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A MEAN-FIELD STOCHASTIC CONTROL PROBLEM WITH PARTIAL OBSERVATIONS

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The authors would like to dedicate this paper to Professor Hans-Jürgen Engelbert, on the occasion of his 70th birthday, for his generous guidance and inspirational discussions throughout the past decades.

In this paper, we are interested in a new type of *mean-field*, non-Markovian stochastic control problems with partial observations. More precisely, we assume that the coefficients of the controlled dynamics depend not only on the paths of the state, but also on the conditional law of the state, given the observation to date. Our problem is strongly motivated by the recent study of the mean field games and the related McKean–Vlasov stochastic control problem, but with added aspects of path-dependence and partial observation. We shall first investigate the well-posedness of the state-observation dynamics, with combined reference probability measure arguments in nonlinear filtering theory and the Schauder fixed-point theorem. We then study the stochastic control problem with a partially observable system in which the conditional law appears nonlinearly in both the coefficients of the system and cost function. As a consequence, the control problem is intrinsically "time-inconsistent", and we prove that the Pontryagin stochastic maximum principle holds in this case and characterize the adjoint equations, which turn out to be a new form of mean-field type BSDEs.

1. Introduction. In this paper, we are interested in the following *mean-field-type* stochastic control problem, on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0})$:

$$
\begin{cases} dX_t = \mathbb{E}\{b(t, \varphi_{\cdot\wedge t}, \mathbb{E}[X_t|\mathcal{G}_t], u)\}|_{\varphi=X,u=u_t}\,dt \\ \qquad\qquad + \mathbb{E}\{\sigma(t, \varphi_{\cdot\wedge t}, \mathbb{E}[X_t|\mathcal{G}_t], u)\}|_{\varphi=X,u=u_t}\,dB_t, \\ X_0 = x, \end{cases} \tag{1.1}
$$

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where B is an $\mathbb{F}$-Brownian motion, b and σ are measurable functions satisfying reasonable conditions, $\varphi_{\cdot\wedge t}$ and $X_{\cdot\wedge t}$ denote the continuous function and process, respectively, "stopped" at t; $\mathbb{G} \stackrel{\triangle}{=} \{\mathcal{G}_t\}_{t\ge 0}$ is a given filtration that could involve the information of X itself, and $u = \{u_t : t \ge 0\}$ is the "control process", assumed to be adapted to a filtration $\mathbb{H} = \{\mathcal{H}_t\}_{t\ge 0}$, where $\mathcal{H}_t \subseteq \mathcal{F}_t^X \vee \mathcal{G}_t$, $t \ge 0$. We note that if $\mathcal{G}_t = \{\varnothing, \Omega\}$, for all $t \ge 0$ [i.e., the conditional expectation in (1.1) becomes expectation], $\mathcal{H}_t = \mathcal{F}_t^X$, and coefficients are "Markovian" (i.e., $\varphi_{\cdot\wedge t} = \varphi_t$), then the problem becomes a stochastic control problem with McKean–Vlasov dynamics and/or a Mean-field game (see, e.g., [5, 7, 9] in its "forward" form, and [2–4] in its "backward" form). On the other hand, when $\mathbb{G}$ is a given filtration, this is the so-called *conditional mean-field SDE* (CMFSDE for short) studied in [8]. We note that in that case the conditioning is essentially "open-looped".

The problem that this paper is particularly focusing on is when $\mathcal{G}_t = \mathcal{F}_t^Y$, $t \ge 0$, where Y is an "observation process" of the dynamics of X, that is, the case when the pair (X, Y) forms a "close-looped" or "coupled" CMFSDE. More precisely, we shall consider the following partially observed controlled dynamics (assuming $b = 0$ for notational simplicity):

$$\begin{cases} dX_t = \mathbb{E}\{\sigma(t, \varphi_{\cdot\wedge t}, \mathbb{E}[X_t|\mathcal{F}_t^Y], u)\}|_{\varphi=X,u=u_t}\, dB_t^1; \\ dY_t = h(t, X_t)\, dt + \hat{\sigma}\, dB_t^2; \qquad X_0 = x, Y_0 = 0. \end{cases} \tag{1.2}$$

Here, X is the "signal" process that can only be observed through Y, (B^1, B^2) is a standard Brownian motion and $\hat{\sigma}$ is a constant. We should note that in SDEs (1.2) the conditioning filtration $\mathbb{F}^Y$ now depends on X itself, therefore, it is much more convoluted than the CMFSDE we have seen in the literature. Furthermore, the path-dependent nature of the coefficients makes the SDE essentially *non-Markovian*. Such form of CMFSDEs, to the best of our knowledge, has not been explored fully in the literature.

Our study of the CMFSDE (1.2) is strongly motivated by the following variation of the mean-field game in a finance context, which would result in a type of stochastic control problem involving a controlled dynamics of such a form. Consider a firm whose *fundamental value*, under the risk neutral measure $\mathbb{P}^0$ with zero interest, evolves as the following SDE with "stochastic volatility" $\sigma = \sigma(t, \omega)$, $(t, \omega) \in [0, \infty) \times \Omega$:

$$X_t = x + \int_0^t \sigma(s, \cdot)\, dB_s^1, \qquad t \ge 0, \tag{1.3}$$

where B^1 is the intrinsic noise from inside the firm. We assume that such fundamental value process cannot be observed directly, but can be observed through a stochastic dynamics (e.g., its stock value) via an SDE:

$$Y_t = \int_0^t h(s, X_s)\, ds + B_t^2, \qquad t \ge 0, \tag{1.4}$$

where B^2 is the noise from the market, which we assume is independent of B^1 (this is by no means necessary, we can certainly consider the filtering problem with correlated noises).

Now let us assume that the volatility σ in (1.3) is affected by the actions of a large number of investors, and all can only make decisions based on the information from the process Y. Therefore, similar to [7] (or [13]) we begin by considering N individual investors, and assume that ith investor's private state dynamics is of the form:

$$dU_t^i = \sigma^i\big(t, U_{\cdot\wedge t}^i, \bar{\nu}_t^N, \alpha_t^i\big)\, dB_t^{1,i}, \qquad t \geq 0, 1 \leq i \leq N, \tag{1.5}$$

where $B^{1,i}$'s are independent Brownian motions, and $\bar{\nu}_t^N$ denotes the empirical conditional distribution of $U = (U^1, \dots, U^N)$, given the (common) observation $Y = \{Y_t : t \geq 0\}$, that is, $\bar{\nu}_t^N \stackrel{\triangle}{=} \frac{1}{N}\sum_{j=1}^N \delta_{\mathbb{E}[U_t^j|\mathcal{F}_t^Y]}$, where δ_x denotes the Dirac measure at x. More precisely, the notation in (1.5) means (see, e.g., [7])

$$\begin{aligned}\sigma^i\big(t, U_{\cdot\wedge t}^i, \bar{\nu}_t^N, \alpha_t^i\big) &\stackrel{\triangle}{=} \int_{\mathbb{R}} \tilde{\sigma}^i\big(t, U_{\cdot\wedge t}^i, y, \alpha_t^i\big)\bar{\nu}_t^N(dy)\\ &= \frac{1}{N}\sum_{j=1}^N \int_{\mathbb{R}} \tilde{\sigma}^i\big(t, U_{\cdot\wedge t}^i, y, \alpha_t^i\big)\delta_{\mathbb{E}[U_t^j|\mathcal{F}_t^Y]}(dy)\\ &= \frac{1}{N}\sum_{j=1}^N \tilde{\sigma}^i\big(t, U_{\cdot\wedge t}^i, \mathbb{E}[U_t^j|\mathcal{F}_t^Y], \alpha_t^i\big).\end{aligned} \tag{1.6}$$

Here, $\tilde{\sigma}^i$'s are the functions defined on appropriate (Euclidean) spaces.

We now assume that each investor chooses an individual strategy to minimize the cost; the cost functional of the ith agent is of the form:

$$J^i(\alpha^i) \stackrel{\triangle}{=} \mathbb{E}\bigg\{\Phi^i(U_T^i) + \int_0^T L^i\big(t, U_{\cdot\wedge t}^i, \bar{\nu}_t^N, \alpha_t^i\big)\, dt\bigg\}, \qquad 1 \leq i \leq N. \tag{1.7}$$

Following the argument of Lasry and Lions [15] (see also [5–8, 13]), if we assume that the game is *symmetric*, that is, $\tilde{\sigma}^i = \tilde{\sigma}$, L^i and $\Phi^i = \Phi$ are independent of i, and that the number of investors N converges to $+\infty$, then under suitable technical conditions, one could find (approximate) Nash equilibriums through a limiting dynamics, and assign a representative investor the unified strategy α, determined by a *conditional* McKean–Vlasov-type SDE:

$$dX_t = \sigma(t, X_{\cdot\wedge t}, \mu_t, \alpha_t)\, dB_t^1, \qquad t \geq 0, \tag{1.8}$$

where μ is the conditional distribution of X_t given $\mathcal{F}_t^Y$, and

$$\begin{aligned}\sigma(t, X_{\cdot\wedge t}, \mu_t, u_t) &\stackrel{\triangle}{=} \int \sigma(t, X_{\cdot\wedge t}, y, u_t)\mu_t(dy)\\ &= \mathbb{E}\big\{\sigma\big(t, \varphi_{\cdot\wedge t}, \mathbb{E}\big[X_t|\mathcal{F}_t^Y\big], u\big)\big\}\big|_{\substack{\varphi = X_\cdot\\ u = u_t}}.\end{aligned}$$

Furthermore, the value function becomes, with similar notation,

$$V(x) = \inf_{\alpha} J(\alpha) \stackrel{\triangle}{=} \mathbb{E}\left\{\Phi(X_T) + \int_0^T L(t, X_{\cdot\wedge t}, \mu_t, \alpha_t)\,dt\right\}. \tag{1.9}$$

We note that (1.8) and (1.9), together with (1.4), form a stochastic control problem involving CMFSDE dynamics and partial observations, as we are proposing.

The main objective of this paper is two-fold: We shall first study the exact meaning as well as the well-posedness of the dynamics, and then investigate the stochastic maximum principle for the corresponding stochastic control problem. For the wellposedness of (1.2), we shall use a scheme that combines the idea of [9] and the techniques of nonlinear filtering, and prove the existence and uniqueness of the solution to SDE (1.8) via Schauder's fixed-point theorem on $\mathscr{P}_2(\Omega)$, the space of probability measures with finite second moment, endowed with the 2-Wasserstein metric. We note that the important elements in this argument include the so-called *reference probability space* that is often seen in the nonlinear filtering theory and the Kallianpur–Striebel formula (cf., e.g., [1, 21]), which enable us to define the solution mapping.

Our next task is to prove Pontryagin's maximum principle for our stochastic control problem. The main idea is similar to earlier works of the first two authors [4, 16], with some significant modifications. In particular, since in the present case the control problem can only be carried out in a weak form, due to the lack of strong solution of CMFSDE, the existence of the common reference probability space is essential. Consequently, extra efforts are needed to overcome the complexity caused by the change of probability measures, which, together with the path-dependent nature of the underlying dynamic system, makes even the first order adjoint equation more complicated than the traditional ones. To the best of our knowledge, the resulting mean-field backward SDE is new.

The paper is organized as follows. In Section 2, we provide all the necessary preparations, including some known facts of nonlinear filtering. In Sections 3 and 4, we prove the well-posedness of the partially observable dynamics. In Section 5, we introduce the stochastic control problem, and in Section 6 we study the variational equations and give some important estimates. Finally, in Section 7 we prove the Pontryagin maximum principle.

2. Preliminaries. Throughout this paper, we consider the *canonical space* $(\Omega, \mathcal{F})$, where $\Omega \stackrel{\triangle}{=} \mathbb{C}_0([0,\infty);\mathbb{R}^{2d}) = \{\omega \in \mathbb{C}([0,\infty);\mathbb{R}^{2d}) : \omega_0 = \mathbf{0}\}$, and $\mathcal{F}$ be its topological σ-field. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}$ be the natural filtration on Ω, that is, for each $t \ge 0$, $\mathcal{F}_t$ is the topological σ-field of the space $\Omega_t \stackrel{\triangle}{=} \{\omega(\cdot\wedge t) : \omega \in \Omega\}$. For simplicity, throughout this paper we assume $d = 1$, and that all the processes are 1-dimensional, although the higher dimensional cases can be argued similarly without substantial difficulties. Furthermore, we let $\mathscr{P}(\Omega)$ denote the space of all probability measures on $(\Omega, \mathcal{F})$, and for each $\mathbb{P} \in \mathscr{P}(\Omega)$, we assume that $\mathbb{F}$ is

$\mathbb{P}$-augmented so that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ satisfies the *usual hypotheses*.

Next, for given $T > 0$ we denote $\mathbb{C}_T = \mathbb{C}([0, T])$ endowed by the supremum norm $\|\cdot\|_{\mathbb{C}_T}$, and let $\mathscr{B}(\mathbb{C}_T)$ be its topological σ-field. Consider now the space of all probability measures on $(\mathbb{C}_T, \mathscr{B}(\mathbb{C}_T))$, denoted by $\mathscr{P}(\mathbb{C}_T)$, and for $p \geq 1$ we let $\mathscr{P}_p(\mathbb{C}_T) \subseteq \mathscr{P}(\mathbb{C}_T)$ be those that have finite pth moment. We recall that the *p-Wasserstein metric* on $\mathscr{P}_p(\mathbb{C}_T)$ is defined as a mapping $W_p : \mathscr{P}_p(\mathbb{C}_T) \times \mathscr{P}_p(\mathbb{C}_T) \mapsto \mathbb{R}_+$ such that, for all $\mu, \nu \in \mathscr{P}_p(\mathbb{C}_T)$,

$$
\begin{aligned}
W_p(\mu, \nu) \stackrel{\triangle}{=} \inf\biggl\{ \biggl(\int_{\mathbb{C}_T^2} \|x - y\|_{\mathbb{C}_T}^p \pi(dx, dy) \biggr)^{\frac{1}{p}} : \\
\pi \in \mathscr{P}_p(\mathbb{C}_T^2) \text{ with marginals } \mu \text{ and } \nu \biggr\}.
\end{aligned} \tag{2.1}
$$

In this paper, we shall use the 2-Wasserstein metric W_2, and abbreviate $(\mathscr{P}_2(\mathbb{C}_T), W_2)$ by $\mathscr{P}_2(\mathbb{C}_T)$. Since $\mathbb{C}_T$ is a separable Banach space, it is known that $\mathscr{P}_2(\mathbb{C}_T)$ is a separable and complete metric space. Furthermore, it is known that (cf., e.g., [19]), for $\mu_n, \mu \in \mathscr{P}_2(\mathbb{C}_T)$

$$
\begin{aligned}
&\lim_{n\to\infty} W_2(\mu_n, \mu) = 0 \\
&\iff \quad \mu_n \xrightarrow{w} \mu \text{ in } \mathscr{P}_2(\mathbb{C}_T) \text{ and, as } N \to +\infty, \\
&\qquad \sup_n \int_\Omega \|\varphi\|_{\mathbb{C}_T}^2 I\{\|\varphi\|_{\mathbb{C}_T} \geq N\} \mu_n(d\varphi) \to 0.
\end{aligned} \tag{2.2}
$$

Next, for any $\mathbb{P} \in \mathscr{P}(\Omega)$, $p, q \geq 1$, any sub-filtration $\mathbb{G} \subseteq \mathbb{F}$, and any Banach space $\mathbb{X}$, we denote $L^p(\mathbb{P}; \mathbb{X})$ to be all $\mathbb{X}$-valued L^p-random variables under $\mathbb{P}$. In particular, we denote by $L^p(\mathbb{P}; \mathbb{R})$ to be all real valued L^p-random variables under $\mathbb{P}$. Further, we denote by $L^p_{\mathbb{G}}(\mathbb{P}; L^q([0, T]))$ the L^p-space of all $\mathbb{G}$-adapted processes η, such that

$$
\|\eta\|_{p,q,\mathbb{P}} \stackrel{\triangle}{=} \biggl\{ \mathbb{E}^{\mathbb{P}} \biggl[\int_0^T |\eta_t|^q \, dt \biggr]^{p/q} \biggr\}^{1/p} < \infty. \tag{2.3}
$$

If $p = q$, we simply write $L^p_{\mathbb{G}}(\mathbb{P}; [0, T]) \stackrel{\triangle}{=} L^p_{\mathbb{G}}(\mathbb{P}; L^p([0, T]))$. Finally, we define $L^{\infty-}_{\mathbb{G}}(\mathbb{P}; [0, T]) \stackrel{\triangle}{=} \bigcap_{p>1} L^p_{\mathbb{G}}(\mathbb{P}; [0, T])$ and $\mathscr{L}^{\infty-}_{\mathbb{G}}(\mathbb{P}; \mathbb{C}_T) \stackrel{\triangle}{=} \bigcap_{p>1} L^p_{\mathbb{G}}(\mathbb{P}; \mathbb{C}_T)$, where $L^p_{\mathbb{G}}(\mathbb{P}; \mathbb{C}_T)$ is the space of all continuous, $\mathbb{F}$-adapted, processes $\xi = \{\xi_t\}$ such that $\|\xi\|_{\mathbb{C}_T} \in L^p(\mathbb{P}; \mathbb{R})$. We will often drop "$\mathbb{P}$" from the subscript/superscript when the context is clear.

We now give a more precise description of the SDEs (1.2), in terms of the standard McKean–Vlasov SDE. Again we consider only the case $b = 0$, and we assume further that $\hat{\sigma} = 1$ in (1.2) for simplicity.

We begin by introducing some notation. Let X be the state process and Y the observation process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, for some $\mathbb{P} \in \mathscr{P}(\Omega)$. We denote the "filtered" state process by $U_t^{X|Y} = \mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]$, $t \ge 0$. Since (as we show in Lemma 3.1 below) the process $U^{X|Y}$ is continuous, we denote its law under $\mathbb{P}$ on $\mathbb{C}_T$ by $\mu^{X|Y} = \mathbb{P} \circ [U^{X|Y}]^{-1} \in \mathscr{P}(\mathbb{C}_T)$. Next, let $P_t(\varphi) = \varphi(t)$, $\varphi \in \mathbb{C}_T$, $t \ge 0$, be the projection mapping, and define $\mu_t^{X|Y} = \mu^{X|Y} \circ P_t{}^{-1}$. Then, for any $\varphi \in \mathbb{C}_T$, and $u \in \mathbb{R}$, we can write

$$\mathbb{E}\big[\sigma\big(t, \varphi_{\cdot\wedge t}, \mathbb{E}[X_t|\mathcal{F}_t^Y], u\big)\big] = \int \sigma(t, \varphi_{\cdot\wedge t}, y, u)\mu_t^{X|Y}(dy) \stackrel{\triangle}{=} \sigma\big(t, \varphi_{\cdot\wedge t}, \mu_t^{X|Y}, u\big).$$

We should note that since the dynamics X is nonobservable, the decision of the controller can only be made based on the information observed from the process Y. Therefore, it is reasonable to assume that the control process u is $\mathbb{F}^Y = \{\mathcal{F}_t^Y\}_{t\ge 0}$ adapted (or progressively measurable). We should remark that, for a given such control, it is by no means clear that the state-observation SDEs will have a strong solution on a prescribed probability space, as we shall see from our well-posedness result in the next sections. We therefore consider a "weak formulation" which we now describe. Consider the pairs $(\mathbb{P}, u)$, where $\mathbb{P} \in \mathscr{P}(\Omega)$, $u \in L^2_{\mathbb{F}}(\mathbb{P}; [0, T])$, such that the following SDEs are well defined:

$$\begin{aligned} X_t &= x + \int_0^t \mathbb{E}^{\mathbb{P}}\big[\sigma\big(s, \varphi_{\cdot\wedge s}, \mathbb{E}^{\mathbb{P}}[X_s|\mathcal{F}_s^Y], z\big)\big]\Big|_{\varphi = X, z = u_s} dB_s^1 \\ &= x + \int_0^t \int_{\mathbb{R}} \sigma(s, X_{\cdot\wedge s}, y, u_s)\mu_s(dy)\, dB_s^1 \\ &= x + \int_0^t \sigma(s, X_{\cdot\wedge s}, \mu_s, u_s)\, dB_s^1, \end{aligned} \tag{2.4}$$

$$Y_t = \int_0^t h(s, X_s)\, ds + B_t^2, \qquad t \ge 0, \tag{2.5}$$

where (B^1, B^2) is a standard 2-dimensional Brownian motion under $\mathbb{P}$, and $\mu_t(\cdot) \stackrel{\triangle}{=} \mathbb{P} \circ \mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]^{-1}(\cdot)$ is the distribution, under $\mathbb{P}$, of the conditional expectation of X_t, given $\mathcal{F}_t^Y$. We note that we *do not* require that the solution to (2.4) and (2.5) (or probability $\mathbb{P}$ for given u) be unique(!). Now let U be a convex subset of $\mathbb{R}^k$. For simplicity, assume $k = 1$.

DEFINITION 2.1. A pair $(\mathbb{P}, u) \in \mathscr{P}(\Omega) \times L^2_{\mathbb{F}}(\mathbb{P}; [0, T])$ is called an "admissible control" if:

(i) $u_t \in U$, for all $t \in [0, T]$, and $B = (B^1, B^2)$ is a $(\mathbb{F}, \mathbb{P})$-Brownian motion;
(ii) there exist processes $(X, Y) \in L^2_{\mathbb{F}}(\mathbb{P}; [0, T])$ satisfying SDEs (2.4) and (2.5); and
(iii) $u \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{P}; [0, T])$.

We shall denote the set of all admissible controls by $\mathscr{U}_{\mathrm{ad}}$. For simplicity, we often write $u \in \mathscr{U}_{\mathrm{ad}}$, and denote the associated probability measure(s) $\mathbb{P}$ by $\mathbb{P}^u$, for $u \in \mathscr{U}_{\mathrm{ad}}$.

REMARK 2.1. As we shall see later, under our standing assumptions to every control $u \in \mathscr{U}_{\mathrm{ad}}$ there is only one probability measure $\mathbb{P}^u$ associated. We should note, however, that unlike the traditional filtering problem, the main difficulty of SDE (2.4)–(2.5) lies in the mutual dependence between the solution pair X^u and Y, via the law of conditional expectation $\mu_t^u = \mathbb{P}^u \circ \mathbb{E}^{\mathbb{P}^u}[X_t^u | \mathcal{F}_t^Y]^{-1}$ in the coefficients. Moreover, the requirement that u is $\mathbb{F}^Y$-adapted adds an additional seemingly "circular" nature to the problem. Thus, the well-posedness of the problem is far from obvious, and will be the main subject of Section 3.

We note that under the weak formulation the state-observation processes (X^u, Y) are often defined on different probability spaces. To facilitate our discussion we shall designate a common space on which all the controlled dynamics can be evaluated. In light of the nonlinear filtering theory, we make the following assumption.

ASSUMPTION 2.1. There exists a probability measure $\mathbb{Q}^0$ on $(\Omega, \mathcal{F})$, such that under $\mathbb{Q}^0$, (B^1, Y) is a 2-dimensional Brownian motion, where Y is the observation process.

We note that the probability measure $\mathbb{Q}^0$ is commonly known as the "reference probability measure" in nonlinear filtering theory. The existence of such measure can be argued once the existence of the weak solution of (2.4)–(2.5) is known.

