Electron. Commun. Probab. **21** (2016), no. 39, 1–10.
DOI: 10.1214/16-ECP4786
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Critical heights of destruction for a forest-fire model on the half-plane*

Robert Graf†

Abstract

Consider the following forest-fire model on the upper half-plane of the triangular lattice: Each site can be "vacant" or "occupied by a tree". At time 0 all sites are vacant. Then the process is governed by the following random dynamics: Trees grow at rate 1, independently for all sites. If an occupied cluster reaches the boundary of the half-plane or if it is about to become infinite, the cluster is instantaneously destroyed, i.e. all of its sites turn vacant.

Let $t_c = \log 2$ denote the critical time after which an infinite cluster first appears in the corresponding pure growth process, where there is only the growth of trees but no destruction mechanism. Choose an arbitrary infinite cone in the half-plane whose apex lies on the boundary of the half-plane and whose boundary lines are non-horizontal. We prove that at time t_c almost surely only finitely many sites inside the cone have been affected by destruction in the forest-fire process.

Keywords: forest-fire model; half-plane; self-organized criticality.
AMS MSC 2010: Primary 60K35; 82C22, Secondary 82B43.
Submitted to ECP on December 29, 2015, final version accepted on March 31, 2016.

1 Introduction

Forest-fire processes were first introduced by B. Drossel and F. Schwabl in [3] as a toy model for self-organized criticality. This concept, coined by P. Bak, C. Tang and K. Wiesenfeld in their seminal paper [1], describes certain dynamical systems which are inherently driven towards a permanent critical state. By now, various forest-fire models with different dynamics and on different graphs have been studied in the physics and mathematics literature. The original motivation of this paper comes from the following forest-fire model on a box of the square lattice $\mathbb{Z}^2$: Each site in the box can be "vacant" or "occupied by a tree". At time 0 all sites are vacant. Then the process is governed by two competing random mechanisms: On the one hand, trees grow according to rate 1 Poisson processes, independently for all sites. On the other hand, if an occupied cluster reaches the boundary of the box, it is instantaneously destroyed, i.e. all of its sites turn vacant. Since the box contains only finitely many sites, the existence and uniqueness of such a process are clear. This model is believed to be self-organized critical when the radius of the box tends to infinity while keeping the centre of the box fixed. If this limit exists in a suitable sense, it yields a limit process on the lattice $\mathbb{Z}^2$. By Prokhorov's

*Supported by the Studienstiftung des deutschen Volkes
†Ludwig-Maximilians-Universität München, Germany. E-mail: robert.graf@math.lmu.de

theorem one can easily show that at least subsequential limit processes exist in a weak sense - compare the arguments in [4] and [5]. However, it has been an open question for many years what the dynamics of such a (subsequential) limit process on $\mathbb{Z}^2$ is like. It seems clear that all sites are vacant at time 0 and that trees grow according to rate 1 Poisson processes, independently for all sites. As for a potential destruction mechanism, one might intuitively expect that clusters get destroyed as soon as they become infinite. Yet this intuition turns out to be wrong as it has recently been proven by D. Kiss, I. Manolescu and V. Sidoravicius in [8] that a process with this dynamics does not exist on the square lattice $\mathbb{Z}^2$. In fact, the non-existence had already been conjectured by J. van den Berg and R. Brouwer in [2].

In [5] the setting described above is modified in the following way: As the radius of the box tends to infinity, the bottom side rather than the centre is fixed so that the subsequential limit processes live on the upper half-plane of the square lattice $\mathbb{Z}^2$ instead of the full plane. Additionally, the destruction mechanism is restricted to the fixed bottom side and edges are added between the left and right side of the box to make the setup invariant under horizontal translations. Theorem 1.2 in [5] states that every subsequential limit process on the upper half-plane of the square lattice $\mathbb{Z}^2$ has the following dynamics: At time 0 all sites are vacant. Then trees grow according to rate 1 Poisson processes, independently for all sites. If an occupied cluster reaches the boundary of the half-plane or if it becomes infinite, it is instantaneously destroyed.

The overall effect of this destruction mechanism on the half-plane is measured by the so-called heights of destruction. For a time t and a site x on the boundary of the half-plane, the corresponding height of destruction is defined as the maximal height at which sites vertically above x have been affected by destruction up to time t. By Theorem 1.5 in [5] there exists a deterministic critical time $t_c = t_c(\mathbb{Z}^2)$ such that almost surely the heights of destruction are finite for $t < t_c$ and infinite for $t > t_c$. In other words, before the critical time, the effect of the destruction mechanism is only felt locally near the boundary of the half-plane whereas after the critical time, it is felt globally on the entire half-plane. The value of the critical time t_c corresponds with the critical probability $p_c = p_c(\mathbb{Z}^2)$ of independent site percolation on $\mathbb{Z}^2$ via $1 - e^{-t_c} = p_c$.

It is the aim of this paper to show that the heights of destruction are also almost surely finite for $t = t_c$. However, the proof we give below needs an inequality which is related to two critical exponents of independent site percolation (Condition 3.1 below) and which is currently known for the triangular lattice but not for the square lattice. We therefore consider an analogous forest-fire model on the upper half-plane of the triangular lattice and generalize the concept of the heights of destruction in such a way that it becomes independent of the lattice.

