

Exit time for a reaction diffusion model: Case of a one well potential

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Abstract. We consider a interacting particle system, the Glauber + Kawasaki model. This model is the result of the combination of a fast stirring, the Kawasaki part, and a spin flip process, the Glauber part. This process has a Reaction–Diffusion equation as hydrodynamic limit, as is proven by De Masi and Presutti (*Mathematical Methods for Hydrodynamic Limits* (1991) Springer). The ergodicity of these dynamics (one well potential) was proven in Brasseco et al. (*Amer. Math. Soc. Transl. Ser. 2* **198** (2000) 37–49), for any dimension. In this article, we prove the asymptotic exponentiality for certain exit time from a subset of the basin of attraction of the well.

1 Introduction

The class of interacting particle systems that we study, was proposed by De Masi, Ferrari and Lebowitz, as alternative models for reaction–diffusion systems. On the configuration space $\{-1, +1\}^{\mathbb{Z}^d}$, they are obtained from the superposition of a Glauber (spin flip) type dynamics, corresponding to the reactive part, and a stirring one, also called Kawasaki dynamics at infinite temperature, that corresponds to the diffusive part, which is speeded by a factor, say ε^{-2} . The kinetic limit, which here corresponds to the hydrodynamical one, involves the simultaneous change of space scaling in the diffusive limit for the stirring, to provide the macro scale (macro = ε micro). Under such limit, the macroscopic description is verified, being given by a reaction diffusion equation of the form

$$\partial_t m = \Delta m + F(m), \tag{1.1}$$

where $m(r, t) \in \mathbb{R}$ represents the magnetization or density. The force term $F(\cdot)$ being determined by the Glauber rates. For example, given any polynomial $F(\cdot)$ we may choose finite range spin flip rates which lead to the above equation (of course there are innumerable choices, though the behaviour should be similar). That is, the empirical magnetization, or the density of particles in small boxes, converges in probability to the solution of the reaction diffusion equation. Indeed one knows more: strong forms of propagation of chaos (i.e., asymptotic independence of different spins) have been proven, initially by De Masi, Ferrari and Lebowitz (1986),

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and later in sharper forms by several authors, see [De Masi and Presutti \(1991\)](#). At the level of propagation of chaos, more general systems, with $m(r, t)$ taking values in \mathbb{R}^n are treated just in the same way, though we restrict ourselves to the case $n = 1$. So there is no loss in assuming $F(m) = -V'(m)$ in the above equation.

For a class of rates leading to a single well potential, the ergodicity for any $d \geq 1$, has been proven by [Brassescio et al. \(2000a\)](#). The same authors also proved ergodicity in the case of a double well potential, provided $V(\cdot)$ has a unique minimizer, cf. [Brassescio et al. \(2000b\)](#). We also refer to the article of [Durrett and Neuhauser \(1994\)](#), where a large classes of spin flip rates were considered, with the characteristic of having a trapping configuration (e.g. the contact process). In such case phase transition might occur, for the same reason which leads to the existence of an invariant measure with magnetization different from the trapping configuration, that is, is related to the minimizer of the potential.

The problem which we want to address refers to the large deviation behaviour of the process, fixing Glauber rates which lead to a one well potential. Our goal is to investigate the asymptotic loss of memory for the exit time from attracting sets contained in the “basin of attraction” of the ergodic measure. Connected to the reaction–diffusion model, related questions have been considered when the system is obtained by the addition of a small random noise to the equation (1.1), as treated by [Martinelli, Olivieri and Scoppola \(1989\)](#), [Brassescio, Olivieri and Vares \(1998\)](#), using coupling arguments, together with estimates of large deviations, which in this infinite dimensional setup were obtained by [Faris and Jona-Lasinio \(1982\)](#), extending Freidlin and Wentzell results. There are many analogous results in the context of low temperature Glauber dynamics on finite volumes, indeed extended to a large class of Markov chains, but they involve rather different methods.

We consider a macroscopically finite volume, that is, a torus of side ε^{-1} , though the result extends to larger volumes, which are tending to infinity in the macro scale, as for instance, ε^{-l} , $l > 1$. Also, for simplicity we prove for dimension $d = 1$, and study the problem when we start the process from the Bernoulli measure, which has the mean in one of the basins of the attraction. There is no problem to generalizing to dimensions $d \geq 1$, and to consider the initial configuration starting from an smooth profile, which is contained also in a basin of attraction. We prove that the properly rescaled exit time follows an exponential law. This result uses more simple ideas than in [Hinojosa \(2004\)](#), because, in the case of a one well potential, it is possible to couple two evolutions of the process starting in the same basin of attraction in a time of order $a|\log \varepsilon|$ with a probability of order ε^n , see Proposition 4.2 below, this fact was proven in [Brassescio et al. \(2000a\)](#). In the case of a double well potential, there exists a similar result, see [Brassescio et al. \(2000b\)](#), but the probability of this event is bounded by $c_\varepsilon \varepsilon^n$, where c_ε is related to the passage from one well to the other, the exponential exit time result is still valid, see [Hinojosa \(2004\)](#), but the proof is more difficult. Finally, we use a result from [Azlarov and Volodin \(1986\)](#), which bound the uniform distance between e^{-t} and $\mathbb{P}(\xi > t)$ (ξ is a random variable) from a bound over $\mathbb{P}(\xi > t + s | \xi > s) - \mathbb{P}(\xi > t)$; this is the loss of memory property.

