

THE WIENER CONDITION AND THE CONJECTURES OF EMBRECHTS AND GOLDIE

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We show that the class of convolution equivalent distributions and the class of locally subexponential distributions are not closed under convolution roots. It gives a negative answer to the classical conjectures of Embrechts and Goldie. Moreover, we establish two sufficient conditions in order that the class of convolution equivalent distributions is closed under convolution roots.

1. Introduction and main results. In what follows, we denote by \mathbb{R} the real line and by \mathbb{R}_+ the half-line $[0, \infty)$. Let \mathbb{N} be the totality of positive integers. The symbol $\delta_a(dx)$ stands for the delta measure at $a \in \mathbb{R}$. Let μ and ρ be finite measures on \mathbb{R} . We denote the convolution of μ and ρ by $\mu * \rho$ and denote the n th convolution power of ρ by ρ^{n*} with the understanding that $\rho^{0*}(dx) = \delta_0(dx)$. For positive functions $f(x)$ and $g(x)$ on $[a, \infty)$ for some $a \in \mathbb{R}$, we define the relation $f(x) \sim g(x)$ by $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ and the relation $f(x) \asymp g(x)$ by $0 < \liminf_{x \rightarrow \infty} f(x)/g(x) \leq \limsup_{x \rightarrow \infty} f(x)/g(x) < \infty$. The tail of a finite measure η on \mathbb{R} is denoted by $\bar{\eta}(x)$, that is, $\bar{\eta}(x) := \eta((x, \infty))$ for $x \in \mathbb{R}$. Let $\gamma \in \mathbb{R}$. The γ -exponential moment of η is denoted by $\hat{\eta}(\gamma)$, namely,

$$\hat{\eta}(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x} \eta(dx).$$

If $\hat{\eta}(\gamma) < \infty$, we define the Fourier–Laplace transform $\hat{\eta}(\gamma + iz)$ for $z \in \mathbb{R}$ as

$$\hat{\eta}(\gamma + iz) := \int_{-\infty}^{\infty} e^{(\gamma + iz)x} \eta(dx).$$

We use the words “increase” and “decrease” in the wide sense allowing flatness. A distribution always means a probability distribution.

DEFINITION 1.1. Let $\gamma \geq 0$.

(i) A distribution ρ on \mathbb{R} belongs to the class $\mathcal{L}(\gamma)$ if $\bar{\rho}(x) > 0$ for every $x \in \mathbb{R}$ and if

$$\bar{\rho}(x + a) \sim e^{-\gamma a} \bar{\rho}(x) \quad \text{for every } a \in \mathbb{R}.$$

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(ii) A distribution ρ on \mathbb{R} belongs to the class $\mathcal{S}(\gamma)$ if $\rho \in \mathcal{L}(\gamma)$ with $\widehat{\rho}(\gamma) < \infty$ and if

$$\overline{\rho^{2*}}(x) \sim 2\widehat{\rho}(\gamma)\bar{\rho}(x).$$

(iii) Let $\gamma_1 \in \mathbb{R}$. A distribution ρ on \mathbb{R} belongs to the class $\mathcal{M}(\gamma_1)$ if $\widehat{\rho}(\gamma_1) < \infty$.

DEFINITION 1.2. Let \mathbf{G} be the totality of nonnegative measurable functions $g(x)$ on \mathbb{R} satisfying $g(x) > 0$ for all sufficiently large $x > 0$. Let $g(x) \in \mathbf{G}$:

(i) We say that $g(x) \in \mathbf{L}$ if $g(x+a) \sim g(x)$ for every $a \in \mathbb{R}$.

(ii) We say that $g(x) \in \mathbf{AD}$ if, for every $a \geq 0$,

$$\limsup_{x \rightarrow \infty} \frac{g(x+a)}{g(x)} \leq 1.$$

DEFINITION 1.3. (i) Let $\Delta := (0, c]$ with $c > 0$. A distribution ρ on \mathbb{R} belongs to the class \mathcal{L}_Δ if $\rho((x, x+c]) \in \mathbf{L}$.

(ii) Let $\Delta := (0, c]$ with $c > 0$. A distribution ρ on \mathbb{R} belongs to the class \mathcal{S}_Δ if $\rho \in \mathcal{L}_\Delta$ and $\rho^{2*}((x, x+c]) \sim 2\rho((x, x+c])$.

(iii) A distribution ρ on \mathbb{R} belongs to the class \mathcal{L}_{loc} if $\rho \in \mathcal{L}_\Delta$ for each $\Delta := (0, c]$ with $c > 0$.

(iv) A distribution ρ on \mathbb{R} belongs to the class \mathcal{S}_{loc} if $\rho \in \mathcal{S}_\Delta$ for each $\Delta := (0, c]$ with $c > 0$.

Rogozin [16] in the one-sided case and Pakes [14] in the two-sided case proved that $\rho \in \mathcal{S}(\gamma)$ with $\gamma \geq 0$ if and only if $\rho \in \mathcal{L}(\gamma)$ and there exists $M > 0$ such that $\overline{\rho^{2*}}(x) \sim M\bar{\rho}(x)$. Distributions in the class $\mathcal{S}(0)$ are called *subexponential*. Those in the class $\mathcal{S}(\gamma)$ are called *convolution equivalent* and those in the class \mathcal{S}_{loc} are called *locally subexponential*. The class $\mathcal{S}(0)$ was introduced by Chistyakov [6] for applications to branching processes. The study of the class $\mathcal{S}(\gamma)$ goes back to Chover et al. [4, 5]. The class \mathcal{S}_Δ was introduced by Asmussen et al. [1] and the class \mathcal{S}_{loc} was by Borovkov and Borovkov [3] and Watanabe and Yamamuro [22]; see also Foss et al. [12]. Applications of those classes include renewal theory, random walks, queues, branching processes, Lévy processes and infinite divisibility. We extend the notion of the Wiener condition for a function in $L^1(\mathbb{R})$ to that for a finite measure as follows.

DEFINITION 1.4. We say that a finite measure η on \mathbb{R} satisfies the *Wiener condition* if $\widehat{\eta}(iz) \neq 0$ for every $z \in \mathbb{R}$. We denote by \mathcal{W} the totality of finite measures on \mathbb{R} which satisfy the Wiener condition.

REMARK 1.1. (i) For every $a \in \mathbb{R}$, $\delta_a(dx) \in \mathcal{W}$. If $\eta_1, \eta_2 \in \mathcal{W}$, then $\eta_1 * \eta_2 \in \mathcal{W}$.

