Electronic Journal of Probability

Electron. J. Probab. **21** (2016), no. 60, 1–52.
ISSN: 1083-6489 DOI: 10.1214/16-EJP4399

Invariant measures of mass migration processes

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Abstract

We introduce the "mass migration process" (MMP), a conservative particle system on $\mathbb{N}^{\mathbb{Z}^d}$. It consists in jumps of k particles ($k \geq 1$) between sites, with a jump rate depending only on the state of the system at the departure and arrival sites of the jump. It generalizes misanthropes processes, hence zero range and target processes. After the construction of MMP, our main focus is on its invariant measures. We derive necessary and sufficient conditions for the existence of translation invariant and invariant product probability measures. In the particular cases of asymmetric mass migration zero range and mass migration target dynamics, these conditions yield explicit solutions. If these processes are moreover attractive, we obtain a full characterization of all translation invariant, invariant probability measures. We also consider attractiveness properties (through couplings), condensation phenomena, and their links for MMP. We illustrate our results on many examples; we prove the coexistence of condensation and attractiveness in one of them.

Keywords: interacting particle systems; multiple jumps; product invariant measures; attractiveness; zero-range process; misanthropes process; target process; mass migration process; condensation.
AMS MSC 2010: 60K35; 82C22.
Submitted to EJP on July 1, 2015, final version accepted on July 20, 2016.
Supersedes arXiv:1507.00778v3.
Supersedes HAL:hal-01172112.

1 Introduction

In the study of an interacting particle system, an essential tool is the explicit knowledge of an invariant measure. Conservative systems such as exclusion or zero-range processes possess a one-parameter family of translation invariant and invariant product probability measures, where the parameter represents the average particle density per

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site [26, 19, 1]. Under conditions on the rates, this is also the case for misanthropes processes, which include the more recently studied target processes [8, 15, 21]. All these dynamics (we call them *single-jump models*) consist in individual jumps of particles between sites, with rates which are the product of two terms: a transition probability giving the direction of the jump, and a function depending on the occupation numbers at the departure and/or arrival sites of the jump. Each dynamics has its particular features, which dictate the precise form of the rates.

However, finding invariant measures is challenging as soon as one departs from these classical models. In this paper we address this question for a class of models generalizing them, that we call *mass migration processes* (abbreviated as MMP). Those dynamics, of state space $\mathbb{N}^{\mathbb{Z}^d}$, allow multiple (simultaneous) jumps of particles between sites, according to rates written as above, but where the function depends also on the number $k \geq 1$ of particles which jump. We distinguish among them the processes for which the rates do not depend on the occupation number of the arrival site and call them *mass migration zero range processes* (abbreviated as MM-ZRP), and the processes for which the rates do not depend on the occupation number of the departure site (as long as it is non-empty) and call them *mass migration target processes* (abbreviated as MM-TP).

Examples of such dynamics have appeared in the literature in various contexts and under various names. Without being exhaustive, we quote some of them that will illustrate our results: The totally asymmetric *stick process* has a constant jump rate (independent of particles' numbers); it was studied in [25] in the context of Ulam's problem, and a generalization with nearest neighbour jumps was considered in [14]; it possesses a one parameter family of product geometric invariant measures. The MMP is a particular case of the multiple particle jump model studied in [14] in the context of attractiveness properties; under some conditions on the rates, the latter possesses a one parameter family of translation invariant and invariant measures, that are not always explicit. Dynamics with zero range interaction and multiple jumps were studied in a finite volume setup in the context of condensation properties: they were called *mass transport models* in [10, 11, 12], and *generalized zero range processes* in [16] (where the number k of particles that may jump together was bounded). In those references, the set of sites is an interval, a general finite graph or a cube; in [11, 12], the authors consider a continuous mass instead of particles moving among sites. A generic form of stationary product measures is exhibited for all those dynamics; such an explicit form is required to analyse condensation phenomena, this is why these models are often chosen as examples (see the reviews [9, 10, 7]). In the context of exactly solvable models and duality, the q-Hahn asymmetric zero range processes are other dynamics with zero range interaction and multiple jumps (see [3] and references therein) with explicit product invariant measures; this knowledge is crucial for exact solvability. Another possible extension of these models are processes in which multiple births and deaths are superimposed to multiple jumps; their attractiveness properties have been established in [5], and then they have been applied to the study of survival and extinction of species in [6].

In all those cases, a specific derivation is performed for each model to find stationary product measures, under conditions on the rates if necessary. In the present paper, our central result unifies and generalizes those derivations, which in particular opens the door to the study of condensation phenomena and exact solvability for new models, or models not tractable up to now: In Theorem 3.1, we obtain necessary and sufficient conditions on the rates of a MMP (which is a dynamics in infinite volume) to guarantee the existence of translation invariant, product invariant probability measures for the process. We can state explicitly how these measures look like for single-jump models, that is, misanthropes, zero-range and target processes (abbreviated as MP, ZRP and TP), as well as for MM-ZRP and MM-TP. For MMP, we then analyse attractiveness properties, as well as condensation issues; our exact formulas for invariant measures enable us to study the relations between those two properties. Finally, we illustrate our results on various examples, including the ones previously mentioned. In particular, we prove for the first time that attractiveness and condensation can coexist, on an example of MM-ZRP (see Proposition 6.3).

The paper is organized as follows. In Section 2, we give a formal description and state an existence theorem for MMP; proofs are done in Section 8. Section 3 is devoted to invariant measures of MMP: Theorem 3.1, single-jump models, MM-ZRP and MM-TP. Proofs are done in Section 4. Section 5 deals with attractiveness of MMP, and contains the determination of the extremal translation invariant and invariant measures for MM-ZRP and MM-TP. We explain in Section 6 how known results on condensation for ZRP can be applied to MMP, and check whether MMP could also be attractive. In Section 7 we bring examples of MMPs and, for each one, we check whether the conditions established in Sections 2, 3, 5, and 6 are satisfied.

2 The model

2.1 Description, existence results

Let us introduce the *mass migration process*, abbreviated as MMP. Particles are located on a countable set X of sites, typically $\mathbb{Z}^d$. At a given time $t \geq 0$, for a configuration $\eta_t = (\eta_t(x) : x \in \mathrm{X})$, $\eta_t(x) \in \mathbb{N}$ is the number of particles on site $x \in \mathrm{X}$. Particles move between sites with respect to the *mass migration dynamics*, that is, $k \geq 1$ particles from the total amount α of particles at a departure site x jump simultaneously to a target site y occupied by β particles with a rate

$$p(x,y)g^k_{\alpha,\beta} \tag{2.1}$$

where $(p(x,y), x, y \in \mathrm{X})$ is a transition probability on X, and where, for all $\alpha,\ \beta \in \mathbb{N}$,

$$g^0_{\alpha,\beta} = 0, \quad g^k_{\alpha,\beta} \text{ are nonnegative for } 0 < k \leq \alpha, \quad \text{and} \quad g^k_{\alpha,\beta} = 0 \text{ for } k > \alpha \text{ or } \alpha = 0. \tag{2.2}$$

This dynamics is *conservative*: the total number of particles involved in a transition is preserved. When X is finite, rates (2.1) define straightforwardly a Markov process $(\eta_t)_{t\geq 0}$ on the state space $\mathbb{N}^{\mathrm{X}}$. When X is countably infinite, some care is required to define $(\eta_t)_{t\geq 0}$ as a Markov process. For this, we rely on methods by Andjel [1] and Liggett & Spitzer [18] and proceed as follows. Details and proofs are given in Section 8.

To avoid initial configurations that could cause explosions, we first restrict the state space to

$$\mathfrak{X} = \{\eta \in \mathbb{N}^{\mathrm{X}} : \|\eta\| < +\infty\} \tag{2.3}$$

where

$$\|\eta\| = \sum_{x\in\mathrm{X}} \eta(x)a_x \tag{2.4}$$

for a positive function a on X such that, for some positive constant M,

$$\sum_{y\in\mathrm{X}} p(x,y)a_y + \sum_{y\in\mathrm{X}} a_y p(y,x) \leq M a_x, \text{ for every } x \in \mathrm{X}, \tag{2.5}$$

and

$$\sum_{x\in\mathrm{X}} a_x < +\infty. \tag{2.6}$$

The following proposition and theorem define the infinitesimal generator $\mathcal{L}$ and the corresponding semigroup $(S(t), t \geq 0)$ for the Markov process $(\eta_t)_{t\geq 0}$. We also introduce the associated markovian coupled process, and finally characterize invariant measures for $(\eta_t)_{t\geq 0}$.

We assume from now on that $(p(x,y) :\ x, y \in \mathrm{X})$ is an irreducible transition probability satisfying

$$\sup_{y\in\mathrm{X}} \sum_{x\in\mathrm{X}} p(x,y) = m_p < +\infty, \tag{2.7}$$

and that the rates satisfy (2.2). Let the set of Lipschitz functions on $\mathfrak{X}$ be

$$\mathbf{L} = \{f : \mathfrak{X} \to \mathbb{R},\ \text{such that for some } L_f > 0, |f(\eta) - f(\zeta)| \leq L_f \|\eta - \zeta\| \text{ for all } \ \eta, \zeta \in \mathfrak{X}\} \tag{2.8}$$

(where $\|\eta - \zeta\| = \sum_{x\in\mathrm{X}} |\eta(x) - \zeta(x)| a_x$).

Proposition 2.1. *Assume that there exists a constant $C > 0$ such that*

$$\sum_{k=1}^{\alpha} k\, g^k_{\alpha,\beta} \leq C(\alpha + \beta) \qquad \textit{for all } \alpha > 0,\ \beta \geq 0, \tag{2.9}$$

then the infinitesimal generator defined for $f \in \mathbf{L}, \eta \in \mathfrak{X}$ by

$$\mathcal{L}f(\eta) = \sum_{x,y\in\mathrm{X}} \sum_{k>0} p(x,y)\ g^k_{\eta(x),\eta(y)} \left(f\left(\mathcal{S}^k_{x,y}\eta\right) - f(\eta)\right) \tag{2.10}$$

where

$$(\mathcal{S}^k_{x,y}\eta)(z) = \begin{cases} \eta(x) - k & \text{if } z = x \text{ and } \eta(x) \geq k \\ \eta(y) + k & \text{if } z = y \text{ and } \eta(x) \geq k \\ \eta(z) & \text{otherwise} \end{cases} \tag{2.11}$$

satisfies

$$|\mathcal{L}f(\eta)| \leq L_f C(1 + M + m_p)\|\eta\|\,. \tag{2.12}$$

We stress that the natural condition (2.9) on rates is not strong enough to prove Theorem 2.2 below (see Section 8), hence we introduce for this the following sufficient condition, which implies (2.9): There exists a constant $C > 0$ such that

$$\sum_{k=1}^{\alpha\vee\gamma} k\,|g^k_{\alpha,\beta} - g^k_{\gamma,\delta}| \ \leq\ C(|\alpha - \gamma| + |\beta - \delta|) \qquad \text{for all } \alpha, \beta, \gamma, \delta \geq 0. \tag{2.13}$$

Theorem 2.2. *Consider the infinitesimal generator $\mathcal{L}$ given by* (2.10)*, and assume that condition* (2.13) *is satisfied. Then there exists a Markov semigroup of operators $(S(t), t \geq 0)$, defined on Lipschitz functions $\mathbf{L}$ on $\mathfrak{X}$, and a constant $c > 0$ such that for all $f \in \mathbf{L},\ t \geq 0$, $\eta \in \mathfrak{X}$, the following items hold:*

(1) $S(t)f \in \mathbf{L}$ and $L_{S(t)f} \leq L_f e^{ct}$;

(2) $S(t)f(\eta) = f(\eta) + \int_0^t \mathcal{L}S(s)f(\eta)\,\mathrm{d}s$;

(3) if $\sum_{x\in\mathrm{X}} \eta(x) < +\infty$, $S(t)f(\eta) = \mathbb{E}^\eta f(\eta_t)$,
where $(\eta_t)_{t\geq 0}$ on the right-hand side is a (countable state space) Markov process with rates (2.1), initial configuration $\eta_0 = \eta$, and $\mathbb{E}^\eta$ denotes its expectation.

Having a Markov semigroup $S(t)$ on $\mathbf{L}$, Daniell-Kolmogorov extension theorem (e.g. [22, Theorem 31.1]) yields the corresponding Markov process, defined by probabilities P^η

on trajectories where projections $P^\eta(\pi_t \in \cdot)$ concentrate on $\mathfrak{X}$ and satisfy $\int f(\xi) P^\eta(\pi_t \in d\xi) = S(t)f(\eta)$.

We construct similarly a (markovian) *coupled process* $(\eta_t, \zeta_t)_{t\geq 0}$ on $\mathfrak{X} \times \mathfrak{X}$ (where we set $\|(\eta,\zeta)\| = \|\eta\| + \|\zeta\|$) whose marginals $(\eta_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$ are copies of the original MMP. As detailed in Section 8, such a process in finite volume enables to prove Theorem 2.2, and it can then be extended analogously to infinite volume, with a Markov semigroup $(\overline{S}(t) : t \geq 0)$ derived from an infinitesimal generator $\overline{\mathcal{L}}$. Whereas basic coupling was the natural coupling to use in [1] and [18], it is not valid anymore here, hence we use the one introduced in [14], which will moreover be helpful dealing later on with attractiveness (the purpose for which it was built), see Section 5 for details.

A probability measure $\bar{\mu}$ on $\mathbb{N}^{\mathrm{X}}$ is called *invariant* for the process with generator $\mathcal{L}$ and Markov semigroup $(S(t) : t \geq 0)$ if

$$\bar{\mu} \text{ is supported on } \mathfrak{X}, \text{ and} \tag{2.14a}$$

$$\int S(t) f \,\mathrm{d}\bar{\mu} = \int f \,\mathrm{d}\bar{\mu} \qquad \text{for every bounded } f \in \mathbf{L} \text{ and every } t \geq 0. \tag{2.14b}$$

Proposition 2.3. *Let $\bar{\mu}$ be a probability measure on $\mathbb{N}^{\mathrm{X}}$ satisfying*

$$\int \|\eta\| \,\mathrm{d}\bar{\mu}(\eta) < +\infty. \tag{2.15}$$

Let us consider a process with generator $\mathcal{L}$ given by (2.10) *where rates satisfy* (2.9)*. Assume that the Markov semigroup $(S(t) : t \geq 0)$ is such that statements (i)–(vii) of Lemma 8.11 hold. Then* (2.14b) *is equivalent to*

$$\int \mathcal{L} f \,\mathrm{d}\bar{\mu} = 0 \;\; \textit{for every bounded cylinder function } f \textit{ on } \mathbb{N}^{\mathrm{X}}. \tag{2.16}$$

2.2 Alternatives

For some models, conditions (2.13) and/or (2.9) will not be valid, and/or the involved probability measures will not satisfy condition (2.15). All these assumptions were sufficient for our construction, so for some examples they could be not necessary, and an alternative construction could work (this includes dynamics in finite volume). Moreover some examples require a state space smaller than $\mathfrak{X}$ and a different construction (see Section 7).

To deal with some of these cases, we shall indicate how to adapt the proofs for invariant measures results in Section 3 for models satisfying the two following assumptions:

- A probability measure $\bar{\mu}$ on $\mathbb{N}^{\mathrm{X}}$ is invariant for the process with generator $\mathcal{L}$ given by (2.10) and Markov semigroup $(S(t) : t \geq 0)$ when (2.14b) is satisfied, and (2.14b) is equivalent to (2.16). (2.17a)

- There exists a constant $C > 0$ such that

$$\sum_{k\leq\alpha} g^k_{\alpha,\beta} \leq C < +\infty. \tag{2.17b}$$

Note that assumption (2.17b) implies (2.9).

2.3 Examples

All along this paper, we shall illustrate our results on the following models, which have only one conservation law. They were all initially defined as single-jump models, that is, with rates $p(x,y)g^k_{\alpha,\beta}$ such that $g^k_{\alpha,\beta}=0$ for $k>1$. We analyse their generalizations to $k\geq 2$, which were either already defined or that we introduce in this work.

• In the *misanthropes process* (MP), introduced by Cocozza in [8],

$$g^k_{\alpha,\beta} \;=\; \mathbb{1}_{\{k=1\}} g^1_{\alpha,\beta} \tag{2.18}$$

for a nonnegative function $g^1_{\cdot,\cdot}$ on $\mathbb{N}\times\mathbb{N}$, non-decreasing (non-increasing) in its first (second) coordinate. In the works of Godrèche et al. [15, 20, 21], arbitrary (that is, without monotonicity properties) nonnegative $g^1_{\cdot,\cdot}$ on $\mathbb{N}\times\mathbb{N}$ are considered in (2.18) and the process is then called the *dynamic urn model* or the *migration process*. However, as per usual, we keep the denomination MP also for those cases; we denote by mass migration processes (MMP) the dynamics extended to multiple jumps.

• *Zero range processes* (ZRP) are single-jump dynamics (that is, with rate (2.18)) introduced by Spitzer in [26], for which the dependence on β is dropped in $g^1_{\alpha,\beta}$. We denote by MM-ZRP their extension to multiple jumps, which was introduced in [16]. To simplify the notation for the rates, g^k_α denotes a function of k and α (the occupation number on the departure site of the jump) only:

$$g^k_{\alpha,\beta} \;=\; \mathbb{1}_{\{k=1\}} g^1_{\alpha} \quad \text{(ZRP)} \tag{2.19}$$

$$g^k_{\alpha,\beta} \;=\; g^k_{\alpha} \qquad \text{(MM-ZRP).} \tag{2.20}$$

• *Target processes* (TP) are single-jump dynamics (that is, with rate (2.18)) introduced in [21], for which the dependence on α is (almost) dropped in $g^1_{\alpha,\beta}$: only $\alpha>0$ is required. We define in this paper their generalization to multiple jumps (MM-TP). To simplify the notation for the rates, $g^k_{*,\beta}$ denotes a function of k and β (the occupation number on the arrival site of the jump) only:

$$g^k_{\alpha,\beta} \;=\; \mathbb{1}_{\{k=1\leq\alpha\}}\, g^1_{*,\beta} \quad \text{(TP)} \tag{2.21}$$

$$g^k_{\alpha,\beta} \;=\; \mathbb{1}_{\{k\leq\alpha\}}\, g^k_{*,\beta} \quad \text{(MM-TP).} \tag{2.22}$$

We shall study in detail various examples of MM-ZRP and MM-TP in Section 7.

3 Product invariant measures

Throughout this section we consider a mass migration process (MMP) $(\eta_t)_{t\geq 0}$ on the set of sites $\mathrm{X}=\mathbb{Z}^d$, with generator (2.10), rates satisfying assumption (2.9), which is translation invariant, that is, with $(p(x,y), x,y\in\mathrm{X})$ a translation invariant transition probability (hence bistochastic, so that $m_p=1$ in (2.7)). Let us denote by $\mathcal{S}$ the set of translation invariant probability measures on $\mathbb{N}^{\mathbb{Z}^d}$, and by $\mathcal{I}$ the set of invariant probability measures for the MMP $(\eta_t)_{t\geq 0}$. We are interested in product, translation invariant and invariant probability measures for $(\eta_t)_{t\geq 0}$. Most proofs are done in Section 4.

3.1 Necessary and sufficient conditions for product invariant measures

In this subsection, we exhibit necessary and sufficient relations between the rates of the MMP $(\eta_t)_{t\geq 0}$ and the single site marginal μ of a *product, translation invariant* probability measure $\bar\mu$ on $\mathbb{N}^{\mathbb{Z}^d}$ which make the latter invariant for $(\eta_t)_{t\geq 0}$.

Theorem 3.1. *Consider a mass migration process with generator $\mathcal{L}$ given by* (2.10)*, whose rates satisfy* (2.9)*. Let $\bar{\mu} \in \mathcal{S}$ be a product measure whose single site marginal μ has a finite first moment $\|\mu\|_1 = \sum_{n\in\mathbb{N}} n\mu(n)$. Let us denote for all $\alpha, \beta \geq 0$*

$$\mathbf{A}(\alpha,\beta) = \begin{cases} \sum\limits_{k\leq\beta} g^k_{\alpha+k,\beta-k} \dfrac{\mu(\alpha+k)\mu(\beta-k)}{\mu(\alpha)\mu(\beta)} - \sum\limits_{k\leq\alpha} g^k_{\alpha,\beta} & \textit{if } \mu(\alpha)\mu(\beta) \neq 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{3.1}$$

A necessary condition for $\bar{\mu}$ to be invariant for the process is

$$\sum_{k\leq\beta} g^k_{\alpha+k,\beta-k}\mu(\alpha+k)\mu(\beta-k) = 0 \textit{ for all } \alpha,\beta \geq 0 \textit{ such that } \mu(\alpha)\mu(\beta) = 0 \tag{3.2}$$

combined with

$$\mathbf{A}(\alpha,\beta) = -\mathbf{A}(\beta,\alpha) \textit{ for all } \alpha,\beta \geq 0. \tag{3.3}$$

These two conditions are also sufficient when $p(\cdot,\cdot)$ is symmetric. When $p(\cdot,\cdot)$ is asymmetric, (3.2) *has to be combined with the following stronger condition to be necessary and sufficient: there exists a function ψ on $\mathbb{N}$ such that*

$$\mathbf{A}(\alpha,\beta) = \psi(\beta) - \psi(\alpha) \quad \textit{for all } \alpha,\beta \geq 0 \quad \textit{such that } \mu(\alpha)\mu(\beta) \neq 0. \tag{3.4}$$

Theorem 3.1 includes dynamics with state space $\{0,\cdots,\gamma\}^{\mathbb{Z}^d}$, for some $\gamma > 0$, as well as measures $\bar{\mu}$ such that $\mu(\alpha) = 0$ for various value(s) of α. We shall deal with such cases in a forthcoming paper, and we shall restrict ourselves in the present paper to measures such that $\mu(\alpha) > 0$ for all $\alpha \in \mathbb{N}$. In this case definition (3.1) reduces to its first line, and condition (3.2) is absent.

Remark 3.2. (a) Conditions (3.3) or (3.4) imply that $\mathbf{A}(\alpha,\alpha) = 0$ for all $\alpha \geq 0$.
(b) Condition (3.4) combined with $\psi(0) = 0$ implies

$$\psi(\alpha) = -\mathbf{A}(\alpha,0) = \mathbf{A}(0,\alpha) \quad \text{for all } \alpha \geq 0. \tag{3.5}$$

It is also possible to deal with a product measure $\bar{\mu}$ whose single site marginal μ has an infinite first moment, either by taking an alternative norm to the one defined in (2.4) in the construction of the process, or if assumptions (2.17a)–(2.17b) are valid.

Remark 3.3. If $\sum_{n\geq0} n\mu(n) = +\infty$ but there exists a non-decreasing function $f : [0,+\infty) \to [0,+\infty)$, with $f(0) = 0$ and increasing to infinity when $n \to +\infty$ in such a way that $\sum_{n\geq0} f(n)\mu(n) < +\infty$, then it is possible to replace the coefficients a_x in the norm $\|\cdot\|$ defined in (2.4) by new coefficients a^*_x for which a new norm $\|\cdot\|^*$ defined by $\|\eta\|^* = \sum_{x\in\mathrm{X}} \eta(x)a^*_x$ satisfies $\bar{\mu}(\|\eta\|^* < +\infty) = 1$.

Corollary 3.4. *Under assumptions (2.17a)–(2.17b), Theorem 3.1 is valid without assuming that μ has a finite first moment.*

From Theorem 3.1 and Corollary 3.4, we deduce that if the product measure $\bar{\mu}$ is invariant for the MMP, the latter possesses a one-parameter family of invariant product probability measures:

Corollary 3.5. *Consider a MMP with generator $\mathcal{L}$ given by* (2.10)*. Let $\bar{\mu} \in \mathcal{S}$ be a product measure invariant for the process. Assume that either the rates of the MMP satisfy assumption* (2.9) *and the single site marginal μ of $\bar{\mu}$ has a finite first moment, or that the rates of the MMP satisfy assumptions (2.17a)–(2.17b). Let $\varphi_c \geq 1$ be the radius of convergence of the series*

$$Z_\varphi = \sum_{n=0}^{+\infty} \varphi^n \mu(n). \tag{3.6}$$

Then for all $\varphi < \varphi_c$, the translation invariant product measure $\bar{\mu}_\varphi$ defined by its single site marginal

$$\mu_\varphi(n) = \bar{\mu}_\varphi(\eta(x) = n) = \frac{1}{Z_\varphi}\varphi^n \mu(n), \qquad n \in \mathbb{N}, \tag{3.7}$$

is invariant for the process. Moreover,

$$\text{either if } \sum_{n\geq 0} n\varphi_c^n \mu(n) < +\infty, \text{ or if } Z_{\varphi_c} < +\infty \text{ and assumptions (2.17a)–(2.17b) are satisfied,} \tag{3.8}$$

then the measure $\bar{\mu}_{\varphi_c}$ is also invariant for the process.

Therefore the latter possesses a one-parameter family of invariant product probability measures, either $\{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z)\}$ or $\{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z')\}$, where $\mathrm{Rad}(Z') \subseteq \mathrm{Rad}(Z)$ are defined by

$$\mathrm{Rad}(Z) = \begin{cases} (0, \varphi_c] & \text{if } Z_{\varphi_c} < +\infty \\ (0, \varphi_c) & \text{if } Z_{\varphi_c} = +\infty \end{cases} \tag{3.9}$$

and

$$\mathrm{Rad}(Z') = \begin{cases} (0, \varphi_c] & \text{if } \sum_{n\geq 0} n\varphi_c^n \mu(n) < +\infty \\ (0, \varphi_c) & \text{otherwise.} \end{cases} \tag{3.10}$$

For further use, when $Z_{\varphi_c} < +\infty$ we define the *critical density* by

$$\rho_c = Z_{\varphi_c}^{-1} \sum_{n\geq 0} n\varphi_c^n \mu(n). \tag{3.11}$$

Moreover, to improve readability later on, we denote by (H) the following hypothesis on the MMP under study, that gathers assumptions from Theorem 3.1 and Corollary 3.4:

(H) *Either the rates satisfy assumption* (2.9) *and the single site marginal of the involved product translation invariant probability measure(s) has a finite first moment, or assumptions (2.17a)–(2.17b) are verified.*

3.2 Applications

In this subsection, we use Theorem 3.1, Corollaries 3.4 and 3.5 to characterize invariant product probability measures for the various examples introduced in Section 2.3. The necessary and sufficient conditions obtained in Theorem 3.1 will be exploited in two ways: 1) the rates being given, check that a measure $\bar{\mu}$ is invariant; 2) a measure $\bar{\mu}$ being given, define rates for which $\bar{\mu}$ is invariant.

This second point of view will enable us in Section 6 to associate many different dynamics to a measure $\bar{\mu}$ satisfying condensation properties.

3.2.1 Single-jump models

The single-jump models we consider are misanthropes processes (MP), for which the generic rate is $p(x,y)g^k_{\alpha,\beta}$ with $g^k_{\alpha,\beta}$ given by (2.18). In this case we derive the following proposition from Theorem 3.1 and Corollary 3.4.

Proposition 3.6. *Consider a MP with rates $p(x,y)g^1_{\alpha,\beta}$. Assume that the rates $(g^1_{1,\alpha})_{\alpha\geq 0}$ and $(g^1_{\alpha,0})_{\alpha\geq 1}$ are positive. Let $\bar{\mu} \in \mathcal{S}$ be a product probability measure whose single site marginal μ satisfies $\mu(\alpha) > 0$ for all $\alpha \in \mathbb{N}$. Assume hypothesis* (H) *is satisfied. Then $\bar{\mu}$ is invariant for the MP if and only if μ is given by*

$$\mu(\alpha) = \mu(0) \prod_{k=1}^{\alpha} \Big[\frac{\mu(1)}{\mu(0)} \frac{g^1_{1,k-1}}{g^1_{k,0}} \Big] \quad \text{for all } \alpha \geq 1, \tag{3.12}$$

and provided that the following compatibility conditions are fulfilled, respectively

$$g^1_{\alpha+1,\beta} = \frac{g^1_{\alpha+1,0}}{g^1_{1,\alpha}} \frac{g^1_{1,\beta}}{g^1_{\beta+1,0}} g^1_{\beta+1,\alpha} \quad \textit{for all} \quad \alpha, \beta \geq 0, \tag{3.13}$$

in the symmetric case, and (3.13) *together with*

$$g^1_{\beta,\alpha} = g^1_{\alpha,\beta} + g^1_{\beta,0} - g^1_{\alpha,0} \quad \textit{for all} \quad \alpha, \beta \geq 0 \tag{3.14}$$

in the asymmetric case.

