Electron. Commun. Probab. **21** (2016), no. 43, 1–18.
DOI: 10.1214/16-ECP4359
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Joint asymptotic distribution of certain path functionals of the reflected process

Aleksandar Mijatović* Martijn Pistorius†

Abstract

Let $\tau(x)$ be the first time that the reflected process Y of a Lévy process X crosses $x > 0$. The main aim of this paper is to investigate the joint asymptotic distribution of $Y(t) = X(t) - \inf_{0\le s\le t} X(s)$ and the path functionals $Z(x) = Y(\tau(x)) - x$ and $m(t) = \sup_{0\le s\le t} Y(s) - y^*(t)$, for a certain non-linear curve $y^*(t)$. We restrict to Lévy processes X satisfying Cramér's condition, a non-lattice condition and the moment conditions that $E[|X(1)|]$ and $E[\exp(\gamma X(1))|X(1)|]$ are finite (where γ denotes the Cramér coefficient). We prove that $Y(t)$ and $Z(x)$ are asymptotically independent as $\min\{t, x\} \to \infty$ and characterise the law of the limit (Y_∞, Z_∞). Moreover, if $y^*(t) = \gamma^{-1}\log(t)$ and $\min\{t, x\} \to \infty$ in such a way that $t\exp\{-\gamma x\} \to 0$, then we show that $Y(t)$, $Z(x)$ and $m(t)$ are asymptotically independent and derive the explicit form of the joint weak limit $(Y_\infty, Z_\infty, m_\infty)$. The proof is based on excursion theory, Theorem 1 in [7] and our characterisation of the law (Y_∞, Z_∞).

Keywords: reflected Lévy process; asymptotic independence; limiting overshoot; Cramér condition.
AMS MSC 2010: 60G51; 60F05; 60G17.
Submitted to ECP on June 12, 2015, final version accepted on May 9, 2016.
Supersedes arXiv:1306.6746.

1 Introduction and main results

The reflected process Y of a Lévy process X is a strong Markov process on $\mathbb{R}_+$ equal to X reflected at its running infimum. The reflected process is of great importance in many areas of probability, ranging from the fluctuation theory for Lévy processes (e.g. [2, Chapter VI] and the references therein) to mathematical statistics (e.g. [19, 22], CUSUM method of cumulative sum), queueing theory (e.g. [1, 20]), mathematical finance (e.g. [11, 17], drawdown as risk measure), mathematical genetics (e.g. [14] and references therein) and many more. The aim of this paper is to study the weak limiting behaviour of the reflected process $Y = (Y(t))_{t\ge 0}$, $Y(t) = X(t) - \inf_{0\le s\le t} X(s)$, and the overshoot $Z(x)$ and the centered running maximum $m(t)$ of Y given by

$$Z(x) \doteq Y(\tau(x)) - x, \qquad m(t) \doteq Y^*(t) - y^*(t), \qquad t, x \in \mathbb{R}_+, \tag{1.1}$$

where y^* is a specific non-linear curve to be specified shortly. Here $\tau(x)$ and $Y^*(t)$ denote the first entry time of Y into the interval (x, ∞), which is finite almost surely, and the

*Department of Mathematics, King's College London E-mail: aleksandar.mijatovic@kcl.ac.uk
†Department of Mathematics, Imperial College London E-mail: m.pistorius@imperial.ac.uk

supremum up to time t of the reflected process respectively,

$$\tau(x) \doteq \inf\{t \geq 0 : Y(t) > x\} \quad (\inf \emptyset \doteq \infty), \qquad Y^*(t) \doteq \sup_{0 \leq s \leq t} Y(s).$$

In this paper we restrict to Lévy processes satisfying the conditions:

Assumption 1.1. *(i) Cramér's condition, $E[e^{\gamma X(1)}] = 1$ for $\gamma > 0$, holds, (ii) $E[|X(1)|] + E[e^{\gamma X(1)}|X(1)|] < \infty$ and (iii) either the Lévy measure of X is non-lattice or 0 is regular for $(0, \infty)$.*

Under Cramér's condition, it is well-known that X tends to $-\infty$ almost surely which yields by a classical time reversal argument that the reflected process Y has a weak limit Y_∞ equal in distribution to the ultimate supremum $\sup_{t\geq 0} X(t)$ (see e.g. [1, Chapter IX] or Section 2 below). The second functional, the overshoot $Z(x)$, also admits a weak limit Z_∞ (the form of which is given in Proposition 3.1 below). The following result addresses the question of the weak asymptotics of the vector $(Y(t), Z(x))$.

Theorem 1.2. *$Y(t)$ and $Z(x)$ are asymptotically independent, as $\min\{t, x\} \to \infty$, in the sense that*

$$\begin{aligned}\lim_{\min\{x,t\}\to\infty} E[\exp(-uY(t) - vZ(x))] &= E[\exp(-uY_\infty)]E[\exp(-vZ_\infty)] \\ &= \lim_{\min\{x,t\}\to\infty} E[\exp(-uY(t))]E[\exp(-vZ(x))],\end{aligned}$$

for $u, v \in \mathbb{R}_+$. $(Y(t), Z(x))$ converges weakly to the law (Y_∞, Z_∞) determined by the Laplace transform

$$E[\exp(-uY_\infty - vZ_\infty)] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)}, \qquad \text{for all} \quad u, v \in \mathbb{R}_+, \tag{1.2}$$

where ϕ is the Laplace exponent of the ascending ladder-height subordinator of X which satisfies $\phi(0) > 0$. In particular, the law of the sum $Y_\infty + Z_\infty$ is exponential with mean $1/\gamma$.

We turn next to the weak asymptotics of the triplet $(Y(t),\ Z(x),\ m(t))$. To avoid degeneracies we specify the centering curve to be given by

$$y^*(t) = \gamma^{-1} \log(t), \qquad t \in \mathbb{R}_+ \backslash \{0\}. \tag{1.3}$$

This choice is informed by Iglehart [12], where in the analogous random walk setting $x(n) = \gamma^{-1} \log n$ was chosen as centering sequence, and by the main result in Doney & Maller [7], which implies that the running maximum $m(t)$ of Y after centering by the curve $y^*(t)$ given in (1.3) converges weakly to a Gumbel distribution (see [8, Chapter 3] for the form of the Gumbel distribution and Section 2.2 below for a simple derivation of the distribution of m_∞ deploying [7, Theorem 1]). A question of interest is if and when the asymptotic independence of $Y(t)$ and $Z(x)$ extends to that of the triplet $Y(t)$, $Z(x)$ and $m(t)$. A priori, it appears unlikely that $Z(x)$ and $m(t)$ are asymptotically independent in general, for x and t tending to infinity in an arbitrary way. In the next result we give a sufficient condition for such asymptotic independence to hold, namely that $\min\{x, t\} \to \infty$ such that

$$x - y^*(t) \to \infty, \qquad \text{or equivalently } t\exp\{-\gamma x\} \to 0. \tag{1.4}$$

Since, by [7], the process Y^* has weakly convergent random fluctuations around the deterministic curve y^*, the assumption $x - y^*(t) \to \infty$ in effect forces the process Y to reach the level x for the first time after time t. The result is as follows.

Theorem 1.3. *Let* $\min\{t,x\} \to \infty$ *such that* $t\exp\{-\gamma x\} \to 0$*. Then* $(Y(t), Z(x), m(t))$ *converges weakly and the law of the weak limit* $(Y_\infty, Z_\infty, m_\infty)$ *is determined by the Fourier-Laplace transform*

$$\begin{aligned} &E\left[\exp\left(-uY_\infty - vZ_\infty + \mathrm{i}\beta m_\infty\right)\right] \\ &\quad = \frac{\gamma}{\gamma+v}\cdot\frac{\phi(v)}{\phi(u)}\cdot\Gamma\left(1-\frac{\mathrm{i}\beta}{\gamma}\right)\cdot\exp\left[\mathrm{i}\beta\gamma^{-1}\log\left(\ell C_\gamma\widehat{\phi}(\gamma)\right)\right] \end{aligned} \tag{1.5}$$

for all $u, v \in \mathbb{R}_+$*,* $\beta \in \mathbb{R}$*, where* $\widehat{\phi}$ *is the Laplace exponent of the decreasing ladder-height process,* $\widehat{L}^{-1}$ *is the decreasing ladder-time processes with* $\ell \doteq 1/E[\widehat{L}^{-1}(1)]$ *(see Section 2 for the definitions of* $\widehat{\phi}$ *and* $\widehat{L}^{-1}$*),* $\Gamma(\cdot)$ *denotes the gamma function and the constant* C_γ *is given by*

$$C_\gamma \doteq \frac{\phi(0)}{\gamma\phi'(-\gamma)}, \tag{1.6}$$

where $\phi'(-\gamma) \in \mathbb{R}_+\backslash\{0\}$*. In particular,* $Y(t)$*,* $Z(x)$ *and* $m(t)$ *are asymptotically independent: for any* $a, b \in \mathbb{R}_+$ *and* $c \in \mathbb{R}$

$$P(Y(t) \le a, Z(x) \le b, m(t) \le c) = P(Y(t) \le a)P(Z(x) \le b)P(m(t) \le c) + o(1).^1$$

The remainder of the paper is devoted to the proofs of Proposition 3.1 (in which the form of the law of the asymptotic overshoot Z_∞ is identified) and Theorems 1.2 and 1.3. Section 2 is concerned with preliminary results, Proposition 3.1 is established in Section 3 and the proof of the asymptotic independence is given in Section 4. The proofs of Theorems 1.2 and 1.3 draw on these results and are presented in Section 5.

