Geometry of stochastic delay differential equations with jumps in manifolds

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Abstract

In this article we propose a model for stochastic delay differential equation with jumps (SDDEJ) in a differentiable manifold M endowed with a connection ∇. In our model, the continuous part is driven by vector fields with a fixed delay and the jumps are assumed to come from a distinct source of (càdlàg) noise, without delay. The jumps occur along adopted differentiable curves with some dynamical relevance (with fictitious time) which allow to take parallel transport along them. In the last section, using a geometrical approach, we show that the horizontal lift of the solution of an SDDEJ is again a solution of an SDDEJ in the linear frame bundle BM with respect to a horizontal connection ∇^H in BM.

1 Introduction

Many natural phenomena exhibit delays with respect to inputs: mathematically, this is a well known and established theory in the literature. In fact, standard delay differential equations have been extensively studied, see e.g. the classical Hale [12] and references therein. Models for these equations in manifolds appear in Oliva [23], and stochastic perturbations are considered in Langevin, Oliva and Oliveira [17], Mohammed [20], Mohammed and Scheutzow [22, 21], Caraballo, Kloeden and Real [2] for SPDE. In Léandre [18], the author consider a stochastic delay equation on a compact manifold, and apply Malliavian calculus to show a positivity theorem on the distribution of the solution. More recently, differential equations with random unbounded delay have been considered in Garrido-Atienza, Ogrowsky and Schmalfuss [11].

Besides delay, another usual feature in systems in biology, physics, economics, climatology, etc is the presence of jumps, both in the input and in the output. In this paper, we put these two characteristics, delays and jumps, together in the same mathematical framework.

In our model of stochastic delay differential equation with jumps (SDDEJ), the continuous part of the solution is driven by vector fields with a fixed delay $d > 0$, and the jumps are assumed to come from a distinct source of (càdlàg) noise, without delay. This

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idea is inspired by the fact that in several phenomena, informations reach a receptor via distinct communication sources or channels, hence it is reasonable that delays on time are dependent on these distinct sources. As a simple example: in a storm, lightnings have instantaneous impact, but thunders come with delay.

Following the ideas behind the so called Marcus equation, as in Kurtz, Pardoux and Protter [16], our jumps in the solution occur along fictitious differentiable curves which allow to take parallel transport along them. Generically speaking, as in Section 2, these fictitious differentiable curves can be taken in many distinct ways (say, randomly or along geodesics, etc). In our model, presented in Section 3, they follow the deterministic flow generated by the vector fields, without delay. We consider that the noise splits into a continuous component which is a Brownian motion with drift (extensible to a class of continuous semimartingales) plus a component given by a sequence of (càdlàg) jumps. The number of jumps is assumed to be finite in a compact time interval. Hence it includes Lévy jump-diffusion, but not Lévy process in general. This idea of finite jumps in bounded intervals has parallel in the theory of chain control sets, see Colonius and Kliemann [5], Patrão and San Martin [24], and references therein.

Another model with some numerical results for delay stochastic systems with jumps in Euclidean spaces can be found in Dareiotis, Kumar and Sabanis [7] for Lévy processes. Also, stochastic geometry with jumps is considered in Cohen [4], where the authors use second order calculus.

In our model, the delay are treated using parallel transport along the solutions, prescribed by a connection ∇ in a differentiable manifold M. Using a geometrical approach, we show that the horizontal lift of a SDDEJ solution to the linear frame bundle BM is again a solution of an associated SDDEJ in BM, extending the result of Catuogno and Ruffino [3] for the continuous case.

The paper is organized as follows: In Section 2 we construct parallel transports along curves with jumps: here, the jumps are taken along a generic family of fictitious curves. In Section 3, our model of delay differential equations with jumps is presented. Compactness of the manifold M is assumed only to guarantee the existence and uniqueness of solution for SDDE (without jumps) for all $t \geq 0$, as in Léandre and Mohammed [19]. Finally, in Section 4, we explore geometrical aspects of the SDDEJ. After a short revision on the geometry of linear frame bundle and on the horizontal connection ∇^H in BM, we present our main result about the horizontal lift of a SDDEJ solution.