The rest of the paper is organized as follows: Section 2 gives the formal statement of our result, Section 3 discusses the tools from percolation theory that we use, in particular with regard to critical exponents, and Section 4 contains the proof of our result.

2 Statement of the main result

Let $i = \sqrt{-1}$ denote the imaginary unit, let

$$\mathbb{T} := \left\{ k + l e^{i\pi/3} : k, l \in \mathbb{Z} \right\}$$

be the set of sites of the triangular lattice, let

$$\mathbb{C}^{\mathrm{u}} := \{ z \in \mathbb{C} : \operatorname{Im} z \geq 0 \}$$

be the upper half-plane and let $\mathbb{T}^{\mathrm{u}} := \mathbb{T} \cap \mathbb{C}^{\mathrm{u}}$ be the set of sites of the half-plane triangular lattice (see Figure 1). Note that according to our definition the relation $\mathbb{Z} \subset \mathbb{T}^{\mathrm{u}} \subset \mathbb{C}^{\mathrm{u}}$

holds, where $\mathbb{Z}$ can be interpreted as the (inner) boundary of $\mathbb{T}^{\mathrm{u}}$ in $\mathbb{T}$. Two sites $v, w \in \mathbb{T}$ of the triangular lattice are said to be **neighbours** if their Euclidean distance is 1. For a subset $S \subset \mathbb{T}^{\mathrm{u}}$ of the half-plane triangular lattice, we write

$$\partial S := \{v \in \mathbb{T}^{\mathrm{u}} \setminus S : \exists w \in S \text{ s.t. } v \text{ and } w \text{ are neighbours}\}$$

for the **outer boundary** of S in $\mathbb{T}^{\mathrm{u}}$. For a site $x \in \mathbb{Z}$, for example, we have $\partial\{x\} = \{x+1, x+e^{i\pi/3}, x+e^{i\cdot 2\pi/3}, x-1\}$.

At every point in time we describe the configuration of the half-plane forest-fire process by an element in $\{0,1\}^{\mathbb{T}^{\mathrm{u}}}$, where "1" corresponds to an occupied site and "0" corresponds to a vacant site. In order to introduce some notation, let $V \in \{\mathbb{T}^{\mathrm{u}}, \mathbb{T}\}$, let $(\alpha_v)_{v\in V} \in \{0,1\}^V$ and let $j \in \{0,1\}$. A j**-path** in $(\alpha_v)_{v\in V}$ from a site $y \in V$ to a site $z \in V$ is a sequence $v_0, v_1, \dots, v_l$ of distinct sites in V (where $l \in \mathbb{N}_0 := \{0,1,2,\dots\}$) such that the following holds:

- $v_0 = y$, $v_l = z$;
- v_{k-1} is a neighbour of v_k for all $k \in \{1,\dots,l\}$;
- $\alpha_{v_k} = j$ for all $k \in \{0,\dots,l\}$.

If $Y, Z \subset V$ are subsets, then a j-path in $(\alpha_v)_{v\in V}$ from Y to Z is simply any j-path in $(\alpha_v)_{v\in V}$ from a site $y \in Y$ to a site $z \in Z$. Moreover, the **cluster** of a site $y \in V$ in $(\alpha_v)_{v\in V}$ is the set of all sites z in V such that there exists a 1-path in $(\alpha_v)_{v\in V}$ from y to z. If $\alpha_y = 0$, then the cluster of y in $(\alpha_v)_{v\in V}$ is just the empty set.

We now give a formal definition of the forest-fire model, which is similar to the definitions in [5] and [4]. Here, if $I \subset \mathbb{R}$ is a left-open interval and $I \ni t \mapsto f_t \in \mathbb{R}$ is a function, we write $f_{t^-} := \lim_{s\uparrow t} f_s$ for the left-sided limit at t, provided the limit exists.

Definition 2.1. *Let $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}}}$ be a process with values in $(\{0,1\} \times \mathbb{N}_0)^{[0,\infty)\times\mathbb{T}^{\mathrm{u}}}$, initial condition $\eta_{0,z} = 0$ for $z \in \mathbb{T}^{\mathrm{u}}$ and boundary condition $\eta_{t,x} = 0$ for $t \in [0,\infty), x \in \mathbb{Z}$, and let $t \mapsto (\eta_{t,z}, G_{t,z})$ be càdlàg for all $z \in \mathbb{T}^{\mathrm{u}}$. For $t \in (0,\infty)$ and $z \in \mathbb{T}^{\mathrm{u}}$, let $C_{t^-,z}$ denote the cluster of z in the configuration $(\eta_{t^-,w})_{w\in\mathbb{T}^{\mathrm{u}}}$.*

*Then $(\eta_{t,z})_{t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}}}$ is called a $\mathbb{T}^{\mathrm{u}}$**-forest-fire process** with **growth processes** $(G_{t,z})_{t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}}}$ if the following conditions are satisfied:*

[POISSON] *The processes $(G_{t,z})_{t\in[0,\infty)}$, $z \in \mathbb{T}^{\mathrm{u}}$, are independent Poisson processes with rate 1.*

[TRANSL-INV] *The distribution of $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}}}$ is invariant under translations along the real line, i.e. the processes $(\eta_{t,z}, G_{t,z})_{t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}}}$ and $(\eta_{t,z+1}, G_{t,z+1})_{t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}}}$ have the same distribution.*

[GROWTH] *For all $t \in (0,\infty)$ and all $z \in \mathbb{T}^{\mathrm{u}} \setminus \mathbb{Z}$ the following implications hold:*

(i) $G_{t^-,z} < G_{t,z} \Rightarrow \eta_{t,z} = 1$;

(ii) $\eta_{t^-,z} < \eta_{t,z} \Rightarrow G_{t^-,z} < G_{t,z}$.