2 Preliminaries

In this section we briefly define the setting and collect results that are deployed throughout. We refer to [2, Chapter I] and [16] for background on the fluctuation theory of Lévy processes.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\ge0}, P)$ be a filtered probability space that carries a Lévy process X satisfying Assumption 1.1. Here $\Omega \doteq D(\mathbb{R}_+, \mathbb{R})$ is the Skorokhod space of real-valued functions that are right-continuous on $\mathbb{R}_+$ and have left-limits on $\mathbb{R}_+\backslash\{0\}$, X is the coordinate process, $\{\mathcal{F}(t)\}_{t\ge0}$ denotes the completed filtration generated by X, which is right-continuous, and $\mathcal{F}$ is the completed σ-algebra generated by $\{X(t)\}_{t\ge0}$. For any $x \in \mathbb{R}$ denote by P_x the probability measure on $(\Omega, \mathcal{F})$ corresponding to the Lévy process X shifted by x and let $P_x^{(\gamma)}$ (with $P^{(\gamma)} \doteq P_0^{(\gamma)}$) be the Cramér measure on $(\Omega, \mathcal{F})$, that is, the unique measure such that its restriction to $\mathcal{F}(t)$ is given by

$$P_x^{(\gamma)}(A) \doteq E_x[\mathrm{e}^{\gamma(X(t)-x)}\mathbf{I}_A], \qquad A \in \mathcal{F}(t), \qquad t \in \mathbb{R}_+,$$

where E_x is the expectation under P_x and $\mathbf{I}_A$ is the indicator of A. Under Assumption 1.1, $P_x^{(\gamma)}$ is a probability measure with $P_x^{(\gamma)}(X(0) = x) = 1$ and the convexity of the Laplace exponent $\theta \mapsto \log E_x[\exp(\theta(X(1)-x))]$ on $[0,\gamma]$ implies

$$E_x[X_1 - x] \in (-\infty, 0), \quad E_x^{(\gamma)}[X(1) - x] \in (0, \infty). \tag{2.1}$$

As $E[X(1)]$ is strictly negative and finite, $X^*(t) \doteq \sup_{0\le s\le t} X(s)$ converges almost surely as $t \uparrow \infty$ to $X^*_\infty \doteq \sup_{t\ge0} X(t)$, which is finite almost surely. Moreover, since $X^*(t)$ and $Y(t)$ have the same distribution for any $t > 0$ (by the duality lemma for Lévy processes—see [2]), it follows that $Y(t)$ converges in distribution to a limit Y_∞ that has the same

[1] Here we use the definiton $f(t,x) = o(1)$ if $\lim_{\min\{t,x-y*(t)\}\to\infty} f(t,x) = 0$.

distribution as X^*_∞. The distribution of the latter can be expressed explicitly in terms of that of the ladder-height process H of X, as we recall below.

Let L be a local time[2] at zero of the reflected process $\widehat{Y} = \{\widehat{Y}(t)\}_{t\geq 0}$ of the dual process $\widehat{X} \doteq -X$, that is, $\widehat{Y}(t) \doteq X^*(t) - X(t)$. The ladder-time process $L^{-1} = \{L^{-1}(t)\}_{t\geq 0}$ is equal to the right-continuous inverse of L. Analogously, let $\widehat{L}$ be a local time of Y at zero, with inverse denoted by $\widehat{L}^{-1}$. Denote by $\kappa(q) \doteq -\log E[\exp\{-qL^{-1}(1)\}\mathbf{I}_{\{L^{-1}(1)<\infty\}}]$ and $\widehat{\kappa}(q) \doteq -\log E[\exp\{-q\widehat{L}^{-1}(1)\}\mathbf{I}_{\{\widehat{L}^{-1}(1)<\infty\}}]$ the Laplace exponents of L^{-1} and $\widehat{L}^{-1}$. For later reference we record that the mean of $\widehat{L}^{-1}(1)$ is finite.

Lemma 2.1. *We have $E^{(\gamma)}[L^{-1}(1)] \in \mathbb{R}_+\backslash\{0\}$ and*

$$1/\ell \doteq E[\widehat{L}^{-1}(1)] = 1/\kappa(0) \in \mathbb{R}_+\backslash\{0\}. \tag{2.2}$$

Proof. From the Wiener-Hopf factorisation of X (see e.g. [16, p. 166]), we have that for some $k \in \mathbb{R}_+\backslash\{0\}$

$$q = k\,\kappa(q)\,\widehat{\kappa}(q), \qquad q \in \mathbb{R}_+\backslash\{0\}. \tag{2.3}$$

Since $X_t \to -\infty$ a.s. under P, $\widehat{Y}$ is transient while Y is recurrent, so that we have $P(L^{-1}(1) < \infty) < 1$ and $P(\widehat{L}^{-1}(1) < \infty) = 1$ and $\kappa(0) > 0 = \widehat{\kappa}(0)$. Differentiating (2.3) at $q \in \mathbb{R}_+\backslash\{0\}$ and letting $q \searrow 0$ yields $1 = \kappa(0)\widehat{\kappa}'(0) \Leftrightarrow E[\widehat{L}_1^{-1}] = 1/\kappa(0)$. By a similar argument it follows that $E^{(\gamma)}[L_1^{-1}] = 1/\widehat{\kappa}^{(\gamma)}(0) < \infty$, where $\widehat{\kappa}^{(\gamma)}$ denotes the Laplace exponent of L^{-1} under $P^{(\gamma)}$. □

By the strong law of large numbers and the fact that $\widehat{L}^{-1}$ is a Lévy process (see e.g. [2, p.92]) we have that

$$\frac{\widehat{L}(t)}{t} \sim \frac{t}{\widehat{L}^{-1}(t)} \sim \ell \quad \text{a.s. as } t \to \infty, \tag{2.4}$$

where we denote $f(x) \sim g(x)$ as $x \uparrow \infty$ if the functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+\backslash\{0\}$ satisfying $\lim_{x\uparrow\infty} f(x)/g(x) = 1$.

The ladder-height process $H = \{H(t)\}_{t\geq 0}$ is given by $H(t) \doteq X(L^{-1}(t))$ for all $t \geq 0$ with $L^{-1}(t)$ finite and by $H(t) \doteq +\infty$ otherwise. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be the Laplace exponent of H, with

$$\phi(\theta) \doteq -\log E[\mathrm{e}^{-\theta H(1)}\mathbf{I}_{\{H(1)<\infty\}}], \quad \theta \in \mathbb{R}_+.$$

The Lévy-Khintchine formula for ϕ and an integration-by-parts imply

$$\begin{aligned} \phi(v) &= \phi(0) + \int_{\mathbb{R}_+\backslash\{0\}} (1 - \mathrm{e}^{-vx})\nu_H(\mathrm{d}x) \\ &= \phi(0) + v\left(m + \int_0^\infty \mathrm{e}^{-vx}\overline{\nu}_H(x)\,\mathrm{d}x\right), \quad v \in \mathbb{R}_+, \end{aligned} \tag{2.5}$$

where $\overline{\nu}_H(x) \doteq \nu_H((x,\infty))$ for $x \in \mathbb{R}_+\backslash\{0\}$ is the tail-function of ν_H, the Lévy measure of H, and

$$m \doteq \lim_{u\to\infty} \phi(u)/u$$

denotes the drift of the ladder-height process H. Note that

$$\phi(0) = \kappa(0) \in \mathbb{R}_+\backslash\{0\}. \tag{2.6}$$

The decreasing ladder-height process $\widehat{H}$ is defined similarly, and its Laplace exponent is denoted by $\widehat{\phi}$. By analytical continuation, continuity and Assumption 1.1(i), the domains

[2] In the case 0 is not regular for $[0,\infty)$, only a finite number of maxima of X are attained in any compact time interval. In this case we work with the right-continuous version of local time L.

of definition of $\phi(\theta)$, $\widehat{\phi}(\theta)$ and the characteristic exponent $\Psi(\theta) = -\log E[\exp\{\mathbf{i}\theta X(1)\}]$, $\theta \in \mathbb{R}$, can be extended to $\{\theta \in \mathbb{C} : \Re(\theta) \in (-\gamma, \infty)\}$, $\{\theta \in \mathbb{C} : \Re(\theta) \in [0, \infty)\}$ and $\{\theta \in \mathbb{C} : \Im(\theta) \in (-\gamma, 0]\}$, respectively. Denoting this extensions again by ϕ, $\widehat{\phi}$ and Ψ, the Wiener-Hopf factorisation of X [2, p. 166] implies that the following holds for some $k' \in \mathbb{R}_+ \backslash \{0\}$:

$$\Psi(-\mathbf{i}\theta) = k'\phi(-\theta)\widehat{\phi}(\theta), \qquad \theta \in \mathbb{C},\ \Re(\theta) = 0. \tag{2.7}$$

By uniqueness of analytical extension, the validity of (2.7) extends to all $\theta \in \mathbb{C}$ with $\Re(\theta) \in (-\gamma, 0]$. From (2.7) we have in particular that the law of X^*_∞ and hence of Y_∞ is characterised (see [2, p. 163]) by its Laplace transform

$$E[\mathrm{e}^{-uY_\infty}] = E[\mathrm{e}^{-uX^*_\infty}] = \frac{\phi(0)}{\phi(u)}, \quad u \in \mathbb{R}_+. \tag{2.8}$$

Furthermore, denoting by $K(x) \doteq X(T(x)) - x$ the overshoot of X at $T(x)$ on the event $\{T(x) < \infty\}$, the second Wiener-Hopf factorisation of X (see e.g. [2, p.183]) implies

$$\int_0^\infty q\mathrm{e}^{-qx} E\left[\mathrm{e}^{-uK(x)}\right] \mathrm{d}x = \frac{q}{\phi(q)} \cdot \frac{\phi(q) - \phi(u)}{q - u}, \qquad q, u > 0, \tag{2.9}$$