2　General aspects of parallel transport

Let M be a differentiable manifold, and ∇ a connection on M. This structure, via parallel transport, allows one to map vectors from a tangent space at a point in a differentiable curve into the tangent space at another point of this curve. To fix notation, consider a differentiable curve $\alpha : I \to M$ defined in an interval $I \subset \mathbf{R}$. For $s, t \in I$, the parallel transport along α from $\alpha(s)$ to $\alpha(t)$ induced by ∇ is the linear isometry denoted by $P_{s,t}^\nabla(\alpha) : T_{\alpha(s)}M \to T_{\alpha(t)}M$, such that the covariant derivative of $t \mapsto P_{s,t}^\nabla(\alpha)$ vanishes. If α is continuous and differentiable by parts, its parallel transport is constructed joining the corresponding parallel transports along each differentiable segment (see e.g. Kobayashi and Numizu [15]).

2.1　Parallel transport along a curve with jumps

Let $\gamma : I \to M$ be a càdlàg curve with discontinuities in a countable, discrete and closed set $D = \{t_1, t_2, \ldots\}$, possibly finite. Suppose that γ is differentiable in $I \setminus D$. Let $\mathcal{B} = (\beta_n)_{n \in \mathbf{N}}$ be a family of differentiable curves $\beta_n : [0, 1] \to M$ such that, for each $n \in \mathbf{N}$, $\beta_n(0) = \lim_{s \to t_n-} \gamma(s)$ and $\beta_n(1) = \gamma(t_n)$. Intuitively, these differentiable curves fills

the gaps along the trajectory of γ. Hence, we can define the parallel transport along γ with respect to $\mathcal{B}$. Precisely, fix a subinterval $[s, t] \subseteq I$. Take $J = D \cap (s, t]$, that is, J is the set of the times that jumps occur in $(s, t]$. By assumption, J is finite and, for sack of simplicity, write $J = \{t_1 < t_2 < \ldots < t_k\}$.

Now, define the curve $(\gamma \vee \mathcal{B})_{s,t} : [s, t + k] \to M$ which concatenates γ with elements of the family $\{\beta_i, i = 1, \ldots, k\}$ in the following way:

$$(\gamma \vee \mathcal{B})_{s,t}(u) = \begin{cases} \gamma(u), \text{ for } u \in [s, t_1) \\ \beta_1(u - t_1), \text{ for } u \in [t_1, t_1 + 1] \\ \gamma(u - 1), \text{ for } u \in [t_1 + 1, t_2 + 1) \\ \beta_2(u - t_2 - 1), \text{ for } u \in [t_2 + 1, t_2 + 2] \\ \quad \vdots \\ \beta_k(u - t_k - k + 1), \text{ for } u \in [t_k + k - 1, t_k + k] \\ \gamma(u - k), \text{ for } u = t_k + k. \end{cases}$$

Since the constructed curve $(\gamma \vee \mathcal{B})_{s,t}$ is continuous and differentiable by parts, we define the parallel transport along γ with respect to $\mathcal{B}$ by:

$$P_{s,t}^{\nabla, \mathcal{B}}(\alpha) := P_{s,t+k}^{\nabla}\Big((\alpha \vee \mathcal{B})_{s,t}\Big).$$

Note that the domain of $(\gamma \vee \mathcal{B})_{s,t}$ is artificially extended due to the 'fictitious' curves in $\mathcal{B}$ which fill the gaps of γ.

The family $\mathcal{B}$ of artificial curves which fill the gaps of the noncontinuous cádlag trajectories comprehend many kinds of jumps: say, e.g., along geodesics in the direction of the vector fields, along their flows, uniformly distributed on the unit geodesic ball around the left limit points, etc. In the next sections, motivated by Marcus equation, see e.g. Kurtz, Pardoux and Protter [16], it is established by the deterministic flow generated by the vector fields of the differential equation.