[DESTRUCTION] *For all $t \in (0,\infty)$ and all $x \in \mathbb{Z}$, $z \in \mathbb{T}^{\mathrm{u}} \setminus \mathbb{Z}$ the following implications hold:*

(i) $\big(G_{t^-,x} < G_{t,x} \Rightarrow \forall v \in \partial\{x\}\, \forall w \in C_{t^-,v} : \eta_{t,w} = 0\big)$ *and* $\big(|C_{t^-,z}| = \infty \Rightarrow \forall w \in C_{t^-,z} : \eta_{t,w} = 0\big)$;

(ii) $\eta_{t^-,z} > \eta_{t,z}$ $\Rightarrow \big(\big(\exists u \in \partial C_{t^-,z} \cap \mathbb{Z} \text{ s.t. } G_{t^-,u} < G_{t,u}\big) \text{ or } |C_{t^-,z}| = \infty\big)$.

The existence of a $\mathbb{T}^{\mathrm{u}}$-forest-fire process can be proved in exactly the same way as Theorem 1.2 in [5], where an analogous process on the upper half-plane of the square lattice $\mathbb{Z}^2$ is shown to exist. As in [5] it is currently unknown whether $\mathbb{T}^{\mathrm{u}}$-forest-fire processes are unique in distribution or whether they are adapted to the filtration generated by the growth processes. The uncertainty regarding uniqueness is the reason why the translation-invariance property [TRANSL-INV] does not necessarily follow from the other parts of Definition 2.1 and is therefore included in the definition. It is also unclear whether the destruction of infinite clusters occurs with positive probability at all. However, below we will see that up to the critical time all of these questions are easily answered.

For the remainder of the paper let $(\eta_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ be a $\mathbb{T}^{\mathrm{u}}$-forest-fire process with growth processes $(G_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ on a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$. An important auxiliary process is the **pure growth process** on $\mathbb{T}^{\mathrm{u}}$ defined by

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t\in[0,\infty), z\in\mathbb{T}^{\mathrm{u}},$$

where $1_{\mathcal{A}}$ denotes the indicator function of an event $\mathcal{A}$. For a fixed time $t\in[0,\infty)$, the configuration $\sigma_t^{\mathrm{u}} := (\sigma_{t,z})_{z\in\mathbb{T}^{\mathrm{u}}}$ has the same distribution as independent site percolation on $\mathbb{T}^{\mathrm{u}}$ where sites are occupied with probability $1-e^{-t}$. It is a well-known fact of percolation theory (see e.g. [6], Section 3.1) that there exists a critical time $t_c = t_c(\mathbb{T}) = \log 2$ such that a.s. the configuration σ_t^{u} contains no infinite cluster for $t\le t_c$ and exactly one infinite cluster for $t>t_c$. Since the $\mathbb{T}^{\mathrm{u}}$-forest-fire process is dominated by the pure growth process in the sense that for $s,t\in[0,\infty)$, $z\in\mathbb{T}^{\mathrm{u}}$

$$s\le t \Rightarrow \eta_{s,z}\le\sigma_{s,z}\le\sigma_{t,z} \tag{2.1}$$

holds, it follows that a.s. for all $t\in(0,t_c]$ and $z\in\mathbb{T}^{\mathrm{u}}$, the cluster $C_{t-,z}$ is finite. On the time interval $(0,t_c]$, the destruction mechanism is therefore a.s. simplified to:

[DESTRUCTION] For all $t\in(0,t_c]$ and all $x\in\mathbb{Z}$, $z\in\mathbb{T}^{\mathrm{u}}\setminus\mathbb{Z}$ the following implications hold:

(i) $G_{t-,x} < G_{t,x} \Rightarrow \forall v\in\partial\{x\}\,\forall w\in C_{t-,v}:\eta_{t,w}=0;$
(ii) $\eta_{t-,z} > \eta_{t,z} \Rightarrow \exists u\in\partial C_{t-,z}\cap\mathbb{Z}$ s.t. $G_{t-,u}<G_{t,u}$.