As the identity (2.7) holds for any $\theta \in \mathbb{C}$ with $\Re(\theta) \in (-\gamma, 0]$, we may define $\phi(-\gamma) \doteq \lim_{\theta \searrow -\gamma} \Psi(-\mathbf{i}\theta)/k'\widehat{\phi}(\theta)$. The values of $\phi(-\gamma)$ and the right-derivative $\phi'(-\gamma)$ of ϕ at $x = -\gamma$ are given as follows:

Lemma 2.2. *The function ϕ is right-differentiable at $-\gamma$ and we have*

$$\text{(i) } \phi(-\gamma) = 0, \qquad \text{(ii) } \phi'(-\gamma) = E^{(\gamma)}[H(1)] \in \mathbb{R}_+ \backslash \{0\}. \tag{2.10}$$

Furthermore, the Laplace exponent $\phi^{(\gamma)}$ of H under $P^{(\gamma)}$ satisfies $\phi^{(\gamma)}(\gamma + u) = \phi(u)$ for any $u \geq -\gamma$ and

$$\int_0^\infty \mathrm{e}^{\gamma y} \overline{\nu}_H(y) \mathrm{d}y = \frac{\phi(0)}{\gamma} - m. \tag{2.11}$$

Proof. **(i)** As $\widehat{H}$ is a non-zero subordinator we have $\widehat{\phi}(\gamma) > 0$ and hence $\phi(-\gamma) = 0$ from (2.7) with $\theta = \gamma$.

(ii) The concavity of ϕ, part (i) and (2.6) imply that the right-derivative $\phi'(-\gamma)$ is strictly positive, and equal to $\phi'(-\gamma) = E[\mathrm{e}^{\gamma H(1)} H(1)] = E^{(\gamma)}[H(1)]$. We next show that $\phi'(-\gamma)$ is finite. As $\widehat{\phi}(\gamma)$ and $\widehat{\phi}'(\gamma)$ are strictly positive and finite, (left-)differentiation of (2.7) at $\theta = \gamma$ yields, by deploying by part (i) and the fact $E[\mathrm{e}^{\gamma X(1)}] = 1$,

$$-E[X(1)\mathrm{e}^{\gamma X(1)}] = k\phi(-\gamma)\widehat{\phi}'(\gamma) - k\phi'(-\gamma)\widehat{\phi}(\gamma) = -k\phi'(-\gamma)\widehat{\phi}(\gamma). \tag{2.12}$$

As the left-hand side of (2.12) is finite (by Assumption 1.1(ii)) it follows that $\phi'(-\gamma) < \infty$. By a change-of-measure argument (see [16, Corollary 3.10]) and (2.10)(i) it follows that $\phi^{(\gamma)}(u) = \phi(u + \gamma) - \phi(-\gamma) = \phi(u + \gamma)$ for $u \geq 0$, while part (i) and (2.5) yield (2.11). □

2.1 Asymptotic first-passage probabilities and overshoot distributions

We next turn to asymptotics of first-passage probabilities and the distribution of overshoots for large starting values of X. Let $T(x)$ and $\widehat{T}(x)$ denote the first-passage times of X into the intervals (x, ∞) and $(-\infty, -x)$ respectively for any $x \in \mathbb{R}_+$,

$$T(x) \doteq \inf\{t \geq 0 : X(t) \in (x, \infty)\}, \qquad \widehat{T}(x) \doteq \inf\{t \geq 0 : X(t) \in (-\infty, -x)\}. \tag{2.13}$$

It is shown in [3] that, under Assumption 1.1, Cramér's estimate, which was first-established for random walks, remains valid for the Lévy process X (with C_γ defined in (1.6)):

$$P(T(y) < \infty) \sim C_\gamma \mathrm{e}^{-\gamma y} \quad \text{as } y \to \infty. \tag{2.14}$$

From (2.14) the following asymptotic results can be derived:

Proposition 2.3. **(i)** *(Asymptotic two-sided exit probability) For any $z > 0$ we have*

$$P(T(x) < \widehat{T}(z)) \sim C_\gamma \mathrm{e}^{-\gamma x} \left(1 - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]\right) \qquad \textit{as} \quad x \to \infty, \tag{2.15}$$

where C_γ is given in (1.6)*.*
(ii) *(Asymptotic overshoot) Let $u \in \mathbb{R}_+$ and fix $z > 0$. Then*

$$E^{(\gamma)}[\mathrm{e}^{-uK(\infty)}] = \phi(u-\gamma)/(u\phi'(-\gamma))$$

and we have

$$\begin{aligned} E\left[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(z)\}}\right] &\sim C(u)\mathrm{e}^{-\gamma x}\left(1 - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]\right), \quad \textit{as } x \to \infty, \\ \textit{with } C(u) &\doteq \frac{\gamma}{\gamma+u}\cdot\frac{\phi(u)}{\phi(0)}\cdot C_\gamma. \end{aligned} \tag{2.16}$$

Proof. **(i)** By the strong Markov property and spatial homogeneity of X it follows from (2.14) that

$$\begin{aligned} P(T(x) < \widehat{T}(z)) = P(T(x) < \infty) \\ - \int_{(-\infty,-z]} P_y(T(x) < \infty) P(X(\widehat{T}(z)) \in \mathrm{d}y, \widehat{T}(z) < T(x)). \end{aligned} \tag{2.17}$$

The translation invariance of X and Cramér's estimate imply the following equality

$$P_y(T(x) < \infty) \quad = \quad C_\gamma \mathrm{e}^{-\gamma x}\mathrm{e}^{\gamma y}\left(1 + r(x-y)\right) \quad \text{for all} \quad x > y, \tag{2.18}$$

where $\lim_{x'\to\infty} r(x') = 0$. Equality (2.18) applied to the identity in (2.17) yields

$$\begin{aligned} C_\gamma^{-1}\mathrm{e}^{\gamma x} P(T(x) < \widehat{T}(z)) &= 1 - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right] \\ &+ r(x) - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))} r(x - X(\widehat{T}(z)))\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right]. \end{aligned} \tag{2.19}$$

Since $X(\widehat{T}(z)) \leq -z < 0$ on the event $\{\widehat{T}(z) < \infty\}$, which satisfies $P(\widehat{T}(z) < \infty) = 1$ by Assumption 1.1, the dominated convergence theorem implies $E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right] = E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right] + o(1)$ as $x \to \infty$. An application of the dominated convergence theorem to the second expectation on the right-hand side of equality (2.19), together with the fact that r vanishes in the limit as $x \to \infty$, proves the first statement in the proposition.
(ii) Since by Assumption 1.1 and Lemma 2.2 H is a non-lattice subordinator with $E^{(\gamma)}[H(1)] \in \mathbb{R}_+\backslash\{0\}$ and the overshoot $K(x)$ is equal to that of H over x, [4, Theorem 1] implies that the weak limit $K(x) \xrightarrow{\mathcal{D}} K_\infty$, as $x \to \infty$, exists under $P^{(\gamma)}$. Hence the continuity theorem [5, p. 16, Theorem 2.1] implies $\lim_{x\uparrow\infty} E^{(\gamma)}[\mathrm{e}^{-uK(x)}] = E^{(\gamma)}[\mathrm{e}^{-uK_\infty}]$ for any fixed $u \geq 0$. Combining this limit, which is bounded, with the second Wiener-Hopf factorisation (2.9) under the measure $P^{(\gamma)}$, which reads as

$$\int_0^\infty q\mathrm{e}^{-qx} E^{(\gamma)}\left[\mathrm{e}^{-uK(x)}\right]\,\mathrm{d}x = \frac{q}{\phi(q-\gamma)}\cdot\frac{\phi(q-\gamma)-\phi(u-\gamma)}{q-u}, \qquad q, u > 0,$$

we have in the limit as $q \downarrow 0$ that $E^{(\gamma)}[\mathrm{e}^{-uK_\infty}] = \phi(u-\gamma)/(u\phi'(-\gamma))$. Thus, changing measure from P to $P^{(\gamma)}$ leads to the following for any $u \geq 0$ ($C(u)$ is defined in (2.16)):

$$E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\infty\}}] = \mathrm{e}^{-\gamma x}\cdot E^{(\gamma)}[\mathrm{e}^{-(\gamma+u)K(x)}] \sim C(u)\mathrm{e}^{-\gamma x} \quad \text{as } x \to \infty. \tag{2.20}$$

Furthermore, since the expectation in (2.20) is bounded as $x \to \infty$, there exists a bounded function $R : \mathbb{R}_+ \to \mathbb{R}$, such that $E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\infty\}}] = C(u)\mathrm{e}^{-\gamma x}(1 + R(x))$ for $x > 0$, and $\lim_{x\to\infty} R(x) = 0$. The strong Markov property at $\widehat{T}(z)$ and an argument analogous to the one used in the proof of part (i) (cf. (2.19)) yields

$$\begin{aligned} &C(u)^{-1}\mathrm{e}^{\gamma x}E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(z)\}}] \\ &= \; 1 - E[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}] + R(x) - E[\mathrm{e}^{\gamma X(\widehat{T}(z))}R(x - X(\widehat{T}(z)))\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}], \end{aligned}$$

which implies equivalence (2.16). □

2.2 Asymptotic distribution of $m(t)$

To establish the existence and forms of the asymptotic distribution of $m(t)$ as t tend to infinity we draw on excursion theory. We refer to [2, Chapters O.5, IV], [16, Chapter 6] and [6, 9] for treatments of excursion theory.