3 Delay differential equations with jumps

In this section, we present our model of delay differential equations with jumps (DDEJ), including the deterministic and stochastic case (SDDEJ). The solution for this equation is constructed by induction on the number of jumps, such that, in the intervals between consecutive jumps, we use the theory of standard delay differential equations. We remark that the existence and uniqueness of solution in stochastic delay differentiable equations (without jumps) is a particular case of the theory of stochastic functional differential equations (see, e.g. Léandre and Mohammed [19]). We start describing the simpler context:

3.1 Deterministic case

Initially, we construct the discontinuous (càdlàg) integrator S_t which drives our model of DDEJ. Let $(t_n)_{n \in \mathbf{N}}$ be an increasing, discrete and closed sequence in $\mathbf{R}_{>0}$ which indicates the points of discontinuities of S_t. Let $(J_n)_{n \in \mathbf{N}}$ be the corresponding sequence in $\mathbf{R}$ of increments at the jumps of S_t. Define the integer function which counts the number of jumps up to time t by $N_t = \max\{n : t_n \leq t\}$, with the convention that the maximum of the empty set is zero. Consider the integrator $S : \mathbf{R}_{\geq 0} \to \mathbf{R}$ in the following way:

$$S_t = t + \sum_{k=0}^{N_t} J_k.$$

The DDEJ in the manifold M is written as:

$$dx(t) = P^{\nabla}_{t-d,t}(x) \, F(x(t-d)) \, dS_t \tag{3.1}$$

with initial condition given by a differentiable curve $\beta_0 : [-d,0] \to M$, where $d \in (0,1]$ is a fixed time delay and F is a smooth vector field in the manifold M. We construct a solution $\gamma(t)$ of equation (3.1) as follows:

• **Solution before the first jump**

For $t \in [0,t_1)$, $\gamma(t)$ is the solution of the delay differential equation (without jumps) given by:

$$\begin{cases} x'(t) = P^{\nabla}_{t-d,t}(x) \, F(x(t-d)) \\ x(t) = \beta_0(t), \ \forall t \in [-d,0]. \end{cases}$$

• **Solution at the jumps**

Suppose the solution has been constructed in the interval $[0,t_m)$. We define the solution at the time t_m, corresponding to the m-th jump. Consider the ordinary differential equations $y'_n(t) = J_n F(y_n)$, for $n \in \mathbf{N}$, $n \leq m$. We denote the solution flows of these equations by φ^n_t. For each $n \leq m$, take $z_n = \lim\limits_{s \to t_n-} \gamma(s)$. Now, let $\mathcal{B}_m = (\beta_n)_{n \leq m}$ be the family of curves (considered in Section 2) given by:

$$\beta_n(t) = \varphi^n_t(z_n)\big|_{[0,1]}, \tag{3.2}$$

and define $\gamma(t_m) = \beta_m(1)$.

• **Solution in the intervals between jumps**

In this case, define the solution using the parallel transport along γ, with respect to the family $\mathcal{B}_m$. So, for $t \in (t_m, t_{m+1})$, $\gamma(t)$ is the solution of the following delay differential equation:

$$\begin{cases} x'(t) = P^{\nabla,\mathcal{B}_m}_{t-d,t}(x) \, F(x(t-d)) \\ x(t) = \gamma(t), \ \forall t \in [-d,t_m]. \end{cases}$$

Note that, although the initial condition of the equation above may have jumps, results of the standard theory of delay differential equations on existence and uniqueness still hold. In fact, this initial condition is used only to parallel transport the vector field, as described in the previous section. Therefore, by induction, the solution is well defined for all $t \geq 0$. Uniqueness comes from the fact that the solution is unique in each step.

The generic fictitious curves at the jumps introduced in Section 2 have been established here as the deterministic flow of the vector field (without delay). We shall refer to this family of curves associated to γ by $\mathcal{B}_F = \{\mathcal{B}_n, n \in \mathbf{N}\}$. The solution is not intrinsic to the trajectory: it depends on the interpolating dynamical system in each jump. The impact on the jumps when one varies these dynamics can be described in the following way. Suppose that the vector field F is parametrized by an element $\lambda \in \Lambda$, an open set of an Euclidean space $\mathbf{R}^d$. If the map $F(\lambda, x) : \Lambda \times M \mapsto TM$ is continuous, then the jumps are also continuous with respect to λ, i.e. at time t_n, the map $\beta_n(0) \mapsto \beta_n(1)$ is continuous with respect to λ. Since $\beta_n(1) = \varphi^n_t(z_n)$, (equation (3.2)) then, naturally, it is also continuous with respect to the left limits z_n. Moreover, if $F(\lambda, x)$ is differentiable, then the jumps are also differentiable. More explicitly, suppose that M is endowed with a Riemannian structure where ∇ is the Levi-Civita connection. The infinitesimal dependence of the flow φ^n_t with respect to λ can be described by the linear map $\dfrac{\partial \varphi^n_t}{\partial \lambda} : \mathbf{R}^d \to T_{\beta(t)}M$, which satisfies the following non-homogeneous linear covariant equation along $\beta(t)$:

$$\frac{D}{\partial t}\left(\frac{\partial \varphi^n_t}{\partial \lambda}\right) = J_n \nabla F(\lambda, \beta(t))\left(\frac{\partial \varphi^n_t}{\partial \lambda}\right) + J_n \frac{\partial F(\lambda, \beta(t))}{\partial \lambda}. \tag{3.3}$$

where $\nabla F(\lambda, \beta(t)) : T_{\beta(t)} \to T_{\beta(t)}$ is the covariant derivative in M and $\frac{\partial F(\lambda, \beta(t))}{\partial \lambda} = \left(\frac{\partial F}{\lambda_1}, \ldots, \frac{\partial F}{\lambda_d} \right)$. This covariant formula can be easily deduced from classical results of dependence of solutions of ODE with respect to initial conditions and basic properties of Riemannian connections (see, e.g. Jost [14]).

Yet, this model reflects the observation that, in most physical situations, the informations arriving at a receptor come from different sources, with corresponding different delays. Here, continuous informations have a fix delay d, but discontinuities in the integrator have no delay.

3.2 Stochastic case

Let $(B_t^1, \ldots, B_t^m)$ be a Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and $(N_t)_{t \geq 0}$ be a random counting process that indicates the number of jumps up to time t, with the properties that $N_0 = 0$ and N_t is finite (almost surely) for all $t \geq 0$. Consider a sequence $(J_k)_{k \in \mathbf{N}}$ of random variables in $\mathbf{R}^{m+1}$. Taking $B_t^0 = t$, the integrator of our model is $L_t = (L_t^0, L_t^1, \ldots, L_t^m)$, given by:

$$L_t^i = B_t^i + \sum_{k=0}^{N_t} J_k^i.$$

An example of this kind of process is the Levy jump-diffusion (see e.g. Applebaum [1]), where N_t is a Poisson process and (J_k) are i.i.d. random variables.

Write the stochastic delay differential equation with jumps (SDDEJ) by:

$$dx_t = P_{t-d,t}^{\nabla} A(x_{t-d}) \diamond dL_t, \tag{3.4}$$

where $A = (A_0, A_1, \ldots, A_m)$ are smooth vector fields in M, considering the initial condition $\beta_0 : [-d, 0] \to M$, a differentiable curve in M.

We define the solution of this equation in an analogous way to the deterministic case. So, fixing $\omega \in \Omega$, in the intervals between the jumps, the solution is given by the corresponding Stratonovich stochastic delay differential equation, that is:

$$dx_t = \sum_{i=0}^m P_{t-d,t}^{\nabla} A^i(x_{t-d}) \circ dL_t^i,$$

with the appropriate initial condition, as we have done in the previous case. Besides that, at the times of jump, the solution hops instantaneously in the direction of the solution at time one of the following ODE (without delay):

$$\begin{cases} y'(t) = \sum_{k=0}^m J_n^k A^k(y) \\ y(0) = \lim_{s \to t_n-} \gamma(s), \end{cases}$$

So, with the same notation as before, we have the following well expected result for SDDEJ:

Theorem 3.1. *There exists a unique solution γ for the SDDEJ (3.4) in a differentiable manifold M endowed with an affine connection ∇, defined in $t \in [-d, \infty)$, with initial condition $\gamma(t) = \beta_0(t)$ when $t \in [-d, 0]$.*

Proof. The existence follows by an analogous construction we have done for the deterministic case. Uniqueness holds since in each step of the construction, the respective solution is unique, by theory of ODE and standard stochastic delay differential equations, as in [19]. $\qquad \square$

In this case, for each $\omega \in \Omega$, we have a family of differentiable curves established by the deterministic flow of the vector field. Again, we call this random family of jumps associated to γ by $\mathcal{B}_F(\omega) = \{\mathcal{B}_n(\omega), n \in \mathbf{N}\}$.