(ii) Define, for a complex number z , $\phi(z) := \sum_{n=0}^{\infty} p_n z^n$ where $p_n \geq 0$ for all $n \geq 0$ and $0 < \sum_{n=0}^{\infty} p_n (1 + \varepsilon)^n < \infty$ for some $\varepsilon > 0$. Let $\eta := \sum_{n=0}^{\infty} p_n \rho^{n*}$ for a distribution ρ on \mathbb{R} . Assume that $\phi(z)$ has no zero in a unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ in the complex plane \mathbb{C} . Then $\eta \in \mathcal{W}$ for every distribution ρ .

DEFINITION 1.5. We say that a class \mathcal{C} of distributions on \mathbb{R} is closed under convolution roots if $\mu^{n*} \in \mathcal{C}$ for some $n \in \mathbb{N}$ implies that $\mu \in \mathcal{C}$.

Embrechts et al. [10] in the one-sided case and Watanabe [21] in the two-sided case proved that the class $\mathcal{S}(0)$ is closed under convolution roots. Embrechts and Goldie stated in [9] that a crucial point for proving limit theorems using $\mathcal{S}(\gamma)$ is the convolution roots closure of $\mathcal{S}(\gamma)$. Further, they gave the following conjectures in [8, 9], respectively.

CONJECTURE I. The class $\mathcal{L}(\gamma)$ with $\gamma \geq 0$ is closed under convolution roots.

CONJECTURE II. The class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is closed under convolution roots.

Embrechts and Goldie [9] in the one-sided case and Pakes [15] in the two-sided case obtained the following.

THEOREM A. Let $\gamma > 0$ and let μ be a distribution on \mathbb{R} . If $\mu \in \mathcal{L}(\gamma)$ and $\mu^{n*} \in \mathcal{S}(\gamma)$ for some $n \in \mathbb{N}$, then $\mu \in \mathcal{S}(\gamma)$.

Moreover, Watanabe showed in Theorem 1.1 of [21] the following.

THEOREM B. Let $\gamma > 0$ and let μ be an infinitely divisible distribution on \mathbb{R} . If $\mu^{n*} \in \mathcal{S}(\gamma)$ for some $n \in \mathbb{N}$, then $\mu \in \mathcal{S}(\gamma)$.

We see from Theorem A that if Conjecture I is true for every $\gamma > 0$, then so is Conjecture II. However, Shimura and Watanabe [19] disproved Conjecture I for every $\gamma \geq 0$ and, recently, Xu et al. [26] also did for $\gamma = 0$. On the other hand, Conjecture II was unsolved for over 30 years. In this paper, we disprove Conjecture II for every $\gamma > 0$ and extend Theorems A and B. Celebrated Wiener's approximation theorem plays a key role for the resolution. We discover an idea that the Wiener condition fails for a finite measure $e^{\gamma x} \mu(dx)$ of a counterexample μ to Conjecture II. Our main results are as follows.

THEOREM 1.1. The class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is not closed under convolution roots.

By using exponential tilts, we have the following corollary.

COROLLARY 1.1. *Let $\Delta := (0, c]$ with $c > 0$ and let $\gamma > 0$. We have the following:*

- (i) *The class \mathcal{S}_{loc} is not closed under convolution roots.*
- (ii) *The class \mathcal{S}_{Δ} is not closed under convolution roots.*
- (iii) *The class $\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)$ is not closed under convolution roots.*
- (iv) *The class \mathcal{L}_{loc} is not closed under convolution roots.*
- (v) *The class \mathcal{L}_{Δ} is not closed under convolution roots.*

Next, we establish an extension of Theorem A. Note that $\mu \in \mathcal{L}(\gamma)$ with $\gamma \geq 0$ if and only if $e^{\gamma x} \bar{\mu}(x) \in \mathbf{L}$. The condition $e^{\gamma x} \bar{\mu}(x) \in \mathbf{AD}$ is found in Theorem 7 of Foss and Korshunov [11] in the one-sided case.

THEOREM 1.2. *Let $\gamma \geq 0$ and let μ be a distribution on \mathbb{R} . Assume that $e^{\gamma x} \bar{\mu}(x) \in \mathbf{AD}$. Then $\mu^{n*} \in \mathcal{S}(\gamma)$ for some $n \in \mathbb{N}$ implies that $\mu \in \mathcal{S}(\gamma)$.*

COROLLARY 1.2. *Let μ be a distribution on \mathbb{R} . Assume that $\hat{\mu}(-\gamma) < \infty$ for some $\gamma > 0$ and that $\mu((x, x + c]) \in \mathbf{AD}$ for every $c > 0$. Then $\mu^{n*} \in \mathcal{S}_{\text{loc}}$ for some $n \in \mathbb{N}$ implies that $\mu \in \mathcal{S}_{\text{loc}}$.*

REMARK 1.2. Let μ be a distribution on \mathbb{R} . The condition $e^{\gamma x} \bar{\mu}(x) \in \mathbf{AD}$ necessarily holds for $\gamma = 0$. If μ is unimodal with a density $p(x) \in \mathbf{G}$, then the condition $\mu((x, x + c]) \in \mathbf{AD}$ holds for every $c > 0$.

Finally, we present an extension of Theorem B. Note that every infinitely divisible distribution on \mathbb{R} satisfies the Wiener condition and that if $\hat{\mu}(\gamma) < \infty$ for an infinitely divisible distribution μ on \mathbb{R} , then $e^{\gamma x} \mu(dx) \in \mathcal{W}$; see Theorem 25.17 of Sato [17].

THEOREM 1.3. *Let $\gamma > 0$ and let μ be a distribution on \mathbb{R} . Assume that $\hat{\mu}(\gamma) < \infty$ and $e^{\gamma x} \mu(dx) \in \mathcal{W}$, that is, $\hat{\mu}(\gamma + iz) \neq 0$ for every $z \in \mathbb{R}$. Then $\mu^{n*} \in \mathcal{S}(\gamma)$ for some $n \in \mathbb{N}$ implies that $\mu \in \mathcal{S}(\gamma)$.*

COROLLARY 1.3. *Let μ be a distribution on \mathbb{R} . Assume that $\hat{\mu}(-\gamma) < \infty$ for some $\gamma > 0$ and $\mu \in \mathcal{W}$. Then $\mu^{n*} \in \mathcal{S}_{\text{loc}}$ for some $n \in \mathbb{N}$ implies that $\mu \in \mathcal{S}_{\text{loc}}$.*

REMARK 1.3. We see from Theorems 1.2 and 1.3 that each counterexample μ on \mathbb{R} to Conjecture II must satisfy that $\hat{\mu}(\gamma) < \infty$ and $\hat{\mu}(\gamma + iz_0) = 0$ for some $z_0 \in \mathbb{R}$ and that $\limsup_{x \rightarrow \infty} e^{\gamma a_0} \bar{\mu}(x + a_0) / \bar{\mu}(x) > 1$ for some $a_0 > 0$.

