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Reconciling the Subjective and Objective Aspects of Probability

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Abstract. Since the early nineteenth century, the concept of objective probability has been dynamic. As we recognize this history, we can strengthen Professor Nozer Singpuwalla's vision of reliability of survival analysis by aligning it with earlier conceptions elaborated by Laplace, Borel, Kolmogorov, Ville and Neyman. By emphasizing testing and recognizing the generality of the vision of Kolmogorov and Neyman, we gain a perspective that does not rely on exchangeability.

Key words and phrases: Objective probability, subjective probability, game-theoretic probability, martingale testing, defensive forecasting.

I would like to thank Professor Singpurwalla for presenting his views on reliability theory so clearly, and I would like to thank the editor for this opportunity to comment.

Professor Singpurwalla proposes to improve our understanding of reliability theory and survival analysis by revisiting the interpretation of probability. The notion of exchangeability, he argues, is key; it allows us to link personal probability and propensity. He also expresses the hope that "the linkage also bring about a rapprochement between the relative frequency and personalistic interpretations of probability".

The desire to reconcile or unify competing interpretations of probability has long been widespread among statisticians. Singpurwalla cites M. G. Kendall's 1949 plea for reconciliation, and he could also have cited numerous contributions to this journal, including A. P. Dawid's "Probability, Causality and the Empirical World: A Bayes–de Finetti–Popper–Borel Synthesis" (2004) and my own "The Unity and Diversity of Probability" (1990). None of these efforts at reconciliation have gained much traction. Our efforts to find unity seem only to multiply the number of competing viewpoints.

I believe that only deeper historical understanding can bring order to this babel. In the nineteenth century, especially among continental writers, there was greater unity in the interpretation of probability—greater linkage between its subjective and objective aspects. If today's statisticians and probabilists were better informed about this older linkage and the ways in which it broke down during the twentieth century, we would be much closer to having a common language for moving forward.

TABLE 1
Words used by four authors to distinguish between subjective and objective probability

	Subjective	**Objective**
Laplace (1812)	probabilité	possibilité facilité
Poisson (1837)	probabilité	chance
Cournot (1843)	probabilité subjective	probabilité objective
Sinpurwalla	survivability probability personal probability	reliability propensity objective chance

As I read the words Singpurwallah uses to distinguish between subjective and objective aspects of probability, I found myself wanting to use instead the language of the nineteenth-century giants, as in Table 1. Singpurwalla's main message could be reframed, I think, as a plea for leaving aside the divisions introduced by twentieth-century authors such as von Mises, de Finetti and Popper, and returning to the insights of Laplace and his nineteenth-century successors.

One of Singpurwalla's points is that the twentieth-century view of objective probability often takes con-

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stant probabilities as basic, whereas objective probabilities or propensities may and usually do change over time. But the notion that objective probabilities vary is hardly novel. From John Graunt onward, statistics has always been predominantly about time series (Klein, 1997). Moreover, many nineteenth-century authors, including Poisson, Cournot and Bienaymé, paid great attention to variation in probabilities (Heyde and Seneta, 1977).

Singpurwalla (Section 6) writes:

> Lapace … claimed that the Bernoulli parameter P was the cause of an observed binary random variable, and that one's knowledge about P changes as more and more binary observations are obtained. With filtering, the added dimension is to make provision for the P itself to change over time due to the physics of the scenario.

Yet this added dimension was already discussed by Laplace (Fischer, 2011, page 30). Would Laplace find any fundamental conceptual novelty in Singpurwalla's models? Would he want to use de Finetti's theorem to interpret these models? Probably not.

When we extend our historical horizon to the nineteenth century, von Mises's frequency theory begins to look like an aberration and Popper's philosophy looks less innovative. We should remember that von Mises's ideas were vigorously rejected by the leading French probabilists of his time, especially Lévy and Fréchet, who also wrote at length about the objective concept of probability. One reason for this rejection was that the convergence of relative frequencies considered by von Mises is not the only almost sure property of a random sequence. As Jean Ville pointed out, the relative frequencies in a sequence and selected subsequences might converge but do so from above or otherwise violate the law of the iterated logarithm (Ville, 1939, Bienvenu, Shafer and Shen, 2009). Many philosophers and statisticians continue to present von Mises's as the canonical account of objective probability, but in order to bring it into the mainstream of mathematical probability, we need to revise it, as Ville did, to accommodate all properties that are given probability one by a probabilistic theory—at least all such properties that are simple or computable. From the more practical perspective of statistics, an objective concept of probability must insist on all simple properties that are given probability close to one, and once we do this there is no need to insist that the objective probabilities for a sequence of events or the objective expectations for a sequence of variables should be constant. When we ask, as Jerzy Neyman did, whether frequencies of a scientific phenomenon are consistent with a stochastic process used to model it (Neyman, 1960), we are not pretending that the stochastic process gives constant probabilities; instead we are asking for consistency with martingale-type generalizations of Bernoulli's theorem.

Following Fréchet, Kolmogorov gave a concise set of axioms and definitions that presented probability theory as a child of measure theory and functional analysis. This removed time, which had been fundamental since Laplace, from the basic probability picture and contributed to the misperception that interpreting probability means interpreting isolated or constant probabilities. But it is clear, from his other work in probability at the time, that Kolmogorov did consider time fundamental to the larger probability picture, and Doob devoted great energy to emphasizing the role of time and the fundamental role of Ville's martingales. In more recent decades, the fundamental role of martingales has often become explicit in survival analysis (Aalen et al., 2009).

In the game-theoretic generalization of Kolmogorov's framework developed by Vladmir Vovk, myself, and others over the past twenty years (Shafer and Vovk, 2001, www.probabilityandfinance.com), martingales and supermartingales are used to test probabilistic forecasts. Consistent with the traditional practice of rejecting a hypothesis when a test statistic selected in advance comes out too large, we reject the model or forecasting system when a nonnegative supermartingale reaches too large a multiple of its initial value. Since a nonnegative supermartingale is the capital process for a strategy that bets at the prices set by the model without risking more than its initial capital, this method of testing can be said to identify validity of the model with its resistance to betting: the model is valid if we cannot multiply our capital by a large factor betting against it. This way of looking at testing brings us back to an attitude shared by Émile Borel and Paul Lévy, who did not see so great a distance between the subjective and objective interpretations of probability. A probabilistic model or forecasting system is objectively valid if no one can beat it. It is subjectively valid if you think no one can beat it. There is a difference between asserting objective validity and asserting subjective validity, but in practice this difference may be small.

This is not the place to list all the contributions game-theoretic probability can make to interpreting probability and developing its applications, but I do want to note one application that may be relevant to Singpurwalla's claim (Section 2.2) that "without exchangeability, it is not possible to justify inductive statistical inference." Suppose, to fix ideas, that we observe in sequence quantities $X_1, X_2, \dots, X_N$, where N is very large and each X_n will be either 0 or 1. Suppose we are required, for $n = 1, \dots, N-1$, to give a probability for $X_n = 1$, having already observed $X_1, \dots, X_{n-1}$. Is it possible to do this so that the probabilities will be valid, that is, will pass all reasonable statistical tests? Yes. One method for giving such valid forecasts, called *defensive forecasting* emerges from the game-theoretic framework. It involves obtaining a single betting strategy by averaging the betting strategies corresponding to the different statistical tests we want to beat and then choosing the probabilities to beat that betting strategy (Vovk, Takemura and Shafer, 2005, Shafer, 2008).

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