_\epsilon})^{-\frac{\gamma+2\sigma}{\gamma}}|(x_s)_{s<\infty}]^{\frac{1}{2}}\Big(\int_{s_\epsilon}^{\ln\frac{1}{\epsilon}} e^{\gamma x_u}\,du\Big)^{\frac{1}{2}}\Big] \\
\leq & C\mathbb{E}[Z_\epsilon(D_{h_\epsilon})^{-\frac{\gamma+2\sigma}{\gamma}}]^{\frac{1}{2}}\mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon}\int_{s_\epsilon}^{\ln\frac{1}{\epsilon}} e^{\gamma x_u}\,du\Big]^{\frac{1}{2}} \\
= & C\mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon}\int_{s_\epsilon}^{\ln\frac{1}{\epsilon}} e^{\gamma x_u}\,du\Big]^{\frac{1}{2}}
\end{aligned}$$

On the set I_ϵ, we have the estimate

$$\int_{s_\epsilon}^{\ln\frac{1}{\epsilon}} e^{\gamma x_u}\,du \leq C\ln\tfrac{1}{\epsilon}e^{-\gamma s_\epsilon^\theta} = C\ln\tfrac{1}{\epsilon}e^{-\gamma(\ln\frac{1}{\epsilon})^{\theta/6}}$$

which implies

$$\lim_{\epsilon\to 0}(\ln\tfrac{1}{\epsilon})^{\frac{1}{2}}B_3(N,\epsilon)=0. \tag{6.7}$$

Finally we focus on $B_2(N,\epsilon)$. We first make the change of variables $y=e^{\gamma c}Z_\epsilon(D_{h_\epsilon})$ to get

$$B_2(N,\epsilon) \leq C\,\mathbb{E}\Big[\mathbf{1}_{B_{N,\epsilon}}\mathbf{1}_{I_\epsilon^c}Z_\epsilon(D_1)^{-\frac{\sigma}{\gamma}}\Big] \tag{6.8}$$

We claim

Lemma 6.2. *Let B be a standard Brownian motion and $\beta > x > 0$ and $\theta \in]0,1/2[$. Then, for some constant $C>0$ (independent of everything)*

$$\begin{aligned}
P_\epsilon(x) := & \mathbb{P}_x\Big(\min_{u\in[s_\epsilon,-\ln\epsilon]}\beta-B_u < s_\epsilon^\theta, \min_{u\in[0,-\ln\epsilon]}\beta-B_u\geq 0\Big) \\
\leq & (\beta-x)(\ln\tfrac{1}{\epsilon})^{-1/2}s_\epsilon^{\theta-1/2}.
\end{aligned}$$

Conditioning (6.8) on the sigma algebra generated by $\{X_{\hat{g},u}(0); u>1\}$, we can use Lemma 6.2 to get

$$(\ln\tfrac{1}{\epsilon})^{\frac{1}{2}}B_2(N,\epsilon) \leq Cs_\epsilon^{\theta-1/2}\,\mathbb{E}\Big[(N-X_{\hat{g},1}(0))_+ Z_\epsilon(D_1)^{-\frac{\sigma}{\gamma}}\Big] \tag{6.9}$$

The last expectation is clearly finite and bounded independently of ϵ so that

$$\lim_{\epsilon\to 0}(\ln\tfrac{1}{\epsilon})^{\frac{1}{2}}B_2(N,\epsilon)=0. \tag{6.10}$$

and, gathering (6.5)+(6.6)+(6.7)+(6.10), the proof of (3.18) and hence Lemma 3.2 is complete. □

Proof of Lemma 6.2. We condition first on the filtration $\mathcal{F}_{s_\epsilon}$ generated by the Brownian motion up to time s_ϵ. From Lemma 5.1, we obtain

$$\begin{aligned}
& \mathbb{P}\Big[\min_{u\in[s_\epsilon,-\ln\epsilon]}\beta-B_u<(s_\epsilon)^\theta, \min_{u\in[s_\epsilon,-\ln\epsilon]}\beta-B_u\geq 0|\mathcal{F}_{s_\epsilon}\Big] \\
& \leq\sqrt{\frac{2}{\pi}}\int_{\frac{(\beta-B_{s_\epsilon}-(s_\epsilon)^\theta)_+}{(\ln\frac{1}{\epsilon}-s_\epsilon)^{1/2}}}^{\frac{(\beta-B_{s_\epsilon})_+}{(\ln\frac{1}{\epsilon}-s_\epsilon)^{1/2}}} e^{-\frac{u^2}{2}}\,du
\end{aligned}$$