Let $\epsilon = \{\epsilon_t\}_{t\geq 0}$ denote the excursion process of Y away from zero, with $\epsilon_t \in \mathcal{E} = \{\varepsilon \in \Omega : \varepsilon \geq 0\} \cup \{\partial\}$, where ∂ denotes an isolated state and where ϵ_t we recall is given by

$$\epsilon_t \doteq \begin{cases} \begin{Bmatrix} \begin{cases} Y\left(s + \widehat{L}^{-1}(t-)\right), & s \in [0, \widehat{L}^{-1}(t) - \widehat{L}^{-1}(t-)) \\ 0, & s \geq \widehat{L}^{-1}(t) - \widehat{L}^{-1}(t-) \end{cases} \end{Bmatrix}, & \text{if } \widehat{L}^{-1}(t-) < \widehat{L}^{-1}(t), \\ \partial, & \text{otherwise,} \end{cases} \tag{2.21}$$

with $\widehat{L}^{-1}(0-) \doteq 0$. Since Y is a recurrent strong Markov process under P, Itô's characterisation [13] implies that ϵ is a Poisson point process under P, with intensity (or excursion) measure n defined on $(\mathcal{E}, \mathcal{G})$, where $\mathcal{G}$ is the Borel sigma-algebra on the Polish space $\mathcal{E}$. Under $P^{(\gamma)}$, Y is transient (as $E^{(\gamma)}[X(1)] > 0$), so that $\widehat{L}_\infty \doteq \lim_{t\in\infty} \widehat{L}(t)$ is finite almost surely and furthermore $\widehat{L}_\infty$ is an exponential random variable independent of the killed Lévy process $\{(\widehat{L}^{-1}(t), \widehat{H}(t))\}_{t\in[0,\widehat{L}_\infty)}$. The process $\epsilon' = \{\epsilon'_t\}_{t\geq 0}$, defined by $\epsilon'_t \doteq \epsilon_t$ for $t < \widehat{L}_\infty$ and by $\epsilon'_t \doteq \partial$ otherwise, is under $P^{(\gamma)}$ a Poisson point process killed at an independent exponential random time with mean $E^{(\gamma)}[\widehat{L}_\infty]$; we denote by $n^{(\gamma)}$ its intensity measure.

For an excursion $\varepsilon \in \mathcal{E}$ and $x \in \mathbb{R}_+\backslash\{0\}$ let $\rho(x, \varepsilon)$ and $\zeta(\varepsilon)$ be the first time that ε enters the interval (x, ∞) and the lifetime of ε respectively:

$$\rho(x, \varepsilon) \doteq \inf\{s \in \mathbb{R}_+ : \varepsilon(s) > x\}, \qquad \zeta(\varepsilon) \doteq \inf\{t \in \mathbb{R}_+\backslash\{0\} : \varepsilon(t) = 0\}. \tag{2.22}$$

For brevity we sometimes write $\rho(x)$ and ζ instead of $\rho(x, \varepsilon)$ and $\zeta(\varepsilon)$. Note that $\zeta(\epsilon(t))$ is given in terms of $\widehat{L}^{-1}$ by $\zeta(\epsilon(t)) = \widehat{L}^{-1}(t) - \widehat{L}^{-1}(t-)$ for any $t \in \mathbb{R}_+$.

The distribution of $\widehat{L}(\tau(x))$ can be expressed in terms of n as follows.

Lemma 2.4. *For any $x \in \mathbb{R}_+\backslash\{0\}$ the random variable $\widehat{L}(\tau(x))$ is exponentially distributed under P (resp. $P^{(\gamma)}$) with parameter $n(\rho(x) < \zeta)$ (resp. $n^{(\gamma)}(\rho(x) < \zeta)$).*

Proof. The definition of the first-passage time $\rho(x, \varepsilon)$ in (2.22) implies the equality $\widehat{L}(\tau(x)) = T_A \doteq \inf\{t \geq 0 : \epsilon(t) \in A\}$ where $A \doteq \{\varepsilon \in \mathcal{E} : \rho(x, \varepsilon) < \zeta(\varepsilon)\}$. The statement follows since T_A is exponentially distributed with parameter $n(A)$ (e.g. [16, Lemma 6.18(i)]). □

In [7, Theorem 1][3] it is shown that the following version of Cramér's estimate holds under under the excursion measure n (with C_γ defined in (1.6)):

$$n(\rho(x) < \zeta) \sim C_\gamma\, \widehat{\phi}(\gamma)\, \mathrm{e}^{-\gamma x} \qquad \text{as} \quad x \to \infty. \tag{2.23}$$

[3] [7, Theorem 1] is established under the same hypotheses as in Assumption 1.1. In particular, the condition $E[|X(1)|] < \infty$ is used in the proof of [7, Theorem 1].

Using the estimate (2.23) the asymptotic distribution of $m(t)$ may be identified as follows:

Proposition 2.5. *If $t \to \infty$ then $m(t)$ converges in distribution to m_∞, which follows a Gumbel distribution,*

$$P(m_\infty < z) = \exp\left(-\ell\, C_\gamma\, \widehat{\phi}(\gamma)\, \mathrm{e}^{-\gamma z}\right), \qquad \textit{for all } z \in \mathbb{R}, \tag{2.24}$$

where $\ell = 1/\phi(0)$ and C_γ is given in (1.6).

Proof. We give a short proof of (2.24) based on [7, Theorem 1]. To establish (2.24) we show that the following holds if $\min\{x, t\} \to \infty$ and $t\mathrm{e}^{-\gamma x} \to 1$:

$$P(Y^*(t) - x < z) \;\; = \;\; \exp(-t\,\ell\, n(\rho(x+z) < \zeta)) + o(1) \quad \text{for any } z \in \mathbb{R}. \tag{2.25}$$

Since (2.23) implies $tn(\rho(x+z) < \zeta) \to C_\gamma \widehat{\phi}(\gamma)\mathrm{e}^{-\gamma z}$ as $\min\{x,t\} \to \infty$ and $t\mathrm{e}^{-\gamma x} \to 1$, the limit in (2.24) follows from (2.25). To complete the proof we now verify the claim in (2.25). Note that as $\tau(x) \to \infty$ P-a.s. as $x \to \infty$, the law of large numbers implies that $\widehat{L}(\tau(x))/\tau(x) \to \ell$ P-a.s. as $t \to \infty$, where $\ell \in \mathbb{R}_+\backslash\{0\}$ (by (2.2), (2.4) and (2.6)). In particular, for any $\delta > 0$ and $z \in \mathbb{R}_+$, we have $P(\widehat{L}(\tau(x+z))/\tau(x+z) \in (\ell - \delta, \ell + \delta)) = 1 + o(1)$ as $x \to \infty$. Hence as $\min\{x,t\} \to \infty$ we have

$$\begin{aligned} P(Y^*(t) < x+z) \;\; &= \;\; P(\tau(x+z) > t, \widehat{L}(\tau(x+z))/\tau(x+z) \geq \ell - \delta) + o(1) \\ &\leq \;\; P(\widehat{L}(\tau(x+z)) > t(\ell - \delta)) + o(1). \end{aligned}$$

Similarly, it follows that as $\min\{x,t\} \to \infty$ we have

$$\begin{aligned} P(Y^*(t) < x+z) \;\; &\geq \;\; P(\widehat{L}(\tau(x+z)) > \widehat{L}(t), \widehat{L}(t) \leq t(\ell+\delta)) \\ &\geq \;\; P(\widehat{L}(\tau(x+z)) > t(\ell+\delta), \widehat{L}(t) \leq t(\ell+\delta)) \\ &= \;\; P(\widehat{L}(\tau(x+z)) > t(\ell+\delta)) + o(1). \end{aligned}$$

By Lemma 2.4, $\widehat{L}(\tau(x+z))$ is exponentially distributed with parameter $n(\rho(x+z) < \zeta)$ and hence we find

$$\exp(-(\ell+\delta)t\, n(\rho(x+z) < \zeta)) + o(1) \leq P(Y^*(t) < x+z) \leq \exp(-(\ell-\delta)t\, n(\rho(x+z) < \zeta)) + o(1).$$

Since this result holds for any $\delta > 0$, the equality in (2.25) follows. □

3 Limiting overshoot of the reflected process

In this section we prove the following result, which also plays a role in the proofs of Theorems 1.2 and 1.3.

Proposition 3.1. **(i)** *The weak limit Z_∞ of $Z(x)$ as $x \to \infty$ has Laplace transform*

$$E[\mathrm{e}^{-vZ_\infty}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \qquad \textit{for all} \quad v \in \mathbb{R}_+. \tag{3.1}$$

(ii) *The law of the asymptotic overshoot Z_∞ is given by*

$$P(Z_\infty > x) = \frac{\gamma}{\phi(0)} \mathrm{e}^{-\gamma x} \int_x^\infty \mathrm{e}^{\gamma y}\, \overline{\nu}_H(y)\, \mathrm{d}y, \quad x \in \mathbb{R}_+, \quad \textit{and} \quad P(Z_\infty = 0) = \frac{\gamma}{\phi(0)} m. \tag{3.2}$$

Remark. Note that Z_∞ is continuous on $\mathbb{R}_+\backslash\{0\}$ and has a non-zero atom at zero precisely if the drift m of H is strictly positive, in which case the probability of creeping of X is strictly positive if X is not a compound Poisson process (see e.g. [16, Lemma 7.10]). In fact, as shown in [10], $\frac{\gamma}{\phi(0)} m$ is equal to the asymptotic (conditional) probability $\lim_{x \to \infty} P(X(T(x)) = x | T(x) < \infty)$ of creeping of X.