Indeed, suppose that $u \in \mathscr{U}_{\mathrm{ad}}$ and $\mathbb{P}^u \in \mathscr{P}(\Omega)$ is the associated probability such that the SDEs (2.4) and (2.5) have a solution (X^u, Y) on $(\Omega, \mathcal{F}, \mathbb{P}^u)$. Consider the following SDE:

$$\bar{L}_t = 1 - \int_0^t h(s, X_s^u) \bar{L}_s \, dB_s^2 = 1 + \int_0^t \bar{L}_s \, dZ_s^u, \tag{2.6}$$

where $Z_t^u = -\int_0^t h(s, X_s^u) \, dB_s^2$. We denote its solution by $\bar{L}^u$. Then, under appropriate conditions on h, both Z^u and $\bar{L}^u$ are $\mathbb{P}^u$-martingales, and $\bar{L}^u$ is the stochastic exponential

$$\bar{L}_t^u = \exp\left\{ -\int_0^t h(s, X_s^u) \, dB_s^2 - \frac{1}{2} \int_0^t |h(s, X_s^u)|^2 \, ds \right\}. \tag{2.7}$$

Thus, the Girsanov theorem suggests that $d\mathbb{Q}^0 = \bar{L}_T^u \, d\mathbb{P}^u$ defines a new probability measure $\mathbb{Q}^0$ under which (B^1, Y) is a Brownian motion, hence a "reference measure".

The essence of Assumption 2.1 is, therefore, to assign a *prior distribution* on the observation process Y *before* the well-posedness of the control system is established. In fact, with such an assumption one can begin by assuming that (B^1, Y) is the canonical process [i.e., $(B_t^1, Y_t)(\omega) = \omega(t)$, $\omega \in \Omega$] and $\mathbb{Q}^0$ the Wiener measure on $(\Omega, \mathcal{F})$, and then proceed to prove the existence of the weak solution of the system (2.4) and (2.5). This scheme will be carried out in details in Section 3.

Continuing with our control problem, for any $u \in \mathscr{U}_{\mathrm{ad}}$, we define the *cost functional* by

$$\begin{aligned} J(t,x;u) &\stackrel{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}\left\{\int_t^T f(s, X^u_{\cdot\wedge s}, \mu^u_s, u_s)\,ds + \Phi(X^u_T, \mu^u_T)\right\} \\ &= \mathbb{E}^{\mathbb{Q}^0}\left\{\int_t^T \mathbb{E}^{\mathbb{P}^u}\big[f(s, \varphi_{\cdot\wedge s}, \mathbb{E}^{\mathbb{P}^u}[X^u_s|\mathcal{F}^Y_s], u)\big]\Big|_{\varphi=X^u, u=u_s} ds \right. \\ &\qquad \left. + \mathbb{E}^{\mathbb{P}^u}\big[\Phi(x, \mathbb{E}^{\mathbb{P}^u}[X^u_T|\mathcal{F}^Y_T])\big]\big|_{x=X^u_T}\right\}, \end{aligned} \tag{2.8}$$

and we denote the value function as

$$V(t,x) \stackrel{\triangle}{=} \inf_{u\in\mathscr{U}_{\mathrm{ad}}} J(t,x;u). \tag{2.9}$$

We shall make use of the following *standing assumptions* on the coefficients.

ASSUMPTION 2.2. (i) The mappings $(t,\varphi,x,y,z) \mapsto \sigma(t,\varphi_{\cdot\wedge t},y,z)$, $h(t,x)$, $f(t,\varphi_{\cdot\wedge t},y,z)$, and $\Phi(x,y)$ are bounded and continuous, for $(t,\varphi,x,y,z) \in [0,T]\times\mathbb{C}_T\times\mathbb{R}\times\mathbb{R}\times U$.

(ii) The partial derivatives $\partial_y\sigma$, $\partial_z\sigma$, $\partial_y f$, $\partial_z f$, $\partial_x h$, $\partial_x\Phi$, $\partial_y\Phi$ are bounded and continuous, for $(\varphi,x,y,z)\in\mathbb{C}_T\times\mathbb{R}\times\mathbb{R}\times U$, uniformly in $t\in[0,T]$.

(iii) The mappings $\varphi\mapsto\sigma(t,\varphi_{\cdot\wedge t},y,z)$, $f(t,\varphi_{\cdot\wedge t},y,z)$, as functionals from $\mathbb{C}_T$ to $\mathbb{R}$, are Fréchet differentiable. Furthermore, there exists a family of measures $\{\ell(t,\cdot)\}|_{t\in[0,T]}$, satisfying $0\le\int_0^T \ell(t,ds)\le C$, for all $t\in[0,T]$, such that both derivatives, denoted by $D_\varphi\sigma = D_\varphi\sigma(t,\varphi_{\cdot\wedge t},y,z)$ and $D_\varphi f = D_\varphi f(t,\varphi_{\cdot\wedge t},y,z)$, respectively, satisfy

$$\big|D_\varphi\sigma(t,\varphi_{\cdot\wedge t},y,z)(\psi)\big| + \big|D_\varphi f(t,\varphi_{\cdot\wedge t},y,z)(\psi)\big| \le \int_0^T \big|\psi(s)\big|\ell(t,ds), \tag{2.10}$$

$\psi\in\mathbb{C}_T$, uniformly in (t,φ,y,z).

(iv) The mapping $y\mapsto y\partial_y\sigma(t,\varphi_{\cdot\wedge t},y,z)$ is uniformly bounded, uniformly in (t,φ,z).

(v) The mapping $x\mapsto x\partial_x h(t,x)$ is bounded, uniformly in $(t,x)\in[0,T]\times\mathbb{R}$.

(vi) The mappings $x\mapsto xh(t,x)$, $x^2\partial_x h(t,x)$ are bounded, uniformly in $(t,x)\in[0,T]\times\mathbb{R}$.

We note that some of the assumptions above are merely technical and can be improved, but we prefer not to dwell on such technicalities and focus on the main ideas instead.

REMARK 2.2. Note that if $(t,\varphi,y,z)\mapsto\phi(t,\varphi_{\cdot\wedge t},y,z)$ is a function defined on $[0,T]\times\mathbb{C}_T\times\mathbb{R}\times\mathbb{R}$ satisfying Assumption 2.2(i), (ii), then for any $\mu\in\mathscr{P}_2(\mathbb{C}_T)$, we can define a function on the space $[0,T]\times\Omega\times\mathbb{C}_T\times\mathscr{P}_2(\mathbb{C}_T)\times U$:

$$\bar\phi(t,\omega,\varphi_{\cdot\wedge t},\mu_t,z) \stackrel{\triangle}{=} \int_{\mathbb{R}} \phi(t,\varphi_{\cdot\wedge t},y,z)\mu_t(dy), \tag{2.11}$$

where $\mu_t = \mu \circ P_t^{-1}$ and $P_t(\varphi) \stackrel{\triangle}{=} \varphi(t)$, $(t, \varphi) \in [0, T] \times \mathbb{C}_T$. Then $\bar{\phi}$ must satisfy the following Lipschitz condition:

$$
\begin{aligned}
&|\bar{\phi}(t, \varphi^1_{\cdot\wedge t}, \mu^1_t, z^1) - \bar{\phi}(t, \varphi^2_{\cdot\wedge t}, \mu^2_t, z^2)| \\
&\qquad \le K\{\|\varphi^1 - \varphi^2\|_{\mathbb{C}_t} + W_2(\mu^1, \mu^2) + |z^1 - z^2|\},
\end{aligned}
\tag{2.12}
$$

where $\|\cdot\|_{\mathbb{C}_t}$ is the sup-norm on $\mathbb{C}([0, t])$ and $W_2(\cdot, \cdot)$ is the 2-Wasserstein metric.

REMARK 2.3. The Fréchet derivatives $D_\varphi \sigma$ and $D_\varphi f$ by definition belong to $\mathbb{C}^*_T \stackrel{\triangle}{=} \mathscr{M}[0, T]$, the space of all finite signed Borel measures on $[0, T]$, endowed with the total variation norm $|\cdot|_{\mathrm{TV}}$ (with a slight abuse of notation, we still denote it by $|\cdot|$). Thus, the Assumption 2.2(iii) amounts to saying that, as measures, for any (t, φ, y, z),

$$
|D_\varphi \sigma(t, \varphi_{\cdot\wedge t}, y, z)(ds)| + |D_\varphi f(t, \varphi_{\cdot\wedge t}, y, z)(ds)| \le \ell(t, ds). \tag{2.13}
$$

This inequality will be crucial in our discussion in Section 7.

To end this section, we recall some basic facts in nonlinear filtering theory, adapted to our situation. We begin by considering the inverse Girsanov kernel of $\bar{L}^u$ defined by (2.7):

$$
L^u_t \stackrel{\triangle}{=} [\bar{L}^u_t]^{-1} = \exp\left\{\int_0^t h(s, X^u_s)\, dY_s - \frac{1}{2}\int_0^t |h(s, X^u_s)|^2\, ds\right\}, \tag{2.14}
$$

$t \in [0, T]$. Then L^u is a $\mathbb{Q}^0$-martingale, $d\mathbb{P}^u = L^u_T\, d\mathbb{Q}^0$, and L^u satisfies the following SDE on $(\Omega, \mathcal{F}, \mathbb{Q}^0)$:

$$
L_t = 1 + \int_0^t h(s, X_s) L_s\, dY_s, \qquad t \in [0, T]. \tag{2.15}
$$

Let us now denote $L = L^u$ for simplicity. An important ingredient that we are going to use frequently is the SDEs known as the *Kushner–Stratonovic* or *Fujisaki–Kallianpur–Kunita* (FKK) equation for the "normalized conditional probability". Let us denote

$$
S_t \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[L_t X_t | \mathcal{F}^Y_t], \qquad S^0_t \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[L_t | \mathcal{F}^Y_t], \qquad t \ge 0. \tag{2.16}
$$

Since under $\mathbb{Q}^0$ the process (B^1, Y) is a Brownian motion, the σ-field $\mathcal{F}^Y_{t,T}$ and $\mathcal{F}^Y_t \vee \mathcal{F}^{B^1}_t$ are independent, where $\mathcal{F}^Y_{t,T} \stackrel{\triangle}{=} \sigma\{Y_r - Y_t : t \le r \le T\}$. It is standard to show that [in light of (2.15)] S and S^0 satisfy the following SDEs:

$$
S^0_t = 1 + \int_0^t \mathbb{E}^{\mathbb{Q}^0}[h(s, X_s) L_s | \mathcal{F}^Y_s]\, dY_s, \qquad t \ge 0, \tag{2.17}
$$

and

$$
S_t = x + \int_0^t \mathbb{E}^{\mathbb{Q}^0}[L_s X_s h(s, X_s) | \mathcal{F}^Y_s]\, dY_s, \qquad t \ge 0. \tag{2.18}
$$

Furthermore, let $U_t \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{P}^u}[X_t|\mathcal{F}_t^Y]$, $t \ge 0$. Then, by the Bayes formula (also known as the Kallianpur–Striebel formula, see, e.g., [1]) we have

$$U_t = \frac{\mathbb{E}^{\mathbb{Q}^0}[L_t X_t|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}^0}[L_t|\mathcal{F}_t^Y]} = \frac{S_t}{S_t^0}, \qquad t \ge 0, \mathbb{Q}^0\text{-a.s.} \tag{2.19}$$

A simple application of Itô's formula and some direct computation then lead to the following FKK equation:

$$\begin{aligned} dU_t &= \{\mathbb{E}^{\mathbb{P}^u}[X_t h(t, X_t)|\mathcal{F}_t^Y] - \mathbb{E}^{\mathbb{P}^u}[X_t|\mathcal{F}_t^Y]\mathbb{E}^{\mathbb{P}^u}[h(t, X_t)|\mathcal{F}_t^Y]\}\,dY_t \\ &\quad + \{\mathbb{E}^{\mathbb{P}^u}[X_t|\mathcal{F}_t^Y]\{\mathbb{E}^{\mathbb{P}^u}[h(t, X_t)|\mathcal{F}_t^Y]\}^2 \\ &\quad - \mathbb{E}^{\mathbb{P}^u}[X_t h(t, X_t)|\mathcal{F}_t^Y]\mathbb{E}^{\mathbb{P}^u}[h(t, X_t)|\mathcal{F}_t^Y]\}\,dt. \end{aligned} \tag{2.20}$$

In fact, one can easily show that

$$S_t = U_t \exp\left\{\int_0^t \mathbb{E}^{\mathbb{P}^u}[h(s, X_s)|\mathcal{F}_s^Y]\,dY_s - \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}^u}[h(s, X_s)|\mathcal{F}_s^Y]^2\,ds\right\}. \tag{2.21}$$

3. Well-posedness of the state-observation dynamics. In this and next sections, we investigate the well-posedness of the controlled state-observation system (2.4) and (2.5). More precisely, we shall argue that the admissible control set $\mathscr{U}_{\mathrm{ad}}$, defined by Definition 2.1, is not empty. We first note that, for a fixed $\mathbb{P} \in \mathscr{P}(\Omega)$ and $u \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{P}, [0, T])$, if we define

$$\phi^u(t, \omega, \varphi_{\cdot\wedge t}, \mu_t) \stackrel{\triangle}{=} \int_{\mathbb{R}} \phi\big(t, \varphi_{\cdot\wedge t}, y, u_t(\omega)\big)\mu_t(dy), \tag{3.1}$$

where $\phi = b, \sigma$, then we can write the control-observation system (2.4) and (2.5) as a slightly more generic form (denoting $b^u = b$ and $\sigma^u = \sigma$ for simplicity):

$$\begin{cases} X_t = x + \displaystyle\int_0^t b\big(s, \cdot, X_{\cdot\wedge s}, \mu_s^{X|Y}\big)\,ds + \int_0^t \sigma\big(s, \cdot, X_{\cdot\wedge s}, \mu_s^{X|Y}\big)\,dB_s^1; \\ Y_t = \displaystyle\int_0^t h(s, X_s)\,ds + B_t^2, \qquad t \ge 0, \end{cases} \tag{3.2}$$

where $B = (B^1, B^2)$ is a $\mathbb{P}$-Brownian motion, and $\mu_t^{X|Y} = \mathbb{P}\circ[\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]]^{-1}$. Our task is to prove the well-posedness of SDE (3.2) in a *weak* sense [i.e., including the existence of the probability measure $\mathbb{P}$(!)]. In light of Remark 2.2, we shall assume that the coefficients b and σ in (3.2) satisfy the following assumptions that are slightly weaker than Assumption 2.2, but sufficient for our purpose in this section.

ASSUMPTION 3.1. The coefficients $b, \sigma : [0, T] \times \mathbb{C}_T \times \mathscr{P}_2(\mathbb{C}_T) \mapsto \mathbb{R}$ enjoy the following properties:

(i) For fixed $(\varphi, \mu) \in \mathbb{C}_T \times \mathscr{P}_2(\mathbb{C}_T)$, the mapping $(t, \omega) \mapsto (b, \sigma)(t, \omega, \varphi, \mu)$ is an $\mathbb{F}$-progressively measurable process.

(ii) For fixed $t \in [0, T]$, and $\mathbb{Q}^0$-a.e. $\omega \in \Omega$, there exists $K > 0$, independent of (t, ω), such that for all $(\varphi^1, \mu^1), (\varphi^2, \mu^2) \in \mathbb{C}_T \times \mathscr{P}_2(\mathbb{C}_T)$, it holds

$$\begin{aligned} &\big|\phi\big(t, \omega, \varphi^1_{\cdot\wedge t}, \mu^1_t\big) - \phi\big(t, \omega, \varphi^2_{\cdot\wedge t}, \mu^2_t\big)\big| \\ &\qquad \le K\Big(\sup_{t\in[0,T]} |\varphi^1_t - \varphi^2_t| + W_2(\mu^1, \mu^2)\Big), \end{aligned} \tag{3.3}$$

for $\phi = b, \sigma$, respectively.

In the rest of the section, we shall still assume $b = 0$, as it does not add extra difficulties. Now assume that (X, Y) satisfies (3.2) under $\mathbb{P}$, and let us denote $U^{X|Y}_t \overset{\triangle}{=} \mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}^Y_t]$, $t \ge 0$. (We note that $U^{X|Y}$ should be understood as the "optional projection" of X onto $\mathbb{F}^Y$!) We first check that $U^{X|Y}$ is indeed a continuous process.

LEMMA 3.1. *Assume that Assumption 2.2 holds. Then $U^{X|Y}$ admits a continuous version.*

PROOF. First, note that $\mathbb{P} \sim \mathbb{Q}^0$, and X has continuous paths, $\mathbb{P}$-a.s. By Bayes formula (2.19), we can write $U^{X|Y}_t = \frac{\mathbb{E}^{\mathbb{Q}^0}[L_t X_t|\mathcal{F}^Y_t]}{\mathbb{E}^{\mathbb{Q}^0}[L_t|\mathcal{F}^Y_t]} = \frac{S_t}{S^0_t}$, where S^0 and S satisfy (2.17) and (2.18), respectively, and L satisfies (2.15). Clearly, the representations (2.17) and (2.18) indicate that both S^0 and S have continuous paths, thus $U^{X|Y}$ must have a continuous version. □

We now define $\mu^{X|Y}(\cdot) = \mathbb{P} \circ [U^{X|Y}]^{-1}(\cdot)$, and $\mu^{X|Y}_t(\cdot) = \mathbb{P} \circ [U^{X|Y}_t]^{-1}(\cdot)$, for any $t \ge 0$. Lemma 3.1 then implies that $\mu^{X|Y} \in \mathscr{P}_2(\mathbb{C}_T)$, justifying the definition of SDE (3.2). In what follows when the context is clear, we shall omit "$X|Y$" from the superscript.

We note that the special circular nature of SDE (3.2) between its solution and its law of the conditional expectation (whence the underlying probability) makes it necessary to specify the meaning of a solution. We have the following definition.

DEFINITION 3.1 (Weak Solution). An eight-tuple $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X, Y, B^1, B^2)$ is called a solution to the filtering equation (3.2) if:

(i) $(\Omega, \mathcal{F})$ is the canonical space, $\mathbb{P} \in \mathscr{P}(\Omega)$, and $\mathbb{F}$ is the canonical filtration;

(ii) (B^1, B^2) is a 2-dimensional $\mathbb{F}$-Brownian motion under $\mathbb{P}$;

(iii) (X, Y) is an $\mathbb{F}$-adapted continuous process such that (3.2) holds for all $t \in [0, T]$, $\mathbb{P}$-almost surely.

To prove the well-posedness, we shall use a generalized version of the Schauder fixed-point theorem (see Cauty [10], or a recent generalization in [11]). To this end, we consider the following subset of $\mathscr{P}_2(\mathbb{C}_T)$:

$$\mathscr{E} \overset{\triangle}{=} \Big\{\mu \in \mathscr{P}_2(\mathbb{C}_T) \Big| \sup_{t\in[0,T]} \int_{\mathbb{R}} |y|^4 \mu_t(dy) < \infty\Big\}. \tag{3.4}$$

In the above $\mu_t = \mu \circ P_t{}^{-1} \in \mathscr{P}_2(\mathbb{R})$, and $P_t(\varphi) = \varphi(t)$, $\varphi \in \Omega$, is the projection mapping. Clearly, $\mathscr{E}$ is a convex subset of $\mathscr{P}_2(\mathbb{C}_T)$.

We now construct a mapping $\mathscr{T} : \mathscr{E} \mapsto \mathscr{E}$, whose fixed point, if exists, would give a solution to the SDE (3.2). We shall begin with the reference probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$, thanks to Assumption 2.1, then (B^1, Y) is a $\mathbb{Q}^0$-Brownian motion. We may assume without loss of generality that (B^1, Y) is the canonical process, and $\mathbb{Q}^0$ is the Wiener measure.

For any $\mu \in \mathscr{E}$, we consider the SDE on the space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$:

$$X_t = x + \int_0^t \sigma(s, \cdot, X_{\cdot \wedge s}, \mu_s)\, dB_s^1, \qquad t \geq 0. \tag{3.5}$$

Note that as the distribution μ is given, (3.5) is an "open-loop" SDE with "functional Lipschitz" coefficient, thanks to Assumption 3.1. Thus, there exists a unique (strong) solution to (3.5), which we denote by $X = X^\mu$.

Now, using X^μ we define the process $L^\mu = \{L_t^\mu\}_{t\geq 0}$ as in (2.14) on probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$, and then we define the probability $d\mathbb{P}^\mu \overset{\triangle}{=} L_T^\mu \, d\mathbb{Q}^0$. By the Kallianpur–Striebel formula (2.19) we can define a process

$$U_t^\mu \overset{\triangle}{=} \mathbb{E}^{\mathbb{P}^\mu}\big[X_t^\mu | \mathcal{F}_t^Y\big] = \frac{\mathbb{E}^{\mathbb{Q}^0}[L_t^\mu X_t^\mu | \mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}^0}[L_t^\mu | \mathcal{F}_t^Y]} = \frac{S_t^\mu}{S_t^{\mu,0}}, \qquad t \geq 0, \tag{3.6}$$

where $S_t^\mu \overset{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[L_t^\mu X_t^\mu | \mathcal{F}_T^Y]$, $S_t^{\mu,0} \overset{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[L_t^\mu | \mathcal{F}_T^Y]$, $t \geq 0$, and then we denote

$$\mathscr{T}(\mu) \overset{\triangle}{=} \nu^\mu = \mathbb{P}^\mu \circ \big[U^\mu\big]^{-1} \in \mathscr{P}(\mathbb{C}_T). \tag{3.7}$$

Our task is to show that the solution mapping $\mathscr{T} : \mu \mapsto \nu^\mu$ satisfies the desired assumptions for Schauder's fixed-point theorem.

THEOREM 3.1. *The solution mapping $\mathscr{T} : \mathscr{E} \to \mathscr{P}_2(\mathbb{C}_T)$ enjoys the following properties*:

(1) $\mathscr{T}(\mathscr{E}) \subseteq \mathscr{E}$.

(2) $\mathscr{T}(\mathscr{E})$ *is compact under* 2-*Wasserstein metric*.

(3) $\mathscr{T} : (\mathscr{E}, W_1(\cdot,\cdot)) \to (\mathscr{P}_2(\mathbb{C}_T), W_2(\cdot,\cdot))$ *is continuous*, *that is*, *whenever* $\mu, \mu^n \in \mathscr{E}, n \geq 1$, *is such that* $W_1(\mu^n, \mu) \to 0$, *we have that* $W_2(\mathscr{T}(\mu^n), \mathscr{T}(\mu)) \to 0$.

We remark that an immediate consequence of (3) is that $\mathscr{T} : \mathscr{E} \to \mathscr{P}_2(\mathbb{C}_T)$ is continuous under both the 1- and the 2-Wasserstein metrics. Moreover, the compactness of $\mathscr{T}(\mathscr{E})$ under the 2-Wasserstein metric stated in (2) implies that in the 1-Wasserstein metric.