As a consequence of the Russo-Seymour-Welsh results on critical percolation we also have

$$\mathbf{P}\left[\sigma_{t_c}^{\mathrm{u}} \text{ contains infinitely many disjoint } 0\text{-paths from } \mathbb{Z}_{<x} \text{ to } \mathbb{Z}_{>x}\right]=1, \tag{2.2}$$
$$\mathbf{P}\left[\sigma_{t_c}^{\mathrm{u}} \text{ contains infinitely many disjoint } 1\text{-paths from } \mathbb{Z}_{<x} \text{ to } \mathbb{Z}_{>x}\right]=1 \tag{2.3}$$

for all $x\in\mathbb{Z}$, where $\mathbb{Z}_{<x} := \{x'\in\mathbb{Z}:x'<x\}$ and $\mathbb{Z}_{>x} := \{x'\in\mathbb{Z}:x'>x\}$. Equations (2.1) and (2.2) imply that on the time interval $[0,t_c]$, $\mathbb{T}^{\mathrm{u}}$-forest-fire processes are unique in distribution and adapted to the filtration generated by the growth processes. Indeed, if we partition $\mathbb{T}^{\mathrm{u}}$ into a random collection of finite sets separated by 0-paths in $\sigma_{t_c}^{\mathrm{u}}$, the different areas of the partition do not interact in the $\mathbb{T}^{\mathrm{u}}$-forest-fire process so that the $\mathbb{T}^{\mathrm{u}}$-forest-fire process can be uniquely constructed from the growth processes by a graphical representation.

In this paper we analyse the total effect of destruction in the $\mathbb{T}^{\mathrm{u}}$-forest-fire process up to a certain time, which is quantified by the heights of destruction:

Definition 2.2. *For $t\in[0,\infty)$ and $S\subset\mathbb{C}^{\mathrm{u}}$, let*

$$Y_t(S) := \sup\left\{\operatorname{Im} z : z\in S\cap\mathbb{T}^{\mathrm{u}} \text{ and } \exists s\in(0,t] \text{ s.t. } \eta_{s-,z}>\eta_{s,z}\right\}\vee 0 \tag{2.4}$$

*be the height up to which sites in S have been destroyed up to time t, where $Y_t(S)$ can take values in $[0,\infty]$. We call $Y_t(S)$ the **height of destruction** in S up to time t.*

Note that $Y_t(S)$ is monotone increasing in t and S in the sense that for $t_1,t_2\in[0,\infty)$ and $S_1,S_2\subset\mathbb{C}^{\mathrm{u}}$ the implication

$$(t_1\le t_2 \text{ and } S_1\subset S_2)\Rightarrow Y_{t_1}(S_1)\le Y_{t_2}(S_2) \tag{2.5}$$

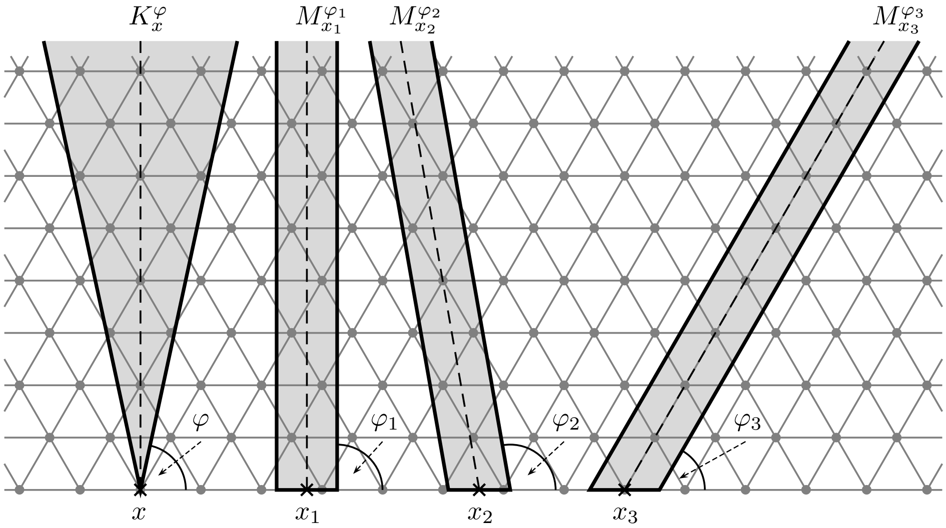

Figure 1: The half-plane triangular lattice $\mathbb{T}^{\mathrm{u}}$, a cone K_x^φ and three semi-infinite tubes $M_{x_j}^{\varphi_j}$ $(j = 1, 2, 3)$.

holds. We will study the heights of destruction in infinite cones and semi-infinite tubes as defined in Definition 2.3 and illustrated in Figure 1.

Definition 2.3. *(i) For $x \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$, let*

$$K_x^\varphi := \left\{ x + ae^{i\varphi} + be^{i(\pi-\varphi)} : a, b \geq 0 \right\}$$

denote the infinite cone whose apex is x and whose boundary lines have angular directions φ and $\pi - \varphi$, respectively.

(ii) For $x \in \mathbb{R}$ and $\varphi \in (0, \pi)$, let $L_x^\varphi := \left\{ x + ye^{i\varphi} : y \geq 0 \right\}$ denote the half-line with starting point x and angular direction φ, and let

$$M_x^\varphi := \left\{ z \in \mathbb{C}^{\mathrm{u}} : \operatorname{dist}(z, L_x^\varphi) \leq \frac{1}{2} \right\}$$

denote the semi-infinite tube with centre line L_x^φ and width 1, where

$$\operatorname{dist}(z, S) := \inf \left\{ |z - z'| : z' \in S \right\} \tag{2.6}$$

is the distance of a point $z \in \mathbb{C}$ from a set $S \subset \mathbb{C}$.

