

Continuum percolation in high dimensions

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Abstract. Consider a Boolean model Σ in \mathbb{R}^d . The centers are given by a homogeneous Poisson point process with intensity λ and the radii of distinct balls are i.i.d. with common distribution ν . The critical covered volume is the proportion of space covered by Σ when the intensity λ is critical for percolation. We study the asymptotic behaviour, as d tends to infinity, of the critical covered volume. It appears that, in contrast to what happens in the constant radii case studied by Penrose, geometrical dependencies do not always vanish in high dimension.

Résumé. Considérons un modèle booléen Σ dans \mathbb{R}^d . Les centres des boules sont donnés par un processus ponctuel de Poisson homogène d'intensité λ , et les rayons par une suite de variables aléatoires indépendantes et identiquement distribuées de loi commune ν . Le volume critique recouvert est la proportion de l'espace recouverte par Σ quand on prend pour λ la valeur critique pour la percolation des boules. Nous étudions le comportement asymptotique, quand la dimension d tend vers $+\infty$, de ce volume critique recouvert. En particulier, nous montrons que contrairement à ce qui se passe dans le cas des boules de rayon constant étudié par Penrose, les dépendances liées à la géométrie ne disparaissent pas toujours en grande dimension.

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1. Introduction and statement of the main results

Consider a homogeneous Poisson point process on \mathbb{R}^d . At each point of this process, we center a ball with random radius, the radii of distinct balls being i.i.d. and independent of the point process: the union Σ of these random balls is called a Boolean model. This Boolean model depends on three parameters: the intensity λ of the point process of centers, the common distribution ν of the radii of the balls and the dimension d . We denote by $\lambda_d^c(\nu)$ the critical intensity for percolation in Σ .

In this paper, we mainly focus on distributions concentrating on two distinct radii. For any $\rho \geq 1$ and $d \geq 1$, consider for instance the probability measure

$$\mu_d^\rho = \frac{1}{1 + \rho^{-d}} (\delta_1 + \rho^{-d} \delta_\rho).$$

The ρ^{-d} normalization will be discussed and motivated below Display (15). In our main result, Theorem 1.1, we will give the asymptotic behavior of the critical intensity $\lambda_d^c(\mu_d^\rho)$ as d tends to ∞ .

Penrose studied in [12] the case of constant radii, which can be obtained by taking $\rho = 1$ and we studied in [8] the case $1 < \rho < 2$. In both cases, the asymptotic behavior of $\lambda_d^c(\mu_d^\rho)$ is given by the asymptotic behavior of an associated Galton–Watson process. This is due to the fact that the geometrical dependencies vanish in high dimension.

We prove here that it is not the case for large values of ρ : it appears that when $\rho > 2$ the asymptotic behavior of $\lambda_d^c(\mu_d^\rho)$ is no longer given by the asymptotic behavior of the associated Galton–Watson process. In other words, when $\rho > 2$, geometry still plays a significant role even when the dimension tends to infinity. Moreover, one observes that the asymptotic behavior of $\lambda_d^c(\mu_d^\rho)$ depends in a complex way on the value of ρ .

The Boolean model

Let us give here an equivalent construction of the Boolean model. Let ν be a finite¹ measure on $(0, +\infty)$. We assume that the mass of ν is positive. Let $d \geq 2$ be an integer, $\lambda > 0$ be a real number and ξ be a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ whose intensity measure is the Lebesgue measure on \mathbb{R}^d times $\lambda\nu$. We define a random subset of \mathbb{R}^d as follows:

$$\Sigma(\lambda\nu) = \bigcup_{(c,r) \in \xi} B(c, r),$$

where $B(c, r)$ is the open Euclidean ball centered at $c \in \mathbb{R}^d$ and with radius $r \in (0, +\infty)$. The random subset $\Sigma(\lambda\nu)$ is a Boolean model driven by $\lambda\nu$.

We say that $\Sigma(\lambda\nu)$ percolates if with positive probability the origin belongs to an unbounded connected component of $\Sigma(\lambda\nu)$. This is equivalent to the almost-sure existence of an unbounded connected component of $\Sigma(\lambda\nu)$. We refer to the book by Meester and Roy [9] for background on continuum percolation. The critical intensity is defined by:

$$\lambda_d^c(\nu) = \inf\{\lambda > 0 : \Sigma(\lambda\nu) \text{ percolates}\}.$$

One easily checks that $\lambda_d^c(\nu)$ is finite. In [6] it is proven that $\lambda_d^c(\nu)$ is positive if and only if

$$\int r^d \nu(dr) < +\infty. \tag{1}$$

We assume from now on that this assumption is fulfilled.

By ergodicity, the Boolean model $\Sigma(\lambda\nu)$ has a deterministic natural density. This is also the probability that a given point belongs to the Boolean model and it is given by:

$$P(0 \in \Sigma(\lambda\nu)) = 1 - \exp\left(-\lambda \int \nu_d r^d \nu(dr)\right),$$

where ν_d denotes the volume of the unit euclidean ball in \mathbb{R}^d . The critical covered volume $c_d^c(\nu)$ is the density of the Boolean model when the intensity is critical:

$$c_d^c(\nu) = 1 - \exp\left(-\lambda_d^c(\nu) \int \nu_d r^d \nu(dr)\right).$$

We define the normalized critical intensity as:

$$\tilde{\lambda}_d^c(\nu) = \lambda_d^c(\nu) \int \nu_d (2r)^d \nu(dr).$$

We then have $c_d^c(\nu) = 1 - \exp(-\frac{\tilde{\lambda}_d^c(\nu)}{2^d})$. The factor 2^d may seem arbitrary here: it will simplify the statement of the next theorems.

We will now give two scaling relations which partly justify our preference for c_d^c or $\tilde{\lambda}_d^c$ over λ_d^c . For all $a > 0$, define $H^a \nu$ as the image of ν under the map $x \mapsto ax$. By scaling, we get:

$$\tilde{\lambda}_d^c(H^a \nu) = \tilde{\lambda}_d^c(\nu). \tag{2}$$

¹There is no greater generality in considering finite measures instead of probability measures; this is simply more convenient.

This is a consequence of Proposition 2.11 in [9], and it may become more obvious when considering the two following facts: a critical Boolean model remains critical when rescaling and the density is invariant by rescaling (here by rescaling we mean rescaling centers and radii by the same constant); therefore the critical covered volume and then the normalized threshold are invariant. One also easily checks the following invariance:

$$\tilde{\lambda}_d^c(av) = \tilde{\lambda}_d^c(v). \tag{3}$$

Constant radii

Assume that the measure ν is a Dirac mass at 1, i.e. that the radii of the balls are all equal to 1. Penrose [12] proved the following result:

$$\lim_{d \rightarrow \infty} \tilde{\lambda}_d^c(\delta_1) = 1. \tag{4}$$

The inequality $\tilde{\lambda}_d^c(\delta_1) > 1$ holds for any $d \geq 2$. The proof is simple, and here is the idea. We consider the following natural genealogy. The deterministic ball $B(0, 1)$ is said to be the ball of generation 0. The random balls of $\Sigma(\lambda\delta_1)$ that touch $B(0, 1)$ are then the balls of generation 1. The random balls that touch one ball of generation 1 without being one of them are then the balls of generation 2; balls of generation 3 are those which intersect a ball of generation 2 and are not from generations 1 or 2, and so on. Let us denote by N_d the number of all balls that are descendants of $B(0, 1)$. There is no percolation if and only if N_d is almost surely finite.

Now denote by m the Poisson distribution with mean $\lambda\nu_d 2^d$: this is the law of the number of balls of $\Sigma(\lambda\delta_1)$ that touch a given ball of radius 1. Therefore, if there were no interference between children of different balls, N_d would be equal to Z , the total population in a Galton–Watson process with offspring distribution m . Because of the interferences due to the fact that the Boolean model lives in \mathbb{R}^d , this is not true: in fact, N_d is only stochastically dominated by Z . Therefore, if $\lambda\nu_d 2^d \leq 1$, then Z is finite almost surely, then N_d is almost surely finite and therefore there is no percolation. This implies

$$\tilde{\lambda}_d^c(\delta_1) = \nu_d 2^d \lambda_d^c(\delta_1) > 1.$$

The difficult part of (4) is to prove that if d is large, then the interferences are small, as a consequence N_d is close to Z and therefore there is percolation as soon as $\nu_d 2^d \lambda = \nu$ for a given $\nu > 1$ and d large enough.

To sum up, at first order, the asymptotic behavior of the critical intensity of the Boolean model with constant radius is given by the threshold of an associated Galton–Watson process: roughly speaking, as the dimension increases, the geometrical constraints of the finite dimension space decrease and in the limit, we recover the non-geometrical case of the corresponding Galton–Watson process.

Random radii with two values: A first simple case

Let $\rho > 1$, consider the measure $\mu = \delta_1 + \delta_\rho$, and set, for $d \geq 2$,

$$\mu_d = \delta_1 + \rho^{-d} \delta_\rho.$$

Let us motivate the definition of μ_d with the following two related properties:

- (1) Consider the Boolean model $\Sigma(\lambda\mu_d)$ on \mathbb{R}^d driven by $\lambda\mu_d$ where $\lambda > 0$. The number of balls of $\Sigma(\lambda\mu_d)$ with radius 1 that contains a given point is a Poisson random variable with intensity $\lambda\nu_d$. The number of balls of $\Sigma(\lambda\mu_d)$ with radius ρ that contains a given point is also a Poisson random variable with intensity $\lambda\nu_d$. Thus the introduction of μ_d is done to keep the relative importance of the two types of radii independent of the dimension d .
- (2) Consider two independent Boolean model Σ and Σ' , both driven by $\lambda\delta_1$. Then $\Sigma \cup \rho\Sigma'$ is a Boolean model driven by $\lambda\mu_d$.

In our previous work [8], we proved the following result in the specific case $1 < \rho < 2$:

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln(\tilde{\lambda}_d^c(\mu_d)) = \ln\left(\frac{2\sqrt{\rho}}{1+\rho}\right). \quad (5)$$

In this case, as in the case of deterministic radii, the first order of the asymptotic behavior of the critical intensity in high dimension is given by the threshold of an associated Galton–Watson process, as we will briefly discuss now. Thinking of the asymptotic (5), take $\lambda = \frac{\kappa^d}{v_d 2^d}$ where $\kappa > 0$ is a given constant.

The associated Galton–Watson process is now two-type, one for each radius. Consider the offspring distribution of type ρ of an individual of type 1: it is the number of balls of a Boolean model directed by $\frac{\lambda}{\rho^d} \delta_\rho$ that intersect a given ball of radius 1. Therefore, this is a Poisson random variable with mean $\frac{\lambda}{\rho^d} v_d (1 + \rho)^d$. The other offspring distributions are defined similarly. We moreover assume that the offspring of type 1 and ρ of a given individual are independent. The matrix of means of offspring distributions is thus given by:

$$M_d = \begin{pmatrix} \lambda v_d (1 + 1)^d & \frac{\lambda}{\rho^d} v_d (1 + \rho)^d \\ \lambda v_d (1 + \rho)^d & \frac{\lambda}{\rho^d} v_d (\rho + \rho)^d \end{pmatrix} = \kappa^d \begin{pmatrix} 1 & (\frac{1+\rho}{2})^d \\ (\frac{1+\rho}{2})^d & 1 \end{pmatrix}. \quad (6)$$

Let r_d denote the largest eigenvalue of M_d . The extinction probability of the two-type Galton–Watson process is 1 if and only if $r_d \leq 1$ (see Theorem 2, p. 186 in the book by Athreya and Ney [1]). We have:

$$r_d \sim \left(\frac{\kappa(1+\rho)}{2\sqrt{\rho}}\right)^d, \quad \text{and thus} \quad \kappa = \frac{2\sqrt{\rho}}{1+\rho} \text{ is the critical parameter.}$$

As before, because of the geometric interferences, the number of all balls that are descendants of $B(0, 1)$ in the Boolean model is only stochastically dominated by the total population of the two-types Galton–Watson process and thus if $\kappa \leq \frac{2\sqrt{\rho}}{1+\rho}$, there is no percolation for the Boolean model with intensity $\lambda = \frac{\kappa^d}{v_d 2^d}$. So, for any dimension $d \geq 2$,

$$\frac{1}{d} \ln(\tilde{\lambda}_d^c(\mu_d)) \geq \ln\left(\frac{2\sqrt{\rho}}{1+\rho}\right). \quad (7)$$

Thus (5) says that the comparison with the two-type Galton–Watson process is asymptotically sharp on a logarithmic scale when $1 < \rho \leq 2$. Here again, as the dimension increases, the geometrical constraints of the finite dimension space decrease and in the limit, we recover the non-geometrical case of the corresponding Galton–Watson process. This is no longer the case when $\rho > 2$, as we will see in our main result Theorem 1.1.