The formula in (3.1) of Proposition 3.1, which characterises the law of the limiting overshoot Z_∞, is implied by the main result in [18]. As this formula constitutes a key step in the proofs of Theorems 1.2 and 1.3, we give in this section an independent proof of Proposition 3.1 based on excursion theory alone. This approach is in the spirit of the present paper and should be contrasted with the result in [18], which crucially relies on the renewal theorem.

The proof relies on the expression of the distribution of $Z(x)$ in terms of the excursion measure n and on representation results for random variables $K_F(x)$ that are defined for $x \in \mathbb{R}_+\backslash\{0\}$ and Borel-measurable and non-negative functions $F : \mathcal{E} \to \mathbb{R}$ by

$$K_F(x) \doteq \sum_g F(\epsilon_{\widehat{L}(g)}) \mathbf{I}_{\{g \le \tau(x) < g + \zeta(\epsilon_{\widehat{L}(g)})\}}, \tag{3.3}$$

where the sum runs over all left-end points g of excursion intervals. We write $n(F) = n(F(\varepsilon)) \doteq \int_{\mathcal{E}} F(\varepsilon)\, n(\mathrm{d}\varepsilon)$ for any Borel-measurable non-negative (or integrable) functional $F : \mathcal{E} \to \mathbb{R}$. In this notation we have $n(A) = n(\mathbf{I}_A)$ for any $A \in \mathcal{G}$.

Lemma 3.2. (i) *We have*

$$P(Z(x) > y) \quad = \quad n(\varepsilon(\rho(x,\varepsilon)) - x > y | \rho(x) < \zeta) \qquad \text{for any} \quad x, y \in \mathbb{R}_+,$$

where $n(B|A) \doteq n(B \cap A)/n(A)$ *for any* $A, B \in \mathcal{G}$ *with* $n(A) \in \mathbb{R}_+\backslash\{0\}$.

(ii) *Define* $\widehat{\mathcal{V}}(x) \doteq E\left[\widehat{L}(\tau(x))\right]$, $\widehat{\mathcal{V}}^{(\gamma)}(x) \doteq E^{(\gamma)}\left[\widehat{L}(\tau(x))\right]$ *and let* $G(\varepsilon) \doteq F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}$. *Then the following hold:*

$$n(G) \quad = \quad \widehat{\mathcal{V}}(x)^{-1}\, E\left[K_F(x)\right], \qquad n^{(\gamma)}(G) = \widehat{\mathcal{V}}^{(\gamma)}(x)^{-1} E^{(\gamma)}\left[K_F(x)\right]. \tag{3.4}$$

(iii) *The following identity holds:* $n^{(\gamma)}(F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}) = n(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}})$. *Hence we have*

$$n^{(\gamma)}(\rho(x,\varepsilon) < \zeta(\varepsilon)) = n(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}). \tag{3.5}$$

(iv) *For any* $z \in \mathbb{R}_+\backslash\{0\}$ *we have as* $x \to \infty$*:*

$$n^{(\gamma)}(\rho(x,\varepsilon) < \zeta(\varepsilon)) \sim \widehat{\phi}(\gamma) \qquad \text{and} \qquad \mathrm{e}^{\gamma x} n(\varepsilon(\rho(z,\varepsilon)) > x, \rho(z,\varepsilon) < \zeta(\varepsilon)) = o(1). \tag{3.6}$$

Proof of Lemma 3.2. **(i)** The assertion is a consequence of the fact that $\epsilon(T_A)$ follows an n-uniform distribution (that is, $P(\epsilon(T_A) \in B) = n(B|A)$ for any $B \in \mathcal{G}$, see e.g. [2, Sec. O.5, Proposition O.2]) and taking B to be equal to $\{\varepsilon \in \mathcal{E} : \rho(x,\varepsilon) < \zeta(\varepsilon), \varepsilon(\rho(x,\varepsilon)) - x > y\}$.

(ii) As the proof of the two identities (3.4) is identical, we derive only the left-hand side of (3.4). Since for every $\varepsilon \in \mathcal{E}$ the process $t \to F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\le\tau(x)\}}$ is left-continuous and $\mathcal{F}(\widehat{L}^{-1}(t-))$-adapted, an application of the compensation formula to the Poisson point process ϵ (see e.g. [2, Chapter O.5] or [15]) yields

$$\begin{aligned} E[K_F(x)] \quad &= \quad E\left[\sum_g \mathbf{I}_{\{g\le\tau(x)\}} \cdot F(\epsilon_{\widehat{L}(g)})\mathbf{I}_{\{\tau(x)-g<\zeta(\epsilon_{\widehat{L}(g)})\}}\right] \\ &= \quad E\left[\sum_{t\ge 0} \mathbf{I}_{\{\widehat{L}^{-1}(t-)\le\tau(x)\}} \cdot \left\{F(\epsilon_t)\mathbf{I}_{\{\tau(x)-\widehat{L}^{-1}(t-)<\zeta(\epsilon_t\}}\right\}\right] = I_1 \cdot I_2, \end{aligned} \tag{3.7}$$

where $I_2 = n(F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}})$ and

$$\begin{aligned} I_1 \quad &= \quad E\left[\int_0^\infty \mathbf{I}_{\{\widehat{L}^{-1}(t-)\le\tau(x)\}}\mathrm{d}t\right] \;=\; E\left[\int_0^\infty \mathbf{I}_{\{t\le\widehat{L}(\tau(x))\}}\mathrm{d}t\right] \\ &= \quad E\left[\widehat{L}(\tau(x))\right] = \widehat{\mathcal{V}}(x), \end{aligned} \tag{3.8}$$

where we used that $\{\widehat{L}^{-1}(t-) \leq \tau(x)\} = \{t \leq \widehat{L}(\tau(x))\}$. Inserting I_1 and I_2 in (3.7) and dividing by I_1 yields the left-hand side in (3.4).

(iii) Another application of the compensation formula yields

$$\begin{aligned} E^{(\gamma)}\left[K_G(x)\right] &= E\left[\mathrm{e}^{\gamma X(\tau(x))}\sum_g F(\epsilon_{\widehat{L}(g)})\mathbf{I}_{\{g\leq\tau(x)<g+\zeta(\epsilon_{\widehat{L}(g)})\}})\right] \\ &= E\left[\sum_g \mathrm{e}^{\gamma X(g)}\mathbf{I}_{\{g\leq\tau(x)\}}\cdot \mathrm{e}^{\gamma\epsilon_{\widehat{L}(g)}(\rho(x,\epsilon_{\widehat{L}(g)}))}\mathbf{I}_{\{\rho(x,\epsilon_{\widehat{L}(g)})<\zeta(\epsilon_{\widehat{L}(g)})\}}\right] \\ &= E\left[\sum_{t\geq 0}\left\{\mathrm{e}^{\gamma X(\widehat{L}^{-1}(t-))}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\right\}\cdot \mathrm{e}^{\gamma\epsilon_t(\rho(x,\epsilon_t))}\mathbf{I}_{\{\rho(x,\epsilon_t)<\zeta(\epsilon_t)\}}\right] \\ &= J_1\cdot J_2, \end{aligned} \tag{3.9}$$

where $J_2 = n\left(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}\right)$ and, by an application of Fubini's theorem,

$$\begin{aligned} J_1 &= E\left[\int_0^\infty \mathrm{e}^{\gamma X(\widehat{L}^{-1}(t-))}\mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\mathrm{d}t\right] = E^{(\gamma)}\left[\int_0^\infty \mathbf{I}_{\{\widehat{L}^{-1}(t-)\leq\tau(x)\}}\mathrm{d}t\right] \\ &= E^{(\gamma)}[\tau(x)] = \widehat{\mathcal{V}}^{(\gamma)}(x). \end{aligned}$$

Combining the right-hand side in (3.4) with (3.9) and the forms of J_1 and J_2 yields the stated identity.

(iv) Since $\widehat{L}(\tau(x))$ under $P^{(\gamma)}$ follows an exponential distribution with mean $1/n^{(\gamma)}(\rho(x) < \zeta))$ (see Lemma 2.4) we have

$$n^{(\gamma)}(\rho(x)<\zeta) = -\log P^{(\gamma)}(\widehat{L}(\tau(x))>1),$$

so that

$$\lim_{x\uparrow\infty} n^{(\gamma)}(\rho(x)<\zeta) = -\log P^{(\gamma)}(\widehat{L}^{-1}(1)<\infty),$$

which is equal to $\widehat{\phi}^{(\gamma)}(0) = \widehat{\phi}(\gamma)$ (as $\widehat{\phi}^{(\gamma)}(u) = \widehat{\phi}(\gamma+u)$, $u\geq 0$). Chebyshev's inequality and part (ii) of the lemma imply

$$\begin{aligned} \mathrm{e}^{\gamma x}n(\varepsilon(\rho(z,\varepsilon))>x, \rho(z,\varepsilon)<\zeta(\varepsilon)) &\leq n(\mathrm{e}^{\gamma\varepsilon(\rho(z,\varepsilon))}\mathbf{I}_{\{\varepsilon(\rho(z,\varepsilon))>x,\rho(z,\varepsilon)<\zeta(\varepsilon)\}}) \\ &= n^{(\gamma)}(\varepsilon(\rho(z,\varepsilon))>x, \rho(z,\varepsilon)<\zeta(\varepsilon)). \end{aligned}$$

As the latter tends to zero as $x\uparrow\infty$, the second assertion in (3.6) follows. □

We next apply Lemma 3.2 to establish the asymptotic behaviour of certain integrals against the excursion measure as $x\to\infty$.

Lemma 3.3. *Let $u\geq 0$. Then, as $x\to\infty$, we have*

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}|\rho(x)<\zeta) \longrightarrow C(u)\cdot C_\gamma^{-1} = \frac{\gamma}{\gamma+u}\cdot\frac{\phi(u)}{\phi(0)}. \tag{3.10}$$

In particular, $Z(x)$ converges weakly to a random variable Z_∞ with Laplace transform $E[\exp(-uZ_\infty)] = C(u)\cdot C_\gamma^{-1}$.