2 Definitions and results

We consider, for each $\varepsilon > 0$, a family of Markov process $\{\sigma_t\}_{t \geq 0}$, $\sigma_t \in \Omega^\varepsilon = \{-1, 1\}^{\mathbb{Z}_\varepsilon}$, where we let \mathbb{Z}_ε be the set of integers mod $[\varepsilon^{-1}]$, $[x]$ is the integer part of $x \in \mathbb{R}$: $\mathbb{Z}_\varepsilon = \mathbb{Z}_{[\varepsilon^{-1}]} = \{0, 1, \dots, [\varepsilon^{-1}]\}$, with both ends identified. The evolution of the process is governed by the generator:

$$L_\mu^{(\varepsilon)} = \varepsilon^{-2} L_0 + L_\mu, \quad (2.1)$$

where L_0 is the generator of the stirring process:

$$L_0 f(\sigma) = \sum_{|x-y|=1} [f(\sigma^{x,y}) - f(\sigma)], \quad (2.2)$$

f is a function in Ω^ε , $x, y \in \mathbb{Z}_\varepsilon$, and

$$\sigma^{x,y}(z) = \begin{cases} \sigma(z), & z \neq x, y; \\ \sigma(y), & z = x; \\ \sigma(x), & z = y, \end{cases}$$

and L_μ is the generator of the spin flip process:

$$L_\mu f(\sigma) = \sum_{x \in \mathbb{Z}_\varepsilon} c_\mu(x, \sigma) [f(\sigma^x) - f(\sigma)], \quad (2.3)$$

again f is a function in Ω^ε , and

$$\sigma^x(z) = \begin{cases} -\sigma(z), & z = x; \\ \sigma(z), & z \neq x. \end{cases}$$

$c_\mu(x, \sigma)$, the spin flip intensity, is:

$$c_\mu(x, \sigma) = c_0(x, \sigma) - \frac{\mu}{2} \sigma(x), \quad (2.4)$$

and $c_0(x, \sigma)$ is defined as:

$$c_0(x, \sigma) = 1 - \gamma \sigma(x) [\sigma(x-1) + \sigma(x+1)] + \gamma^2 \sigma(x-1) \sigma(x+1), \quad (2.5)$$

μ is restricted to $0 \leq \mu < 2(1-\gamma)^2$ for $c_\mu(x, \sigma)$ to be positive, we will consider in this article the case of $\mu = 0$, and $\gamma \in (0, \frac{1}{2})$, so $c_\mu(x, \sigma) = c_0(x, \sigma)$. The process defined above is called the Glauber + Kawasaki process.

If ν is a probability on Ω^ε (resp. a single configuration σ), we will denote by $E_\nu^{(\varepsilon)}$ (resp. by $E_\sigma^{(\varepsilon)}$) the expectation of the above process starting with law ν (resp. from the configuration σ); also we write $P_\nu^{(\varepsilon)}$ (resp. $P_\sigma^{(\varepsilon)}$) when we refer to the law of the process. In [De Masi, Ferrari and Lebowitz \(1986\)](#), it was proved that, as $\varepsilon \rightarrow 0$, the process, in infinite volume, converges to the solution of the reaction-diffusion equation:

$$\frac{\partial m}{\partial t} = \Delta m - V'_\mu(m), \quad (2.6)$$

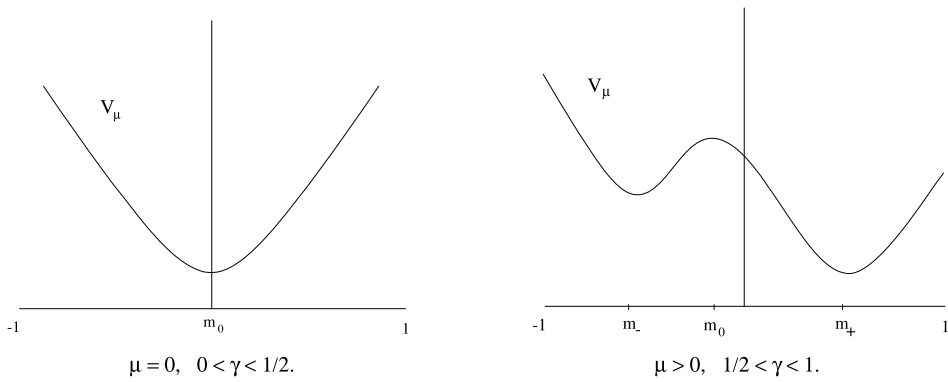


Figure 1 One well potential on the left. Two well potential on the right.

where:

$$-V'_\mu = \mathbb{E}_{\nu_m}(-2\sigma(0)c_\mu(0, \sigma)) = -\alpha m - \beta m^3 + \mu, \tag{2.7}$$

ν_m denoting the Bernoulli product measure on Ω^ε with average m and

$$\alpha = 2(1 - 2\gamma), \quad \beta = 2\gamma^2. \tag{2.8}$$

In the case $\mu = 0$, the derivative of the potential is:

$$-V' = \mathbb{E}_{\nu_m}(-2\sigma(0)c(0, \sigma)) = -\alpha m - \beta m^3.$$

The polynomial V'_μ is the derivative of the, in general double well, potential V_μ (see Figure 1), and in the case we will consider ($\mu = 0$, and $\gamma \in (0, \frac{1}{2})$) that potential V has only one well, with a minimum at 0.