Proof. For MP, definition (3.1) becomes:

$$\mathbf{A}(\alpha,\beta) = \frac{\mu(\alpha+1)}{\mu(\alpha)} \frac{\mu(\beta-1)}{\mu(\beta)} g^1_{\alpha+1,\beta-1} - g^1_{\alpha,\beta} \qquad \text{for all } \alpha \geq 0, \beta \geq 1 \tag{3.15}$$

$$\mathbf{A}(\alpha,0) = -g^1_{\alpha,0} \qquad \text{for all } \alpha \geq 0. \tag{3.16}$$

Here, condition (3.3) of Theorem 3.1 reduces to well-known *detailed balance conditions*, also called *pairwise balance conditions* in the asymmetric case [20, 24]. Indeed, using (3.15), condition (3.3) for $\alpha \geq 1$, $\beta \geq 1$ can be written as:

$$g^1_{\alpha+1,\beta-1}\mu(\alpha+1)\mu(\beta-1) - g^1_{\beta,\alpha}\mu(\alpha)\mu(\beta) = g^1_{\alpha,\beta}\mu(\alpha)\mu(\beta) - g^1_{\beta+1,\alpha-1}\mu(\alpha-1)\mu(\beta+1) \tag{3.17}$$

where both sides of the equality have the same form and differ only by a shift $(\alpha,\beta) \to (\alpha-1,\beta+1)$. For $\alpha = 0$, $\beta \geq 1$, using (3.15), (3.16), condition (3.3) reads

$$g^1_{1,\beta-1}\mu(1)\mu(\beta-1) - g^1_{\beta,0}\mu(0)\mu(\beta) = 0. \tag{3.18}$$

Iterating (3.17) then using (3.18), one gets for $\alpha \geq 0$, $\beta \geq 1$,

$$\begin{aligned} & g^1_{\alpha+1,\beta-1}\mu(\alpha+1)\mu(\beta-1) - g^1_{\beta,\alpha}\mu(\alpha)\mu(\beta) \\ & \quad = g^1_{1,\alpha+\beta-1}\mu(1)\mu(\alpha+\beta-1) - g^1_{\alpha+\beta,0}\mu(\alpha+\beta)\mu(0) \\ & \quad = 0. \end{aligned} \tag{3.19}$$

Equation (3.19) expresses detailed balance condition, which can be written in a more symmetric form as (cf. [20]):

$$g^1_{\alpha+1,\beta}\mu(\alpha+1)\mu(\beta) = g^1_{\beta+1,\alpha}\mu(\beta+1)\mu(\alpha) \quad \text{for all } \alpha, \beta \geq 0. \tag{3.20}$$

On one hand, (3.20) for $\beta = 0$ gives a relation between the invariant product measure and the subsets of rates $(g^1_{1,\alpha})_{\alpha\geq 0}$ and $(g^1_{\alpha,0})_{\alpha\geq 1}$. One has

$$\frac{g^1_{\alpha+1,0}}{g^1_{1,\alpha}} = \frac{\mu(1)}{\mu(0)} \frac{\mu(\alpha)}{\mu(\alpha+1)} \quad \text{for all } \alpha \geq 0. \tag{3.21}$$

Remark 3.7. Equation (3.21) allows to derive the measure μ, given the rates, up to the ratio $\mu(1)/\mu(0)$. This is consistent with Corollary 3.5 which defines a family of invariant product measures from a given one.

In other words, the existence of a product invariant, translation invariant probability measure for the dynamics implies the existence of a family of such measures. Indeed, from (3.21), one gets (3.12).

On the other hand, inserting expression (3.21) in (3.20) leads to an expression of detailed balance condition in terms of the jump rates only, that is, (3.13). Condition (3.13) has to be supplemented by condition (3.4) in the asymmetric case: the latter, after using Remark 3.2(b) and (3.16), (3.19), writes (3.14). □

In [8], conditions (3.13), (3.14) (denoted there by (2.3), (2.4)) were proved to be sufficient to get product invariant measures, satisfying (3.21) (denoted there by (2.6)).

We now apply Proposition 3.6 to ZRP and TP, that is, we check what become the compatibility conditions (3.13), (3.14), and the form of the marginal (3.12) of a product invariant measure, when it exists. We also explicit Corollary 3.5 in these cases.

• *Zero range process (ZRP).*

Here, both conditions (3.13) and (3.14) are always satisfied, so that there are no compatibility conditions on the rates.

Equation (3.21) expresses that for a product invariant measure with marginal μ, there is a constant $\varphi_\mu > 0$ such that for all $\alpha \geq 1$,

$$g^1_\alpha = \varphi_\mu \frac{\mu(\alpha-1)}{\mu(\alpha)}. \tag{3.22}$$

Consistently with Remark 3.7, a convenient expression for φ_μ is thus

$$\varphi_\mu = g^1_1 \frac{\mu(1)}{\mu(0)} > 0. \tag{3.23}$$

Let $\varphi_c \geq 1$ be the radius of convergence of the series (3.6) which writes here

$$Z_\varphi = \mu(0) \sum_{k=0}^{+\infty} \frac{(\varphi\varphi_\mu)^k}{(g^1_k)!} \tag{3.24}$$

where

$$(g^1_k)! = \begin{cases} 1 & \text{if } k = 0 \\ g^1_k g^1_{k-1} \cdots g^1_1 & \text{if } k > 0. \end{cases} \tag{3.25}$$

By Corollary 3.5, there exists a family of product, translation invariant, invariant measures, $\{\bar{\mu}_\varphi, 0 < \varphi < \varphi_c\}$ or $\{\bar{\mu}_\varphi, 0 < \varphi \leq \varphi_c\}$ if (3.8) is satisfied, with single site marginal (cf. (3.12)):

$$\mu_\varphi(\alpha) = \frac{\mu(0)}{Z_\varphi} \frac{(\varphi\varphi_\mu)^\alpha}{(g^1_\alpha)!} \text{ for } \alpha \geq 0. \tag{3.26}$$

Remark 3.8. Instead of considering equation (3.14), one can insert equation (3.22) in (3.15), (3.16) and get the following expression for $\mathbf{A}(\alpha, \beta)$,

$$\mathbf{A}(\alpha, \beta) = g^1_\beta - g^1_\alpha \quad \text{for all } \alpha, \beta \geq 0 \tag{3.27}$$

which is directly the form of condition (3.4) with $\psi(\alpha) = g^1_\alpha$. This implies that, for the zero range process, conditions for the existence of product invariant measures are the same for both $p(\cdot, \cdot)$ symmetric and $p(\cdot, \cdot)$ asymmetric.

The invariant measures derived here are those studied by [26, 1].

• *Target process (TP).*

For a target process, conditions (3.13) are always satisfied, hence there are no compatibility conditions on the rates in the symmetric case. Condition (3.14) becomes here

$$g^1_{*,\beta} = g^1_{*,1} \qquad \text{for } \beta > 0. \tag{3.28}$$

Thus, only under condition (3.28) does an asymmetric target process admit translation invariant, invariant product probability measures.

Equation (3.21) expresses that, if $\bar{\mu}$ is a product invariant measure with single site marginal μ, there is a constant

$$\tilde{\varphi}_\mu = \frac{1}{g^1_{*,0}} \frac{\mu(1)}{\mu(0)} > 0 \tag{3.29}$$

such that for all $\beta \geq 0$,

$$g^1_{*,\beta} \tilde{\varphi}_\mu = \frac{\mu(\beta+1)}{\mu(\beta)}. \tag{3.30}$$

Let $\varphi_c \geq 1$ be the radius of convergence of the series

$$Z_\varphi = \mu(0)\Big(1 + \sum_{k=1}^{+\infty} (\varphi\tilde{\varphi}_\mu)^k \, g^1_{*,k-1}\, g^1_{*,k-2} \cdots g^1_{*,0}\Big). \tag{3.31}$$

Then, by Corollary 3.5, there exists a one-parameter family of product, translation invariant, invariant probability measures, $\{\bar{\mu}_\varphi, 0 < \varphi < \varphi_c\}$ or $\{\bar{\mu}_\varphi, 0 < \varphi \leq \varphi_c\}$ if (3.8) is satisfied, with single site marginal given by (cf. (3.12)):

$$\mu_\varphi(0) = \frac{\mu(0)}{Z_\varphi}, \qquad \mu_\varphi(\alpha) = \frac{\mu(0)}{Z_\varphi}(\varphi\tilde{\varphi}_\mu)^\alpha \; g^1_{*,\alpha-1}\, g^1_{*,\alpha-2} \cdots g^1_{*,0} \;\text{ for } \alpha \geq 1. \tag{3.32}$$

Then, only under condition (3.28), an asymmetric target process admits a family of invariant, translation invariant, product probability measures, for which the expression (3.32) for the single site marginal depends on the two distinct jump rates, and becomes:

$$\mu_\varphi(\alpha) = \mu_\varphi(0)(\varphi\tilde{\varphi}_\mu)^\alpha g^1_{*,0}(g^1_{*,1})^{\alpha-1} \quad \text{for } \alpha \geq 1, \quad \mu_\varphi(0) = \frac{(1-\varphi\tilde{\varphi}_\mu)g^1_{*,1}}{g^1_{*,1} + \varphi\tilde{\varphi}_\mu(g^1_{*,0} - g^1_{*,1})}. \tag{3.33}$$

Remark 3.9. Comparing (3.32) with the analogous formula (3.26) for zero range process, we see that there is a family of invariant, translation invariant, product measures, common to both a zero range process and a target process if and only if there exists a constant $c_0 > 0$ such that

$$g^1_{*,\alpha} = \frac{c_0^2}{g^1_{\alpha+1}} \tag{3.34}$$

holds for all $\alpha \geq 0$. In [21], (3.34) is called a *duality relation* between ZRP and TP.

The invariant measures derived here were studied in [21].

3.2.2 Mass migration zero range process (MM-ZRP)

In this subsection and the following one, we shall see that the possibility for k to be larger than 1 yields a very different behavior than for single-jump models.

Proposition 3.10. *Let $\bar{\mu} \in \mathcal{S}$ be a product measure whose single site marginal μ satisfies $\mu(0) > 0$. Consider a MM-ZRP with rates $p(x,y)g^k_\alpha$ for an asymmetric $p(\cdot,\cdot)$. Assume hypothesis* (H) *is satisfied. Then $\bar{\mu}$ is invariant for the MM-ZRP if and only if*

$$g^k_{\alpha+k}\mu(\alpha+k) = \mu(\alpha)\frac{\mu(k)}{\mu(0)} g^k_k \qquad \textit{for all } k \geq 1, \alpha \geq 1. \tag{3.35}$$

If $p(\cdot,\cdot)$ is symmetric then condition (3.35) *is only sufficient for $\bar{\mu}$ to be invariant.*

The form (3.35) for the rate is the generic one exhibited in [11, 16]. We shall study such models in detail along the paper, in Subsection 5.2, Examples 7.4 and 7.5.

An additional assumption on the rates yields the following corollary.

Corollary 3.11. *Let $\bar{\mu} \in \mathcal{S}$ be a product measure whose single site marginal μ satisfies $\mu(0) > 0$. Consider a MM-ZRP with rates $p(x,y)g_\alpha^k$ for an asymmetric $p(\cdot,\cdot)$. Assume hypothesis* (H) *is satisfied. Assume that*

$$\text{for all } \alpha \geq 1, \quad g_\alpha^1 > 0. \tag{3.36}$$

Then $\bar{\mu}$ is invariant for the MM-ZRP if and only if

$$\frac{\mu(\alpha)}{\mu(0)} = \frac{1}{(g_\alpha^1)!}\Big(\frac{\mu(1)}{\mu(0)}g_1^1\Big)^\alpha \quad \text{for all } \alpha \geq 1, \quad \text{and} \tag{3.37}$$

$$g_\alpha^k = \frac{(g_\alpha^1)!}{(g_{\alpha-k}^1)!(g_k^1)!}g_k^k \quad \text{for all } 1 \leq k \leq \alpha, \tag{3.38}$$

where $(g_k^1)!$ was defined in (3.25).

Indeed putting $k = 1$ in (3.35) gives (3.37), and then inserting (3.37) back in (3.35) for $k \geq 2$ gives the compatibility condition (3.38) on the rates. The converse is straightforward.

Remark 3.12. Unlike for ZRP, the rates have here to satisfy a compatibility condition, namely (3.38).

The result of Proposition 3.10 can be used on one hand to find invariant measures when the rates of a process are given, and on the other hand to set rates of a process such that it has a prescribed invariant measure. Hence we rephrase Proposition 3.10 and Corollary 3.11 with Corollary 3.5 as follows.

Proposition 3.13.
(a) If μ is a probability measure on $\mathbb{N}$ with $\mu(\alpha) > 0$ for all $\alpha \in \mathbb{N}$, then formula (3.35) *gives transition rates for a MM-ZRP for which the product measure $\bar{\mu}$ with single site marginal μ is invariant, provided assumption* (H) *is satisfied. Here $g_k^k > 0$, $k \geq 1$ can be arbitrarily chosen such that the rates $g_{\alpha+k}^k$ for all $\alpha, k \geq 1$ satisfy* (2.9)*.*

(b) If a MM-ZRP has rates g_α^k which satisfy (2.9)*,* (3.36) *and* (3.38)*, then this MM-ZRP possesses a one-parameter family of product, translation invariant, invariant probability measures $\bar{\mu}_\varphi$ with single site marginal given by* (3.23)–(3.26)*, for $\varphi \in \mathrm{Rad}(Z')$ or $\varphi \in \mathrm{Rad}(Z)$, provided assumption* (H) *is satisfied.*

Note that both (a) and (b) hold independently of $p(\cdot,\cdot)$. If we assume $p(\cdot,\cdot)$ asymmetric then condition (3.38) *is also necessary for the existence of product invariant measures.*

Remark 3.14. In (b) above, the measure $\bar{\mu}_\varphi$, which depends only on the rates g_α^k for $k = 1$, is the same as in the single-jump case.

We conclude that an asymmetric MM-ZRP has the set of product, translation invariant, invariant measures either empty (if (3.35) is not satisfied for any μ) or given explicitly by either $\{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z')\}$ or $\{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z)\}$ where $\bar{\mu}_\varphi$ are product measures with marginals given by (3.23)–(3.26), provided assumption (H) is satisfied.

The situation is rather different for a symmetric MM-ZRP. In general, the only equivalence result follows from Theorem 3.1 and says that a product, translation invariant probability measure $\bar{\mu}$ (provided assumption (H) is satisfied) is invariant for this process with rates g_α^k if and only if

$$\begin{aligned}&\sum_{k\leq\beta} g_{\alpha+k}^k\mu(\alpha+k)\mu(\beta-k) - \sum_{k\leq\alpha} g_\alpha^k\mu(\alpha)\mu(\beta)\\ &= -\sum_{k\leq\alpha} g_{\beta+k}^k\mu(\beta+k)\mu(\alpha-k) + \sum_{k\leq\beta} g_\beta^k\mu(\beta)\mu(\alpha) \quad \text{for all } \alpha \geq 0, \beta \geq 1.\end{aligned} \tag{3.39}$$

If (3.35) is satisfied, then all product, translation invariant probability measures $\bar{\mu}_\varphi$ with marginals given by (3.26) are invariant for the symmetric MM-ZRP with rates g^k_α. But there could exist other product translation invariant, invariant probability measures, satisfying (3.39) but not (3.35).

3.2.3 Mass migration target process (MM-TP)

Theorem 3.15. *Let $\bar{\mu} \in \mathcal{S}$ be a product measure whose single site marginal μ satisfies $\mu(\alpha) > 0$ for all $\alpha \in \mathbb{N}$. Consider a MM-TP with rates $p(x,y)g^k_{*,\alpha}$ for an asymmetric $p(\cdot,\cdot)$. Assume hypothesis* (H) *is satisfied. Then $\bar{\mu}$ is invariant for the MM-TP if and only if*

$$g^\alpha_{*,\beta} = g^\alpha_{*,0} + \frac{1}{\mu(\beta)} \sum_{k=1}^{\beta} H_\alpha(\beta,k) g^k_{*,0} \qquad \text{for all } \alpha \geq 1, \beta \geq 1 \tag{3.40}$$

where the $H_\alpha(\beta,k)$ depend only on $\mu(\cdot)$ and are solution of the following recurrence relations:

$$\begin{cases} H_\alpha(\beta,\beta) = \Delta_\alpha(\beta)\mu(0) \quad \text{for } \beta \geq 1, \\ H_\alpha(\beta,k) = \Delta_\alpha(k)\mu(\beta-k) + \sum_{l=1}^{\beta-k} \Delta_\alpha(l) H_l(\beta-l,k) \quad \text{for } 1 \leq k \leq \beta-1, \end{cases} \tag{3.41}$$

with for all $r > 0$ and all $s \geq 0$

$$\Delta_r(s) = \frac{\mu(r+s)}{\mu(r)} - \frac{\mu(r+s-1)}{\mu(r-1)}. \tag{3.42}$$

By iterating (3.41), we obtain the simplest terms $H_\alpha(\beta,k)$:

$$\begin{aligned} H_\alpha(\beta,\beta-1) &= \Delta_\alpha(\beta-1)\mu(1) + \Delta_\alpha(1)\Delta_1(\beta-1)\mu(0) \\ H_\alpha(\beta,\beta-2) &= \Delta_\alpha(\beta-2)\mu(2) + \Delta_\alpha(1)\Delta_1(\beta-2)\big(\mu(1) + \Delta_1(1)\mu(0)\big) \\ &\quad + \Delta_\alpha(2)\Delta_2(\beta-2)\mu(0). \end{aligned}$$

The general term can be guessed:

$$\begin{aligned} &H_\alpha(\beta,k) = \Delta_\alpha(k)\mu(\beta-k) \\ &+ \sum_{r=1}^{\beta-k} \sum_{\substack{k_1,\dots,k_r \geq 1 \\ k_1+\dots+k_r \leq \beta-k}} \Delta_\alpha(k_1)\Delta_{k_1}(k_2)\dots\Delta_{k_{r-1}}(k_r)\Delta_{k_r}(k)\mu(\beta-k-\sum_{j=1}^{r} k_j). \end{aligned} \tag{3.43}$$

Like for MM-ZRP, the result of Theorem 3.15 can be used on one hand to find invariant measures of a MM-TP with given rates, and on the other hand to set rates of a MM-TP such that it has a prescribed invariant measure. As in Corollary 3.11 and Proposition 3.13, an additional assumption on the rates enables to write a compatibility condition on the rates.

Proposition 3.16.
(a) Let μ be a probability measure on $\mathbb{N}$ with $\mu(\alpha) > 0$ for all $\alpha \in \mathbb{N}$, and $g^\alpha_{*,0} > 0$ be arbitrarily chosen rates for all $\alpha \geq 1$ satisfying condition (2.9)*. Then equations* (3.40)–(3.42) *give transition rates for a MM-TP for which the product measure with single site marginal μ is invariant, provided that all such rates are nonnegative, and that assumption* (H) *is satisfied. A sufficient condition for the $g^\alpha_{*,\beta}$ to be nonnegative is that $\Delta_r(s) \geq 0$ for all $r > 0$, $s \geq 0$, or equivalently that the ratio $\dfrac{\mu(n+1)}{\mu(n)}$ is a nondecreasing function of n.*

(b) Consider a MM-TP with the following rates. For $k \geq 1$, $g^k_{*,0}$ and $g^k_{*,1}$ are given such that they satisfy (2.9) *and that*

$$g^1_{*,0} > 0, \quad g^1_{*,1} > 0; \qquad \textit{and} \quad \textit{for} \;\; \alpha \geq 2, \quad g^1_{*,1} + \sum_{k=2}^{\alpha} (g^k_{*,1} - g^k_{*,0}) > 0. \tag{3.44}$$

For the weights $w(\alpha)$, $\alpha \geq 0$, defined as

$$\begin{cases} w(0) = 1 \\ w(1) = g^1_{*,0} \\ w(2) = g^1_{*,1} g^1_{*,0} \\ w(\alpha) = w(\alpha-1)\left(g^1_{*,1} + \sum\limits_{i=2}^{\alpha-1} (g^i_{*,1} - g^i_{*,0}) \right) \qquad \textit{for} \;\; \alpha \geq 3 \end{cases} \tag{3.45}$$

then setting for all $r > 0$ and all $s \geq 0$,

$$\Delta^*_r(s) = \frac{w(r+s)}{w(r)} - \frac{w(r+s-1)}{w(r-1)} \tag{3.46}$$

and

$$\begin{cases} H^*_\alpha(\beta,\beta) = \dfrac{w(0)}{w(\beta)} \Delta^*_\alpha(\beta) & \textit{for} \;\; \alpha \geq 1, \beta \geq 1, \\ H^*_\alpha(\beta,k) = \dfrac{w(\beta-k)}{w(\beta)} \Delta^*_\alpha(k) & \\ \qquad + \sum\limits_{l=1}^{\beta-k} \dfrac{w(\beta-l)}{w(\beta)} \Delta^*_\alpha(l) H^*_l(\beta-l,k) & \textit{for} \;\; \alpha \geq 1, \, 1 \leq k \leq \beta-1, \end{cases} \tag{3.47}$$

the rates $g^\alpha_{*,\beta}$, for all $\alpha \geq 1$, $\beta > 1$, are given by

$$g^\alpha_{*,\beta} = g^\alpha_{*,0} + \sum_{k=1}^{\beta} H^*_\alpha(\beta,k) g^k_{*,0}\,. \tag{3.48}$$

Then there is a one-parameter family of product invariant, translation invariant, probability measures $\{\bar\mu_\varphi, \varphi \in \mathrm{Rad}(Z')\}$ or $\{\bar\mu_\varphi : \varphi \in \mathrm{Rad}(Z)\}$ with single site marginal given by, for $\tilde\varphi_\mu$ defined in (3.29),

$$\mu_\varphi(\alpha) \;=\; \frac{\mu(0)}{Z_\varphi} (\varphi\tilde\varphi_\mu)^\alpha \, w(\alpha) \;\; \textit{for} \;\; \alpha \geq 0, \tag{3.49}$$

$$Z_\varphi \;=\; \mu(0) \sum_{n \geq 0} (\varphi\tilde\varphi_\mu)^n w(n), \tag{3.50}$$

provided that the rates (3.48) *satisfy* (2.9) *and assumption* (H) *is satisfied.*

Note that both (a) and (b) hold independently of $p(\cdot,\cdot)$. If we assume $p(\cdot,\cdot)$ asymmetric then conditions (3.44)–(3.48) *on rates are also necessary for the existence of product invariant probability measures.*

Remark 3.17. (a) In Proposition 3.16(b), if $g^k_{*,1} \geq g^k_{*,0}$ for $k \geq 2$, then (3.44) is satisfied, but the process will not be attractive unless $g^k_{*,1} = g^k_{*,0}$, see (5.9a) in Lemma 5.1 below. Similarly, in Proposition 3.16(a), if $\mu(n+1)/\mu(n)$ is strictly nondecreasing, the process will not be attractive, see Proposition 6.2 below.
(b) Taking $g^k_{*,1} = g^k_{*,0} = 0$ for $k \geq 2$ in Proposition 3.16(b) yields $g^\alpha_{*,\beta} = 0$, for all $\alpha > 1, \beta \geq 0$ by (3.48), and

$$\Delta^*_1(1) \;=\; g^1_{*,1} - g^1_{*,0} \quad ; \quad \Delta^*_1(l) = 0 \quad \text{for } l \geq 2$$

$$\begin{aligned} H_1^*(\beta,1) &= \frac{g^1_{*,1}-g^1_{*,0}}{g^1_{*,0}} \quad \text{for } \beta\geq 1 \quad \text{thus} \\ g^1_{*,\beta} &= g^1_{*,0}+H_1^*(\beta,1)g^1_{*,0}=g^1_{*,1} \quad \text{for } \beta\geq 1, \end{aligned}$$

that is, we recover (3.28).

Therefore an asymmetric MM-TP has the set of all product, translation invariant, invariant measures either empty (when rates do not satisfy (3.40)–(3.41) for any μ) or given explicitly as the set $\{\bar\mu_\varphi : \varphi \in \mathrm{Rad}(Z')\}$ or $\{\bar\mu_\varphi : \varphi \in \mathrm{Rad}(Z)\}$ where $\bar\mu_\varphi$ are product measures with marginals given by (3.49)–(3.50), provided assumption (H) is satisfied.

Note that for a symmetric MM-TP, the set of all product, translation invariant, invariant measures is in general bigger. In addition to the measures $\bar\mu_\varphi$ with single site marginal (3.49) when (3.40)–(3.41) are satisfied, there may exist other product translation invariant, invariant measures satisfying (3.3) but not (3.40)–(3.41).

4 Proofs for Section 3

4.1 Proofs for Subsection 3.1

Hereafter, we prove Theorem 3.1, Remark 3.3, and Corollaries 3.4 and 3.5. We first prove the following lemma.