Equation (2.3) indicates that $Y_{t_c}(K_x^\varphi)$ could potentially be equal to ∞. We prove that this case a.s. does not occur:

Theorem 2.4. *For all $x \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$ we have $\mathbf{P}\left[Y_{t_c}(K_x^\varphi) < \infty \right] = 1$.*

Roughly speaking, Theorem 2.4 means that up to and including the critical time t_c, the influence of [DESTRUCTION] in Definition 2.1 is confined to areas close to the boundary of the half-plane. Combining Theorem 2.4 with results in [5], we can determine the behaviour of the heights of destruction for all times:

Corollary 2.5. *(i) For all $x \in \mathbb{R}$ and $\varphi \in (0, \pi/2)$ we have*

$$\mathbf{P}\left[\forall t \in [0, t_c] : Y_t(K_x^\varphi) < \infty \text{ and } \forall t \in (t_c, \infty) : Y_t(K_x^\varphi) = \infty \right] = 1.$$

(ii) For all $x \in \mathbb{R}$ and $\varphi \in (0, \pi)$ we have

$$\mathbf{P}\left[\forall t \in [0, t_c] : Y_t(M_x^\varphi) < \infty \text{ and } \forall t \in (t_c, \infty) : Y_t(M_x^\varphi) = \infty \right] = 1.$$

In other words, the heights of destruction experience a phase transition in the sense that they are a.s. finite until the critical time t_c and become infinite immediately after t_c. Note that a priori, statements about the finiteness of the heights of destruction are stronger for cones than for tubes whereas statements about the infinity of the heights of destruction are stronger for tubes than for cones. With regards to the unknown uniqueness of $\mathbb{T}^{\mathrm{u}}$-forest-fire processes after t_c, let us emphasize that Corollary 2.5 holds for any process satisfying Definition 2.1.

Sketch of the proof of Corollary 2.5. In view of Theorem 2.4 and equation (2.5) it suffices to show

$$\mathbf{P}\left[\forall t \in (t_c, \infty) : Y_t(M_x^\varphi) = \infty\right] = 1$$

for $x \in \mathbb{R}$ and $\varphi \in (0, \pi)$. This equation can be proved along the lines of Theorem 1.5 in [5], where a corresponding statement is derived for the upper half-plane of the square lattice $\mathbb{Z}^2$ and for $x \in \mathbb{Z}$, $\varphi = \pi/2$. (The associated height of destruction up to time t is denoted by $Y_{t,x}$ in [5].) A crucial property of M_x^φ in the course of the proof is the fact that any 1-path which crosses from the left of M_x^φ to the right of M_x^φ has at least one site in M_x^φ; this is the reason why we have defined M_x^φ to have width 1. □

3 Tools from percolation theory needed for Theorem 2.4

The proof of Theorem 2.4 is based on tools from percolation theory for which it is convenient to have the pure growth process available on the whole triangular lattice $\mathbb{T}$ and not just on $\mathbb{T}^{\mathrm{u}}$ (e.g. equations (3.1) and (3.2) below). For the remainder of the paper we therefore extend the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ in such a way that it also contains processes $(G_{t,z})_{t\in[0,\infty), z\in\mathbb{T}\setminus\mathbb{T}^{\mathrm{u}}}$ on the lower half-plane and that $(G_{t,z})_{t\in[0,\infty)}$, $z \in \mathbb{T}$, are independent Poisson processes with rate 1. We define the corresponding pure growth process on $\mathbb{T}$ by

$$\sigma_{t,z} := 1_{\{G_{t,z}>0\}}, \qquad t \in [0, \infty), z \in \mathbb{T}.$$

For $t \in [0, \infty)$, we henceforth abbreviate $\eta_t := (\eta_{t,z})_{z\in\mathbb{T}^{\mathrm{u}}}$, $\sigma_t^{\mathrm{u}} := (\sigma_{t,z})_{z\in\mathbb{T}^{\mathrm{u}}}$ and $\sigma_t := (\sigma_{t,z})_{z\in\mathbb{T}}$.

We will frequently use the following terminology: Let $V \in \{\mathbb{T}^{\mathrm{u}}, \mathbb{T}\}$, let $(\alpha_v)_{v\in V} \in \{0,1\}^V$ be a configuration and let $w \in V$, $S \subset \mathbb{C}$. Then we write $\{w \leftrightarrow S \text{ in } (\alpha_v)_{v\in V}\}$ (in words: w is connected to S in $(\alpha_v)_{v\in V}$) for the event that there exists a 1-path in $(\alpha_v)_{v\in V}$ from a site $y \in V$ to a site $z \in V$ such that y is a neighbour of w and $\mathrm{dist}(z, S) \le 1$ holds, where $\mathrm{dist}(z, S)$ is defined as in (2.6). Note that our definition of $\{w \leftrightarrow S \text{ in } (\alpha_v)_{v\in V}\}$ does not impose any condition on the site w itself.