PROOF. (1) Given $\mu \in \mathscr{E}$, we need only show that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} |y|^4 \nu_t^\mu(dy) < \infty. \tag{3.8}$$

To see this, we note that for $t \in [0, T]$, by Jensen's inequality,

$$\int_{\mathbb{R}} |y|^4 \nu_t^\mu(dy) = \int_{\mathbb{R}} |y|^4 \mathbb{P}^\mu \circ [U^\mu]^{-1}(dy) = \mathbb{E}^{\mathbb{P}^\mu}[|\mathbb{E}^{\mathbb{P}^\mu}[X_t^\mu|\mathcal{F}_t^Y]|^4] \le \mathbb{E}^{\mathbb{P}^\mu}[|X_t^\mu|^4].$$

Since under $\mathbb{Q}^0$, B^1 is also a Brownian motion, it is standard to argue that, as X^μ is the solution to the SDE (3.5), it holds that

$$\sup_{0\le t\le T} \mathbb{E}^{\mathbb{Q}^0}[|X_t^\mu|^{2n}] \le C(1+|x|^{2n}) \qquad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Furthermore, noting that the process L^μ is an L^2-martingale under $\mathbb{Q}^0$, we have

$$\begin{aligned}\sup_{0\le t\le T} \int_{\mathbb{R}^d} |y|^4 \nu_t^\mu(dy) &\le \sup_{0\le t\le T} \mathbb{E}^{\mathbb{P}^\mu}[|X_t^\mu|^4] = \sup_{0\le t\le T} \mathbb{E}^{\mathbb{Q}^0}[L_T^\mu |X_t^\mu|^4] \\ &\le (\mathbb{E}^{\mathbb{Q}^0}[|L_T^\mu|^2])^{\frac{1}{2}} \sup_{0\le t\le T} \mathbb{E}^{\mathbb{Q}^0}[|X_t^\mu|^8]^{\frac{1}{2}} < \infty,\end{aligned}$$

thanks to (3.9). In other words, $\nu^\mu = \mathscr{T}(\mu) \in \mathscr{E}$, proving (1).

(2) We shall prove that for any sequence $\{\mu_t^n\} \subseteq \mathscr{E}$, there exists a subsequence, denoted by $\{\mu_t^n\}$ itself, such that $\lim_{n\to\infty} \mathscr{T}(\mu^n) = \nu$ in 2-Wasserstein metric, for some $\nu \in \mathscr{T}(\mathscr{E})$.

In light of the equivalence relation (2.2), we shall first argue that the family $\{\mathscr{T}(\mu^n)\}_{n\ge 1}$ is tight. To this end, recall that

$$U_t^n = \mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^Y] = \frac{S_t^n}{S_t^{n,0}}, \tag{3.10}$$

where $S_t^n \overset{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[L_t^n X_t^n|\mathcal{F}_t^Y]$, $S_t^{n,0} \overset{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[L_t^n|\mathcal{F}_t^Y]$, $t \ge 0$, and $d\mathbb{P}^n \overset{\triangle}{=} L_T^n\, d\mathbb{Q}^0$. It then follows from the FKK equation (2.20) that

$$\begin{aligned} dU_t^n = {} & \{\mathbb{E}^{\mathbb{P}^n}[X_t^n h(t, X_t^n)|\mathcal{F}_t^Y] - \mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^Y]\mathbb{E}^{\mathbb{P}^n}[h(t, X_t^n)|\mathcal{F}_t^Y]\}\, dY_t \\ & + \{\mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^Y](\mathbb{E}^{\mathbb{P}^n}[h(t, X_t^n)|\mathcal{F}_t^Y])^2 \\ & - \mathbb{E}^{\mathbb{P}^n}[X_t^n h(t, X_t^n)|\mathcal{F}_t^Y]\mathbb{E}^{\mathbb{P}^n}[h(t, X_t^n)|\mathcal{F}_t^Y]\}\, dt. \end{aligned} \tag{3.11}$$

Now denote $B_t^{2,n} \overset{\triangle}{=} Y_t - \int_0^t h(s, X_{\cdot\wedge s}^n)\, ds$. Then $(B^1, B^{2,n})$ is a 2-dimensional standard $\mathbb{P}^n$-Brownian motion. Since h is bounded, so is $\mathbb{E}^{\mathbb{P}^n}[h(t, X_{\cdot\wedge t}^n)|\mathcal{F}_t^Y]$. We thus have the following estimate:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^n}[|U_t^n - U_s^n|^4] &\le C\mathbb{E}^{\mathbb{P}^n}\left[\left(\int_s^t \mathbb{E}^{\mathbb{P}^n}[|X_s^n|^2|\mathcal{F}_s^Y]\, ds\right)^2\right] \\ &\le C\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le s\le T} |\mathbb{E}^{\mathbb{P}^n}[|X_s^n|^2|\mathcal{F}_s^Y]|^2\Big]|t-s|^2 \\ &\le C\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le s\le T} \Big|\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le r\le T} |X_r^n|^2|\mathcal{F}_s^Y\Big]\Big|^2\Big]|t-s|^2 \\ &\le C\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le s\le T} |X_s^n|^4\Big]|t-s|^2 \le C|t-s|^2. \end{aligned} \tag{3.12}$$

Thus, as $U_0^n = x$, $n \geq 1$, the sequence of continuous processes $\{U^n\}$ is relatively compact (cf., e.g., Ethier–Kurtz [12]). Therefore, the sequence of their laws $\{\mathscr{T}(\mu^n) \stackrel{\triangle}{=} \mathbb{P}^n \circ [U^n]^{-1}, n \geq 1\} \subseteq \mathscr{P}(\mathbb{C}_T)$ is tight. Consequently, we can find a subsequence, we may assume itself, that converges weakly to a limit $\nu \in \mathscr{P}_2(\mathbb{C}_T)$. Furthermore, for each $n \geq 1$, we apply the Jensen, Burkholder–Davis-Gundy, and Hölder inequalities to get, with $\nu^n \stackrel{\triangle}{=} \mathscr{T}(\mu^n)$,

$$\begin{aligned}\int_{\mathbb{C}_T} \|\varphi\|_{\mathbb{C}_T}^4 \nu^n(d\varphi) &= \mathbb{E}^{\mathbb{P}^n}[\|U^n\|_{\mathbb{C}_T}^4] = \mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le t\le T}|\mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^n]|^4\Big] \\ &\le \mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le t\le T}\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le r\le T}|X_r^n||\mathcal{F}_t^n\Big]^4\Big] \\ &\le C\Big[\mathbb{E}^{\mathbb{P}^n}\Big[\sup_{0\le r\le T}|X_r^n|^6\Big]\Big]^{2/3} = C\Big[\mathbb{E}^{\mathbb{Q}_0}\Big[L_T^n\sup_{0\le r\le T}|X_r^n|^6\Big]\Big]^{2/3} \\ &\le C[\mathbb{E}^{\mathbb{Q}_0}[(L_T^n)^4]]^{1/6}\Big[\mathbb{E}^{\mathbb{Q}_0}\Big[\sup_{0\le r\le T}|X_r^n|^8\Big]\Big]^{1/2} < +\infty.\end{aligned} \tag{3.13}$$

But noting that h is bounded, one deduces from (3.9) that

$$\sup_{n\ge 1}\int_{\mathbb{C}_T}\|\varphi\|_{\mathbb{C}_T}^4\nu^n(d\varphi) < \infty, \tag{3.14}$$

and thus,

$$\sup_{n\ge 1}\int_{\mathbb{C}_T}\|\varphi\|_{\mathbb{C}_T}^2 I\{|\varphi||_{\mathbb{C}_T}\ge N\}\nu^n(d\varphi)\to 0 \qquad \text{as } N\to+\infty.$$

This, together with the fact that $\nu^n = \mathscr{T}(\mu^n) \xrightarrow{w} \nu$, implies that $W_2(\nu^n, \nu) \to 0$, and $\nu \in \mathscr{E}$, as $n \to \infty$, where $W_2(\cdot,\cdot)$ is the 2-Wasserstein metric on $\mathscr{P}_2(\mathbb{C}_T)$. This proves (2).

(3) We now check that the mapping $\mathscr{T} : (\mathscr{E}, W_1(\cdot,\cdot)) \to (\mathscr{P}_2(\mathbb{C}_T), W_2(\cdot,\cdot))$ is continuous. To this end, for each $\mu \in \mathscr{E}$, we consider the following SDE on the probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$:

$$\begin{cases} dX_t = \sigma(t, X_{\cdot\wedge t}, \mu_t)\, dB_t^1, & X_0 = x; \\ dB_t^2 = dY_t - h(t, X_t)\, dt, & B_0^2 = 0; \\ dL_t = h(t, X_t) L_t\, dY_t, & L_0 = 1. \end{cases} \tag{3.15}$$

Now let $\{\mu^n\} \subseteq \mathscr{E}$ be any sequence such that $\mu^n \to \mu$, as $n \to \infty$, in the 1-Wasserstein metric, and denote by $(X^n, B^{n,2}, L^n)$ the corresponding solutions to (3.15). Define

$$\sigma^n(t, \omega_{\cdot\wedge t}) \stackrel{\triangle}{=} \sigma(t, \omega_{\cdot\wedge t}, \mu_t^n), \qquad (t,\omega) \in [0,T]\times\Omega.$$

Then by Assumption 3.1(ii), the σ^n's are functional Lipschitz deterministic functions, with Lipschitz constant independent of n. This and standard SDE arguments lead to that, as $n \to \infty$,

$$\mathbb{E}^{\mathbb{Q}^0}\Big\{\sup_{0\le t\le T}\big|X_t^n - X_t\big|^p + \sup_{0\le t\le T}\big|L_t^n - L_t\big|^p\Big\} \to 0, \qquad \text{in } L^p(\mathbb{Q}^0),\ p \ge 1. \tag{3.16}$$

We deduce that $U_t^n = \mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^Y] = S_t^n/S_t^{n,0}$ converges in probability under $\mathbb{Q}^0$ to $\frac{\mathbb{E}^{\mathbb{Q}^0}[L_t X_t|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}^0}[L_t|\mathcal{F}_t^Y]} = \mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]$, where $d\mathbb{P} \stackrel{\triangle}{=} L_T\, d\mathbb{Q}^0$.

Now for any $\psi \in \mathbb{C}_b(\mathbb{R})$, letting $n \to \infty$ we have

$$\begin{aligned}\langle \psi, \mathscr{T}(\mu^n)_t\rangle &= \mathbb{E}^{\mathbb{P}^n}\big[\psi(\mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^Y])\big] = \mathbb{E}^{\mathbb{Q}^0}\big[L_T^n\psi(\mathbb{E}^{\mathbb{P}^n}[X_t^n|\mathcal{F}_t^Y])\big]\\ \longrightarrow\quad &\mathbb{E}^{\mathbb{Q}^0}\big[L_T\psi(\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y])\big] = \mathbb{E}^{\mathbb{P}}\big[\psi(\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y])\big]\\ &= \langle \psi, \mathbb{P}\circ[\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]]^{-1}\rangle \qquad \text{as } n\to\infty.\end{aligned} \tag{3.17}$$

This implies that $\nu_t = \mathbb{P}\circ[\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]]^{-1} = \mathscr{T}(\mu)_t$, for all $t \in [0,T]$. With the same argument, one shows that, for any $0 \le t_1 < t_2 < \cdots < t_k < \infty$,

$$\mathscr{T}(\mu^n)_{t_1,\dots,t_k} \stackrel{\triangle}{=} \mathbb{P}\circ\big(\mathbb{E}^{\mathbb{P}}[X_{t_1}^n|\mathcal{F}_{t_1}^Y],\dots,\mathbb{E}^{\mathbb{P}}[X_{t_k}^n|\mathcal{F}_{t_k}^Y]\big)^{-1} \xrightarrow{d} \nu_{t_1,\dots,t_k}, \qquad \text{as } n\to\infty.$$

That is, the finite dimensional distributions of $\mathscr{T}(\mu^n)$ converge to those of ν, and as $\{\mathscr{T}(\mu^n)\}_{n\ge1}$ is tight by part (2), we conclude that $\mathscr{T}(\mu^n) \stackrel{w}{\to} \nu$ in $\mathscr{P}(\mathbb{C}_T)$. This, together with (3.13), further shows that $W_2(\mathscr{T}(\mu^n),\mathscr{T}(\mu)) \to 0$, as $n\to\infty$, proving the continuity of $\mathscr{T}$, whence (3). The proof is now complete. □

As a consequence of Theorem 3.1, we have the following existence result for SDE (3.2).

PROPOSITION 3.1. *Let Assumption* 3.1 *hold. Then SDE* (3.2) *has at least one solution in the sense of Definition* 3.1.

PROOF. The proof follows from Theorem 3.1 and a generalization of the Schauder fixed-point theorem by Cauty (see [10], or a recent generalization [11]). To do this, we must check: (i) $\mathscr{E}$ is a convex subset of a Hausdorff topological linear space, (ii) $\mathscr{T}$ is continuous and $\mathscr{T}(\mathscr{E}) \subseteq \mathscr{E}$ and (iii) $\mathscr{T}(\mathscr{E}) \subset K$, for some compact K in $\mathscr{P}_2(\mathbb{C}_T)$.

To imbed $\mathscr{E}$ into a Hausdorff topological linear space, we borrow the argument of Li-Min [17]. Let $\mathscr{M}_1(\mathbb{C}_T)$ be the space of all bounded signed Borel measures $\nu(\cdot)$ on $\mathbb{C}_T$ such that $|\int_{\mathbb{C}_T}\|\varphi\|_{\mathbb{C}_T}\nu(d\varphi)| < +\infty$, endowed with the norm

$$\|\nu\|_1 := \sup\Big\{\Big|\int_{\mathbb{C}_T} h\, d\nu\Big| : h \in \mathrm{Lip}_1(\mathbb{C}_T),\ \big|h(0)\big| \le 1\Big\}.[5]$$

[5] $\mathrm{Lip}_1(\mathbb{C}_T)$ denotes the set of all real-valued Lipschitz functions over $\mathbb{C}_T$ with Lipschitz constant 1.

Clearly, $(\mathscr{M}_1(\mathbb{C}_T), \|\cdot\|_1)$ is a normed (hence Hausdorff topological) linear space. Since $\mathscr{P}_2(\mathbb{C}_T) \subset \mathscr{P}_1(\mathbb{C}_T) \subset \mathscr{M}_1(\mathbb{C}_T)$, and by the Kantorovich–Rubinstein formula,

$$W_1(\nu^1, \nu^2) = \sup\left\{\left|\int_{\mathbb{C}_T} h\, d(\nu^1 - \nu^2)\right| : h \in \mathrm{Lip}_1(\mathbb{C}_T), |h(0)| \le 1\right\}$$
$$= \|\nu^1 - \nu^2\|_1,$$

for all $\nu^1, \nu^2 \in \mathcal{P}_1(\mathbb{C}_T)$, the topology generated by the norm $\|\cdot\|_1$ on $\mathscr{P}_2(\mathbb{C}_T)$ coincides with the one generated by the 1-Wasserstein metric on $\mathscr{P}_2(\mathbb{C}_T)$. Thus, $\mathscr{E} \subset \mathscr{P}_2(\mathbb{C}_T)$ is a convex subset of $\mathscr{M}_1(\mathbb{C}_T)$, proving (i). Further, note that $\mathscr{T} : \mathscr{E} \to \mathscr{P}_2(\mathbb{C}_T)$ is continuous under the 1-Wasserstein metric, hence also under the $\|\cdot\|_1$-norm, verifying (ii). Finally, since $\mathscr{T}(\mathscr{E}) \subset \mathscr{E}$, and $\mathscr{E}$ is compact under the 2-Wasserstein metric, hence also under the $\|\cdot\|_1$-norm, proving (iii). We can now apply Cauty's theorem to conclude the existence of a fixed point $\nu \in \mathscr{E} \subset \mathscr{P}_2(\mathbb{C}_T)$ such that $\mathscr{T}(\nu) = \nu$.

We note that the existence of the fixed point μ amounts to saying that SDE (3.15) has a solution on the probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$, with $\mu = \mu^{X|Y} = \mathbb{P} \circ [U]^{-1}$, and $U_t = \mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_t^Y]$, $t \ge 0$, where $d\mathbb{P} = L_T\, d\mathbb{Q}^0$ by construction. But this in turn defines a solution of (3.2) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, thanks to the Girsanov transformation. However, since under $\mathbb{P}$, (B^1, B^2) constructed in (3.15) is a Brownian motion, $(\Omega, \mathcal{F}, \mathbb{P}, X, Y, B^1, B^2)$ defines a (weak) solution of SDE (3.2). $\square$

4. Uniqueness. In this section, we investigate the uniqueness of the solution to SDE (3.2). We note that the general uniqueness for the weak solution for this problem is quite difficult, we will content ourselves with a version that is relatively more amendable.

To begin with, and let $\mathbb{Q}^0$ be the reference probability measure under which (B^1, Y) is a Brownian motion. For each $u \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{Q}^0, [0, T])$, consider the SDE on $(\Omega, \mathcal{F}, \mathbb{Q}^0)$:

$$\begin{cases} dX_t^u = \sigma(t, X^u_{\cdot\wedge t}, \mu_t^{X^u|Y}, u_t)\, dB_t^1, & X_0^u = x; \\ dB_t^2 = dY_t - h(t, X_t^u)\, dt, & B_0^2 = 0; \\ dL_t^u = h(t, X_t^u) L_t^u\, dY_t, & L_0 = 1, t \ge 0, \end{cases} \tag{4.1}$$

where $\mu_t^{X^u|Y} := \mathbb{P}^u \circ [\mathbb{E}^{\mathbb{P}^u}[X_t^u|\mathcal{F}_t^Y]]^{-1}$, and $d\mathbb{P}^u := L_T^u\, d\mathbb{Q}^0$. We shall argue that, under Assumption 2.2, the solution of the SDE (4.1) is pathwisely unique.

REMARK 4.1. It is clear that if $u \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{Q}^0, [0, T])$, and (X^u, B^2, L^u) is a solution to (4.1) under $\mathbb{Q}^0$, then $u \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{P}^u, [0, T])$ [since $\frac{d\mathbb{P}^u}{d\mathbb{Q}^0} \in L^p(\Omega)$ for all $p > 1$, thanks to Assumption 2.2], and the process (X^u, Y, B^1, B^2) is a solution to (2.4) and (2.5) on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^u, \mathbb{F})$ in the sense of

Definition 3.1, where $\mathbb{F} := \mathbb{F}^{B^1,Y}$. Conversely, if $(\Omega, \mathcal{F}, \mathbb{P}^u, \mathbb{F}, B^1, B^2, X, Y)$ is a weak solution of (2.4)–(2.5), then following the argument of Section 2, we see that $d\mathbb{Q}^0 = [L^u_T]^{-1}\, d\mathbb{P}^u$ defines a reference measure, where L^u is defined by (2.6) or (2.7), and $(X, B^2, [L^u]^{-1})$ will be a solution of (4.1) with respect to the $\mathbb{Q}^0$-Brownian motion (B^1, Y). In what follows, we shall call the solution to (4.1) the $\mathbb{Q}^0$-dynamics of the system (2.4) and (2.5).

Bearing Remark 4.1 in mind, let us first try to establish a result in the spirit of the Yamada–Watanabe theorem: *the pathwise uniqueness of* (4.1) *implies the uniqueness in law for the original SDEs* (2.4) *and* (2.5). To do this, we begin by noting that, given the "regular" nature of the canonical space Ω, a process $u \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{P}^u, [0, T])$ amounts to saying that (cf., e.g., [18, 20]) there exists a progressively measurable functional $\mathbf{u} : [0, T] \times \mathbb{C}_T \mapsto U$ such that $u_t(\omega) = \mathbf{u}(t, Y_{\cdot\wedge t}(\omega))$, $dt d\mathbb{P}^u$-a.s., such that u has all the finite moments under $\mathbb{P}^u$ (hence also true under $\mathbb{Q}^0 \sim \mathbb{P}^u$!). We have the following proposition.

PROPOSITION 4.1. *Assume that Assumption* 2.2 *is in force*, *and that the pathwise uniqueness holds for SDE* (4.1). *Let* $\mathbf{u} : [0, T] \times \Omega \mapsto U$ *be a given progressively measurable functional*, *and* $(\Omega, \mathcal{F}, \mathbb{P}^i, \mathbb{F}, B^{1,i}, B^{2,i}, X^i, Y^i)$, $i = 1, 2$, *be two* (*weak*) *solutions of* (2.4)–(2.5) *corresponding to the controls* $u^i = \mathbf{u}(\cdot, Y^i)$, $i = 1, 2$, *respectively. Then it holds that*

$$\mathbb{P}^1 \circ \big[\big(B^{1,1}, B^{2,1}, X^1, Y^1\big)\big]^{-1} = \mathbb{P}^2 \circ \big[\big(B^{1,2}, B^{2,2}, X^2, Y^2\big)\big]^{-1}.$$

PROOF. Following the argument of Section 2.2, we define $d\mathbb{Q}^{0,i} = [L^i_T]^{-1}\, d\mathbb{P}^i$, where $L^i = [\bar{L}^i]^{-1}$ and $\bar{L}^i$ is the unique solution of the SDE (2.6) with respect to $(X^i, B^{1,i}, Y^i)$, $i = 1, 2$. Then, as the $\mathbb{Q}^{0,i}$-dynamics, $(X^i, B^{2,i}, L^i)$ satisfies (4.1), $i = 1, 2$, $\mathbb{Q}^{0,i}$-a.s. In particular, we recall (3.6) that

$$U^{X^i|Y^i}_t = \mathbb{E}^{\mathbb{P}^i}\big[X^i_t|\mathcal{F}^{Y^i}_t\big] = \frac{\mathbb{E}^{\mathbb{Q}^{0,i}}[L^i_t X^i_t|\mathcal{F}^{Y^i}_t]}{\mathbb{E}^{\mathbb{Q}^{0,i}}[L^i_t|\mathcal{F}^{Y^i}_t]}, \qquad \mathbb{Q}^{0,i}\text{-a.s., } t \in [0, T].$$

Thus, there exist two progressively measurable functionals $\Phi^i : [0, T] \times \Omega \mapsto \mathbb{R}$ such that $U^{X^i|Y^i}_t = \Phi^i(t, Y^i_{\cdot\wedge t})$, $dt d\mathbb{Q}^{0,i}$-a.s., $i = 1, 2$. We now consider an intermediate SDE on $(\Omega, \mathcal{F}, \mathbb{Q}^{0,2})$:

$$\text{(4.2)} \quad \begin{cases} d\widehat{X}^2_t = \sigma\big(t, \widehat{X}^2_{\cdot\wedge t}, \Phi^1\big(t, Y^2_{\cdot\wedge t}\big), \mathbf{u}\big(t, Y^2_{\cdot\wedge t}\big)\big)\, dB^{1,2}_t, & \widehat{X}^2_0 = x; \\ d\widehat{L}^2_t = h\big(t, \widehat{X}^2_t\big)\widehat{L}^2_t\, dY^2_t, & \widehat{L}^2_0 = 1, t \in [0, T]. \end{cases}$$

Clearly, comparing with (4.1) for $\mathbb{Q}^{0,1}$-dynamics $(X^1, B^{2,1}, L^1)$, this SDE has the same coefficient $\widehat{\sigma}(t, \omega, \varphi_{\cdot\wedge t}) := \sigma(t, \varphi_{\cdot\wedge t}, \Phi^1(t, \omega^2_{\cdot\wedge t}), \mathbf{u}(t, \omega^2_{\cdot\wedge t}))$, and $h(t, x)\ell$, which is jointly measurable, uniformly Lipschitz in φ with linear growth (in ℓ), uniformly in $(t, \omega, \varphi, \ell)$, thanks to Assumption 2.2, except that it is driven by the $\mathbb{Q}^{0,2}$-Brownian motion $(B^{1,2}, Y^2)$. Thus, by the classical SDE theory (cf., e.g., [14]) we know that there exists a (unique) measurable functional $\Psi : \mathbb{C}_T \times \mathbb{C}_T \to \mathbb{C}_T \times \mathbb{C}_T$ such that $(X^1, L^1) = \Psi(B^{1,1}, Y^1)$, $\mathbb{Q}^{0,1}$-a.s., and $(\widehat{X}^2, \widehat{L}^2) = \Psi(B^{1,2}, Y^2)$, $\mathbb{Q}^{0,2}$-a.s. Since $\mathbb{Q}^{0,1} \circ (B^{1,1}, Y^1)^{-1} = \mathbb{Q}^{0,2} \circ (B^{1,2}, Y^2)^{-1} = \mathbb{Q}^0$, the Wiener measure on $(\Omega, \mathcal{F})$, we deduce that

$$\mathbb{Q}^{0,1} \circ \big(B^{1,1}, Y^1, X^1, L^1\big)^{-1} = \mathbb{Q}^{0,2} \circ \big(B^{1,2}, Y^2, \widehat{X}^2, \widehat{L}^2\big)^{-1}. \tag{4.3}$$

We now claim that $(\widehat{X}^2, B^{2,2}, \widehat{L}^2)$ coincides with the $\mathbb{Q}^{0,2}$-dynamics of (2.4)–(2.5). Indeed, it suffices to argue that in SDE (4.2),

$$\Phi^1\big(t, Y^2_{\cdot\wedge t}\big) = \mathbb{E}^{\widehat{\mathbb{P}}^2}\big[\widehat{X}^2_t \big| \mathbb{F}^{Y^2}_t\big] = U_t^{\widehat{X}^2|Y^2}, \qquad \mathbb{Q}^{0,2}\text{-a.s.,} \tag{4.4}$$

where $d\widehat{\mathbb{P}}^2 := \widehat{L}^2\, d\mathbb{Q}^{0,2}$. To see this, we note that, for all $t \in [0, T]$ and any bounded Borel measurable function $f : \mathbb{C}_T \to \mathbb{R}$, it follows from (4.3) and the definition of $U_t^{X|Y}$ that

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{P}}^2}\big[f\big(Y^2_{\cdot\wedge t}\big)\Phi^1\big(t, Y^2_{\cdot\wedge t}\big)\big] &= \mathbb{E}^{\mathbb{Q}^{0,2}}\big[\widehat{L}^2_t f\big(Y^2_{\cdot\wedge t}\big)\Phi^1\big(t, Y^2_{\cdot\wedge t}\big)\big] \\
&= \mathbb{E}^{\mathbb{Q}^{0,1}}\big[L^1_t f\big(Y^1_{\cdot\wedge t}\big)\Phi^1\big(t, Y^1_{\cdot\wedge t}\big)\big] \\
&= \mathbb{E}^{\mathbb{P}^1}\big[f\big(Y^1_{\cdot\wedge t}\big)U_t^{X^1|Y^1}\big] \\
&= \mathbb{E}^{\mathbb{P}^1}\big[f\big(Y^1_{\cdot\wedge t}\big)X^1_t\big] = \mathbb{E}^{\mathbb{Q}^{0,1}}\big[L^1_t f\big(Y^1_{\cdot\wedge t}\big)X^1_t\big] \\
&= \mathbb{E}^{\mathbb{Q}^{0,2}}\big[\widehat{L}^2_t f\big(Y^2_{\cdot\wedge t}\big)\widehat{X}^2_t\big] \\
&= \mathbb{E}^{\widehat{\mathbb{P}}^2}\big[f\big(Y^2_{\cdot\wedge t}\big)\widehat{X}^2_t\big] = \mathbb{E}^{\widehat{\mathbb{P}}^2}\big[f\big(Y^2_{\cdot\wedge t}\big)U_t^{\widehat{X}^2|Y^2}\big],
\end{aligned}$$

proving (4.4), whence the claim.