Nonclosure under convolution roots for the other distribution classes was shown by Shimura and Watanabe [18] for the class \mathcal{OS} of O -subexponential distributions, and by Watanabe and Yamamuro [24] for the class \mathcal{S}_{ac} of distributions with subexponential densities. The organization of this paper is as follows. In Section 2, we give preliminaries for the proofs of the main results. In Sections 3, 4 and 5, we prove Theorems 1.1, 1.2 and 1.3 and their corollaries, respectively.

2. Preliminaries. The following lemma is a direct consequence of Corollary 2 of Cline [7].

LEMMA 2.1. *Let $\gamma > 0$ and μ be a distribution on \mathbb{R}_+ . If $\bar{\mu}(x) \sim ce^{-\gamma x}x^{-\alpha}$ with $c > 0$ and $\alpha > 1$, then $\mu \in \mathcal{S}(\gamma)$.*

LEMMA 2.2. *Let $\gamma \geq 0$. We have the following:*

(i) *Let $\mu \in \mathcal{L}(\gamma)$ with $\hat{\mu}(\gamma) < \infty$. Then $\mu \in \mathcal{S}(\gamma)$ if and only if*

$$\lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{\int_{A+}^{(x-A)+} \bar{\mu}(x-u)\mu(du)}{\bar{\mu}(x)} = 0.$$

(ii) *Let μ_1 and μ_2 be distributions on \mathbb{R} . If $\mu_1 \in \mathcal{S}(\gamma)$ and $\bar{\mu}_2(x) \sim c\bar{\mu}_1(x)$ with $c > 0$, then $\mu_2 \in \mathcal{S}(\gamma)$.*

PROOF. First, we prove assertion (i). Let $\mu \in \mathcal{L}(\gamma)$ with $\hat{\mu}(\gamma) < \infty$ and let $A > 0$. We have, for $x > 2A$,

$$\bar{\mu}^{2*}(x) = \sum_{j=1}^3 H_j(x),$$

where

$$H_1(x) := 2 \int_{-\infty}^{A+} \bar{\mu}(x-u)\mu(du), \quad H_2(x) := \bar{\mu}(x-A)\bar{\mu}(A),$$

and

$$H_3(x) := \int_{A+}^{(x-A)+} \bar{\mu}(x-u)\mu(du).$$

Since we see that

$$\sup_{u \in (-\infty, A]} \frac{\bar{\mu}(x-u)}{\bar{\mu}(x)} \leq \frac{\bar{\mu}(x-A)}{\bar{\mu}(x)} \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\bar{\mu}(x-A)}{\bar{\mu}(x)} = e^{\gamma A},$$

we obtain from the dominated convergence theorem that

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{H_1(x)}{\bar{\mu}(x)} = 2 \int_{-\infty}^{A+} \lim_{x \rightarrow \infty} \frac{\bar{\mu}(x-u)}{\bar{\mu}(x)} \mu(du) = 2 \int_{-\infty}^{A+} e^{\gamma u} \mu(du).$$

We have

$$(2.2) \quad \lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{H_2(x)}{\bar{\mu}(x)} = \lim_{A \rightarrow \infty} e^{\gamma A} \bar{\mu}(A) \leq \lim_{A \rightarrow \infty} \int_{A+}^{\infty} e^{\gamma x} \mu(dx) = 0.$$

Thus, we see from (2.1) and (2.2) that

$$\begin{aligned} 0 &= \lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{H_1(x)}{\bar{\mu}(x)} - 2\hat{\mu}(\gamma) \leq \liminf_{x \rightarrow \infty} \frac{\overline{\mu^{2*}}(x)}{\bar{\mu}(x)} - 2\hat{\mu}(\gamma) \\ &\leq \limsup_{x \rightarrow \infty} \frac{\overline{\mu^{2*}}(x)}{\bar{\mu}(x)} - 2\hat{\mu}(\gamma) \\ &= \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{H_3(x)}{\bar{\mu}(x)}. \end{aligned}$$

Thus, we find that $\mu \in \mathcal{S}(\gamma)$, that is,

$$\lim_{x \rightarrow \infty} \frac{\overline{\mu^{2*}}(x)}{\bar{\mu}(x)} = 2\hat{\mu}(\gamma)$$

if and only if

$$\lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{H_3(x)}{\bar{\mu}(x)} = 0.$$

The proof of assertion (ii) is due to Lemma 2.4 of Pakes [14]. \square

Let $\gamma \in \mathbb{R}$. For $\mu \in \mathcal{M}(\gamma)$, we define the *exponential tilt* $\mu_{\langle \gamma \rangle}$ of μ as

$$\mu_{\langle \gamma \rangle}(dx) := \frac{1}{\hat{\mu}(\gamma)} e^{\gamma x} \mu(dx).$$

Exponential tilts preserve convolutions, that is, $(\mu * \rho)_{\langle \gamma \rangle} = \mu_{\langle \gamma \rangle} * \rho_{\langle \gamma \rangle}$ for distributions $\mu, \rho \in \mathcal{M}(\gamma)$. Let \mathcal{C} be a distribution class. For a class $\mathcal{C} \subset \mathcal{M}(\gamma)$, we define the class $\mathfrak{E}_{\gamma}(\mathcal{C})$ by

$$\mathfrak{E}_{\gamma}(\mathcal{C}) := \{\mu_{\langle \gamma \rangle} : \mu \in \mathcal{C}\}.$$

It is obvious that $\mathfrak{E}_{\gamma}(\mathcal{M}(\gamma)) = \mathcal{M}(-\gamma)$ and that $(\mu_{\langle \gamma \rangle})_{\langle -\gamma \rangle} = \mu$ for $\mu \in \mathcal{M}(\gamma)$. The class $\mathfrak{E}_{\gamma}(\mathcal{S}(\gamma))$ is determined by Watanabe and Yamamuro as follows.