The idea of jumping in the direction of the deterministic flow at time one comes from Marcus SDE, where the integrator is a semimartingale with jumps (for more details see Kurtz, Pardoux e Protter [16]). The behaviour of jumps when changing the dynamics at the fictitious interval is the same as in equation (3.3).

4 Geometrical aspects of SDDEJ

In this section, we show that the parallel transport, i.e., the horizontal lift of a solution of an SDDEJ in a manifold (M, ∇) can be described as a solution of a SDDEJ in the linear frame bundle BM, with respect to a horizontal connection in BM (described below). The equation for this horizontal lift corresponds to an extension of the results on stochastic geometry started with Itô [13] and Dynkin [8] to our model of SDDEJ (see also [3]).

4.1 Horizontal lifts to the frame bundle

For reader's convenience we recall briefly some geometrical facts about the frame bundle of a manifold (for more details, see e.g., among many others, Elworthy [9], Kobayashi and Nomizu [15]). Let M be a differentiable manifold, with dimension n. The frame bundle BM of M is the set of all linear isomorphisms $p : R^n \to T_xM$ for $x \in M$. The projection $\pi : BM \to M$ maps p to the corresponding $x \in M$. BM is a principal bundle over M, with right action of the Lie group $GL(n, \mathbf{R})$, given by the composition with the linear isomorphisms.

Given p in the manifold BM, each tangent space T_pBM can be decomposed as a direct sum of the vertical and a horizontal subspace, $T_pBM = V_pBM \oplus H_pBM$. The vertical subspace is determined by $V_pBM = Ker(\pi_*(p))$, where π_* denotes the derivative of the projection π. The horizontal subspace H_pBM is established by the connection ∇ in M, namely it is generated by the derivative of parallel frames along curves in M passing at $\pi(p)$. In this context, one can consider the horizontal lift of a vector $v \in T_xM$ at $p \in \pi^{-1}(x)$ as the unique tangent vector $v^H \in H_pBM$ such that $\pi_*(p)(v^H) = v$.

We say that a differentiable curve $\alpha : I \to BM$ is horizontal when its derivative belongs to $H_{\alpha(t)}BM$ for all $t \in I$. In fact, given a differentiable curve $\beta : [0, T] \to M$, and $p \in \pi^{-1}(\beta(0))$, there exists a unique horizontal curve $\beta^H : [0, T] \to BM$, with the property that $\pi(\beta^H(t)) = \beta(t)$ for all t in the domain. The curve β^H is called the horizontal lift of the curve β (see e.g. [15]).

Putting together the technique of Section 2 and the horizontal lift described above, one can define the horizontal lift of a curve with jumps in M. Let $\gamma : [0, \infty) \to M$ be a càdlàg curve, $D = \{t_1, t_2, \ldots\}$ the countable, closed and discrete set of points of discontinuity, and $\mathcal{B} = (\beta_n)_{n \in \mathbf{N}}$ be a family of differentiable curves $\beta_n : [0, 1] \to M$, such that, for all $n \in \mathbf{N}$, $\beta_n(0) = \lim_{s \to t_n-} \gamma(s)$ and $\beta_n(1) = \gamma(t_n)$. Fix $p \in \pi^{-1}(0)$. Under these conditions, we define the horizontal lift of γ in p with respect to the family $\mathcal{B}$ by the càdlàg curve in BM given by:

$$\gamma_p^{H,\mathcal{B}}(t) := P_{0,t}^{\nabla,\mathcal{B}}(\gamma) \circ p.$$

Each element A in the Lie algebra $\mathcal{G}l(n, \mathbb{R})$ of the Lie group $GL(n, \mathbf{R})$ determines a vertical vector field in BM given by, at a point $p \in BM$,