Now, by pathwise uniqueness of SDE (4.1), we conclude that $(X^2, L^2) = (\widehat{X}^2, \widehat{L}^2)$, $\mathbb{Q}^{0,2}$-a.s. Thus, (4.3) implies that $\mathbb{Q}^{0,1} \circ [(B^{1,1}, Y^1, X^1, L^1)]^{-1} = \mathbb{Q}^{0,2} \circ [(B^{1,2}, Y^2, X^2, L^2)]^{-1}$, and consequently, $\mathbb{Q}^{0,1} \circ [(B^{1,1}, B^{2,1}, X^1, Y^1)]^{-1} = \mathbb{Q}^{0,2} \circ [(B^{1,2}, B^{2,2}, X^2, Y^2)]^{-1}$. This proves the uniqueness in law for the system (2.4)–(2.5). $\square$

We now turn our attention to the main result of this section: the pathwise uniqueness of (4.1). We shall establish some fundamental estimates which will be useful in our future discussions. Since all controlled dynamics are constructed via the reference probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$, we shall consider only their $\mathbb{Q}^0$-dynamics, namely the solution to (4.1). Recall the space $L^p(\mathbb{Q}^0; L^2([0, T]))$, $p > 1$, and the norm $\|\cdot\|_{p,2,\mathbb{Q}^0}$ defined by (2.3). We have the following important result.

PROPOSITION 4.2. *Assume that Assumption 2.2 is in force. Let $u, v \in \mathscr{U}_{\mathrm{ad}}$ be given. Then, for any $p > 2$, there exists a constant $C_p > 0$, such that the following* *estimates hold*:

$$\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le s\le T}\big(\big|X_s^u - X_s^v\big|^2 + \big|L_s^u - L_s^v\big|^2 + \big|X_s^u L_s^u - X_s^v L_s^v\big|^2\big)\Big] \le C\|u-v\|^2_{2,2,\mathbb{Q}^0}; \tag{4.5}$$

$$\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le s\le T}\big|X_s^u - X_s^v\big|^p\Big] \le C_p\|u-v\|^p_{p,2,\mathbb{Q}^0}. \tag{4.6}$$

PROOF. We split the proof into several steps. Throughout this proof, we let $C>0$ be a generic constant, depending only on the bounds and Lipschitz constants of the coefficients and the time duration $T>0$, and it is allowed to vary from line to line.

Step 1 (*Estimate for X*). First, let us denote, for any $u \in \mathscr{U}_{\mathrm{ad}}$,

$$\sigma^u\big(t,\varphi_{\cdot\wedge t},\mu_t^u\big) \stackrel{\triangle}{=} \int_{\mathbb{R}} \sigma(t,\varphi_{\cdot\wedge t},y,u_t)\mu_t^u(dy), \qquad (t,\varphi)\in[0,T]\times\mathbb{C}_T, \tag{4.7}$$

and $\mu_t^u \stackrel{\triangle}{=} \mu^{X^u|Y}\circ P_t^{-1} = \mathbb{P}^u\circ(\mathbb{E}^{\mathbb{P}^u}[X_t^u|\mathcal{F}_t^Y])^{-1}$, $t\ge 0$. Then, we have

$$\begin{aligned}
&\big|\sigma^u\big(t,X^u_{\cdot\wedge t},\mu_t^u\big) - \sigma^v\big(t,X^v_{\cdot\wedge t},\mu_t^v\big)\big| \\
&\quad= \left|\int_{\mathbb{R}}\sigma\big(t,X^u_{\cdot\wedge t},y,u_t\big)\mu_t^u(dy) - \int_{\mathbb{R}}\sigma\big(t,X^v_{\cdot\wedge t},y,v_t\big)\mu_t^v(dy)\right| \\
&\quad\le C\Bigg\{|u_t-v_t| + \sup_{0\le s\le t}\big|X_s^u - X_s^v\big| \\
&\qquad + \left|\int_{\mathbb{R}}\sigma\big(t,X^v_{\cdot\wedge t},y,v_t\big)\big[\mu_t^u(dy)-\mu_t^v(dy)\big]\right|\Bigg\}.
\end{aligned} \tag{4.8}$$

Next, let us denote $S_t^u = \mathbb{E}^{\mathbb{Q}^0}[L_t^u X_t^u|\mathcal{F}_t^Y]$ and $S_t^{u,0} = \mathbb{E}^{\mathbb{Q}^0}[L_t^u|\mathcal{F}_t^Y]$, and define S_t^v, $S_t^{v,0}$ in a similar way. By (2.19) and the fact that $d\mathbb{P}^u = L_T^u\,d\mathbb{Q}^0$, we see

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\sigma\big(t,X^v_{\cdot\wedge t},y,v_t\big)\big[\mu^u(dy)-\mu^v(dy)\big]\right| \\
&\quad= \big|\mathbb{E}^u\big[\sigma\big(t,\varphi_{\cdot\wedge t},\mathbb{E}^u\big[X_t^u|\mathcal{F}_t^Y\big],u\big)\big] - \mathbb{E}^v\big[\sigma\big(t,\varphi_{\cdot\wedge t},\mathbb{E}^v\big[X_t^v|\mathcal{F}_t^Y\big],u\big)\big]|_{\substack{\varphi=X^v,\\ u=v_t}}\big| \\
&\quad= \left|\mathbb{E}^{\mathbb{Q}^0}\left\{L_t^u\sigma\left(t,\varphi_{\cdot\wedge t},\frac{\mathbb{E}^{\mathbb{Q}^0}[L_t^uX_t^u|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}^0}[L_t^u|\mathcal{F}_t^Y]},u\right)\right.\right. \\
&\qquad\left.\left. - L_t^v\sigma\left(t,\varphi_{\cdot\wedge t},\frac{\mathbb{E}^{\mathbb{Q}^0}[L_t^vX_t^v|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}^0}[L_t^v|\mathcal{F}_t^Y]},u\right)\right\}\Big|_{\substack{\varphi=X^v,\\ u=v_t}}\right| \\
&\quad\le I_1 + I_2,
\end{aligned} \tag{4.9}$$

where (noting the definition of S^u, $S^{u,0}$ and the fact that they are both $\mathbb{F}^Y$-adapted)

$$
\begin{aligned}
I_1 &= \left|\mathbb{E}^{\mathbb{Q}^0}\left\{L_t^u \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u}{S_t^{u,0}}, u\right) - L_t^v \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u}{S_t^{v,0}}, u\right)\right\}\Big|_{\varphi=X^v, u=v_t}\right| \\
&= \left|\mathbb{E}^{\mathbb{Q}^0}\left\{S_t^{u,0} \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u}{S_t^{u,0}}, u\right) - S_t^{v,0} \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u}{S_t^{v,0}}, u\right)\right\}\Big|_{\varphi=X^v, u=v_t}\right|;
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \left|\mathbb{E}^{\mathbb{Q}^0}\left\{L_t^v \left[\sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u}{S_t^{v,0}}, u\right) - \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^v}{S_t^{v,0}}, u\right)\right]\right\}\Big|_{\varphi=X^v, u=v_t}\right| \\
&= \left|\mathbb{E}^{\mathbb{Q}^0}\left\{S_t^{v,0} \left[\sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u}{S_t^{v,0}}, u\right) - \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^v}{S_t^{v,0}}, u\right)\right]\right\}\Big|_{\varphi=X^v, u=v_t}\right|.
\end{aligned}
$$

Clearly, we have

$$
I_2 \le C\mathbb{E}^{\mathbb{Q}^0}\left\{S_t^{v,0}\frac{|S_t^u - S_t^v|}{S_t^{v,0}}\right\} \le C\mathbb{E}^{\mathbb{Q}^0}[|L_t^u X_t^u - L_t^v X_t^v|]. \tag{4.10}
$$

To estimate I_1, we write $\hat{\sigma}(t, \omega, \varphi_{\cdot\wedge t}, y, z) = y\sigma(t, \varphi_{\cdot\wedge t}, \frac{S_t^u(\omega)}{y}, z)$. Since

$$
\begin{aligned}
&\partial_y \hat{\sigma}(t, \omega, \varphi_{\cdot\wedge t}, y, z) \\
&\quad = \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u(\omega)}{y}, z\right) - \frac{S_t^u(\omega)}{y}\partial_y \sigma\left(t, \varphi_{\cdot\wedge t}, \frac{S_t^u(\omega)}{y}, z\right),
\end{aligned} \tag{4.11}
$$

we see that $y \mapsto \partial_y \hat{\sigma}(t, \varphi_{\cdot\wedge t}, y, z)$ is uniformly bounded thanks to Assumption 2.2(iv). Thus, we have

$$
I_1 \le C\|\partial_y \hat{\sigma}\|_\infty \mathbb{E}^{\mathbb{Q}^0}|S_t^{u,0} - S_t^{v,0}| \le C\mathbb{E}^{\mathbb{Q}^0}|L_t^u - L_t^v|. \tag{4.12}
$$

From (4.1), we have $X_t^u - X_t^v = \int_0^t [\sigma^u(s, X_{\cdot\wedge s}^u, \mu_s^u) - \sigma^v(s, X_{\cdot\wedge s}^v, \mu_s^v)]\,dB_s^1$. Combining (4.8)–(4.12), we see that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le s\le t}|X_s^u - X_s^v|^p\Big] \\
&\quad \le C\mathbb{E}^{\mathbb{Q}^0}\bigg\{\bigg[\int_0^t \Big[\sup_{r\in[0,s]}|X_r^u - X_r^v|^2 + |u_s - v_s|^2 \\
&\qquad + (\mathbb{E}^{\mathbb{Q}^0}|L_s^u - L_s^v|)^2 + (\mathbb{E}^{\mathbb{Q}^0}|L_s^u X_s^u - L_s^v X_s^v|)^2\Big]\,ds\bigg]^{p/2}\bigg\}.
\end{aligned} \tag{4.13}
$$

Applying the Gronwall inequality, we obtain that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le s\le t}|X_s^u - X_s^v|^p\Big] &\le C\mathbb{E}^{\mathbb{Q}^0}\bigg\{\bigg[\int_0^t [|u_s - v_s|^2 + \mathbb{E}^{\mathbb{Q}^0}[|L_s^u - L_s^v|^2] \\
&\qquad + \mathbb{E}^{\mathbb{Q}^0}[|L_s^u X_s^u - L_s^v X_s^v|^2]]\,ds\bigg]^{p/2}\bigg\}.
\end{aligned} \tag{4.14}
$$

Step 2 (*Estimate for* L). We first note that, for $t \in [0, T]$,

$$
\begin{aligned}
&\big|L_t^u h(t, X_t^u) - L_t^v h(t, X_t^v)\big| \\
&\quad = \left| L_t^u h\left(t, \frac{L_t^u X_t^u}{L_t^u}\right) - L_t^v h\left(t, \frac{L_t^v X_t^v}{L_t^v}\right)\right| \\
&\quad \le \left| L_t^u h\left(t, \frac{L_t^u X_t^u}{L_t^u}\right) - L_t^u h\left(t, \frac{L_t^v X_t^v}{L_t^u}\right)\right| \\
&\qquad + \left| L_t^u h\left(t, \frac{L_t^v X_t^v}{L_t^u}\right) - L_t^v h\left(t, \frac{L_t^v X_t^v}{L_t^v}\right)\right| \\
&\quad \le C\big|L_t^u X_t^u - L_t^v X_t^v\big| + \left| L_t^u h\left(t, \frac{L_t^v X_t^v}{L_t^u}\right) - L_t^v h\left(t, \frac{L_t^v X_t^v}{L_t^v}\right)\right|.
\end{aligned}
\tag{4.15}
$$

To estimate the second term above, we still define $\hat{h}(t, \omega, x) \stackrel{\triangle}{=} xh(t, \frac{L_t^v(\omega) X_t^v(\omega)}{x})$. Then, similar to (4.11), one shows that $x \mapsto \partial_x \hat{h}(t, \omega, x)$ is uniformly bounded, thanks to Assumption 2.2(v). Consequently, we have

$$
\left| L_t^u h\left(t, \frac{L_t^v X_t^v}{L_t^u}\right) - L_t^v h\left(t, \frac{L_t^v X_t^v}{L_t^v}\right)\right| \le \|\partial_x \hat{h}\|_\infty \big|L_t^u - L_t^v\big|.
\tag{4.16}
$$

Now, combining (4.15) and (4.16) we obtain

$$
\big|L_t^u h(t, X_t^u) - L_t^v h(t, X_t^v)\big| \le C\big(\big|L_t^u - L_t^v\big| + \big|L_t^u X_t^u - L_t^v X_t^v\big|\big).
\tag{4.17}
$$

Therefore, noting that $L_t^u = 1 + \int_0^t h(s, X_s^u) L_s^u \, dY_s$, we deduce from (4.17) and Gronwall's inequality that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0 \le s \le t} \big|L_s^u - L_s^v\big|^2\Big] \\
&\quad \le C \mathbb{E}^{\mathbb{Q}^0}\left[\int_0^t \big|L_s^u X_s^u - L_s^v X_s^v\big|^2 \, ds\right], \qquad \mathbb{Q}^0\text{-a.s., } 0 \le t \le T.
\end{aligned}
\tag{4.18}
$$

Step 3 (*Estimate for* $L_t X_t$). It is clear from (4.14) and (4.18) that it suffices to find the estimate of $L_t^u X_t^u - L_t^v X_t^v$ in terms of $u - v$. To see this, we note that

$$
\begin{aligned}
L_t^u X_t^u &= x + \int_0^t L_s^u X_s^u h(s, X_s^u) \, dY_s \\
&\quad + \int_0^t L_s^u \mathbb{E}^{\mathbb{P}^u}\big[\sigma\big(s, \varphi_{\cdot \wedge s}, \mathbb{E}^{\mathbb{P}^u}\big[X_s^u | \mathcal{F}_s^Y\big], v\big)\big]\Big|_{\substack{\varphi = X^u, \\ v = u_s}} \, dB_s^1.
\end{aligned}
\tag{4.19}
$$

Now define $\tilde{h}(t, x) \stackrel{\triangle}{=} xh(t, x)$. Then it is easily seen that as h satisfies Assumption 2.2(vi), $\tilde{h}$ satisfies Assumption 2.2(v). Thus, similar to (4.17) we have

$$
\begin{aligned}
\big|L_s^u X_s^u h(s, X_s^u) - L_s^v X_s^v h(s, X_s^v)\big| &= \big|L_s^u \tilde{h}(s, X_s^u) - L_s^v \tilde{h}(s, X_s^v)\big| \\
&\le C\big(\big|L_s^u - L_s^v\big| + \big|L_s^u X_s^u - L_s^v X_s^u\big|\big).
\end{aligned}
\tag{4.20}
$$

On the other hand, for any $u \in \mathscr{U}_{\mathrm{ad}}$, recalling (4.7) for the notation σ^u and μ^u, we have

$$\begin{aligned}\Delta_t^{u,v} &\triangleq \big|L_s^u \mathbb{E}^{\mathbb{P}^u}\big[\sigma\big(s, \varphi_{\cdot\wedge s}, \mathbb{E}^{\mathbb{P}^u}\big[X_s^u|\mathcal{F}_s^Y\big], z\big)\big]\big|_{\substack{\varphi=X^u;\\ z=u_s}} \\ &\quad - L_s^v \mathbb{E}^{\mathbb{P}^v}\big[\sigma\big(s, \varphi_{\cdot\wedge s}, \mathbb{E}^{\mathbb{P}^v}\big[X_s^v|\mathcal{F}_s^Y\big], z\big)\big]\big|_{\substack{\varphi=X^v\\ z=v_s}}\big| \\ &= \big|L_t^u \sigma^u\big(t, X_{\cdot\wedge t}^u, \mu_t^u\big) - L_t^v \sigma^v\big(t, X_{\cdot\wedge t}^v, \mu_t^v\big)\big|.\end{aligned}$$

Then, following a similar argument as in Step 1, we have

$$\begin{aligned}\Delta_t^{u,v} &\le C L_t^v \big(\mathbb{E}^{\mathbb{Q}^0}\big[\big|L_t^u - L_t^v\big|\big] + \mathbb{E}^{\mathbb{Q}^0}\big[\big|X_t^u L_t^u - X_t^v L_t^v\big|\big]\big) \\ &\quad + C\big(\big|L_t^u - L_t^v\big| + \big|L_t^u X_t^u - L_t^v X_t^v\big|\big) + C L_t^v |u_t - v_t|.\end{aligned}$$

Squaring both sides above and then taking the expectations, we deduce that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^0}\big[\big|\Delta_t^{u,v}\big|^2\big] &\le C\big(\mathbb{E}^{\mathbb{Q}^0}\big[\big|L_s^u - L_s^v\big|^2\big] + \mathbb{E}^{\mathbb{Q}^0}\big[\big|X_t^u L_t^u - X_t^v L_t^v\big|^2\big]\big) \\ &\quad + C\mathbb{E}^{\mathbb{Q}^0}\big[\big(L_t^v\big)^2 |u_t - v_t|^2\big].\end{aligned} \tag{4.21}$$

Now, combining (4.19)–(4.21), for $p > 2$ we can find $C_p > 0$ such that

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le s\le t} \big|L_s^u X_s^u - L_s^v X_s^v\big|^2\Big] \\ &\quad \le C\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t \big|L_s^u X_s^u h\big(s, X_s^u\big) - L_s^v X_s^v h\big(s, X_s^v\big)\big|^2 ds\Big] \\ &\qquad + C\mathbb{E}^{\mathbb{Q}^0}\int_0^t \big|\Delta_s^{u,v}\big|^2 ds \\ &\quad \le C_p\Big\{\mathbb{E}^{\mathbb{Q}^0}\Big[\Big(\int_0^t |u_s - v_s|^2 ds\Big)^{p/2}\Big]\Big\}^{2/p} + C\mathbb{E}^{\mathbb{Q}^0}\int_0^t \big|L_s^u - L_s^v\big|^2 ds \\ &\qquad + C\mathbb{E}^{\mathbb{Q}^0}\int_0^t \big|L_s^u X_s^u - L_s^v X_s^v\big|^2 ds.\end{aligned} \tag{4.22}$$

Hence, applying Gronwall's inequality we obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le s\le t} \big|L_s^u X_s^u - L_s^v X_s^v\big|^2\Big] \\ &\quad \le C_p \|u - v\|_{p,2,\mathbb{Q}^0}^2 + C\mathbb{E}^{\mathbb{Q}^0}\int_0^t \big|L_s^u - L_s^v\big|^2 ds.\end{aligned} \tag{4.23}$$

Combining (4.23) with (4.18) and applying the Gronwall inequality again, we conclude that

$$\mathbb{E}^{\mathbb{Q}^0}\Big\{\sup_{0\le s\le t} \big|L_s^u - L_s^v\big|^2\Big\} \le C_p \|u - v\|_{p,2,\mathbb{Q}^0}^2. \tag{4.24}$$

This, together with (4.14) and (4.23), implies (4.5). (4.6) then follows easily from (4.5) and (4.13), proving the proposition. □

A direct consequence of Proposition 4.2 is the following uniqueness result.

COROLLARY 4.1. *Assume that Assumption 2.2 holds. Then the solution to SDE (4.1) is pathwisely unique.*

PROOF. Setting $u = v$ in Proposition 4.2, we obtain the result. □

5. A stochastic control problem with partial observation. We are now ready to study the stochastic control problem with partial observation. We first note that in theory for each $(\mathbb{P}^u, u) \in \mathscr{U}_{\mathrm{ad}}$ our state-observation dynamics (X^u, Y^u) lives on probability space $(\Omega, \mathcal{F}, \mathbb{P}^u)$, which varies with control u. We shall consider their $\mathbb{Q}^0$-dynamics so that our analysis can be carried out on a common probability space, thanks to Assumption 2.1. Therefore, in what follows, for each $(\mathbb{P}^u, u) \in \mathscr{U}_{\mathrm{ad}}$ we consider only the $\mathbb{Q}^0$-dynamics (X^u, Y, L^u), which satisfies the following SDE:

$$
\begin{cases} dX_t^u = \sigma^u\big(t, X_{\cdot\wedge t}^u, \mu_t^u\big)\, dB_t^1, & X_0^u = x; \\ dB_t^{2,u} = dY_t - h\big(t, X_t^u\big)\, dt, & B_0^{2,u} = 0; \\ dL_t^u = h\big(t, X_t^u\big) L_t^u \, dY_t, & L_0^u = 1, t \geq 0, \end{cases} \tag{5.1}
$$

where (B^1, Y) is a $\mathbb{Q}^0$-Brownian motion, $d\mathbb{P}^u = L_T^u d\mathbb{Q}^0$, and $\mu_t^{X^u|Y} = \mathbb{P}^u \circ [\mathbb{E}^{\mathbb{P}^u}[X_t|\mathcal{F}_t^Y]]^{-1}$. For simplicity, we denote $\mathbb{E}^u[\cdot] \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{P}^u}[\cdot]$ and $\mathbb{E}^0[\cdot] \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{Q}^0}[\cdot]$.