The case of infinite volume (i.e., $\{-1, 1\}^{\mathbb{Z}^d}$) was considered in [Brassesco et al. \(2000a\)](#) (one well) and in [Brassesco et al. \(2000b\)](#) (two wells), with the same dynamics. It was proved there that for ε small enough the process is ergodic (in our case the process is immediately ergodic, since we are in finite volume). Observe that (2.6) is not ergodic (in the sense that it has two invariant solutions), also that $m = 0$ is the minimum in the case that we consider.

We shall consider the set of configurations with empirical magnetization in small intervals (of size ε^{-a} , where $0 < a < 1$ is close to one), belongs to $[-\delta, \delta]$, with δ small enough, that is,

$$A_\delta^{(\varepsilon)} = \left\{ \sigma \in \Omega^\varepsilon : \left| \varepsilon^a \sum_{x:|x| \leq \frac{\varepsilon^{-a}}{2}} \sigma(x+y) \right| < \delta, \forall y \in \mathbb{Z}_\varepsilon \right\}. \tag{2.9}$$

When it is clear from the context we drop the superscript ε and write simply A_δ . Now consider the Bernoulli product measure, ν_m , on Ω^ε with average $m \in [-\delta, \delta]$, and define the following stopping time

$$\tau^\varepsilon = \inf\{t > 0 : \sigma_t \notin A_\delta^{(\varepsilon)}\}.$$

Also, define β_ε as: $P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon > \beta_\varepsilon) = e^{-1}$. Regarding the asymptotic distribution of $\tau^\varepsilon/\beta_\varepsilon$, the main result is the following theorem.

Theorem 2.1. For τ^ε defined above and $m \in [-\delta, \delta]$ it holds:

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \geq 0} |P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon) - e^{-t}| = 0. \quad (2.10)$$

3 Proof of Theorem 2.1

We shall prove that if we start the process from the Bernoulli product measure, ν_{m-} , then, under $P^{(\varepsilon)}$, $\tau^\varepsilon/\beta_\varepsilon$ converges, in the Levy metric, to a mean one exponential r.v. First, we prove that $\tau^\varepsilon/\beta_\varepsilon$ is uniformly close, in the Levy metric, to a family of exponentials r.v.; and then that this family converges also in the Levy metric to a mean one exponential r.v. As convergence in Levy metric, with an absolutely continuous limit, implies convergence in Kolmogorov metric, then the theorem is proved.

Let $k \geq 1$ be fixed. In Hinojosa (2004), it is proved the following lower bound for β_ε : for any $n \geq 1$, there exists $c_n > 0$, such that for $\varepsilon > 0$ small enough it holds:

$$P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon < \varepsilon^{-k}) \leq c_n \varepsilon^n. \quad (3.1)$$

Note that, by the Cramer–Chernov theorem for large deviations for i.i.d. (Bernoulli) random variables, for any $\delta' > 0$, at time zero

$$\begin{aligned} P_{\nu_m}^{(\varepsilon)}(A_{\delta'}^c) &\leq \varepsilon^{-1} P_{\nu_m}^{(\varepsilon)}\left(\left|\varepsilon^a \sum_{|x| \leq \frac{\varepsilon-a}{2}} \sigma(x)\right| \geq \delta'\right) \\ &\leq \varepsilon^{-1} e^{-\tilde{c}\varepsilon^{-a}} \leq e^{-c\varepsilon^{-a}}. \end{aligned} \quad (3.2)$$

For some $c > 0$. Then we shall consider that with large ν_m -probability the initial configuration starts from $A_{\delta'}$, for any $\delta' < \delta$. Also, since (3.1), if ε is small enough we have that:

$$P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon > \varepsilon^{-k}) > e^{-1}.$$

But by definition $P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon > \beta_\varepsilon) = e^{-1}$, therefore $\varepsilon^{-k} < \beta_\varepsilon$, for any $k \geq 1$, and ε small enough.

The main part of the proof is contained in Proposition 3.1. It says that the family of r.v. $\tau^\varepsilon/\beta_\varepsilon$ has, uniformly in ε , loss of memory property (note that this feature characterizes the exponential distribution). From this proposition, we shall prove that, in fact, for each ε we have an exponential random variable that is close, in the Levy metric sense, to $\tau^\varepsilon/\beta_\varepsilon$.