LEMMA 2.3 (Theorem 2.1 of [22]). *Let $\gamma > 0$. We have the following:*

(i) *We have $\mathfrak{E}_{\gamma}(\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)) = \mathcal{L}_{\text{loc}} \cap \mathcal{M}(-\gamma)$. Moreover, if $\rho \in \mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)$, then we have*

$$\rho_{\langle \gamma \rangle}((x, x+c]) \sim \frac{c\gamma}{\hat{\rho}(\gamma)} e^{\gamma x} \bar{\rho}(x) \quad \text{for all } c > 0.$$

(ii) *We have $\mathfrak{E}_{\gamma}(\mathcal{S}(\gamma)) = \mathcal{S}_{\text{loc}} \cap \mathcal{M}(-\gamma)$.*

A straightforward consequence of the above lemma is the following.

LEMMA 2.4. *Let $\gamma > 0$. We have the following:*

- (i) *The class $\mathcal{S}(\gamma)$ is closed under convolution roots if and only if so is the class $\mathcal{S}_{\text{loc}} \cap \mathcal{M}(-\gamma)$.*
- (ii) *The class $\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)$ is closed under convolution roots if and only if so is the class $\mathcal{L}_{\text{loc}} \cap \mathcal{M}(-\gamma)$.*

The following lemma is Wiener's approximation theorem in [25].

LEMMA 2.5 (Theorem 4.8.4 of [2] or Theorem 8.1 of [13]). *For $f(x) \in L^1(\mathbb{R})$, the following are equivalent:*

- (1) *Linear combinations of the translates of $f(x)$ are dense in $L^1(\mathbb{R})$.*
- (2) *$\int_{-\infty}^{\infty} \exp(izx) f(x) dx \neq 0$ for every $z \in \mathbb{R}$.*
- (3) *If, for a bounded measurable function $g(x)$ on \mathbb{R} ,*

$$\int_{-\infty}^{\infty} g(x-t) f(t) dt = 0 \quad \text{for every } x \in \mathbb{R},$$

then $g(x) = 0$ for almost every $x \in \mathbb{R}$.

We shall use the following extension of the above lemma for a finite measure.

LEMMA 2.6. *Let η be a finite measure on \mathbb{R} . The following are equivalent:*

- (1) *$\eta \in \mathcal{W}$.*
- (2) *If, for a bounded measurable function $g(x)$ on \mathbb{R} ,*

$$\int_{-\infty}^{\infty} g(x-t) \eta(dt) = 0 \quad \text{for almost every } x \in \mathbb{R},$$

then $g(x) = 0$ for almost every $x \in \mathbb{R}$.

PROOF. Suppose that $\eta \in \mathcal{W}$ and, for a bounded measurable function $g(x)$ on \mathbb{R} ,

$$(2.3) \quad \int_{-\infty}^{\infty} g(x-t) \eta(dt) = 0 \quad \text{for almost every } x \in \mathbb{R}.$$

Define $f(x) := 2(1-x)1_{[0,1)}(x)$ and an absolutely continuous finite measure η_1 as $\eta_1 := (f(x) dx) * \eta$. Note that

$$(2.4) \quad \int_{-\infty}^{\infty} \exp(izx) f(x) dx \neq 0 \quad \text{for every } z \in \mathbb{R}.$$

Then we have $\eta_1 \in \mathcal{W}$ and see from (2.3) that, for every $y \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} g(y-t) \eta_1(dt) = \int_{-\infty}^{\infty} f(y-x) dx \int_{-\infty}^{\infty} g(x-t) \eta(dt) = 0.$$

Thus, we obtain from Lemma 2.5 that $g(x) = 0$ for almost every $x \in \mathbb{R}$. Conversely, assume that assertion (2) holds and that, for a bounded measurable function $g(x)$ on \mathbb{R} , we have

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} g(x-t)\eta_1(dt) \\ &= \int_{-\infty}^{\infty} \eta(dt) \int_{-\infty}^{\infty} g(x-u-t)f(u)du \quad \text{for every } x \in \mathbb{R}. \end{aligned}$$

Then we find from assertion (2) that

$$\int_{-\infty}^{\infty} g(x-t)f(t)dt = 0 \quad \text{for almost every } x \in \mathbb{R},$$

and hence for every $x \in \mathbb{R}$ by the continuity in x of this integral. Thus, we see from Lemma 2.5 and (2.4) that $g(x) = 0$ for almost every $x \in \mathbb{R}$. It follows from Lemma 2.5 that $\eta_1 \in \mathcal{W}$ and, by (2.4), $\eta \in \mathcal{W}$. \square

3. Proofs of Theorem 1.1 and its corollary. We prove Theorem 1.1 only for $\gamma = 1$. The general case for $\gamma > 0$ is similar and omitted. The symbol $[x]$ stands for the largest integer not exceeding a real number x and the symbol $1_B(x)$ does for the indicator function of a subset B of \mathbb{R} . Let Λ_0 be the totality of increasing sequences $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n \rightarrow \infty} \lambda_n = \infty$ such that the following $\lambda \in [0, 2\pi]$ exists:

$$\lambda := \lim_{n \rightarrow \infty} (\lambda_n - 2\pi[\lambda_n/(2\pi)]).$$

For any sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n \rightarrow \infty} x_n = \infty$, there exists a subsequence $\{\lambda_n\} \in \Lambda_0$ of $\{x_n\}$. We define two positive right-continuous functions $\phi_1(x)$ and $\phi_2(x)$ on \mathbb{R}_+ as

$$\phi_1(x) = e^{-x} \left(3\pi + 1 + \sqrt{2} \sin \left(x - \frac{\pi}{4} \right) \right) 1_{[0, \infty)}(x)$$

and

$$\phi_2(x) = \frac{1}{3\pi} 1_{[0, 2\pi)}(x) + \sum_{n=1}^{\infty} \frac{1}{\pi^3 n^2} 1_{[2n\pi, 2(n+1)\pi)}(x).$$

Note that the two functions $\phi_1(x)$ and $\phi_2(x)$ are decreasing on \mathbb{R}_+ and $\phi_1(0) \times \phi_2(0) = 1$ and that $\int_0^{\infty} \phi_2(x) dx = 1$ and $\int_0^{\infty} \exp(inx)\phi_2(x) dx = 0$ for all $n \in \mathbb{N}$. Thus, we can define a distribution ξ on \mathbb{R}_+ by using its tail $\bar{\xi}(x)$ as $\bar{\xi}(x) := \phi_1(x)\phi_2(x)$ on \mathbb{R}_+ . In the following lemma, we show that the Wiener condition fails for the measure $e^x \xi(dx)$.