Let us now give some heuristics to explain why the behavior is different when $1 \leq \rho < 2$ and when $\rho > 2$. We can see that the main contribution to the Galton–Watson process, in the limit when d tends to ∞ , comes from lines in which the two-types alternate. Indeed, let us keep only those lines. Let us assume that the ancestor is a ball of radius ρ centered at 0. The mean number of grandchildren, in those alternating lines, of a the ancestor (centers of a ball of radius ρ which touches a ball of radius 1 which touches the ancestor) is

$$\frac{\kappa^d (1 + \rho)^d}{(2\rho)^d} \frac{\kappa^d (1 + \rho)^d}{2^d} = \left(\frac{\kappa^2 (1 + \rho)^2}{4\rho}\right)^d. \quad (8)$$

In particular, this is larger than one as soon as

$$\kappa > \frac{2\sqrt{\rho}}{1 + \rho} \quad (9)$$

which corresponds, in the limit when $d \rightarrow \infty$, to the critical threshold of the Galton–Watson process. Let us now consider geometric constraints. Each grandchild (center of a ball of radius ρ which touches a ball of radius 1 which touches the ancestor) is located at $U + V$ where U and V are independent and uniformly distributed on $B(0, 1 + \rho)$. But in high dimension, one typically has $\|U\| \approx 1 + \rho$, $\|V\| \approx 1 + \rho$ and $U \cdot V \approx 0$ (see Lemma 3 in [12] for a close

statement). Therefore, in high dimension, $\|U + V\| \approx \sqrt{2}(1 + \rho)$. Let us develop this heuristic argument by assuming that each grandchild is centered in $B(0, \sqrt{2}(1 + \rho))$. The mean number of grandchildren is then bounded from above by the mean number of balls of radius ρ centered in $B(0, \sqrt{2}(1 + \rho))$, that is

$$\frac{\kappa^d (\sqrt{2}(1 + \rho))^d}{(2\rho)^d} = \left(\frac{\kappa(1 + \rho)}{\sqrt{2}\rho} \right)^d. \tag{10}$$

Considering (9) and comparing (8) and (10), one can guess that the geometrical constraints will be harmless when

$$\frac{\kappa^2(1 + \rho)^2}{4\rho} \leq \frac{\kappa(1 + \rho)}{\sqrt{2}\rho} \quad \text{for some } \kappa > \frac{2\sqrt{\rho}}{1 + \rho}.$$

This happens when $\rho < 2$.

Random radii with two values: General case

To state our main result, we need some further notation. Fix $\rho > 1$ and $k \geq 1$. Set $r_1 = r_{k+1} = 1 + \rho$, and for $i \in \{2, \dots, k\}$, $r_i = 2$. For $(a_i)_{2 \leq i \leq k+1} \in [0, 1)^k$, we build an increasing sequence of distances $(d_i)_{1 \leq i \leq k+1}$ by setting (see Figure 1):

$$\begin{aligned} d_1 &= r_1 = 1 + \rho, \\ \forall i \in \{2, \dots, k + 1\} \quad d_i^2 &= d_{i-1}^2 + 2r_i a_i d_{i-1} + r_i^2. \end{aligned} \tag{11}$$

Note that the sequence $(d_i)_{1 \leq i \leq k+1}$ depends on ρ, k , and the a_i 's. Let then the function \mathcal{D} be defined by the relation

$$\mathcal{D}(a_2, \dots, a_{k+1}) = d_{k+1}.$$

Now set, for every $k \geq 1$,

$$\kappa_\rho^c(k) = \inf_{0 \leq a_2, \dots, a_{k+1} < 1} \max \left(\left(\frac{4\rho}{(1 + \rho)^2 \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}} \right)^{\frac{1}{k+1}}, \frac{2\rho}{\mathcal{D}(a_2, \dots, a_{k+1})} \right). \tag{12}$$

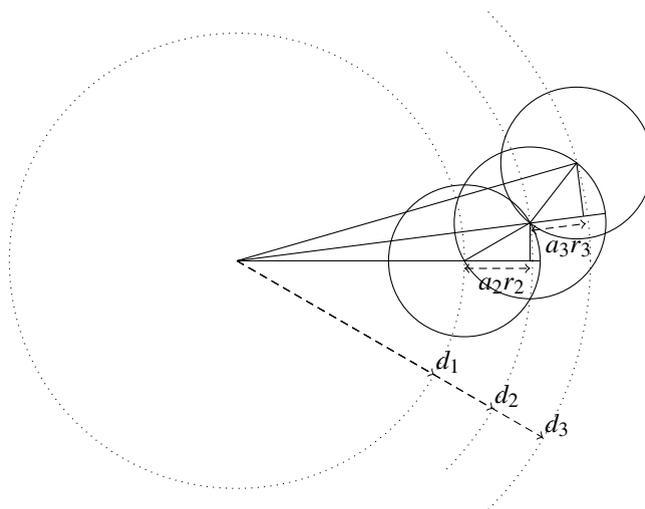


Fig. 1. Dotted circles are of respective radii $d_i, i \in \{1, 2, 3\}$. Circles in plain line are of radius 2.

Finally, let:

$$\kappa_\rho^c = \inf_{k \geq 1} \kappa_\rho^c(k). \tag{13}$$

We give some intuition on κ_ρ^c and the distances d_i in Section 2.2. Our main result says that κ_ρ^c gives the asymptotic behavior of $\tilde{\lambda}_d^c(\mu_d)$ when μ charges two distinct points:

Theorem 1.1. *Let $b > a > 0$, $\alpha > 0$ and $\beta > 0$. Set $\mu = \alpha\delta_a + \beta\delta_b$, $\rho = b/a > 1$ and, for $d \geq 2$, $\mu_d = \alpha a^{-d}\delta_a + \beta b^{-d}\delta_b$. Then*

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln(\tilde{\lambda}_d^c(\mu_d)) = \ln(\kappa_\rho^c) < 0. \tag{14}$$

Note that if one considers μ instead of μ_d , one has² $\tilde{\lambda}_d^c(\alpha\delta_a + \beta\delta_b) \rightarrow 1$ and thus $\tilde{\lambda}_d^c(\alpha\delta_a + \beta\delta_b) \sim \tilde{\lambda}_d^c(\delta_1)$. This behavior is due to the fact that, without normalization, the influence of the small balls vanishes in high dimension.

In the next lemma we collect some properties of the $\kappa_\rho^c(k)$'s and κ_ρ^c . In Figure 2, we plot $\kappa_\rho^c(k)$, for $k \in \{1, 2, 3\}$. The data come from the formulas in Lemma 1.2 for $k = 1$ and from numerical estimations for $k \geq 2$.

Lemma 1.2. *Let $\rho > 1$.*

(i) $0 < \kappa_\rho^c(1) < 1$. *More precisely:*

$$\text{if } 1 < \rho \leq 2 \text{ then } \kappa_\rho^c(1) = \frac{2\sqrt{\rho}}{1 + \rho}, \quad \text{while if } \rho \geq 2 \text{ then } \kappa_\rho^c(1) = \frac{\sqrt{4 + \rho^2}}{1 + \rho}.$$

(ii) $0 < \kappa_\rho^c < 1$.

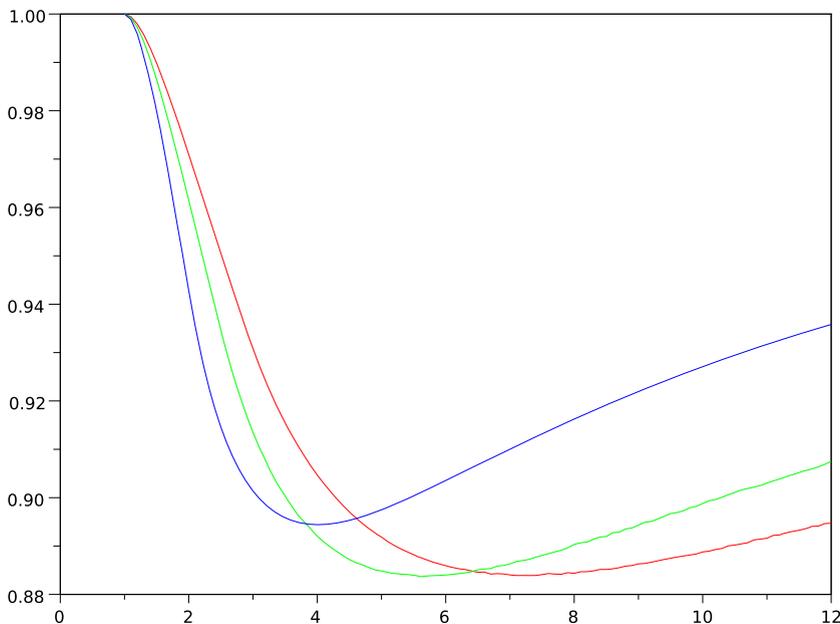


Fig. 2. $\kappa_\rho^c(k)$ as a function of ρ ($k \in \{1, 2, 3\}$, from left to right).

²The upper bound can be proven using $\lambda_d^c(\alpha\delta_a + \beta\delta_b) \leq \lambda_d^c(\beta\delta_b)$. The lower bound can be proven using the easy part of the comparison with a two-type Galton–Watson process.

(iii) There exists $\rho_0 > 2$ such that if $\rho \leq \rho_0$, then $\kappa_\rho^c = \kappa_\rho^c(1)$. This implies:

$$\text{if } 1 < \rho \leq 2 \text{ then } \kappa_\rho^c = \frac{2\sqrt{\rho}}{1 + \rho}, \quad \text{while if } 2 \leq \rho \leq \rho_0 \text{ then } \kappa_\rho^c = \frac{\sqrt{4 + \rho^2}}{1 + \rho}.$$

(iv) As ρ goes to $+\infty$, $\kappa_\rho^c(k) = 1 - \frac{k}{\rho} + o(1/\rho)$. Thus one can not restrict the infimum in (13) to a finite number of k .

As we see in Lemma 1.2(iii), our previous result (5) for $1 < \rho \leq 2$ is a particular case of Theorem 1.1, where κ_ρ has a simple expression obtained by comparison with the associated two-type Galton–Watson process.

But for $\rho > 2$, the same comparison does not give the good value of κ_ρ^c any more: as we can see in (12) and (13), the value of κ_ρ^c depends in a much more complex way of the radii ratio ρ . This expresses that in the case $\rho > 2$, geometrical constraints still play a significant role in percolation properties even in the limit when d tends to ∞ .

The proofs of Theorem 1.1 and of Lemma 1.2 are given in Section 2. The main ideas of the proofs are given in Section 2.2. The general structure of the proof of Theorem 1.1 is the same as the proof of (5) in [8]: the lower bound for κ_ρ^c follows from the comparison with a well-chosen branching process which takes into account some geometrical constraints, while the upper bound is obtained by showing that when $\kappa > \kappa_\rho^c$, we can embed in the Boolean model with intensity $\lambda = \frac{\kappa^d}{v_d 2^d}$ a super-critical 2-dimensional oriented percolation that implies the existence of infinite paths of balls. But let us be more specific and explain the differences between [8] and this work.

In the special case $1 < \rho \leq 2$ that we studied in [8], it appears that the lower bound (7) obtained by comparison with the two-types Galton–Watson process gives the right value for κ_ρ^c . To obtain the upper bound, we prove that if $\kappa > \kappa_\rho^c$ and if the dimension d is large enough, there exists with positive probability an infinite path along which 1-balls alternate with ρ -balls. To do so, we build an embedded oriented percolation: the existence of an open edge is linked to the existence in the Boolean model of a well-positioned path composed of one 1-ball and one ρ -ball. The existence of an infinite open path in the oriented percolation implies the existence of an infinite path of balls in the Boolean model, and we can prove that for d large enough the oriented percolation is supercritical. This comparison with oriented percolation was already the last step in the paper of Penrose [12].

But in the general case, this comparison with the two-type Galton–Watson process is too crude. To study the existence of an infinite path of balls in the Boolean model, we first fix $k \geq 1$ and look at k -alternating paths of balls: a path is k -alternating if it is a path of balls along which balls with radius ρ alternate with sequences of k balls with radius 1 – see (17) for a precise definition. We prove that the critical parameter for the existence of infinite k -alternating paths of balls is $\kappa_\rho^c(k)$, and this is done in two steps. The lower bound for $\kappa_\rho^c(k)$ is, as before, obtained by comparison with a well-chosen branching process. To obtain a sharp bound, we optimize in the positions of the centers of the balls, and this is the role played by the $(a_i)_{2 \leq i \leq k+1}$ in the definition (12) of $\kappa_\rho^c(k)$ – see also Figure 1 and Section 2.2. Then we prove in Proposition 2.7 that if $\kappa > \kappa_\rho^c(k)$ and if the dimension d is large enough, there exists with positive probability an infinite k -alternating path. As before, this step is done by embedding a super-critical 2-dimensional oriented percolation in the Boolean model: an open edge corresponds here to the existence of a well-positioned path composed of one ρ -ball followed by k 1-balls. The proof of the fact that the oriented percolation is supercritical when d is large enough is more intricate as it has to include the optimization in the $(a_i)_{2 \leq i \leq k+1}$. Finally, we deduce the critical parameter κ_ρ^c for the existence of infinite paths of balls in the Boolean model from the critical parameters $\kappa_\rho^c(k)$ for the existence of infinite k -alternating paths of balls.

To summarize, this works provides a proof in whole generality, and the strategy is already present in our previous work [8] on a particularly simple subcase.