Before stating the loss of memory property, we let $\gamma_\varepsilon = \tilde{a}|\log \varepsilon|$, where \tilde{a} is a positive constant that will be fixed later. Observe that

$$\frac{\gamma_\varepsilon}{\beta_\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

and, since (3.1)

$$P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon < \gamma_\varepsilon) \leq c_n \varepsilon^n. \tag{3.3}$$

Also let $F_\varepsilon(t) = P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon)$ and

$$h_\varepsilon(t, s) = \frac{F_\varepsilon(t + s)}{F_\varepsilon(s)} - F_\varepsilon(t).$$

Proposition 3.1. *For any $n \geq 0$, there exists c_n such that for ε small enough it holds*

$$\sup_{s \geq 0, t \geq \frac{\gamma_\varepsilon}{\beta_\varepsilon}} |h_\varepsilon(t, s)| \leq c_n \varepsilon^n.$$

We shall prove this proposition in the next section.

This result will allow us to prove the following proposition.

Proposition 3.2. *For any $n \geq 1$ there exists c_n such that for any ε small enough the following holds*

$$d_L\left(\frac{\tau^\varepsilon}{\beta_\varepsilon}, \mathcal{E}(\lambda_\varepsilon)\right) \leq c_n \varepsilon^n,$$

where $d_L(\cdot, \cdot)$ refers to the Levy metric between two random variables, that is,

$$d_L(X, Y) = \inf\{\varrho > 0 : P(X > t + \varrho) - \varrho \leq P(Y > t) \leq P(X > t - \varrho) + \varrho, \forall t \in \mathbb{R}\},$$

and $\mathcal{E}(\lambda)$ is a exponential random variable with mean λ^{-1} , recall that $\lambda_\varepsilon^{-1} = \mathbb{E}_{\nu_m}^{(\varepsilon)} \frac{\tau^\varepsilon}{\beta_\varepsilon}$.

We defer the proof of this proposition to the end of this section.

Finally, the Theorem 2.1 will be proved after showing that $\lambda_\varepsilon \rightarrow 1$, since this implies that $d_L(\mathcal{E}(1), \mathcal{E}(\lambda_\varepsilon)) \rightarrow 0$, and then

$$d_L\left(\frac{\tau^\varepsilon}{\beta_\varepsilon}, \mathcal{E}(1)\right) \rightarrow 0.$$

The proof that $\lambda_\varepsilon \rightarrow 1$ follows easily. We give some details of the proof of this result, we follow the exponential characterization from [Azlarov and Volodin \(1986\)](#). We have that $\frac{\tau^\varepsilon}{\beta_\varepsilon}$ has finite moments of all orders, in particular $\lambda_\varepsilon^{-1} = \mathbb{E}_{\nu_m}^{(\varepsilon)} \frac{\tau^\varepsilon}{\beta_\varepsilon} < \infty$. By definition $P_{\nu_m}^{(\varepsilon)}(\tau^\varepsilon > \beta_\varepsilon) = e^{-1}$, and this implies that $e^{-1} \leq \lambda_\varepsilon^{-1}$, so that $0 < \lambda_\varepsilon \leq e$, for all $\varepsilon > 0$.

Take any $\varrho \geq 0$ in the the defining property of $d_L(\frac{\tau^\varepsilon}{\beta_\varepsilon}, \mathcal{E}(\lambda_\varepsilon))$ and $t = 1$, so that:

$$e^{-\lambda_\varepsilon(1+\varrho)} - \varrho \leq e^{-1} \leq e^{-\lambda_\varepsilon(1-\varrho)} + \varrho,$$

then

$$\begin{aligned} e^{-1} - e^{-\lambda_\varepsilon} &\leq e^{-\lambda_\varepsilon} (e^{\lambda_\varepsilon \varrho} - 1) + \varrho \\ &\leq \varrho (e^{e\varrho+1} + 1), \end{aligned}$$

also

$$e^{-1} - e^{-\lambda_\varepsilon} \geq (e^{-e\varrho} - 1) - \varrho \geq -\varrho (e^{1-e\varrho} + 1),$$

but the infimum of such ϱ is less than $c_n \varepsilon^n$, so that

$$|e^{-1} - e^{-\lambda_\varepsilon}| \leq c'_n \varepsilon^n,$$

and then $\lambda_\varepsilon \rightarrow 1$, as ε goes to zero.

Proof of Proposition 3.2. It's enough to prove that the two inequalities in the definition of the Levy distance, $d_L(\frac{\tau^\varepsilon}{\beta_\varepsilon}, \mathcal{E}(\lambda_\varepsilon))$, hold for $\varrho = c_n \varepsilon^n$, with $n \geq 1$.