Lemma 4.1. *Under the assumptions of Theorem 3.1, for all $\alpha,\beta\in\mathbb{N}$, the three quantities $\sum_{\gamma\geq 0}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)$, $\sum_{\gamma\geq 0}|\mathbf{A}(\gamma,\beta)|\mu(\gamma)$, and $\sum_{\alpha\geq 0}\sum_{\gamma\geq 0}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)\mu(\alpha)$ are finite. Moreover if* (3.4) *is satisfied, we have*

$$\sum_{\beta\geq 0}|\psi(\beta)|\mu(\beta) < +\infty. \tag{4.1}$$

Proof. We define the set

$$S_\mu = \{\alpha\in\mathbb{N}, \mu(\alpha)\neq 0\}. \tag{4.2}$$

For N fixed, we deal simultaneously with the term $\sum_{\gamma\leq N}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)$ for a given value of $\alpha\in\mathbb{N}$ with $\alpha\in S_\mu$ and for its sum over $\alpha\leq N$, that is, $\sum_{\alpha\leq N}\sum_{\gamma\leq N}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)\mu(\alpha)$. We omit a similar proof for $\sum_{\gamma\geq 0}|\mathbf{A}(\gamma,\beta)|\mu(\gamma)$.

$$\sum_{\gamma\leq N}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma) \leq \frac{1}{\mu(\alpha)}\sum_{\gamma\leq N}\sum_{k\leq\gamma} g^k_{\alpha+k,\gamma-k}\mu(\alpha+k)\mu(\gamma-k) + \sum_{\gamma\leq N}\sum_{k\leq\alpha} g^k_{\alpha,\gamma}\mu(\gamma) \tag{4.3}$$

$$\begin{aligned} \sum_{\alpha\leq N}\sum_{\gamma\leq N}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)\mu(\alpha) \leq & \sum_{\alpha\leq N}\sum_{\gamma\leq N}\sum_{k\leq\gamma} g^k_{\alpha+k,\gamma-k}\mu(\alpha+k)\mu(\gamma-k) \\ & + \sum_{\alpha\leq N}\sum_{\gamma\leq N}\sum_{k\leq\alpha} g^k_{\alpha,\gamma}\mu(\alpha)\mu(\gamma). \end{aligned} \tag{4.4}$$

We have by (2.9)

$$\sum_{\gamma\leq N}\mu(\gamma)\sum_{k\leq\alpha} g^k_{\alpha,\gamma} \leq \sum_{\gamma\leq N}\mu(\gamma)C(\alpha+\gamma) \leq C(\alpha+\|\mu\|_1) \tag{4.5}$$

$$\sum_{\alpha\leq N}\mu(\alpha)\sum_{\gamma\leq N}\mu(\gamma)\sum_{k\leq\alpha} g^k_{\alpha,\gamma} \leq \sum_{\alpha\leq N}\mu(\alpha)C(\alpha+\|\mu\|_1) \leq 2C\|\mu\|_1. \tag{4.6}$$

We have also

$$\begin{aligned} \sum_{\gamma\leq N}\sum_{k\leq\gamma} g^k_{\alpha+k,\gamma-k}\mu(\alpha+k)\mu(\gamma-k) &= \sum_{k\leq N}\sum_{k\leq\gamma\leq N} g^k_{\alpha+k,\gamma-k}\mu(\alpha+k)\mu(\gamma-k) \\ &\leq \sum_{k\leq N}\sum_{\alpha\leq N}\sum_{k\leq\gamma\leq N} g^k_{\alpha+k,\gamma-k}\mu(\alpha+k)\mu(\gamma-k) \end{aligned} \tag{4.7}$$

and finally

$$\begin{aligned}\sum_{k\le N}\sum_{\alpha\le N}\sum_{k\le\gamma\le N} g^k_{\alpha+k,\gamma-k}\mu(\alpha+k)\mu(\gamma-k) &= \sum_{k\le N}\ \sum_{k\le\alpha^*\le k+N}\ \sum_{\gamma^*\le N-k} g^k_{\alpha^*,\gamma^*}\mu(\alpha^*)\mu(\gamma^*)\\ &\le \sum_{k\le N}\ \sum_{k\le\alpha^*\le 2N}\ \sum_{0\le\gamma^*\le N} g^k_{\alpha^*,\gamma^*}\mu(\alpha^*)\mu(\gamma^*)\\ &\le \sum_{0\le\alpha^*\le 2N}\ \sum_{0\le\gamma^*\le N}\ \sum_{k\le\alpha^*} g^k_{\alpha^*,\gamma^*}\mu(\alpha^*)\mu(\gamma^*)\\ &\le \sum_{0\le\alpha^*\le 2N}\ \sum_{0\le\gamma^*\le N}\mu(\alpha^*)\mu(\gamma^*)C(\alpha^*+\gamma^*)\\ &\le 2C\|\mu\|_1. \end{aligned}\tag{4.8}$$

Combining (4.3)–(4.8) we get

$$\begin{aligned}\sum_{\gamma\le N}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma) &\le 2\frac{C}{\mu(\alpha)}\|\mu\|_1 + C(\alpha+\|\mu\|_1)\\ \sum_{\alpha\le N}\sum_{\gamma\le N}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)\mu(\alpha) &\le 4C\|\mu\|_1\end{aligned}$$

and we can let $N\to+\infty$ to conclude that for any $\alpha\ge 0$ with $\alpha\in S_\mu$, $\sum_{\gamma\ge 0}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)$ is finite, and so is $\sum_{\alpha\ge 0}\sum_{\gamma\ge 0}|\mathbf{A}(\alpha,\gamma)|\mu(\gamma)\mu(\alpha)$.

Finally, if ψ is a function on $\mathbb{N}$ such that $\psi(\beta)-\psi(\alpha)=\mathbf{A}(\alpha,\beta)$ for all $\alpha,\beta\in S_\mu$, then for every $\alpha\in S_\mu$,

$$\sum_{\beta\ge 0}|\psi(\beta)|\mu(\beta)\le\sum_{\beta\ge 0}|\mathbf{A}(\alpha,\beta)|\mu(\beta)+|\psi(\alpha)|<+\infty$$

and (4.1) is also satisfied. □

Proof of Theorem 3.1.
Step 1. We first note that, using (2.6),

$$\int_{\mathfrak{X}}\|\eta\|\,\mathrm{d}\bar\mu(\eta)\le\int\|\eta\|\,\mathrm{d}\bar\mu(\eta)=\sum_{x\in\mathrm{X}}a_x\int\eta(x)\,\mathrm{d}\mu(\eta(x))=\left(\sum_{x\in\mathrm{X}}a_x\right)\left(\sum_{n\ge 0}n\mu(n)\right).\tag{4.9}$$

Hence, if μ has a finite first moment, then $\bar\mu(\mathfrak{X})=\bar\mu(\|\eta\|<+\infty)=1$ is satisfied.

Now, as a consequence of Proposition 2.3, Lemmas 8.8 and 8.2, the probability measure $\bar\mu$ is invariant if and only if for every bounded cylinder function f on $\mathbb{N}^{\mathbb{Z}^d}$,

$$0=\int\mathcal{L}f(\eta)\,\mathrm{d}\bar\mu(\eta)=\lim_{n\to+\infty}\int\mathcal{L}_nf(\eta)\,\mathrm{d}\bar\mu(\eta),\tag{4.10}$$

where $\mathcal{L}_n$ is the finite volume approximation of $\mathcal{L}$ defined by (8.3).

Let f be a bounded cylinder function, and denote by $V_f\in\mathbb{Z}^d$ its finite support, and by m_f an upper bound for $|f|$. We want to derive an expression for $\int\mathcal{L}_nf(\eta)\,\mathrm{d}\bar\mu(\eta)$, involving $\mathbf{A}(\cdot,\cdot)$ defined in (3.1). We have

$$\int\mathcal{L}_nf(\eta)\,\mathrm{d}\bar\mu(\eta)=\sum_{\substack{x,y\in\mathrm{X}_n,x\ne y,\\ \{x,y\}\cap V_f\ne\emptyset}}p(x,y)\int\sum_{k>0}g^k_{\eta(x),\eta(y)}(f(\mathcal{S}^k_{x,y}\eta)-f(\eta))\,\mathrm{d}\bar\mu(\eta).\tag{4.11}$$

For further use, note that since the probability measure μ has a finite first moment $\|\mu\|_1$, we have by (2.9), for $x, y \in \mathrm{X}_n, x \neq y$

$$\left.\begin{array}{l}\int \sum\limits_{k>0} g^k_{\eta(x),\eta(y)} |f(\mathcal{S}^k_{x,y}\eta)| \,\mathrm{d}\bar{\mu}(\eta) \\ \int \sum\limits_{k>0} g^k_{\eta(x),\eta(y)} |f(\eta)| \,\mathrm{d}\bar{\mu}(\eta)\end{array}\right\} \leq m_f C \int (\eta(x)+\eta(y))\,\mathrm{d}\bar{\mu}(\eta) \leq 2m_f C\|\mu\|_1. \tag{4.12}$$

Step 2. We now prove that condition (3.2) is necessary. Let α_0, β_0 be such that $\mu(\alpha_0)\mu(\beta_0) = 0$ and consider the function $f_0(\eta) = \mathbb{1}_{\{\eta(x_0)=\alpha_0, \eta(y_0)=\beta_0\}}$ for some x_0, y_0 with $p(x_0, y_0) \neq 0$. If $\bar{\mu}$ is an invariant product measure, we have, since $\mu(\alpha_0)\mu(\beta_0) = 0$,

$$\begin{aligned} 0 &= \int \mathcal{L}_n f_0(\eta)\,\mathrm{d}\bar{\mu}(\eta) \\ &= \int \sum_{x,y\in\mathrm{X}_n, x\neq y} p(x,y) \sum_{k>0} g^k_{\eta(x),\eta(y)} (f_0(S^k_{x,y}\eta) - f_0(\eta))\,\mathrm{d}\bar{\mu}(\eta) \\ &= \int \sum_{x,y\in\mathrm{X}_n, x\neq y} p(x,y) \sum_{k>0} g^k_{\eta(x),\eta(y)} f_0(S^k_{x,y}\eta)\,\mathrm{d}\bar{\mu}(\eta). \end{aligned}$$

The last line is a sum of nonnegative terms, thus each term is zero and in particular the one for $x = x_0$, $y = y_0$, which reads

$$\sum_{k\leq\beta_0} g^k_{\alpha_0+k,\beta_0-k}\mu(\alpha_0+k)\mu(\beta_0-k) = 0$$

that is, condition (3.2).

Step 3. For $x, y \in \mathrm{X}_n, x \neq y, \{x, y\} \cap V_f \neq \emptyset$, thanks to (4.12), the integral in the right hand side of (4.11) reads

$$\begin{aligned} &\int \sum_{k>0} g^k_{\eta(x),\eta(y)} (f(\mathcal{S}^k_{x,y}\eta) - f(\eta))\,\mathrm{d}\bar{\mu}(\eta) \\ &\quad= \sum_{k>0} \int_{\{\eta(x)\geq k\}} g^k_{\eta(x),\eta(y)} f(\mathcal{S}^k_{x,y}\eta)\,\mathrm{d}\bar{\mu}(\eta) - \sum_{k>0} \int_{\{\eta(x)\geq k\}} g^k_{\eta(x),\eta(y)} f(\eta)\,\mathrm{d}\bar{\mu}(\eta) \\ &\quad= \sum_{k>0} \int_{\{\eta(y)\geq k\}} g^k_{\eta(x)+k,\eta(y)-k} f(\eta)\,\mathrm{d}\bar{\mu}(S^k_{y,x}\eta) - \sum_{k>0} \int_{\{\eta(x)\geq k\}} g^k_{\eta(x),\eta(y)} f(\eta)\,\mathrm{d}\bar{\mu}(\eta) \\ &\quad= \sum_{k>0} \int_{\{\eta(y)\geq k\}} \mathbb{1}_{\{\eta(x)\in S_\mu\}} \mathbb{1}_{\{\eta(y)\in S_\mu\}} g^k_{\eta(x)+k,\eta(y)-k} f(\eta)\,\mathrm{d}\bar{\mu}(S^k_{y,x}\eta) \\ &\qquad - \sum_{k>0} \int_{\{\eta(x)\geq k\}} g^k_{\eta(x),\eta(y)} f(\eta)\,\mathrm{d}\bar{\mu}(\eta) \\ &\quad= \sum_{k>0} \int_{\{\eta(y)\geq k\}} \mathbb{1}_{\{\eta(x)\in S_\mu\}} \mathbb{1}_{\{\eta(y)\in S_\mu\}} g^k_{\eta(x)+k,\eta(y)-k} f(\eta) D^k_\mu(\eta(x),\eta(y))\,\mathrm{d}\bar{\mu}(\eta) \\ &\qquad - \sum_{k>0} \int_{\{\eta(x)\geq k\}} g^k_{\eta(x),\eta(y)} f(\eta)\,\mathrm{d}\bar{\mu}(\eta) \\ &\quad= \int \mathbb{1}_{\{\eta(x)\in S_\mu\}} \mathbb{1}_{\{\eta(y)\in S_\mu\}} \Big(\sum_{k=1}^{\eta(y)} g^k_{\eta(x)+k,\eta(y)-k} D^k_\mu(\eta(x),\eta(y)) \\ &\qquad - \sum_{k=1}^{\eta(x)} g^k_{\eta(x),\eta(y)}\Big) f(\eta)\,\mathrm{d}\bar{\mu}(\eta) \end{aligned}$$

$$= \int \mathbb{1}_{\{\eta(x)\in S_\mu\}}\mathbb{1}_{\{\eta(y)\in S_\mu\}}\mathbf{A}(\eta(x),\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta)$$
$$= \int \mathbf{A}(\eta(x),\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta) \tag{4.13}$$

where we used condition (3.2) in the third equality and introduced in the next one the quantity

$$D^k_\mu(\alpha,\beta) = \begin{cases} \dfrac{\mu(\alpha+k)\mu(\beta-k)}{\mu(\alpha)\mu(\beta)} & \text{if } k\le\beta \text{ and } \mu(\alpha)\mu(\beta)\neq 0,\\ 1 & \text{if } \mu(\alpha)\mu(\beta)=0. \end{cases}$$

Step 4. We prove *necessity* of condition (3.4) and condition (3.3), respectively for the asymmetric and symmetric cases. Let us denote

$$(\mathbf{A}\mu)(\alpha) = \sum_{\gamma\ge 0}\mathbf{A}(\alpha,\gamma)\mu(\gamma), \qquad (\mu\mathbf{A})(\beta) = \sum_{\gamma\ge 0}\mathbf{A}(\gamma,\beta)\mu(\gamma). \tag{4.14}$$

Using (4.13), we have for any n large enough so that $V_f\subset \mathrm{X}_n$

$$\begin{aligned}\int \mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta) &= \sum_{x,y\in V_f,x\neq y} p(x,y)\int \mathbf{A}(\eta(x),\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\ &\quad + \sum_{x\in V_f,y\in \mathrm{X}_n\setminus V_f} p(x,y)\int\Big(\sum_{\beta\ge 0}\mathbf{A}(\eta(x),\beta)\mu(\beta)\Big)f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\ &\quad + \sum_{x\in \mathrm{X}_n\setminus V_f,y\in V_f} p(x,y)\int\Big(\sum_{\alpha\ge 0}\mathbf{A}(\alpha,\eta(y))\mu(\alpha)\Big)f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\ &= \sum_{x,y\in V_f,x\neq y} p(x,y)\int \mathbf{A}(\eta(x),\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\ &\quad + \sum_{x\in V_f,y\in \mathrm{X}_n\setminus V_f} p(x,y)\int(\mathbf{A}\mu)(\eta(x))f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\ &\quad + \sum_{x\in \mathrm{X}_n\setminus V_f,y\in V_f} p(x,y)\int(\mu\mathbf{A})(\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta). \end{aligned}\tag{4.15}$$

In the first equality, we have split the sum over x and y in three terms, depending on whether they belong to the support of f, and used the fact that $\bar{\mu}$ is a product measure. In the second equality, we introduced notation (4.14).

Consider $f(\eta)=\mathbb{1}_{\{\eta(x_0)=\alpha\}}$ for some fixed $x_0\in\mathbb{Z}^d$ and $\alpha\in S_\mu$. We get in this case from (4.15),

$$\int \mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta) = \sum_{y\in \mathrm{X}_n\setminus\{x_0\}}\Big(p(x_0,y)(\mathbf{A}\mu)(\alpha)+p(y,x_0)(\mu\mathbf{A})(\alpha)\Big)\mu(\alpha).$$

Taking the limit $n\to+\infty$ gives by (4.10)

$$(\mathbf{A}\mu)(\alpha)+(\mu\mathbf{A})(\alpha)=0. \tag{4.16}$$

Now consider $f(\eta)=\mathbb{1}_{\{\eta(x_0)=\alpha,\eta(y_0)=\beta\}}$ for some fixed x_0, y_0 in $\mathbb{Z}^d$, $x_0\neq y_0$ and $\alpha,\beta\in S_\mu$. We get from (4.15),

$$\begin{aligned}\int \mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta) = \mu(\alpha)\mu(\beta)\,\Big(& p(x_0,y_0)\mathbf{A}(\alpha,\beta)+p(y_0,x_0)\mathbf{A}(\beta,\alpha)\\ &+\sum_{z\in \mathrm{X}_n\setminus\{x_0,y_0\}}\Big(p(x_0,z)(\mathbf{A}\mu)(\alpha)+p(z,x_0)(\mu\mathbf{A})(\alpha)\Big)\\ &+\sum_{z\in \mathrm{X}_n\setminus\{x_0,y_0\}}\Big(p(y_0,z)(\mathbf{A}\mu)(\beta)+p(z,y_0)(\mu\mathbf{A})(\beta)\Big)\Big).\end{aligned}$$

Taking the limit $n \to +\infty$ gives by (4.10)

$$\begin{aligned}\mu(\alpha)\mu(\beta)\Big(&p(x_0,y_0)\mathbf{A}(\alpha,\beta)+p(y_0,x_0)\mathbf{A}(\beta,\alpha)\\&+(1-p(x_0,y_0))\big((\mathbf{A}\mu)(\alpha)+(\mu\mathbf{A})(\beta)\big)\\&+(1-p(y_0,x_0))\big((\mu\mathbf{A})(\alpha)+(\mathbf{A}\mu)(\beta)\big)\Big)=0.\end{aligned}\tag{4.17}$$

Using (4.16) and the fact that $\mu(\alpha)\mu(\beta)\neq 0$, (4.17) becomes

$$\begin{aligned}&p(x_0,y_0)\Big(\mathbf{A}(\alpha,\beta)+(\mu\mathbf{A})(\alpha)-(\mu\mathbf{A})(\beta)\Big)\\&+p(y_0,x_0)\Big(\mathbf{A}(\beta,\alpha)+(\mu\mathbf{A})(\beta)-(\mu\mathbf{A})(\alpha)\Big)=0.\end{aligned}\tag{4.18}$$

The same relation being valid under exchange of x_0 and y_0, both the symmetric and antisymmetric parts of equation (4.18) are separately zero. The symmetric part reads

$$\Big(p(x_0,y_0)+p(y_0,x_0)\Big)\Big(\mathbf{A}(\alpha,\beta)+\mathbf{A}(\beta,\alpha)\Big)=0\tag{4.19}$$

which implies (3.3) since there is some (x_0,y_0) such that $p(x_0,y_0)+p(y_0,x_0)\neq 0$.

Assuming (3.3), the antisymmetric part gives

$$(p(x_0,y_0)-p(y_0,x_0))\Big(\mathbf{A}(\alpha,\beta)-(\mu\mathbf{A})(\beta)+(\mu\mathbf{A})(\alpha)\Big)=0.$$

If $p(\cdot,\cdot)$ is symmetric, this equation is identically zero for any choice of $\mathbf{A}$, and we are left with (3.3) alone. If $p(\cdot,\cdot)$ is asymmetric, there is a choice of (x_0,y_0) such that $p(x_0,y_0)\neq p(y_0,x_0)$ and we obtain

$$\mathbf{A}(\alpha,\beta)=\psi(\beta)-\psi(\alpha).\tag{4.20}$$

Since (4.20) defines ψ up to a constant, we may take $\psi(0)=0$ and write $\psi(\cdot)$ in terms of $\mathbf{A}$ as

$$\psi(\gamma)=(\mu\mathbf{A})(\gamma)-(\mu\mathbf{A})(0).\tag{4.21}$$

Step 5. Let us show now that conditions (3.2)–(3.3) and conditions (3.2)–(3.4) *are sufficient* for $\bar{\mu}$ to be invariant, respectively in the symmetric and in the asymmetric case.

Assume first $p(.,.)$ symmetric and conditions (3.2)–(3.3) satisfied. Equation (4.13) gives

$$\begin{aligned}\int\mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta)&=\frac{1}{2}\int\sum_{\substack{x,y\in\mathrm{X}_n,x\neq y,\\\{x,y\}\cap V_f\neq\emptyset}}p(x,y)\left(\mathbf{A}(\eta(x),\eta(y))+\mathbf{A}(\eta(y),\eta(x))\right)f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\&=0.\end{aligned}\tag{4.22}$$

Assume now $p(.,.)$ asymmetric and conditions (3.2)–(3.4) satisfied. Thus, recalling that f is a cylinder function, and using that by (4.1), ψ is integrable, we have

$$\sum_{x,y\in\mathrm{X}_n,\{x,y\}\cap V_f\neq\emptyset}p(x,y)\int\Big|\psi(\eta(y))f(\eta)\Big|\,\mathrm{d}\bar{\mu}(\eta)\leq m_f|V_f|\sum_{\alpha\in\mathbb{N}}|\psi(\alpha)|\mu(\alpha)<+\infty\tag{4.23}$$

then

$$\begin{aligned}\int\mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta)&=\int\sum_{x,y\in\mathrm{X}_n,\{x,y\}\cap V_f\neq\emptyset}p(x,y)\Big(\psi(\eta(y))-\psi(\eta(x))\Big)f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\&=\int\sum_{x,y\in\mathrm{X}_n,\{x,y\}\cap V_f\neq\emptyset}(p(x,y)-p(y,x))\psi(\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta).\end{aligned}$$

For every n such that $V_f \subset \mathrm{X}_n$,

$$\begin{aligned}
\int \mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta) &= \sum_{y\in V_f}\sum_{x\in \mathrm{X}_n}\big(p(x,y)-p(y,x)\big)\int \psi(\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\
&\quad + \sum_{y\in \mathrm{X}_n\setminus V_f}\sum_{x\in V_f}\big(p(x,y)-p(y,x)\big)\Big(\sum_{\alpha\in\mathbb{N}}\psi(\alpha)\mu(\alpha)\Big)\int f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\
&= \sum_{y\in V_f}\sum_{x\in \mathrm{X}_n}\big(p(x,y)-p(y,x)\big)\int \psi(\eta(y))f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\
&\quad - \sum_{y\in V_f}\sum_{x\in \mathrm{X}_n\setminus V_f}\big(p(x,y)-p(y,x)\big)\Big(\sum_{\alpha\in\mathbb{N}}\psi(\alpha)\mu(\alpha)\Big)\int f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\
&= \sum_{y\in V_f}\sum_{x\in \mathrm{X}_n}\big(p(x,y)-p(y,x)\big)\int \Big(\psi(\eta(y))-\sum_{\alpha\in\mathbb{N}}\psi(\alpha)\mu(\alpha)\Big)f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\\
&= \sum_{y\in V_f}\Big[\int\Big(\psi(\eta(y))-\sum_{\alpha\in\mathbb{N}}\psi(\alpha)\mu(\alpha)\Big)f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\Big]\sum_{x\notin \mathrm{X}_n}\big(p(y,x)-p(x,y)\big).
\end{aligned}$$

In the first equality, we have split the sum on y in two terms and used the fact that $\bar{\mu}$ is a product measure. In the second equality, we have exchanged the roles of x and y in the second term. The third equality uses that

$$\sum_{y\in V_f}\sum_{x\in V_f}(p(x,y)-p(y,x)) = 0.$$

In the fourth equality, we used the fact that $p(\cdot,\cdot)$ is bistochastic,

$$\sum_{x\in \mathrm{X}_n}(p(x,y)-p(y,x)) = \sum_{x\notin \mathrm{X}_n}(p(y,x)-p(x,y)).$$

Thus, using (4.23), we have

$$\Big|\int \mathcal{L}_n f(\eta)\,\mathrm{d}\bar{\mu}(\eta)\Big| \le 2\sum_{y\in V_f}\Big|\sum_{x\notin \mathrm{X}_n}\big(p(x,y)-p(y,x)\big)\Big| m_f \sum_{\alpha\in\mathbb{N}}|\psi(\alpha)|\mu(\alpha)$$

which converges to 0 when $n\to+\infty$.
Thus equation (4.10) is satisfied. □

Proof of Remark 3.3. Define, for $x\in\mathrm{X}$, $a_x^* = a_x/f^{-1}(\|x\|_1)$ where f^{-1} is the inverse function of f and $\|x\|_1 = \sum_{i=1}^d |x_i|$. Define, for all $x\in\mathrm{X}, k\in\mathbb{N}$, the events

$$A_x = \{\eta(x)\ge f^{-1}(\|x\|_1)\} = \{\eta(x)a_x^*\ge a_x\},\quad B_k = \{\eta(0)\ge f^{-1}(k)\}.$$

Since $\bar{\mu}$ is translation invariant, $\bar{\mu}(A_x) = \bar{\mu}\left(\eta(0)\ge f^{-1}(\|x\|_1)\right) = \bar{\mu}(B_k)$ whenever $\|x\|_1 = k$. Then we have

$$\sum_{k\in\mathbb{N}}\bar{\mu}(B_k) = \sum_{k\in\mathbb{N}}\bar{\mu}(\eta(0)\ge f^{-1}(k)) = \sum_{k\in\mathbb{N}}\bar{\mu}(f(\eta(0))\ge k) = \sum_{n\in\mathbb{N}} f(n)\mu(n) < +\infty.$$

It follows from Borel-Cantelli Lemma that $\bar{\mu}(B_k$ holds for infinitely many $k\in\mathbb{N}) = 0$. Therefore also $\bar{\mu}(A_x$ holds for at most finitely many $x\in\mathbb{Z}^d) = 1$. But it means that $\bar{\mu}(\sum_{x\in\mathbb{Z}^d}\eta(x)a_x^* < +\infty) = 1$. □

Proof of Corollary 3.4. Note first that in the proof of Lemma 4.1, replacing each bound involving the first moment of μ by the bound (2.17b) yields the results of the lemma.

We now explain where and how to modify the proof of Theorem 3.1 to obtain Corollary 3.4 under assumptions (2.17a)–(2.17b). By assumption (2.17a), the first remark in Step 1 is useless. The first equality in (4.10) comes from assumption (2.17a); to derive the second one thanks to assumption (2.17b), we need to go into the proofs of Subsections 8.2, 8.3.

Let f be a bounded cylinder function, where V_f denotes the support of f and m_f an upper bound for $|f|$. Let $\eta \in \mathbb{N}^X$. Using (2.17b), we replace the computation (8.18) done in the proof of Lemma 8.2 by

$$\begin{aligned} |\mathcal{L}_n f(\eta)| &\leq \sum_{x,y\in \mathrm{X}_n, x\neq y, \{x,y\}\cap V_f \neq \emptyset} p(x,y) \sum_{k=1}^{\eta(x)} g^k_{\eta(x),\eta(y)} |f(\mathcal{S}^k_{x,y}\eta) - f(\eta)| \\ &\leq 2m_f \sum_{x,y\in \mathrm{X}_n, x\neq y, \{x,y\}\cap V_f \neq \emptyset} p(x,y) \sum_{k=1}^{\eta(x)} g^k_{\eta(x),\eta(y)} \\ &\leq 2m_f\, C\, |V_f|. \end{aligned} \tag{4.24}$$

This computation (4.24) enables to adapt Lemmas 8.2, 8.4, 8.6, 8.8, and (2.12) in Proposition 2.1, thus we obtain (4.10).

Then by assumption (2.17b), the two terms considered in (4.12) are bounded directly by m_f, which enables to do Step 3. No other modification is needed. □

Proof of Corollary 3.5. For every $\varphi < \varphi_c$, and for $\varphi = \varphi_c$ if $\sum_{n\geq 0} n\varphi_c^n \mu(n) < +\infty$, if the rates of the MMP satisfy assumption (2.9) and the single site marginal μ of $\bar{\mu}$ has a finite first moment, then the measure $\bar{\mu}_\varphi$ has a finite first moment, and Theorem 3.1 may be applied to $\bar{\mu}_\varphi$.

For every $\varphi < \varphi_c$, and for $\varphi = \varphi_c$ if $\sum_{n\geq 0} n\varphi_c^n \mu(n) = +\infty$ with $Z_{\varphi_c} < +\infty$, if the rates of the MMP satisfy assumptions (2.17a)–(2.17b), then Corollary 3.4 may be applied to $\bar{\mu}_\varphi$.

It follows from the definition (3.7) of $\bar{\mu}_\varphi$ that the quantities $\mathbf{A}(.,.)$ defined in (3.1) coincide for both $\bar{\mu}$ and $\bar{\mu}_\varphi$ for all $a, \beta \in \mathbb{N}$. Since the necessary and sufficient conditions given in Theorem 3.1 are expressed through these quantities, they are thus satisfied for $\bar{\mu}_\varphi$ as soon as they are satisfied for $\bar{\mu}$. If $\bar{\mu}$ is invariant for the MMP, then these necessary and sufficient conditions are satisfied.