$$A^*(p) = \frac{d}{dt}\left(p \cdot \exp At\right)|_{t=0}.$$

The map $\mathcal{G}l(n, \mathbb{R}) \mapsto V_pBM$ is surjective. In order to define a SDDEJ in BM, one needs a connection in this manifold as well. There are many ways of extending a connection ∇

of M to BM. In this section we are interested in the so called horizontal lift ∇^H which is defined (for a torsion free connection ∇, see e.g. Cordero et al. [6, Chap. 6]) as the unique connection in BM which satisfies:

$$
\begin{cases}
\nabla^H_{A^*} B^* & = & (AB)^* \\
\nabla^H_{A^*} X^H & = & 0 \\
\nabla^H_{X^H} A^* & = & 0 \\
\nabla^H_{X^H} Y^H & = & (\nabla_X Y)^H
\end{cases}
\tag{4.1}
$$

This extension has the property that the parallel transport commutes with the horizontal lift. Precisely, given a curve α in BM and a vector $v \in T_{\pi \circ \alpha(0)} M$, we have that $P^{\nabla^H}_{0,t}(\alpha)(v^H) = (P^{\nabla}_{0,t}(\pi \circ \alpha)(v))^H$, see [3, Lemma 2.1].

4.2 Main results

As in the previous section, initially we deal with deterministic systems. Catuogno and Ruffino [3, Prop. 2.1] used the extended connection ∇^H to prove that, if a curve γ is the solution of a deterministic DDE in M, then its horizontal lift is the solution of the corresponding DDE in BM. Now, we extend this result to the discontinuous case.

Let $F : M \to TM$ be a smooth vector field and $\beta_0 : [-d, 0] \to M$ a differentiable curve (initial condition). Let γ be the solution of the following delay differential equation with jumps in M:

$$
\begin{cases}
dx(t) = P^{\nabla}_{t-d,t}(x) \, F(x(t-d)) \, dS_t \\
x(t) = \beta_0(t) \text{ for } t \in [-d, 0].
\end{cases}
$$

This solution induces canonically the family $\mathcal{B}_F$ of curves along the deterministic flow at the jumps, as defined in Section 3.1. Fix $p \in \pi^{-1}(\gamma(-d))$. Consider $\beta_0^H : [-d, 0] \to BM$ the horizontal lift of β_0 at the point p, and let F^H be the horizontal lift of the vector field F. Under these conditions:

Theorem 4.1. *The horizontal lift $\gamma_p^{H, \mathcal{B}_F}$ is the solution of the following DDEJ in BM with respect to the connection ∇^H:*

$$
\begin{cases}
du(t) = P^{\nabla^H}_{t-d,t}(u) \, F^H(u(t-d)) \, dS_t \\
u(t) = \beta_0^H(t) \ \text{for } t \in [-d, 0].
\end{cases}
\tag{4.2}
$$

Proof. Let $(t_n)_{n \in \mathbf{N}}$ be the increasing sequence of discontinuities of the integrator S_t. Let $u : [0, \infty) \to BM$ be the solution of equation (4.2), whose existence and uniqueness is guaranteed in Section 3.1. We show, by induction on the number of jumps, that $u(t) = \gamma_p^{H, \mathcal{B}_F}(t)$ for all $t \geq 0$.

For $t \in [0, t_1)$, the solution $u(t)$ is given by a delay differential equation in the continuous case, so we can apply directly the result in [3, Prop. 2.1] to obtain the equality. In particular, define $p_1 := \lim_{s \to t_1^-} u(s) = \lim_{s \to t_1^-} \gamma_p^{H, \mathcal{B}_F}(s)$.

At time t_1, when the first jump occurs, take the curve $\beta_1 \in \mathcal{B}_F$ (we recall that this is the solution of the ODE $y'(t) = J_1 F(y)$, with initial condition $y(0) = \pi(p_1)$). Consider its horizontal lift β_1^H at p_1. As β_1^H is the solution of $z'(t) = J_1 F^H(z)$, with initial condition $z(0) = p_1$, we have that $u(t_1) = \gamma_p^{H, \mathcal{B}_F}(t_1)$.