Take $\theta_\varepsilon(t) = \lambda_\varepsilon \int_0^\infty P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon) h_\varepsilon(t, s) ds$, note that $|\theta_\varepsilon(t)| \leq 1$, for all t , and by Proposition 3.1, for $t \geq \frac{\gamma_\varepsilon}{\beta_\varepsilon}$, $|\theta_\varepsilon(t)| \leq c_n \varepsilon^n$. From Azlarov and Volodin (1986), the following representation of $F_\varepsilon(t)$ holds:

$$F_\varepsilon(t) = e^{-\lambda_\varepsilon t} + \lambda_\varepsilon \int_0^t e^{-\lambda_\varepsilon(t-u)} \theta_\varepsilon(u) du - \theta_\varepsilon(t).$$

We begin with the first inequality of the Levy metric. Since, for $k > n$ large and ε small enough $\frac{\gamma_\varepsilon}{2c_n \varepsilon^n} \leq \varepsilon^{-k} < \beta_\varepsilon$, then $\frac{\gamma_\varepsilon}{\beta_\varepsilon} \ll 2c_n \varepsilon^n = \varrho$, So that $t + \varrho > \frac{\gamma_\varepsilon}{\beta_\varepsilon}$, and we can use the Proposition 3.1 to get

$$\begin{aligned} F_\varepsilon(t + \varrho) &= e^{-\lambda_\varepsilon(t+\varrho)} + \lambda_\varepsilon \int_0^{(t+\varrho)} e^{-\lambda_\varepsilon(t+\varrho-u)} \theta_\varepsilon(u) du - \theta_\varepsilon(t + \varrho) \\ &\leq e^{-\lambda_\varepsilon(t+\varrho)} + \lambda_\varepsilon \int_0^{\frac{\gamma_\varepsilon}{\beta_\varepsilon}} e^{-\lambda_\varepsilon(t+\varrho-u)} du \\ &\quad + c_n \varepsilon^n \lambda_\varepsilon \int_{\frac{\gamma_\varepsilon}{\beta_\varepsilon}}^{(t+\varrho)} e^{-\lambda_\varepsilon(t+\varrho-u)} du + c_n \varepsilon^n \tag{3.4} \\ &\leq e^{-\lambda_\varepsilon t} e^{-\lambda_\varepsilon(\varrho - \frac{\gamma_\varepsilon}{\beta_\varepsilon})} + \varrho \\ &\leq e^{-\lambda_\varepsilon t} + \varrho. \end{aligned}$$

Now for the other inequality in the Levy metric, we must analyse various cases

- (i) If $t < \varrho$, then $F_\varepsilon(t - \varrho) = 1$ so obviously $F_\varepsilon(t - \varrho) \geq e^{-\lambda_\varepsilon t} - \varrho$.
- (ii) If $t > \varrho$ and $t - \varrho < \frac{\gamma_\varepsilon}{\beta_\varepsilon}$ then $F_\varepsilon(t - \varrho) \geq F_\varepsilon(\frac{\gamma_\varepsilon}{\beta_\varepsilon}) \geq 1 - c_n \varepsilon^n \geq e^{-\lambda_\varepsilon t} - \varrho$, by Eq. (3.3).

(iii) If $t > \varrho$ and $t - \varrho > \frac{\gamma_\varepsilon}{\beta_\varepsilon}$ then

$$\begin{aligned}
 F_\varepsilon(t - \varrho) &= e^{-\lambda_\varepsilon(t-\varrho)} + \lambda_\varepsilon \int_0^{(t-\varrho)} e^{-\lambda_\varepsilon(t-\varrho-u)} \theta_\varepsilon(u) du - \theta_\varepsilon(t - \varrho) \\
 &\geq e^{-\lambda_\varepsilon(t-\varrho)} - \lambda_\varepsilon \int_0^{\frac{\gamma_\varepsilon}{\beta_\varepsilon}} e^{-\lambda_\varepsilon(t-\varrho-u)} du \\
 &\quad - c_n \varepsilon^n \lambda_\varepsilon \int_{\frac{\gamma_\varepsilon}{\beta_\varepsilon}}^{(t+\varrho)} e^{-\lambda_\varepsilon(t+\varrho-u)} du - c_n \varepsilon^n \tag{3.5} \\
 &\geq e^{-\lambda_\varepsilon(t-\varrho)} - (e^{-\lambda_\varepsilon(t-\varrho-\frac{\gamma_\varepsilon}{\beta_\varepsilon})} - e^{-\lambda_\varepsilon(t-\varrho)}) - \varrho \\
 &= e^{-\lambda_\varepsilon t} [2e^{\lambda_\varepsilon \varrho} - e^{\lambda_\varepsilon(\varrho+\frac{\gamma_\varepsilon}{\beta_\varepsilon})}] - \varrho.
 \end{aligned}$$

All we need to prove is that $2e^{\lambda_\varepsilon \varrho} - e^{\lambda_\varepsilon(\varrho+\frac{\gamma_\varepsilon}{\beta_\varepsilon})} > 1$. To this end, first observe that for z close to 1 we have that $2z - 1 > z^{3/2}$. Since $\varrho = 2c_n \varepsilon^n$ and $0 \leq \lambda_\varepsilon \leq e$ we have that $\lambda_\varepsilon \varrho \rightarrow 0$ as $\varepsilon \rightarrow 0$, so that taking $z = e^{\lambda_\varepsilon \varrho}$, for ε small enough,

$$2e^{\lambda_\varepsilon \varrho} - e^{3/2 \lambda_\varepsilon \varrho} > 1.$$

But $\frac{\gamma_\varepsilon}{\beta_\varepsilon} < \varrho/2$, so $e^{\lambda_\varepsilon(\varrho+\frac{\gamma_\varepsilon}{\beta_\varepsilon})} < e^{3/2 \lambda_\varepsilon \varrho}$, using this in (3.5), we can conclude that

$$F_\varepsilon(t - \varrho) \geq e^{-\lambda_\varepsilon t} - \varrho.$$

So, the infimum in the Levy distance, is less than $\varrho = c_n \varepsilon^n$, and the proposition is proved. □

4 Loss of memory property

In this section we prove the loss of memory property of τ^ε .