Thus, by Theorem 3.1 or by Corollary 3.4 for $\varphi < \varphi_c$, and, for $\varphi = \varphi_c$, either by Theorem 3.1 if $\sum_{n\geq 0} n\varphi_c^n \mu(n) < +\infty$ under assumption (2.9), or by Corollary 3.4 if $\sum_{n\geq 0} n\varphi_c^n \mu(n) = +\infty$ with $Z_{\varphi_c} < +\infty$ under assumptions (2.17a)–(2.17b), we have that $\bar{\mu}_\varphi$ is also invariant for the process. □

4.2 Proofs for Subsection 3.2

Proof of Proposition 3.10. We apply Theorem 3.1 with Corollary 3.4. The product measure $\bar{\mu}$ is invariant for the MM-ZRP if and only if conditions (3.3) and (3.4) are satisfied, provided assumption (H) is satisfied. For the MM-ZRP, we have from (3.1),

$$\mathbf{A}(\alpha,\beta) = \sum_{k\leq\beta} \frac{\mu(\alpha+k)\mu(\beta-k)}{\mu(\alpha)\mu(\beta)} g^k_{\alpha+k} - \sum_{k\leq\alpha} g^k_\alpha \quad \text{and} \quad \mathbf{A}(\alpha,0) = -\sum_{k\leq\alpha} g^k_\alpha.$$

Then the invariance condition (3.4) writes, using (3.5),

$$\sum_{k\leq\beta} \frac{\mu(\alpha+k)\mu(\beta-k)}{\mu(\alpha)\mu(\beta)} g^k_{\alpha+k} = \sum_{k\leq\beta} g^k_\beta \qquad \text{for all } \alpha,\beta \geq 0. \tag{4.25}$$

To prove that condition (3.35) implies (4.25) is straightforward. For the converse, let us denote $\phi^k_\alpha = g^k_\alpha \mu(\alpha)$, so (4.25) writes

$$\sum_{k\le\beta}\mu(\beta-k)\phi^k_{\alpha+k} = \mu(\alpha)\sum_{k\le\beta}\phi^k_\beta \qquad \text{for all } \ \alpha,\beta\ge 0. \tag{4.26}$$

Taking $\beta=1$ in (4.26) gives

$$\phi^1_{\alpha+1} = \frac{\mu(\alpha)}{\mu(0)}\phi^1_1 \quad \text{for } \alpha\ge 1. \tag{4.27}$$

We prove by induction on (n,i), with $0<i<n$, that we have

$$\phi^{n-i}_n = \frac{\mu(i)}{\mu(0)}\phi^{n-i}_{n-i} \qquad \text{for all } \ 0<i<n,\, n\ge 2, \tag{4.28}$$

that is, (3.35) after the change of variables $n=\alpha+k$, $i=\alpha$. To initiate the induction for $n=2, i=1$, we choose $\beta=\alpha=1$ in (4.26). To complete the induction with $n>2, i=n-1$, we choose $\alpha=n-1$ in (4.27).

For the induction step, let us fix $n>2$, $0<i<n$ and assume that (4.28) holds for all (n',i') such that $2\le n'<n$ and $0<i'<n'$. Use (4.26) for $\alpha=i$, $\beta=n-i$ (that is, $\alpha+\beta=n$).

$$\sum_{k=1}^{n-i}\mu(n-i-k)\phi^k_{k+i} = \mu(i)\sum_{k=1}^{n-i}\phi^k_{n-i}\,.$$

After a change of variable $l=n-i-k$, we have

$$\mu(0)\phi^{n-i}_n + \sum_{l=1}^{n-i-1}\mu(l)\phi^{n-i-l}_{n-l} = \mu(i)\sum_{l=0}^{n-i-1}\phi^{n-i-l}_{n-i}\,. \tag{4.29}$$

Using the induction assumption for each term in the sum of the l.h.s. of (4.29) yields

$$\mu(0)\phi^{n-i}_n = \mu(i)\phi^{n-i}_{n-i}\,.$$

The induction is proved. □

Proof of Theorem 3.15. We apply Theorem 3.1 with Corollary 3.4. The product measure $\bar\mu$ is invariant for the MM-TP if and only if conditions (3.3) and (3.4) are satisfied, provided assumption (H) is satisfied. Formula (3.1) writes

$$\mathbf{A}(\alpha,\beta) = \sum_{k\le\beta}\frac{\mu(\alpha+k)\mu(\beta-k)}{\mu(\alpha)\mu(\beta)}g^k_{*,\beta-k} - \sum_{k\le\alpha}g^k_{*,\beta}, \qquad \text{for all } \alpha\ge 0,\beta\ge 0. \tag{4.30}$$

Using Remark 3.2, we set $\psi(0)=0$ and equation (3.5) reads

$$\psi(\alpha) = -\mathbf{A}(\alpha,0) = \sum_{k\le\alpha}g^k_{*,0} \ \text{ for } \alpha>0. \tag{4.31}$$

Thus the set of equations (3.4) is equivalent to

$$\mathbf{A}(0,\beta) \;=\; \psi(\beta) \qquad \text{for all } \ \beta\ge 0 \ \text{ and} \tag{4.32}$$

$$\mathbf{A}(\alpha,\beta)-\mathbf{A}(\alpha-1,\beta) \;=\; \psi(\alpha-1)-\psi(\alpha) \qquad \text{for all } \ \alpha>0,\beta\ge 0. \tag{4.33}$$

We first consider equations (4.33). Using (4.30), (4.31) and notation (3.42), they can be written as, for all $\alpha>0,\beta>0$,

$$g^\alpha_{*,\beta} = g^\alpha_{*,0} + \sum_{k=1}^{\beta}\frac{\mu(\beta-k)}{\mu(\beta)}\Delta_\alpha(k)g^k_{*,\beta-k}\,. \tag{4.34}$$

For $\beta = 1$, it reads

$$\begin{aligned} g^{\alpha}_{*,1} &= g^{\alpha}_{*,0} + \frac{\mu(0)}{\mu(1)}\Delta_\alpha(1) g^1_{*,0} = g^{\alpha}_{*,0} + \frac{\mu(0)}{\mu(1)}\Big(\frac{\mu(\alpha+1)}{\mu(\alpha)} - \frac{\mu(\alpha)}{\mu(\alpha-1)}\Big) g^1_{*,0} \\ &= g^{\alpha}_{*,0} + \frac{1}{\mu(1)} H_\alpha(1,1) g^1_{*,0} \end{aligned} \tag{4.35}$$

which is equation (3.40) for $\beta = 1$, using Definition (3.41). We now prove by induction on $\beta \geq 1$ that equation (3.40) is satisfied. Suppose that the rates fulfil equations (3.41) for all values of β up to some value $\tilde{\beta} - 1 \geq 1$. We derive (3.40) for $\beta = \tilde{\beta}$ as follows. We have

$$\begin{aligned} g^{\alpha}_{*,\tilde{\beta}} &= g^{\alpha}_{*,0} + \frac{\mu(0)}{\mu(\tilde{\beta})}\Delta_\alpha(\tilde{\beta}) g^{\tilde{\beta}}_{*,0} + \sum_{k=1}^{\tilde{\beta}-1} \frac{\mu(\tilde{\beta}-k)}{\mu(\tilde{\beta})}\Delta_\alpha(k) g^k_{*,\tilde{\beta}-k} \\ &= g^{\alpha}_{*,0} + \frac{\mu(0)}{\mu(\tilde{\beta})}\Delta_\alpha(\tilde{\beta}) g^{\tilde{\beta}}_{*,0} + \sum_{k=1}^{\tilde{\beta}-1} \frac{\mu(\tilde{\beta}-k)}{\mu(\tilde{\beta})}\Delta_\alpha(k)\Big(g^k_{*,0} + \frac{1}{\mu(\tilde{\beta}-k)} \sum_{l=1}^{\tilde{\beta}-k} H_k(\tilde{\beta}-k,l) g^l_{*,0}\Big) \\ &= g^{\alpha}_{*,0} + \frac{\mu(0)}{\mu(\tilde{\beta})}\Delta_\alpha(\tilde{\beta}) g^{\tilde{\beta}}_{*,0} + \sum_{k=1}^{\tilde{\beta}-1} \frac{\mu(\tilde{\beta}-k)}{\mu(\tilde{\beta})}\Delta_\alpha(k) g^k_{*,0} \\ &\qquad + \frac{1}{\mu(\tilde{\beta})} \sum_{l=1}^{\tilde{\beta}-1}\sum_{k=1}^{\tilde{\beta}-l} \Delta_\alpha(k) H_k(\tilde{\beta}-k,l) g^l_{*,0} \\ &= g^{\alpha}_{*,0} + \frac{\mu(0)}{\mu(\tilde{\beta})}\Delta_\alpha(\tilde{\beta}) g^{\tilde{\beta}}_{*,0} + \frac{1}{\mu(\tilde{\beta})} \sum_{k=1}^{\tilde{\beta}-1}\Big(\Delta_\alpha(k)\mu(\tilde{\beta}-k) + \sum_{l=1}^{\tilde{\beta}-k} \Delta_\alpha(l) H_k(\tilde{\beta}-l,k)\Big) g^k_{*,0} \\ &= g^{\alpha}_{*,0} + \frac{1}{\mu(\tilde{\beta})} \sum_{k=1}^{\tilde{\beta}} H_\alpha(\tilde{\beta},k)\; g^k_{*,0} \end{aligned}$$

where we used (4.34) for the first equality and the induction hypothesis for the second one, we exchanged the order of the two sums then the names of indexes in the third and fourth ones, and we used Definition (3.41) to conclude.

Conversely, suppose that equations (3.40) are valid. We now prove that equations (4.33) or equivalently equations (4.34) are verified. We have

$$\begin{aligned} g^{\alpha}_{*,\beta} &= g^{\alpha}_{*,0} + \frac{1}{\mu(\beta)} \sum_{k=1}^{\beta} H_\alpha(\beta,l) g^k_{*,0} \\ &= g^{\alpha}_{*,0} + \frac{1}{\mu(\beta)} H_\alpha(\beta,\beta) g^{\beta}_{*,0} + \frac{1}{\mu(\beta)} \sum_{k=1}^{\beta-1}\Big(\Delta_\alpha(k)\mu(\beta-k) + \sum_{l=1}^{\beta-k} \Delta_\alpha(l) H_l(\beta-l,k)\Big) g^k_{*,0} \\ &= g^{\alpha}_{*,0} + \frac{1}{\mu(\beta)}\Big(H_\alpha(\beta,\beta) g^{\beta}_{*,0} + \sum_{k=1}^{\beta-1} \Delta_\alpha(k)\mu(\beta-k) g^k_{*,0} \\ &\qquad + \sum_{k=1}^{\beta-1} \Delta_\alpha(k) \sum_{l=1}^{\beta-k} H_k(\beta-k,l) g^l_{*,0}\Big) \\ &= g^{\alpha}_{*,0} + \frac{1}{\mu(\beta)}\Big(\Delta_\alpha(\beta)\mu(0) g^{\beta}_{*,0} + \sum_{k=1}^{\beta-1} \Delta_\alpha(k)\mu(\beta-k) g^{\alpha}_{*,\beta-k}\Big) \\ &= g^{\alpha}_{*,0} + \sum_{k=1}^{\beta} \Delta_\alpha(k)\mu(\beta-k) g^{\alpha}_{*,\beta-k} \end{aligned}$$

where we used Definitions (3.41) for the second and fourth equalities, and we exchanged the order of the two sums then the names of indexes in the third one.

It remains to show that solutions of (3.40) also verify (4.32). We do it in Lemma 4.4 below which finishes this proof. □

We need two preparatory lemmas to derive Lemma 4.4.

Lemma 4.2. *The coefficients* $H_\alpha(\beta,k)$ *solution of* (3.41) *are also solution of the following (dual) recurrence relation*

$$\begin{cases} H_\alpha(\beta,\beta)=\Delta_\alpha(\beta)\mu(0) \quad \text{for } \beta\geq 1, \\ H_\alpha(\beta,k)=\Delta_\alpha(k)\mu(\beta-k)+\sum_{l=1}^{\beta-k} H_\alpha(\beta-k,l)\Delta_l(k) \quad \text{for } 1\leq k\leq \beta-1. \end{cases} \tag{4.36}$$

Proof. We do an induction on (β,k), with $1\leq k\leq\beta-1$. Notice first that for all integers β,m, we have, by (3.41),

$$\Delta_\alpha(m)H_m(\beta,\beta)=\Delta_\alpha(m)\Delta_m(\beta)\mu(0)=\Delta_m(\beta)H_\alpha(m,m). \tag{4.37}$$

To initiate the induction with $\beta=2,k=1$, and to complete it for $k=\beta-1,\beta>2$, we compute, for $\beta\geq 2$, using (3.41), (4.37):

$$H_\alpha(\beta,\beta-1)=\Delta_\alpha(\beta-1)\mu(1)+\Delta_\alpha(1)H_1(\beta-1,\beta-1)=\Delta_\alpha(\beta-1)\mu(1)+\Delta_1(\beta-1)H_\alpha(1,1)$$

which is the expression of $H_\alpha(\beta,\beta-1)$ given by (4.36). Then we fix $1\leq k<\beta-1$, and we assume (4.36) satisfied for all (β',k') such that $\beta'<\beta$ and $1\leq k'<\beta'-1$. We have

$$\begin{aligned}
&\Delta_\alpha(k)\mu(\beta-k)+\sum_{l=1}^{\beta-k} H_\alpha(\beta-k,l)\Delta_l(k)\\
&=\Delta_\alpha(k)\mu(\beta-k)+\sum_{l=1}^{\beta-k-1}\Big(\Delta_\alpha(l)\mu(\beta-l-k)+\sum_{m=1}^{\beta-l-k}\Delta_\alpha(m)H_m(\beta-k-m,l)\Big)\Delta_l(k)\\
&\quad+H_\alpha(\beta-k,\beta-k)\Delta_{\beta-k}(k)\\
&=\Delta_\alpha(k)\mu(\beta-k)+\sum_{l=1}^{\beta-k-1}\Delta_\alpha(l)\Big(\Delta_l(k)\mu(\beta-l-k)+\sum_{m=1}^{\beta-l-k}H_l(\beta-l-k,m)\Delta_m(k)\Big)\\
&\quad+\mu(0)\Delta_\alpha(\beta-k)\Delta_{\beta-k}(k)\\
&=\Delta_\alpha(k)\mu(\beta-k)+\sum_{l=1}^{\beta-k}\Delta_\alpha(l)H_l(\beta-l,k)\\
&=H_\alpha(\beta,k)
\end{aligned}$$

where we applied first (3.41) to $H_\alpha(\beta-k,l)$ to get the first equality, exchanged the order of the two sums then the names of indexes to get the second one, applied the induction hypothesis with $\beta'=\beta-l<\beta$ to get the third one and finally applied (3.41) in the other direction to conclude. □

Lemma 4.3. *The function defined by* $\psi(\beta,k)=\sum_{l=1}^{\beta-k}\mu(l)H_l(\beta-l,k)$ *for* $\beta>1$ *and* $k<\beta$ *satisfies*

$$\psi(\beta,k)=\sum_{l=1}^{\beta-k}\mu(l)H_l(\beta-l,k)=\mu(0)\mu(\beta)-\mu(k)\mu(\beta-k). \tag{4.38}$$

Proof. We do an induction on (β,k), with $1\leq k\leq\beta-1$. To initiate the induction with $\beta=2,k=1$, and to complete it for $k=\beta-1,\beta>2$, we compute, for $\beta\geq 2$, using (3.41), (3.42):

$$\psi(\beta,\beta-1)=\mu(1)H_1(\beta-1,\beta-1)=\mu(1)\mu(0)\Delta_1(\beta-1)=\mu(1)\mu(0)\left(\frac{\mu(\beta)}{\mu(1)}-\frac{\mu(\beta-1)}{\mu(0)}\right).$$

Then we fix $1 \le k < \beta - 1$, and we assume (4.38) satisfied for all (β', k') such that $\beta' < \beta$ and $1 \le k' < \beta' - 1$. We have

$$\begin{aligned}
\psi(\beta,k) &= \sum_{l=1}^{\beta-k-1} \mu(l)\Big(\Delta_l(k)\mu(\beta-l-k) + \sum_{m=1}^{\beta-l-k} H_l(\beta-l-k,m)\Delta_m(k)\Big) \\
&\quad + \mu(\beta-k)H_{\beta-k}(k,k) \\
&= \sum_{l=1}^{\beta-k-1} \Big(\mu(l)\mu(\beta-l-k) + \sum_{m=1}^{\beta-l-k} \mu(m)H_m(\beta-m-k,l)\Big)\Delta_l(k) \\
&\quad + \mu(\beta-k)\Delta_{\beta-k}(k)\mu(0) \\
&= \sum_{l=1}^{\beta-k-1} \Big(\mu(l)\mu(\beta-l-k) + \psi(\beta-k,l)\Big)\Delta_l(k) + \mu(\beta-k)\Delta_{\beta-k}(k)\mu(0) \\
&= \sum_{l=1}^{\beta-k} \mu(0)\mu(\beta-k)\Delta_l(k) \\
&= \mu(0)\mu(\beta-k)\Big(\frac{\mu(\beta)}{\mu(\beta-k)} - \frac{\mu(k)}{\mu(0)}\Big) \\
&= \mu(0)\mu(\beta) - \mu(k)\mu(\beta-k)
\end{aligned}$$

where we applied (4.36) to get the first equality, we exchanged the order of sums then the names of indexes to get the second one, we used the definition of $\psi(.,.)$ to get the third one, we applied the induction hypothesis for the fourth one and (3.42) for the fifth one. $\square$

Now we are ready to show that

Lemma 4.4. *Equation* (4.32) *holds with rates* $g^{\alpha}_{*,\beta}$ *as in* (3.40)–(3.41).

Proof. We have, for $\beta \ge 1$,

$$\begin{aligned}
\mathbf{A}(0,\beta) &= \mathbb{1}_{\{\beta\ge 2\}} \sum_{k=1}^{\beta-1} \frac{\mu(k)\mu(\beta-k)}{\mu(0)\mu(\beta)} g^k_{*,\beta-k} + \frac{\mu(\beta)\mu(0)}{\mu(0)\mu(\beta)} g^{\beta}_{*,0} \\
&= \mathbb{1}_{\{\beta\ge 2\}} \sum_{k=1}^{\beta-1} \frac{\mu(k)\mu(\beta-k)}{\mu(0)\mu(\beta)} \Big(g^k_{*,0} + \frac{1}{\mu(\beta-k)} \sum_{l=1}^{\beta-k} H_k(\beta-k,l) g^l_{*,0}\Big) + g^{\beta}_{*,0} \\
&= \mathbb{1}_{\{\beta\ge 2\}} \frac{1}{\mu(0)\mu(\beta)} \sum_{k=1}^{\beta-1} \Big(\mu(k)\mu(\beta-k) + \sum_{l=1}^{\beta-k} \mu(l)H_l(\beta-l,k)\Big) g^k_{*,0} + g^{\beta}_{*,0} \\
&= \frac{1}{\mu(0)\mu(\beta)} \sum_{k\le\beta} \Big(\mu(0)\mu(\beta)\Big) g^k_{*,0} = \sum_{k\le\beta} g^k_{*,0} = -\mathbf{A}(\beta,0)
\end{aligned}$$

where we used (4.30) for the first equality, (3.40) for the second one, we exchanged the order of sums then the names of indexes to get the third one, we applied Lemma 4.3 for the fourth one and (4.31) for the last one. $\square$

Proof of Proposition 3.16. Point (a) follows directly from Theorem 3.15. As a preliminary for point (b), take a MM-TP whose rates satisfy (2.13) and (3.44), for which a product probability measure $\bar{\mu} \in \mathcal{S}$ with single-site marginal μ (with $\mu(\alpha) > 0$ for all $\alpha \ge 0$) is invariant. Then by Theorem 3.15, equations (3.40)–(3.41) are satisfied, so that for $\beta = 1, \alpha \ge 1$ we obtain again equation (4.35) that we may write as

$$\frac{\mu(\alpha+1)}{\mu(\alpha)} = \frac{\mu(\alpha)}{\mu(\alpha-1)} + \frac{(g^{\alpha}_{*,1} - g^{\alpha}_{*,0})}{g^1_{*,0}} \frac{\mu(1)}{\mu(0)} \tag{4.39}$$

which implies that

$$\begin{aligned}\frac{\mu(2)}{\mu(1)} &= \frac{\mu(1)}{\mu(0)}\frac{g^1_{*,1}}{g^1_{*,0}} \qquad \text{and} \\ \mu(\alpha+1) &= \mu(\alpha)\frac{\mu(1)}{\mu(0)}\frac{1}{g^1_{*,0}}\left(g^1_{*,1}+\sum_{i=2}^{\alpha}(g^i_{*,1}-g^i_{*,0})\right) \qquad \text{for } \alpha\geq 2\end{aligned}$$

that is, using notations (3.29), (3.45),

$$\mu(\alpha) = \mu(0)(\tilde{\varphi}_\mu)^\alpha w(\alpha) \qquad \text{for } \alpha \geq 0. \tag{4.40}$$

Defining, for $\alpha \geq 1, 1 \leq k \leq \beta$, $\Delta^*_\alpha(\beta)$ and $H^*_\alpha(\beta,k)$ by

$$\Delta_\alpha(k) = (\tilde{\varphi}_\mu)^k \Delta^*_\alpha(k), \qquad H_\alpha(\beta,k) = (\tilde{\varphi}_\mu)^\beta w(\beta)\mu(0)H^*_\alpha(\beta,k) \tag{4.41}$$

we have that $H^*_\alpha(\beta,k)$ and $\Delta^*_\alpha(\beta)$ satisfy (3.47) and (3.46).

We now come to point (b). We consider a MM-TP where the $g^\alpha_{*,\beta}$ satisfy (2.9) and conditions (3.44), (3.48) for $H^*_\alpha(\beta,k)$, $\alpha\geq 1, 1\leq k\leq\beta$ given in (3.47), $\Delta^*_r(s)$, $r>0, s\geq 0$ given in (3.46) and $w(\alpha)$, $\alpha \geq 0$ given in (3.45). Then, consistently with (4.40)–(4.41), formulas (3.49)–(3.50) define a one-parameter family of product invariant measures, provided assumption (H) is satisfied. □

5 Attractiveness and coupling rates

In this section, we take advantage of the work done in [14] on multiple-jump conservative dynamics of misanthrope type, a class of models including MMP, to analyze attractiveness properties of MMP. In Subsection 5.1, we derive the extremal translation invariant and invariant probability measures for MM-ZRP and MM-TP. In Lemma 5.1 and in Subsection 5.2 we give necessary and sufficient conditions for attractiveness of MMP, MM-ZRP and MM-TP based on the conditions established in [14].

Let us first recall the set-up for attractiveness, for which we refer to [19]. We consider the following partial order on the state space $\mathfrak{X} \subset \mathbb{N}^{\mathrm{X}}$. For $\eta,\zeta\in\mathfrak{X}$,

$$\eta \leq \zeta \ \text{ if and only if } \eta(x) \leq \zeta(x) \text{ for every } x \in \mathrm{X}.$$

A bounded, continuous function f on $\mathfrak{X}$ is called *monotone* if $f(\eta) \leq f(\zeta)$ whenever $\eta \leq \zeta$. We call a particle system *attractive* if for every bounded, monotone continuous function f on $\mathfrak{X}$ and every time $t > 0$, the function $S(t)f$ is again a bounded, monotone, continuous function on $\mathfrak{X}$. To check attractiveness of a particle system, the first step is to derive necessary conditions on the rates. The second step is to check that they are sufficient by constructing an *increasing*, markovian coupling $(\eta_t,\zeta_t)_{t\geq 0}$ of two copies of the process, that is, a coupling which preserves the ordering of its marginals:

$$\eta_0 \leq \zeta_0 \text{ implies } \eta_t \leq \zeta_t,\ P^{(\eta_0,\zeta_0)}\text{-almost surely, for every } t > 0.$$

Usually, for dynamics with transitions consisting in the jump, birth or death of a single particle, the so-called *basic coupling* is used: for jumps, it means that coupled particles try to jump together as much as possible from the same departure site to the same arrival site. But basic coupling is useless when $k \geq 1$ particles jump simultaneously, as in the class of conservative dynamics analyzed in [14]. Let us rewrite the results from [14] we need in the context of models with generator (2.10).

For all $\alpha,\beta,k\geq 0$, we denote

$$\mathbf{\Sigma}^k_{\alpha,\beta} = \sum_{k'>k} g^{k'}_{\alpha,\beta} = \sum_{k'=k+1}^{\alpha} g^{k'}_{\alpha,\beta}. \tag{5.1}$$

The following necessary and sufficient conditions for attractiveness were established in [14, Theorem 2.9]: For all $\alpha, \beta, \gamma, \delta \geq 0$, with $\alpha \leq \gamma,\ \beta \leq \delta$,

$$\text{for all } l \geq 0, \quad \mathbf{\Sigma}^{\delta-\beta+l}_{\alpha,\beta} \ \leq \ \mathbf{\Sigma}^{l}_{\gamma,\delta} \quad \text{and} \tag{5.2a}$$

$$\text{for all } k \geq 0, \quad \mathbf{\Sigma}^{k}_{\alpha,\beta} \ \geq \ \mathbf{\Sigma}^{\gamma-\alpha+k}_{\gamma,\delta}. \tag{5.2b}$$

Sufficiency was proved by constructing the following (markovian) increasing coupling process $(\eta_t, \zeta_t)_{t\geq 0}$ on $\mathfrak{X} \times \mathfrak{X}$, with semigroup $(\overline{S}(t) : t \geq 0)$ and infinitesimal generator $\overline{\mathcal{L}}$.

$$\overline{\mathcal{L}}h(\eta,\zeta) = \sum_{x\in \mathrm{X}} \sum_{y\in \mathrm{X}} p(x,y) \sum_{k\geq 0} \sum_{l\geq 0} G^{k,l}_{\eta(x),\eta(y),\zeta(x),\zeta(y)} \left(h\left(\mathcal{S}^k_{x,y}\eta, \mathcal{S}^l_{x,y}\zeta\right) - h\left(\eta,\zeta\right)\right) \tag{5.3}$$

is defined for $(\eta,\zeta) \in \mathfrak{X} \times \mathfrak{X}$ on the set of functions

$$\begin{aligned} \overline{\mathbf{L}} \ &= \ \{h : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R} \text{ such that for some } \overline{L_h} > 0, \\ &\quad |h(\eta_1,\zeta_1) - h(\eta_2,\zeta_2)| \leq \overline{L_h}(\|\eta_1 - \eta_2\| + \|\zeta_1 - \zeta_2\|), \text{ for all } \eta_1,\eta_2,\zeta_1,\zeta_2 \in \mathfrak{X}\}. \end{aligned} \tag{5.4}$$

The rates for coupled jumps are defined in [14, Proposition 2.11]: for all $\alpha,\beta,\gamma,\delta \geq 0$, for all $1 \leq k \leq \alpha$ when $1 \leq \alpha$, for all $1 \leq l \leq \gamma$ when $1 \leq \gamma$,

$$\begin{aligned} G^{k,l}_{\alpha,\beta,\gamma,\delta} &= \left(g^k_{\alpha,\beta} - g^k_{\alpha,\beta} \wedge \left(\mathbf{\Sigma}^l_{\gamma,\delta} - \mathbf{\Sigma}^l_{\gamma,\delta} \wedge \mathbf{\Sigma}^k_{\alpha,\beta}\right)\right) \wedge \left(g^l_{\gamma,\delta} - g^l_{\gamma,\delta} \wedge \left(\mathbf{\Sigma}^k_{\alpha,\beta} - \mathbf{\Sigma}^l_{\gamma,\delta} \wedge \mathbf{\Sigma}^k_{\alpha,\beta}\right)\right) \\ G^{k,0}_{\alpha,\beta,\gamma,\delta} &= g^k_{\alpha,\beta} - g^k_{\alpha,\beta} \wedge \left(\mathbf{\Sigma}^0_{\gamma,\delta} - \mathbf{\Sigma}^0_{\gamma,\delta} \wedge \mathbf{\Sigma}^k_{\alpha,\beta}\right) \\ G^{0,l}_{\alpha,\beta,\gamma,\delta} &= g^l_{\gamma,\delta} - g^l_{\gamma,\delta} \wedge \left(\mathbf{\Sigma}^0_{\alpha,\beta} - \mathbf{\Sigma}^0_{\alpha,\beta} \wedge \mathbf{\Sigma}^l_{\gamma,\delta}\right). \end{aligned} \tag{5.5}$$

In other words, for a given time t and a departure site $x \in \mathrm{X}$, a target site y is chosen with probability $p(x,y)$, and the simultaneous jump of k particles in the first coordinate together with l particles in the second coordinate from x to y occurs with a rate $p(x,y)G^{k,l}_{\alpha,\beta,\gamma,\delta}$ where $\alpha = \eta_{t_-}(x)$, $\beta = \eta_{t_-}(y)$, $\gamma = \zeta_{t_-}(x)$, $\delta = \zeta_{t_-}(y)$, $0 \leq k \leq \alpha$ and $0 \leq l \leq \gamma$. Other jumps are not allowed.