General random radii

If μ is a finite measure on $(0, +\infty)$ and if $d \geq 2$ is an integer, we define a measure μ_d on $(0, +\infty)$ by setting:

$$\mu_d(dr) = r^{-d} \mu(dr). \tag{15}$$

Note that, for any d , the assumption (1) is fulfilled by μ_d , and that $(\delta_1)_d = \delta_1$. Note also that μ_d is not necessarily a finite measure. However the definitions of $\lambda_d^c(\mu_d)$, $c_d^c(\mu_d)$ and $\tilde{\lambda}_d^c(\mu_d)$ made above still make sense in this case and we still have $\lambda_d^c(\mu_d) \in (0, +\infty)$ thanks to Theorem 1.1 in [7]. Theorem 1.3 is an easy consequence of Theorem 1.1. Its proof is given in Section 3.

Theorem 1.3. *Let μ be a finite measure on $(0, +\infty)$. We assume that the mass of μ is positive and that μ is not concentrated on a singleton. Then:*

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(\tilde{\lambda}_d^c(\mu_d)) < 0.$$

A straightforward consequence of Theorem 1.3 and (4) – or, actually, of the much weaker and easier convergence of $\ln(\tilde{\lambda}_d^c(\delta_1))$ to 0 – is the following result:

Corollary 1.4. *Let μ be a finite measure on $(0, +\infty)$. We assume that the mass of μ is positive and that μ is not concentrated on a singleton. Then, for any d large enough, we have:*

$$\tilde{\lambda}_d^c(\mu_d) < \tilde{\lambda}_d^c(\delta_1), \quad \text{or equivalently} \quad c_d^c(\mu_d) < c_d^c(\delta_1).$$

In the physical literature, it is strongly believed that, at least when $d = 2$ and $d = 3$, the critical covered volume is minimum in the case of a deterministic radius, when the distribution of radius is a Dirac measure. This conjecture is supported by numerical evidence (to the best of our knowledge, the most accurate estimations are given in a paper by Quintanilla and Ziff [13] when $d = 2$ and in a paper by Consiglio, Baker, Paul and Stanley [3] when $d = 3$). The conjecture was also supported by some heuristic arguments in any dimension (see for example Dhar [4], Balram and Dhar [2] and Meester, Roy and Sarkar [10]). We refer to [8] for more details. The asymptotic (5), combined with (4), disproves the conjecture in high dimension. Corollary 1.4 states that, in some specific sense, the conjecture is generically false in high dimension.

2. The case when the radii take two values

Before proving Theorem 1.1, we begin with the proof of Lemma 1.2:

2.1. *Proof of Lemma 1.2*

(i) By definition, $\kappa_\rho^c(1) = \inf_{0 \leq a < 1} \max(\phi_1(a), \phi_2(a))$, where $\phi_1, \phi_2 : [0, 1) \rightarrow \mathbb{R}$ are defined by:

$$\phi_1(a) = \frac{2\sqrt{\rho}}{(1 + \rho)(1 - a^2)^{1/4}} \quad \text{and} \quad \phi_2(a) = \frac{\rho\sqrt{2}}{(1 + \rho)\sqrt{1 + a}}.$$

If $\rho \leq 2$ then $\phi_1(0) \geq \phi_2(0)$. As ϕ_1 is increasing and ϕ_2 is decreasing, we get:

$$\kappa_\rho^c(1) = \inf_{0 \leq a < 1} \phi_1(a) = \phi_1(0) = \frac{2\sqrt{\rho}}{1 + \rho}.$$

Assume, on the contrary, $\rho \geq 2$. Set

$$a = \frac{\rho^2 - 4}{\rho^2 + 4} \in [0, 1).$$

Then $\phi_1(a) = \phi_2(a)$. As ϕ_1 is increasing and ϕ_2 is decreasing, we get:

$$\kappa_\rho^c(1) = \phi_1(a) = \phi_2(a) = \frac{\sqrt{4 + \rho^2}}{1 + \rho}.$$

(ii) Clearly we have, for every $k \geq 1$:

$$\kappa_\rho^c(k) \geq \inf_{0 \leq a_2, \dots, a_{k+1} < 1} \left(\frac{4\rho}{(1 + \rho)^2 \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}} \right)^{1/(k+1)} = \left(\frac{4\rho}{(1 + \rho)^2} \right)^{1/(k+1)}.$$

Therefore, as $\kappa_\rho^c = \inf_{k \geq 1} \kappa_\rho^c(k)$,

$$0 < \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/2} \leq \kappa_\rho^c \leq \kappa_\rho^c(1) < 1.$$

(iii) The last inequalities imply that $\kappa_\rho^c = \kappa_\rho^c(1)$ if $1 < \rho \leq 2$. In fact,

$$\text{if } \kappa_\rho^c(1) \leq \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/3}, \tag{16}$$

$$\text{then } \forall k \geq 2 \quad \kappa_\rho^c(1) \leq \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/3} \leq \left(\frac{4\rho}{(1+\rho)^2} \right)^{1/(k+1)} \leq \kappa_\rho^c(k),$$

and thus it is true that $\kappa_\rho^c = \kappa_\rho^c(1)$. As the inequality in (16) is strict for $\rho = 2$, we obtain by continuity the existence of $\rho_0 > 2$ such that for every $\rho \in (1, \rho_0)$, $\kappa_\rho^c = \kappa_\rho^c(1)$.

(iv) Fix $k \geq 1$. The lower bound follows easily from the following observation: by construction, we have $\mathcal{D}(a_2, \dots, a_{k+1}) \leq 2(\rho + k)$. This implies

$$\kappa_\rho^c(k) \geq \inf_{a_2, \dots, a_{k+1}} \frac{2\rho}{\mathcal{D}(a_2, \dots, a_{k+1})} \geq \frac{\rho}{\rho + k} = 1 - \frac{k}{\rho} + o(1/\rho).$$

To obtain the upper bound, note that for $1 \leq i \leq k - 1$, we have

$$d_{i+1}^2 = (d_i + 2a_{i+1})^2 + 4(1 - a_{i+1}^2) = (d_i + 2a_{i+1})^2 \left(1 + \frac{4(1 - a_{i+1}^2)}{(d_i + 2a_{i+1})^2} \right)$$

and then

$$d_i + 2a_{i+1} \leq d_{i+1} \leq (d_i + 2a_{i+1}) \left(1 + \frac{2(1 - a_{i+1}^2)}{(d_i + 2a_{i+1})^2} \right) = d_i + 2a_{i+1} + \frac{2(1 - a_{i+1}^2)}{d_i + 2a_{i+1}} \leq d_i + 2a_{i+1} + \frac{2}{1 + \rho}.$$

In the last step, we use the fact that the sequence $(d_j)_j$ is non decreasing and that $d_1 = 1 + \rho$. By summation, we get

$$d_k = d_1 + \sum_{i=1}^{k-1} (d_{i+1} - d_i) = 1 + \rho + 2 \sum_{i=2}^k a_i + R_k, \quad \text{with } 0 \leq R_k \leq \frac{2(k-1)}{1 + \rho}.$$

Take now $\mu > 1/2$ and $\varepsilon > 0$ such that $\mu + (k - 1)\varepsilon < 1$. Take, for $2 \leq i \leq k$, the specific values $a_i = \cos(\rho^{-\varepsilon})$ and $a_{k+1} = \cos(\rho^{-\mu})$. Hence,

$$d_k = (1 + \rho) \left(1 + \frac{2(k-1)}{\rho} + o(\rho^{-1}) \right),$$

$$d_{k+1}^2 = d_k^2 + 2(1 + \rho)d_k \cos(\rho^{-\mu}) + (1 + \rho)^2 = 4(1 + \rho)^2 \left(1 + \frac{2(k-1)}{\rho} + o(\rho^{-1}) \right) \quad \text{and thus}$$

$$\frac{2\rho}{d_{k+1}} = 1 - \frac{k}{\rho} + o(\rho^{-1}).$$

On the other hand, we have

$$\frac{4\rho}{(1 + \rho)^2 \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}} \sim 4\rho^{-1+(k-1)\varepsilon+\mu} = o(1).$$

Finally, $\kappa_\rho^c(k) \leq \max\left(\left(\frac{4\rho}{(1+\rho)^2 \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}} \right)^{\frac{1}{k+1}}, \frac{2\rho}{d_{k+1}} \right) = 1 - \frac{k}{\rho} + o(\rho^{-1})$. This ends the proof. □

2.2. Notations and ideas of the proof of Theorem 1.1

In the whole proof, we fix $\rho > 1$ and $\kappa > 0$.

Once the dimension $d \geq 1$ is given, we consider two independent stationary Poisson point processes on \mathbb{R}^d : χ_1 and χ_ρ , with respective intensities

$$\lambda_1 = \frac{\kappa^d}{v_d 2^d} \quad \text{and} \quad \lambda_\rho = \frac{\kappa^d}{v_d 2^d \rho^d}.$$

With χ_1 and χ_ρ , we respectively associate the two Boolean models

$$\Sigma_1 = \bigcup_{x \in \chi_1} B(x, 1) \quad \text{and} \quad \Sigma_\rho = \bigcup_{x \in \chi_\rho} B(x, \rho).$$

Note that Σ_ρ is an independent copy of $\rho \Sigma_1$. Note also that the expected number of balls of Σ_1 that touches a given ball of radius 1 is κ^d . Thus the expected number of balls of Σ_ρ that touches a given ball of radius ρ is also κ^d .

We focus on the percolation properties of the following two-type Boolean model

$$\Sigma = \Sigma_1 \cup \Sigma_\rho.$$

We begin by studying the existence of infinite k -alternating paths. For $k \geq 1$, an infinite k -alternating path is an infinite path made of balls such that the radius of the first ball is ρ , the radius of the next k balls is 1, the radius of the next ball is ρ and so on. For a fixed $k \geq 1$, we wonder whether infinite k -alternating paths exist and seek the critical threshold for their existence. A natural first step is to study the following quantities:

$$N_0 = \#\{x_1 \in \chi_\rho : \|x_1\| < 2\rho\}, \tag{17}$$

$$\text{and for } k \geq 1, \quad N_k = \#\left\{ \begin{array}{l} x_{k+1} \in \chi_\rho : \exists (x_i)_{1 \leq i \leq k} \in \chi_1 \text{ distinct such that} \\ \|x_1\| < 1 + \rho, \forall i \in \{1, \dots, k-1\} \|x_{i+1} - x_i\| < 2, \\ \|x_{k+1} - x_k\| < 1 + \rho \end{array} \right\}.$$

Fix $k \geq 1$. Remember that $\kappa_\rho^c(k)$ is defined in (12).

A lower bound for $\kappa_\rho^c(k)$

In Section 2.3, we obtain lower bounds for $\kappa_\rho^c(k)$ by looking for upper bounds for $E(N_k)$. On one side, a natural genealogy is associated to the definition of N_k : see the comments below (4). In particular, in this genealogy,

- The ancestor is 0. The children of 0 are the points of $\chi_1 \cap B(0, 1 + \rho)$.
- For any $j \in \{1, \dots, k-1\}$ and any individual x of generation j , the children of x belongs to $\chi_1 \cap B(x, 2)$.
- For any individual x of generation k , the children of x belongs to $\chi_\rho \cap B(x, 1 + \rho)$.

On the other side, the process lives in \mathbb{R}^d and the geometry induces dependences: if x_1 and x'_1 are two individuals of the first generation, their children are a priori dependent. If we forget geometry and only consider genealogy, we get the following upper bound:

$$E(N_k) \leq \lambda_1 |B(\cdot, 1 + \rho)| \left(\prod_{i=2}^k \lambda_1 |B(\cdot, 2)| \right) \lambda_\rho |B(\cdot, 1 + \rho)|,$$

where $|\cdot|$ stands for the volume. But the points of the last generation are in $B(0, 2\rho + 2k)$. So if we forget genealogy and only consider geometry we get the following upper bound:

$$E(N_k) \leq \lambda_\rho |B(0, 2\rho + 2k)|.$$

Expliciting the two previous bounds and combining them together, we get:

$$E(N_k) \leq \min \left(\frac{\kappa^{k+1} (1 + \rho)^2}{4\rho}, \frac{\kappa(\rho + k)}{\rho} \right)^d.$$

In this upper bound, the first argument of the minimum is due to genealogy while the second one is due to geometry. To get the geometrical term, we considered the worst case: the one in which, at each generation i , x_i is as far from the origin as possible. This gives a very poor bound. To get a better bound, we proceed as follows. Fix $a_2, \dots, a_{k+1} \in [0, 1)$. As before, we set $r_1 = r_{k+1} = 1 + \rho$, and for $i \in \{2, \dots, k\}$, $r_i = 2$ and we build the increasing sequence of distances $(d_i)_{1 \leq i \leq k+1}$ as in (11).