Proof. Fix $M>0$ and recall that, under the probability measure $n(\cdot|\rho(M)<\zeta)$, the coordinate process has the same law as the first excursion of Y away from zero with height larger than M. For any $x>M$, the following identity holds:

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}|\rho(x)<\zeta) = n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}\mathbf{I}_{\{\rho(x)<\zeta\}}|\rho(M)<\zeta)\frac{n(\rho(M)<\zeta)}{n(\rho(x)<\zeta)}. \tag{3.11}$$

The strong Markov property under the probability measure $n(\cdot|\rho(M)<\zeta)$, implies that $\varepsilon\circ\theta_{\rho(M)}$ has the same law as the process X with entrance law $n(\varepsilon(\rho(M,\varepsilon))\in \mathrm{d}z|\rho(M)<\zeta)$ and killed at the epoch of the first passage into the interval $(-\infty,0]$. We therefore find

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}\mathbf{I}_{\{\rho(x)<\zeta\}}|\rho(M)<\zeta)=n\left(\mathrm{e}^{-u(\varepsilon(\rho(M,\varepsilon))-x)}\mathbf{I}_{\{\varepsilon(\rho(M,\varepsilon))>x\}}|\rho(M)<\zeta\right)$$
$$+\int_{[M,x]}E_z\left[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(0)\}}\right]n(\varepsilon(\rho(M,\varepsilon))\in\mathrm{d}z|\rho(M)<\zeta). \quad (3.12)$$

By the second equality in (3.6) of Lemma 3.2, we have as $x\uparrow\infty$:

$$\mathrm{e}^{\gamma x}n\left(\mathrm{e}^{-u(\varepsilon(\rho(M,\varepsilon))-x)}\mathbf{I}_{\{\varepsilon(\rho(M,\varepsilon))>x\}}\middle|\rho(M)<\zeta\right)$$
$$\le \mathrm{e}^{\gamma x}\frac{n\left(\varepsilon(\rho(M,\varepsilon))>x,\rho(M,\varepsilon)<\zeta(\varepsilon)\right)}{n(\rho(M)<\zeta)}=o(1).$$

This estimate, spatial homogeneity of X and equations (3.11) and (3.12) yield as $x\to\infty$:

$$\begin{aligned}&n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta)\\ =\;&\int_{[M,x]}E\left[\mathrm{e}^{-uK(x-z)}\mathbf{I}_{\{T(x-z)<\widehat{T}(z)\}}\right]\frac{n(\varepsilon(\rho(M,\varepsilon))\in\mathrm{d}z,\rho(M)<\zeta)}{n(\rho(x)<\zeta)}\\ +\;&o(1).\end{aligned} \quad (3.13)$$

Formula (2.16) of Proposition 2.3 implies the following equality:

$$E\left[\mathrm{e}^{-uK(x-z)}\mathbf{I}_{\{T(x-z)<\widehat{T}(z)\}}\right]=C(u)\mathrm{e}^{-\gamma x}\left(1-G(z)+R(x-z)\right)\mathrm{e}^{\gamma z}, \quad (3.14)$$

where $G,R:\mathbb{R}_+\to\mathbb{R}$ are bounded functions such that $G(z)=E[\mathrm{e}^{\gamma X(\widehat{T}(z))}]$ and $\lim_{x'\to\infty}R(x')=0$. Therefore the equality in (3.13), the asymptotic behaviour of $n(\rho(x)<\zeta)$ given in (2.23) and Lemma 3.2 (ii) imply the following identity as $x\to\infty$:

$$\begin{aligned}&n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta)=A_\gamma(u)n^{(\gamma)}(\varepsilon(\rho(M,\varepsilon))\in[M,x],\rho(M,\varepsilon)<\zeta(\varepsilon))\\ &+\;A_\gamma(u)n^{(\gamma)}\left(\left[R(x-\varepsilon(\rho(M,\varepsilon)))-G(\varepsilon(\rho(M,\varepsilon)))\right]I_{\{\varepsilon(\rho(M,\varepsilon))\in[M,x],\rho(M,\varepsilon)<\zeta(\varepsilon)\}}\right)\\ &+\;o(1),\end{aligned} \quad (3.15)$$

where $A_\gamma(u)\doteq C(u)/(C_\gamma\widehat{\phi}(\gamma))$. By (3.15) the limit $\lim_{x\to\infty}n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta)$ exists and the dominated convergence theorem yields

$$\lim_{x\to\infty}n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta)$$
$$=A_\gamma(u)\left(n^{(\gamma)}(\rho(M)<\zeta)-n^{(\gamma)}\left(G(\varepsilon(\rho(M,\varepsilon)))I_{\{\rho(M,\varepsilon)<\zeta(\varepsilon)\}}\right)\right).$$

Since this equality holds for any $M>0$ and the left-hand side does not depend on M, if the right-hand side has a limit as $M\to\infty$, then the equality also holds in this limit. Note that (3.6) of Lemma 3.2 (iii) implies $\lim_{M\to\infty}n^{(\gamma)}(\rho(M)<\zeta)=\widehat{\phi}(\gamma)$. Since $G(\varepsilon(\rho(M,\varepsilon)))\le\mathrm{e}^{-\gamma M}$ on $\{\rho(M,\varepsilon)<\zeta(\varepsilon)\}$, an application of the dominated convergence theorem yields (3.10). By combining with Lemma 3.2(i) we find the stated form of Laplace transform of Z_∞. □

With the previous results in hand we complete next the proof of Proposition 3.1.

Proof of Proposition 3.1. **(i)** Equation (3.1) is established in Lemma 3.3.

(ii) Straightforward algebra, starting from (3.1), shows that the Laplace transform of $x \mapsto \exp(\gamma x)P(Z_\infty > x)$ is given by

$$\int_0^\infty \mathrm{e}^{-vx}\mathrm{e}^{\gamma x}P(Z_\infty > x)\mathrm{d}x = \frac{1}{v-\gamma}\left(1 - \frac{\gamma}{\phi(0)}\frac{\phi(v-\gamma)}{v}\right)$$
$$= \frac{\gamma}{\phi(0)}\left[\frac{1}{v}\left(\frac{\phi(0)}{\gamma} - m\right) - \frac{1}{v}\left(\frac{\phi(v-\gamma)-\phi(0)-(v-\gamma)m}{v-\gamma}\right)\right],$$

for $v > \gamma$. A direct Laplace inversion, based on the representation (2.5) of ϕ and (2.11) in Lemma 2.2, yields the left-hand side of formula (3.2). The atom at zero is obtained by taking the limit in (3.1) of part (i) as $v \to \infty$. □

4 Asymptotic independence

In this section we establish the asymptotic independence of $Y(t)$, $Z(x+y)$ and $M(t,x)$ as $\min\{t,x,y\} \to \infty$, *i.e.* for any $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$

$$P(Y(t) \le a, Z(x+y) \le b, M(t,x) \le c) = P(Y(t) \le a)P(Z(x+y) \le b)P(M(t,x) \le c) + o(1),$$ [4]

where $M(t,x) \doteq Y^*(t) - x$, $t, x \in \mathbb{R}_+$. From this we deduce (see Lemma 4.4 below) the asymptotic independence of $(Y(t), X(x), m(t))$ as $\min\{t,x\} \to \infty$ and $x - y^*(t) \to \infty$, described in Theorem 1.3. We start with the following observations concerning the large-time behaviour of the local time $\widehat{L}$:

Lemma 4.1. *The following statements hold true:*
(i) *As in Theorem 1.3 denote $\ell = 1/E[\widehat{L}^{-1}(1)]$. For any $\delta \in (0, \ell/2)$ we have*

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) \le \frac{4}{\mathrm{e}\ell}\delta.$$

(ii) *The following limit holds: $P(\widehat{L}(t) = \widehat{L}(\tau(x))) \longrightarrow 0$ as $\min\{x,t\} \to \infty$;*
(iii) *For any $\delta_1, \delta_2 \in [0, 1/4)$ we have*

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(t(1-\delta_1)) \le \widehat{L}(\tau(x)) \le \widehat{L}(t(1+\delta_2))) \le \frac{8}{\mathrm{e}}\max\{\delta_1, \delta_2\}. \tag{4.1}$$

For any $s \in \mathbb{R}_+\backslash\{0\}$ it holds $P(\widehat{L}((t-s)\vee 0) \le \widehat{L}(\tau(x)) < \widehat{L}(t)) \longrightarrow 0$ as $\min\{x,t\} \to \infty$.

Proof. **(i)** Recall ℓ is finite (Lemma 2.1). For any $x, t \in \mathbb{R}_+\backslash\{0\}$, Lemma 2.4 implies $P(\widehat{L}(\tau(x)) > t) = \mathrm{e}^{-t\,n(B(x))}$ for all $t \in \mathbb{R}_+\backslash\{0\}$, where $B(x) \doteq \{\rho(x) < \zeta\}$ with ρ defined in (2.22). Therefore for any $\delta \in (0, \ell/2)$ the following holds:

$$P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) = \mathrm{e}^{-t\,\ell\,n(B(x))}\left(\mathrm{e}^{\delta\,t\,n(B(x))} - \mathrm{e}^{-\delta\,t\,n(B(x))}\right).$$

The Mean-Value Theorem implies that there exists $\xi_{t,x} \in (-\delta, \delta)$ such that

$$\begin{aligned} P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) &= 2\delta t n(B(x))\mathrm{e}^{(\xi_{t,x}-\ell)tn(B(x))} \\ &\le 2\delta t n(B(x))\mathrm{e}^{-tn(B(x))\ell/2} \le \delta 4/(\mathrm{e}\ell), \end{aligned}$$

where the inequality follows from $|\xi_{t,x}| < \ell/2$. Since $t, x \in \mathbb{R}_+\backslash\{0\}$ are arbitrary, this concludes the proof of part (i).
(ii) As the ratio $t/\widehat{L}^{-1}(t)$ tends to ℓ almost surely (see 2.4), we have for any $\delta \in (0, \ell/2)$,

$$P\left(\widehat{L}(t)/t \in [\ell-\delta, \ell+\delta]\right) = 1 + o(1), \qquad \text{as } t \to \infty. \tag{4.2}$$

[4] Here again $f(t,x,y) = o(1)$ ($\min\{x,y,t\} \to \infty$) if $\lim_{\min\{t,x,y\}\to\infty} f(t,x,y) = 0$.