Sections 3.1 and 3.2 briefly explain the major results from percolation theory which are needed for the proof of Theorem 2.4.

3.1 Exponential decay of the radius

For $z \in \mathbb{T}$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, let

$$H_n(z) := \bigcup_{j=1}^{6} \left\{ z + (1-a)ne^{i(j-1)\pi/3} + ane^{ij\pi/3} : a \in [0,1] \right\}$$

denote the boundary of the regular hexagon with vertices $z + ne^{ij\pi/3}$, $j = 1, \ldots, 6$. There exists a function $\xi : (0, t_c) \to (0, \infty)$ such that for all $t \in (0, t_c)$ the full-plane one-arm event $\{0 \leftrightarrow H_n(0) \text{ in } \sigma_t\}$ satisfies

$$\lim_{n\to\infty} -\frac{\log \mathbf{P}\left[0 \leftrightarrow H_n(0) \text{ in } \sigma_t\right]}{n} = \frac{1}{\xi(t)}; \tag{3.1}$$

$\xi(t)$ is called the correlation length of the configuration σ_t. Moreover, there exists a universal constant $c \in (0,\infty)$ such that for all $t \in (0,t_c)$ and $n \in \mathbb{N}$

$$\mathbf{P}\left[0 \leftrightarrow H_n(0) \text{ in } \sigma_t\right] \leq cn \exp\left(-\frac{n}{\xi(t)}\right) \tag{3.2}$$

holds. For the proof of (3.1) and (3.2) the reader is referred to [6], Section 6.1. (In this reference, analogous statements are proven for bond percolation on the square lattice $\mathbb{Z}^2$ and boxes rather than hexagons but the main arguments of the proofs are transferable to the present setting.)

3.2 Critical exponents

As we will see in Section 4, the core of the proof of Theorem 2.4 depends on the following condition:

Condition 3.1. *There exist $\nu, \rho > 0$ such that the correlation length satisfies*

$$\xi(t) \leq (t_c - t)^{-\nu + o(1)} \qquad \text{for } t \uparrow t_c, \tag{3.3}$$

the one-arm half-plane event at criticality satisfies

$$\mathbf{P}\left[0 \leftrightarrow H_n(0) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma^{\mathrm{u}}_{t_c}\right] \leq n^{-\rho + o(1)} \qquad \text{for } n \to \infty, \tag{3.4}$$

and $1/\nu + \rho > 1$.

Since we work on the triangular lattice, equations (3.3) and (3.4) are known to hold with equality for $\nu = 4/3$ and $\rho = 1/3$ so that Condition 3.1 is satisfied in the present setting. The existence and values of these and other critical percolation exponents for the triangular lattice were proven by S. Smirnov and W. Werner in [14] (Theorems 1(iv) and 3) and are also discussed in the survey article [12] (Theorems 33(i) and 22). The proof in [14] is based on a combination of several deep results, namely the scaling relations proven by H. Kesten in [7], the determination of critical exponents for the Schramm-Loewner evolution by G. Lawler, O. Schramm and W. Werner in [9], [10], [11], and the proof of Cardy's formula by S. Smirnov in [13]. As the current proof of Cardy's formula only works for the triangular lattice, rigorous results for critical exponents are largely restricted to the triangular lattice although these exponents are widely believed to be universal. In particular, it is unknown whether Condition 3.1 holds for the square lattice.

4 Proof of Theorem 2.4

Let $x \in \mathbb{R}$ and $\varphi \in (0,\pi/2)$; we need to show $\mathbf{P}[Y_{t_c}(K_x^\varphi) = \infty] = 0$. To keep the notation simple, we assume $x = 0$ in the following and abbreviate $K := K_0^\varphi$. The proof for general, possibly non-integer x only differs in technical details. Since the $\mathbb{T}^{\mathrm{u}}$-forest-fire process $(\eta_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ is dominated by the corresponding pure growth process $(\sigma_{t,z})_{t\in[0,\infty),z\in\mathbb{T}^{\mathrm{u}}}$ in the sense of equation (2.1), a.s. all clusters that are destroyed in the $\mathbb{T}^{\mathrm{u}}$-forest-fire process in the time interval $(0,t_c]$ are finite. Hence, if $Y_{t_c}(K) = \infty$ holds, then a.s. infinitely many clusters which reach from K to the inner boundary $\mathbb{Z}$ must have been destroyed up to the critical time t_c. Moreover, since there are only finitely many jumps in a rate 1 Poisson process up to time t_c, every site on the inner boundary $\mathbb{Z}$ can only be the origin of finitely many destruction events up to time t_c. This implies the inclusion

$$\{Y_{t_c}(K) = \infty\} \overset{\text{a.s.}}{\subset} \limsup_{n\to\infty} \mathcal{A}_n \cup \limsup_{n\to\infty} \mathcal{A}_{-n},$$

where the limit superior is defined as

$$\limsup_{n\to\infty} \mathcal{A}_n := \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \mathcal{A}_n$$

and

$$\mathcal{A}_n := \{\exists t \in [0, t_c) \text{ s.t. } n \leftrightarrow K \text{ in } \eta_t \text{ and } G_{t,n} < G_{t_c,n}\}$$