REMARK 5.1. A convenient and practical way to identify admissible control is to simply consider the space $L_{\mathbb{F}^Y}^{\infty-}(\mathbb{Q}^0; [0, T])$ (cf. Definition 2.1), which is independently well defined, thanks to Assumption 2.1. It is easy to check that, under Assumption 2.2, $u \in L_{\mathbb{F}^Y}^{\infty-}(\mathbb{Q}^0; [0, T])$ if and only if $u \in L_{\mathbb{F}^Y}^{\infty-}(\mathbb{P}^u; [0, T])$. Therefore, in what follows by $u \in \mathscr{U}_{\mathrm{ad}}$ we mean that $u \in L_{\mathbb{F}^Y}^{\infty-}(\mathbb{Q}^0; [0, T])$.

We recall that for $u \in \mathscr{U}_{\mathrm{ad}}$ and $\mu \in \mathscr{P}_2(\mathbb{C}_T)$, the coefficient σ^u in (5.1) is defined by (4.7). Thus, we can write the cost functional as

$$
J(u) \stackrel{\triangle}{=} \mathbb{E}^0\left\{\Phi\big(X_T^u, \mu_T^u\big) + \int_0^T f^u\big(s, X_s^u, \mu_s^u\big)\, ds\right\}. \tag{5.2}
$$

An admissible control $u^* \in \mathscr{U}_{\mathrm{ad}}$ is said to be optimal if

$$
J\big(u^*\big) = \inf_{u \in \mathscr{U}_{\mathrm{ad}}} J(u). \tag{5.3}
$$

We remark that the cost functional $J(\cdot)$ involves the law of the conditional expectation of the solution in a nonlinear way. Therefore, such a control problem is intrinsically "*time-inconsistent*", and thus, the dynamic programming approach in general does not apply. For this reason, we shall consider only the necessary condition of the optimal solution, that is, Pontryagin's maximum principle.

To this end, we let $u^* \in \mathscr{U}_{\mathrm{ad}}$ be an *optimal control*, and consider the convex variations of u^*:

$$u_t^{\theta,v} := u_t^* + \theta(v_t - u_t^*), \qquad t \in [0,T], 0 < \theta < 1, v \in \mathscr{U}_{\mathrm{ad}}. \tag{5.4}$$

Here, we assume that $u^*, v \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{Q}^0; [0,T])$. Since U is convex, $u_t^{\theta,v} \in U$, for all $t \in [0,T]$, $v \in \mathscr{U}_{\mathrm{ad}}$, and $\theta \in (0,1)$. We denote $(X^{\theta,v}, Y, L^{\theta,v})$ to be the corresponding $\mathbb{Q}^0$-dynamics that satisfies (5.1), with control $u^{\theta,v}$. Applying Proposition 4.2 [(4.5) and (4.6)] and noting that Y is a Brownian motion under $\mathbb{Q}^0$, we get, for $p > 2$,

$$\lim_{\theta\to 0} \mathbb{E}^0\Big[\sup_{0\le t\le T} |X_t^{\theta,v} - X_t^{u^*}|^2\Big] \le C_p \lim_{\theta\to 0} \|u^{\theta,v} - u^*\|^2_{p,2,\mathbb{Q}^0} = 0; \tag{5.5}$$

$$\lim_{\theta\to 0} \mathbb{E}^0\Big[\sup_{0\le t\le T} |L_t^{\theta,v} - L_t^{u^*}|^2\Big] = 0. \tag{5.6}$$

In the rest of the section, we shall derive, heuristically, the "variational equations" which play a fundamental role in the study of maximum principle. The complete proof will be given in the next section. For notational simplicity, we shall denote $u = u^*$, the optimal control, from now on, bearing in mind that all discussions will be carried out for the $\mathbb{Q}^0$-dynamics, therefore, on the same probability space.

Now for $u^1, u^2 \in \mathscr{U}_{\mathrm{ad}}$, let (X^1, L^1) and (X^2, L^2) denote the corresponding solutions of (5.1). We define $\delta X = \delta X^{1,2} = \delta X^{u^1,u^2} \stackrel{\triangle}{=} X^{u^1} - X^{u^2}$ and $\delta L = \delta L^{1,2} = \delta L^{u^1,u^2} \stackrel{\triangle}{=} L^{u^1} - L^{u^2}$, and will often drop the superscript "1,2" if the context is clear. Then δX and δL satisfy the equations

$$\begin{cases} \delta X_t = \displaystyle\int_0^t [\sigma^{u^1}(s, X^1_{\cdot\wedge s}, \mu^1_s) - \sigma^{u^2}(s, X^2_{\cdot\wedge s}, \mu^2_s)]\, dB^1_s; \\ \delta L_t = \displaystyle\int_0^t [L^1_s h(s, X^1_s) - L^2_s h(s, X^2_s)]\, dY_s. \end{cases} \tag{5.7}$$

As before, let $U_t^i \stackrel{\triangle}{=} \mathbb{E}^{u^i}[X_t^i | \mathcal{F}_t^Y]$ and $\mu_t^i = \mathbb{P}^{u^i} \circ [U_t^i]^{-1}$, $t \ge 0$, $i = 1, 2$. We can easily check that

$$\begin{aligned} &\sigma^{u^1}(t, X^1_{\cdot\wedge t}, \mu^1_t) - \sigma^{u^2}(t, X^2_{\cdot\wedge t}, \mu^2_t) \\ &\quad = \mathbb{E}^0\{L^1_t \sigma(t, \varphi^1_{\cdot\wedge t}, U^1_t, z^1) - L^2_t \sigma(t, \varphi^2_{\cdot\wedge t}, U^2_t, z^2)\}|_{\substack{\varphi^1 = X^1, \varphi^2 = X^2; \\ z^1 = u^1_t, z^2 = u^2_t}} \\ &\quad = \mathbb{E}^0\bigg\{\delta L^{1,2}_t \sigma(t, \varphi^1_{\cdot\wedge t}, U^1_t, z^1) \\ &\qquad + L^2_t \bigg[\int_0^1 D_\varphi \sigma(t, \varphi^2_{\cdot\wedge t} + \lambda(\varphi^1_{\cdot\wedge t} - \varphi^2_{\cdot\wedge t}), U^1_t, z^1)(\varphi^1_{\cdot\wedge t} - \varphi^2_{\cdot\wedge t})\, d\lambda \end{aligned} \tag{5.8}$$

$$
\begin{aligned}
&+\int_0^1 \partial_y\sigma\big(t,\varphi^2_{\cdot\wedge t},U^2_t+\lambda(U^1_t-U^2_t),z^1\big)\,d\lambda\cdot(U^1_t-U^2_t)\\
&+\int_0^1 \partial_z\sigma\big(t,\varphi^2_{\cdot\wedge t},U^2_t,z^2+\lambda(z^1-z^2)\big)\,d\lambda\cdot(z^1-z^2)\Big]\Big\}\Big|_{\substack{\varphi^1=X^1,\varphi^2=X^2;\\ z^1=u^1_t,z^2=u^2_t}}.
\end{aligned}
$$

Now let $u^1=u^{\theta,v}$ and $u^2=u^*=u$, and denote

$$
\delta_\theta X\triangleq\delta_\theta X^{u,v}=\frac{X^{\theta,v}-X^u}{\theta},\qquad \delta_\theta L\triangleq\delta_\theta L^{u,v}=\frac{L^{\theta,v}-L^u}{\theta},
$$

$$
\delta_\theta U\triangleq\delta_\theta U^{u,v}=\frac{U^{\theta,v}-U^u}{\theta}.
$$

Combining (5.7) and (5.8), we have

$$
\begin{aligned}
\delta_\theta X_t=\int_0^t\big\{&\mathbb{E}^0\big\{\delta_\theta L_s\cdot\sigma\big(s,\varphi^1_{\cdot\wedge s},U^{\theta,v}_s,z^1\big)\big\}\big|_{\substack{\varphi^1=X^{\theta,v},\\ z^1=u^{\theta,v}_s}}+[D\sigma]^{\theta,u,v}_s(\delta_\theta X_{\cdot\wedge s})\\
&+\mathbb{E}^0\big\{B^{\theta,u,v}\big(s,\varphi^2_{\cdot\wedge s},z^1\big)\delta_\theta U_s\big\}\big|_{\substack{\varphi^2=X^u;\\ z^1=u^{\theta,v}_s}}+C^{\theta,u,v}_\sigma(s)(v_s-u_s)\big\}\,dB^1_s,
\end{aligned}
\tag{5.9}
$$

where

$$
\begin{aligned}
&[D\sigma]^{\theta,u,v}_t(\psi)\\
&\quad=\mathbb{E}^0\bigg\{L^u_t\int_0^1 D_\varphi\sigma\big(t,\varphi^2_{\cdot\wedge t}+\lambda(\varphi^1_{\cdot\wedge t}-\varphi^2_{\cdot\wedge t}),U^{\theta,v}_t,z^1\big)(\psi)\,d\lambda\bigg\}\bigg|_{\substack{\varphi^1=X^{\theta,v},\varphi^2=X^u,\\ z^1=u^{\theta,v}_t}},\\
&B^{\theta,u,v}\big(t,\varphi^2_{\cdot\wedge t},z^1\big)=L^u_t\int_0^1\partial_y\sigma\big(t,\varphi^2_{\cdot\wedge t},U^u_t+\lambda(U^{\theta,v}_t-U^u_t),z^1\big)\,d\lambda,\\
&C^{\theta,u,v}_\sigma(t)=\mathbb{E}^0\bigg\{L^u_t\int_0^1\partial_z\sigma\big(t,\varphi^2_{\cdot\wedge t},U^u_t,z^2+\lambda(z^1-z^2)\big)\,d\lambda\bigg\}\bigg|_{\substack{\varphi^2=X^u;z^1=u^{\theta,v}_t,\\ z^2=u_t}}.
\end{aligned}
\tag{5.10}
$$

Here, the integral involving the Fréchet derivative $D_\varphi\sigma$ is in the sense of Bochner. Noting that $U^{\theta,v}_t=\frac{\mathbb{E}^0[L^{\theta,v}_t X^{\theta,v}_t|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^{\theta,v}_t|\mathcal{F}^Y_t]}$ and $U^u_t=\frac{\mathbb{E}^0[L^u_t X^u_t|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^u_t|\mathcal{F}^Y_t]}$, we can easily check that

$$
\begin{aligned}
\delta_\theta U_t&=\frac{\mathbb{E}^0[L^u_t|\mathcal{F}^Y_t]\mathbb{E}^0[L^{\theta,v}_t X^{\theta,v}_t|\mathcal{F}^Y_t]-\mathbb{E}^0[L^{\theta,v}_t|\mathcal{F}^Y_t]\mathbb{E}^0[L^u_t X^u_t|\mathcal{F}^Y_t]}{\theta\mathbb{E}^0[L^{\theta,v}_t|\mathcal{F}^Y_t]\mathbb{E}^0[L^u_t|\mathcal{F}^Y_t]}\\
&=\frac{\mathbb{E}^0[L^u_t|\mathcal{F}^Y_t]\mathbb{E}^0[\delta_\theta L_t X^{\theta,v}_t+L^u_t\delta_\theta X_t|\mathcal{F}^Y_t]-\mathbb{E}^0[\delta_\theta L_t|\mathcal{F}^Y_t]\mathbb{E}^0[L^u_t X^u_t|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^{\theta,v}_t|\mathcal{F}^Y_t]\mathbb{E}^0[L^u_t|\mathcal{F}^Y_t]}\\
&=\frac{\mathbb{E}^0[\delta_\theta L_t X^{\theta,v}_t+L^u_t\delta_\theta X_t|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^{\theta,v}_t|\mathcal{F}^Y_t]}\\
&\qquad-\frac{\mathbb{E}^0[\delta_\theta L_t|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^{\theta,v}_t|\mathcal{F}^Y_t]}U^u_t.
\end{aligned}
\tag{5.11}
$$

Now, sending $\theta \to 0$, and assuming that

$$K_t = K_t^{u,v} \stackrel{\triangle}{=} \lim_{\theta \to 0} \delta_\theta X_t^{u,v}; \qquad R_t = R_t^{u,v} \stackrel{\triangle}{=} \lim_{\theta \to 0} \delta_\theta L_t^{u,v} \tag{5.12}$$

both exist in $L^2(\mathbb{Q}^0)$, then from (5.7)–(5.11) we have, at least formally,

$$\begin{aligned} K_t = \int_0^t \Big\{ & \mathbb{E}^0 \big[R_s \sigma\big(s, \varphi_{\cdot\wedge s}, U_s^u, z\big)\big]\big|_{\varphi = X^u, z = u_s} + [D\sigma]_s^{u,v}(K_{\cdot\wedge s}) \\ & + \mathbb{E}^0 \bigg[B^{u,v}(s, \varphi_{\cdot\wedge s}, z) \bigg(\frac{\mathbb{E}^0[R_s X_s^u + L_s^u K_s | \mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u | \mathcal{F}_s^Y]} \\ & - \frac{\mathbb{E}^0[R_s | \mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u | \mathcal{F}_s^Y]} U_s^u \bigg) \bigg] \bigg|_{\substack{\varphi = X^u; \\ z = u_s}} + C_\sigma^{u,v}(s)(v_s - u_s) \Big\} \, dB_s^1, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} [D\sigma]_t^{u,v}(\psi) &\stackrel{\triangle}{=} \mathbb{E}^0 \big\{ L_t^u D_\varphi \sigma\big(t, \varphi_{\cdot\wedge t}, U_t^u, z\big)(\psi)\big\}\big|_{\varphi = X^u; z = u_t}, \\ B^{u,v}(t, \varphi_{\cdot\wedge t}, z) &\stackrel{\triangle}{=} L_t^u \partial_y \sigma\big(t, \varphi_{\cdot\wedge t}, U_t^u, z\big), \\ C_\sigma^{u,v}(t) &\stackrel{\triangle}{=} \mathbb{E}^0 \big\{ L_t^u \partial_z \sigma\big(t, \varphi_{\cdot\wedge t}, U_t^u, z\big)\big\}\big|_{\varphi = X^u; z = u_t}. \end{aligned} \tag{5.14}$$

Observing also that U_t^u is $\mathcal{F}_t^Y$-measurable, we have

$$\begin{aligned} & \mathbb{E}^0 \bigg[B^{u,v}(s, \varphi_{\cdot\wedge s}, z) \bigg(\frac{\mathbb{E}^0[R_s X_s^u + L_s^u K_s | \mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u | \mathcal{F}_s^Y]} - \frac{\mathbb{E}^0[R_s | \mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u | \mathcal{F}_s^Y]} U_s^u \bigg) \bigg] \bigg|_{\substack{\varphi = X^u; \\ z = u_s}} \\ & \quad = \mathbb{E}^u \big[\partial_y \sigma\big(s, \varphi_{\cdot\wedge s}, U_s^u, z\big) \mathbb{E}^u \big\{ \big(L_s^u\big)^{-1} R_s \big[X_s^u - U_s^u\big] + K_s | \mathcal{F}_s^Y \big\}\big]\big|_{\substack{\varphi = X^u; \\ z = u_s}} \\ & \quad = \mathbb{E}^u \big[\big(L_s^u\big)^{-1} \partial_y \sigma\big(s, \varphi_{\cdot\wedge s}, U_s^u, z\big) \big\{ R_s \big[X_s^u - U_s^u\big] + L_s^u K_s \big\}\big]\big|_{\substack{\varphi = X^u; \\ z = u_s}} \\ & \quad = \mathbb{E}^0 \big[\partial_y \sigma\big(s, \varphi_{\cdot\wedge s}, U_s^u, z\big)\big(R_s X_s^u + L_s^u K_s\big) \\ & \qquad - U_s^u \partial_y \sigma\big(s, \varphi_{\cdot\wedge s}, U_s^u, z\big) R_s \big]\big|_{\substack{\varphi = X^u; \\ z = u_s}}. \end{aligned} \tag{5.15}$$

Consequently, if we define

$$\Psi(t, \varphi_{\cdot\wedge t}, x, y, z) \stackrel{\triangle}{=} \sigma(t, \varphi_{\cdot\wedge t}, y, z) + \partial_y \sigma(t, \varphi_{\cdot\wedge t}, y, z)(x - y), \tag{5.16}$$

then we can rewrite (5.13) as

$$\begin{aligned} K_t = \int_0^t \{ & \mathbb{E}^0 \big[\Psi\big(s, \varphi_{\cdot\wedge s}, X_s^u, U_s^u, z\big) R_s + \partial_y \sigma\big(s, \varphi_{\cdot\wedge s}, U_s^u, z\big) L_s^u K_s \big]\big|_{\substack{\varphi = X^u; \\ z = u_s}} \\ & + [D\sigma]_s^{u,v}(K_{\cdot\wedge s}) + C_\sigma^{u,v}(s)(v_s - u_s) \} \, dB_s^1. \end{aligned} \tag{5.17}$$

Similarly, we can formally write down the SDE for R:

$$R_t = \int_0^t \big[R_s h\big(s, X_s^u\big) + L_s^u \partial_x h\big(s, X_s^u\big) K_s \big] \, dY_s, \qquad t \geq 0. \tag{5.18}$$

The following theorem is regarding the well-posedness of the SDEs (5.17) and (5.18).

THEOREM 5.1. *Assume that Assumption 2.2 is in force, and let* $u, v \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{Q}^0; [0,T])$ *be given. Then there is a unique solution* $(K, R) \in \mathscr{L}^{\infty-}_{\mathbb{F}}(\mathbb{Q}^0; \mathbb{C}^2_T)$ *to SDEs* (5.17) *and* (5.18).

PROOF. Let $u, v \in L^{\infty-}_{\mathbb{F}^Y}(\mathbb{Q}^0; [0,T])$ be given. We define $F^1_t(K,R)$ and $F^2_t(K,R)$, $t \in [0,T]$, to be the right-hand side of (5.17) and (5.18), respectively.

We first observe that $F^1_t(0,0) = \int_0^t C^{u,v}_\sigma(s)(v_s - u_s)\,dB^1_s$, and $F^2_t(0,0) \equiv 0$, $t \in [0,T]$. Then, for any $p > 2$, it holds that

$$\mathbb{E}^u\Big[\sup_{0\le s\le t} \big|F^1_s(0,0)\big|^p\Big] \le C_p \mathbb{E}^u\bigg[\bigg(\int_0^t |v_s - u_s|^2\,ds\bigg)^{p/2}\bigg], \qquad t \in [0,T]. \tag{5.19}$$

Now let $(K^i, R^i) \in \mathscr{L}^{\infty-}_{\mathbb{F}}(\mathbb{Q}^0; \mathbb{C}_T)$, $i = 1,2$. We define $\widetilde{K}^i \triangleq F_1(K^i, R^i)$, $\widetilde{R}^i \triangleq F_1(K^i, R^i)$, $i = 1,2$ and $\bar{K} \triangleq K^1 - K^2$, $\bar{R} \triangleq R^1 - R^2$, $\hat{K} \triangleq \widetilde{K}^1 - \widetilde{K}^2$, and $\hat{R} \triangleq \widetilde{R}^1 - \widetilde{R}^2$. Then, noting that σ, $\partial_y\sigma$, $y\partial_y\sigma$, and $\partial_z\sigma$ are all bounded, thanks to Assumption 2.2, we see that

$$\big|\Psi(t, \varphi_{\cdot\wedge t}, x, y, z)\big| \le C\big(1 + |x|\big), \qquad (t,x,y,z) \in [0,T] \times \mathbb{R}^3, \varphi \in \mathbb{C}_T,$$

where, and in what follows, $C > 0$ is some generic constant which is allowed to vary from line to line. It then follows that

$$\begin{aligned} &\big|\mathbb{E}^0\big[\Psi\big(t, \varphi_{\cdot\wedge t}, X^u_t, U^u_t, z\big)\bar{R}_s + \partial_y\sigma\big(t, \varphi_{\cdot\wedge t}, U^u_t, z\big)L^u_t\bar{K}_t\big]\big| \\ &\quad \le C\mathbb{E}^0\big[\big(1 + |X^u_t|\big)|\bar{R}_t| + \big|L^u_t\bar{K}_t\big|\big] \le C\big[\mathbb{E}^0\big[|\bar{K}_t|^2 + |\bar{R}_t|^2\big]\big]^{1/2}. \end{aligned} \tag{5.20}$$

Since $D_\varphi\sigma$ is also bounded, we have $|[D\sigma]^{u,v}_t(\psi)| \le C\sup_{0\le s\le t}|\psi(s)|$, for $\psi \in \mathbb{C}_T$. Then from the definition of $\hat{K}$ and (5.20) we have, for any $p \ge 2$ and $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}^0\Big[\sup_{0\le s\le t}|\hat{K}_s|^{2p}\Big] &\le C_p\int_0^t \big(\mathbb{E}^0\big[|\bar{R}_s|^2 + |\bar{K}_s|^2\big]\big)^p\,ds \\ &\quad + C_p\int_0^t \mathbb{E}^0\Big[\sup_{0\le r\le s}|\bar{K}_r|^{2p}\Big]\,ds. \end{aligned} \tag{5.21}$$

On the other hand, the boundedness of h and $\partial_x h$ implies that, recalling the definition of $\hat{R}$, for $p \ge 2$ and $t \in [0,T]$,

$$\begin{aligned} \Big(\mathbb{E}^0\Big[\sup_{s\le t}|\hat{R}_s|^p\Big]\Big)^2 &\le C_p\int_0^t \mathbb{E}^0\big[|\bar{R}_s|^p\big]^2\,ds + C_p\int_0^t \mathbb{E}^0\big[|L^u_s\bar{K}_s|^p\big]^2\,ds \\ &\le C_p\int_0^t \big(\mathbb{E}^0\big[|\bar{R}_s|^p\big]\big)^2\,ds + C_p\int_0^t \mathbb{E}^0\big[|\bar{K}_s|^{2p}\big]\,ds. \end{aligned} \tag{5.22}$$

Combining (5.21) and (5.22) we have, for $t \in [0, T]$,

$$\mathbb{E}^0\Big[\sup_{0\le s\le t} |\hat{K}_s|^{2p}\Big] + \Big(\mathbb{E}^0\Big[\sup_{0\le s\le t} |\hat{R}_s|^{p}\Big]\Big)^2$$
$$\le C_p \int_0^t (\mathbb{E}^0\Big[\sup_{0\le r\le s} |\bar{K}_r|^{2p}\Big] + \Big(\mathbb{E}^0\Big[\sup_{0\le r\le s} |\bar{R}_r|^{p}\Big]^2\Big)\,ds.$$

This, together with (5.19), enables us to apply standard SDE arguments to deduce that there is a unique solution $(K, R) \in \mathscr{L}_{\mathbb{F}}^{\infty-}(\mathbb{P}; \mathbb{C}_T)$ of (5.17) and (5.18), such that for all $p \ge 2$, it holds that

$$\mathbb{E}^0[\|K\|_{\mathbb{C}_T}^{2p}] + \mathbb{E}^0[\|R\|_{\mathbb{C}_T}^{2p}] \le C_p \|v_s - u_s\|_{p,2,\mathbb{Q}^0}^2. \tag{5.23}$$

We leave it to the interested reader, and this completes the proof. □

6. Variational equations. In this section, we validate the heuristic arguments in the previous section and derive the variational equation of the optimal trajectory rigorously. Recall the processes $\delta_\theta X = \delta_\theta X^{u,v}$, $\delta_\theta L = \delta_\theta L^{u,v}$, and (K, R) defined in the previous section. Denote

$$\eta_t^\theta \stackrel{\triangle}{=} \delta_\theta X_t - K_t, \qquad \tilde{\eta}_t^\theta \stackrel{\triangle}{=} \delta_\theta L_t - R_t, \qquad t \in [0, T]. \tag{6.1}$$

Our main purpose of this section is to prove the following result.