LEMMA 3.1. We have $\widehat{\xi}(1) < \infty$ and

$$(3.1) \quad \widehat{\xi}(1+i) = 0.$$

PROOF. Note that $\phi_1(x) \asymp e^{-x}$ and $\phi_2(x) \sim 4\pi^{-1}x^{-2}$, and hence $\bar{\xi}(x) \asymp e^{-x}x^{-2}$. Thus, $\hat{\xi}(1) < \infty$. We have by using integration by parts

$$\hat{\xi}(1+i) = \bar{\xi}(0) + (1+i) \int_0^\infty e^{(1+i)x} \bar{\xi}(x) dx = 1 - \int_0^\infty \phi_2(x) dx = 0.$$

Thus, the lemma is true. \square

LEMMA 3.2. *We have $\xi \notin \mathcal{L}(1)$, and hence $\xi \notin \mathcal{S}(1)$.*

PROOF. For every $\{\lambda_n\} \in \Lambda_0$ and every $a \in \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} \frac{e^{a\bar{\xi}(\lambda_n + a)}}{\bar{\xi}(\lambda_n)} = \frac{3\pi + 1 + \sqrt{2} \sin(\lambda + a - \frac{\pi}{4})}{3\pi + 1 + \sqrt{2} \sin(\lambda - \frac{\pi}{4})},$$

which is not constant in a . Thus, we see that $\xi \notin \mathcal{L}(1)$ and hence $\xi \notin \mathcal{S}(1)$. \square

LEMMA 3.3. *We have $\xi^{2*} \in \mathcal{S}(1)$.*

PROOF. Let $g(x) := 1_{[1,\infty)}(x)x^{-2}e^{-x}$ and $A > 1$. Then we have

$$\begin{aligned} & \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{\int_A^{x-A} g(x-u)g(u) du}{g(x)} \\ (3.2) \quad &= \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{2 \int_A^{x/2} g(x-u)g(u) du}{g(x)} \leq 8 \lim_{A \rightarrow \infty} \int_A^\infty u^{-2} du = 0 \end{aligned}$$

and

$$(3.3) \quad \lim_{A \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{g(x-A)g(A)}{g(x)} = \lim_{A \rightarrow \infty} A^{-2} = 0.$$

We see that, for $x > 2A$,

$$\overline{\xi^{2*}}(x) = I_1(x) + I_2(x),$$

where

$$I_1(x) := 2 \int_{0-}^{A+} \bar{\xi}(x-u)\xi(du)$$

and

$$I_2(x) := \int_{A+}^{(x-A)+} \bar{\xi}(x-u)\xi(du) + \bar{\xi}(x-A)\bar{\xi}(A).$$

Note that $\bar{\xi}(x) \asymp g(x)$, and hence $\bar{\xi}(x) \leq c_1 g(x)$ with some $c_1 > 0$ for $x > 1$ and that $-g'(x) \leq 3g(x)$ for $x > 1$. By using integration by parts, we obtain that, for

$x > 2A$,

$$\begin{aligned}
 & \int_{A+}^{(x-A)+} \bar{\xi}(x-u)\xi(du) \\
 & \leq c_1 \int_{A+}^{(x-A)+} g(x-u)\xi(du) \\
 & \leq c_1 g(x-A)\bar{\xi}(A) - c_1 \int_A^{x-A} g'(x-u)\bar{\xi}(u) du \\
 & \leq c_1^2 g(x-A)g(A) + 3c_1^2 \int_A^{x-A} g(x-u)g(u) du.
 \end{aligned}$$

Thus, we find that, for $x > 2A$,

$$I_2(x) \leq 2c_1^2 g(x-A)g(A) + 3c_1^2 \int_A^{x-A} g(x-u)g(u) du,$$

and hence by (3.2) and (3.3)

$$(3.4) \quad \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{I_2(x)}{g(x)} = 0.$$

For every $\{\lambda_n\} \in \Lambda_0$, we have

$$\lim_{n \rightarrow \infty} \frac{I_1(\lambda_n)}{g(\lambda_n)} = 8\pi^{-1} \int_{0-}^{A+} \left(3\pi + 1 + \sqrt{2} \sin\left(\lambda - u - \frac{\pi}{4}\right) \right) e^u \xi(du).$$

Thus, we see from (3.1) of Lemma 3.1 and (3.4) that, for every $\{\lambda_n\} \in \Lambda_0$,

$$\lim_{n \rightarrow \infty} \frac{\overline{\xi^{2*}}(\lambda_n)}{g(\lambda_n)} = \lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{I_1(\lambda_n)}{g(\lambda_n)} = \frac{8(3\pi + 1)}{\pi} \widehat{\xi}(1),$$

which is independent of the choice of $\{\lambda_n\} \in \Lambda_0$. Recall that any sequence converging to infinity has a subsequence in Λ_0 . Thus, we have

$$\overline{\xi^{2*}}(x) \sim 8(3\pi + 1)\pi^{-1} \widehat{\xi}(1) e^{-x} x^{-2}.$$

Hence, by Lemma 2.1, we establish that $\xi^{2*} \in \mathcal{S}(1)$. \square

PROOF OF THEOREM 1.1. The proof is due to Lemmas 3.2 and 3.3. \square

PROOF OF COROLLARY 1.1. The proof of assertion (i) is due to Theorem 1.1 and (i) of Lemma 2.4. If the class \mathcal{S}_Δ is closed under convolution roots for some Δ , then so is for every Δ and thereby the class \mathcal{S}_{loc} is closed under convolution roots. Thus, assertion (ii) is due to assertion (i). The proof of assertion (iii) is due to Lemmas 3.2 and 3.3. We see from assertion (iii) and (ii) of Lemma 2.4 that $\mathcal{L}_{\text{loc}} \cap \mathcal{M}(-\gamma)$ with some $\gamma > 0$ is not closed under convolution roots and hence so is \mathcal{L}_{loc} . If the class \mathcal{L}_Δ is closed under convolution roots for some Δ , then so is for every Δ and thereby the class \mathcal{L}_{loc} is closed under convolution roots. Thus, assertion (v) is due to assertion (iv). \square

4. Proofs of Theorem 1.2 and its corollary. In Sections 4 and 5, let μ be a distribution on \mathbb{R} satisfying $\bar{\mu}(x) > 0$ for all $x \in \mathbb{R}$. Let $\{X_j\}_{j=1}^\infty$ be IID random variables with the same distribution μ . We define $J_k(x)$ for $1 \leq k \leq 3$ and $\varepsilon(A)$ for $A > 0$ and $n \geq 2$ as

$$\begin{aligned} J_1(x) &:= \int_{-\infty}^{A+} \bar{\mu}(x-u) \mu^{(n-1)*}(du), \\ J_2(x) &:= \int_{A+}^{(x-A)+} \bar{\mu}(x-u) \mu^{(n-1)*}(du) + \bar{\mu}(A) \overline{\mu^{(n-1)*}}(x-A), \\ J_3(x) &:= \int_{A+}^{(x-A)+} \bar{\mu}(x-u) \mu(du) + \bar{\mu}(A) \bar{\mu}(x-A) \end{aligned}$$

and

$$\varepsilon(A) := \limsup_{x \rightarrow \infty} \frac{J_2(x) + J_3(x)}{\overline{\mu^{n*}}(x)}.$$

The following lemma is important for the proofs of Theorems 1.2 and 1.3.