Even if this coupling was originally built in connection with attractiveness, it is of course always valid, thus is used in Sections 2 and 8.

Attractiveness conditions (5.2a)–(5.2b) can be reset in the following explicit form for MMP.

Lemma 5.1. *Consider in (a) the MMP given by generator (2.10) with rates $g^k_{\alpha,\beta}$ and, respectively in (b), (b') and (c), the MM-ZRP with rates g^k_α and the MM-TP with rates $g^k_{*,\beta}$.*

(a) The MMP is attractive if and only if for all $\alpha \geq 1,\ \beta \geq 0,\ k \geq 0$,

$$\mathbf{\Sigma}^{k+1}_{\alpha+1,\beta} \quad \leq \quad \mathbf{\Sigma}^{k}_{\alpha,\beta} \quad \leq \quad \mathbf{\Sigma}^{k}_{\alpha+1,\beta} \quad \textit{and} \tag{5.6a}$$

$$\mathbf{\Sigma}^{k+1}_{\alpha,\beta} \quad \leq \quad \mathbf{\Sigma}^{k}_{\alpha,\beta+1} \quad \leq \quad \mathbf{\Sigma}^{k}_{\alpha,\beta}. \tag{5.6b}$$

(b) The MM-ZRP is attractive if and only if for all $\alpha \geq 1,\ k \geq 0$,

$$\sum_{k'>k+1}^{\alpha+1} g^{k'}_{\alpha+1} \quad \leq \quad \sum_{k'>k}^{\alpha} g^{k'}_{\alpha} \quad \leq \quad \sum_{k'>k}^{\alpha+1} g^{k'}_{\alpha+1}. \tag{5.7}$$

(b') The MM-ZRP is attractive if and only if for all $\alpha \geq 1$, $1 \leq m \leq \alpha$

$$0 \ \leq \ \sum_{j=0}^{m-1} \left(g^{\alpha-j}_{\alpha} - g^{\alpha+1-j}_{\alpha+1}\right) \quad \leq g^{\alpha+1-m}_{\alpha+1}. \tag{5.8}$$

(c) The MM-TP is attractive if and only if for all $\alpha \geq 1,\ \beta \geq 0,\ k \geq 0,$

$$g_{*,\beta}^{\alpha+1} \quad \leq \quad g_{*,\beta}^{\alpha} \quad ; \quad g_{*,\beta+1}^{\alpha} \quad \leq \quad g_{*,\beta}^{\alpha} \quad \textit{and} \tag{5.9a}$$

$$\sum_{k'>k+1}^{\alpha} g_{*,\beta}^{k'} \quad \leq \quad \sum_{k'>k}^{\alpha} g_{*,\beta+1}^{k'}. \tag{5.9b}$$

Proof.
(a) Taking $(\gamma,\delta)=(\alpha+1,\beta)$ in (5.2a) (resp. in (5.2b)) gives the second (resp. first) inequality in (5.6a). Taking $(\gamma,\delta)=(\alpha,\beta+1)$ in (5.2a) (resp. in (5.2b)) gives the first (resp. second) inequality in (5.6b).

For the converse, iterate $\gamma-\alpha$ times the second (resp. first) inequality in (5.6a), then $\delta-\beta$ times the first (resp. second) inequality in (5.6b), to get (5.2a) (resp. (5.2b)).
(b) Writing (5.6a) for MM-ZRP gives (5.7), and (5.6b) becomes trivial for MM-ZRP.
(b') To rewrite (5.7) as (5.8), make the change of variables $k'=\alpha-j$ (resp. $k'=\alpha+1-j$) in the middle (resp. right and left) sum, then subtract the left sum from every term of the inequalities.
(c) Taking $k=\alpha-1$ in (5.6a)–(5.6b) gives (5.9a). Taking $k<\alpha-1$ in the first inequality of (5.6b) gives (5.9b). For the converse, the second inequality in (5.6b) (resp. the first inequality in (5.6a)) corresponds to the replacement of α by k' followed by a sum over k' of the second (resp. the first) inequality in (5.9a). □

Remark 5.2. For single-jump models, all the inequalities in Lemma 5.1 are restricted to $k=0$. First, we recover well-known results: for ZRP, attractiveness means that g_{α}^{1} is a nondecreasing function of α (and basic coupling is an increasing coupling process, see [1] for a detailed construction). The misanthrope process was introduced in [8] as an attractive process, that is, with the function $g_{\alpha,\beta}^{1}$ nondecreasing in its first coordinate α and nonincreasing in its second one β (thus explaining the name *misanthrope*).

For TP, attractiveness means that the function $g_{*,\beta}^{1}$ is nonincreasing in β. When a ZRP and a TP are dual single jump models (see Remark 3.9), attractiveness of the first is equivalent to attractiveness of the second.

5.1 Characterization of $(\mathcal{I}\cap\mathcal{S})_e$ for MM-ZRP and MM-TP

The fact that attractiveness is a very strong property of conservative particle systems is illustrated by the characterization of the set of all translation invariant and invariant measures of MMP. The latter follows from [14, Theorem 5.13], which requires attractiveness and irreducibility conditions for the coupling transition rates, denoted by *(IC)*. Under conditions *(IC)*, if an initial coupled configuration (ξ_0,ζ_0) contains a pair of discrepancies of opposite signs (that is, on two sites x,y, $(\xi_0(x)-\zeta_0(x))(\xi_0(y)-\zeta_0(y))<0$), there is a positive probability (for the coupled evolution) that it evolves into a locally ordered coupled configuration (that is, for some $t\geq 0$, $(\xi_t(x)-\zeta_t(x))(\xi_t(y)-\zeta_t(y))\geq 0$; we refer to [14, Section 2.3] for a detailed analysis of the evolution of discrepancies).

We derive here $(\mathcal{I}\cap\mathcal{S})_e$ for the particular cases of MM-ZRP in the context of Proposition 3.13 and MM-TP in the context of Proposition 3.16. To check conditions *(IC)*, we explicit the values of $G_{\alpha,\beta,\gamma,\delta}^{k,l}$ for $\alpha,\beta,\gamma,\delta\geq 0$ and $(k,l)\in\{(0,1),(1,0),(1,1)\}$, according to expressions (5.1), (5.5).

$$G_{\alpha,\beta,\gamma,\delta}^{0,1}=\begin{cases} g_{\gamma,\delta}^{1} & \text{if } \ \mathbf{\Sigma}_{\alpha,\beta}^{0}\leq\mathbf{\Sigma}_{\gamma,\delta}^{1} \\ \mathbf{\Sigma}_{\gamma,\delta}^{0}-\mathbf{\Sigma}_{\alpha,\beta}^{0} & \text{if } \ \mathbf{\Sigma}_{\gamma,\delta}^{1}<\mathbf{\Sigma}_{\alpha,\beta}^{0}\leq\mathbf{\Sigma}_{\gamma,\delta}^{0} \\ 0 & \text{if } \ \mathbf{\Sigma}_{\gamma,\delta}^{0}<\mathbf{\Sigma}_{\alpha,\beta}^{0} \end{cases} \tag{5.10}$$

$$G^{1,0}_{\alpha,\beta,\gamma,\delta} = \begin{cases} 0 & \text{if } \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} \\ \boldsymbol{\Sigma}^0_{\alpha,\beta} - \boldsymbol{\Sigma}^0_{\gamma,\delta} & \text{if } \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} < \boldsymbol{\Sigma}^0_{\alpha,\beta} \\ g^1_{\alpha,\beta} & \text{if } \boldsymbol{\Sigma}^0_{\gamma,\delta} < \boldsymbol{\Sigma}^1_{\alpha,\beta} \end{cases} \tag{5.11}$$

$$G^{1,1}_{\alpha,\beta,\gamma,\delta} = \begin{cases} 0 & \text{if } \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} \\ \boldsymbol{\Sigma}^0_{\alpha,\beta} - \boldsymbol{\Sigma}^1_{\gamma,\delta} & \text{if } \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} < \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} \\ g^1_{\gamma,\delta} & \text{if } \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} < \boldsymbol{\Sigma}^0_{\alpha,\beta} \\ g^1_{\alpha,\beta} & \text{if } \boldsymbol{\Sigma}^1_{\gamma,\delta} < \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} \\ \boldsymbol{\Sigma}^0_{\gamma,\delta} - \boldsymbol{\Sigma}^1_{\alpha,\beta} & \text{if } \boldsymbol{\Sigma}^1_{\gamma,\delta} < \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} < \boldsymbol{\Sigma}^0_{\alpha,\beta} \\ 0 & \text{if } \boldsymbol{\Sigma}^0_{\gamma,\delta} \le \boldsymbol{\Sigma}^1_{\alpha,\beta} \end{cases} \tag{5.12}$$

Proposition 5.3. *Consider a MM-ZRP for which the rates g^k_α satisfy* (2.9)*,* (3.36) *and* (3.38)*. Assume that the attractiveness condition* (5.7) *is satisfied. Then*

$$(\mathcal{I} \cap \mathcal{S})_e = \{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z')\} \textit{ or } \{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z)\} \tag{5.13}$$

where each $\bar{\mu}_\varphi$ is a product probability measure with single site marginal μ_φ given by (3.26)*, provided that assumption* (H) *is satisfied.*

Proof. By Proposition 3.13, the MM-ZRP has a one-parameter family of product, invariant, translation invariant probability measures. As explained above, due to attractiveness, we have to check conditions *(IC)* of [14, Theorem 5.13]. For $x, y \in \mathbb{Z}^d$, let $\alpha, \beta, \gamma, \delta \ge 0$ be such that $\xi_0(x) = \alpha, \xi_0(y) = \beta, \zeta_0(x) = \gamma, \zeta_0(y) = \delta$ with $\alpha < \gamma, \beta > \delta$. Condition (5.7) for $k = 0$ implies

$$\boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} \qquad \text{and} \qquad \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta}.$$

This combined with (3.36) enables to simplify (5.10)–(5.12), which become

$$\begin{aligned} G^{0,1}_{\alpha,\beta,\gamma,\delta} &= \begin{cases} g^1_{\gamma,\delta} & \text{if } \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} \\ \boldsymbol{\Sigma}^0_{\gamma,\delta} - \boldsymbol{\Sigma}^0_{\alpha,\beta} & \text{if } \boldsymbol{\Sigma}^1_{\gamma,\delta} < \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} \end{cases} \\ G^{1,0}_{\alpha,\beta,\gamma,\delta} &= 0 \\ G^{1,1}_{\alpha,\beta,\gamma,\delta} &= \begin{cases} 0 & \text{if } \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} \\ \boldsymbol{\Sigma}^0_{\alpha,\beta} - \boldsymbol{\Sigma}^1_{\gamma,\delta} & \text{if } \boldsymbol{\Sigma}^1_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} < \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^0_{\gamma,\delta} \\ 0 & \text{if } \boldsymbol{\Sigma}^0_{\gamma,\delta} \le \boldsymbol{\Sigma}^1_{\alpha,\beta} \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} &\text{if} && \boldsymbol{\Sigma}^0_{\alpha,\beta} \le \boldsymbol{\Sigma}^1_{\gamma,\delta} && \text{then} && G^{0,1}_{\alpha,\beta,\gamma,\delta} = g^1_{\gamma,\delta} > 0; \\ &\text{while if} && \boldsymbol{\Sigma}^0_{\alpha,\beta} > \boldsymbol{\Sigma}^1_{\gamma,\delta} && \text{then} && G^{1,1}_{\alpha,\beta,\gamma,\delta} = \boldsymbol{\Sigma}^0_{\alpha,\beta} - \boldsymbol{\Sigma}^1_{\gamma,\delta} > 0. \end{aligned} \tag{5.14}$$

Then, proceeding as in the proof of [14, Proposition 6.3], we build a path of coupled transitions which leads to a locally ordered coupled configuration. In a similar way, if $\alpha < \gamma, \beta = \delta$, we build a path of coupled transitions which creates a discrepancy on y. Hence conditions *(IC)* are satisfied. □

Remark 5.4. The path of coupled transitions built previously is a priori not unique. For instance, for the stick process (see Example 7.3 in Section 7), other paths are exhibited in [14, Subsection 6.4].

Proposition 5.5. *Consider a MM-TP for which the rates $g^k_{*,\beta}$ satisfy* (2.9)*, relations* (3.44)–(3.48)*, and are such that $g^1_{*,\alpha} > 0$ for every $\alpha \ge 0$. Assume moreover the attractiveness conditions* (5.9a)–(5.9b) *satisfied. Then*

$$(\mathcal{I} \cap \mathcal{S})_e = \{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z')\} \textit{ or } \{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z)\} \tag{5.15}$$

where each $\bar{\mu}_\varphi$ is the product measure with single site marginal μ_φ given by (3.49)–(3.50)*, provided that assumption* (H) *is satisfied.*

Proof. By Proposition 3.16, the MM-TP has a one-parameter family of product, translation invariant, invariant probability measures. Due to attractiveness, we have to check conditions *(IC)* of [14, Theorem 5.13]. Since we assumed that $g^1_{*,\alpha} > 0$ for every $\alpha \geq 0$, we can proceed exactly as in the proof of Proposition 5.3 to check conditions *(IC)*, restricting ourselves to coupled jumps with $k, l \in \{0,1\}$. □

5.2 Attractiveness of MM-ZRP with product invariant measures

In this paragraph we give equivalent forms of condition (5.8) for attractiveness of MM-ZRP for the following rate, which is generic for MM-ZRP with product invariant measures (cf. (3.35)), as will be developed in Examples 7.4 and 7.5.

$$g^k_\alpha = \frac{\pi(\alpha-k)}{\pi(\alpha)}\, h(k) \quad \text{for } \ 1 \leq k \leq \alpha \tag{5.16}$$

where π is a positive function on $\mathbb{N}$ and h is a nonnegative function on $\mathbb{N}\setminus\{0\}$; we set $\pi(i) = 0$ for $i \in \mathbb{Z}^-$ and $h(0) = 0$.

Here, condition (5.8) for attractiveness writes: For all $\alpha \geq 1$, $1 \leq m \leq \alpha$,

$$0 \overset{(a)}{\leq} \sum_{j=0}^{m-1} \pi(j)\Big(\frac{h(\alpha-j)}{\pi(\alpha)} - \frac{h(\alpha+1-j)}{\pi(\alpha+1)}\Big) \overset{(b)}{\leq} \frac{\pi(m)h(\alpha+1-m)}{\pi(\alpha+1)}. \tag{5.17}$$

For each $n \geq 0$ denote

$$r(n) = \frac{\pi(n)}{\pi(n+1)}. \tag{5.18}$$

Lemma 5.6. *(a) Assume that r is a nonincreasing function. Then condition (5.17) for attractiveness is equivalent to*

$$\frac{h(\alpha+1)}{\pi(\alpha+1)} \leq \frac{h(\alpha)}{\pi(\alpha)} \quad \textit{for all } \ \alpha \geq 1 \quad \textit{and} \tag{5.19a}$$

$$\sum_{k=1}^{\alpha} h(k)\Big(\frac{\pi(\alpha-k)}{\pi(\alpha)} - \frac{\pi(\alpha-k+1)}{\pi(\alpha+1)}\Big) \leq \pi(0)\frac{h(\alpha+1)}{\pi(\alpha+1)} \quad \textit{for all } \ \alpha \geq 1 \tag{5.19b}$$

and the following condition is necessary for the attractiveness of MM-ZRP with g^k_α given by (5.16)*.*

$$S(\alpha) \leq S(\alpha+1) \quad \textit{for all } \ \alpha \geq 2, \quad \textit{where} \quad S(\alpha) = \frac{1}{\pi(\alpha)} \sum_{i=1}^{\alpha-1} \pi(i)\pi(\alpha-i). \tag{5.20}$$

(a') If we moreover assume that $h(\alpha) = h_0\pi(\alpha)$ for all $\alpha \geq 1$ for some constant $h_0 > 0$ then condition (5.20) *is sufficient and necessary for the attractiveness of MM-ZRP with g^k_α of the form*

$$g^k_\alpha = h_0 \frac{\pi(\alpha-k)}{\pi(\alpha)}\, \pi(k) \quad \textit{for } \ 1 \leq k \leq \alpha. \tag{5.21}$$

(b) Assume that r is a nondecreasing function. Then, to have attractiveness, condition (5.19a) *is necessary, and the condition*

$$\frac{\pi(\alpha+1)}{\pi(\alpha)} \geq \max_{1\leq k\leq\alpha}\Big(\frac{h(k+1)}{h(k)}\Big) \quad \textit{for all } \ \alpha \geq 1 \tag{5.22}$$

is sufficient.

Proof.
Preliminaries: Note that the function

$$m \mapsto \phi_\alpha(m) = \frac{\pi(m)h(\alpha+1-m)}{\pi(\alpha+1)} - \sum_{j=0}^{m-1} \pi(j)\Big(\frac{h(\alpha-j)}{\pi(\alpha)} - \frac{h(\alpha+1-j)}{\pi(\alpha+1)}\Big) \qquad (5.23)$$

defined for $1 \le m \le \alpha$, which represents the r.h.s. of (5.17b) minus its l.h.s., has the same monotonicity as r: if r is nonincreasing (resp. nondecreasing), then ϕ_α is also nonincreasing (resp. nondecreasing).

Part (a).
• Taking $m = 1$ in (5.17a) yields (5.19a). Then (5.19a) combined with the fact that r is nonincreasing implies that, for $0 \le j \le m-1$, $1 \le m \le \alpha$,

$$\frac{h(\alpha-j)}{h(\alpha-j+1)} \ge \frac{\pi(\alpha)}{\pi(\alpha+1)}$$

hence (5.17a) is valid. Therefore (5.19a) is equivalent to (5.17a).
• Because r is nonincreasing, so is $\phi_\alpha(.)$ (by the preliminaries). Hence condition (5.17b) for every $1 \le m \le \alpha$ is equivalent to (5.17b) for $m = \alpha$. The latter is equivalent, by the change of variables $k = \alpha - j$, to condition (5.19b).
• Using (5.19a) twice in (5.19b), first through inequality $h(k) \ge \pi(k)\dfrac{h(\alpha)}{\pi(\alpha)}$ for each summand on its left hand side, then on its r.h.s., we obtain

$$\frac{h(\alpha)}{\pi(\alpha)} \sum_{k=1}^{\alpha} \frac{\pi(k)}{\pi(0)}\Big(\frac{\pi(\alpha-k)}{\pi(\alpha)} - \frac{\pi(\alpha-k+1)}{\pi(\alpha+1)}\Big) \le \frac{h(\alpha+1)}{\pi(\alpha+1)} \le \frac{h(\alpha)}{\pi(\alpha)}$$

which is trivial for $\alpha = 1$, and implies (5.20) for $\alpha \ge 2$.

Part (a').
If moreover the equality in (5.19a) holds for all α (that is, $h(\alpha) = h_0\pi(\alpha)$), then (5.20) implies (5.19b).

Part (b).
As in Part (a), taking $m = 1$ in (5.17a) yields (5.19a). For the converse, condition (5.22) implies (5.17a). To get (5.17b), we use that because r is nondecreasing, so is $\phi_\alpha(.)$ (by the preliminaries). Hence condition (5.17b) for every $1 \le m \le \alpha$ is equivalent to (5.17b) for $m = 1$, which comes from the fact that r is nondecreasing. □

Remark 5.7. Taking $k = \alpha$ in (5.21) implies that for all $\alpha \ge 1$, $g_\alpha^\alpha = h_0\pi(0)$.

Because (5.20) does not depend on h anymore, Lemma 5.6 implies the following corollary.

Corollary 5.8. *Assume r nonincreasing. If the MM-ZRP with $g_\alpha^k = \dfrac{\pi(\alpha-k)}{\pi(\alpha)}\pi(k)$ is not attractive then the MM-ZRP with $g_\alpha^k = \dfrac{\pi(\alpha-k)}{\pi(\alpha)}h(k)$ is not attractive for any other choice of $h(\cdot)$.*

Therefore, for attractiveness properties when r is nonincreasing, it is enough to restrict ourselves to the MM-ZRP with g_α^k given by (5.21), for which we arrange condition (5.20). By (5.18), we can write for $2 \le i \le \alpha$,

$$S(\alpha) = \frac{\pi(i)\pi(\alpha-i)}{\pi(\alpha)} = \pi(1)\frac{r(\alpha-1)r(\alpha-2)r(\alpha-3)\dots r(\alpha-i)}{r(1)r(2)\dots r(i-1)}. \qquad (5.24)$$

6 Condensation

It follows from Section 3 that we are able, for a probability measure $\bar{\mu}$ which is product, translation invariant on $\mathbb{Z}^d$, with single site marginal μ satisfying some conditions, to define rates of a MMP for which $\bar{\mu}$ is invariant. Moreover this yields a one-parameter family $\{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z')\}$ or $\{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z)\}$ of invariant measures for this process; the single site partition function Z_φ, defined in (3.6), has radius of convergence φ_c, and the critical density ρ_c was defined in (3.11).

The point is that, as we explain below in subsection 6.1, condensation phenomena amount to convergence results concerning only stationary distributions, and not the details of the dynamics. So it is enough to verify convergence properties of the partition function and the average density of particles with respect to the involved probability measure to know whether condensation occurs in the stationary states. Then having a stationary state which produces condensation we can study different MMPs which lead to this stationary state: We give some examples at the end of this section, and others in Section 7.

We end this section checking whether attractiveness and condensation could be compatible for MMPs. We answer 'no' for particular cases. We then turn to this question for the second set-up for condensation (see below for a definition), where we show that the answer 'yes' is possible.

6.1 Known results

Consider a finite set $\Lambda \subset \mathbb{Z}$ with L sites where N particles evolve according to the restriction on Λ of the infinite volume dynamics, with a periodic boundary condition. To stress dependence on finite volume we denote by $\bar{\mu}_\varphi^\Lambda$ the product measure on $\mathbb{N}^\Lambda$ with marginals μ_φ. Such a finite volume process has a unique stationary distribution $\mu_{N,L}$ (independent of φ) which can be expressed for $\eta \in \mathbb{N}^\Lambda$ as

$$\mu_{N,L}(\eta) = \bar{\mu}_\varphi^\Lambda\left(\eta | \sum_{x\in\Lambda}\eta(x) = N\right). \tag{6.1}$$

The first set-up for condensation [2, 9, 17] concerns, for a system with finite critical density $\rho_c < +\infty$, the thermodynamic limit of the above restricted dynamics when $N/L \to \rho > \rho_c$ when $N, L \to +\infty$. Then there is no appropriate invariant measure and the system can be seen at two levels: *a homogeneous background* (fluid phase) where occupation numbers per site are distributed according to the product measure at critical density $\bar{\mu}_{\varphi_c}$, and *a condensate* (condensed phase) in which the excess mass is accumulated on a single site. We say that the process (or dynamics) *allows condensation*.

In the second set-up for condensation [4, 13, 23], the number L of sites is fixed and only the total number N of particles goes to infinity. This set-up requires only φ_c to be finite, but ρ_c can be infinite. We talk about *condensation on a fixed finite volume*.

We mention both approaches since we shall present examples for each one. In either case, we say that the process *exhibits condensation*. Note that both cases require that $Z_{\varphi_c} < +\infty$.

In the first set-up for condensation, assuming $\rho_c < +\infty$, the following *equivalence of ensembles* result was proved in [17] in the context of zero range processes: For all $\rho \geq 0$, $n \in \mathbb{N}$, we have (for μ_φ given by (3.26))

$$\lim_{\substack{L,N\to+\infty \\ N/L\to\rho}} \mu_{N,L}(\eta(x_1) = k_1, ..., \eta(x_n) = k_n) = \prod_{i=1}^{n} \mu_\varphi(k_i) \qquad \text{for all } k_1, \ldots, k_n \in \mathbb{N}. \tag{6.2}$$

In the particular case of a ZRP with rate

$$g_n^1 = 1 + \frac{b}{n} \qquad \text{for some} \quad b > 2, \tag{6.3}$$

by (3.26), the product invariant measures $\bar{\mu}_\varphi$ have single site marginals given by

$$\mu_\varphi(k) = Z_\varphi^{-1} \frac{\varphi^k}{\prod_{i=1}^k \left(1 + \frac{b}{i}\right)}. \tag{6.4}$$

The local convergence in (6.2) was improved in [2] in (strong) convergence in the total variation norm: for $\rho > \rho_c$

$$\lim_{\substack{L,N\to+\infty \\ N/L\to\rho}} \left\| \mu_{N,L} - \frac{1}{L} \sum_{x\in\Lambda} \left(\mu_{\varphi_c}^{\Lambda\setminus\{x_L\}} * \delta_{N-\sum_{\Lambda\setminus\{x_L\}}\eta} \right) \circ \sigma^{x,x_L} \right\|_{\text{t.v.}} = 0, \tag{6.5}$$

where x_L is the last site of linearly ordered Λ, $\mu_{\varphi_c}^{\Lambda\setminus\{x_L\}} * \delta_{N-\sum_{\Lambda\setminus\{x_L\}}\eta}$ is a probability measure on $\mathbb{N}^\Lambda$ with marginal on $\mathbb{N}^{\Lambda\setminus\{x_L\}}$ given by $\mu_{\varphi_c}^{\Lambda\setminus\{x_L\}}$ and one site marginal on $\{x_L\}$ given by the Dirac measure at $N - \sum_{x\in\Lambda\setminus\{x_L\}} \eta(x)$, and σ^{x,x_L} is a permutation of configurations given by

$$\left(\sigma^{x,x_L}\eta\right)(z) = \begin{cases} \eta(z) & \text{for } z \neq x, x_L \\ \eta(x_L) & \text{for } z = x \\ \eta(x) & \text{for } z = x_L. \end{cases}$$

If moreover $b > 2$, (6.5) implies a stable limit theorem for $M_L(\eta) = \max_{x\in\Lambda} \eta(x)$ (see [2]).

For the second set-up for condensation, consider the case $b \in (1, 2]$ for the measure $\bar{\mu}_\varphi$ defined by (6.4). As mentioned in [17], $Z_\varphi < +\infty$ for every $\varphi \in [0, 1]$, that is, $\varphi_c = 1$, and there is no finite critical density but

$$\mu_1(0) = \frac{b-1}{b} \quad \text{and} \quad \mu_1(k) \sim \Gamma(b)(b-1)k^{-b} \text{ for } k > 0.$$

Quoting from [17], "a typical configuration for this stationary distribution has a hierarchical structure which can be understood as a precursor for a condensation phenomenon". It is proved in [13] that for a fixed number of sites $L \geq 2$, the probability measures

$$\hat{\mu}_{N,L-1} \text{ converge weakly to } \hat{\mu}_1^{L-1} \text{ as } N \to +\infty \tag{6.6}$$

where $\hat{\mu}_{N,L-1}$, $\hat{\mu}_1^{L-1}$ are measures on $\mathbb{N}^{\{1,\ldots,L-1\}}$ derived respectively from $\mu_{N,L}$ and $\bar{\mu}_1^{\{1,\ldots,L\}}$ on $\mathbb{N}^{\{1,\ldots,L\}}$, applying an operator on them which orders the coordinates of a configuration and removes the last one (the maximal one). It means that all but finitely many particles accumulate at one site.

6.2 Back to MMP

As indicated at the beginning of this section, let us give some examples of MMPs exhibiting condensation. For instance, consider again the generic model for condensation used firstly in [9] and later in [17] and [2], that is, ZRP with rate (6.3) and invariant measures $\bar{\mu}_\varphi$ given by their single site marginals (6.4). Then many possible MMPs have this stationary state and hence they may exhibit condensation.

• If we choose a MM-ZRP, by (3.35) in Proposition 3.10, it has rates

$$g_{\alpha+k}^k = c(k) \frac{\mu(\alpha)}{\mu(\alpha+k)} = c(k) \prod_{i=\alpha+1}^{\alpha+k} \left(1 + \frac{b}{i}\right) \tag{6.7}$$

for $k \geq 1$, $\alpha \geq 0$ where c is an arbitrary positive function on $\mathbb{N} \setminus \{0\}$ and μ is the special case of (6.4) for $\varphi = 1$. We study this case in details later on, as Example 7.5.