Denote by $\tilde{N}_k(a_2, \dots, a_{k+1})$ the number of points $x_{k+1} \in \chi_\rho$ for which there exists a path x_1, \dots, x_k fulfilling the same requirement as for N_k and such that $\|x_i\| \approx d_i$ for all i (a proper definition is provided in the proof of Lemma 2.4). Proceeding as before, we obtain the following upper bound:

$$E(\tilde{N}_k(a_2, \dots, a_{k+1})) \lesssim \min\left(\kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i)^2}, \frac{\kappa \mathcal{D}(a_2, \dots, a_{k+1})}{2\rho}\right)^d.$$

Here again, the first argument of the minimum is due to genealogy while the second one is due to geometry.^{3,4} Optimizing then on the a_i 's, we get:

$$E(N_k) \lesssim \sup_{a_2, \dots, a_{k+1}} \min\left(\kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i)^2}, \frac{\kappa \mathcal{D}(a_2, \dots, a_{k+1})}{2\rho}\right)^d.$$

A precise statement is given in Lemma 2.4. The precise value of the threshold $\kappa_\rho^c(k)$ given in (12) is then the value such that the above upper bound converges to 0 when $\kappa < \kappa_\rho^c(k)$. This heuristic will be developed in Section 2.3: we will prove there that when $\kappa < \kappa_\rho^c(k)$, $E(N_k)$ converges to 0 as d tends to infinity, and this will imply that there exists no infinite k -alternating paths.

An upper bound for $\kappa_\rho^c(k)$

If, on the contrary, $\kappa > \kappa_\rho^c(k)$ then we will prove that $E(N_k)$ does not converge to 0. Actually, to prove that when $\kappa > \kappa_\rho^c(k)$ there exist infinite k -alternating paths, we will show, in Section 2.4, the following stronger property: with a probability that converges to 1 as d tends to infinity, we can find a path which fulfills the requirements of the definition of N_k – or more precisely of $\tilde{N}_k(a_2, \dots, a_{k+1})$ for some a_2, \dots, a_{k+1} nearly optimal – and which fulfills some extra conditions on the positions of the balls. This is Proposition 2.8 and this is the main technical part of this paper. Those extra conditions provide independence properties between the existence of different paths of the same kind. We can then show the existence of many such paths and concatenate some of them to build an infinite k -alternating path. Technically, the last step is achieved by comparing our model with a supercritical oriented percolation process on \mathbb{Z}^2 .

³Genealogical and geometrical constraints are essentially the only constraints. Here is a closely related statement in an easier framework. Let $n \geq 1$ be an integer. Let A_1, \dots, A_n be independent random subsets of an Euclidean sphere \mathcal{S} in \mathbb{R}^d . We assume the existence of a small p such that $P(x \in A_\ell) = p$ for all x in the sphere and all $\ell \in \{1, \dots, n\}$. Let m denote the area on the sphere. Then $E(m(\bigcup_\ell A_\ell)) = m(\mathcal{S})P(x \in \bigcup_\ell A_\ell)$ for any given $x \in \mathcal{S}$. Therefore:

$$E\left(m\left(\bigcup_\ell A_\ell\right)\right) \geq m(\mathcal{S})(nP(x \in A_1) - n(n - 1)P(x \in A_1)P(x \in A_2)) \geq m(\mathcal{S})np(1 - np).$$

Note that $m(\mathcal{S})p = E(m(A_1))$. If $np \leq 1/2$ we get $E(m(\bigcup_\ell A_\ell)) \geq nE(m(A_1))/2$. If $np > 1/2$ we throw away some of the A_i . More precisely, we keep n' of them where n' is the largest integer such that $n'p \leq 1/2$. Using the fact that p is small, we get $E(m(\bigcup_\ell A_\ell)) \geq m(\mathcal{S})n'p/2 \geq m(\mathcal{S})/5$. Therefore, we always have:

$$\min(m(\mathcal{S}), nE(m(A_1)))/5 \leq E\left(m\left(\bigcup_\ell A_\ell\right)\right) \leq \min(m(\mathcal{S}), nE(m(A_1))). \tag{18}$$

In our setup, \mathcal{S} will be the region of the space where we look for the individuals of a given generation i ; the A_ℓ will be the regions of the space where we look for the children of a given individual ℓ of generation $i - 1$. If we multiply by the intensity the inequalities that replace (18) in our setup, we get that the mean number of children at generation i is roughly given by the minimum between geometrical and genealogical constraints.

⁴There is essentially no geometrical constraint in generations 2 to k . Very roughly, this is due to the fact that, when i increases from 1 to k : there is more and more space (the d_i are increasing) and the intensity of the relevant Poisson point process is the same; the expected number of individuals in the i th generation of the Galton–Watson process decreases. In other words, geometrical constraints decrease while genealogical constraints increase.

In this comparison, an open bond in the oriented percolation process corresponds to one of the above paths in our model.

From infinite k-alternating paths to infinite paths

Recall $\kappa_\rho^c = \inf_{k \geq 1} \kappa_\rho^c(k)$. With the previous results, it is rather easy to show that there is no percolation for d large enough as soon as $\kappa < \kappa_\rho^c$. When $\kappa > \kappa_\rho^c$ then $\kappa > \kappa_\rho^c(k)$ for a $k \geq 1$. Therefore there is k -alternating percolation and therefore there is percolation.

2.3. Subcritical phase

Let $\rho > 1$ be fixed. We consider, in \mathbb{R}^d , the two-type Boolean model Σ introduced in Section 2.2, with radii 1 and ρ and respective intensities

$$\lambda_1 = \frac{\kappa^d}{v_d 2^d} \quad \text{and} \quad \lambda_\rho = \frac{\kappa^d}{v_d 2^d \rho^d}$$

depending on some $\kappa \in (0, 1)$. The aim of this subsection is to prove the following proposition:

Proposition 2.1. *Let $\rho > 1$ be fixed. If $\kappa < \kappa_\rho^c$, then, as soon as the dimension d is large enough, percolation does not occur in the two-type Boolean model Σ .*

In the following of this subsection, we fix $\rho > 1$ and $0 < \kappa < \kappa_\rho^c$.

We start with an elementary upper bound, in which we do not take into account the geometrical constraints. We recall that the N_k have been introduced in (17).

Lemma 2.2. *$E(N_0) = \kappa^d$ and, for $k \geq 1$, $E(N_k) \leq (\frac{\kappa^{k+1}(1+\rho)^2}{4\rho})^d$.*

Proof. The result for N_0 follows directly from the equality $E(N_0) = \lambda_\rho |B(0, 2\rho)|$.

Take now $k \geq 1$. We have:

$$E(N_k) \leq \lambda_1 |B(\cdot, 1 + \rho)| \left(\prod_{i=2}^k \lambda_1 |B(\cdot, 2)| \right) \lambda_\rho |B(\cdot, 1 + \rho)|, \tag{19}$$

where $B(\cdot, r)$ stands for a ball with radius r and center unspecified. This can for instance be seen as follows (we use Slivnyak’s theorem, see Proposition 4.1.1 in [11]):

$$\begin{aligned} E(N_k) &\leq E \left(\sum_{x_1, \dots, x_k \in \mathcal{X}_1 \text{ distinct}, x_{k+1} \in \mathcal{X}_\rho} 1_{x_1 \in B(0, 1+\rho)} \cdots 1_{x_{k+1} \in B(x_k, 1+\rho)} \right) \\ &= \lambda_1^k \lambda_\rho \int_{\mathbb{R}^{d(k+1)}} dx_1 \cdots dx_{k+1} 1_{x_1 \in B(0, 1+\rho)} \cdots 1_{x_{k+1} \in B(x_k, 1+\rho)}, \end{aligned}$$

which gives (19). The lemma follows. □

To give a more accurate upper bound for the N_k ’s, we are going to cut the balls into slices and to estimate which slices give the main contribution. For $x \in \mathbb{R}^d \setminus \{0\}$, $0 \leq a < b \leq 1$ and $r > 0$, we now define:

$$\text{if } a > 0: \quad B(x, r, a, b) = \left\{ y \in \mathbb{R}^d : \|y - x\| \leq r \text{ and } ar < \left\langle y - x, \frac{x}{\|x\|} \right\rangle \leq br \right\},$$

$$\text{if } a = 0: \quad B(x, r, 0, b) = \left\{ y \in \mathbb{R}^d : \|y - x\| \leq r \text{ and } \left\langle y - x, \frac{x}{\|x\|} \right\rangle \leq br \right\}.$$

The next lemma gives asymptotics for the volume of these sets:

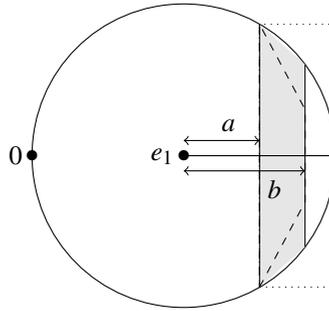


Fig. 3. The ball slice $B(e_1, 1, a, b)$ (in grey) is contained in the cylinder (dotted) and contains the the slice of cone (dashed).

Lemma 2.3. For $x \in \mathbb{R}^d \setminus \{0\}$, $0 \leq a < b \leq 1$ and $r > 0$,

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln \left(\frac{|B(x, r, a, b)|}{v_d} \right) = \ln(r\sqrt{1 - a^2}).$$

Proof. Note that it is sufficient to prove the lemma for $x = e_1$, the first vector of the canonical basis, and $r = 1$.

First, if $a = 0$, the result follows directly from the inequality $v_d/2 \leq |B(e_1, 1, 0, b)| \leq v_d$.

Assume next that $a > 0$. On the one hand, $B(e_1, 1, a, b)$ is included in the cylinder (see Figure 3)

$$\{x = (x_1, \dots, x_d) : x_1 \in [a, 1] \text{ and } \|(0, x_2, \dots, x_d)\| \leq \sqrt{1 - a^2}\},$$

which implies

$$|B(e_1, 1, a, b)| \leq v_{d-1} \sqrt{1 - a^2}^{d-1} (1 - a). \tag{20}$$

On the other end, by convexity, $B(e_1, 1, a, b)$ contains the following slice of cone (see Figure 3):

$$\left\{ x = (x_1, \dots, x_d) : x_1 \in [a, b] \text{ and } \|(0, x_2, \dots, x_d)\| \leq \sqrt{1 - a^2} \frac{1 - x_1}{1 - a} \right\},$$

which implies

$$\frac{v_{d-1} \sqrt{1 - a^2}^{d-1}}{d} ((1 - a) - (1 - b)) \leq |B(e_1, 1, a, b)|. \tag{21}$$

The lemma follows from (20) and (21). □

We can now improve the control given in Lemma 2.2:

Lemma 2.4. For every $k \geq 1$,

$$\begin{aligned} & \limsup_{d \rightarrow \infty} \frac{1}{d} \ln(E(N_k)) \\ & \leq \ln \left(\sup_{0 \leq a_2, \dots, a_{k+1} < 1} \min \left(\kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}, \frac{\kappa \mathcal{D}(a_2, \dots, a_{k+1})}{2\rho} \right) \right) \end{aligned}$$

Proof. • Fix $N \geq 1$. Note that the ball $B(x, r)$ is the disjoint union of the slices $B(x, r, n/N, (n + 1)/N)$ for $n \in \{0, \dots, N - 1\}$. For any $n_2, \dots, n_{k+1} \in \{0, \dots, N - 1\}$, we set

$$a_i = \frac{n_i}{N} \quad \text{and} \quad a_i^+ = \frac{n_i + 1}{N}.$$

We focus on the contribution of a specific product of slices:

$$N_k(n_2, \dots, n_{k+1}) = \# \left\{ \begin{array}{l} x_{k+1} \in \mathcal{X}_\rho : \exists (x_i)_{1 \leq i \leq k} \in \mathcal{X}_1 \text{ distinct with} \\ \|x_1\| < 1 + \rho, \forall i \in \{1, \dots, k-1\} x_{i+1} \in B(x_i, 2, a_{i+1}, a_{i+1}^+), \\ x_{k+1} \in B(x_k, 1 + \rho, a_{k+1}, a_{k+1}^+) \end{array} \right\}.$$

Then we have:

$$N_k \leq \sum N_k(n_2, \dots, n_{k+1}), \tag{22}$$

where the sum is over $(n_2, \dots, n_{k+1}) \in \{0, \dots, N-1\}^k$.