$$\leq \mathbf{1}_{\{\beta - B_{s_\epsilon} \in [0,(s_\epsilon)^\theta]\}} \sqrt{\frac{2}{\pi}} \frac{(\beta - B_{s_\epsilon})_+}{(\ln \frac{1}{\epsilon} - s_\epsilon)^{1/2}} + \mathbf{1}_{\{\beta - B_{s_\epsilon} > (s_\epsilon)^\theta\}} \frac{(s_\epsilon)^\theta}{(\ln \frac{1}{\epsilon} - s_\epsilon)^{1/2}}.$$

Integrating, we get that

$$\begin{aligned} P_\epsilon(x) \leq & \sqrt{\frac{2}{\pi}} (\ln \frac{1}{\epsilon} - s_\epsilon)^{-1/2} \mathbb{E}\Big[\mathbf{1}_{\{\min_{u \in [0,s_\epsilon]} \beta - B_u \geq 0\}} (\beta - B_{s_\epsilon}) \mathbf{1}_{\{\beta - B_{s_\epsilon} \in [0,(s_\epsilon)^\theta]\}} \Big] \\ & + \frac{(s_\epsilon)^\theta}{(\ln \frac{1}{\epsilon} - s_\epsilon)^{1/2}} \mathbb{E}\Big[\mathbf{1}_{\{\min_{u \in [0,s_\epsilon]} \beta - B_u \geq 0\}} \Big]. \end{aligned}$$

The second expectation is estimated with Lemma 5.1. Concerning the first one, we use the fact under the probability measure $\frac{1}{\beta - x} \mathbf{1}_{\{\min_{u \in [0,s_\epsilon]} \beta - B_u \geq 0\}} (\beta - B_{s_\epsilon})$, the process $(\beta - B_u)_{u \leq s_\epsilon}$ is a $3d$-Bessel process, call it Bess_t. Hence, using the Markov inequality, the scale invariance of a Bessel process and the fact that the mapping $x \mapsto \mathbb{E}^x \Big[\frac{1}{\mathrm{Bess}_1} \Big]$ is decreasing, we deduce

$$\begin{aligned} P_\epsilon(x) \leq & \sqrt{\frac{2}{\pi}} (\ln \frac{1}{\epsilon} - s_\epsilon)^{-1/2} (\beta - x) \mathbb{E}^{\beta - x} \Big[\mathbf{1}_{\{\mathrm{Bess}_{s_\epsilon} \leq (s_\epsilon)^\theta\}} \Big] + \sqrt{\frac{2}{\pi}} \frac{(s_\epsilon)^{\theta - 1/2}}{(\ln \frac{1}{\epsilon} - s_\epsilon)^{1/2}} (\beta - x) \\ \leq & \sqrt{\frac{2}{\pi}} (\ln \frac{1}{\epsilon} - s_\epsilon)^{-1/2} (\beta - x) (s_\epsilon)^{\theta - 1/2} \mathbb{E}^0 \Big[\frac{1}{\mathrm{Bess}_1} \Big] + \sqrt{\frac{2}{\pi}} \frac{(s_\epsilon)^{\theta - 1/2}}{(\ln \frac{1}{\epsilon} - s_\epsilon)^{1/2}} (\beta - x). \qquad \square \end{aligned}$$

A Riemann surfaces with conical singularities and cusps

A metric g on a Riemann surface M has a conical singularity of order α (α real number > -1) at a point $x \in M$ if in some neighbourhood of x

$$g = e^u |dz|^2$$

where z is a local complex coordinate defined in the neighbourhood of x with $u - 2\alpha \ln |z - z(x)|$ continuous in the neighbourhood of x.