Equation (4.2) yields the following as $\min\{x,t\} \to \infty$:

$$\begin{aligned} P(\widehat{L}(t) = \widehat{L}(\tau(x))) &= P(\widehat{L}(t) = \widehat{L}(\tau(x)), \widehat{L}(t) \in t[\ell-\delta, \ell+\delta]) + o(1) \\ &\le P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) + o(1). \end{aligned}$$

Hence part (ii) yields $\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(t) = \widehat{L}(\tau(x))) \le \delta 4/(\mathrm{e}\ell)$. Since $\delta \in (0, \ell/2)$ was arbitrary and probabilities are non-negative quantities, the limit in part (ii) follows. **(iii)** Note that for any $\alpha \ge 0$ the quotient $\widehat{L}(t\alpha)/t$ tends to $\ell\alpha$ P-a.s. as $t \to \infty$. For any $\delta_1, \delta_2 \in [0, 1/4)$ we therefore find that the probability of the event

$$A_{\delta_1,\delta_2}(t,x) = \{\widehat{L}(t(1-\delta_1)) \le \widehat{L}(\tau(x)) \le \widehat{L}(t(1+\delta_2))\}$$

satisfies the following as $\min\{x,t\} \to \infty$:

$$\begin{aligned} P(A_{\delta_1,\delta_2}(t,x)) &= P(A_{\delta_1,\delta_2}(t,x), \widehat{L}(t(1-\delta_1)), \widehat{L}(t(1+\delta_2)) \in t[\ell(1-\delta), \ell(1+\delta)]) + o(1) \\ &\le P(\widehat{L}(\tau(x)) \in t[\ell(1-\delta), \ell(1+\delta)]) + o(1), \end{aligned} \tag{4.3}$$

for any $\delta \in (2\max\{\delta_1, \delta_2\}, 1/2)$. Since $0 < \delta\ell < \ell/2$, part (ii) of the lemma and inequality (4.3) imply that $\limsup_{\min\{x,t\}\to\infty} P(A_{\delta_1,\delta_2}(t,x)) \le \delta 4/\mathrm{e}$. Therefore the first inequality in part (iv) is satisfied. The second limit in part (iv) follows by noting that, for any $s \in \mathbb{R}_+$ and $\delta_1 \in (0, 1/4)$, the inclusion $\{\widehat{L}((t-s)\vee 0) \le \widehat{L}(\tau(x)) < \widehat{L}(t)\} \subset A_{\delta_1,0}(t,x)$ holds for all (t,x) with large $\min\{x,t\}$. Hence by (4.1) we have

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}((t-s)\vee 0) \le \widehat{L}(\tau(x)) < \widehat{L}(t)) \le \delta_1 8/\mathrm{e}.$$

Since δ_1 can be chosen arbitrarily small, this proves part (iv) and hence the lemma. □

Before moving to the proof of the asymptotic independence of $Y(t), Z(x+y)$ and $M(x,t)$, we establish the asymptotic behaviour of certain convolutions that will arise in the proof.

Lemma 4.2. *For $a \in [0,\infty)$ and any family of sets $F(t) \in \mathcal{F}$, $t \in \mathbb{R}_+$, we have as $\min\{y,t\} \to \infty$*

$$\begin{aligned} &\int_{[0,t]} P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))\, P(T(a) \in \mathrm{d}s) \\ &\qquad = P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t))\, P(Y(t) > a) + o(1). \end{aligned} \tag{4.4}$$

Proof. The proof of this lemma is based on Lemma 4.1. Since $Y(t)$ and $\sup_{0\le s\le t} X(s)$ are equal in law (by time reversal) and $P(T(a) = t) \to 0$ as $t \to \infty$[5] (as $X_t \to -\infty$ by Assumption 1.1), it follows that $P(T(a) \le t) = P(Y(t) > a) + o(1)$ as $t \to \infty$. Thus, to prove equality (4.4), it is sufficient to establish

$$\int_{[0,t]} \Big(P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t)) - P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s)) \Big) P(T(a) \in \mathrm{d}s) = o(1) \tag{4.5}$$

as $\min\{y,t\} \to \infty$. Since the local time $\widehat{L}$ is non-decreasing, the integrand in (4.5) satisfies

$$|P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t)) - P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))| \le P(\widehat{L}(t-s) \le \widehat{L}(\tau(y)) < \widehat{L}(t)).$$

Hence Lemma 4.1(iv) and the dominated convergence theorem imply that (4.5) holds. □

[5] Note $P(T(a) = t) = 0$ if X is not equal to the sum of a compound Poisson process and a deterministic drift. Indeed, in this case $X(t)$ is continuous (see [21, Theorem 27.4]), so that $P(T(a) = t) \le P(X(t) = a) = 0$.

We move next to the asymptotic independence of $Y(t), Z(x+y)$ and $M(t,x)$.

Lemma 4.3. *For any* $t,x \in \mathbb{R}_+\backslash\{0\}$, $a,b \in \mathbb{R}_+$, $c \in \mathbb{R}$, $y \in [0,x]$ *and Borel sets* $A,B,C \in \mathcal{B}(\mathbb{R})$ *with* $A = (-\infty, a]$, $B = (-\infty, b]$ *and* $C = (-\infty, c]$ *denote*

$$\pi_1(t,A) = P(Y(t) \in A), \quad \pi_2(x,B) = P(Z(x) \in B), \quad \pi_3(t,y) = P(\widehat{L}(\tau(y)) < \widehat{L}(t)).$$

We have as $\min\{t,y,x-y\} \to \infty$

$$P(Y(t) \in A, Z(x) \in B) = \pi_1(t,A)\pi_2(x,B) + o(1), \tag{4.6}$$

$$P(Y(t) \in A, Z(x) \in B, \widehat{L}(\tau(y)) < \widehat{L}(t)) = \pi_1(t,A)\pi_2(x,B)\pi_3(t,y) + o(1), \tag{4.7}$$

$$P(Y(t) \in A, Z(x) \in B, M(t,y) \in C) = \pi_1(t,A)\pi_2(x,B)P(M(t,y) \in C) + o(1), \tag{4.8}$$

$$P(Y(t) \in A, Z(x) \in B, m(t) \in C) = \pi_1(t,A)\pi_2(x,B)P(m(t) \in C) + o(1). \tag{4.9}$$

Proof. Fix $t,x \in \mathbb{R}_+\backslash\{0\}$, $y \in [0,x]$, $a,b \in \mathbb{R}_+$ arbitrary, with $A = (-\infty,a]$, $B = (-\infty,b]$. As a first step we note that by a classical application of excursion theory[6] involving $G(\tau(x)) = \sup\{s < \tau(x) : Y(s) = 0\} = \widehat{L}^{-1}(\widehat{L}(\tau(x))-)$ the random elements $\mathcal{A} := \{Y(s) : 0 \le s \le G(\tau(x))\}$ and $\mathcal{A}' := \epsilon(\widehat{L}(\tau(x)))$ are independent. Hence the sets $\{Z(x) \in B\}$ and $\{\widehat{L}(\tau(y)) > \widehat{L}(t), Y(t) \in A\}$, which are measurable with respect to $\sigma(\mathcal{A}')$ and $\sigma(\mathcal{A})$ respectively, are independent, that is,

$$\begin{aligned} &P(\widehat{L}(\tau(y)) > \widehat{L}(t), Y(t) \in A, Z(x) \in B) \\ &\quad = P(\widehat{L}(\tau(y)) > \widehat{L}(t), Y(t) \in A)P(Z(x) \in B). \end{aligned} \tag{4.10}$$

Next we establish additional (asymptotic) factorisation. Let $A' \in \mathcal{B}(\mathbb{R})$ arbitrary. Since $s \mapsto \mathbf{I}_{\{\tau(x)<s\le t, Z(x)\in B\}}$ is left-continuous and adapted an application of the compensation formula of excursion theory (see e.g. [2, Cor. IV.11]) yields

$$\begin{aligned} &P\left(\widehat{L}(\tau(x)) < \widehat{L}(t), Y(t) \in A', Z(x) \in B\right) \\ &= E\left[\sum_g \mathbf{I}_{\{\tau(x)<g\le t, Z(x)\in B\}}\mathbf{I}_{\{\epsilon_{\widehat{L}(g)}(t-g)\in A', t-g<\zeta(\epsilon_{\widehat{L}(g)})\}}\right] \\ &= E\left[\int_{[0,t]} \mathbf{I}_{\{\tau(x)<s\le t, Z(x)\in B\}}\, n(\varepsilon(t-s)\in A', t-s<\zeta(\varepsilon))\mathrm{d}\widehat{L}(s)\right], \end{aligned} \tag{4.11}$$

where the sum is over all left-end points of excursion intervals. Denote by $e(q)$ an exponential random time with mean $1/q$, defined by extending the probability space to $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$. Replacing t by the exponential time $e(q)$ in (4.11) and denoting $\mathbb{P} = P \times P'$ we have by the lack of memory property of $e(q)$ (taking $A' = \mathbb{R}$ and $B = \mathbb{R}$, $x = 0$, respectively)

$$\begin{aligned} &\mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Z(x) \in B\right) \\ &\quad = \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\le e(q), Z(x)\in B\}}\mathrm{d}\widehat{L}(s)\right]\mathbb{E}[n(e(q) < \zeta(\varepsilon))], \end{aligned} \tag{4.12}$$

$$\mathbb{P}\left(Y(e(q) \in A'\right) = \mathbb{E}\left[\widehat{L}(e(q)\right]\mathbb{E}[n(\varepsilon(e(q)) \in A', e(q) < \zeta(\varepsilon))] \tag{4.13}$$

$$= \frac{\mathbb{E}[n(\varepsilon(e(q)) > y, e(q) < \zeta(\varepsilon))]}{\mathbb{E}[n(e(q) < \zeta(\varepsilon))]}, \tag{4.14}$$