- As we can check that the points contributing to $N_k(n_2, \dots, n_{k+1})$ are in $B(0, \mathcal{D}(a_2^+, \dots, a_{k+1}^+))$, we get:

$$E(N_k(n_2, \dots, n_{k+1})) \leq \lambda_\rho v_d \mathcal{D}(a_2^+, \dots, a_{k+1}^+)^d,$$

this leads to:

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(E(N_k(n_2, \dots, n_{k+1}))) \leq \ln\left(\frac{\kappa \mathcal{D}(a_2^+, \dots, a_{k+1}^+)}{2\rho}\right). \tag{23}$$

- Besides, proceeding as in the proof of Lemma 2.2, we obtain:

$$E(N_k(n_2, \dots, n_{k+1})) \leq \lambda_1 |B(0, 1 + \rho)| \left(\prod_{i=2}^k \lambda_1 |B(\cdot, 2, a_i, a_i^+)| \right) \lambda_\rho |B(\cdot, 1 + \rho, a_{k+1}, a_{k+1}^+)|.$$

With Lemma 2.3, we deduce:

$$\limsup_{d \rightarrow \infty} \frac{1}{d} \ln E(N_k(n_2, \dots, n_{k+1})) \leq \ln\left(\kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}\right). \tag{24}$$

- From (22), (23) and (24) we finally get:

$$\begin{aligned} & \limsup_{d \rightarrow +\infty} \frac{\ln(E(N_k))}{d} \\ & \leq \ln\left(\max_{a_2, \dots, a_{k+1} \in \{0, \dots, \frac{N-1}{N}\}} \min\left(\kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq i \leq k+1} (1 - a_i^2)}, \frac{\kappa \mathcal{D}(a_2^+, \dots, a_{k+1}^+)}{2\rho}\right)\right). \end{aligned}$$

As \mathcal{D} is uniformly continuous on $[0, 1]^k$, we end the proof by taking the limit when N goes to $+\infty$. □

The next step consists in taking into account all $k \geq 0$ simultaneously; we thus introduce

$$N = \# \left(\left\{ \begin{array}{l} y \in \mathcal{X}_\rho : \exists k \geq 1, \exists (x_i)_{1 \leq i \leq k} \in \mathcal{X}_1 \text{ distinct with} \\ \|x_1\| < 1 + \rho, \forall i \in \{1, \dots, k-1\} \|x_{i+1} - x_i\| < 2, \\ \|y - x_k\| < 1 + \rho \end{array} \right\} \cup \{y \in \mathcal{X}_\rho : \|y\| < 2\rho\} \right). \tag{25}$$

Lemma 2.5. *If $\kappa < \kappa_\rho^c$, then $\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(E(N)) < 0$.*

Proof. We have:

$$E(N) \leq \sum_{k \geq 0} E(N_k). \tag{26}$$

As $\kappa < \kappa_\rho^c$, Lemma 2.4 ensures that for every $k \geq 1$:

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(E(N_k)) < 0. \tag{27}$$

Moreover, the assumption $\kappa < \kappa_\rho^c$ also implies, thanks to Lemma 1.2, that $\kappa < 1$. We can then choose k_0 large enough to have:

$$\frac{\kappa^{k_0+1}(1 + \rho)^2}{4\rho} \leq \exp(-1).$$

With Lemma 2.2, we thus get:

$$E(N_0) + \sum_{k \geq k_0} E(N_k) \leq \kappa^d + \exp(-d) \sum_{k \geq 0} \kappa^{kd} = \kappa^d + \exp(-d) \frac{1}{1 - \kappa^d}.$$

With (26) and (27), this ends the proof. □

The next lemma is elementary

Lemma 2.6. *Assume $\kappa < 1$. Then the connected components of $\bigcup_{x \in \mathcal{X}_1} B(x, 1)$ are bounded with probability 1.*

Proof. For any integer $k \geq 0$, denote by M_k the number of balls with radius 1 linked to $B(0, 1)$ by a chain of k distinct balls with radius 1. Proceeding as in the proof of Lemma 2.2, we get:

$$E(M_k) \leq \kappa^{d(k+1)}.$$

Now denote by M the number of balls with radius 1 linked to $B(0, 1)$ by a chain of (perhaps no) balls with radius 1. Then:

$$E(M) \leq E\left(\sum_{k \geq 0} M_k\right) = \frac{\kappa^d}{1 - \kappa^d} < +\infty.$$

Therefore, M is finite with probability 1. So the connected components that touch $B(0, 1)$ are bounded with probability 1. So with probability 1, every connected component is bounded. □

Proof of Proposition 2.1. Remember that we proved in Lemma 1.2 that $\kappa_\rho^c < 1$. Take κ such that $0 < \kappa < \kappa_\rho^c < 1$.

Let ξ_1 be the set of balls with radius ρ that can be connected to $B(0, \rho)$ through a chain of balls with radius 1 (we consider the condition as fulfilled if the ball touches $B(0, \rho)$ directly). Let ξ_2 be the set of random balls with radius ρ that are not in ξ_1 , but that can be connected to $B(0, \rho)$ through a path of random balls in which there is only one ball with radius ρ . Let ξ_3 be the set of random balls with radius ρ that are not in ξ_1 nor in ξ_2 , but that can be connected to $B(0, \rho)$ through a path of random balls in which there is two distinct balls with radius ρ . We define similarly ξ_4, ξ_5 and so on and denote by ξ the disjoint union of all these sets.

We have $\#\xi_1 = N$. (Remember that N has been defined in (25).) By Lemma 2.5, we have:

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(E(\#\xi_1)) < 0.$$

Take some $\mu > 0$ and assume from now on that d is large enough to have

$$\frac{1}{d} \ln(E(\#\xi_1)) \leq -\mu.$$

For every $k \geq 1$, we have

$$E(\#\xi_k) \leq (E(\#\xi_1))^k \leq \exp(-dk\mu). \tag{28}$$

This can be proven as follows. For $x_0, x_1 \in \mathbb{R}^d$, write $x_0 \leftrightarrow x_1$ if there exists $n \geq 0$ and $c_1, \dots, c_n \in \chi_1$ such that, in the sequence $B(x_0, \rho), B(c_1, 1), B(c_2, 1), \dots, B(c_n, 1), B(x_1, \rho)$, each ball touches the next one. Remember that χ_1 and χ_ρ are two independent Poisson point processes, introduced in Section 2.2. For $x_0, x_1, \dots, x_k \in \mathbb{R}^d$ we denote by

$$\{x_0 \leftrightarrow x_1\} \square \{x_1 \leftrightarrow x_2\} \square \dots \square \{x_{k-1} \leftrightarrow x_k\}$$

the following event: each of the k properties $x_i \leftrightarrow x_{i+1}$ can be realized using different points $c \in \chi_1$. We have

$$\begin{aligned} E(\#\xi_k) &\leq E\left(\sum_{x_1, \dots, x_k \in \chi_\rho \text{ distinct}} 1_{\{0 \leftrightarrow x_1\} \square \{x_1 \leftrightarrow x_2\} \square \dots \square \{x_{k-1} \leftrightarrow x_k\}}\right) \\ &= E_\rho\left(\sum_{x_1, \dots, x_k \in \chi_\rho \text{ distinct}} P_1(\{0 \leftrightarrow x_1\} \square \{x_1 \leftrightarrow x_2\} \square \dots \square \{x_{k-1} \leftrightarrow x_k\})\right). \end{aligned}$$

Here, P_1 denotes the law of χ_1 and E_ρ the expectation with respect to the law of χ_ρ : the last equality follows from the independence of χ_1 and χ_ρ . Iterating BK inequality [14]⁵ to χ_1 and then using stationarity we get, for any $x_1, \dots, x_k \in \mathbb{R}^d$,

$$\begin{aligned} &P_1(\{x_0 \leftrightarrow x_1\} \square \{x_1 \leftrightarrow x_2\} \square \dots \square \{x_{k-1} \leftrightarrow x_k\}) \\ &\leq P_1(0 \leftrightarrow x_1) P_1(x_1 \leftrightarrow x_2) \dots P_1(x_{k-1} \leftrightarrow x_k) \\ &= P_1(0 \leftrightarrow x_1) P_1(0 \leftrightarrow x_2 - x_1) \dots P_1(0 \leftrightarrow x_k - x_{k-1}). \end{aligned}$$

Thus,

$$E(\#\xi_k) \leq E_\rho\left(\sum_{x_1, \dots, x_k \in \chi_\rho \text{ distinct}} P_1(0 \leftrightarrow x_1) P_1(0 \leftrightarrow x_2 - x_1) \dots P_1(0 \leftrightarrow x_k - x_{k-1})\right).$$

Slivnyak equality, applied to χ_ρ , then yields

$$\begin{aligned} E(\#\xi_k) &\leq \lambda_\rho^k \int_{(\mathbb{R}^d)^k} P_1(0 \leftrightarrow x_1) P_1(0 \leftrightarrow x_2 - x_1) \dots P_1(0 \leftrightarrow x_k - x_{k-1}) dx_1 \dots dx_k \\ &\leq \left(\lambda_\rho \int_{\mathbb{R}^d} P_1(0 \leftrightarrow x) dx\right)^k = E(\#\xi_1)^k. \end{aligned}$$

This proves (28). As $\xi = \bigcup_{k \geq 1} \xi_k$, we deduce now from (28) that ξ is finite with probability 1. So if an unbounded connected component of $\Sigma_1 \cup \Sigma_\rho$ touches $B(0, \rho)$ then there is an unbounded component in Σ_1 . As $\kappa < 1$, Lemma 2.6 rules out the possibility of an unbounded connected component in Σ_1 . So with probability 1, the connected components of $\Sigma_1 \cup \Sigma_\rho$ that touch $B(0, \rho)$ are bounded, which ends the proof. \square

2.4. Supercritical phase

We fix here $\rho > 1$. We consider once again the two-type Boolean model Σ introduced in Section 2.2 and we fix an integer $k \geq 1$.

For every $n \geq 0$, we set $R_n = \rho$ if $k + 1$ divides n and $R_n = 1$ otherwise. We say that percolation by k -alternation occurs if there exists an infinite sequence of distinct points $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that, for every $n \geq 0$:

- $x_n \in \chi_{R_n}$.
- $B(x_n, R_n) \cap B(x_{n+1}, R_{n+1}) \neq \emptyset$.

⁵In [14] the result is stated for events depending on a point process in a bounded set. To apply the result to our setting, we can restrict our events to the ball $B(0, n)$ by requiring that the centers of the balls of radius 1 belongs to $B(0, n)$, apply BK inequality and then take the limit as $n \rightarrow \infty$.

In other words, percolation by k -alternation occurs if there exists an infinite path along which k balls of radius 1 alternate with one ball of radius ρ , i.e. if there exists an infinite k -alternating path. The aim of this subsection is to prove the following proposition:

Proposition 2.7. *Let $\rho > 1$ and $k \geq 1$ be fixed. Assume that $\kappa \in (\kappa_\rho^c(k), 1)$. If the dimension d is large enough, then percolation by k -alternation occurs with probability one.*

As announced in Section 2.2, percolation by k -alternation of the two-type Boolean model in the supercritical case will be proved by embedding in the model a supercritical 2-dimensional oriented percolation process.

We thus specify the two first coordinates, and introduce the following notations. When $d \geq 3$, for any $x \in \mathbb{R}^d$, we write

$$x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{d-2}.$$

We write $B'(c, r)$ for the open Euclidean balls of \mathbb{R}^2 with center $c \in \mathbb{R}^2$ and radius $r > 0$. In the same way we denote by $B''(c, r)$ the open Euclidean balls of \mathbb{R}^{d-2} with center $c \in \mathbb{R}^{d-2}$ and radius $r > 0$.

2.4.1. *One step in the 2-dimensional oriented percolation model*

The point here is to define the event that will govern the opening of the edges in the 2-dimensional oriented percolation process: it is naturally linked to the existence of a finite path composed of k balls of radius 1 and a ball of radius ρ , whose positions of centers are specified.

We define, for a given dimension d , the two following subsets of \mathbb{R}^d :

$$W = d^{-1/2}((-1, 1) \times (-1, 0) \times \mathbb{R}^{d-2}),$$

$$W^+ = d^{-1/2}((0, 1) \times (0, 1) \times \mathbb{R}^{d-2}).$$

For $x \in W$ we set:

$$\mathcal{G}_k^+(x) = \left\{ \begin{array}{l} \text{There exist distinct } x_1, \dots, x_k \in \chi_1 \cap W^+ \text{ and } x_{k+1} \in \chi_\rho \cap W^+ \\ \text{such that } x, x_1, \dots, x_{k+1} \text{ is a path} \end{array} \right\}. \tag{29}$$

Our goal here is to prove that the probability of occurrence of this event is asymptotically large:

Proposition 2.8. *Let $\rho > 1$ and $k \geq 1$ be fixed. Assume that $\kappa \in (\kappa_\rho^c(k), 1)$ and choose $p \in (0, 1)$. If the dimension d is large enough, then for every $x \in W$,*

$$P(\mathcal{G}_k^+(x)) \geq p.$$

Note already that by translation invariance, $P(\mathcal{G}_k^+(x))$ does not depend on x'' , so we can assume without loss of generality that $x'' = 0$. In the sequel of this subsection, $\rho > 1$ and $k \geq 1$ are fixed.

Remember the definitions of the $(d_i)_{1 \leq i \leq k+1}$ and of $\kappa_\rho^c(k)$ we give in the [Introduction](#) (see Figure 1). The first step consists in choosing a nearly optimal sequence $(a_i)_{2 \leq i \leq k+1} \in [0, 1)^k$ that satisfies some extra inequalities:

Lemma 2.9. *We can choose $(a_i)_{2 \leq i \leq k+1} \in [0, 1)^k$ such that:*

$$1 < \kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq j \leq k+1} (1 - a_j^2)} < \kappa \frac{d_{k+1}}{2\rho}. \tag{30}$$

Proof. As $\kappa_\rho^c(k) < \kappa$, we can choose $(a_i^*)_{2 \leq i \leq k+1} \in (0, 1)^k$ such that the two following conditions

$$\kappa > \left(\frac{4\rho}{(1 + \rho)^2 \sqrt{1 - a_2^2} \cdots \sqrt{1 - a_{k+1}^2}} \right)^{\frac{1}{k+1}}, \tag{31}$$

$$\kappa > \frac{2\rho}{d_{k+1}} \tag{32}$$

are fulfilled for $(a_i)_i = (a_i^*)_i$. We fix $(a_3, \dots, a_{k+1}) = (a_3^*, \dots, a_{k+1}^*)$. Note that

$$f : a_2 \mapsto \kappa \frac{d_{k+1}}{2\rho} \text{ is continuous and increasing,}$$

$$g : a_2 \mapsto \kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq j \leq k+1} (1 - a_j^2)} \text{ is continuous and decreasing,}$$

and that $\lim_{a_2 \rightarrow 1} g(a_2) = 0$. Moreover, Conditions (32) and (31) ensure that $f(a_2^*) > 1$ and $g(a_2^*) > 1$.