LEMMA 4.1. *Let $\gamma \geq 0$ and let $n \geq 2$. Then we have the following:*

(i) *We have, for $x > nA$,*

$$(4.1) \quad \overline{\mu^{n*}}(x) \leq nJ_1(x) + nJ_2(x)$$

and

$$(4.2) \quad nJ_1(x) - 2^{-1}n(n-3)J_2(x) - 2^{-1}n(n-1)J_3(x) \leq \overline{\mu^{n*}}(x).$$

(ii) *If $\mu^{n*} \in \mathcal{S}(\gamma)$, then $\lim_{A \rightarrow \infty} \varepsilon(A) = 0$, and hence we have*

$$(4.3) \quad \lim_{A \rightarrow \infty} \liminf_{x \rightarrow \infty} \frac{nJ_1(x)}{\overline{\mu^{n*}}(x)} = \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{nJ_1(x)}{\overline{\mu^{n*}}(x)} = 1.$$

PROOF. We have, for $x > nA$,

$$\begin{aligned} \overline{\mu^{n*}}(x) &= P\left(\sum_{j=1}^n X_j > x\right) \\ &\leq \sum_{k=1}^n P\left(X_k > A, \sum_{j=1}^n X_j > x\right) \\ &= nJ_1(x) + nJ_2(x). \end{aligned}$$

Thus, (4.1) of assertion (i) is true. On the other hand, we see that, for $x > nA$,

$$\begin{aligned}
 & P\left(X_1 > A, X_2 > A, \sum_{j=1}^n X_j > x\right) \\
 &= P\left(X_1 > A, X_2 > A, \sum_{j=1}^n X_j > x, \sum_{j=3}^n X_j \geq 0\right) \\
 &\quad + P\left(X_1 > A, X_2 > A, \sum_{j=1}^n X_j > x, \sum_{j=3}^n X_j < 0\right) \\
 &\leq P\left(X_1 > A, \sum_{j=2}^n X_j > A, \sum_{j=1}^n X_j > x\right) \\
 &\quad + P(X_1 > A, X_2 > A, X_1 + X_2 > x) \\
 &= J_2(x) + J_3(x),
 \end{aligned}$$

with the understanding that $\sum_{j=3}^n X_j = 0$ for $n = 2$. By using Bonferroni inequality, we have, for $x > nA$,

$$\begin{aligned}
 & P\left(\sum_{j=1}^n X_j > x\right) \\
 &\geq \sum_{k=1}^n P\left(X_k > A, \sum_{j=1}^n X_j > x\right) \\
 &\quad - \sum_{1 \leq k < l \leq n} P\left(X_k > A, X_l > A, \sum_{j=1}^n X_j > x\right) \\
 &= nJ_1(x) + nJ_2(x) - 2^{-1}n(n-1)P\left(X_1 > A, X_2 > A, \sum_{j=1}^n X_j > x\right) \\
 &\geq nJ_1(x) - 2^{-1}n(n-3)J_2(x) - 2^{-1}n(n-1)J_3(x).
 \end{aligned}$$

Hence, (4.2) of assertion (i) is true. Next, suppose that $\mu^{n*} \in \mathcal{S}(\gamma)$. Let $d := \mu([0, \infty))$. We obtain that, for $x > nA$,

$$\begin{aligned}
 & d^n J_2(x) + d^{2n-2} J_3(x) \\
 &= P\left(X_1 > A, \sum_{j=2}^n X_j > A, \sum_{j=1}^n X_j > x, X_k \geq 0 \text{ for } n+1 \leq k \leq 2n\right) \\
 &\quad + P(X_1 > A, X_2 > A, X_1 + X_2 > x, X_k \geq 0 \text{ for } 3 \leq k \leq 2n)
 \end{aligned}$$

$$\begin{aligned}
&\leq 2P\left(X_1 + \sum_{j=n+2}^{2n} X_j > A, \sum_{j=2}^{n+1} X_j > A, \sum_{j=1}^{2n} X_j > x\right) \\
&= 2 \int_{A+}^{(x-A)+} \overline{\mu^{n*}}(x-u) \mu^{n*}(du) + 2 \overline{\mu^{n*}}(A) \overline{\mu^{n*}}(x-A).
\end{aligned}$$

Note from $\widehat{\mu}(\gamma) < \infty$ that

$$\lim_{A \rightarrow \infty} e^{\gamma A} \overline{\mu^{n*}}(A) \leq \lim_{A \rightarrow \infty} \int_{A+}^{\infty} e^{\gamma x} \mu^{n*}(dx) = 0.$$

Thus, we see from $\mu^{n*} \in \mathcal{S}(\gamma)$ and (i) of Lemma 2.2 that

$$\begin{aligned}
\lim_{A \rightarrow \infty} d^{2n-2} \varepsilon(A) &\leq \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{d^n J_2(x) + d^{2n-2} J_3(x)}{\overline{\mu^{n*}}(x)} \\
&\leq 2 \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{\int_{A+}^{(x-A)+} \overline{\mu^{n*}}(x-u) \mu^{n*}(du)}{\overline{\mu^{n*}}(x)} \\
&\quad + 2 \lim_{A \rightarrow \infty} e^{\gamma A} \overline{\mu^{n*}}(A) = 0.
\end{aligned}$$

Hence, we obtain (4.3) from (4.1) and (4.2). \square

PROOF OF THEOREM 1.2. Let $\gamma \geq 0$, $n \geq 2$, and $A > 0$. We continue to use $J_k(x)$ for $1 \leq k \leq 3$ and $\varepsilon(A)$ defined above. Define C^* and C_* as

$$C^* := \limsup_{x \rightarrow \infty} \frac{\bar{\mu}(x)}{\overline{\mu^{n*}}(x)}, \quad C_* := \liminf_{x \rightarrow \infty} \frac{\bar{\mu}(x)}{\overline{\mu^{n*}}(x)}.$$