Now, arguing by induction, suppose that $u(t) = \gamma_p^{H, \mathcal{B}_F}(t)$ for all $t \in [-d, t_m]$ for $m \geq 1$. We claim that this equality also holds in the interval $(t_m, t_{m+1}]$. In fact, let k be the number of jumps that occur in the interval $(t_m - d, t_m)$.

Firstly, for the case $(t_{m+1} - t_m) > d$, we have:

$$
t_m < t_{m-k} + d < \ldots < t_{m-1} + d < t_m + d < t_{m+1},
$$

and it is enough to analyse each of these $(k+2)$ subintervals. For the first subinterval, that is, $(t_m, t_{m-k}+d)$, we consider a DDE with delay $(d+k+1)$, where the initial condition is the concatenation of the following curves:

- u_t, in the interval $[t_m - d, t_{m-k})$;
- β_{m-k}^H, in the interval $[0,1]$;
- u_t, in the interval $[t_{m-k}, t_{m-k+1})$;
- β_{m-k+1}^H, in the interval $[0,1]$;

  $\vdots$

- u_t, in the interval $[t_{m-1}, t_m)$;
- β_m^H, in the interval $[0,1]$.

Therefore, for $t \in (t_m, t_{m-k}+d)$, we have a delay differential equation, with a fictitious bigger delay, but without jumps. Applying the result in [3, Prop. 2.1] the equality in this subinterval holds. For the n-th subinterval, with $n < (k+2)$, take the respective DDE (without jumps), where the initial condition concatenates appropriately u_t and β_i^H, with $i \in \{m-k+n-1, \ldots, m\}$ and use [3, Prop. 2.1]. And for the last subinterval, that is, for $t \in (t_m + d, t_{m+1})$, consider the DDE with delay d and initial condition given by u_t, and again we have the equality.

Secondly, for the case $(t_{m+1} - t_m) \leq d$ the argument is essentially the same, we just have to consider a smaller number of subintervals. Therefore, we have that $u(t) = \gamma_p^{H, \mathcal{B}_F}(t)$ for $t \in (t_m, t_{m+1})$. In particular, define:

$$p_{m+1} := \lim_{s \to t_{m+1}^-} u(s) = \lim_{s \to t_{m+1}^-} \gamma_p^{H, \mathcal{B}_F}(s).$$

Finally, consider β_{m+1}^H, the horizontal lift of the differentiable curve β_{m+1} at the point p_{m+1}. The equality at t_{m+1} also holds: $u(t_{m+1}) = \gamma_p^{H, \mathcal{B}_F}(t_{m+1})$ and, by induction, the proof is complete. $\qquad\square$

The same result holds in the stochastic case due to the transfer principle (see, e.g. Emery [10]). In this sense, with the same notation as before, let α be the solution of the SDDEJ

$$\begin{cases} dx(t) = P_{t-d,t}^{\nabla}(x) \, F(x(t-d)) \diamond dL_t \\ x(t) = \beta_0(t) \ \text{for } t \in [-d, 0]. \end{cases}$$

This solution induces the random family of curves $\mathcal{B}_F(\omega)$ along the deterministic flow at the jumps, as defined in Section 3.2. Hence, we have:

Theorem 4.2. *The horizontal lift $\alpha_p^{H, \mathcal{B}_F(\omega)}$ of α (solution of the SDDEJ in the manifold M) is solution of the following SDDEJ in BM, with respect to the connection ∇^H:*

$$\begin{cases} du(t) = P_{t-d,t}^{\nabla^H}(u) \, F^H(u(t-d)) \diamond dL_t \\ u(t) = \beta_0^H(t) \ \text{for } t \in [-d, 0]. \end{cases} \tag{4.3}$$

Proof. In the proof of the deterministic case (Theorem 4.1), in each step, the solution is given by a standard delay differential equation (without jumps). The corresponding lift is the solution of the lifted equation in BM. So, applying successively the transfer principle: before the first jump, at the jumps and in the intervals between jumps, we have the result (cf. [3, Thm. 2.2]). $\qquad\square$

This geometrical approach opens the possibility of exploring other aspects of this model of SDDEJ, e.g. holonomies, invariant measures, Lyapunov exponents, rotation numbers, positivity of the distribution, among others.

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