PROPOSITION 6.1. *Let $(\mathbb{P}^u, u) = (\mathbb{P}^{u^*}, u^*) \in \mathscr{U}_{\mathrm{ad}}$ be an optimal control, (X^u, L^u) be the corresponding solution of (5.1) and let $U_t^u = \mathbb{E}^u[X_t^u|\mathcal{F}_t^Y]$, $t \ge 0$. For any $v \in \mathscr{U}_{\mathrm{ad}}$, let $(K, R) = (K^{u,v}, R^{u,v})$ be the solution of the linear equations (5.17) and (5.18). Then, for all $p > 1$, it holds that*

$$\lim_{\theta\to 0} \mathbb{E}^0[\|\eta^\theta\|_{\mathbb{C}_T}^p] = \lim_{\theta\to 0} \mathbb{E}^0\Big[\sup_{s\in[0,T]}\Big|\frac{X_s^{\theta,v} - X_s^u}{\theta} - K_s\Big|^p\Big] = 0; \tag{6.2}$$

$$\lim_{\theta\to 0} \mathbb{E}^0[\|\tilde{\eta}^\theta\|_{\mathbb{C}_T}^p] = \lim_{\theta\to 0} \mathbb{E}^0\Big[\sup_{s\in[0,T]}\Big|\frac{L_s^{\theta,v} - L_s^u}{\theta} - R_s\Big|^p\Big] = 0. \tag{6.3}$$

The proof of Proposition 6.1 is quite lengthy, and we shall split it into two parts.

PROOF OF (6.3). This part is relatively easy. We note that with a direct calculation using the equations (5.7) and (5.18) it is readily seen that $\tilde{\eta}^\theta$ satisfies the following SDE:

$$\begin{aligned}\tilde{\eta}_t^\theta &= \int_0^t \tilde{\eta}_r^\theta h(r, X_r^{\theta,v})\,dY_r \\ &\quad + \int_0^t L_r^u \int_0^1 \partial_x h(r, X_r^u + \lambda\theta(\eta_r^\theta + K_r))\eta_r^\theta\,d\lambda\,dY_r + I_t^{1,\theta} + I_t^{2,\theta},\end{aligned} \tag{6.4}$$

where

$$
\begin{aligned}
I_t^{1,\theta} &= \int_0^t R_r\big(h\big(r, X_r^{\theta,v}\big) - h\big(r, X_r^u\big)\big)\, dY_r; \\
I_t^{2,\theta} &= \int_0^t L_r^u \int_0^1 \partial_x h\big(r, X_r^u + \lambda\theta\big(\eta_r^\theta + K_r\big)\big) K_r \, d\lambda \, dY_r \\
&\quad - \int_0^t L_r^u \partial_x h\big(r, X_r^u\big) K_r \, dY_r.
\end{aligned}
$$

We claim that, for all $p > 1$,

$$
\begin{aligned}
&\lim_{\theta\to 0} \mathbb{E}^u\Big[\sup_{t\in[0,T]} |I_t^{1,\theta}|^p\Big] = 0, \\
&\lim_{\theta\to 0} \mathbb{E}^u\Big[\sup_{t\in[0,T]} |I_t^{2,\theta}|^p\Big] = 0.
\end{aligned}
\tag{6.5}
$$

Indeed, note that $dY_t = dB_t^2 - h(t, X_t^u)\, dt$, and B^2 is a $\mathbb{P}^u$-Brownian motion. Proposition 4.2, together with the bounded and continuity of h and $\partial_x h$, leads to that, for all $p \geq 2$,

$$
\begin{aligned}
&\mathbb{E}^u\Big\{\sup_{t\in[0,T]} |I_t^{1,\theta}|^p\Big\} \\
&\quad = \mathbb{E}^0\bigg\{L_T^u \sup_{t\in[0,T]} \bigg|\int_0^t R_s\big[h\big(s, X_s^{\theta,v}\big) - h\big(s, X_s^u\big)\big]\, dY_s\bigg|^p\bigg\} \\
&\quad \leq 2\mathbb{E}^u\bigg\{\sup_{t\in[0,T]} \bigg|\int_0^t R_s\big[h\big(s, X_s^{\theta,v}\big) - h\big(s, X_s^u\big)\big]\, dB_s^2\bigg|^p\bigg\} \\
&\qquad + 2\mathbb{E}^0\bigg\{L_T^u \sup_{t\in[0,T]} \bigg|\int_0^t R_s\big[h\big(s, X_s^{\theta,v}\big) - h\big(s, X_s^u\big)\big] h\big(s, X_s^u\big)\, ds\bigg|^p\bigg\} \\
&\quad \leq C_p \mathbb{E}^0\bigg\{L_T^u \int_0^T R_s^p\big(|X_s^{\theta,v} - X_s^u|^p \wedge 1\big)\, ds\bigg\} \\
&\quad \leq C_p\{\mathbb{E}^0[(L_T^u)^3]\}^{\frac{1}{3}}\Big\{\mathbb{E}^0\Big[\sup_{s\in[0,T]} |R_s|^{3p}\Big]\Big\}^{\frac{1}{3}}\Big\{\mathbb{E}^0\Big[\sup_{s\in[0,T]} \big(|X_s^{\theta,v} - X_s^u|^2 \wedge 1\big)\Big]\Big\}^{\frac{1}{3}} \\
&\quad \leq C_p\|u - u^{\theta,v}\|_{p,2,\mathbb{Q}^0}^{\frac{2}{3}} \leq C|\theta|^{\frac{2}{3}},
\end{aligned}
$$

where we used the following estimate for any function $f \in L^\infty(\mathbb{R})$ bounded by $C_0 \geq 1$:

$$
\begin{aligned}
\big|f(x) - f(x')\big|^{3p} &\leq \big(2C_0\big(\big|f(x) - f(x')\big| \wedge 1\big)\big)^{3p} \\
&\leq (2C_0)^{3p}\big(\big|f(x) - f(x')\big|^2 \wedge 1\big), \qquad \forall p \geq 2.
\end{aligned}
\tag{6.6}
$$

Similarly, we have

$$
\begin{aligned}
\mathbb{E}^u\Big\{\sup_{t\in[0,T]}|I_t^{2,\theta}|^p\Big\} &= \mathbb{E}^0\Big\{L_T^u \sup_{t\in[0,T]}\Big|\int_0^t L_r^u K_r\Big[\int_0^1 \big[\partial_x h\big(r, X_r^u+\lambda\theta(\eta_r^\theta+K_r)\big) \\
&\qquad - \partial_x h\big(r, X_r^u\big)\big]\,d\lambda\Big]\,dY_r\Big|^p\Big\} \\
&\le C_p\mathbb{E}^0\Big\{L_T^u\int_0^T |L_r^u|^p|K_r|^p\Big[\int_0^1 \big|\partial_x h\big(r, X_r^u+\lambda\theta(\eta_r^\theta+K_r)\big) \\
&\qquad - \partial_x h\big(r, X_r^u\big)\big|\,d\lambda\Big]^p\,dr\Big\} \\
&\le C_p\mathbb{E}^0\Big\{\int_0^T\Big[\int_0^1 \big[\big|\partial_x h\big(r, X_r^u+\lambda\theta(\eta_r^\theta+K_r)\big) \\
&\qquad - \partial_x h\big(r, X_r^u\big)\big|^2\wedge 1\big]\,d\lambda\Big]\,dr\Big\}^{1/3}.
\end{aligned}
$$

Here, in the above the second inequality follows from (6.6) applied to $\partial_x h$, the Hölder inequality, and the fact that $L^u, K \in \mathscr{L}_{\mathbb{F}}^{\infty-}(\mathbb{Q}^0; \mathbb{C}_T)$ (see Theorem 5.1), and the last inequality follows from the L^p-estimate (5.23). Now, from (4.5), (5.17) and (5.18) we see that

$$
\mathbb{E}^0\Big\{\sup_{t\in[0,T]}\big(|\eta_t^\theta|^2+|K_t|^2\big)\Big\}\le C, \qquad \theta\in(0,1).
$$

Hence, since $\theta[\|\eta^\theta\|_{\mathbb{C}_T}+\|K\|_{\mathbb{C}_T}]\to 0$, in probability $\mathbb{Q}^0$, as $\theta\to 0$, the continuity of $\partial_x h$ and the bounded convergence theorem then imply (6.5), proving the claim. Recalling (6.4), we see that (6.3) follows from (6.5), provided (6.2) holds, which we now substantiate. $\square$

PROOF OF (6.2). This part is more involved. We first rewrite (5.9) as follows:

$$
\begin{aligned}
\delta_\theta X_t = \int_0^t &\{\mathbb{E}^0\{(\tilde{\eta}_s^\theta+R_s)\sigma(s,\varphi_{\cdot\wedge s},U_s^{\theta,v},z)\}|_{\substack{\varphi=X^{\theta,v},\\ z=u_s^{\theta,v}}} \\
&+[D\sigma]_s^{\theta,u,v}\big(\eta_{\cdot\wedge s}^\theta+K_{\cdot\wedge s}\big)+\mathbb{E}^0\{B^{\theta,u,v}(s,\varphi_{\cdot\wedge s},z)\delta_\theta U_s\}|_{\substack{\varphi=X^u;\\ z=u_s^{\theta,v}}} \\
&+C_\sigma^{\theta,u,v}(s)(v_s-u_s)\}\,dB_s^1.
\end{aligned}\tag{6.7}
$$

Here, $[D\sigma]^{\theta,u,v}$, $B^{\theta,u,v}$ and $C^{\theta,u,v}$ are defined by (5.10). Furthermore, in light of (5.11), we can also write

$$
\delta_\theta U_t = \frac{\mathbb{E}^0[(\tilde{\eta}_t^\theta+R_t)X_t^{\theta,v}+L_t^u(\eta_t^\theta+K_t)|\mathcal{F}_t^Y]}{\mathbb{E}^0[L_t^{\theta,v}|\mathcal{F}_t^Y]}-\frac{\mathbb{E}^0[(\tilde{\eta}_t^\theta+R_t)|\mathcal{F}_t^Y]}{\mathbb{E}^0[L_t^{\theta,v}|\mathcal{F}_t^Y]}U_t^u.
$$

Plugging this into (6.7), we have

$$
\begin{aligned}
\delta_\theta X_t = \int_0^t \Big\{ & \mathbb{E}^0\{\tilde{\eta}^\theta_s \sigma(s, \varphi_{\cdot\wedge s}, U^{\theta,v}_s, z)\}|_{\substack{\varphi = X^{\theta,v}, \\ z = u^{\theta,v}_s}} + [D\sigma]^{\theta,u,v}_s(\eta^\theta_{\cdot\wedge s}) \\
& + \mathbb{E}^0\Big\{ B^{\theta,u,v}(s, \varphi_{\cdot\wedge s}, z)\Big[\frac{\mathbb{E}^0[\tilde{\eta}^\theta_s X^{\theta,v}_s + L^u_s \eta^\theta_s | \mathcal{F}^Y_s]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} \\
& - \frac{\mathbb{E}^0[\tilde{\eta}^\theta_s|\mathcal{F}^Y_s]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} U^u_s \Big]\Big\}\Big|_{\substack{\varphi = X^u; \\ z = u^{\theta,v}_s}} \Big\} dB^1_s \\
& + \int_0^t \Big\{ \mathbb{E}^0\{R_s \sigma(s, \varphi_{\cdot\wedge s}, U^\theta_s, z)\}|_{\substack{\varphi = X^{\theta,v}, \\ z = u^{\theta,v}_s}} + [D\sigma]^{\theta,u,v}_s(K_{\cdot\wedge s}) \\
& + \mathbb{E}^0\Big\{ B^{\theta,u,v}(s, \varphi_{\cdot\wedge s}, z)\Big[\frac{\mathbb{E}^0[R_s X^{\theta,v}_s + L^u_s K_s | \mathcal{F}^Y_s]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} \\
& - \frac{\mathbb{E}^0[R_s|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} U^u_s \Big]\Big\}\Big|_{\substack{\varphi = X^{\theta,v}; \\ z = u^{\theta,v}_s}} + C^{\theta,u,v}_\sigma(s)(v_s - u_s) \Big\} dB^1_s.
\end{aligned}
$$

Now, recalling (5.17) [or more conveniently, (5.13)] we have

$$
\begin{aligned}
\eta^\theta_t &= \delta_\theta X_t - K_t \\
&= \int_0^t \Big\{ \mathbb{E}^0\{\tilde{\eta}^\theta_s \sigma(s, \varphi_{\cdot\wedge s}, U^{\theta,v}_s, z)\}|_{\substack{\varphi = X^{\theta,v}, \\ z = u^{\theta,v}_s}} + [D\sigma]^{\theta,u,v}_s(\eta^\theta_{\cdot\wedge s}) \\
&\quad + \mathbb{E}^0\Big\{ B^{\theta,u,v}(s, \varphi_{\cdot\wedge s}, z)\Big[\frac{\mathbb{E}^0[\tilde{\eta}^\theta_s X^{\theta,v}_s + L^u_s \eta^\theta_s | \mathcal{F}^Y_s]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} \\
&\quad - \frac{\mathbb{E}^0[\tilde{\eta}^\theta_s|\mathcal{F}^Y_s]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} U^u_s \Big]\Big\}\Big|_{\substack{\varphi = X^u; \\ z = u^{\theta,v}_s}} \Big\} dB^1_s \\
&\quad + I^{3,\theta,1}_t + I^{3,\theta,2}_t + I^{3,\theta,3}_t + I^{3,\theta,4}_t,
\end{aligned}
\tag{6.8}
$$

where, for $t \in [0, T]$,

$$
\begin{aligned}
I^{3,\theta,1}_t &\triangleq \int_0^t \mathbb{E}^0\{R_s[\sigma(s, \varphi^1_{\cdot\wedge s}, U^{\theta,v}_s, z^1) - \sigma(s, \varphi^2_{\cdot\wedge s}, U^u_s, z^2)]\}\Big|_{\substack{\varphi^1 = X^{\theta,v}, z^1 = u^{\theta,v}_s \\ \varphi^2 = X^u, z^2 = u_s}} dB^1_s; \\
I^{3,\theta,2}_t &\triangleq \int_0^t \mathbb{E}^0\{[D\sigma]^{\theta,u,v}_s(K_{\cdot\wedge s}) - [D\sigma]^{u,v}_s(K_{\cdot\wedge s})\} dB^1_s; \\
I^{3,\theta,3}_t &\triangleq \int_0^t \Big\{ \mathbb{E}^0\Big\{ B^{\theta,u,v}(s, \varphi_{\cdot\wedge s}, z)\Big(\frac{\mathbb{E}^0[R_s X^{\theta,v}_s + L^u_s K_s|\mathcal{F}^Y_s]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} \\
&\qquad - \frac{\mathbb{E}^0[R_s|\mathcal{F}^Y_t]}{\mathbb{E}^0[L^{\theta,v}_s|\mathcal{F}^Y_s]} U^u_s\Big)\Big\}\Big|_{\substack{\varphi = X^u; \\ z = u^{\theta,v}_s}}
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
&- \mathbb{E}^0\bigg\{B^{u,v}(s,\varphi_{\cdot\wedge s},z)\bigg(\frac{\mathbb{E}^0[R_sX_s^u+L_s^uK_s|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}\\
&- \frac{\mathbb{E}^0[R_s|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}U_s^u\bigg)\bigg\}\bigg|_{\substack{\varphi=X^u;\\ z=u_s}}\bigg\}\,dB_s^1;
\end{aligned}
$$

$$
I_t^{3,\theta,4} \triangleq \int_0^t \mathbb{E}^0\big[C_\sigma^{\theta,u,v}(s)(v_s-u_s)-C_\sigma^{u,v}(s)(v_s-u_s)\big]\,dB_s^1. \qquad \square
$$

We have the following lemma.

LEMMA 6.1. *Suppose that Assumption* 2.2 *holds. Then, for all* $p>1$,

$$
\lim_{\theta\to 0}\mathbb{E}^0\Big\{\sup_{0\le t\le T}|I_t^{3,\theta,i}|^p\Big\}=0, \qquad i=1,\dots,4. \tag{6.10}
$$

PROOF. We first recall that $U_s^{\theta,v}\triangleq\mathbb{E}^{\theta,v}[X_s^{\theta,v}|\mathcal{F}_s^Y]$ and $U_s^u\triangleq\mathbb{E}^u[X_s^u|\mathcal{F}_s^Y]$. Using the Kallianpur–Strieble formula, we have

$$
\begin{aligned}
&\mathbb{E}^0\int_0^T|U_s^{\theta,v}-U_s^u|^p\,ds\\
&\quad\le C_p\bigg\{\mathbb{E}^0\int_0^T\bigg|\frac{\mathbb{E}^0[L_s^{\theta,v}X_s^{\theta,v}|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]}-\frac{\mathbb{E}^0[L_s^uX_s^u|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]}\bigg|^p\,ds\\
&\qquad+\mathbb{E}^0\int_0^T\bigg|\frac{\mathbb{E}^0[L_s^uX_s^u|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]}-\frac{\mathbb{E}^0[L_s^uX_s^u|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}\bigg|^p\,ds\bigg\}\\
&\quad\triangleq C_p\{J_\theta^1+J_\theta^2\}.
\end{aligned}\tag{6.11}
$$

We now estimate J_θ^1 and J_θ^2, respectively. First, note that, for any $p>1$, we can find a constant $C_p>0$ such that for any $\theta\in(0,1)$ and $u\in\mathscr{U}_{\mathrm{ad}}$,

$$
\mathbb{E}^0\big[(L_s^{\theta,v})^p\big]+\mathbb{E}^0\big[(L_s^{\theta,v})^{-p}\big]+\mathbb{E}^0\big[(L_s^u)^p\big]\le C_p.
$$

Thus, applying the Hölder and Jensen inequalities as well as Proposition 4.2, we have, for any $p>1$, and $\theta\in(0,1)$,

$$
\begin{aligned}
&\mathbb{E}^0\int_0^T\bigg|\frac{\mathbb{E}^0[L_s^{\theta,v}X_s^{\theta,v}|\mathcal{F}_s^Y]-\mathbb{E}^0[L_s^uX_s^u|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]}\bigg|^p\,ds\\
&\quad\le\int_0^T\mathbb{E}^0\bigg\{\frac{|L_s^{\theta,v}X_s^{\theta,v}-L_s^uX_s^u|^p}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]^p}\bigg\}\,ds\\
&\quad\le\int_0^T\Big\{\{\mathbb{E}^0|L_s^{\theta,v}X_s^{\theta,v}-L_s^uX_s^u|^2\}^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{\mathbb{E}^0\left[\frac{|L_s^{\theta,v}X_s^{\theta,v} - L_s^u X_s^u|^{2p-2}}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]^{2p}}\right]\right\}^{1/2}\Bigg\} ds \\
&\le \int_0^t \{\mathbb{E}^0|L_s^{\theta,v}X_s^{\theta,v} - L_s^u X_s^u|^2\}^{1/2} \\
&\quad \times \{\mathbb{E}^0[|L_s^{\theta,v}X_s^{\theta,v} - L_s^u X_s^u|^{2p-2}]\mathbb{E}^0[[L_s^{\theta,v}]^{-2p}|\mathcal{F}_s^Y]]\}^{1/2} ds \\
&\le C_p\theta\|u-v\|_{2,2,\mathbb{Q}^0}.
\end{aligned} \tag{6.12}
$$

Similarly, one can also argue that, for any $p>1$, the following estimates hold:

$$
\begin{aligned}
&\mathbb{E}^0\int_0^T \left|\frac{1}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]} - \frac{1}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}\right|^p ds \\
&\qquad \le C_p\theta\|u-v\|_{2,2,\mathbb{Q}^0}, \qquad \theta\in(0,1).
\end{aligned} \tag{6.13}
$$

Clearly, (6.12) and (6.13) imply that $J_\theta^1 + J_\theta^2 \le C_p\theta\|u-v\|_{2,2,\mathbb{Q}^0}$, for some constant $C_p>0$, depending only on p, the Lipschitz constant of the coefficients, and T. Therefore, we have

$$
\mathbb{E}^0\int_0^T |U_s^{\theta,v} - U_s^u|^p ds \le C_p\theta\|u-v\|_{2,2,\mathbb{Q}^0} \to 0, \qquad \text{as } \theta\to 0. \tag{6.14}
$$

We can now prove (6.10) for $i=1,\dots,4$. First, by the Burkholder–Davis–Gundy inequality, we have

$$
\begin{aligned}
\mathbb{E}^0\Big[\sup_{0\le t\le T}|I_t^{3,\theta,1}|^2\Big] &\le C\int_0^T \mathbb{E}^0\big|\mathbb{E}^0\{R_s[\sigma(s,\varphi^1_{\cdot\wedge s},U_s^{\theta,v},z^1) \\
&\qquad - \sigma(s,\varphi^2_{\cdot\wedge s},U_s^u,z^2)]\}\big|_{\substack{\varphi^1=X^{\theta,v},z^1=u_s^{\theta,v}\\ \varphi^2=X^u,z^2=u_s}}\big|^2 ds.
\end{aligned}
$$

Since σ is bounded and Lipschitz continuous in (φ,y,z), it follows from Proposition 4.2 and (6.14) that $\lim_{\theta\to 0}\mathbb{E}^0[\sup_{0\le t\le T}|I_t^{3,\theta,1}|^2]=0$. By the similar arguments using the continuity of $D_\varphi\sigma$ and that of $\partial_z\sigma$, respectively, it is not hard to show that, for all $p>1$,

$$
\lim_{\theta\to 0}\mathbb{E}^0\Big[\sup_{0\le t\le T}|I_t^{3,\theta,2}|^p\Big]=0; \qquad \lim_{\theta\to 0}\mathbb{E}^0\Big[\sup_{0\le t\le T}|I_t^{3,\theta,4}|^p\Big]=0.
$$

It remains to prove the convergence of $I^{3,\theta,3}$. To this end, we note that, for any $p>1$,

$$
\mathbb{E}^0\Big[\sup_{s\in[0,T]}\big(|R_s|^p + |K_s|^p\big)\Big] \le C_p, \tag{6.15}
$$

and by (6.14) we have, for $p>1$,

$$
\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}^0\int_0^T \big|\mathbb{E}^0\{|B^{\theta,u,v}(s,\varphi_{\cdot\wedge s},z) - B^{u,v}(s,\varphi_{\cdot\wedge s},z^1)|^2\}\big|_{\substack{\varphi=X^u,z=u_s^{\theta,v}\\ z^1=u_s}}\big|^p ds \\
&\quad = 0.
\end{aligned} \tag{6.16}
$$