Thus if $f(a_2^*) > g(a_2^*)$ the proof is over. If $f(a_2^*) \leq g(a_2^*)$, we can take $a_2 > a_2^*$ such that $1 < g(a_2) < f(a_2^*)$: then $f(a_2) \geq f(a_2^*) > g(a_2) > 1$ and the lemma is proved. \square

Note that (30) implies (31) and (32).

As explained in Section 2.3, the main contribution to the number N_k of centers x_{k+1} of balls of radius ρ that are linked to a ball of radius ρ centered at the origin by a chain $(x_i)_{1 \leq i \leq k}$ of k balls of radius 1 – see the precise definition (17) – is obtained for $\|x_i\| \sim d_i$, where the d_i 's are build from a (nearly) optimal sequence $(a_i)_{2 \leq i \leq k+1} \in [0, 1)^k$. So we fix a nearly optimal family $(a_i)_{2 \leq i \leq k+1} \in (0, 1)^k$ satisfying (30), we build the associated family of distances $(d_i)_{1 \leq i \leq k+1} \in (0, 1)^k$ and we are going to look for a good sequence of centers $(x_i)_{1 \leq i \leq k+1} \in (0, 1)^k$ with $\|x_i\| \sim d_i$.

We thus introduce the following subsets of \mathbb{R}^2 :

$$D'_0 = (-d^{-1/2}, d^{-1/2}) \times (-d^{-1/2}, 0),$$

$$\forall i \in \{1, \dots, k + 1\} \quad D'_i = (0, d^{-1/2})^2,$$

and the following sets in \mathbb{R}^{d-2}

$$C''_0 = \{0\},$$

$$\forall i \in \{1, \dots, k + 1\} \quad C''_i = B''(0, d_i - 2d^{-1}) \setminus B''(0, d_i - 3d^{-1}).$$

Finally, for $i \in \{0, \dots, k + 1\}$, we set $C_i = D'_i \times C''_i$. Note that for d large enough, these sets are disjoint. The next lemma controls the asymptotics in the dimension d of the volume of these sets

Lemma 2.10. *For every $i \in \{1, \dots, k + 1\}$:*

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln \frac{|C''_i|}{v_{d-2}} = \lim_{d \rightarrow +\infty} \frac{1}{d} \ln \frac{|C_i|}{v_d} = \ln d_i.$$

Proof. This can be proven by elementary computations. \square

Each x_i will be taken in C_i , but we also have to ensure that the $(x_i)_{1 \leq i \leq k+1}$ form a path. Note that for $i \in \{2, \dots, k + 1\}$, we have $0 < d_{i-1} + a_i r_i < d_i$, which legitimates the following definition. See also Figure 1. For $i \in \{2, \dots, k + 1\}$ and d large enough, we denote by θ_i the unique real number in $(0, \pi/2)$ such that

$$\cos \theta_i = \frac{d_{i-1} + a_i r_i}{d_i} + d^{-1/2}.$$

We introduce next, for $y = (y', y'') \in C_{i-1}$, the following subset of \mathbb{R}^{d-2} :

$$D''_i(y'') = \{z'' \in C''_i : \langle z'', y'' \rangle \geq \|y''\| \cdot \|z''\| \cdot \cos \theta_i\}.$$

We also set $D''_0 = C''_0$ and $D''_1(y'') = C''_1$ for every $y \in C_0$. Finally, we define for every $i \in \{1, \dots, k + 1\}$ and $y \in C_{i-1}$:

$$D_i(y) = D'_i \times D''_i(y'') \subset C_i,$$

and $D_0 = D'_0 \times D''_0$.

Lemma 2.11. • *If the dimension d is large enough, for every $i \in \{1, \dots, k + 1\}$ and $y \in C_{i-1}$,*

$$D_i(y) \subset B(y, r_i) \cap C_i.$$

• *Let $x_0 \in D_0$. If there exist $X_1, \dots, X_k \in \chi_1$ and $X_{k+1} \in \chi_\rho$ such that $X_1 \in D_1(x_0), X_2 \in D_2(X_1), \dots, X_{k+1} \in D_{k+1}(X_k)$, then the event $\mathcal{G}_+(x_0)$ occurs.*

Proof. • The inclusion $D_i(y) \subset C_i$ is clear for every $i \in \{1, \dots, k + 1\}$. Let $i \in \{2, \dots, k + 1\}$, $y \in C_{i-1}$ and $z \in D_i(y)$. Then, as soon as d is large enough,

$$\begin{aligned} \|z - y\|^2 &= \|z' - y'\|^2 + \|z'' - y''\|^2 \\ &\leq \frac{2}{d} + \|y''\|^2 + \|z''\|^2 - 2\langle y'', z'' \rangle \\ &\leq \frac{2}{d} + (d_{i-1} - 2d^{-1})^2 + (d_i - 2d^{-1})^2 - 2(d_{i-1} - 3d^{-1})(d_i - 3d^{-1}) \cos \theta_i \\ &\leq d_i^2 + d_{i-1}^2 - 2d_{i-1}(d_{i-1} + a_i r_i) - 2d^{-1/2} d_i d_{i-1} + O_i(d^{-1}) \\ &\leq r_i^2 - 2d^{-1/2} d_i d_{i-1} + O_i(d^{-1}) \leq r_i^2. \end{aligned}$$

Let now $y \in C_0$ and $z \in D_1(y)$. As $d_1 = 1 + \rho = r_1 > 2$, we obtain, for d large enough:

$$\|z - y\|^2 = \|z' - y'\|^2 + \|z'' - y''\|^2 \leq \frac{8}{d} + (d_1 - 2d^{-1})^2 \leq r_1^2.$$

• The second point is a simple consequence of the first point, of the fact that the sets $D_i(x_{i-1})$, as the sets C_i , are disjoint and of the definition of the event $\mathcal{G}_+(x_0)$. □

Note that for $i \in \{1, \dots, k + 1\}$, $|D_i(y)|$ and $|D''_i(y'')|$ do not depend on the choice of $y \in C_{i-1}$. We thus denote by $|D_i|$ and $|D''_i|$ these values. We now give asymptotic estimates for these values:

Lemma 2.12. *For every $i \in \{2, \dots, k + 1\}$,*

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln \frac{|D''_i|}{v_{d-2}} = \lim_{d \rightarrow +\infty} \frac{1}{d} \ln \frac{|D_i|}{v_d} = \ln(r_i \sqrt{1 - a_i^2}).$$

Proof. We have, by homogeneity and isotropy:

$$|D''_i| = ((d_i - 2d^{-1})^{d-2} - (d_i - 3d^{-1})^{d-2})|S|, \tag{33}$$

where $S = \{x = (x_1, \dots, x_{d-2}) \in B''(0, 1) : x_1 \geq \|x\| \cos(\theta_i)\}$. But, as illustrated on Figure 4, S is included in the cylinder

$$\{(x_i)_{1 \leq i \leq d-2} \in \mathbb{R}^{d-2} : x_1 \in [0, 1], \|(x_2, \dots, x_{d-2})\| \leq \sin(\theta_i)\}$$

and S contains the cone

$$\{(x_i)_{1 \leq i \leq d-2} \in \mathbb{R}^{d-2} : x_1 \in [0, \cos(\theta_i)], \|(x_2, \dots, x_{d-2})\| \leq x_1 \sin(\theta_i) \cos(\theta_i)^{-1}\}.$$

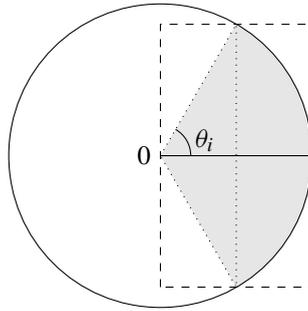


Fig. 4. In light grey, the portion S of the unit ball $B''(0, 1) \subset \mathbb{R}^{d-2}$ is included in the cylinder (dashed) and contains the cone (dotted).

Therefore:

$$v_{d-3} \cos(\theta_i) \sin(\theta_i)^{d-3} (d-2)^{-1} \leq |S| \leq v_{d-3} \sin(\theta_i)^{d-3}. \tag{34}$$

From (33), (34), and the limits $\cos(\theta_i) \rightarrow (d_{i-1} + a_i r_i) d_i^{-1} \neq 0$ and $d_i \sin(\theta_i) \rightarrow r_i \sqrt{1 - a_i^2}$, we get

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln \left(\frac{|D_i''|}{v_{d-2}} \right) = \ln \left(r_i \sqrt{1 - a_i^2} \right).$$

The lemma follows. Note that a direct calculation with spherical coordinates can also give the announced estimates. \square

Everything is now in place to prove Proposition 2.8.

Proof of Proposition 2.8. Choose $p < 1$ and $x \in W$ such that $x'' = 0$.

• Remember that by construction, for d large enough, the C_i 's are disjoint. We start with a single individual, encoded by its position $\zeta_0 = \{x\} \subset C_0$, and we build, generation by generation, its offspring: we set, for $1 \leq i \leq k$,

$$\zeta_i = \chi_1 \cap \bigcup_{y \in \zeta_{i-1}} D_i(y) \subset C_i,$$

and for the $(k + 1)$ th generation, we finally set

$$\zeta_{k+1} = \chi_\rho \cap \bigcup_{y \in \zeta_k} D_{k+1}(y) \subset C_{k+1}.$$

By Lemma 2.11, if $\zeta_{k+1} \neq \emptyset$ then the event $\mathcal{G}_k^+(x)$ occurs, at least for d large enough. To bound from below the probability that $\zeta_{k+1} \neq \emptyset$, we now build a simpler process ξ , stochastically dominated by ζ .

• Before building the simpler process ξ , let us explain a possible way to generate ζ . Let $X_0 = x$ be the position of the first individual. We set $\alpha_i = \lambda_1 |D_i|$ for $i \in \{1, \dots, k\}$ and $\alpha_{k+1} = \lambda_\rho |D_{k+1}|$: thus, α_i is the mean number of children of a point of the $(i - 1)$ th generation. Let $(N_i^j)_{i \geq 1, j \geq 1}$ be independent Poisson variables such that N_i^j has parameter α_i . We now proceed generation by generation. First we throw N_1^1 random point $(X_1^j)_{1 \leq j \leq N_1^1}$ uniformly in $D_1(X_0)$ to obtain the first generation $\zeta_1 \subset C_1$. Then if we have obtain the $(i - 1)$ th generation $\zeta_{i-1} = (X_{i-1}^j)_{1 \leq j \leq M_{i-1}} \subset C_{i-1}$ for some $i \leq k + 1$, we throw, for each $j \leq M_{i-1}$, N_i^j random points uniformly in $D_i(X_{i-1}^j) \subset C_i$, we remove the ones that fell in $\bigcup_{k \leq j-1} D_i(X_{i-1}^k)$ and we order all these points according to some deterministic rule to obtain the i th generation $\zeta_i = (X_i^j)_{1 \leq j \leq M_i} \subset C_i$.

• The idea to obtain the simpler process ξ is roughly to keep, from generation 1 on, only the first child of each individual. To do so, we will generate $N = N_1^1$ independent branches of descendance for x , and then we will perform a rejection procedure to take into account the geometric constraints.

Let us now do the construction in detail. Consider a random vector $X = (X_0, X_1, \dots, X_{k+1})$ of points in \mathbb{R}^d defined as follows: X_0 is defined by $X_0 = x$, X_1 is taken uniformly in $D_1(X_0)$, then X_2 is taken uniformly in $D_2(X_1)$, and so on. We think of X as a potential single branch of descendance of x . Let then $(X^j)_{j \geq 1}$ be independent copies of X , independent of N . The $(X^j)_{j \leq N}$ represent the N potential branches of descendance of x . Note that X_0 's progeny $\tilde{\xi}_1 = (X^j)_{j \leq N}$ is distributed as ζ_1 , while, for the moment, the j th individual X_i^j in generation i , with $1 \leq i \leq k$, has exactly one child X_{i+1}^j .