Proof of Proposition 3.1. We shall prove that

$$\begin{aligned}
 &|P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > (t + s)\beta_\varepsilon) - P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon)P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)| \\
 &\leq c_n \varepsilon^n P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon), \tag{4.1}
 \end{aligned}$$

for $t \geq \frac{\gamma_\varepsilon}{\beta_\varepsilon}$. To do this, we shall condition on the first summand at time $s\beta_\varepsilon + \gamma_\varepsilon$, and after some algebra we will get the difference between terms like $P_\sigma^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon)$, $\sigma \in A_{\delta'}^{(\varepsilon)}$, multiplied by $P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)$, with $0 < \delta' < \delta$. So, we need to bound such differences. The main tool is the following proposition, that permits us to couple two processes before the exit time of $A_\delta^{(\varepsilon)}$. Recall that $\gamma_\varepsilon = \tilde{a} |\log \varepsilon|$.

Proposition 4.1. *Let $0 < \delta' < \delta$. For any $n \geq 0$ there exists $c_n > 0$ such that for any $\varepsilon > 0$ small enough, the following holds:*

$$\begin{aligned} \sup_{t \geq 0} |P_{\sigma}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon}) - P_{\sigma'}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon})| \\ \leq P_{\sigma}^{(\varepsilon)}(\tau^{\varepsilon} < \gamma_{\varepsilon}) + P_{\sigma'}^{(\varepsilon)}(\tau^{\varepsilon} < \gamma_{\varepsilon}) + c_n \varepsilon^n, \end{aligned} \quad (4.2)$$

for any $\sigma, \sigma' \in A_{\delta'}^{(\varepsilon)}$.

We defer the proof to the end of the section. Turning to the proof of (4.1), note that $\gamma_{\varepsilon} < \beta_{\varepsilon}$ and for $n \geq 1$ there exists $c_n > 0$ such that

$$P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} < \gamma_{\varepsilon}) \leq c_n \varepsilon^n,$$

for ε small enough. We begin by bounding from above

$$P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > (t+s)\beta_{\varepsilon}) - P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon})P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}).$$

Since, by hypothesis, $t\beta_{\varepsilon} \geq \gamma_{\varepsilon}$, we can condition the process up to time $s\beta_{\varepsilon} + \gamma_{\varepsilon}$ and after using the Markov property, we get that this expression is less than

$$\begin{aligned} \mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^{\varepsilon} > s\beta_{\varepsilon}} P_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}}}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon})) - P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon})P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}) \\ + P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon})P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}) - P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon})P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}). \end{aligned}$$

Decomposing the first summand according to $\{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}} \in A_{\delta'}\}$, or not (recall that $\delta' < \delta$), the last expression is

$$\begin{aligned} \mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^{\varepsilon} > s\beta_{\varepsilon}} \mathbf{1}_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}} \in A_{\delta'}} [P_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}}}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon}) - P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon})]) \\ + \mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^{\varepsilon} > s\beta_{\varepsilon}} \mathbf{1}_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}} \notin A_{\delta'}} [P_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}}}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon}) - P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon})]) \quad (4.3) \\ + P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}) [P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon} - \gamma_{\varepsilon}) - P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > t\beta_{\varepsilon})]. \end{aligned}$$

By Proposition 4.1, the first term in (4.3) can be bounded by:

$$\mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^{\varepsilon} > s\beta_{\varepsilon}} \mathbf{1}_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}} \in A_{\delta'}} P_{\sigma_{s\beta_{\varepsilon} + \gamma_{\varepsilon}}}^{(\varepsilon)}(\tau^{\varepsilon} \leq \gamma_{\varepsilon})) + c_n \varepsilon^n P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}).$$

The first summand could be expressed, after conditioning up to $s\beta_{\varepsilon} + \frac{\gamma_{\varepsilon}}{2}$, as

$$\begin{aligned} \mathbb{E}_{v_m}^{(\varepsilon)}\left(\mathbf{1}_{\tau^{\varepsilon} > s\beta_{\varepsilon}} \mathbf{1}_{\sigma_{s\beta_{\varepsilon} + \frac{\gamma_{\varepsilon}}{2}} \in A_{\delta'}} P_{\sigma_{s\beta_{\varepsilon} + \frac{\gamma_{\varepsilon}}{2}}}^{(\varepsilon)}\left(\exists u, \frac{\gamma_{\varepsilon}}{2} \leq u \leq \frac{3\gamma_{\varepsilon}}{2}, \sigma_u \notin A_{\delta'}\right)\right) \\ + c_n \varepsilon^n P_{v_m}^{(\varepsilon)}(\tau^{\varepsilon} > s\beta_{\varepsilon}). \end{aligned}$$

The conditional probability that appears inside, could be bounded by $c_n \varepsilon^n$, since if $\sigma \in A_{\delta}$ then, at times of order γ_{ε} , it belongs to $A_{\delta'}$, $\delta > \delta'$ with probability of order $c_n \varepsilon^n$. For the proof of this we can follow the same steps of Proposition 4.3 in Hinojosa (2004).