Recall that an Euclidean cone of angle θ is

$$C_\theta = \big\{ (r,t); r \geq 0, t \in \mathbb{R}/\theta\mathbb{Z} \big\}_{(0,t) \sim (0,t')}$$

equipped with the metric $ds^2 = dr^2 + r^2\, dt^2$ and that $\mathbb{C}$ equipped with the metric $|z|^{2\beta} |dz|^2$ is isometric to C_θ where $\theta = 2\pi(\beta + 1)$. Therefore, if a surface has at some point a conical singularity of order β, then this surface admits at this point a "tangent cone" of angle $\theta = 2\pi(\beta + 1)$.

The boundary case of conical singularities is the case $\alpha = -1$ and this is the threshold at which the singularity ceases to be integrable, in which case the singularity becomes a cusp and has a somewhat different structure. More precisely, a metric g on a Riemann surface M has a cusp singularity at a point $x \in M$ if in some neighbourhood of x

$$g = e^u |dz|^2$$

where z is a local complex coordinate defined in the neighbourhood of x and $u(z) + 2 \ln |z - z(x)| = o(\ln |z - z(x)|)$ (with the Landau notation) in the neighbourhood of x.

The prototype of cusp model is

$$C = \big\{ (r,t); r > 0, t \in \mathbb{R}/\mathbb{Z} \big\}$$

equipped with the hyperbolic metric $ds^2 = r^{-2}(dr^2 + dt^2)$ and the punctured disk equipped with the metric $\frac{|dz|^2}{|z|^2 (\ln |z|)^2}$ is isometric to C. $\square$

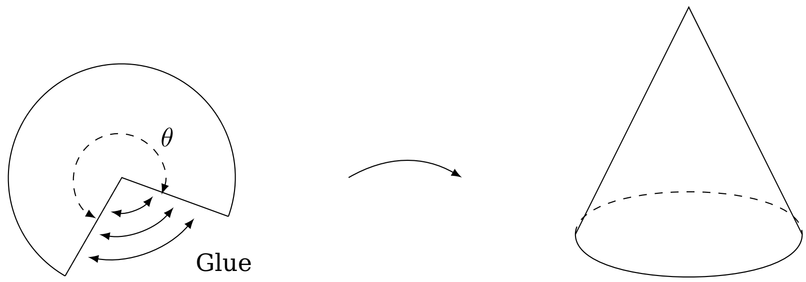

Figure 1: Cone with angle θ. Glue isometrically the two boundary segments of the left-hand side figure to get the cone of the right-hand side figure. Such a cone is isometric to the complex plane equipped with the metric $ds^2 = |z|^{2-\frac{\theta}{\pi}} dzd\bar{z}$.

A.1 Conjecture on the Ising model on random triangulations

By a *triangulation* of the unit sphere we mean a finite connected graph T s.t. there is an embedding of T to S^2 s.t. each connected component of $S^2 \setminus T$ (a *face*) has a boundary consisting of 3 edges (we denote the embedding of T by T again). We identify two triangulations if there is an orientation preserving homeomorphism of S^2 mapping the one to the other. A *marked* triangulation is a triangulation together with a choice of 3 vertices v_1, v_2, v_3. We denote by $\mathcal{T}$ the set of marked triangulations and by $|T|$ the number of faces in T.

We will consider a two-parameter family of probability measures $\mathbb{P}_{\mu_0,\gamma}$ on $\mathcal{T}$ defined by

$$\mathbb{P}_{\mu_0,\beta}(T) = \frac{1}{Z_{\mu_0,\beta}} e^{-\mu_0|T|} Z(T,\beta) \tag{A.1}$$

where $Z_{\mu_0,\beta}$ is a normalization constant and $Z(T,\beta)$ is the partition function of the Ising model on T at inverse temperature β

$$Z(T,\beta) = \sum_{\sigma\in\{-1,1\}^{\#V(T)}} e^{\frac{\beta}{2}\sum_{i\sim j}\sigma_i\sigma_j}.,$$

where $V(T)$ stands for the set of vertices of T and $i \sim j$ means that the vertices i, j are neighbors. These Boltzmann weights depend on some parameter denoted β, which we now tune to its critical point $\beta = \beta_c = \ln 2$. It is known that

$$Z_N(\beta_c) := \sum_{T\in\mathcal{T}:|T|=N} Z(T,\beta_c) = N^{-1/6} e^{\bar{\mu}N}(1 + o(1)) \tag{A.2}$$

so that $\mathbb{P}_{\mu_0,\beta_c}$ is defined for $\mu_0 > \bar{\mu}$ and $\lim_{\mu_0\downarrow\bar{\mu}} Z_{\mu_0,\beta_c} = \infty$. Hence as $\mu_0 \to \mu$ the measure samples large triangulations.