for $n \in \mathbb{Z}$. By symmetry, we have $\mathbf{P}[\mathcal{A}_{-n}] = \mathbf{P}[\mathcal{A}_n]$ for all $n \in \mathbb{N}$; consequently, it suffices to prove

$$\mathbf{P}\left[\limsup_{n\to\infty} \mathcal{A}_n\right] = 0. \tag{4.1}$$

Applying equation (2.1) once more and using the topological fact that any connection $n \leftrightarrow K$ necessarily contains a connection $n \leftrightarrow H_{\lfloor n \sin\varphi\rfloor}(n) \cap \mathbb{C}^{\mathrm{u}}$, we obtain the inclusions

$$\begin{aligned} \mathcal{A}_n &\subset \{\exists t \in [0, t_c) \text{ s.t. } n \leftrightarrow K \text{ in } \sigma_t^{\mathrm{u}} \text{ and } G_{t,n} < G_{t_c,n}\} \\ &\subset \left\{\exists t \in [0, t_c) \text{ s.t. } n \leftrightarrow H_{\lfloor n \sin\varphi\rfloor}(n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_t^{\mathrm{u}} \text{ and } G_{t,n} < G_{t_c,n}\right\} =: \mathcal{B}_n. \end{aligned} \tag{4.2}$$

Now recall the exponents ν, ρ of Condition 3.1, choose an arbitrary $\delta > 0$ satisfying $1/\nu + \rho > 1 + \delta$ and $1/\nu > \delta$, and set $n_0 := \min\left\{n \in \mathbb{N} : t_c - n^{-1/\nu+\delta} > 0\right\}$. For $n \geq n_0$, we consider the event

$$\mathcal{C}_n := \left\{\exists t \in [0, t_c - n^{-1/\nu+\delta}) \text{ s.t. } n \leftrightarrow H_{\lfloor n \sin\varphi\rfloor}(n) \cap \mathbb{C}^{\mathrm{u}} \text{ in } \sigma_t^{\mathrm{u}}\right\}$$

that the connection $n \leftrightarrow H_{\lfloor n \sin\varphi\rfloor}(n) \cap \mathbb{C}^{\mathrm{u}}$ in the pure growth process already occurs before time $t_c - n^{-1/\nu+\delta}$. The probability of this event can be estimated from above as follows:

$$\begin{aligned} \mathbf{P}\left[\mathcal{C}_n\right] &\leq \mathbf{P}\left[\exists t \in [0, t_c - n^{-1/\nu+\delta}) \text{ s.t. } 0 \leftrightarrow H_{\lfloor n \sin\varphi\rfloor}(0) \text{ in } \sigma_t\right] \\ &= \mathbf{P}\left[0 \leftrightarrow H_{\lfloor n \sin\varphi\rfloor}(0) \text{ in } \sigma_{t_c - n^{-1/\nu+\delta}}\right] \\ &\leq cn \exp\left(-\frac{\lfloor n \sin\varphi\rfloor}{\xi\left(t_c - n^{-1/\nu+\delta}\right)}\right) \\ &\leq cn \exp\left(-\frac{\lfloor n \sin\varphi\rfloor}{\left(n^{-1/\nu+\delta}\right)^{-\nu+o(1)}}\right) \qquad \text{for } n \to \infty \\ &= cn \exp\left(-n^{\delta\nu+o(1)}\right) \qquad \text{for } n \to \infty. \end{aligned}$$

Here we first drop the condition that the connection occurs in the upper half-plane $\mathbb{C}^{\mathrm{u}}$ and use the translation-invariance of the pure growth process; then we employ the fact that σ_t is monotone increasing in t; finally we successively apply equations (3.2) and (3.3). In particular, this estimate implies

$$\sum_{n=n_0}^{\infty} \mathbf{P}\left[\mathcal{C}_n\right] < \infty$$

and hence

$$\mathbf{P}\left[\limsup_{n\to\infty} \mathcal{C}_n\right] = 0$$

by the Borel-Cantelli lemma. Regarding the limit superior of the events $\mathcal{B}_n$ (defined in (4.2)), we thus conclude

$$\mathbf{P}\left[\limsup_{n\to\infty}\mathcal{B}_n\right]=\mathbf{P}\left[\limsup_{n\to\infty}(\mathcal{B}_n\setminus\mathcal{C}_n)\right]\le\mathbf{P}\left[\limsup_{n\to\infty}\mathcal{D}_n\right], \tag{4.3}$$

where we abbreviate

$$\mathcal{D}_n := \left\{\exists t\in[t_c-n^{-1/\nu+\delta},t_c) \text{ s.t. } n\leftrightarrow H_{\lfloor n\sin\varphi\rfloor}(n)\cap\mathbb{C}^{\mathrm{u}} \text{ in } \sigma_t^{\mathrm{u}} \text{ and } G_{t,n}<G_{t_c,n}\right\}$$

for $n\ge n_0$. The probability of the event $\mathcal{D}_n$ can be bounded from above as follows:

$$\begin{aligned}\mathbf{P}[\mathcal{D}_n] &\le \mathbf{P}\left[n\leftrightarrow H_{\lfloor n\sin\varphi\rfloor}(n)\cap\mathbb{C}^{\mathrm{u}} \text{ in } \sigma_{t_c}^{\mathrm{u}} \text{ and } G_{t_c-n^{-1/\nu+\delta},n}<G_{t_c,n}\right]\\ &= \mathbf{P}\left[0\leftrightarrow H_{\lfloor n\sin\varphi\rfloor}(0)\cap\mathbb{C}^{\mathrm{u}} \text{ in } \sigma_{t_c}^{\mathrm{u}}\right]\ \mathbf{P}\left[G_{n^{-1/\nu+\delta},0}>0\right]\\ &\le n^{-\rho+o(1)}\cdot\left(1-\exp\left(-n^{-1/\nu+\delta}\right)\right) \qquad \text{for } n\to\infty\\ &\le n^{-\rho+o(1)}\cdot n^{-1/\nu+\delta} \qquad \text{for } n\to\infty\\ &= n^{-\rho-1/\nu+\delta+o(1)} \qquad \text{for } n\to\infty.\end{aligned}$$

Here we first relax the condition on the times at which the connection and the growth event occur, resorting to the fact that σ_t^{u} is monotone increasing in t; then we use the independence and translation-invariance of the events $\{n\leftrightarrow H_{\lfloor n\sin\varphi\rfloor}(n)\cap\mathbb{C}^{\mathrm{u}} \text{ in } \sigma_{t_c}^{\mathrm{u}}\}$ and $\{G_{t_c-n^{-1/\nu+\delta},n}<G_{t_c,n}\}$; in the next step we apply equation (3.4); finally we use the inequality $1-e^{-y}\le y$ which is valid for all $y\in\mathbb{R}$. Since $-\rho-1/\nu+\delta<-1$ holds by our choice of δ, the previous estimate shows

$$\sum_{n=n_0}^{\infty}\mathbf{P}[\mathcal{D}_n]<\infty.$$

Invoking the Borel-Cantelli lemma again, we get

$$\mathbf{P}\left[\limsup_{n\to\infty}\mathcal{D}_n\right]=0. \tag{4.4}$$

Together with (4.2) and (4.3), equation (4.4) yields the proof of (4.1) and hence of Theorem 2.4. □

Remark 4.1. In the proof we used that if the height of destruction is infinite, the forest-fire process contains infinitely many clusters which connect the cone to the boundary of the half-plane, and hence exhibits infinitely many half-plane one-arm events. By monotonicity, these events also occur in the pure growth process. Since forest-fire clusters can touch the boundary at one point only, in fact, the forest-fire process exhibits infinitely many three-arm half-plane events. However, the first and third arm consist of vacant sites so that these events do not necessarily occur in the pure growth process. This observation therefore does not yield a simple way of strengthening the proof.

References

[1] P. Bak, C. Tang, and K. Wiesenfeld, *Self-organized criticality: An explanation of 1/f noise*, Phys. Rev. Lett. **59** (1987), 381–384. MR-0949160

[2] J. van den Berg and R. Brouwer, *Self-destructive percolation*, Random Struct. Algorithms **24** (2004), 480–501. MR-2060632

[3] B. Drossel and F. Schwabl, *Self-organized critical forest-fire model*, Phys. Rev. Lett. **69** (1992), 1629–1632.

[4] M. Dürre, *Existence of multi-dimensional infinite volume self-organized critical forest-fire models*, Electron. J. Probab. **11** (2006), 513–539. MR-2242654

[5] R. Graf, *A forest-fire model on the upper half-plane*, Electron. J. Probab. **19** (2014), 1–27. MR-3164761

[6] G. Grimmett, *Percolation*, 2nd ed., Grundlehren der mathematischen Wissenschaften, Springer, 1999. MR-1707339

[7] H. Kesten, *Scaling relations for 2D-percolation*, Comm. Math. Phys. **109** (1987), no. 1, 109–156. MR-0879034

[8] D. Kiss, I. Manolescu, and V. Sidoravicius, *Planar lattices do not recover from forest fires*, Ann. Probab. **43** (2015), no. 6, 3216–3238. MR-3433580

[9] G. Lawler, O. Schramm, and W. Werner, *Values of Brownian intersection exponents I: Half-plane exponents*, Acta Math. **187** (2001), 237–273. MR-1879850

[10] G. Lawler, O. Schramm, and W. Werner, *Values of Brownian intersection exponents II: Plane exponents*, Acta Math. **187** (2001), 275–308. MR-1879851

[11] G. Lawler, O. Schramm, and W. Werner, *Values of Brownian intersection exponents III: Two-sided exponents*, Ann. Inst. Henri Poincaré Probab. Stat. **38** (2002), no. 1, 109–123. MR-1899232

[12] P. Nolin, *Near-critical percolation in two dimensions*, Electron. J. Probab. **13** (2008), 1562–1623. MR-2438816

[13] S. Smirnov, *Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sr. I Math. **333** (2001), no. 3, 239–244. MR-1851632

[14] S. Smirnov and W. Werner, *Critical exponents for two-dimensional percolation*, Math. Res. Lett. **8** (2001), 729–744. MR-1879816

Acknowledgments. I am grateful to Franz Merkl for helpful discussions and remarks.