• If we choose a MM-TP, we obtain from Proposition 3.16(a), using that the ratio $\dfrac{\mu(n+1)}{\mu(n)}$ is nondecreasing, a recursive formula for its rates

$$\begin{aligned} g_{*,0}^{\alpha} &= c^*(\alpha) \\ g_{*,\beta}^{\alpha} &= c^*(\alpha) + \sum_{k=1}^{\beta} \frac{H_\alpha(\beta,k)}{\mu(\beta)}\, c^*(k), \qquad \beta \geq 1, \end{aligned} \tag{6.8}$$

for every $\alpha \geq 1$, where c^* is an arbitrary positive function on $\mathbb{N} \setminus \{0\}$ and the function H is given by formulas (3.42)–(3.43).

For both these MM-ZRP and MM-TP, the results (6.2), (6.5), (6.6) hold, that is, they may exhibit condensation in both set-ups.

Remark 6.1. We observe a duality between MM-ZRP and MM-TP having rates g_α^k and $g_{*,\beta}^k$ which lead to the same family of product, translation invariant, and invariant measures. But it is no longer a simple formula as it was the case for the duality between ZRP and TP explicited in Remark 3.9. Combining (6.7) and (3.42) we obtain that

$$\Delta_\alpha(k) = c(k) \left(\frac{1}{g_{\alpha+k}^k} - \frac{1}{g_{\alpha-1+k}^k} \right).$$

By (6.8), and using (3.43), we then obtain a formula for $g_{*,\beta}^{\alpha}$

$$\begin{aligned} g_{*,\beta}^{\alpha} &= c^*(\alpha) + \sum_{k=1}^{\beta} c^*(k) \Big[g_\beta^k \left(\frac{1}{g_{\alpha+k}^k} - \frac{1}{g_{\alpha-1+k}^k} \right) \\ &\quad + \sum_{r=1}^{\beta-k} \sum_{\substack{k_1,\dots,k_r \geq 1 \\ k_1+\dots+k_r \leq \beta-k}} \frac{c(k_1)\cdots c(k_r) c(k)}{c(k+\sum_{j=1}^r k_j)} g_\beta^{k+\sum_{j=1}^r k_j} \\ &\quad \times \left(\frac{1}{g_{\alpha+k_1}^{k_1}} - \frac{1}{g_{\alpha-1+k_1}^{k_1}} \right) \prod_{j=1}^{r-1} \left(\frac{1}{g_{k_i+k_{i+1}}^{k_{i+1}}} - \frac{1}{g_{k_i-1+k_{i+1}}^{k_{i+1}}} \right) \left(\frac{1}{g_{k_r+k}^{k_r}} - \frac{1}{g_{k_r-1+k}^{k_r}} \right) \Big] \end{aligned}$$

which is a function of $(g_{\alpha+k}^k, g_{\alpha+k-1}^k, g_k^l : 1 \leq k \leq \beta, 1 \leq l \leq k)$, $(c(k) : 1 \leq k \leq \beta)$ and $(c^*(\alpha), c^*(k) : 1 \leq k \leq \beta)$ only.

6.3 Condensation vs. attractiveness

An intriguing question is whether an MMP could be at the same time attractive and exhibit condensation.

The answer is ‘no’ in the two following cases. The first one was the standard one analyzed up to now, which induced the impression that coexistence of both properties could not be possible.

Proposition 6.2. *(a) A (single jump) ZRP or MP cannot at the same time be attractive and exhibit condensation.*

(b) An MM-TP cannot at the same time be attractive and exhibit condensation.

Proof. (a) Recall Remark 5.2. We also saw in Section 3.2.1 that if there exist product invariant measures of the form (3.7) then (see (3.21) and (3.22)),

$$r(n) = \frac{\mu(n)}{\mu(n+1)} = \varphi_\mu\, g_{n+1}^1 \quad \text{for ZRP,} \qquad r(n) = \frac{\mu(n)}{\mu(n+1)} = \overline{\varphi}_\mu\, \frac{g_{n+1,0}^1}{g_{1,n}^1} \quad \text{for MP.}$$

We conclude that an attractive ZRP or an attractive MP has $r(n)$ nondecreasing and so $Z_{\varphi_c} = +\infty$ by usual criteria for convergence of series. Thus they cannot exhibit condensation.
(b) Assume a MM-TP has an invariant product measure with single site marginal μ. If the rates $g^k_{*,\beta}$ satisfy the conditions (5.9a)–(5.9b) for attractiveness then $r(n) = \mu(n)/\mu(n+1)$ is a nondecreasing function and therefore $Z_{\varphi_c} = +\infty$. Indeed, from (5.9a)–(5.9b) we obtain in particular that $g^\alpha_{*,\beta} - g^\alpha_{*,0} \leq 0$ for all $\alpha, \beta \geq 1$. Combining this with (4.35) gives the result. □

Let us consider the second set-up for condensation. For the latter it is proven in [23] that: "All spatially homogeneous processes with stationary product measures that exhibit condensation with a finite critical density are necessarily not attractive". However, processes with an infinite critical density could be also attractive. In [23] an example is numerically studied to support this assertion. In Section 7 below, we prove, for a particular case of Example 7.5, that this coexistence is indeed possible:

Proposition 6.3. *The MM-ZRP with rate*

$$\begin{cases} g^k_\alpha = \pi(1) \dfrac{\prod_{i=\alpha-k}^{\alpha-1}\left(1+\dfrac{b}{i}\right)}{\prod_{i=1}^{k-1}\left(1+\dfrac{b}{i}\right)} \mathbb{1}_{\{k<\alpha\}} \quad \text{for } \alpha \geq 1, k \geq 1, \\ g^\alpha_\alpha = \pi(0) \quad \text{for } \alpha \geq 1, \end{cases} \tag{6.9}$$

where π is a positive function on $\mathbb{N}$ with $\pi(0) \geq (1+b)\pi(1)$, is attractive and exhibits condensation in the second set-up for $b = 3/2$.

7 Examples

In this section, we present some examples of MMPs on $\mathbb{Z}^d$, for which we check existence conditions, whether they have product invariant, translation invariant, probability measures, whether they are attractive, and whether they exhibit condensation.

For each dynamics, we thus verify the following conditions:

(i) *sufficient condition* (2.13) *for existence;*

The following condition is equivalent to (2.13): there exists $C > 0$ such that

$$\sum_{k=1}^{\alpha+1} k\,|g^k_{\alpha+1,\beta} - g^k_{\alpha,\beta}| + \sum_{k=1}^{\alpha} k\,|g^k_{\alpha,\beta+1} - g^k_{\alpha,\beta}| \leq C \quad \text{for all } \alpha, \beta \geq 0. \tag{7.1}$$

It means that there exists $C > 0$ such that

$$\sum_{k=1}^{\alpha} k\,|g^k_{\alpha+1,\beta} - g^k_{\alpha,\beta}| \leq C, \quad \sum_{k=1}^{\alpha} k\,|g^k_{\alpha,\beta+1} - g^k_{\alpha,\beta}| \leq C \text{ and } \alpha g^\alpha_{\alpha,\beta} \leq C \text{ for all } \alpha, \beta \geq 0. \tag{7.2}$$

(i)' *sufficient condition* (2.9) *(which implies* (2.12)*) and condition* (2.17b);

(ii) *necessary and sufficient conditions for attractiveness*, by Lemma 5.1, (5.6a)-(5.9b);

(iii) *necessary and sufficient conditions for* the existence of product, translation invariant, *invariant measures*, by the appropriate result in Section 3.

If there are such product invariant measures, we compute their single site marginals μ_φ, and φ_c, ρ_c. We also check condensation properties.

Remark 7.1. For the various conditions in (i) and (i)', we have that (2.13) as well as (2.17b) implies (2.9), but converses are wrong. None of the implications between (2.13) and (2.17b) hold. Indeed, taking $g^k_{\alpha,\beta} = 0$ for $k > 1$, and $g^1_{\alpha,\beta} = \alpha + \beta$, we have that (2.13) and (2.9) hold, but not (2.17b). Considering a MM-ZRP with g^k_α as in Example 7.3 below, with $r(\alpha) = 1/\alpha$ for α even, and $r(\alpha) = 1/\alpha^2$ for α odd, we have that (2.17b) and (2.9) hold, but not (2.13).

We now study four examples of MM-ZRP, then one of MM-TP, with the following features: condensation is impossible in Example 7.2; there are no product translation invariant invariant probability measures in Example 7.3 (with the exception of the stick process); Example 7.4 possesses a one-parameter family of product translation invariant invariant probability measures; Example 7.5 is a particular case of Example 7.4, for which attractiveness coexists with condensation for one value of its parameter; Example 7.10 is an MM-TP dual to the MM-ZRP of Example 7.5.

Example 7.2. MM-ZRP with $g^k_\alpha = h(k)\,\mathbb{1}_{\{k\le\alpha\}}$ for $\alpha \ge 1, k \ge 1$, h a nonnegative function.

(i) If there exists $C > 0$ such that $\alpha h(\alpha) \le C$ for all $\alpha \ge 0$, then condition (2.13) is satisfied.

(i)' If there exists $C > 0$ such that $\sum_{k=1}^{\alpha} kh(k) \le C\alpha$ (resp. $\sum_{k=1}^{\alpha} h(k) \le C$) for all $\alpha \ge 0$, then condition (2.9) (resp. (2.17b)) is satisfied.

(ii) The process is attractive if and only if

$$h(k) \text{ is nonincreasing for } k \ge 1. \tag{7.3}$$

(iii) Conditions (3.36) and (3.38) are satisfied, hence by Proposition 3.13(b) there are product, translation invariant, invariant probability measures given by (3.26), which are geometric: for every $\varphi \in (0,1)$,

$$\mu_\varphi(\eta(x) = n) = \varphi^n(1-\varphi) \quad \text{for all } x \in \mathbb{Z}^d, n \in \mathbb{N}.$$

They do not depend on h. The normalizing constant is $Z_\varphi = (1-\varphi)^{-1}$; we have $\varphi_c = 1$ and $Z_{\varphi_c} = +\infty$; so there is no possibility of condensation.

Particular cases:

• The process with $h(k) = 1/k$ is attractive;

• the process with $h(k) = k$ for $k < k_0$, $h(k) = 1/k$ for $k \ge k_0$, for some $k_0 \in \mathbb{N}$, is not attractive;

and both satisfy the existence condition (2.13).

• The totally asymmetric q-Hahn ZRP is one of the models studied in [3]; there $p(0,1) = 1$, with $h(k) = \dfrac{q^{k-1}(1-q)}{1-q^k}$ for $q \in (0,1)$. This process is attractive, and condition (2.13) is satisfied.

Example 7.3. MM-ZRP with $g^k_\alpha = R(\alpha)\,\mathbb{1}_{\{k\le\alpha\}}$ for $\alpha \ge 1, k \ge 1$, with R a nonnegative function.

(i) Condition (2.13) is satisfied if there exists $C > 0$ such that for $\alpha \ge 0$,

$$\alpha^2\,|R(\alpha+1) - R(\alpha)| + \alpha R(\alpha) \le C.$$

(i)' Conditions (2.9) and (2.17b) are satisfied if there exists $C > 0$ such that for $\alpha \ge 0$, $\alpha R(\alpha) \le C$.

Note that if R is a constant function, both conditions (2.13), (2.9) and (2.17b) fail. It is the case for the “stick process” introduced in [25] and also studied in [14]; for this example, another construction was provided in [25], on a state space smaller than $\mathfrak{X}$.

(ii) The process is attractive if and only if

$$R(\alpha) \text{ is nonincreasing} \quad \text{and} \quad \alpha R(\alpha) \text{ nondecreasing} \quad \text{for } \alpha \geq 1. \tag{7.4}$$

(iii) Unless R is constant (that is, for the stick process), condition (3.38) is never satisfied (take $k = \alpha - 1 \geq 1$ in (3.38)), so there are no product translation invariant and invariant probability measures.

Particular cases:
• The process with $R(\alpha) = 1/\alpha$ is attractive;
• The process with $R(\alpha) = 1/\alpha^2$ is not attractive;
and both satisfy the existence condition (2.13).

Example 7.4. Generic MM-ZRP with product invariant measures: we consider the rates (5.16) for which we began to investigate attractiveness in Section 5.2, that is,

$$g_\alpha^k = \frac{\pi(\alpha - k)}{\pi(\alpha)}\, h(k)\, \mathbb{1}_{\{k \leq \alpha\}} \quad \text{for } \ \alpha \geq 1, k \geq 1, \tag{7.5}$$

where π is a positive function on $\mathbb{N}$ and h is a nonnegative function on $\mathbb{N} \setminus \{0\}$. For technical reasons we define $\pi(i) = 0$ for $i \in \mathbb{Z}^-$ and $h(0) = 0$.

(i) Condition (2.13) is satisfied if there exists $C > 0$ such that for $\alpha \geq 0$,

$$\sum_{k=1}^{\alpha} k\, h(k) \left| \frac{\pi(\alpha + 1 - k)}{\pi(\alpha + 1)} - \frac{\pi(\alpha - k)}{\pi(\alpha)} \right| + \alpha \frac{h(\alpha)}{\pi(\alpha)} \leq C \tag{7.6}$$

(i)' Condition (2.9) writes: There exists $C > 0$ such that for every $\alpha \geq 1$,

$$\sum_{k=1}^{\alpha} k h(k) \pi(\alpha - k) \leq C \alpha \pi(\alpha). \tag{7.7}$$

Condition (2.17b) writes: There exists $C > 0$ such that for every $\alpha \geq 1$,

$$\sum_{k=1}^{\alpha} \pi(\alpha - k) h(k) \leq C \pi(\alpha). \tag{7.8}$$

(ii) In Section 5.2 we proved that the process is attractive if and only if condition (5.17) is satisfied. In Lemma 5.6 we simplified this condition when the function $r(n) = \pi(n)/\pi(n+1)$ (see (5.18)) is monotone.

(iii) The rate (7.5) is the generic one to satisfy assumption (3.35) hence we have a one-parameter family of product and translation invariant measures

$$\begin{array}{c} \{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z')\}, \quad \text{or } \{\bar{\mu}_\varphi : \varphi \in \mathrm{Rad}(Z)\}, \\ \text{where} \\ \bar{\mu}_\varphi(\eta : \eta(x) = n) = Z_\varphi^{-1} \varphi^n \pi(n) \text{ for all } x \in \mathbb{Z}^d, n \in \mathbb{N} \end{array} \tag{7.9}$$

and $\mathrm{Rad}(Z')$, $\mathrm{Rad}(Z)$ are given by (3.10) and (3.9), provided assumption (H) is satisfied. These measures do not depend on h. Properties of the ratio $r(n)$ yield the existence of a critical value φ_c such that $Z_{\varphi_c} < +\infty$ or moreover $\rho_c < +\infty$. If $\varphi_c \in \mathrm{Rad}(Z) \setminus \mathrm{Rad}(Z')$ then the measure $\bar{\mu}_{\varphi_c}$ is well defined by (7.9) but its first moment is infinite. If assumption (3.8) is satisfied, then $\bar{\mu}_{\varphi_c}$ is invariant.

Example 7.5. A special case of Example 7.4: MM-ZRP with product invariant measures where

$$r(n) = \frac{\pi(n)}{\pi(n+1)} = 1 + \frac{b}{n}, \quad b > 1, \text{ for all } n \geq 1\,; \qquad r(0) = \frac{\pi(0)}{\pi(1)} \geq 1 + b. \tag{7.10}$$

We are particularly interested in the case $h(k) = \pi(k),\ k \geq 1$, which corresponds to example (5.21) with $h_0 = 1$, to (6.7), or to (6.9) (in which the second line comes from Remark 5.7).

We have for $n \geq 1$,

$$\pi(n) = \frac{\pi(1)}{\prod_{i=1}^{n-1}\left(1 + \frac{b}{i}\right)} = \pi(1)\frac{(n-1)!}{\prod_{i=1}^{n-1}(b+i)} = \pi(1)\frac{\Gamma(n)\Gamma(b+1)}{\Gamma(b+n)}. \tag{7.11}$$

By Stirling approximation, when $n \to +\infty$ we have $\pi(n) \sim \pi(1)\Gamma(b+1)n^{-b}$. Therefore there exist positive constants ω_1, ω_2 depending on b such that, for all $n > 0$,

$$\omega_1 n^{-b} \leq \pi(n) \leq \omega_2 n^{-b}. \tag{7.12}$$

(i) Condition (2.13) is equivalent to (cf. (7.6), (7.2)): there exists $C > 0$ such that, for every $\alpha \geq 2$,

$$\sum_{k=1}^{\alpha-1} k\, h(k) \left(\frac{\pi(\alpha-k)}{\pi(\alpha)} - \frac{\pi(\alpha+1-k)}{\pi(\alpha+1)}\right) \leq C \quad \text{and} \quad \alpha\frac{h(\alpha)}{\pi(\alpha)} \leq C. \tag{7.13}$$

Since we have from (7.11)

$$\frac{\pi(\alpha-k)}{\pi(\alpha)} - \frac{\pi(\alpha+1-k)}{\pi(\alpha+1)} \quad = \quad kb\frac{\Gamma(\alpha-k)\Gamma(b+\alpha)}{\Gamma(\alpha+1)\Gamma(b+\alpha+1-k)},$$

then, using (7.12), condition (7.13) is equivalent to : there exists $C > 0$ such that, for every $\alpha \geq 2$,

$$\alpha^{b-1}\sum_{k=1}^{\alpha-1} k^2 h(k)\frac{1}{(\alpha-k)^{b+1}} \leq C \ \text{ and } \ \alpha h(\alpha) \leq C\pi(\alpha). \tag{7.14}$$

With the choice $h(k) = \pi(k)$, the sufficient condition (7.13) leads to $\alpha h(\alpha) \leq C\pi(\alpha)$ which is never satisfied. But with the choice $h(k) = \pi(k)/k$ that we put in (7.14) combined with (7.12) for $\pi(k)$, then condition (7.13) writes:

$$\text{For all } \alpha \geq 2, \quad \sum_{k=1}^{\alpha-1} f_{k,\alpha}(b) \leq C, \qquad \text{where} \qquad f_{k,\alpha}(b) = \frac{\alpha^{b-1}}{k^{b-1}(\alpha-k)^{b+1}} \tag{7.15}$$

for some $C > 0$. It is satisfied for $b \geq 1$ since we have that

Lemma 7.6. *For every $b \geq 1$ there exists $C > 0$ such that* (7.15) *is valid.*

Proof. For $b = 1$, we have that

$$\sum_{k=1}^{\alpha-1} f_{k,\alpha}(1) = \sum_{k=1}^{\alpha-1} \frac{1}{k^2}\,.$$

For $b > 1$, we have that for $1 \leq k \leq \alpha - 1$, $f_{k,\alpha}(b) \leq \hat{f}_{k,\alpha}(b-1)$ (see (7.17) below), hence the result follows from the proof of Lemma 7.7 below. □

(i)' In the case $h(k) = \pi(k)$, condition (2.9) is satisfied if and only if for some $C > 0$

$$\sum_{k=0}^{\lfloor \alpha/2 \rfloor} g_\alpha^k = \sum_{k=0}^{\lfloor \alpha/2 \rfloor} \frac{\pi(k)\pi(\alpha-k)}{\pi(\alpha)} \leq C \quad \text{for all } \alpha \geq 2. \tag{7.16}$$

Indeed, (2.9) writes (7.7), and if we use that the term $\frac{\pi(k)\pi(\alpha-k)}{\pi(\alpha)}$ is symmetric we obtain (7.16), which, by (7.11), is equivalent to: for some $C > 0$,

$$\frac{\Gamma(b+\alpha)}{\Gamma(\alpha)} \sum_{k=1}^{\lfloor \alpha/2 \rfloor} \frac{\Gamma(\alpha-k)}{\Gamma(b+\alpha-k)} \frac{\Gamma(k)}{\Gamma(b+k)} \leq C$$

which, using Stirling approximation (7.12), is equivalent to:

$$\text{For all } \alpha \geq 2, \quad \sum_{k=1}^{\lfloor \alpha/2 \rfloor} \hat{f}_{k,\alpha}(b) \leq C, \qquad \text{where} \qquad \hat{f}_{k,\alpha}(b) = \frac{\alpha^b}{k^b(\alpha-k)^b} \tag{7.17}$$

for some $C > 0$. We have that

Lemma 7.7. *For every $b > 0$ there exists $C > 0$ such that (7.17) is satisfied.*

Proof. For $b > 0$ and $\alpha \geq 2$, the function $x \mapsto x^{-b}(\alpha-x)^{-b}$ is decreasing on $(0, \alpha/2)$. Therefore

$$\sum_{k=1}^{\alpha-1} \hat{f}_{k,\alpha}(b) \leq 2 \sum_{k=1}^{\lfloor \alpha/2 \rfloor} \hat{f}_{k,\alpha}(b) \leq \left(\frac{\alpha}{\alpha-1}\right)^b + 2\int_{1/(2\alpha)}^{1/2} \frac{1}{u^b(1-u)^b}\,\mathrm{d}u,$$

which implies the lemma. □

Let us now check condition (2.17b), which writes here (7.8). With the choice $h(k) = \pi(k)$, it becomes (7.16). If it is satisfied, it will also be the case for (2.17b) with the choice $h(k) = \pi(k)/k$.

(ii) The process is attractive if and only if (5.19a)–(5.19b) from Lemma 5.6 hold, since $r(\alpha)$ is decreasing. In the case $h(k) = \pi(k)$ on which we now focus, Lemma 5.6*(a')* gives a necessary and sufficient condition for the process to be attractive, namely (5.20). By (5.24) and (7.11), for $\alpha \geq 2$ we have, writing from now on $S_b(\alpha)$ instead of $S(\alpha)$ to stress the dependence in b,

$$\begin{aligned} S_b(\alpha) &= \pi(1) \sum_{j=1}^{\alpha-1} \frac{r(\alpha-1)\cdots r(\alpha-j)}{r(j-1)\cdots r(1)} \\ &= \pi(1) \frac{\Gamma(b+\alpha)\,\Gamma(b+1)}{\Gamma(\alpha)} \sum_{j=1}^{\alpha-1} \frac{\Gamma(j)\,\Gamma(\alpha-j)}{\Gamma(j+b)\,\Gamma(\alpha-j+b)}\,. \end{aligned} \tag{7.18}$$

Proposition 7.8. *The process is attractive for $b = 1$ and $b = 3/2$ and not attractive for all values of b larger or equal to 2.*

Proof. First, for all $\alpha \geq 2$ and all $b > 0$, using that $\Gamma(z+1) = z\Gamma(z)$ for all $z > 0$, we compute the increment

$$\begin{aligned} &S_b(\alpha+1) - S_b(\alpha) \\ &= \pi(1) \left[\frac{b+\alpha}{\alpha} + \sum_{j=1}^{\alpha-1} \frac{\Gamma(b+1)\Gamma(j)}{\Gamma(j+b)} \Big(\frac{\Gamma(b+\alpha+1)\Gamma(\alpha+1-j)}{\Gamma(\alpha+1)\Gamma(\alpha+1-j+b)} - \frac{\Gamma(b+\alpha)\Gamma(\alpha-j)}{\Gamma(\alpha)\Gamma(\alpha-j+b)} \Big) \right] \\ &= \pi(1) \left[\frac{b+\alpha}{\alpha} + \sum_{j=1}^{\alpha-1} \frac{\Gamma(j)\Gamma(\alpha-j)}{\Gamma(\alpha)} \; \frac{\Gamma(b+1)\Gamma(b+\alpha)}{\Gamma(j+b)\Gamma(\alpha-j+b)} \Big(\frac{(b+\alpha)(\alpha-j)}{\alpha(\alpha-j+b)} - 1 \Big) \right] \end{aligned}$$

$$= \pi(1)\left[\frac{b+\alpha}{\alpha}+\sum_{j=1}^{\alpha-1}\frac{\Gamma(j)\Gamma(\alpha-j)}{\Gamma(\alpha)}\;\frac{\Gamma(b+1)\Gamma(b+\alpha)}{\Gamma(j+b)\Gamma(\alpha-j+b)}\Big(\frac{-bj}{\alpha(\alpha-j+b)}\Big)\right]$$
$$= \pi(1)\left[\frac{b+\alpha}{\alpha}-b\;\sum_{j=1}^{\alpha-1}\frac{j!(\alpha-j-1)!}{\alpha!}\;\frac{\Gamma(b+1)\Gamma(b+\alpha)}{\Gamma(j+b)\Gamma(\alpha-j+b+1)}\right].$$

For $\alpha=2$ we get $S_b(3)-S_b(2)=\pi(1)\geq 0$ for all $b>0$. For all $\alpha\geq 3$, we get

$$\begin{aligned}&S_b(\alpha+1)-S_b(\alpha)\\&=\pi(1)\left[\Big(1+\frac{b(\alpha-2)}{\alpha(\alpha-1)}\Big)-b\;\sum_{j=2}^{\alpha-1}\frac{j!(\alpha-j-1)!}{\alpha!}\;\prod_{k=1}^{j-1}\frac{b+\alpha-k}{b+j-k}\right].\end{aligned} \tag{7.19}$$

• For $b=1$, the expression (7.19) becomes

$$S_1(\alpha+1)-S_1(\alpha)=\pi(1)\Big(1+\frac{(\alpha-2)}{\alpha(\alpha-1)}-\frac{\alpha-2}{\alpha-1}\Big)=\frac{2\pi(1)}{\alpha}\geq 0\,. \tag{7.20}$$

Therefore the inequality in (5.20) is satisfied for all $\alpha\geq 2$, and the process is attractive.

• We now consider $b=3/2$. The expression (7.19) reads

$$\begin{aligned}&S_{3/2}(\alpha+1)-S_{3/2}(\alpha)\\&=\pi(1)\left[\Big(1+\frac{3(\alpha-2)}{2\alpha(\alpha-1)}\Big)-9\frac{(2\alpha+1)!}{(\alpha!)^2}\sum_{j=2}^{\alpha-1}\frac{(j!)^2(\alpha-j-1)!\,(\alpha-j+1)!}{(2j+1)!(2\alpha-2j+3)!}\right]\end{aligned} \tag{7.21}$$

We define for all $\alpha\geq 3$ and all $j\in\{0,\cdots,\alpha-1\}$

$$\varphi_\alpha(j)=\frac{(3\alpha-2j+2)}{6(\alpha+1)(\alpha+2)}\;\frac{j!\,(j+1)!}{(2j+1)!}\;\frac{(\alpha-j)!\,(\alpha-j-1)!}{(2\alpha-2j+1)!}\,.$$

In particular

$$\varphi_\alpha(1) = \frac{\alpha!\,(\alpha-2)!}{6(\alpha+1)(\alpha+2)(2\alpha-1)!}\,, \tag{7.22}$$

$$\varphi_\alpha(\alpha-1) = \frac{\alpha!\,(\alpha-1)!(\alpha+4)}{36(\alpha+1)(\alpha+2)(2\alpha-1)!}\,. \tag{7.23}$$

Now for all $j\in\{1,\cdots,\alpha-1\}$, we compute

$$\begin{aligned}&\varphi_\alpha(j)-\varphi_\alpha(j-1)\\&\quad=\frac{1}{3(\alpha+1)(\alpha+2)}\;\frac{(j!)^2(\alpha-j-1)!\,(\alpha-j+1)!}{(2j+1)!(2\alpha-2j+3)!}\\&\qquad\times\Big((j+1)(3\alpha-2j+2)(2\alpha-2j+3)-(2j+1)(\alpha-j)(3\alpha-2j+4)\Big)\\&\quad=\frac{(j!)^2(\alpha-j-1)!\,(\alpha-j+1)!}{(2j+1)!(2\alpha-2j+3)!}\;.\end{aligned} \tag{7.24}$$

Using this expression (7.24) in (7.21), and then (7.22), (7.23) gives

$$\begin{aligned}
&S_{3/2}(\alpha+1) - S_{3/2}(\alpha)\\
&\quad = \pi(1)\left[\Big(1+\frac{3(\alpha-2)}{2\alpha(\alpha-1)}\Big) - 9\frac{(2\alpha+1)!}{(\alpha!)^2}\sum_{j=2}^{\alpha-1}\big(\varphi_\alpha(j)-\varphi_\alpha(j-1)\big)\right]\\
&\quad = \pi(1)\left[\Big(1+\frac{3(\alpha-2)}{2\alpha(\alpha-1)}\Big) - 9\frac{(2\alpha+1)!}{(\alpha!)^2}\big(\varphi_\alpha(\alpha-1)-\varphi_\alpha(1)\big)\right]\\
&\quad = \pi(1)\left[1+\frac{3(\alpha-2)}{2\alpha(\alpha-1)} - \frac{(2\alpha+1)(\alpha^2+3\alpha-10)}{2(\alpha-1)(\alpha+1)(\alpha+2)}\right]\\
&\quad = \frac{3\pi(1)(3\alpha+2)}{\alpha(\alpha+1)(\alpha+2)}\\
&\quad \geq 0\,.
\end{aligned} \tag{7.25}$$

Thus, the inequalities (5.20) are satisfied and the process is attractive for $b = 3/2$.