So that the first summand in (4.3) can be bounded by

$$P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)c_n\varepsilon^n.$$

In similar way for the second term of (4.3), after conditioning up to time $s\beta_\varepsilon$:

$$\begin{aligned} 2\mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^\varepsilon > s\beta_\varepsilon} \mathbf{1}_{\sigma_{s\beta_\varepsilon + \gamma_\varepsilon} \notin A_{\delta'}}) &\leq 2\mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^\varepsilon > s\beta_\varepsilon} P_{\sigma_{s\beta_\varepsilon}}^{(\varepsilon)}(\sigma_{\gamma_\varepsilon} \notin A_{\delta'})) \\ &\leq 2P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)c_n\varepsilon^n, \end{aligned} \tag{4.4}$$

where we used again Hinojosa (2004), in the second inequality.

Finally, the third term in (4.3), after decomposing according to the event $\{\sigma_{\gamma_\varepsilon} \in A_{\delta'}\}$, and using the Proposition 4.1. As above, we get the same bound:

$$\begin{aligned} &P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)(P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon - \gamma_\varepsilon) - \mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\tau^\varepsilon > \gamma_\varepsilon} P_{\sigma_{\gamma_\varepsilon}}^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon - \gamma_\varepsilon))) \\ &\leq P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon) \\ &\quad \times (2P_{v_m}^{(\varepsilon)}(\tau^\varepsilon \leq \gamma_\varepsilon) + c_n\varepsilon^n + \mathbb{E}_{v_m}^{(\varepsilon)}(\mathbf{1}_{\sigma_{\gamma_\varepsilon} \in A_{\delta'}} P_{\sigma_{\gamma_\varepsilon}}^{(\varepsilon)}(\tau^\varepsilon \leq \gamma_\varepsilon))) \\ &\leq P_{v_{m-}}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)c_n\varepsilon^n. \end{aligned} \tag{4.5}$$

So, until now we proved that

$$P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > (t + s)\beta_\varepsilon) - P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > t\beta_\varepsilon)P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon) \tag{4.6}$$

is bounded from above by $P_{v_m}^{(\varepsilon)}(\tau^\varepsilon > s\beta_\varepsilon)c_n\varepsilon^n$, for $s > 0$ and $t > \frac{\gamma_\varepsilon}{\beta_\varepsilon}$. Now, proceeding similarly we shall get the same bound by below, and the proposition is proved. \square

Proof of Proposition 4.1. Note that, if we take $\sigma, \sigma' \in A_{\delta'}^{(\varepsilon)}$, and consider the coupling $P_{\sigma, \sigma'}^{(\varepsilon)}$ of two process starting at σ, σ' , see the Appendix below, then:

$$\begin{aligned} &|P_{\sigma}^{(\varepsilon)}(\tau_\sigma^\varepsilon > t\beta_\varepsilon) - P_{\sigma'}^{(\varepsilon)}(\tau_{\sigma'}^\varepsilon > t\beta_\varepsilon)| \\ &\leq P_{\sigma, \sigma'}^{(\varepsilon)}(\tau_\sigma^\varepsilon > t\beta_\varepsilon, \tau_{\sigma'}^\varepsilon \leq t\beta_\varepsilon) + P_{\sigma, \sigma'}^{(\varepsilon)}(\tau_{\sigma'}^\varepsilon > t\beta_\varepsilon, \tau_\sigma^\varepsilon \leq t\beta_\varepsilon) \\ &\leq P_{\sigma, \sigma'}^{(\varepsilon)}(\tau_\sigma^\varepsilon > t\beta_\varepsilon, \tau_{\sigma'}^\varepsilon < \gamma_\varepsilon) + P_{\sigma, \sigma'}^{(\varepsilon)}(\tau_\sigma^\varepsilon > t\beta_\varepsilon, \gamma_\varepsilon \leq \tau_{\sigma'}^\varepsilon \leq t\beta_\varepsilon) \\ &\quad + P_{\sigma, \sigma'}^{(\varepsilon)}(\tau_{\sigma'}^\varepsilon > t\beta_\varepsilon, \tau_\sigma^\varepsilon < \gamma_\varepsilon) + P_{\sigma, \sigma'}^{(\varepsilon)}(\tau_{\sigma'}^\varepsilon > t\beta_\varepsilon, \gamma_\varepsilon \leq \tau_\sigma^\varepsilon \leq t\beta_\varepsilon) \\ &\leq P_{\sigma}^{(\varepsilon)}(\tau^\varepsilon < \gamma_\varepsilon) + P_{\sigma'}^{(\varepsilon)}(\tau^\varepsilon < \gamma_\varepsilon) + 2P_{\sigma, \sigma'}^{(\varepsilon)}(\sigma_{\gamma_\varepsilon} \neq \sigma'_{\gamma_\varepsilon}). \end{aligned} \tag{4.7}$$