This, together with (6.13), (6.14), an estimate similar to (6.12), and Proposition 4.2, yields that $\lim_{\theta\to 0}\mathbb{E}^0[\sup_{0\le t\le T}|I_t^{3,\theta,3}|^2]=0$, proving the lemma. □

We now continue the proof of (6.2). First, we rewrite (6.8) as

$$\begin{aligned}\eta_t^\theta=\int_0^t\Bigl\{&\mathbb{E}^0\{\tilde\eta_s^\theta\sigma(s,\varphi_{\cdot\wedge s},U_s^{\theta,v},z)\}|_{\substack{\varphi=X^{\theta,v},\\ z=u_s^{\theta,v}}}+[D\sigma]_s^{\theta,u,v}(\eta^\theta_{\cdot\wedge s})\\ &+\mathbb{E}^0\Bigl\{B^{\theta,u,v}(s,\varphi_{\cdot\wedge s},z)\Bigl[\frac{\mathbb{E}^0[\tilde\eta_s^\theta X_s^{\theta,v}+L_s^u\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}\\ &-\frac{\mathbb{E}^0[\tilde\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}U_s^u\Bigr]\Bigr\}\Big|_{\substack{\varphi=X^u;\\ z=u_s^{\theta,v}}}\Bigr\}\,dB_s^1+I_t^{3,\theta,0}+\sum_{i=1}^4 I_t^{3,\theta,i},\end{aligned}\tag{6.17}$$

where

$$\begin{aligned}I_t^{3,\theta,0}\triangleq\int_0^t\mathbb{E}^0\Bigl\{&B^{\theta,u,v}(s,\varphi_{\cdot\wedge s},z)\Bigl[\frac{\mathbb{E}^0[\tilde\eta_s^\theta X_s^{\theta,v}+L_s^u\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]}\\ &-\frac{\mathbb{E}^0[\tilde\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^{\theta,v}|\mathcal{F}_s^Y]}U_s^u\Bigr]\Big|_{\substack{\varphi=X^u;\\ z=u_s^{\theta,v}}}\\ &-B^{\theta,u,v}(s,\varphi_{\cdot\wedge s},z)\Bigl[\frac{\mathbb{E}^0[\tilde\eta_s^\theta X_s^{\theta,v}+L_s^u\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}\\ &-\frac{\mathbb{E}^0[\tilde\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}U_s^u\Bigr]\Big|_{\substack{\varphi=X^u;\\ z=u_s^{\theta,v}}}\Bigr\}\,dB_s^1.\end{aligned}$$

With the same argument as before one shows that $\lim_{\theta\to 0}\mathbb{E}^0[\sup_{0\le t\le T}|I_t^{3,\theta,0}|^2]=0$. On the other hand, similar to (5.15) one can argue that

$$\begin{aligned}&\mathbb{E}^0\Bigl[B^{\theta,u,v}(s,\varphi_{\cdot\wedge s},z)\Bigl(\frac{\mathbb{E}^0[\tilde\eta_s^\theta X_s^{\theta,v}+L_s^u\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}-\frac{\mathbb{E}^0[\tilde\eta_s^\theta|\mathcal{F}_s^Y]}{\mathbb{E}^0[L_s^u|\mathcal{F}_s^Y]}U_s^u\Bigr)\Bigr]\Big|_{\substack{\varphi=X^u;\\ z=u_s^{\theta,v}}}\\ &\quad=\mathbb{E}^0\Bigl[\int_0^1\partial_y\sigma\bigl(s,\varphi_{\cdot\wedge s},U_s^u+\lambda(U_s^{\theta,v}-U_s^u),z\bigr)\,d\lambda\\ &\qquad\times(\tilde\eta_s^\theta X_s^{\theta,v}+L_s^u\eta_s^\theta-U_s^u\tilde\eta_s^\theta)\Bigr]\Big|_{\substack{\varphi=X^u;\\ z=u_s^{\theta,v}}}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\eta_t^\theta=\int_0^t&\{\mathbb{E}^0\{\alpha_s^{1,\theta}(\varphi^1_{\cdot\wedge s},\varphi^2_{\cdot\wedge s},z)\tilde\eta_s^\theta\}|_{\substack{\varphi^1=X^{\theta,v},\varphi^2=X^u,\\ z=u_s^{\theta,v}}}\\ &+\mathbb{E}^0\{\alpha_s^{2,\theta}(\varphi^2_{\cdot\wedge s},z)\tilde\eta_s^\theta\}|_{\substack{\varphi^2=X^u,\\ z=u_s^{\theta,v}}}\}\,dB_s^1\\ &+\int_0^t\{\mathbb{E}^0\{\beta_s^\theta(\varphi^2_{\cdot\wedge s},z)\eta_s^\theta\}|_{\substack{\varphi^2=X^u\\ z=u_s^{\theta,v}}}+[D\sigma]_s^{\theta,u,v}(\eta^\theta_{\cdot\wedge s})\}\,dB_s^1+I_t^{3,\theta},\end{aligned}$$

where $I_t^{3,\theta} = \sum_{i=0}^4 I_t^{3,\theta,i}$, and

$$\alpha_s^{1,\theta}(\varphi^1_{\cdot\wedge s}, \varphi^2_{\cdot\wedge s}, z) \stackrel{\triangle}{=} \int_0^1 D_\varphi \sigma(s, \varphi^2_{\cdot\wedge s} + \lambda(\varphi^1_{\cdot\wedge s} - \varphi^2_{\cdot\wedge s}), U_s^{\theta,v}, z)(\varphi^1_{\cdot\wedge s} - \varphi^2_{\cdot\wedge s})\, d\lambda;$$

$$\begin{aligned}\alpha_s^{2,\theta}(\varphi^2_{\cdot\wedge s}, z) &\stackrel{\triangle}{=} \sigma(s, \varphi^2_{\cdot\wedge s}, U_s^{\theta,v}, z) \\ &\qquad + \int_0^1 \partial_y \sigma(s, \varphi^2_{\cdot\wedge s}, U_s^u + \lambda(U_s^{\theta,v} - U_s^u), z)\, d\lambda (U_s^{\theta,v} - U_s^u); \\ \beta_s^\theta(\varphi^2_{\cdot\wedge s}, z) &\stackrel{\triangle}{=} L_s^u \int_0^1 \partial_y \sigma(s, \varphi^2_{\cdot\wedge s}, U_s^u + \lambda(U_s^{\theta,v} - U_s^u), z)\, d\lambda.\end{aligned}$$

Notice that

$$\begin{aligned}|\alpha_s^{1,\theta}(\varphi^1_{\cdot\wedge s}, \varphi^2_{\cdot\wedge s}, z)| + |\alpha_s^{2,\theta}(\varphi^2_{\cdot\wedge s}, z)| &\le C(1 + |\varphi^1_{\cdot\wedge s}| + |\varphi^2_{\cdot\wedge s}| + |U_s^{\theta,v}| + |U_s^u|), \\ |\beta_s^\theta(\varphi_{\cdot\wedge s}, z)| &\le C L_s^u.\end{aligned}$$

Now by the Burkholder and Cauchy–Schwarz inequalities we have, for all $p \ge 2$, $t \in [0, T]$,

$$\begin{aligned}&\mathbb{E}^0\Big[\sup_{s\in[0,t]} |\eta_s^\theta|^{2p}\Big] \\ &\quad \le C_p \bigg\{\mathbb{E}^0[\|I^{3,\theta}\|^{2p}_{\mathbb{C}_T}] + \mathbb{E}^0\bigg\{\bigg[\int_0^t \Big(\mathbb{E}^0[|\eta_s^\theta|^2 + |\tilde{\eta}_s^\theta|^2] + \sup_{r\in[0,s]} |\eta_s^\theta|^2\Big) ds\bigg]^p\bigg\}\bigg\},\end{aligned}$$

and from Gronwall's inequality one has

$$\begin{aligned}&\mathbb{E}^0\Big[\sup_{s\in[0,t]} |\eta_s^\theta|^{2p}\Big] \\ &\quad \le C_p \bigg\{\mathbb{E}^0[\|I^{3,\theta}\|^{2p}_{\mathbb{C}_T}] + \int_0^t (\mathbb{E}^0[|\tilde{\eta}_s^\theta|^p])^2\, ds\bigg\}, \qquad t \in [0, T].\end{aligned} \tag{6.18}$$

On the other hand, setting $I_t^\theta \stackrel{\triangle}{=} I_t^{1,\theta} + I_t^{2,\theta}$, $t \in [0, T]$, we have from (6.4) that, for $p \ge 2$, $t \in [0, T]$,

$$\mathbb{E}^0\Big[\sup_{s\in[0,t]} |\tilde{\eta}_s^\theta|^p\Big] \le C_p \bigg\{\mathbb{E}^0[\|I^\theta\|^p_{\mathbb{C}_T}] + \int_0^t \mathbb{E}^0[|\tilde{\eta}_s^\theta|^p]\, ds + \int_0^t (\mathbb{E}^0[|\eta_s^\theta|^{2p}])^{1/2}\, ds\bigg\}.$$

Then Gronwall's inequality leads to that

$$\begin{aligned}&\Big(\mathbb{E}^0\Big[\sup_{s\in[0,t]} |\tilde{\eta}_s^\theta|^p\Big]\Big)^2 \\ &\quad \le C_p \bigg\{(\mathbb{E}^0\|I^\theta\|^p_{\mathbb{C}_T})^2 + \int_0^t \mathbb{E}^0[|\eta_s^\theta|^{2p}]\, ds\bigg\}, \qquad t \in [0, T].\end{aligned} \tag{6.19}$$

Combining (6.18), (6.19), applying (6.5) and Lemma 6.1 as well as the Gronwall inequality, we can easily deduce (6.2) by sending $\theta \to 0$. Consequently, (6.3) holds as well.

From Proposition 6.1, (5.11) and the above development we also obtain the following corollary.

COROLLARY 6.1. *We assume that Assumption 2.2 holds. Then, for all $p>1$*,

$$\lim_{\theta\to 0}\mathbb{E}^0\big[\|\delta_\theta U-\overline{V}\|_{\mathbb{C}_T}^p\big]=\lim_{\theta\to 0}\mathbb{E}^0\bigg[\sup_{0\le s\le T}\bigg|\frac{U_s^{\theta,v}-U_s^u}{\theta}-\overline{V}_s\bigg|^p\bigg]=0,$$

where

$$\overline{V}_t\triangleq\frac{\mathbb{E}^0[R_tX_t^u+L_t^uK_t|\mathcal{F}_t^Y]}{\mathbb{E}^0[L_t^u|\mathcal{F}_t^Y]}-\frac{\mathbb{E}^0[R_t|\mathcal{F}_t^Y]}{\mathbb{E}^0[L_t^u|\mathcal{F}_t^Y]}U_t^u,\qquad t\in[0,T].$$

7. Stochastic maximum principle. We are now ready to study the stochastic maximum principle. The main task will be to determine the appropriate *adjoint equation*, which we expect to be a backward stochastic differential equation of mean-field type. We begin with a simple analysis. Suppose that $u=u^*$ is an optimal control, and for any $v\in\mathscr{U}_{\mathrm{ad}}$, we define $u^{\theta,v}$ by (5.4). Then we have

$$\begin{aligned}
0&\le\frac{J(u^{\theta,v})-J(u)}{\theta}\\
&=\frac{1}{\theta}\mathbb{E}^0\bigg\{\mathbb{E}^0\big[L_T^{\theta,v}\Phi\big(x,U_T^{\theta,v}\big)\big]\big|_{x=X_T^\theta}-\mathbb{E}^0\big[L_T^u\Phi\big(x,U_T^u\big)\big]\big|_{x=X_T^u}\\
&\quad+\int_0^T\big[\mathbb{E}^0\big[L_s^{\theta,v}f\big(s,\varphi_{\cdot\wedge s},U_s^{\theta,v},z\big)\big]\big|_{\substack{\varphi=X^{\theta,v},\\ z=u_s^{\theta,v}}}\\
&\quad-\mathbb{E}^0\big[L_s^uf\big(s,\varphi_{\cdot\wedge s},U_s^u,z\big)\big]\big|_{\substack{\varphi=X^u,\\ z=u_s}}\big]\,ds\bigg\}.
\end{aligned}\tag{7.1}$$

Now, repeating the same analysis as that in Proposition 4.2, then sending $\theta\to 0$, it follows from Propositions 4.2, 6.1 and the continuity of the functions Φ and f that

$$\begin{aligned}
0&\le\mathbb{E}^0[K_T\xi]+\mathbb{E}^0[R_T\Theta]+\mathbb{E}^0\bigg\{\int_0^T\big\{\mathbb{E}^0\big[R_sf\big(s,\varphi_{\cdot\wedge s},U_s^u,z\big)\big]\big|_{\varphi=X^u,z=u_s}\\
&\quad+\mathbb{E}^0\big[\partial_yf\big(s,\varphi_{\cdot\wedge s},U_s^u,z\big)\big(X_s^u-U_s^u\big)R_s+L_s^uK_s\big]\big|_{\varphi=X^u,z=u_s}\\
&\quad+\mathbb{E}^0\big[L_s^uD_\varphi f\big(s,\varphi_{\cdot\wedge s},U_s^u,z\big)(\psi_{\cdot\wedge s})\big]\big|_{\varphi=X^u,z=u_s,\psi=K}\\
&\quad+\mathbb{E}^0\big[L_s^u\partial_zf\big(s,\varphi_{\cdot\wedge s},U_s^u,z\big)\big]\big|_{\varphi=X^u,z=u_s}(v_s-u_s)\big\}\,ds\bigg\},
\end{aligned}\tag{7.2}$$

where

$$\begin{aligned}
\xi&\triangleq\mathbb{E}^0\big[L_T^u\partial_x\Phi\big(x,U_T^u\big)\big]\big|_{x=X_T^u}+L_T^u\mathbb{E}^0\big[\partial_y\Phi\big(X_T^u,y\big)\big]\big|_{y=U_T^u},\\
\Theta&\triangleq\mathbb{E}^0\big[\Phi\big(X_T^u,y\big)\big]\big|_{y=U_T^u}+\big(X_T^u-U_T^u\big)\mathbb{E}^0\big[\partial_y\Phi\big(X_T^u,y\big)\big]\big|_{y=U_T^u}.
\end{aligned}\tag{7.3}$$

We now consider the adjoint equations that take the following form of backward SDEs on the reference space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$:

$$\begin{cases} dp_t = -\alpha_t\,dt + d\Gamma_t + q_t\,dB_t^1 + \widetilde{q}_t\,dY_t, & p_T = \xi, \\ dQ_t = -\beta_t\,dt + d\Sigma_t + M_t\,dB_t^1 + \widetilde{M}_t\,dY_t, & Q_T = \Theta. \end{cases} \tag{7.4}$$

Here, the coefficients α, β as well as the two bounded variation processes Γ and Σ are to be determined. Applying Itô's formula and recalling the variational equations (5.17) and (5.18), we can easily derive (denote $U_t^u = \mathbb{E}^u[X_t^u|\mathcal{F}_t^Y]$, $t \in [0,T]$)

$$\begin{aligned} &\mathbb{E}^0[\xi K_T] + \mathbb{E}^0[\Theta R_T] \\ &\quad = \int_0^T \{-\mathbb{E}^0[K_s\alpha_s] - \mathbb{E}^0[R_s\beta_s] + \mathbb{E}^0[q_s\mathbb{E}^0[R_s\sigma(s,\varphi_{\cdot\wedge s}, U_s^u, z)]|_{\substack{\varphi=X^u,\\ z=u_s}}] \\ &\qquad + \mathbb{E}^0[q_s\mathbb{E}^0[\partial_y\sigma(s,\varphi_{z\cdot\wedge s}, U_s^u, z)[(X_s^u - U_s^u)R_s + L_s^u K_s]]|_{\substack{\varphi=X^u,\\ z=u_s}}] \\ &\qquad + \mathbb{E}^0[q_s[D\sigma]_s^{u,v}(K_{\cdot\wedge s}) + q_s C_\sigma^{u,v}(s)(v_s - u_s) + \widetilde{M}_s R_s h(s, X_s^u) \\ &\qquad + \widetilde{M}_s K_s L_s^u \partial_x h(s, X_s^u)]\}\,ds + \mathbb{E}^0\left\{\int_0^T [K_s d\Gamma_s + R_s d\Sigma_s]\right\}, \end{aligned} \tag{7.5}$$

where $[D\sigma]^{u,v}$ and $C^{u,v}$ are defined by (5.14).

By Fubini's theorem, we see that

$$\begin{cases} \mathbb{E}^0[q_s\mathbb{E}^0[R_s\sigma(s,\varphi\cdot\wedge s, U_s^u, z)]|_{\substack{\varphi=X^u,\\ z=u_s}}] \\ \qquad = \mathbb{E}^0[R_s\mathbb{E}^0[q_s\sigma(s, X_{\cdot\wedge s}, y, u_s)]|_{y=U_s^u}]; \\ \mathbb{E}^0[q_s\mathbb{E}^0[\partial_y\sigma(s,\varphi_{z\cdot\wedge s}, U_s^u, z)[(X_s^u - U_s^u)R_s + L_s^u K_s]]|_{\substack{\varphi=X^u,\\ z=u_s}}] \\ \qquad = \mathbb{E}^0[\mathbb{E}^0 z[q_s\partial_y\sigma(s, X_{z\cdot\wedge s}, y, u_s)]|_{y=U_s^u}[(X_s^u - U_s^u)R_s + L_s^u K_s]]. \end{cases} \tag{7.6}$$

Furthermore, in light of definition of $[D\sigma]^{u,v}$ (5.14), if we denote, for fixed (t,φ,z),

$$\mu_\sigma^0(t,\varphi_{\cdot\wedge t}, z)(\cdot) \stackrel{\triangle}{=} \mathbb{E}^0[L_t^u D_\varphi\sigma(t,\varphi_{\cdot\wedge t}, U_t^u, z)](\cdot) \in \mathscr{M}[0,T], \tag{7.7}$$

where $\mathscr{M}[0,T]$ denotes all the Borel measures on $[0,T]$, then we can write

$$\begin{aligned} [D\sigma]_t^{u,v}(K_{\cdot\wedge t}) &= \mathbb{E}^0[L_t^u D_\varphi\sigma(t,\varphi_{\cdot\wedge t}, U_t^u, z)(\psi)]|_{\substack{\varphi=X^u, z=u_t,\\ \psi=K_{\cdot\wedge t}}} \\ &= \int_0^t K_r\mu_\sigma^0(r, X_{\cdot\wedge r}^u, u_r)(dr). \end{aligned} \tag{7.8}$$

Let us now argue that a similar Fubini theorem argument holds for the random measure $\mu_\sigma^0(t, X_{\cdot\wedge t}^u, u_t)(\cdot)$. First, for a given process $q \in L_{\mathbb{F}}^2(\mathbb{Q}^0; [0,T])$, consider the following finite variation (FV) process [in fact, under Assumption 2.2, *integrable variation* (IV) process]:

$$A_t^\sigma \stackrel{\triangle}{=} \int_0^T \int_0^{t\wedge s} q_s\mu_\sigma^0(s, X_{\cdot\wedge s}^u, u_s)(dr)\,ds, \qquad t \in [0,T]. \tag{7.9}$$

It is easy to check, as a (randomized) signed measure on $[0,T]$, it holds $\mathbb{Q}^0$-almost surely that $dA_t^\sigma = \int_t^T q_s\mu_\sigma^0(s, X^u_{\cdot\wedge s}, u_s)(dt)\,ds$. We note that being a "raw FV" process, the process A^σ is not $\mathbb{F}$-adapted. We now consider its *dual predictable projection*:

$$^p\biggl(\int_t^T q_s\mu_\sigma^0\bigl(s, X^u_{\cdot\wedge s}, u_s\bigr)(dt)\,ds\biggr) \stackrel{\triangle}{=} d\bigl[{}^pA_t^\sigma\bigr], \qquad t\in[0,T]. \tag{7.10}$$

We remark that $d[{}^pA_t]$ is a predicable random measure that can be formally understood as

$$\begin{aligned} d\bigl[{}^pA_t^\sigma\bigr] &= \mathbb{E}^0\bigl[dA_t^\sigma|\mathcal{F}_{t-}\bigr] \\ &= \mathbb{E}^0\biggl[\int_t^T q_s\mu_\sigma^0\bigl(s, X^u_{\cdot\wedge s}, u_s\bigr)(dt)\,ds\bigg|\mathcal{F}_{t-}\biggr], \qquad t\in[0,T]. \end{aligned}$$

Using the definition of dual predicable projection and (7.8), we see that, for the continuous process $K\in L^2_{\mathbb{F}}(\mathbb{Q}^0;\mathbb{C}_T)$,

$$\begin{aligned} \int_0^T \mathbb{E}^0\bigl[q_s[D\sigma]_s^{u,v}(K_{\cdot\wedge s})\bigr]\,ds &= \int_0^T \mathbb{E}^0\biggl[q_s\int_0^s K_r\mu_\sigma^0\bigl(r, X^u_{\cdot\wedge r}, u_r\bigr)(dr)\biggr]\,ds \\ &= \mathbb{E}^0\biggl[\int_0^T K_r\,dA_r^\sigma\biggr] = \mathbb{E}^0\biggl[\int_0^T K_r\,d\bigl[{}^pA_r^\sigma\bigr]\biggr] \\ &= \mathbb{E}^0\biggl[\int_0^T K_r^p\biggl(\int_r^T q_s\mu_\sigma^0\bigl(s, X^u_{\cdot\wedge s}, u_s\bigr)(dr)\,ds\biggr)\biggr]. \end{aligned} \tag{7.11}$$

Similarly, we denote $A_t^f \stackrel{\triangle}{=} \int_0^T\int_0^{t\wedge s}\mu_f^0(s, X^u_{\cdot\wedge s}, u_s)(dr)\,ds$, $t\in[0,T]$; and denote its dual predicable projection by ${}^p(\int_t^T \mu_f^0(s, X^u_{\cdot\wedge s}, u_s)(dt)\,ds) = d[{}^pA_t^f]$, $t\in[0,T]$.