• For the last case, for any given $b \geq 2$, we just compute

$$\begin{aligned}
S_b(11) - S_b(10) \quad &= \quad \pi(1)\Big[1+\frac{4b}{45} - \frac{37b(b+9)}{360(b+1)} - \frac{31b(b+9)(b+8)}{2520(b+1)(b+2)}\\
&\qquad - \frac{b(b+9)(b+8)(b+7)}{280(b+1)(b+2)(b+3)}\\
&\qquad - \frac{b(b+9)(b+8)(b+7)(b+6)}{504(b+1)(b+2)(b+3)(b+4)}\Big]\\
&= \quad \pi(1)\frac{-750+(2-b)(4155+3039b+955b^2+151b^3+10b^4)}{315(b+1)(b+2)(b+3)(b+4)}\\
&< \quad 0\,.
\end{aligned} \tag{7.26}$$

Therefore the process is not attractive for $b \geq 2$. □

Remark 7.9. Equation (7.26) shows that the process is not attractive also for values of b slightly smaller than 2. In fact, further numerical computations also indicate that the quantity (7.19) for larger values of α gets negative for smaller values of b. For instance, $S_b(201) - S_b(200)$ is found to be negative for $b \geq 1.55$. Thus we conjecture that the process is attractive for all values of $b \in (0, 3/2]$ and not attractive for all values of $b > 3/2$.

(iii) By part (iii) in Example 7.4, we have a family (7.9) of product, translation invariant, invariant probability measures where

$$\begin{aligned}
&\varphi_c = 1,\\
&Z_1 < +\infty \ \text{ if and only if } \ b > 1,\\
&\rho_c < +\infty \ \text{ if and only if } \ b > 2.
\end{aligned}$$

Since $\sum_{n\geq 1} n\pi(n) = +\infty$ when $b \leq 2$, we have $\mathrm{Rad}(Z') = (0,1]$ for $b > 2$ and $\mathrm{Rad}(Z') = (0,1)$ for $1 < b \leq 2$.

Let us consider the case $1 < b \leq 2$. If moreover $h(k) = \pi(k)$, we have by (7.16) with Lemma 7.7 that $\sum_{k\leq\alpha} g_\alpha^k$ is bounded, hence, by (7.16), assumption (2.17b) is satisfied. Therefore, by Corollary 3.5, if assumption (2.17a) is satisfied, then the probability measure $\bar{\mu}_1$ is invariant, and $\{\bar{\mu}_\varphi : \varphi \in (0,1]\}$ (by (7.9)) with π given by (7.11) are invariant measures.

In this example, we chose h, π such that the process with $b > 2$ exhibits condensation in the first set-up whereas the process with $1 < b \le 2$ exhibits condensation in the second set-up (that is, on a fixed finite volume), which follows from results in [13, 17] (see Section 6). By combining this last fact with Proposition 7.8, we prove coexistence of attractiveness and condensation in the second set-up for $b = 3/2$ (as summarized in Proposition 6.3).

Example 7.10. MM-TP with $g^k_{\alpha,\beta} = \mathbb{1}_{\{k\le\alpha\}} g^k_{*,\beta}$.

(i) Condition (2.13) writes: there exists $C > 0$ such that, for all $\beta, \alpha \ge 0$,

$$\sum_{k=1}^{\alpha} k\,|g^k_{*,\beta+1} - g^k_{*,\beta}| \; + \; \alpha g^{\alpha}_{*,\beta} \; \le C.$$

Note that if $g^k_{*,\beta}$ is independent of k, this condition fails.

(i)' Condition (2.9) writes: for all $\beta, \alpha \ge 0$, $\sum_{k=1}^{\alpha} k g^k_{*,\beta} \le C(\alpha+\beta)$.

Condition (2.17b) writes: for all $\beta, \alpha \ge 0$, $\sum_{k=1}^{\alpha} g^k_{*,\beta} \le C$.

(ii) The process is attractive if and only if conditions (5.9a)–(5.9b) are satisfied. Note that $g^{\alpha}_{*,\beta} \le g^{\alpha-1}_{*,\beta+1}$ for $\alpha \ge 1,\ \beta \ge 0,\ k \ge 0$, is sufficient for (5.9b).

(iii) If conditions (3.40)–(3.42) or conditions (3.44) on rates are satisfied then there exist product invariant probability measures given by formulas (3.49)–(3.50).

The MM-TP dual of Example 7.5 has rates (6.8) (see Remark 6.1). Contrary to duality for single-jump dynamics which conserves attractiveness (see Remark 5.2), we have here a non-attractive dynamics (by Proposition 6.2) dual of a one which can be attractive (by Proposition 7.8).

8 Appendix: construction of the process

In this section we shall prove Proposition 2.1, Theorem 2.2, and Proposition 2.3.

For our construction, consider an approximation of the infinite set X by an increasing sequence of finite sets

$$\mathrm{X}_1 \subset \mathrm{X}_2 \subset \ldots \subset \mathrm{X}, \qquad \bigcup_n \mathrm{X}_n = \mathrm{X} \tag{8.1}$$

and let us define (as in [18]) a restriction of the transition probabilities $(p(x,y), x, y \in \mathrm{X})$ on X_n by

$$p_n(x,y) = \begin{cases} p(x,x) + \sum_{z\notin \mathrm{X}_n} p(x,z) & x = y,\ x \in \mathrm{X}_n \\ 1 & x = y,\ x \notin \mathrm{X}_n \\ p(x,y) & x \ne y,\ x, y \in \mathrm{X}_n \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

The transition probabilities $(p_n(x,y), x, y \in \mathrm{X}_n)$ satisfy (2.5) with M replaced by $M+1$, because $p_n(x,y) \le p(x,y) + \mathbb{1}_{\{x=y\}}$. They induce a restriction of the dynamics of MMP to X_n, whose generator $\mathcal{L}_n$ is given by (2.10) with $p(\cdot,\cdot)$ replaced by $p_n(\cdot,\cdot)$: for $f : \mathbb{N}^{\mathrm{X}} \to \mathbb{R}$, $f \in \mathbf{L}, \eta \in \mathbb{N}^{\mathrm{X}}$,

$$\begin{aligned} \mathcal{L}_n f(\eta) &= \sum_{x,y\in \mathrm{X}} \sum_{k>0} p_n(x,y)\; g^k_{\eta(x),\eta(y)} \left(f\left(\mathcal{S}^k_{x,y}\eta\right) - f(\eta)\right) \\ &= \sum_{x\ne y\in \mathrm{X}_n} \sum_{k>0} p(x,y)\; g^k_{\eta(x),\eta(y)} \left(f\left(\mathcal{S}^k_{x,y}\eta\right) - f(\eta)\right). \end{aligned} \tag{8.3}$$

Since particles do not move outside X_n, this corresponds to a Markov process on the countable state space $\mathbb{N}^{\mathrm{X}_n}$. Hence there is a well defined Markov semigroup of operators $(S_n(t), t \geq 0)$ on functions $f \in \mathbf{L}$, associated to $\mathcal{L}_n$:

$$S_n(t)f(\eta) = f(\eta) + \int_0^t \mathcal{L}_n S_n(s) f(\eta)\,\mathrm{d}s. \tag{8.4}$$

Similarly, the dynamics of the coupled process restricted to X_n has generator $\overline{\mathcal{L}}_n$ given by (5.3) with $p(\cdot,\cdot)$ replaced by $p_n(\cdot,\cdot)$: for $(\eta,\zeta) \in \mathfrak{X} \times \mathfrak{X}$, $h \in \overline{\mathbf{L}}$ (cf. (5.4)),

$$\overline{\mathcal{L}}_n h(\eta,\zeta) = \sum_{x \neq y \in \mathrm{X}_n} p_n(x,y) \sum_{k \geq 0} \sum_{l \geq 0} G^{k,l}_{\eta(x),\eta(y),\zeta(x),\zeta(y)} \left(h\left(\mathcal{S}^k_{x,y}\eta, \mathcal{S}^l_{x,y}\zeta\right) - h\left(\eta,\zeta\right)\right). \tag{8.5}$$

Our goal is to define, in Subsection 8.3, limits of the generators $\mathcal{L}_n$ and semigroups $S_n(t)$, as $n \to +\infty$. For this we shall as a first step, in Subsection 8.2, analyze the restriction of $(\eta_t)_{t\geq 0}$ to the finite set of sites X_n, keeping by an abuse of language the same notations $\mathcal{L}_n, S_n(t), (\eta_t)_{t\geq 0}, \overline{\mathcal{L}}_n$ for the dynamics of state space $\mathbb{N}^{\mathrm{X}_n}$. Subsection 8.1 is devoted to preliminary results.

8.1 Preliminaries

In this subsection we explicit some statements done in Section 2, and derive a key inequality about the coupling rates.

• Let us exhibit a function a on X satisfying (2.5) and (2.6). Choose any $M > 1 + m_p$ (cf. (2.7)) and define, for all $x, y \in \mathrm{X}$,

$$\tilde{p}(x,y) = \frac{1}{1+m_p}\big(p(x,y) + p(y,x)\big).$$

Fix a reference site $x_0 \in \mathrm{X}$ and set

$$a_x = \sum_{n \geq 0} \left(\frac{1+m_p}{M}\right)^n \tilde{p}^{(n)}(x,x_0) \tag{8.6}$$

where $\tilde{p}^{(n)}(.,.)$ are the n-step transition probabilities corresponding to $\tilde{p}(.,.)$:

$$\tilde{p}^{(0)}(x,x_0) = \mathbb{1}_{\{x=x_0\}} \quad \text{and} \quad \tilde{p}^{(n)}(x,x_0) = \sum_{x_1 \in \mathrm{X}} \tilde{p}(x,x_1)\tilde{p}^{(n-1)}(x_1,x_0) \text{ for } n \geq 1. \tag{8.7}$$

Then (2.5) holds, as well as (2.6).

• Note that the set $\mathbf{L}$ includes all bounded cylinder functions. More precisely let f on $\mathbb{N}^{\mathrm{X}}$ be such a function. It is bounded by a value m_f, and its support is a finite set $V_f \subset \mathrm{X}$ such that f depends only on values $(\eta(x) : x \in V_f)$ and not on the whole $\eta \in \mathbb{N}^{\mathrm{X}}$. We can write

$$\|f(\eta) - f(\zeta)\| \leq 2m_f \sum_{x \in V_f} |\eta(x) - \zeta(x)| \leq 2m_f \|\eta - \zeta\| / \min_{x \in V_f} a_x. \tag{8.8}$$

• Next Lemma provides the analogue for the coupled process of (2.13) for the MMP, and is crucial for the following results.

Lemma 8.1. *Under assumption* (2.13)*, the rates* $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ *satisfy, for all* $\alpha,\beta,\gamma,\delta,k,l \geq 0$*,*

$$\sum_{k=0}^{\alpha} \sum_{l=0}^{\gamma} |k-l|\, G^{k,l}_{\alpha,\beta,\gamma,\delta} \;\leq\; 2C\big(|\alpha-\gamma| + |\beta-\delta|\big). \tag{8.9}$$

Proof. We have

$$\begin{aligned}
\sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma}|k-l|G^{k,l}_{\alpha,\beta,\gamma,\delta} &= \sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma}\mathbb{1}_{\{l<k\}}(k-l)G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma}\mathbb{1}_{\{l\geq k\}}(k-l)G^{k,l}_{\alpha,\beta,\gamma,\delta} \\
&= \sum_{l=0}^{\gamma} l\left(\sum_{k=0}^{l}\mathbb{1}_{\{k\leq\alpha\}}G^{k,l}_{\alpha,\beta,\gamma,\delta} - \mathbb{1}_{\{l<\alpha\}}\sum_{k=l+1}^{\alpha}G^{k,l}_{\alpha,\beta,\gamma,\delta}\right) \\
&\quad -\sum_{k=0}^{\alpha} k\left(\mathbb{1}_{\{k\leq\gamma\}}\sum_{l=k}^{\gamma}G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l=0}^{k-1}\mathbb{1}_{\{l\leq\gamma\}}G^{k,l}_{\alpha,\beta,\gamma,\delta}\right) \\
&= \sum_{l=1}^{(\alpha-1)\wedge\gamma} l\left(\sum_{k=0}^{l}G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k=l+1}^{\alpha}G^{k,l}_{\alpha,\beta,\gamma,\delta}\right) + \mathbb{1}_{\{\alpha\leq\gamma\}}\sum_{l=\alpha}^{\gamma} l g^{l}_{\gamma,\delta} \\
&\quad - \sum_{k=1}^{\alpha\wedge\gamma} k\left(\sum_{l=k}^{\gamma}G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l=0}^{k-1}G^{k,l}_{\alpha,\beta,\gamma,\delta}\right) + \mathbb{1}_{\{\alpha>\gamma\}}\sum_{k=\gamma+1}^{\alpha} k g^{k}_{\alpha,\beta} \\
&= \sum_{i=1}^{\alpha\wedge\gamma} iR_i + \mathbb{1}_{\{\alpha<\gamma\}}\sum_{i=\alpha+1}^{\gamma} iR_i - \sum_{i=1}^{\alpha\wedge\gamma} iR^*_i - \mathbb{1}_{\{\alpha>\gamma\}}\sum_{i=\gamma+1}^{\alpha} iR^*_i
\end{aligned}$$

where we have denoted

$$R_l = \begin{cases} \sum_{k=0}^{l} G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k=l+1}^{\alpha} G^{k,l}_{\alpha,\beta,\gamma,\delta}, & \text{for } l=0,\dots,\alpha-1, \\ g^{l}_{\gamma,\delta} & \text{for } l=\alpha,\dots,\gamma \text{ if } \alpha\leq\gamma, \end{cases}$$

$$R^*_k = \begin{cases} \sum_{l=k}^{\gamma} G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l=0}^{k-1} G^{k,l}_{\alpha,\beta,\gamma,\delta}, & \text{for } k=0,\dots,\gamma, \\ -g^{k}_{\alpha,\beta} & \text{for } k=\gamma+1,\dots,\alpha \text{ if } \alpha>\gamma. \end{cases}$$

Therefore using (2.2)

$$\begin{aligned}
&\sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma}|k-l|G^{k,l}_{\alpha,\beta,\gamma,\delta} = \\
&\quad \sum_{i=1}^{\alpha\wedge\gamma} i(R_i-R^*_i) + \mathbb{1}_{\{\alpha<\gamma\}}\sum_{i=(\alpha\wedge\gamma)+1}^{\alpha\vee\gamma} i\left(g^{i}_{\gamma,\delta}-g^{i}_{\alpha,\beta}\right) + \mathbb{1}_{\{\alpha>\gamma\}}\sum_{i=(\alpha\wedge\gamma)+1}^{\alpha\vee\gamma} i\left(g^{i}_{\alpha,\beta}-g^{i}_{\gamma,\delta}\right).
\end{aligned} \tag{8.10}$$

Using notation $\mathbf{\Sigma}^{i}_{\alpha,\beta} = \sum_{k>i} g^{k}_{\alpha,\beta}$ and a telescopic argument [14, Lemma 3.6] for partial sums of $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ gives, for $\alpha,\beta,\gamma,\delta\geq 0$ and $1\leq k\leq\alpha$, $1\leq l\leq\gamma$,

$$\begin{aligned}
G^{k,l}_{\alpha,\beta,\gamma,\delta} &= g^{k}_{\alpha,\beta}\wedge\left(\mathbf{\Sigma}^{l-1}_{\gamma,\delta}-\mathbf{\Sigma}^{k}_{\alpha,\beta}\wedge\mathbf{\Sigma}^{l-1}_{\gamma,\delta}\right) - g^{k}_{\alpha,\beta}\wedge\left(\mathbf{\Sigma}^{l}_{\gamma,\delta}-\mathbf{\Sigma}^{k}_{\alpha,\beta}\wedge\mathbf{\Sigma}^{l}_{\gamma,\delta}\right) \\
\text{or} & \\
G^{k,l}_{\alpha,\beta,\gamma,\delta} &= g^{l}_{\gamma,\delta}\wedge\left(\mathbf{\Sigma}^{k-1}_{\alpha,\beta}-\mathbf{\Sigma}^{k-1}_{\alpha,\beta}\wedge\mathbf{\Sigma}^{l}_{\gamma,\delta}\right) - g^{l}_{\gamma,\delta}\wedge\left(\mathbf{\Sigma}^{k}_{\alpha,\beta}-\mathbf{\Sigma}^{k}_{\alpha,\beta}\wedge\mathbf{\Sigma}^{l}_{\gamma,\delta}\right).
\end{aligned}$$

Then, for $i=1,\dots,\alpha\wedge\gamma$, we obtain

$$\begin{aligned}
R_i &= g^{i}_{\gamma,\delta} - 2\Big[g^{i}_{\gamma,\delta}\wedge\left(\mathbf{\Sigma}^{i}_{\alpha,\beta}-\mathbf{\Sigma}^{i}_{\alpha,\beta}\wedge\mathbf{\Sigma}^{i}_{\gamma,\delta}\right)\Big] \\
&= \begin{cases} g^{i}_{\gamma,\delta} & \text{if } \mathbf{\Sigma}^{i}_{\gamma,\delta}\geq\mathbf{\Sigma}^{i}_{\alpha,\beta} \\ g^{i}_{\gamma,\delta}-2\left(\mathbf{\Sigma}^{i}_{\alpha,\beta}-\mathbf{\Sigma}^{i}_{\gamma,\delta}\right) & \text{if } \mathbf{\Sigma}^{i}_{\gamma,\delta}<\mathbf{\Sigma}^{i}_{\alpha,\beta}<\mathbf{\Sigma}^{i-1}_{\gamma,\delta} \\ -g^{i}_{\gamma,\delta} & \text{if } \qquad \mathbf{\Sigma}^{i}_{\alpha,\beta}\geq\mathbf{\Sigma}^{i-1}_{\gamma,\delta} \end{cases}
\end{aligned}$$

$$\begin{aligned} R_i^* &= 2\Big[g^i_{\alpha,\beta} \wedge \Big(\mathbf{\Sigma}^{i-1}_{\gamma,\delta} - \mathbf{\Sigma}^{i}_{\alpha,\beta} \wedge \mathbf{\Sigma}^{i-1}_{\gamma,\delta}\Big)\Big] - g^i_{\alpha,\beta} \\ &= \begin{cases} -g^i_{\alpha,\beta} & \text{if } \mathbf{\Sigma}^{i}_{\alpha,\beta} \geq \mathbf{\Sigma}^{i-1}_{\gamma,\delta} \\ 2\big(\mathbf{\Sigma}^{i-1}_{\gamma,\delta} - \mathbf{\Sigma}^{i}_{\alpha,\beta}\big) - g^i_{\alpha,\beta} & \text{if } \mathbf{\Sigma}^{i}_{\alpha,\beta} < \mathbf{\Sigma}^{i-1}_{\gamma,\delta} < \mathbf{\Sigma}^{i-1}_{\alpha,\beta} \\ g^i_{\alpha,\beta} & \text{if } \qquad \mathbf{\Sigma}^{i-1}_{\gamma,\delta} \geq \mathbf{\Sigma}^{i-1}_{\alpha,\beta}. \end{cases} \end{aligned}$$

Hence

$$R_i - R_i^* = \begin{cases} g^i_{\alpha,\beta} - g^i_{\gamma,\delta} & \text{if } \mathbf{\Sigma}^{i}_{\alpha,\beta} \geq \mathbf{\Sigma}^{i-1}_{\gamma,\delta} \\ g^i_{\gamma,\delta} - g^i_{\alpha,\beta} & \text{if } \mathbf{\Sigma}^{i}_{\alpha,\beta} \leq \mathbf{\Sigma}^{i}_{\gamma,\delta} \text{ and } \mathbf{\Sigma}^{i-1}_{\alpha,\beta} \leq \mathbf{\Sigma}^{i-1}_{\gamma,\delta} \\ g^i_{\alpha,\beta} - g^i_{\gamma,\delta} & \text{if } \mathbf{\Sigma}^{i}_{\gamma,\delta} < \mathbf{\Sigma}^{i}_{\alpha,\beta} < \mathbf{\Sigma}^{i-1}_{\gamma,\delta} < \mathbf{\Sigma}^{i-1}_{\alpha,\beta} \\ g^i_{\gamma,\delta} - g^i_{\alpha,\beta} - 2(\mathbf{\Sigma}^{i}_{\alpha,\beta} - \mathbf{\Sigma}^{i}_{\gamma,\delta}) & \text{if } \mathbf{\Sigma}^{i}_{\gamma,\delta} < \mathbf{\Sigma}^{i}_{\alpha,\beta} < \mathbf{\Sigma}^{i-1}_{\gamma,\delta} \text{ and } \mathbf{\Sigma}^{i-1}_{\alpha,\beta} \leq \mathbf{\Sigma}^{i-1}_{\gamma,\delta} \\ g^i_{\alpha,\beta} - g^i_{\gamma,\delta} - 2(\mathbf{\Sigma}^{i}_{\gamma,\delta} - \mathbf{\Sigma}^{i}_{\alpha,\beta}) & \text{if } \mathbf{\Sigma}^{i}_{\alpha,\beta} \leq \mathbf{\Sigma}^{i}_{\gamma,\delta} \ \& \ \mathbf{\Sigma}^{i}_{\alpha,\beta} < \mathbf{\Sigma}^{i-1}_{\gamma,\delta} < \mathbf{\Sigma}^{i-1}_{\alpha,\beta}. \end{cases}$$

Conditions in the last but one line imply that $0 < \mathbf{\Sigma}^{i}_{\alpha,\beta} - \mathbf{\Sigma}^{i}_{\gamma,\delta} \leq g^i_{\gamma,\delta} - g^i_{\alpha,\beta}$ and similarly conditions in the last line imply that $0 < \mathbf{\Sigma}^{i}_{\gamma,\delta} - \mathbf{\Sigma}^{i}_{\alpha,\beta} \leq g^i_{\alpha,\beta} - g^i_{\gamma,\delta}$. Therefore we can conclude that $|R_i - R_i^*| \leq 2|g^i_{\alpha,\beta} - g^i_{\gamma,\delta}|$. From (8.10) and using assumption (2.13) then we obtain the desired inequality

$$\sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma} |k-l|\, G^{k,l}_{\alpha,\beta,\gamma,\delta} \;\leq\; 2\sum_{i=1}^{\alpha\vee\gamma} i\,|g^i_{\alpha,\beta} - g^i_{\gamma,\delta}| \;\leq\; 2C\left(|\alpha-\gamma| + |\beta-\delta|\right). \tag{8.11}$$

□

8.2 Finite volume

Because X_n is finite, for $\eta,\eta' \in \mathbb{N}^{\mathrm{X}_n}$, if $N = \sum_{x\in\mathrm{X}_n}\eta(x), N' = \sum_{x\in\mathrm{X}_n}\eta'(x)$, then the Markov process $(\eta_t)_{t\geq0}$ starting from $\eta_0 = \eta$ (resp. the coupled process $(\eta_t,\eta'_t)_{t\geq0}$ starting from $(\eta_0,\eta'_0) = (\eta,\eta')$) has the finite state space $\mathrm{S} = \{\xi \in \mathbb{N}^{\mathrm{X}_n} : \sum_{x\in\mathrm{X}_n}\xi(x) = N\}$ (resp. $\overline{\mathrm{S}} = \{(\xi,\xi') \in \mathbb{N}^{\mathrm{X}_n} \times \mathbb{N}^{\mathrm{X}_n} : \sum_{x\in\mathrm{X}_n}\xi(x) = N, \sum_{x\in\mathrm{X}_n}\xi'(x) = N'\}$). Hence we can write the semi-group as an exponential of a Q-matrix, that is,

$$S_n(t)f(\eta) = \mathbb{E}^{\eta}(f(\eta_t)) = \sum_{\zeta\in\mathrm{S}} f(\zeta)\mathbf{p}_t(\eta,\zeta) \tag{8.12}$$

$$\overline{S}_n(t)h(\eta,\eta') = \mathbb{E}^{(\eta,\eta')}(h(\eta_t,\eta'_t)) = \sum_{(\zeta,\zeta')\in\overline{\mathrm{S}}} h(\zeta,\zeta')\overline{\mathbf{p}}_t(\eta,\eta';\zeta,\zeta'), \tag{8.13}$$

where

$$\mathbf{p}_t(\eta,\zeta) = \sum_{j=0}^{+\infty}\frac{t^j}{j!}q^{(j)}(\eta,\zeta) \tag{8.14}$$

$$q(\eta,\zeta) = \sum_{x,y\in\mathrm{X}_n}\sum_{k=1}^{\eta(x)} p_n(x,y)\, g^k_{\eta(x),\eta(y)}\left(\mathbb{1}_{\{\zeta=\mathcal{S}^k_{x,y}\eta\}} - \mathbb{1}_{\{\zeta=\eta\}}\right) \tag{8.15}$$

$$\overline{\mathbf{p}}_t(\eta,\eta';\zeta,\zeta') = \sum_{j=0}^{+\infty}\frac{t^j}{j!}\overline{q}^{(j)}(\eta,\eta';\zeta,\zeta') \tag{8.16}$$

$$\begin{aligned} \overline{q}(\eta,\eta';\zeta,\zeta') = &\sum_{x,y\in\mathrm{X}_n}\sum_{k=1}^{\eta(x)}\sum_{l=1}^{\eta'(x)} p_n(x,y)\, G^{k,l}_{\eta(x),\eta(y),\eta'(x),\eta'(y)} \\ &\times\left(\mathbb{1}_{\{(\zeta,\zeta')=(\mathcal{S}^k_{x,y}\eta,\mathcal{S}^k_{x,y}\eta')\}} - \mathbb{1}_{\{(\zeta,\zeta')=(\eta,\eta')\}}\right). \end{aligned} \tag{8.17}$$

Recall the constants m_p, C, M introduced in (2.7), (2.5), (2.9) and (2.13).