Suppose that $e^{\gamma x} \bar{\mu}(x) \in \mathbf{AD}$ and $\mu^{n*} \in \mathcal{S}(\gamma)$. Then we have, for $-A \leq u \leq A$,

$$(4.4) \quad \limsup_{x \rightarrow \infty} \frac{\bar{\mu}(x + A - u)}{\bar{\mu}(x)} \leq e^{\gamma(-A+u)}$$

and

$$(4.5) \quad \liminf_{x \rightarrow \infty} \frac{\bar{\mu}(x - A - u)}{\bar{\mu}(x)} \geq e^{\gamma(A+u)}.$$

Note that, for $u \leq -A$, $\bar{\mu}(x + A - u) \leq \bar{\mu}(x + 2A)$ and, for $-A \leq u \leq A$,

$$\frac{\bar{\mu}(x + A - u)}{\bar{\mu}(x)} \leq 1.$$

Hence, by using Fatou's lemma, we find from (4.4) that

$$\begin{aligned}
 & \limsup_{x \rightarrow \infty} \int_{-\infty}^{A+} \frac{\bar{\mu}(x + A - u)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 & \leq \int_{(-A)+}^{A+} \limsup_{x \rightarrow \infty} \frac{\bar{\mu}(x + A - u)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 & \quad + \int_{-\infty}^{(-A)+} \limsup_{x \rightarrow \infty} \frac{\bar{\mu}(x + 2A)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 & \leq \int_{(-A)+}^{A+} e^{\gamma(-A+u)} \mu^{(n-1)*}(du) + \int_{-\infty}^{(-A)+} e^{-2\gamma A} \mu^{(n-1)*}(du).
 \end{aligned}$$

Note from the assumption $\mu^{n*} \in \mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$ that $\bar{\mu}^{n*}(x + A) \sim e^{-\gamma A} \bar{\mu}^{n*}(x)$ and $\bar{\mu}^{n*}(x - A) \sim e^{\gamma A} \bar{\mu}^{n*}(x)$. Thus, we see from (4.1) of Lemma 4.1 that

$$\begin{aligned}
 1 - n\varepsilon(A) & \leq \liminf_{x \rightarrow \infty} \frac{nJ_1(x)}{\mu^{n*}(x)} \\
 & = \liminf_{x \rightarrow \infty} \frac{nJ_1(x + A)}{\mu^{n*}(x + A)} \\
 & = \liminf_{x \rightarrow \infty} \frac{n\bar{\mu}(x)}{e^{-\gamma A} \bar{\mu}^{n*}(x)} \int_{-\infty}^{A+} \frac{\bar{\mu}(x + A - u)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 & \leq nC_* e^{\gamma A} \limsup_{x \rightarrow \infty} \int_{-\infty}^{A+} \frac{\bar{\mu}(x + A - u)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 (4.6) \quad & \leq nC_* \int_{(-A)+}^{A+} e^{\gamma u} \mu^{(n-1)*}(du) + nC_* \int_{-\infty}^{(-A)+} e^{-\gamma A} \mu^{(n-1)*}(du).
 \end{aligned}$$

By using Fatou's lemma in the last inequality, we obtain from (4.2) of Lemma 4.1 and (4.5) that

$$\begin{aligned}
 1 + 2^{-1}n(n-1)\varepsilon(A) & \geq \limsup_{x \rightarrow \infty} \frac{nJ_1(x)}{\mu^{n*}(x)} \\
 & = \limsup_{x \rightarrow \infty} \frac{nJ_1(x - A)}{\mu^{n*}(x - A)} \\
 & \geq \limsup_{x \rightarrow \infty} \frac{n\bar{\mu}(x)}{e^{\gamma A} \bar{\mu}^{n*}(x)} \int_{(-A)+}^{A+} \frac{\bar{\mu}(x - A - u)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 & \geq nC^* e^{-\gamma A} \liminf_{x \rightarrow \infty} \int_{(-A)+}^{A+} \frac{\bar{\mu}(x - A - u)}{\bar{\mu}(x)} \mu^{(n-1)*}(du) \\
 (4.7) \quad & \geq nC^* \int_{(-A)+}^{A+} e^{\gamma u} \mu^{(n-1)*}(du).
 \end{aligned}$$

As $A \rightarrow \infty$ in (4.6) and (4.7), we have, by (ii) of Lemma 4.1,

$$C^* = C_* = n^{-1} \widehat{\mu}(\gamma)^{1-n}.$$

Hence, we establish that

$$\bar{\mu}(x) \sim n^{-1} \widehat{\mu}(\gamma)^{1-n} \overline{\mu^{n*}}(x).$$

Thus, we conclude from (ii) of Lemma 2.2 that $\mu \in \mathcal{S}(\gamma)$. \square

PROOF OF COROLLARY 1.2. Let $\gamma > 0$. Suppose that $\widehat{\mu}(-\gamma) < \infty$ and $g_c(x) := \mu((x, x+c]) \in \mathbf{AD}$ for every $c > 0$. Let $G(x) := e^{\gamma x} \overline{\mu(-\gamma)}(x)$ for $x \in \mathbb{R}$. Then we have

$$\sum_{k=1}^{\infty} e^{-kc\gamma} g_c(x + (k-1)c) \leq \widehat{\mu}(-\gamma) G(x) \leq e^{c\gamma} \sum_{k=1}^{\infty} e^{-kc\gamma} g_c(x + (k-1)c).$$

Let $a \geq 0$ and $\varepsilon > 0$. Since $g_c(x) \in \mathbf{AD}$, we see that there is $N > 0$ such that, for $y > N$,

$$\frac{g_c(y+a)}{g_c(y)} \leq 1 + \varepsilon.$$

Thus, we obtain that, for $x > N$,

$$\frac{G(x+a)}{G(x)} \leq e^{c\gamma} (1 + \varepsilon).$$

Since ε and c can be arbitrarily small, we have $G(x) \in \mathbf{AD}$. Thus, the proof of the corollary is clear from Theorem 1.2 and Lemma 2.3. \square

5. Proofs of Theorem 1.3 and its corollary. Let Λ_1 be the totality of increasing sequences $\{\lambda_k\}_{k=1}^{\infty}$ with $\lim_{k \rightarrow \infty} \lambda_k = \infty$ such that, for every $x \in \mathbb{R}$, the following $m(x; \{\lambda_k\})$ exists and is finite:

$$(5.1) \quad m(x; \{\lambda_k\}) := \lim_{k \rightarrow \infty} \frac{\overline{\mu}(\lambda_k + x)}{\mu^{n*}(\lambda_k)}.$$

The idea of the use of the function $m(x; \{\lambda_k\})$ goes back to Teugels [20] and is extensively employed in Watanabe and Yamamuro [23].