Having in mind that in ζ , it is possible for an individual to have no child, we now perform a first decimation of our process. Let $Y = (Y_i^j)_{2 \leq i \leq k+1, j \geq 1}$ be a family, independent of the previous random variables, of independent random variables, such that Y_i^j follows the Bernoulli law with parameter $1 - \exp(-\alpha_i)$, which is the probability that a Poisson random variable with parameter α_i is different from 0. For $j \leq N$ and $2 \leq i \leq k + 1$, we keep the individual X_i^j if and only if $Y_2^j = \dots = Y_i^j = 1$. We thus set $J_1 = \{1, \dots, N\}$ and, for every $i \in \{2, \dots, k + 1\}$:

$$J_i = \{1 \leq j \leq N : Y_2^j = \dots = Y_i^j = 1\} \quad \text{and} \quad \tilde{\xi}_i = \{X_i^j, j \in J_i\}.$$

Until now, we did not take into account the geometrical constraints between individuals. For every $i \in \{2, \dots, k + 1\}$ and every $j \geq 1$, we set

$$Z_i^j = 1 \quad \text{if } X_i^j \notin \bigcup_{j' \in J_{i-1} \setminus \{j\}} D_i(X_{i-1}^{j'}) \quad \text{and} \quad Z_i^j = 0 \quad \text{otherwise;}$$

$$\xi_i = \{X_i^j, j \in J_i : Z_2^j = \dots = Z_i^j = 1\}.$$

We thus reject an individual X_i^j and its descendance as soon as $Z_i^j = 0$. Recall that, when building generation i from generation $i - 1$, we explore the Poisson point processes in the area $\bigcup_{j \in J_{i-1}} D_i(X_{i-1}^j) \subset C_i$. By construction of the C_i 's, these areas are disjoint for different generations. Therefore, one can check that, for every $i \in \{2, \dots, k + 1\}$, ξ_i is stochastically dominated by ζ_i . Thus to prove Proposition 2.8, we now need to bound from below the probability that ξ_{k+1} is not empty.

• Let T be the smallest integer j such that $Y_2^j = \dots = Y_{k+1}^j = 1$: in other words, T is the smallest exponent of a branch that lives till generation $k + 1$. To ensure that $\xi_{k+1} \neq \emptyset$, it is sufficient that $T \leq N$ and that $Z_2^T = \dots = Z_{k+1}^T = 1$. So:

$$\begin{aligned} 1 - P(\mathcal{G}_k^+(x)) &\leq P(\xi_{k+1} = \emptyset) \\ &\leq P(\#J_{k+1} = 0) + P\left(\{T \leq N\} \cap \bigcup_{2 \leq i \leq k+1} \{Z_i^T = 0\}\right) \\ &\leq P(\#J_{k+1} = 0) + \sum_{2 \leq i \leq k+1} P(T \leq N \text{ and } Z_i^T = 0). \end{aligned}$$

For every $2 \leq i \leq k + 1$, we have by construction:

$$\begin{aligned} P(T \leq N \text{ and } Z_i^T = 0) &= P(T \leq N, \exists j \in J_{i-1} \setminus \{T\} \text{ such that } X_i^T \in D_i(X_{i-1}^j)) \\ &\leq \sum_{j \geq 1} P(T \leq N \text{ and } j \in J_{i-1} \setminus \{T\} \text{ and } X_i^T \in D_i(X_{i-1}^j)) \\ &= \sum_{j \geq 1} E(1_{T \leq N} 1_{j \in J_{i-1} \setminus \{T\}} P(X_i^T \in D_i(X_{i-1}^j) | Y, N)) \\ &= \sum_{j \geq 1} E(1_{T \leq N} 1_{j \in J_{i-1} \setminus \{T\}} P(X_i^1 \in D_i(X_{i-1}^2))) \\ &\leq E(\#J_{i-1}) P(X_i^1 \in D_i(X_{i-1}^2)). \end{aligned}$$

Conditionally to X_{i-1} , X_i is uniformly distributed on $D_i(X_{i-1}) = D'_i \times D''_i(X''_{i-1})$. Therefore, conditionally to X''_{i-1} , X_i is uniformly distributed on $D''_i(X''_{i-1})$ which is the intersection of C''_i and of a cone of given aperture and axis directed by X''_{i-1} . But, by isotropy of the model, the distribution of X''_{i-1} is isotropic. As a consequence, X''_i is uniformly distributed on C''_i .⁶ So $(X^1_i)''$ is uniformly distributed on C''_i and is independent of $(X^2_{i-1})''$, which leads to

$$P(X^1_i \in D_i(X^2_{i-1})) = P((X^1_i)'' \in D''_i((X^2_{i-1})'')) = \frac{|D''_i|}{|C''_i|}. \tag{35}$$

This leads to

$$1 - P(\mathcal{G}_k^+(x)) \leq P(\#J_{k+1} = 0) + \sum_{i=2}^{k+1} E(\#J_{i-1}) \frac{|D''_i|}{|C''_i|}. \tag{36}$$

- For $1 \leq i \leq k + 1$, the cardinality of J_i follows a Poisson law with parameter

$$\eta_i = \alpha_1 \prod_{i'=2}^i (1 - \exp(-\alpha_{i'})).$$

Remember that $\alpha_i = \lambda_1 |D_i|$ for $i \in \{1, \dots, k\}$ and $\alpha_{k+1} = \lambda_\rho |D_{k+1}|$. By Lemmas 2.10 and 2.12, we have the following limits:

$$\begin{aligned} \lim_{d \rightarrow +\infty} \frac{1}{d} \ln \alpha_1 &= \ln \frac{\kappa(1 + \rho)}{2} > 0, \\ \lim_{d \rightarrow +\infty} \frac{1}{d} \ln \alpha_i &= \ln(\kappa \sqrt{1 - a_i^2}) < 0 \quad \text{for } 2 \leq i \leq k, \\ \lim_{d \rightarrow +\infty} \frac{1}{d} \ln \alpha_{k+1} &= \ln\left(\kappa \sqrt{1 - a_{k+1}^2} \frac{1 + \rho}{2\rho}\right) < 0. \end{aligned}$$

To see the signs of the limits, note that $\kappa < 1$, that $\frac{1+\rho}{2\rho} < 1$ and that (30) implies that

$$\kappa > \kappa^{k+1} > \frac{4\rho}{(1 + \rho)^2 \sqrt{1 - a_2^2} \cdots \sqrt{1 - a_{k+1}^2}} > \frac{2}{1 + \rho}.$$

Consequently, we first see that

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln(\eta_{k+1}) = \lim_{d \rightarrow +\infty} \frac{1}{d} \ln(\alpha_1 \cdots \alpha_{k+1}) = \ln\left(\kappa^{k+1} \frac{(1 + \rho)^2}{4\rho} \sqrt{\prod_{2 \leq j \leq k+1} (1 - a_j^2)}\right) > 0 \quad \text{with (30);}$$

therefore, $\lim_{d \rightarrow +\infty} P(\#J_{k+1} = 0) = 0. \tag{37}$

⁶For any Borel map $\phi : \mathbb{R}^{d-2} \rightarrow \mathbb{R}_+$ we have

$$\begin{aligned} E(\phi(X''_i)) &= \int_{\mathbb{R}^d} dP_{X''_{i-1}}(x''_{i-1}) \int_{\mathbb{R}^d} dx''_i \frac{1}{|D''_i|} 1_{D''_i(x''_{i-1})}(x''_i) \phi(x''_i) \\ &= \int_{\mathbb{R}^d} dP_{X''_{i-1}}(x''_{i-1}) \int_{C''_i} dx''_i \frac{1}{|D''_i|} 1_{\langle x''_i, x''_{i-1} \rangle \geq \|x''_i\| \|x''_{i-1}\| \cos(\theta_i)} \phi(x''_i) \\ &= \int_{C''_i} dx''_i \phi(x''_i) \int_{\mathbb{R}^d} dP_{X''_{i-1}}(x''_{i-1}) \frac{1}{|D''_i|} 1_{\langle x''_i, x''_{i-1} \rangle \geq \|x''_i\| \|x''_{i-1}\| \cos(\theta_i)}. \end{aligned}$$

By isotropy of $P_{X''_{i-1}}$, the inner integral does not depend on x''_i . This constant can only be $|C''_i|^{-1}$. This implies the result.

Similarly, for $2 \leq i \leq k + 1$, we have

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln(\eta_{i-1}) = \ln\left(\kappa^{i-1} \frac{1+\rho}{2} \sqrt{\prod_{2 \leq i' \leq i-1} (1-a_{i'}^2)}\right).$$

Lemmas 2.10 and 2.12 ensure that:

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln\left(\frac{|D_i''|}{|C_i''|}\right) = \ln\left(\frac{r_i \sqrt{1-a_i^2}}{d_i}\right).$$

Thus, from the two previous inequalities, we see that for $2 \leq i \leq k + 1$, we have:

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln\left(E(\#J_{i-1}) \frac{|D_i''|}{|C_i''|}\right) = \ln\left(\frac{r_i(1+\rho)\kappa^{i-1}}{2d_i} \sqrt{\prod_{2 \leq i' \leq i} (1-a_{i'}^2)}\right).$$

Now,

$$\text{for } 2 \leq i \leq k, \quad \limsup_{d \rightarrow +\infty} \frac{1}{d} \ln\left(E(\#J_{i-1}) \frac{|D_i''|}{|C_i''|}\right) \leq \ln\left(\frac{1+\rho}{d_i}\right) < 0, \tag{38}$$

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln\left(E(\#J_k) \frac{|D_{k+1}''|}{|C_{k+1}''|}\right) \leq \ln\left(\frac{(1+\rho)^2 \kappa^k}{2d_{k+1}} \sqrt{\prod_{2 \leq i' \leq k+1} (1-a_{i'}^2)}\right) < 0 \tag{39}$$

with (30). To end the proof, we put estimates (37), (38) and (39) in (36). □

2.4.2. *Several steps in the 2-dimensional oriented percolation model*

We prove here Proposition 2.7 by building the supercritical 2-dimensional oriented percolation process embedded in the two-type Boolean Model.

Proof of Proposition 2.7. We first define an oriented graph in the following manner: the set of sites is

$$S = \{(a, n) \in \mathbb{Z} \times \mathbb{N} : |a| \leq n, a + n \text{ is even}\};$$

from any point $(a, n) \in S$, we put an oriented edge from (a, n) to $(a + 1, n + 1)$, and an oriented edge from (a, n) to $(a - 1, n + 1)$. We denote by $\vec{p}_c(2) \in (0, 1)$ the critical parameter for Bernoulli percolation on this oriented graph – see Durrett [5] for results on oriented percolation in dimension 2.

For any $(a, n) \in S$, we define the following subsets of \mathbb{R}^d

$$\begin{aligned} W_{a,n} &= d^{-1/2} (]a - 1, a + 1[\times]n - 1, n[\times \mathbb{R}^{d-2}), \\ W_{a,n}^- &= d^{-1/2} (]a - 1, a[\times]n, n + 1[\times \mathbb{R}^{d-2}), \\ W_{a,n}^+ &= d^{-1/2} (]a, a + 1[\times]n, n + 1[\times \mathbb{R}^{d-2}). \end{aligned}$$

Note that the $(W_{a,n})_{(a,n) \in S}$ are disjoint and that $W_{a,n}^+ \cup W_{a+2,n}^- \subset W_{a+1,n+1}$.

We now fix $k \geq 1$ and $\kappa \in (\kappa_\rho^c(k), 1)$, and for $x_0 \in W_{a,n}$, we introduce the events:

$$\begin{aligned} \mathcal{G}_{a,n}^+(x_0) &= \left\{ \begin{array}{l} \text{There exist distinct } x_1, \dots, x_k \in \chi_1 \cap W_{a,n}^+ \text{ and } x_{k+1} \in \chi_\rho \cap W_{a,n}^+ \\ \text{such that } x_0, x_1, \dots, x_{k+1} \text{ is a path} \end{array} \right\}, \\ \mathcal{G}_{a,n}^-(x_0) &= \left\{ \begin{array}{l} \text{There exist distinct } x_1, \dots, x_k \in \chi_1 \cap W_{a,n}^- \text{ and } x_{k+1} \in \chi_\rho \cap W_{a,n}^- \\ \text{such that } x_0, x_1, \dots, x_{k+1} \text{ is a path} \end{array} \right\}. \end{aligned}$$

Note that $\mathcal{G}_{0,0}^+(x)$ is exactly the event $\mathcal{G}^+(x)$ introduced in (29), and that the other events are obtained from this one by symmetry and/or translation.

Next we choose $p \in (\bar{p}_c(2), 1)$. With Proposition 2.8, and by translation and symmetry invariance, we know that for every large enough dimension d , for every $(a, n) \in S$, for every $x \in W_{a,n}$:

$$P(\mathcal{G}_{a,n}^\pm(x)) \geq p. \tag{40}$$

We fix then a dimension d large enough to satisfy (40). We can now construct the random states, open or closed, of the edges of our oriented graph. We denote by ∞ a virtual site.

Definition of the site on level 0. Almost surely, $\chi_\rho \cap W_{0,0} \neq \emptyset$. We take then some $x(0, 0) \in \chi_\rho \cap W_{0,0}$.