We now plug (7.6) and (7.11) into (7.5) to get

$$\begin{aligned} &\mathbb{E}^0[\xi K_T] + \mathbb{E}^0[\Theta R_T] \\ &\quad= \mathbb{E}^0\biggl\{\int_0^T \bigl\{K_s\bigl[-\alpha_s + L_s^u\mathbb{E}^0\bigl[q_s\partial_y\sigma\bigl(s, X^u_{\cdot\wedge s}, y, u_s\bigr)\bigr]|_{y=U_s^u} \\ &\qquad + M_sL_s^u\partial_xh\bigl(s, X_s^u\bigr)\bigr] + R_s\bigl[-\beta_s + \mathbb{E}^0\bigl[q_s\sigma\bigl(s, X_{\cdot\wedge s}, y, u_s\bigr)\bigr]|_{y=U_s^u} \\ &\qquad + \widetilde{M}_sh\bigl(s, X_s^u\bigr)\bigr] + q_sC_s^{u,v}(v_s-u_s) \\ &\qquad + R_s\mathbb{E}^0\bigl[q_s\partial_y\sigma\bigl(s, X^u_{\cdot\wedge s}, y, u_s\bigr)\bigr]|_{y=U_s^u}\bigl(X_s^u-U_s^u\bigr)\bigr\}\,ds + \int_0^T K_s\,d\bigl[{}^pA_s^\sigma\bigr]\biggr\} \\ &\qquad + \mathbb{E}^0\biggl\{\int_0^T [K_sd\Gamma_s + R_sd\Sigma_s]\biggr\} \\ &\quad= \mathbb{E}^0\biggl\{\int_0^T \bigl[-K_s\hat{\alpha}_s - R_s\hat{\beta}_s + q_sC_\sigma^{u,v}(s)(v_s-u_s)\bigr]\,ds + K_s\,d\bigl[{}^pA_s^\sigma\bigr] \\ &\qquad + [K_sd\Gamma_s + R_sd\Sigma_s]\biggr\}, \end{aligned} \tag{7.12}$$

where

$$
(7.13)\quad \begin{cases} \hat{\alpha}_t \stackrel{\triangle}{=} \alpha_t - L_t^u \mathbb{E}^0[q_t \partial_y \sigma(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} - \widetilde{M}_t L_t^u \partial_x h(t, X_t^u); \\ \hat{\beta}_t \stackrel{\triangle}{=} \beta_t - \mathbb{E}^0[q_t \sigma(t, X_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} - \widetilde{M}_t h(t, X_t^u) \\ \qquad - \mathbb{E}^0[q_t \partial_y \sigma(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u}(X_t^u - U_t^u). \end{cases}
$$

Combining (7.2) and (7.12) and using the processes dA^σ, dA^f and their dual predicable projections, we have

$$
\begin{aligned}
0 \le\ & \mathbb{E}^0\left\{\int_0^T [-K_s\hat{\alpha}_s - R_s\hat{\beta}_s + q_s C_\sigma^{u,v}(s)(v_s - u_s)]\,ds + \int_0^T K_s\, d[{}^pA_s^\sigma]\right\} \\
& + \mathbb{E}^0\left\{\int_0^T [R_s[\mathbb{E}^0[f(s, X_{\cdot\wedge s}, y, u_s)]|_{y=U_s^u}\right. \\
(7.14)\quad & + \mathbb{E}^0[\partial_y f(s, X^u_{\cdot\wedge s}, y, u_s)]|_{y=U_s^u}(X_s^u - U_s^u)] \\
& + L_s^u K_s \mathbb{E}^0[\partial_y f(s, X^u_{\cdot\wedge s}, y, u_s)]|_{y=U_s^u} + C_f^{u,v}(s)(v_s - u_s)]\,ds \\
& \left. + \int_0^T K_s\, d[{}^pA_s^f]\right\} + \mathbb{E}^0\left\{\int_0^T [K_s d\Gamma_s + R_s d\Sigma_s]\right\},
\end{aligned}
$$

where $C_f^{u,v}(s) \stackrel{\triangle}{=} \mathbb{E}^0[L_s^u \partial_z f(s, \varphi_{\cdot\wedge s}, U_s^u, z)]|_{\substack{\varphi=X^u,\\ z=u_s}}$. Now, if we set $\Sigma_t = 0$, and

$$
(7.15)\quad \begin{aligned}
\hat{\alpha}_t &= L_t^u \mathbb{E}^0[\partial_y f(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u}; \\
\hat{\beta}_t &= \mathbb{E}^0[f(t, X_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} \\
&\quad + \mathbb{E}^0[\partial_y f(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u}(X_t^u - U_t^u); \\
d\Gamma_t &= -d[{}^pA_t^\sigma] - d[{}^pA_t^f],
\end{aligned}
$$

then (7.14) becomes

$$
(7.16)\qquad 0 \le \mathbb{E}^0\left\{\int_0^T [q_s C_\sigma^{u,v}(s) + C_f^{u,v}(s)](v_s - u_s)\,ds\right\}, \qquad v \in \mathscr{U}_{\mathrm{ad}}.
$$

From this, we should be able to derive the maximum principle, provided that the adjoint equation (7.4) with coefficients α, β, and Γ determined by (7.13) and (7.15) is well defined.

REMARK 7.1. (1) We remark that the process Γ in (7.15) should be considered as a mapping from the space $L^2_{\mathbb{F}}([0, T] \times \Omega) \times L^2_{\mathbb{F}}(\Omega; \mathbb{C}_T) \times L^2_{\mathbb{F}}([0, T] \times \Omega; U)$ to $\mathscr{M}_{\mathbb{F}}([0, T])$, the space of all the random measures on $[0, T]$, such that:

(i) $(t, \omega) \mapsto \mu(t, \omega, A)$ is $\mathbb{F}$-progressively measurable, for all $A \in \mathscr{B}([0, T])$;

(ii) $\mu(t, \omega, \cdot) \in \mathscr{M}([0, T])$ is a finite Borel measure on $[0, T]$.

(2) Assumption 2.2(iii) implies that the random measure $\mathbb{D}_\sigma[q, X^u, u](t, dt)$ satisfies the following estimate: for any $q \in L^2_{\mathbb{F}}([0,T] \times \Omega)$ and $u \in \mathscr{U}_{\mathrm{ad}}$,

$$\begin{aligned}\mathbb{E}^0\Big[\int_0^T |d^p A_t^\sigma|\Big] &= \mathbb{E}^0\Big\{\int_0^T \Big|^p\Big(\int_t^T q_s \mu_\sigma^0(s, X^u_{\cdot\wedge s}, u_s)(dt)\, ds\Big)\Big|\Big\} \\ &\le \mathbb{E}^0\Big\{\int_0^T \int_0^s |q_s||\mu_\sigma^0(s, X^u_{\cdot\wedge s}, u_s)(dt)|\, ds\Big\} \\ &\le \mathbb{E}^0\Big\{\int_0^T |q_s| \int_0^s \ell(s, dt)\, ds\Big\} \le C\mathbb{E}^0\Big\{\int_0^T |q_s|\, ds\Big\} \\ &\le C\|q\|_{2,2,\mathbb{Q}^0}.\end{aligned} \tag{7.17}$$

The same estimate holds for $\mathbb{D}_f[X^u, u](t, dt)$ as well.

(3) Clearly, the processes A^σ and A^f are originated from the Fréchet derivatives of σ and f, respectively, with respect to the path $\varphi_{\cdot\wedge t}$. If σ and f are of Markovian type, then they will be absolutely continuous with respect to the Lebesgue measure.

We shall now validate all the arguments presented above. To begin with, we note that the choice of α, β, and Γ via by (7.13) and (7.15), together with the terminal condition (ξ, Θ) by (7.3), amounts to saying that the processes $(p, q, \tilde{q})$ and $(Q, M, \tilde{M})$ solve the BSDE

$$\begin{cases} dp_t = -L_t^u\{\mathbb{E}^0[\partial_y f(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} \\ \qquad\qquad + \mathbb{E}^0[q_t \partial_y \sigma(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} \\ \qquad\qquad + \widetilde{M}_t \partial_x h(t, X_t^u)\}\, dt - d^p A_t^\sigma - d^p A_t^f + q_t\, dB_t^1 + \widetilde{q}_t\, dY_t, \\ dQ_t = -\{\mathbb{E}^0[q_t \sigma(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} - \widetilde{M}_t h(t, X_t^u) \\ \qquad\qquad + \mathbb{E}^0[q_t \partial_y \sigma(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u}(X_t^u - U_t^u) \\ \qquad\qquad + \mathbb{E}^0[f(t, X_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u} \\ \qquad\qquad + \mathbb{E}^0[\partial_y f(t, X^u_{\cdot\wedge t}, y, u_t)]|_{y=U_t^u}(X_t^u - U_t^u)\}\, dt \\ \qquad\qquad + M_t\, dB_t^1 + \widetilde{M}_t\, dY_t, \\ p_T = \xi, \qquad Q_T = \Theta. \end{cases} \tag{7.18}$$

Now if we denote $\eta = (p, Q)^T$, $W = (B^1, Y)^T$, $\Xi = \left[\begin{smallmatrix} q & \tilde{q} \\ M & \tilde{M}\end{smallmatrix}\right]$, then we can rewrite (7.18) in a more abstract (vector) form:

$$\begin{cases} d\eta_t = -\{A_t + \mathbb{E}^0[G_t \Xi_t g(t, y)]|_{y=U_t^u} + H_t \Xi_t h_t\}\, dt \\ \qquad\qquad - \Gamma(\Xi)(t, dt) - \Gamma_0(t, dt) + \Xi_t\, dW_t, \\ \eta_T = \Upsilon, \end{cases} \tag{7.19}$$

where $\Upsilon \in L^2_{\mathbb{F}^W_T}(\Omega; \mathbb{Q}^0)$; A, G, H and h are bounded, vector or matrix-valued $\mathbb{F}^W$-adapted processes with appropriate dimensions, g is an $\mathbb{R}^2$-valued progressively measurable random field and U is an $\mathbb{F}^Y$-adapted process. Moreover, the

$\mathbb{R}^2$-valued finite variation processes $\Gamma(\Xi)(t,dt)$ and $\Gamma_0(t,dt)$ take the form

$$\Gamma(\Xi)(t,dt) = {}^p\Big(\int_t^T \Xi_r \mu_r^1(dt)\,dr\Big),$$
$$\Gamma_0(t,dt) = {}^p\Big(\int_t^T \mu_r^2(dt)\,dr\Big), \tag{7.20}$$

where $r \mapsto \mu_r^i(\cdot)$, $i=1,2$, are $\mathscr{M}[0,T]$-valued measurable random processes satisfying, as measures with respect to the total variation norm,

$$|\mu_r^1(dt)| + |\mu_r^2(dt)| \le \ell(r,dt), \qquad r\in[0,T], \mathbb{Q}^0 \text{ a.s.} \tag{7.21}$$

We note that $\Gamma(\Xi)(dt)$ and $\Gamma_0(dt)$ are representing $d[{}^pA_t^\sigma]$ and $[{}^pA_t^f]$ in (7.18), respectively, and can be substantiated by (7.9) and (7.10). Furthermore, by Assumption 2.2, they both satisfy (7.21). To the best of our knowledge, BSDE (7.19) is beyond all the existing frameworks of BSDEs, and we shall give a brief proof for its well-posedness.

THEOREM 1. *Assume that the Assumption 2.2 is in force. Then the BSDE (7.19) has a unique solution* (η,Ξ).

PROOF. The proof is more or less standard, we shall only point out a key estimate. For any given $\widetilde{\Xi}^i \in L^2_{\mathbb{F}^W}([0,T]\times\Omega;\mathbb{R}^4)$, obviously we have a unique solution (η^i,Ξ^i) of (7.19), $i=1,2$, respectively, that is,

$$\begin{cases} d\eta_t^i = -\{A_t + \mathbb{E}^0[G_t\widetilde{\Xi}_t^i g(t,y)]|_{y=U_t^u} + H_t\widetilde{\Xi}_t^i h_t\}\,dt \\ \qquad\quad - \Gamma(\widetilde{\Xi}^i)(t,dt) - \Gamma_0(t,dt) + \Xi_t^i\,dW_t, \\ \eta_T^i = \Upsilon. \end{cases}$$

We define $\widehat{\xi} = \xi^1 - \xi^2$, $\xi^i = \eta^i, \Xi^i$, $i=1,2$, respectively. $\widehat{\widetilde{\Xi}} = \widetilde{\Xi}^1 - \widetilde{\Xi}^2$. Noting the linearity of BSDE (7.19), we see that $\widehat{\eta}$ satisfies

$$\widehat{\eta}_t = \int_t^T \{\mathbb{E}^0[G_s\widehat{\widetilde{\Xi}}_s g(s,y)]|_{y=U_s^u} + H_s\widehat{\widetilde{\Xi}}_s h_s\}\,ds$$
$$+ \int_t^T \Gamma(\widehat{\widetilde{\Xi}})(s,ds) - M_t^T, \tag{7.22}$$

where $M_t^T \triangleq \int_t^T \widehat{\Xi}_s\,dW_s$. Therefore,

$$|\widehat{\eta}_t + M_t^T|^2 \le 2\bigg\{\bigg|\int_t^T \{\mathbb{E}^0[G_s\widehat{\widetilde{\Xi}}_s g(s,y)]|_{y=U_s^u} + H_s\widehat{\widetilde{\Xi}}_s h_s\}\,ds\bigg|^2$$
$$+ \bigg|\int_t^T \Gamma(\widehat{\widetilde{\Xi}})(s,ds)\bigg|^2\bigg\}.$$

Taking expectation on both sides above and noting that $\mathbb{E}^0[\widehat{\eta}_t M_t^T]=0$ and

$$\mathbb{E}^0\bigg\{\bigg|\int_t^T\{\mathbb{E}^0[G_s\widehat{\widehat{\Xi}}_s g(s,y)]|_{y=U_s^u}+H_s\widehat{\widehat{\Xi}}_s h_s\}\,ds\bigg|^2\bigg\}$$
$$\le C(T-t)\mathbb{E}^0\bigg[\int_t^T|\widehat{\widehat{\Xi}}_s|^2\,ds\bigg],$$

we have

$$\begin{aligned}&\mathbb{E}^0[|\widehat{\eta}_t|^2]+\mathbb{E}^0\bigg[\int_t^T|\widehat{\Xi}_s|^2\,ds\bigg]\\&\quad\le C(T-t)\mathbb{E}^0\bigg[\int_t^T|\widehat{\widehat{\Xi}}_s|^2\,ds\bigg]+\mathbb{E}^0\bigg\{\bigg|\int_t^T\Gamma(\widehat{\widehat{\Xi}})(s,ds)\bigg|^2\bigg\}.\end{aligned}\tag{7.23}$$

To estimate the term involving $\Gamma(\widehat{\widehat{\Xi}})$, we note that [recall (7.20)] if a square-integrable process V is increasing and continuous, then so is its dual predictable projection ${}^pV$. Thus, by the definition of ${}^pV$ we have

$$\begin{aligned}\mathbb{E}^0\bigg[\bigg|\int_t^T d[{}^pV_s]\bigg|^2\bigg]&=2\mathbb{E}^0\bigg[\int_t^T({}^pV_s-{}^pV_t)\,d[{}^pV_s]\bigg]\\&=2\mathbb{E}^0\bigg[\int_t^T({}^pV_s-{}^pV_t)\,dV_s\bigg]\le 2\mathbb{E}^0[({}^pV_T-{}^pV_t)(V_T-V_t)]\\&\le 2\bigg(\mathbb{E}^0\bigg[\bigg|\int_t^T d[{}^pV_s]\bigg|^2\bigg]\bigg)^{1/2}\bigg(\mathbb{E}^0\bigg[\bigg|\int_t^T dV_s\bigg|^2\bigg]\bigg)^{1/2}.\end{aligned}$$

That is,

$$\mathbb{E}^0\bigg[\bigg|\int_t^T d[{}^pV_s]\bigg|^2\bigg]\le 4\mathbb{E}^0\bigg[\bigg|\int_t^T dV_s\bigg|^2\bigg].\tag{7.24}$$

Applying this to $V_t\triangleq\int_0^T\int_0^{t\wedge r}|\widehat{\widehat{\Xi}}_r||\mu_r^1(ds)|\,dr$, $t\in[0,T]$, we have

$$\begin{aligned}\mathbb{E}^0\bigg[\bigg|\int_t^T\Gamma(\widehat{\widehat{\Xi}})(s,ds)\bigg|^2\bigg]&\le\mathbb{E}^0\bigg[\bigg|\int_t^T {}^p\bigg(\int_s^T|\widehat{\widehat{\Xi}}_r||\mu_r^1(ds)|\,dr\bigg)\bigg|^2\bigg]\\&\le 4\mathbb{E}^0\bigg[\bigg|\int_t^T\int_s^T|\widehat{\widehat{\Xi}}_r||\mu_r^1(ds)|\,dr\bigg|^2\bigg]\\&\le 4\mathbb{E}^0\bigg[\bigg|\int_t^T\int_s^T|\widehat{\widehat{\Xi}}_r|\ell(r,ds)\,dr\bigg|^2\bigg\}\\&\le C\mathbb{E}^0\bigg[\bigg|\int_t^T|\widehat{\widehat{\Xi}}_r|\,dr\bigg|^2\bigg\}\le C(T-t)\mathbb{E}^0\bigg[\int_0^T|\widehat{\widehat{\Xi}}_s|^2\,ds\bigg]\end{aligned}$$

and, therefore, (7.23) becomes

$$\mathbb{E}^0[|\widehat{\eta}_t|^2]+\mathbb{E}^0\bigg[\int_t^T|\widehat{\Xi}_s|^2\,ds\bigg]\le C(T-t)\mathbb{E}^0\bigg[\int_t^T|\widehat{\widehat{\Xi}}_s|^2\,ds\bigg].\tag{7.25}$$

With this estimate, and following the standard argument one shows that BSDE (7.18) is well-posed on $[T-\delta, T]$ for some (uniform) $\delta > 0$. Iterating the argument, one can then obtain the well-posedness on $[0, T]$. We leave the details to the interested reader. $\square$

We are now ready to prove the main result of this paper. Let us define the *Hamiltonian*: for $(\varphi, \mu) \in \mathbb{C}_T \times \mathscr{P}(\mathbb{C}_T)$, and $k : [0, T] \times \Omega \to \mathbb{R}$ adapted process, $(t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}$,

$$\mathscr{H}(t, \omega, \varphi_{\cdot\wedge t}, \mu, z; k) \stackrel{\triangle}{=} k_t(\omega) \cdot \sigma(t, \varphi_{\cdot\wedge t}, \mu, z) + f(t, \varphi_{\cdot\wedge t}, \mu, z). \tag{7.26}$$

We have the following theorem.

THEOREM 2 (Stochastic Maximum Principle). *Assume that the Assumptions 2.2 and 3.1 hold. Assume further the mapping $z \mapsto \mathscr{H}(t, \varphi_{\cdot\wedge t}, \mu, z)$ is convex. Let $u = u^* \in \mathscr{U}_{\mathrm{ad}}$ be an optimal control and X^u the corresponding trajectory. Then, for $dt \times d\mathbb{Q}^0$-a.e. $(t, \omega) \in [0, T] \times \Omega$ it holds that*

$$\mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t; q_t\big) = \inf_{v \in U} \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, v; q_t\big), \tag{7.27}$$

where $(p, q, \tilde{q})$ and $(Q, M, \tilde{M})$ are the unique solution of the BSDE (7.18).

PROOF. We first recall from (5.14) that

$$C^{u,v}_f(t) = \mathbb{E}^0\big[L^u_t \partial_z f\big(t, \varphi_{\cdot\wedge t}, U^u_t, z\big)\big]\big|_{\varphi = X^u, z = u_t} = \partial_z f\big(t, X^u_{\cdot\wedge t}, \mu^u_t, u_t\big);$$

$$C^{u,v}_\sigma(t) = \mathbb{E}^0\big\{L^u_t \partial_z \sigma\big(t, \varphi_{\cdot\wedge t}, U^u_t, z\big)\big]\big\}\big|_{\varphi = X^u; z = u_t} = \partial_z \sigma\big(t, X^u_{\cdot\wedge t}, \mu^u_t, u_t\big).$$

Then (7.16) implies that

$$\begin{aligned} 0 &\le \mathbb{E}^0\bigg[\int_0^T \big[q_t C^{u,v}_\sigma(t) + C^{u,v}_f(t)\big](v_t - u_t)\, dt\bigg] \\ &= \mathbb{E}^0\bigg[\int_0^T \partial_z \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t; q_t\big)(v_t - u_t)\, dt\bigg]. \end{aligned} \tag{7.28}$$

Therefore, for $dt \times d\mathbb{Q}^0$-a.e. $(t, \omega) \in [0, T] \times \Omega$, and any $v \in U$, it holds that

$$\partial_z \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t; q_t\big)(v - u_t) \ge 0. \tag{7.29}$$

Now, for any $v \in U$, one has, $dt \times d\mathbb{Q}^0$-a.e. on $[0, T] \times \Omega$,

$$\begin{aligned} &\mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, v; q_t\big) - \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t; q_t\big) \\ &\quad = \int_0^1 \partial_z \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t + \lambda(v - u_t); q_t\big)(v - u_t)\, d\lambda \\ &\quad = \int_0^1 \big[\partial_z \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t + \lambda(v - u_t); q_t\big) \\ &\qquad - \partial_z \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t; q_t\big)\big](v - u_t)\, d\lambda \\ &\qquad + \partial_z \mathscr{H}\big(t, \omega, X^u_{\cdot\wedge t}, \mu^u_t, u_t; q_t\big)(v - u_t) \ge 0. \end{aligned}$$

Here the first integral on the right-hand side above is nonnegative due to the convexity of $\mathscr{H}$ in variable z, and the last term is nonnegative because of (7.29). The identity (7.27) now follows immediately. □

REMARK 7.2. In stochastic control literature, inequality (7.28) is sometimes referred to as *the stochastic maximum principle in integral form*, which in many applications is useful, as it does not require the convexity assumption on the Hamiltonian $\mathscr{H}$.

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REFERENCES

[1] BENSOUSSAN, A. (1992). *Stochastic Control of Partially Observable Systems*. Cambridge Univ. Press, Cambridge. MR1191160

[2] BUCKDAHN, R., DJEHICHE, B. and LI, J. (2011). A general stochastic maximum principle for SDEs of mean-field type. *Appl. Math. Optim.* **64** 197–216. MR2822408

[3] BUCKDAHN, R., DJEHICHE, B., LI, J. and PENG, S. (2009). Mean-field backward stochastic differential equations: A limit approach. *Ann. Probab.* **37** 1524–1565. MR2546754

[4] BUCKDAHN, R., LI, J. and PENG, S. (2009). Mean-field backward stochastic differential equations and related partial differential equations. *Stochastic Process. Appl.* **119** 3133–3154. MR2568268

[5] CARMONA, R. and DELARUE, F. (2013). Probabilistic analysis of mean-field games. *SIAM J. Control Optim.* **51** 2705–2734. MR3072222

[6] CARMONA, R. and DELARUE, F. (2015). Forward-backward stochastic differential equations and controlled McKean–Vlasov dynamics. *Ann. Probab.* **43** 2647–2700. MR3395471

[7] CARMONA, R., DELARUE, F. and LACHAPELLE, A. (2013). Control of McKean–Vlasov dynamics versus mean field games. *Math. Financ. Econ.* **7** 131–166. MR3045029

[8] CARMONA, R. and ZHU, X. (2016). A probabilistic approach to mean field games with major and minor players. *Ann. Appl. Probab.* **26** 1535–1580. MR3513598

[9] CARNOMA, R. and DELARUE, F. (2012). Optimal control of McKean-Vlasov stochastic dynamics. Technical Report.

[10] CAUTY, R. (2001). Solution du problème de point fixe de Schauder. *Fund. Math.* **170** 231–246. MR1880901

[11] CAUTY, R. (2012). Une généralisation de la conjecture de point fixe de Schauder. Preprint. Available at arXiv:1201.2586 [math.AT].

[12] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[13] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–251. MR2346927

[14] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR0637061

[15] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621

[16] LI, J. (2012). Stochastic maximum principle in the mean-field controls. *Automatica J. IFAC* **48** 366–373. MR2889429

[17] LI, J. and MIN, H. (2016). Weak solutions of mean-field stochastic differential equations and application to zero-sum stochastic differential games. *SIAM J. Control Optim*. **54** 1826–1858. MR3517577

[18] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

[19] VILLANI, C. (2003). *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483

[20] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls*: *Hamiltonian Systems and HJB Equations*. *Applications of Mathematics* (*New York*) **43**. Springer, New York. MR1696772

[21] ZEITOUNI, O. and BOBROVSKY, B. Z. (1986). On the reference probability approach to the equations of nonlinear filtering. *Stochastics* **19** 133–149. MR0870617

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