LEMMA 5.1. Assume that $\mu^{n*} \in \mathcal{S}(\gamma)$ with $\gamma > 0$ for $n \geq 2$. Define $d := \mu([0, \infty))$. For any sequence $\{x_k\}_{k=1}^{\infty}$ with $\lim_{k \rightarrow \infty} x_k = \infty$, there exists a subsequence $\{\lambda_k\} \in \Lambda_1$ of $\{x_k\}$. Moreover, $m(x; \{\lambda_k\})$ is decreasing and finite, and we have

$$(5.2) \quad M(x; \{\lambda_k\}) := e^{\gamma x} m(x; \{\lambda_k\}) \leq d^{1-n}.$$

PROOF. Let $A > 0$ and

$$T_k(y) := \frac{\overline{\mu}(x_k + y)}{\overline{\mu^{n*}}(x_k)}.$$

Then we see that $T_k(y)$ is decreasing and

$$\sup_{y \in [-A, A]} T_k(y) \leq \frac{\overline{\mu}(x_k - A)}{\overline{\mu^{n*}}(x_k)} \leq d^{1-n} \frac{\overline{\mu^{n*}}(x_k - A)}{\overline{\mu^{n*}}(x_k)}$$

and

$$\lim_{k \rightarrow \infty} \frac{\overline{\mu^{n*}}(x_k - A)}{\overline{\mu^{n*}}(x_k)} = e^{\gamma A}.$$

Thus, $T_k(y)$ is uniformly bounded on all finite intervals. By virtue of Helly's selection principle, there exists an increasing subsequence $\{\lambda_k\} \in \Lambda_1$ of $\{x_k\}$. Since $T_k(x)$ is decreasing, so is $m(x; \{\lambda_k\})$. Moreover,

$$M(x; \{\lambda_k\}) \leq d^{1-n} \lim_{k \rightarrow \infty} \frac{e^{\gamma x} \overline{\mu^{n*}}(\lambda_k + x)}{\overline{\mu^{n*}}(\lambda_k)} = d^{1-n}.$$

Thus, the lemma is true. \square

PROOF OF THEOREM 1.3. Suppose that $\mu^{n*} \in \mathcal{S}(\gamma)$ with $\gamma > 0$ for $n \geq 2$ and that $\widehat{\mu}(\gamma) < \infty$ and $\widehat{\mu}(\gamma + iz) \neq 0$ for every $z \in \mathbb{R}$. Let $A > 0$ and $a \in \mathbb{R}$. Define, for $\{\lambda_k\} \in \Lambda_1$,

$$U_k(y) := \frac{\overline{\mu}(\lambda_k + y)}{\overline{\mu^{n*}}(\lambda_k)}.$$

Since $U_k(a - u)$ is uniformly bounded on $(-\infty, A]$, we obtain from the dominated convergence theorem and (4.3) of Lemma 4.1 that

$$\begin{aligned} e^{-\gamma a} &= \lim_{k \rightarrow \infty} \frac{\overline{\mu^{n*}}(\lambda_k + a)}{\overline{\mu^{n*}}(\lambda_k)} = \lim_{A \rightarrow \infty} n \int_{-\infty}^{A+} \lim_{k \rightarrow \infty} U_k(a - u) \mu^{(n-1)*}(du) \\ &= n \int_{-\infty}^{\infty} m(a - u; \{\lambda_k\}) \mu^{(n-1)*}(du). \end{aligned}$$

Thus, we find that, for every $a \in \mathbb{R}$,

$$(5.3) \quad 1 = n \int_{-\infty}^{\infty} M(a - u; \{\lambda_k\}) e^{\gamma u} \mu^{(n-1)*}(du).$$

Hence, we have, for every $a, b \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} (M(a + b - u; \{\lambda_k\}) - M(b - u; \{\lambda_k\})) e^{\gamma u} \mu^{(n-1)*}(du) = 0.$$

Note that, by (5.2), $M(x; \{\lambda_k\})$ is bounded and that the Wiener condition holds for the finite measure $e^{\gamma x} \mu^{(n-1)*}(dx)$, namely, for every $z \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{izx} e^{\gamma x} \mu^{(n-1)*}(dx) = \widehat{\mu}(\gamma + iz)^{n-1} \neq 0.$$

It follows from Lemma 2.6 that, for every $a \in \mathbb{R}$,

$$M(a+b; \{\lambda_k\}) - M(b; \{\lambda_k\}) = 0 \quad \text{for a.e. } b \in \mathbb{R}.$$

Since the function $m(x; \{\lambda_k\})$ is decreasing, the functions $M(x-; \{\lambda_k\})$ and $M(x+; \{\lambda_k\})$ exist for all $x \in \mathbb{R}$. Taking $b_n = b_n(a) \uparrow 0$ and $b_n = b_n(a) \downarrow 0$, we have, for every $a \in \mathbb{R}$,

$$(5.4) \quad M(a-; \{\lambda_k\}) = M(0-; \{\lambda_k\}) \quad \text{and} \quad M(a+; \{\lambda_k\}) = M(0+; \{\lambda_k\}).$$

Then, taking $a \downarrow 0$ in the first equality of (5.4), we have

$$C(\{\lambda_k\}) := M(0+; \{\lambda_k\}) = M(0-; \{\lambda_k\}).$$

Thus, we obtain from (5.4) that, for every $a \in \mathbb{R}$,

$$M(a; \{\lambda_k\}) = M(a-; \{\lambda_k\}) = M(a+; \{\lambda_k\}) = C(\{\lambda_k\}).$$

Therefore, we see from (5.3) that

$$C(\{\lambda_k\}) = \lim_{k \rightarrow \infty} \frac{\overline{\mu}(\lambda_k)}{\overline{\mu}^{n*}(\lambda_k)} = n^{-1} \widehat{\mu}(\gamma)^{1-n},$$

which is independent of the choice of $\{\lambda_k\} \in \Lambda_1$. Thus, we find from Lemma 5.1 that

$$\overline{\mu}(x) \sim n^{-1} \widehat{\mu}(\gamma)^{1-n} \overline{\mu}^{n*}(x)$$

and, by (ii) of Lemma 2.2, $\mu \in \mathcal{S}(\gamma)$. \square

PROOF OF COROLLARY 1.3. If $\widehat{\mu}(-\gamma) < \infty$ for some $\gamma > 0$ and $\mu \in \mathcal{W}$, then $e^{\gamma x} \mu_{(-\gamma)} \in \mathcal{W}$, too. Thus, the proof is obvious from Theorem 1.3 and Lemma 2.3. \square

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