Definition of the edges between levels n and $n + 1$. Fix $n \geq 0$ and assume we built a site $x(a, n) \in W_{a,n} \cup \{\infty\}$ for every a such that $(a, n) \in S$. Consider $(a, n) \in S$:

- If $x(a, n) = \infty$: we decide that each of the two edges starting from (a, n) is open with probability p and closed with probability $1 - p$, independently of everything else; we set $z^-(a, n) = z^+(a, n) = \infty$.
- Otherwise:
 - Edge to the left-hand side:
 - * if the event $\mathcal{G}_{a,n}^-(x(a, n))$ occurs: we take for $z^-(a, n)$ some point $x_{k+1} \in \chi_\rho \cap W_{a,n}^- \subset W_{a-1,n+1}$ given by the occurrence of the event, and we open the edge from (a, n) to $(a - 1, n + 1)$;
 - * otherwise: we set $z^-(a, n) = \infty$ and we close the edge from (a, n) to $(a - 1, n + 1)$.
 - Edge to the right-hand side:
 - * if the event $\mathcal{G}_{a,n}^+(x(a, n))$ occurs: we take for $z^+(a, n)$ some point $x_{k+1} \in \chi_\rho \cap W_{a,n}^+ \subset W_{a+1,n+1}$ given by the occurrence of the event, and we open the edge from (a, n) to $(a + 1, n + 1)$;
 - * otherwise: we set $z^+(a, n) = \infty$ and we close the edge from (a, n) to $(a + 1, n + 1)$.

For (a, n) outside S , we set $z^\pm(a, n) = \infty$.

Definition of the sites at level $n + 1$. Fix $n \geq 0$ and assume we determined the state of every edge between levels n and $n + 1$. Consider $(a, n + 1) \in S$:

- If $z^+(a - 1, n) \neq \infty$: set $x(a, n + 1) = z^+(a - 1, n) \in W_{a,n+1}$.
- Otherwise:
 - if $z^-(a + 1, n) \neq \infty$: set $x(a, n + 1) = z^-(a + 1, n) \in W_{a,n+1}$;
 - otherwise: set $x(a, n + 1) = \infty$.

Assume that there exists an open path of length n starting from the origin in this oriented percolation: we can check that the leftmost open path of length n starting from the origin gives a path in the two-type Boolean model with n alternating sequences of k balls with radius 1 and one ball with radius ρ . Thus, percolation in this oriented percolation model implies percolation by k -alternation in the two-type Boolean model. Let us check that percolation occurs indeed with positive probability.

For every n , denote by \mathcal{F}_n the σ -field generated by the restrictions of the Poisson point processes χ_1 and χ_ρ to the set

$$d^{-1/2}(\mathbb{R} \times (-\infty, n) \times \mathbb{R}^{d-2}).$$

By definition of the events \mathcal{G} – remember that the $(W_{a,n})_{(a,n) \in S}$ are disjoint – and by (40), the states of the different edges between levels n and $n + 1$ are independent given \mathcal{F}_n . Moreover, given \mathcal{F}_n , the edges between levels n and $n + 1$ has a probability at least p to be open. Therefore, the oriented percolation model we built stochastically dominates Bernoulli oriented percolation with parameter p . As $p > \bar{p}_c(\mathbb{Z}^2)$, with positive probability, there exists an infinite open path in the oriented percolation model we built; this ends the proof of Proposition 2.7. \square

2.5. Proof of Theorem 1.1

We first prove how Propositions 2.1 and 2.7 give (14) when $a = 1, b > 1$ and $\alpha = \beta = 1$, and then we see how we can deduce the general case by scaling and coupling.

When $a = 1$, $b > 1$ and $\alpha = \beta = 1$

Set $\rho = b$. In this case, $v = \delta_1 + \delta_\rho$, so $v_d = \delta_1 + \frac{1}{\rho^d} \delta_\rho$.

Note then that the two-type Boolean model Σ introduced in Section 2.2 and whose intensities depend on $\kappa \in (0, 1)$ coincides with the Boolean model directed by the measure

$$\frac{\kappa^d}{v_d 2^d} v_d$$

as defined in the [Introduction](#).

If $\kappa < \kappa_\rho^c$ then, by Proposition 2.1, there is no percolation for d large enough. Therefore, for any such κ and for any large enough d we have:

$$\lambda_d^c(v_d) \geq \frac{\kappa^d}{v_d 2^d} \quad \text{and then} \quad \tilde{\lambda}_d^c(v_d) = \lambda_d^c(v_d) v_d 2^d \int r^d v_d(dr) \geq 2\kappa^d.$$

Letting d goes to $+\infty$ and then κ goes to κ_ρ^c , we then obtain

$$\liminf_{d \rightarrow +\infty} \frac{1}{d} \ln(\lambda_d^c(v_d)) \geq \ln(\kappa_\rho^c). \quad (41)$$

As $\kappa_\rho^c < 1$ by Lemma 1.2, choose now κ such that $\kappa_\rho^c < \kappa < 1$. Then, there exists $k \geq 1$ such that $\kappa_\rho^c(k) < \kappa$. Therefore, by Proposition 2.7, there is percolation for d large enough in Σ ; by coupling, this remains true for larger κ . Therefore, for any $\kappa > \kappa_\rho^c$ and for any large enough d we have, as before:

$$\lambda_d^c(v_d) \leq \frac{\kappa^d}{v_d 2^d} \quad \text{and then} \quad \tilde{\lambda}_d^c(v_d) \leq 2\kappa^d.$$

Letting d goes to $+\infty$ and then κ goes to κ_ρ^c , we then obtain

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(\lambda_d^c(v_d)) \leq \ln(\kappa_\rho^c). \quad (42)$$

Bringing (41) and (42) together, we get (14) when $a = 1$, $b = \rho > 1$ and $\alpha = \beta = 1$.

When $b > a > 0$ and $\alpha = \beta = 1$

Set $\rho = b/a$. Here, $v = \delta_a + \delta_b$; set $\mu = \delta_1 + \delta_\rho$. With the notation of the [Introduction](#),

$$v_d = \frac{1}{a^d} \left(\delta_a + \frac{1}{\rho^d} \delta_b \right) = \frac{1}{a^d} H^a \left(\delta_1 + \frac{1}{\rho^d} \delta_\rho \right) = \frac{1}{a^d} H^a \mu_d.$$

By the scaling relations (2) and (3), we obtain

$$\tilde{\lambda}_d^c(v_d) = \tilde{\lambda}_d^c(\mu_d).$$

The result when $b > a > 0$ and $\alpha = \beta = 1$ follows then from the previous case.

When $b > a > 0$ and $\alpha, \beta > 0$

Here $v = \alpha \delta_a + \beta \delta_b$. Set $\mu = \delta_a + \delta_b$, $m = \min(\alpha, \beta)$ and $M = \max(\alpha, \beta)$. Then $m\mu_d \leq v_d \leq M\mu_d$ and so

$$m \int r^d d\mu_d(r) \leq \int r^d dv_d(r) \leq M \int r^d d\mu_d(r).$$

Moreover, once again by coupling,

$$\frac{1}{M} \lambda_d^c(\mu_d) = \lambda_d^c(M\mu_d) \leq \lambda_d^c(v_d) \leq \lambda_d^c(m\mu_d) = \frac{1}{m} \lambda_d^c(\mu_d).$$

The two previous inequalities give:

$$\frac{m}{M} \tilde{\lambda}_d^c(\mu_d) \leq \tilde{\lambda}_d^c(v_d) \leq \frac{M}{m} \tilde{\lambda}_d^c(\mu_d),$$

and the theorem follows from the previous case. □

3. Proof of Theorem 1.3

Theorem 1.3 follows from Theorem 1.1 by coupling and scaling. By assumption, μ is a measure on $(0, +\infty)$ whose support is not a singleton. We can therefore choose $b' > a' > 0$ in the support, set $\rho = b'/a'$ and then take a small enough $\varepsilon > 0$ such that

$$\begin{aligned} a'(1 + \varepsilon) < b'(1 - \varepsilon), \quad \mu([a'(1 - \varepsilon), a'(1 + \varepsilon)]) > 0, \\ \mu([b'(1 - \varepsilon), b'(1 + \varepsilon)]) > 0, \quad (1 + \varepsilon)(1 - \varepsilon)^{-1} \kappa_\rho^c < 1. \end{aligned}$$

Set $a = a'(1 - \varepsilon)$, $b = b'(1 - \varepsilon)$ and $\tau = (1 + \varepsilon)(1 - \varepsilon)^{-1} > 1$. We have

$$a\tau < b, \quad \mu([a, a\tau]) > 0, \quad \mu([b, b\tau]) > 0 \quad \text{and} \quad \tau \kappa_\rho^c < 1.$$

Set $\nu = \mu([a, a\tau])\delta_a + \mu([b, b\tau])\delta_b$ and $S = [a, a\tau] \cup [b, b\tau]$. For all $d \geq 1$ we have

$$\tau^{-d} \nu_d(\{a\}) = \mu([a, a\tau]) (a\tau)^{-d} \leq \int_{[a, a\tau]} r^{-d} \mu(dr) = \mu_d([a, a\tau])$$

and, similarly, $\tau^{-d} \nu_d(\{b\}) \leq \mu_d([b, b\tau])$. By coupling, this implies that $\lambda_d^c(1_S \mu_d) \leq \lambda_d^c(\tau^{-d} \nu_d)$, and then that

$$\lambda_d^c(\mu_d) \leq \lambda_d^c(1_S \mu_d) \leq \lambda_d^c(\tau^{-d} \nu_d) = \tau^d \lambda_d^c(\nu_d).$$

But $\tilde{\lambda}_d^c(\mu_d) = \lambda_d^c(\mu_d) 2^d \nu_d \mu((0, +\infty))$ and, similarly, $\tilde{\lambda}_d^c(\nu_d) = \lambda_d^c(\nu) 2^d \nu_d \nu((0, +\infty))$, which leads to

$$\tilde{\lambda}_d^c(\mu_d) \leq \tau^d \frac{\mu((0, +\infty))}{\nu((0, +\infty))} \tilde{\lambda}_d^c(\nu_d).$$

But by Theorem 1.1 we have

$$\lim_{d \rightarrow +\infty} \frac{1}{d} \ln(\tilde{\lambda}_d^c(\nu_d)) = \ln(\kappa_\rho^c), \quad \text{and then} \quad \limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(\tilde{\lambda}_d^c(\mu_d)) \leq \ln(\tau \kappa_\rho^c) < 0,$$

which ends the proof. □

Note that as a byproduct of the proof, we obtain the following upper bound:

$$\limsup_{d \rightarrow +\infty} \frac{1}{d} \ln(\tilde{\lambda}_d^c(\mu_d)) \leq \inf_{0 < a < b < +\infty, a, b \in \text{Supp}(\mu)} \ln(\kappa_{b/a}^c).$$

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References

- [1] K. B. Athreya and P. E. Ney. *Branching Processes. Die Grundlehren der mathematischen Wissenschaften* **196** Springer-Verlag, New York, 1972. [MR0373040](#)
- [2] A. Balram and D. Dhar. Scaling relation for determining the critical threshold for continuum percolation of overlapping discs of two sizes. *Pramāna* **74** (2010) 109–114.
- [3] R. Consiglio, D. R. Baker, G. Paul and H. E. Stanley. Continuum percolation thresholds for mixtures of spheres of different sizes. *Phys. A* **319** (2003) 49–55.
- [4] D. Dhar. On the critical density for continuum percolation of spheres of variable radii. *Phys. A* **242** (3–4) (1997) 341–346.
- [5] R. Durrett. Oriented percolation in two dimensions. *Ann. Probab.* **12** (4) (1984) 999–1040. [MR757768](#)
- [6] J.-B. Gouéré. Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Probab.* **36** (4) (2008) 1209–1220. [MR2435847](#)
- [7] J.-B. Gouéré. Subcritical regimes in some models of continuum percolation. *Ann. Appl. Probab.* **19** (4) (2009) 1292–1318. [MR2538071](#)
- [8] J.-B. Gouéré and R. Marchand. Nonoptimality of constant radii in high dimensional continuum percolation. *Ann. Probab.* **44** (1) (2016) 307–323. [MR3456339](#)
- [9] R. Meester and R. Roy. *Continuum Percolation. Cambridge Tracts in Mathematics* **119**. Cambridge University Press, Cambridge, 1996. [MR1409145](#)
- [10] R. Meester, R. Roy and A. Sarkar. Nonuniversality and continuity of the critical covered volume fraction in continuum percolation. *J. Stat. Phys.* **75** (1–2) (1994) 123–134. [MR1273055](#)
- [11] J. Møller. *Lectures on Random Voronoï Tessellations. Lecture Notes in Statistics* **87**. Springer-Verlag, New York, 1994. [MR1295245](#)
- [12] M. D. Penrose. Continuum percolation and Euclidean minimal spanning trees in high dimensions. *Ann. Appl. Probab.* **6** (2) (1996) 528–544. [MR1398056](#)
- [13] J. A. Quintanilla and R. M. Ziff. Asymmetry in the percolation thresholds of fully penetrable disks with two different radii. *Phys. Rev. E* **76** (5) (2007) 051115.
- [14] J. van den Berg. A note on disjoint-occurrence inequalities for marked Poisson point processes. *J. Appl. Probab.* **33** (2) (1996) 420–426. [MR1385351](#)