Lemma 8.2. *Assume that the rates satisfy* (2.9)*. We have, for all* $f \in \mathbf{L}$*,* $\eta \in \mathbb{N}^{\mathrm{X}_n}$*,* $n \geq 1$*,*

$$|\mathcal{L}_n f(\eta)| \leq L_f C\,(1+m_p+M)\,\|\eta\|.$$

Proof. We have

$$\begin{aligned}
|\mathcal{L}_n f(\eta)| &\leq L_f \sum_{x,y\in\mathrm{X}_n} p(x,y) \sum_{k=1}^{\eta(x)} g^k_{\eta(x),\eta(y)} \|\mathcal{S}^k_{x,y}\eta - \eta\| \\
&= L_f \sum_{x,y\in\mathrm{X}_n} p(x,y) \sum_{k=1}^{\eta(x)} k g^k_{\eta(x),\eta(y)} (a_x + a_y) \\
&\leq L_f\, C \sum_{x,y\in\mathrm{X}_n} p(x,y)\,(\eta(x)+\eta(y))\,(a_x+a_y) \qquad (8.18)\\
&= L_f\, C \sum_{x\in\mathrm{X}_n} \eta(x) \sum_{y\in\mathrm{X}_n} \Big(p(x,y)+p(y,x)\Big)(a_x+a_y) \\
&\leq L_f\, C\,(1+m_p+M) \sum_{x\in\mathrm{X}_n} a_x \eta(x)
\end{aligned}$$

where $f \in \mathbf{L}$ implies the first inequality, the second one comes from (2.9), and the third one from (2.7), (2.5). □

Lemma 8.3. *Assume that the rates satisfy* (2.13)*. We have, for all* $\eta, \eta' \in \mathbb{N}^{\mathrm{X}_n}$*,* $n \geq 1$*,*

$$\left|\overline{\mathcal{L}}_n \Big(\|\eta - \eta'\|\Big)\right| \leq 2C\,(1+m_p+M)\,\|\eta-\eta'\|.$$

Proof. We have

$$\begin{aligned}
\left|\overline{\mathcal{L}}_n \Big(\|\eta - \eta'\|\Big)\right| &= \Big|\sum_{x,y\in\mathrm{X}_n} p(x,y) \sum_{k=1}^{\eta(x)} \sum_{l=1}^{\eta'(x)} G^{k,l}_{\eta(x),\eta(y),\eta'(x),\eta'(y)} \Big(\|\mathcal{S}^k_{x,y}\eta - \mathcal{S}^l_{x,y}\eta'\| - \|\eta-\eta'\|\Big)\Big| \\
&\leq \sum_{x,y\in\mathrm{X}_n} p(x,y) \sum_{k=1}^{\eta(x)} \sum_{l=1}^{\eta'(x)} G^{k,l}_{\eta(x),\eta(y),\eta'(x),\eta'(y)} (a_x+a_y)|k-l| \\
&\leq 2C \sum_{x,y\in\mathrm{X}_n} p(x,y)(a_x+a_y)\,(|\eta(x)-\eta'(x)| + |\eta(y)-\eta'(y)|) \\
&\leq 2C \sum_{x\in\mathrm{X}_n} |\eta(x)-\eta'(x)| \sum_{y\in\mathrm{X}_n} (p(x,y)+p(y,x))(a_y+a_x)
\end{aligned}$$

where the second inequality comes from Lemma 8.1. This implies the result by (2.7), (2.5). □

Based on Lemmas 8.2 and 8.3, the following technical lemmas (corresponding to [1, Lemmas 2.1–2.4]) can be proved for the MMP $(\eta_t)_{t\geq 0}$ on X_n finite. We rely on (8.12)–(8.17).

Lemma 8.4. *Fix* $\eta \in \mathbb{N}^{\mathrm{X}_n}$*. Assume that the rates satisfy* (2.9)*. We have for the Markov process* $(\eta_t)_{t\geq 0}$ *starting from* $\eta_0 = \eta$*, and for* $y \in \mathrm{X}_n$*,*

$$\mathbb{E}^\eta\Big(\eta_t(y)\Big) \leq e^{Cm_p t} \sum_{x\in\mathrm{X}_n} \eta(x) \sum_{l=0}^{+\infty} \frac{(Ct)^l}{l!} p^{(l)}(x,y).$$

Proof. Fix $\eta \in \mathbb{N}^{\mathrm{X}_n}$, $y \in \mathrm{X}_n$. Since by (2.9), (2.7),

$$\begin{aligned}\sum_{\zeta\in\mathbb{N}^{\mathrm{X}_n}} \zeta(y) q(\eta,\zeta) &= \sum_{x,z\in\mathrm{X}_n}\sum_{k=1}^{\eta(x)} p(x,z) g^k_{\eta(x),\eta(z)}(\mathcal{S}^k_{x,z}\eta(y)-\eta(y))\\ &\le \sum_{x\in\mathrm{X}_n}\sum_{k=1}^{\eta(x)} p(x,y) g^k_{\eta(x),\eta(y)} k \le C\sum_{x\in\mathrm{X}_n}\eta(x)p(x,y) \,+\, Cm_p\eta(y)\end{aligned}$$

and $q^{(j)}(\eta,\zeta)=\sum_{\xi\in\mathbb{N}^{\mathrm{X}_n}} q^{(j-1)}(\eta,\xi)q(\xi,\zeta)$ an induction proof gives that for $j\ge 1$,

$$\sum_{\zeta\in\mathbb{N}^{\mathrm{X}_n}} \zeta(y) q^{(j)}(\eta,\zeta) \le C^j \sum_{l=0}^{j}\binom{j}{l} m_p^{j-l}\sum_{z\in\mathrm{X}_n}\eta(z)p^{(l)}(z,y). \tag{8.19}$$

We conclude by (8.14) that

$$\mathbb{E}^\eta\Big(\eta_t(y)\Big) = \sum_{\zeta\in\mathbb{N}^{\mathrm{X}_n}} \zeta(y)\mathbf{p}_t(\eta,\zeta) \le \sum_{j=0}^{+\infty}\frac{t^j}{j!}C^j\sum_{l=0}^{j}\binom{j}{l} m_p^{j-l}\sum_{x\in\mathrm{X}_n}\eta(x)p^{(l)}(x,y)$$

and exchanging summations gives the result. □

Lemma 8.5. *Let $f\in\mathbf{L}$. Assume that the rates satisfy* (2.13). *Then*

$$\begin{aligned}&(a)\qquad S_n(t)f\in\mathbf{L}\ \text{ and } L_{S_n(t)f}\le L_f e^{2C(M+m_p+1)t},\\ &(b)\qquad \text{if } |f(\eta)|\le c_0\|\eta\| \ \text{ then } |S_n(t)f(\eta)|\le c_0 e^{C(M+m_p)t}\|\eta\|.\end{aligned}$$

Proof. It is based on the following consequences of Lemmas 8.2 and 8.3: For every $\eta,\eta'\in\mathbb{N}^{\mathrm{X}_n}$, $j\in\mathbb{N}$,

$$\left|\sum_{\zeta\in\mathrm{S}} q^{(j)}(\eta,\zeta)\|\zeta\|\right| \le (C\,(M+m_p+1))^j\ \|\eta\|, \tag{8.20}$$

$$\left|\sum_{\zeta,\zeta'\in\overline{\mathrm{S}}} \overline{q}^{(j)}(\eta,\eta',\zeta,\zeta')\|\zeta-\zeta'\|\right| \le (2C\,(M+m_p+1))^j\ \|\eta-\eta'\|. \tag{8.21}$$

Let us derive for instance (8.20) (it is similar for (8.21)), by induction on j. For $j=1$, using (8.3), (8.15) and Lemma 8.2 for the function $f(\eta)=\|\eta\|$, which belongs to $\mathbf{L}$ with $L_f=1$, we have

$$\left|\sum_{\zeta\in\mathrm{S}} q(\eta,\zeta)\|\zeta\|\right| = |\mathcal{L}_n f(\eta)| \le C\,(1+m_p+M)\,\|\eta\|.$$

Then for $j>1$, using the computation for $j=1$ we have

$$\left|\sum_{\zeta\in\mathrm{S}} q^{(j+1)}(\eta,\zeta)\|\zeta\|\right| = \left|\sum_{\xi\in\mathrm{S}} q^{(j)}(\eta,\xi)\sum_{\zeta\in\mathrm{S}} q(\xi,\zeta)\|\zeta\|\right| \le C\,(1+m_p+M)\sum_{\xi\in\mathrm{S}} q^{(j)}(\eta,\xi)\|\xi\|.$$

(a) For $f\in\mathbf{L}$, we define $g(\eta,\eta')=f(\eta)-f(\eta')$ and $h(\eta,\eta')=\|\eta-\eta'\|$ for all η,η'. Then by (8.21),

$$\begin{aligned}&|S_n(t)f(\eta)-S_n(t)f(\eta')| = |\overline{S}_n(t)g(\eta,\eta')| \le L_f|\overline{S}_n(t)h(\eta,\eta')|\\ &= L_f\left|\sum_{j=0}^{+\infty}\frac{t^j}{j!}\sum_{\zeta,\zeta'\in\overline{\mathrm{S}}}\overline{q}^{(j)}(\eta,\eta',\zeta,\zeta')h(\zeta,\zeta')\right| \le L_f e^{2C\,(M+m_p+1)t}\|\eta-\eta'\|.\end{aligned}$$

(b) For $f \in \mathbf{L}$, by (8.14), (8.20),

$$|S_n(t)f(\eta)| \leq c_0 \sum_{\zeta\in\mathrm{S}} \mathbf{p}_t(\eta,\zeta)\|\zeta\| = c_0 \left|\sum_{j=0}^{+\infty} \frac{t^j}{j!} \sum_{\zeta\in\mathrm{S}} q^{(j)}(\eta,\zeta)\|\zeta\|\right| \leq c_0 e^{C\,(M+m_p+1)t}\|\eta\|. \quad \square$$

Lemma 8.6. *Let $m < n$ and $f \in \mathbf{L}$. Assume that the rates satisfy* (2.9)*. Then*

$$|(\mathcal{L}_n - \mathcal{L}_m)f(\eta)| \leq CL_f \sum_{\substack{x,y\in\mathrm{X}\\ x\neq y}} |p_n(x,y) - p_m(x,y)|(\eta(x)+\eta(y))(a_x + a_y). \tag{8.22}$$

Proof. We have, using the proof of Lemma 8.2,

$$\begin{aligned}|(\mathcal{L}_n - \mathcal{L}_m)f(\eta)| &\leq \sum_{\substack{x,y\in\mathrm{X}\\ x\neq y}} \sum_{k=1}^{\eta(x)} g^k_{\eta(x),\eta(y)}|p_n(x,y) - p_m(x,y)|L_f\|\mathcal{S}^k_{x,y}\eta - \eta\| \\ &\leq L_f \sum_{\substack{x,y\in\mathrm{X}\\ x\neq y}} |p_n(x,y) - p_m(x,y)| \sum_{k=1}^{\eta(x)} k g^k_{\eta(x),\eta(y)}(a_x + a_y)\end{aligned}$$

and we conclude by (2.9) and the proof of Lemma 8.2. $\square$

Corollary 8.7. *Let $f \in \mathbf{L}$, $\eta \in \mathfrak{X}$ and $t \geq 0$. Assume that the rates satisfy* (2.13)*. Then*

$$|S_n(t)f(\eta) - S_m(t)f(\eta)| \leq CL_f e^{2Cm_pt} \int_0^t H_{n,m}(\eta)\,\mathrm{d}s \tag{8.23}$$

where

$$\begin{aligned}H_{n,m}(\eta) \;&=\; e^{2C(M+1)(t-s)} \sum_{z\in\mathrm{X}_n} \eta(z) \sum_{l=0}^{+\infty} \frac{(Cs)^l}{l!} \\ &\qquad \times \sum_{\substack{x,y\in\mathrm{X}\\ x\neq y}} \big(p_n^{(l)}(z,x) + p_n^{(l)}(z,y)\big)(a_x + a_y)|p_n(x,y) - p_m(x,y)|.\end{aligned} \tag{8.24}$$

Proof. The result follows from the successive use of the integration by parts formula

$$S_n(t)f(\eta) - S_m(t)f(\eta) = \int_0^t S_n(s)(\mathcal{L}_n - \mathcal{L}_m)S_m(t-s)f(\eta)\,\mathrm{d}s$$

and Lemmas 8.6, 8.5(a) and 8.4. $\square$

8.3 Infinite volume

Through this paragraph X is infinite. We consider its approximation introduced in (8.1). Recall notation p_n, $\mathcal{L}_n$ and $S_n(t)$.

We obtained in Subsection 8.2 some properties of finite volume processes with generator $\mathcal{L}_n$ and semigroup $S_n(t)$, $t \geq 0$ that we shall now use to take the limit n to infinity.

Lemma 8.8. *Assume that the rates satisfy* (2.9)*. For all $f \in \mathbf{L}$, $\eta \in \mathfrak{X}$, $\mathcal{L}_nf(\eta) \longrightarrow \mathcal{L}f(\eta)$ when $n \to +\infty$.*

Proof. We use Lemma 8.6 and the proof of Lemma 8.2 to write

$$|(\mathcal{L}_n - \mathcal{L}_m)f(\eta)| \leq CL_f \sum_{x,y\in\mathrm{X}} p(x,y)(\eta(x)+\eta(y))(a_x+a_y) \leq CL_f(1+m_p+M)\|\eta\|$$

and each term on the right-hand side of (8.22) goes to 0 when $n, m \to +\infty$, we have a Cauchy sequence that converges using the dominated convergence theorem, and the fact that $p_n(x,y) \to p(x,y)$ when $n \to +\infty$ yields the limit. $\square$

Proof of Proposition 2.1. It is a consequence of Lemmas 8.8 and 8.2, which also imply (2.12). $\square$

Lemma 8.9. *Assume that the rates satisfy* (2.13). *For all* $f \in \mathbf{L}$, $\eta \in \mathfrak{X}$ *and* $t \geq 0$, $S_n(t)f(\eta)$ *converges when* $n \to +\infty$.

Proof. Let us fix $t > 0$ and $\eta \in \mathfrak{X}$. From Corollary 8.7 we have that $H_{n,m}(\eta)$ given by (8.24) satisfies the upper bound

$$H_{n,m}(\eta) \quad \leq \quad 2(M+2)e^{2C(M+1)t}\|\eta\|. \tag{8.25}$$

Indeed, on the one hand, by induction on l using (2.5) we get, for $z \in \mathrm{X}_n$,

$$\sum_{x \in \mathrm{X}_n} a_x p_n^{(l)}(z,x) \leq a_z (M+1)^l. \tag{8.26}$$

On the other hand, for $x, y \in X_n$ we have

$$|p_n(x,y) - p_m(x,y)| = p(x,y)(\mathbb{1}_{\{x,y \in X_n \setminus X_m\}} + \mathbb{1}_{\{x \in X_n \setminus X_m,\, y \in X_m\}} + \mathbb{1}_{\{y \in X_n \setminus X_m,\, x \in X_m\}}). \tag{8.27}$$

Hence using (8.27), (2.5) and (8.26), we bound the second line of (8.24) by

$$\begin{aligned}
& \sum_{x \in X_n \setminus X_m} a_x p_n^{(l)}(z,x) \sum_{\substack{y \in X_n \\ y \neq x}} p(x,y) + \sum_{x \in X_m} a_x p_n^{(l)}(z,x) \sum_{y \in X_n \setminus X_m} p(x,y) \\
& + \sum_{x \in X_n \setminus X_m} p_n^{(l)}(z,x) \sum_{\substack{y \in X_n \\ y \neq x}} a_y p(x,y) + \sum_{x \in X_m} p_n^{(l)}(z,x) \sum_{y \in X_n \setminus X_m} a_y p(x,y) \\
& + \sum_{y \in X_n \setminus X_m} p_n^{(l)}(z,y) \sum_{\substack{x \in X_n \\ x \neq y}} a_x p(x,y) + \sum_{y \in X_m} p_n^{(l)}(z,y) \sum_{x \in X_n \setminus X_m} a_x p(x,y) \\
& + \sum_{y \in X_n \setminus X_m} a_y p_n^{(l)}(z,y) \sum_{\substack{x \in X_n \\ x \neq y}} p(x,y) + \sum_{y \in X_m} a_y p_n^{(l)}(z,y) \sum_{x \in X_n \setminus X_m} p(x,y) \\
\leq \quad & \sum_{x \in X_n} a_x p_n^{(l)}(z,x) + \sum_{x \in X_n} p_n^{(l)}(z,x)(M+1)a_x \\
& + \sum_{y \in X_n} p_n^{(l)}(z,y)(M+1)a_y + \sum_{y \in X_n} a_y p_n^{(l)}(z,y) \\
\leq \quad & 2(M+2) \sum_{x \in X_n} a_x p_n^{(l)}(z,x) \leq 2(M+2)a_z(M+1)^l.
\end{aligned}$$

Combining (8.24) and (8.23), by the dominated convergence theorem we obtain that $(S_n(t)f(\eta))_{n \geq 0}$ is a Cauchy sequence. $\square$

Definition 8.10. *For all* $f \in \mathbf{L}$, $\eta \in \mathfrak{X}$ *and* $t \geq 0$ *we define*

$$S(t)f(\eta) = \lim_{n \to +\infty} S_n(t)f(\eta). \tag{8.28}$$

The following lemma shows that $(S(t) : t \geq 0)$ thus defined is a semigroup with infinitesimal generator $\mathcal{L}$ given by (2.10).

Lemma 8.11. *Let $f \in \mathbf{L}$, $t, t_1, t_2 \geq 0$, $\eta \in \mathfrak{X}$*

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$, $S(0) = I$;

(ii) $S(t)f(\eta) = f(\eta) + \int_0^t \mathcal{L}S(s)f(\eta)\mathrm{d}s$;

(iii) $|S(t)f(\eta) - f(\eta)| \leq \|\eta\| L_f (e^{2C(M+m_p+1)t} - 1)$;

(iv) $S(t)f(\eta)$ *is continuous in* t;

(v) $\mathcal{L}S(t)f(\eta)$ *is continuous in* t;

(vi) $\lim_{t \searrow 0} \dfrac{S(t)f(\eta) - f(\eta)}{t} = \mathcal{L}f(\eta)$;

(vii) $\mathcal{L}S(t)f(\eta) = S(t)\mathcal{L}f(\eta)$.

Proof. The properties given in the previous Lemmas enable to use exactly the same arguments as in the paper [18] to derive [18, Lemma 2.16]. Hence we refer to it, and just note that for *(iii)*, using Lemmas 8.2 and 8.5*(a)* we have:

$$|\mathcal{L}_n S_n(t) f(\eta)| \leq L_{S_n f} C(1 + m_p + M)\|\eta\| \leq L_f e^{2C(M+m_p+1)t} C(1 + m_p + M)\|\eta\|. \qquad \square$$

Now we are ready to prove Theorem 2.2 and Proposition 2.3.

Proof of Theorem 2.2. We have that *(1)* follows from Lemma 8.5*(a)* and the definition (8.28) of $S(t)$, that *(2)* follows from Lemma 8.11*(ii)*, and that *(3)* is again a consequence of definition (8.28). $\square$

Proof of Proposition 2.3.

• Properties *(i)–(vii)* of Lemma 8.11 and the same arguments as in [18, Lemma 2.17] (see also [1, Lemma 2.9]) yield that $\bar{\mu}$ is invariant if and only if

$$\int \mathcal{L}f \, \mathrm{d}\bar{\mu} = 0 \text{ for every bounded } f \in \mathbf{L}. \tag{8.29}$$

Namely we write that, for a bounded $f \in \mathbf{L}$,

$$\begin{aligned} \int (S(t)f(\eta) - f(\eta)) \, \mathrm{d}\bar{\mu}(\eta) &= \int \Big(\int_0^t \mathcal{L}S(s)f(\eta)\mathrm{d}s \Big) \, \mathrm{d}\bar{\mu}(\eta) \\ &= \int_0^t \int \Big(\mathcal{L}S(s)f(\eta) \, \mathrm{d}\bar{\mu}(\eta) \Big) \mathrm{d}s = 0 \end{aligned} \tag{8.30}$$

where (2.12) enables to exchange limits.

• Now consider an arbitrary $f \in \mathbf{L}$ and assume (8.29) holds. The sequence of bounded Lipschitz functions defined by, for $\eta \in \mathfrak{X}$,

$$f_m(\eta) = \begin{cases} f(\eta) & \text{if } |f(\eta)| \leq m \\ m & \text{if } f(\eta) > m \\ -m & \text{if } f(\eta) < -m \end{cases}$$

satisfies, for every $\eta \in \mathfrak{X}$, $\lim_{m \to +\infty} f_m(\eta) = f(\eta)$, as well as $\lim_{m \to +\infty} \mathcal{L}f_m(\eta) = \mathcal{L}f(\eta)$. Indeed, the Lipschitz constant of f_m is L_f for every m, and we have

$$|\mathcal{L}f_m(\eta) - \mathcal{L}f(\eta)| \leq \sum_{x,y \in \mathrm{X}} p(x,y) \sum_{k=1}^{\eta(x)} g^k_{\eta(x),\eta(y)} |f_m(\mathcal{S}^k_{x,y}\eta) - f(\mathcal{S}^k_{x,y}\eta) + f(\eta) - f_m(\eta)|$$

which is dominated (using (2.9)) by $2CL_f \sum_{x,y\in \mathrm{X}} p(x,y)(a_x+a_y)(\eta(x)+\eta(y))$, which is finite (cf. the proof of Lemma 8.2). Because of (2.12), we know that $|\mathcal{L}f_m(\eta)|$ is dominated by $L_f\|\eta\|$ (up to a multiplicative constant) which is $\bar{\mu}$-integrable, we can use the dominated convergence theorem and conclude that

$$\int \mathcal{L}f(\eta)\,\mathrm{d}\bar{\mu}(\eta) = \lim_{m\to+\infty}\int \mathcal{L}f_m(\eta)\,\mathrm{d}\bar{\mu}(\eta) = 0.$$

We have therefore proved that $\bar{\mu}$ is invariant if and only if

$$\int \mathcal{L}f\,\mathrm{d}\bar{\mu} = 0 \quad \text{for every } f \in \mathbf{L}. \tag{8.31}$$

• Now fix a bounded $f \in \mathbf{L}$ and assume that $\int \mathcal{L}g\,\mathrm{d}\bar{\mu} = 0$ for every bounded cylinder function g. Given the sequence (8.1) of finite sets, one can define a sequence of bounded cylinder functions

$$f_n(\eta) = f(\eta\restriction_n) \quad \text{where} \quad \eta\restriction_n(x) = \begin{cases} \eta(x) & \text{if } x \in \mathrm{X}_n \\ 0 & \text{if } x \notin \mathrm{X}_n \end{cases}$$

that satisfy $\lim_{n\to+\infty} f_n(\eta) = f(\eta)$ for every η. Using the same arguments as before we obtain that

$$\int \mathcal{L}f(\eta)\,\mathrm{d}\bar{\mu}(\eta) = \int \lim_{n\to+\infty}\mathcal{L}f_n(\eta)\,\mathrm{d}\bar{\mu}(\eta) = \lim_{n\to+\infty}\int \mathcal{L}f_n(\eta)\,\mathrm{d}\bar{\mu}(\eta) = 0.$$

Hence the proposition is proved. □

References

[1] Andjel, E.D.: Invariant Measures for the Zero Range Process. *Ann. Probab.* **10** (1982), no. 3, 525–547. MR-0659526

[2] Armendáriz, I., Loulakis, M.: Thermodynamic Limit for the Invariant Measures in Supercritical Zero Range Processes. *Probab. Theory Related Fields,* **145** (2009) no. 1–2, 175–188. MR-2520125

[3] Barraquand, G., Corwin, I.: The q-Hahn asymmetric exclusion process. To appear in *Ann. Appl. Probab.* arXiv:1501.03445

[4] Beltran, J., Landim, C.: Metastability of reversible condensed zero range processes on a finite set. *Probab. Theory Related Fields,* **152** (2012), no. 3–4, 781–807. MR-2892962

[5] Borrello, D.: Stochastic order and attractiveness for particle systems with multiple births, deaths and jumps. *Electron. J. Probab.* **16** (2011), no. 4, 106–151. MR-2754800

[6] Borrello, D.: On the role of Allee effect and mass migration in survival and extinction of a species. *Ann. Appl. Probab.* **22** (2012), no. 2, 670–701. MR-2953566

[7] Chleboun P., Großkinsky S.: Condensation in stochastic particle systems with stationary product measures. *J. Stat. Phys.* **154** (2014), no. 1–2, 432–465. MR-3162548

[8] Cocozza, C. T.: Processus des misanthropes. *Z. Wahrsch. Verw. Gebiete* **70** (1985), no. 4, 509–523. MR-0807334

[9] Evans, M. R.: Phase transitions in one-dimensional nonequilibrium systems. *Braz. J. Phys.* **30**, no. 1 (2000), 42–57.

[10] Evans, M. R. and Hanney, T.: Nonequilibrium Statistical Mechanics of the Zero-Range Process and Related Models. *J. Phys. A: Math. Gen.* **38** (2005), no. 19, R195–R240. MR-2145800

[11] Evans, M. R., Majumdar, S.N. and Zia, R.K.P.: Factorized steady states in mass transport models. *J. Phys. A: Math. Gen.* **37** (2004), no. 25, L275–L280. MR-2073204

[12] Evans, M. R., Majumdar, S.N. and Zia, R.K.P.: Factorised steady states in mass transport models on an arbitrary graph. *J. Phys. A* **39** (2006), no. 18, 4859–4873. MR-2243199

[13] Ferrari, P.A., Landim, C., Sisko, V.: Condensation for a fixed number of independent random variables. *J. Stat. Phys.* **128** (2007), no. 5, 1153–1158. MR-2348789

[14] Gobron, T., Saada, E.: Couplings, attractiveness and hydrodynamics for conservative particle systems. *Ann. Inst. H. Poincaré Probab. Statist.* **46**, no. 4 (2010), 1132–1177. MR-2744889

[15] Godrèche, C.: From urn models to zero-range processes: statics and dynamics. *Ageing and the glass transition*, 261–294, *Lect. Notes Phys.* **716**, Springer, Berlin, 2007. MR-2428023

[16] Greenblatt, R.L., Lebowitz, J.L.: Product Measure Steady States of Generalized Zero Range Processes. *J. Phys. A: Math. Gen.* **39** (2006), no. 7, 1565–1573. MR-2210166

[17] Großkinsky S., Schütz G.M. and Spohn H.: Condensation in the zero range process: stationary and dynamical properties. *J. Statist. Phys.*, **113** (2003), no. 3–4, 389–410. MR-2013129

[18] Liggett, T. M., Spitzer, F.: Ergodic theorems for coupled random walks and other systems with locally interacting components. *Z. Wahrsch. Verw. Gebiete* **56** (1981), no. 4, 443 – 468. MR-0621659

[19] Liggett, T.M.: *Interacting Particle Systems.* Reprint of the 1985 original. Classics in Mathematics. Springer-Verlag, Berlin, 2005. MR-2108619

[20] Luck, J.M., Godrèche, C.: Nonequilibrium stationary states with Gibbs measure for two and three species of interacting particles. *J. Stat. Mech. Theory Exp.* (2006), no. 8, P08009, 18 pp. MR-2249769

[21] Luck, J.M., Godrèche, C.: Structure of the stationary state of the asymmetric target process *J. Stat. Mech. Theory Exp.* (2007), no. 8, P08005, 30 pp. MR-2338276

[22] Rogers, L.C.G., Williams, D.: *Diffusions, Markov Processes and Martingales.* Vol. 1. Foundations. Reprint of the second (1994) edition. Cambridge University Press, UK, 2000. MR-1796539

[23] Rafferty T., Chleboun P., Großkinsky S.: Monotonicity and condensation in homogeneous stochastic particle systems. Preprint (2015). arXiv:1505.02049

[24] Schütz, G. M., Ramaswamy, R. , Barma, M.: Pairwise balance and invariant measures for generalized exclusion processes. *J. Phys. A: Math. Gen.* **29** (1996), no. 4, 837 – 843. MR-1385613

[25] Seppäläinen, T.: A microscopic model for the Burgers equation and longest increasing subsequences. *Electron. J. Probab.* **1** (1996), no. 5, approx. 51 pp. MR-1386297

[26] Spitzer, F.: Interaction of Markov Processes. *Advances in Math.* **5** (1970), 247–290. MR-0268959

Acknowledgments. We thank the anonymous referee for an extremely careful reading of our manuscript, which helped us to improve it. We are grateful to Stefan Grosskinsky for useful discussions. This work was supported by the French Ministry of Education through the grants ANR-07-BLAN-0230 and ANR-2010-BLAN-0108, by the Grant Agency of the Czech Republic under grants P201/12/2613 and 16-15238S. We thank MAP5 at Université Paris Descartes and ÚTIA in Prague for hospitality and financial support.