For each T we may associate a conformal structure on S^2 as follows. Assign to each face f a copy Δ_f of an equilateral triangle Δ of unit area and let $M_T = \sqcup\Delta_f/\sim$ be the disjoint union of the Δ_f where we identify the common edges. M_T is a topological manifold homeomorphic to S^2. We can even equip it with a complex structure with the help of the following atlas. It contains the interiors of Δ_f, mapped by identity to Δ, the interiors of $\Delta_f \cup \Delta_{f'}$ where f and f' share an edge, mapped by identity to two copies of Δ next to each other in $\mathbb{C}$ and neighbourhoods of each vertex $v \in M$ mapped to $\mathbb{C}$ as follows. List faces sharing v in consecutive order $f_0, \dots, f_{n-1}$ and parametrize $\Delta_{f_j} \cap U$ by $z_j = re^{2\pi i\theta}$ with $\theta_j \in [6j/n, 6(j+1)/n]$. Then $z \to z^{n/6}$ provides a complex coordinate for a neigborhood of v. This atlas makes M_T a complex manifold homemorphic to S^2. Picking three points $z_1, z_2, z_3 \in \mathbb{C}$, there is a unique conformal map $\psi_T : M_T \to \hat{\mathbb{C}}$ s.t. $\psi(v_i) = z_i$.

Let λ_T be the area measure on M_T i.e. λ_T is Lebesque measure in the local coordinates on Δ_f. Let ν_T be its image under ψ_T. In the standard coordinate of $\hat{\mathbb{C}}$ it is given by $\nu_T = g_T(z)dz$ where the density g_T is singular at the images of the vertices with $n \neq 6$.

Consider now a *scaling limit* as follows. Recalling that as $\mu_0 \downarrow \bar{\mu}$ the typical size of triangulations, we quantify the gap between $\mu_0, \bar{\mu}$ by setting (for $a > 0$ and fixed $\mu > 0$)

$$\mu_0 = \bar{\mu} + a^2 \mu. \tag{A.3}$$

Now we define observables of the spin field. Let $D_\epsilon(x)$ be the disk with center x and radius ϵ in $\mathbb{C}$. For a triangulation $T \in \mathcal{T}$ (uniformized by ψ_T onto the sphere) together with a spin configuration σ on T we define the total magnetic field inside the disc $D_\epsilon(x)$ by

$$\Phi^{(\epsilon)}_{T,\sigma}(x) = \epsilon^{-2} a^{5/3} \sum_{v \in T} \mathbf{1}_{D_\epsilon(x)}(\psi(v))\sigma(v).$$

Let $(x_i)_{4 \leq i \leq n}$ be some arbitrary points on $\mathbb{C}$.

Conjecture A.1. *Under the relation* (A.3)*, the following convergence holds (for some irrelevant constant C, which may depend on n)*

$$\lim_{\epsilon \to 0} \lim_{a \to 0} a^{4/3} \mathbb{E}_{\mu_0, \beta_c} \Big[\prod_{i=4}^{n} \Phi^{(\epsilon)}_{T,\sigma}(x_i) \Big] \nu_{\bar{\mu}} = C \langle \theta(x_4) \dots \theta(x_n) \rangle_{\hat{g}} \times \Pi_{\alpha, \mathbf{z}}$$

where $\Pi_{\alpha,\mathbf{z}}$ is the correlation function of the Liouville QFT studied in this paper with cosmological constant μ, parameters $\gamma = \sqrt{3}$, $Q = \frac{7}{2\sqrt{3}}$ and n vertex operators at the locations $(x_i)_{i=1,\dots,n}$ with respective weights $\alpha_i = \gamma$ for $i = 1, 2, 3$ and $\alpha_i = \frac{5}{6}\gamma$ for $i > 3$. Here $\langle \theta(x_4) \dots \theta(x_n) \rangle_{\hat{g}}$ stands for the correlation functions of the spin field in the critical Ising model (standard, i.e. not coupled to gravity) on the sphere.

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