For the last term, we have

$$\begin{aligned} P_{\sigma, \sigma'}^{(\varepsilon)}(\sigma_{\gamma_\varepsilon} \neq \sigma'_{\gamma_\varepsilon}) &\leq \sum_{x \in \mathbb{Z}_\varepsilon} P_{\sigma, \sigma'}^{(\varepsilon)}(\sigma_{\gamma_\varepsilon}(x) \neq \sigma'_{\gamma_\varepsilon}(x)) \\ &\leq \varepsilon^{-1} \mathbb{E}_{\sigma, \sigma'}^{(\varepsilon)}|\sigma_{\gamma_\varepsilon}(0) - \sigma'_{\gamma_\varepsilon}(0)| \end{aligned} \tag{4.8}$$

in the last inequality by translation invariance.

Thus, the main step of the proof is to get a bound for the last term. For this part we use Theorem 2.1 in [Brassasco et al. \(2000a\)](#), as stated below, where it is found a bound of order ε^n for the discrepancies between the processes σ_T^+ and σ_T^- , which is defined as the expected value of the site differences for the coupling $P_{\mathbf{1},-\mathbf{1}}^{(\varepsilon)}$ of the processes that starts from $\mathbf{1}$ and $-\mathbf{1}$ and at times $T := \tilde{a}|\ln \varepsilon|$: $\mathbb{E}_{\mathbf{1},-\mathbf{1}}^{(\varepsilon)}(\sigma_T^+(0) - \sigma_T^-(0))$. Using this result and that the coupling is order preserving, see the [Appendix](#), we have the following proposition.

Proposition 4.2. *For any $\gamma < \frac{1}{2}$, $n \geq 1$ and $\varepsilon > 0$ small enough there exists positive constants \tilde{a} and c_n such that:*

$$0 \leq \mathbb{E}_{\sigma,\sigma'}^{(\varepsilon)}|\sigma_T(0) - \sigma'_T(0)| \leq c_n \varepsilon^n,$$

where $T := \tilde{a}|\ln \varepsilon|$, and $\sigma, \sigma' \in A_\delta$.

Observe that by attractiveness of the coupling at time T : $\sigma_T^+(x) \geq \sigma_T(x)$ and $\sigma_T(x) \geq \sigma_T^-(x)$ for any $x \in \mathbb{Z}_\varepsilon$ and any processes σ_t .

Using this result, the right-hand side of (4.8) could be bounded by $c_n \varepsilon^{n-1}$ uniformly in A_δ , and the Proposition 4.1 is proved. \square

Appendix: Graphical representation

In this section, we will describe the graphical construction of the process, due to Harris (see [Haken \(1978\)](#)), the same that appears in [Brassasco et al. \(2000b\)](#), see also [Brassasco et al. \(2000a\)](#). This allows us to exploit the comparison between different evolutions of the process.

For each $x \in \mathbb{Z}_\varepsilon$ let $\mathcal{N}^{x,+}$, $\mathcal{N}^{x,-}$, be Poisson process of intensity $c_{\max} := (1 + \gamma)^2 + \frac{\mu}{2}$ (i.e., the maximum flip rate for the Glauber process), and for each bond $(x, x + e_i)$ (where $e_i = (0, \dots, 1, \dots, 0)$, one in the i th position), let $\mathcal{N}_{x,e_i}^\varepsilon$ be a Poisson process of intensity ε^{-2} . Let also $U_n^{x,+}, U_n^{x,-}, n \geq 1$ be i.i.d. Uniform(0, 1) random variables. We also assume that all the process $\mathcal{N}^{x,+}, \mathcal{N}^{x,-}, \mathcal{N}_{x,e_i}^\varepsilon$ and $\{U_n^{x,+}\}_{n \geq 1}, \{U_n^{x,-}\}_{n \geq 1}, n \geq 1, x \in \mathbb{Z}_\varepsilon$ are independent of each other.

The *graphical representation* is a realization of the process which applies for any initial configuration. Whenever a mark $\mathcal{N}_{x,e_i}^\varepsilon$ appears then we exchange the spins at x and $x + e_i$. At the time of the n th mark of $\mathcal{N}^{x,+}$ ($\mathcal{N}^{x,-}$), we flip the spin at x if $\sigma(x) = 1$ and $c_\mu(x, \sigma) > U_n^{x,+} c_{\max}$ ($\sigma(x) = -1$ and $c_\mu(x, \sigma) > U_n^{x,-} c_{\max}$). This prescription defines the evolution of the Glauber + Kawasaki process.

In the marked process, consider a realization ω . For each $x \in \mathbb{Z}_\varepsilon$, and $t \geq 0$, there are functions $f_{x,t}(\sigma, \omega)$ with the property that, once σ is fixed, the law of the variables $\sigma_t(x) := f_{x,t}(\sigma, \omega)$, is the same as the law of the Glauber + Kawasaki process which starts from σ ; moreover, $f_{x,t}(\sigma, \omega) \leq f_{x,t}(\sigma', \omega)$ if $\sigma \leq \sigma'$. This coupling preserves the order since the rates are attractive.

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