[6] This can be seen to follow directly as a consequence of the splitting property [2, Section O.5, Proposition O.2] of the Poisson point process ϵ at the first entrance time $\mathbb{H}_{B'} = \inf\{s \le 0 : \epsilon(s) \in B'\}$ of ϵ into the set $B' = \{\varepsilon \in \mathcal{E} : \rho(x,\varepsilon) < \zeta(\varepsilon)\}$.

where the last equality follows by taking $A' = \mathbb{R}_+$ in (4.13). Using (4.12) and (4.14) and again replacing t by $e(q)$ in (4.11) we have

$$\begin{aligned}
&\mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Y(e(q)) \in A', Z(x) \in B\right)\\
&= \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\le e(q), Z(x)\in B\}} \mathrm{d}\widehat{L}(s)\right] \mathbb{E}[n(\varepsilon(e(q)) \in A', e(q) < \zeta(\varepsilon))]\\
&= \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\le e(q), Z(x)\in B\}} \mathrm{d}\widehat{L}(s)\right] \mathbb{E}[n(e(q) < \zeta(\varepsilon))]\\
&\quad \times \frac{\mathbb{E}[n(\varepsilon(e(q)) \in A', e(q) < \zeta(\varepsilon))]}{\mathbb{E}[n(e(q) < \zeta(\varepsilon))]}\\
&= \mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Z(x) \in B\right) \mathbb{P}(Y(e(q)) \in A'). \qquad (4.15)
\end{aligned}$$

Dividing the left-hand and right-hand sides of (4.15) by q with $A' = A^c = (a, \infty)$, inverting the Laplace transform in q, noting $q^{-1}\mathbb{P}(Y(e(q)) \in A^c) = q^{-1}\mathbb{P}(X^*(e(q)) > a) = q^{-1}\mathbb{P}(T(a) \le e(q)) = \int_0^\infty \mathrm{e}^{-qt} P(T(a) \in \mathrm{d}t)$, and deploying (4.4) in Lemma 4.2 we have

$$\begin{aligned}
&P(\widehat{L}(\tau(x)) < \widehat{L}(t), Y(t) \in A^c, Z(x) \in B) \qquad (4.16)\\
&= P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B) P(Y(t) \in A^c) + o(1), \qquad \text{as } \min\{x,t\} \to \infty.
\end{aligned}$$

Subtracting $P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B)$ on the left-hand and right-hand sides of (4.16) shows that (4.16) is also valid with A^c replace by A.

Taking note of the following equality for any $y, t \in \mathbb{R}_+\backslash\{0\}$ and set $E \in \mathcal{F}$:

$$\begin{aligned}
&P(E, \widehat{L}(\tau(y)) > \widehat{L}(t)) + P(E, \widehat{L}(\tau(y)) = \widehat{L}(t)) \qquad (4.17)\\
&= P(E) - P(E, \widehat{L}(\tau(y)) < \widehat{L}(t)),
\end{aligned}$$

and applying (4.10) and (4.16) with $B = \mathbb{R}_+$ yields as $\min\{x,t\} \to \infty$

$$\begin{aligned}
&P(Y(t) \in A, Z(x) \in B)\\
&= \pi_1(t,A) P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B) + P(\widehat{L}(\tau(x)) > \widehat{L}(t), Y(t) \in A)\pi_2(x,B)\\
&\quad + P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A, Z(x) \in B) + o(1)\\
&= \pi_1(t,A)\pi_2(x,B) + R(t,x) + o(1),
\end{aligned}$$

where $R(t,x) = P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A, Z(x) \in B) - P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A)\pi_2(x,B) - P(\widehat{L}(\tau(x)) = \widehat{L}(t), Z(x) \in B)\pi_1(t,A) + \pi_1(t,A)\pi_2(x,B)P(\widehat{L}(\tau(x)) = \widehat{L}(t))$. Observing that $R(t,x) = o(1)$ when $\min\{x,t\} \to \infty$ by Lemma 4.1(iii) the proof of (4.6) is complete.

Equation (4.7) follows similarly, by combining the equality (4.17) (with $E = \{Y(t) \in A, Z(x) \in B\}$) with Lemma 4.1(iii) and the identities (4.6), (4.10), and (4.16) (with $B = \mathbb{R}_+$).

Finally, take $C = (-\infty, c]$ for an arbitrary fixed $c \in \mathbb{R}$. In order to prove equality (4.8) note that the following inclusions hold for any $y \in \mathbb{R}_+$:

$$\begin{aligned}
\{M(t,y) \in C\} = \{Y^*(t) \le y + c\} &\subset \{\widehat{L}(t) \le \widehat{L}(\tau((y+c)^+))\} \quad \text{and}\\
\{\widehat{L}(t) \le \widehat{L}(\tau((y+c)^+))\} \cap \{M(t,y) \notin C\} &\subset \{\widehat{L}(\tau((y+c)^+)) = \widehat{L}(t)\}
\end{aligned}$$

(recall that $\tau(x)$ is defined for $x \in \mathbb{R}_+$). These inclusions, together with Lemma 4.1(iii), imply that the following equality holds for any family of events $E(t,x) \in \mathcal{F}$, $t, x \in \mathbb{R}_+$, as $\min\{t, y, x-y\} \to \infty$:

$$P\left(E(t,x), \widehat{L}(t) \le \widehat{L}(\tau((y+c)^+))\right) = P\left(E(t,x), M(t,y) \in C\right) + o(1). \qquad (4.18)$$

Since $\min\{t, y, x-y\} \to \infty$, for the fixed $c \in \mathbb{R}$ the inequalities $0 \le y + c \le x$ hold for all large y and x. In particular (4.7), applied to the complement $\{\widehat{L}(\tau(y+c)) < \widehat{L}(t)\}^c = \{\widehat{L}(\tau(y+c)) \ge \widehat{L}(t)\}$, Lemma 4.1(iii) and (4.18) yield the following equalities:

$$\begin{aligned} & P(Y(t) \in A, Z(x) \in B, M(t,y) \in C) \\ & = \quad P(Y(t) \in A, Z(x) \in B, \widehat{L}(t) \le \widehat{L}(\tau(y+c))) + o(1) \\ & = \quad P(Y(t) \in A) P(Z(x) \in B) P(\widehat{L}(t) \le \widehat{L}(\tau(y+c))) + o(1) \\ & = \quad P(Y(t) \in A) P(Z(x) \in B) P(M(t,y) \in C) + o(1) \end{aligned}$$

as $\min\{t, y, x-y\} \to \infty$, which establishes (4.8). Taking $y = y^*(t)$ in (4.8) (recalling $m(t) = M(t, y^*(t))$ and using (1.4)) yields (4.9), and the proof is complete. □

Lemma 4.4. (i) *As $\min\{x,t\} \to \infty$, $Y(t)$ and $Z(x)$ satisfy*

$$E[\exp(-uY(t) - vZ(x))] = E[\exp(-uY(t))] E[\exp(-vZ(x))] + o(1), \quad \text{for any } u, v \in \mathbb{R}_+\backslash\{0\}. \quad (4.19)$$

(ii) *As $\min\{x,t\} \to \infty$ such that $t\exp(-\gamma x) \to 0$, $Y(t)$, $Z(x)$ and $m(t)$ satisfy*

$$\begin{aligned} E[\exp(-uY(t) - vZ(x) \pm \beta m(t)) \mathbf{I}_{\{-m(t) \in \mathbb{R}_\pm\}}] &= E[\exp(-uY(t))]\, E[\exp(-vZ(x))] \\ \times\ E[\exp(\pm\beta m(t)) \mathbf{I}_{\{-m(t)\in\mathbb{R}_\pm\}}] + o(1), &\quad \text{for any } u, v, \beta \in \mathbb{R}_+\backslash\{0\}, \quad (4.20) \end{aligned}$$

where $\mathbb{R}_- = \mathbb{R}\backslash\mathbb{R}_+$. In particular, we have

$$\begin{aligned} E[\exp(-uY(t) - vZ(x) - \beta|m(t)| - b\, s(m(t)))] &= E[\exp(-uY(t))]\, E[\exp(-vZ(x))] \\ \times\ E[\exp(-\beta|m(t)| - b\, s(m(t)))] + o(1), &\quad \text{for any } u, v, \beta, b \in \mathbb{R}_+\backslash\{0\}, \quad (4.21) \end{aligned}$$

where $s : \mathbb{R} \to (-\infty, \infty]$ is given by $s(x) = \pm 1$ for $-x \in \mathbb{R}_\mp$.

Proof. **(i)** Fix $u, v \in \mathbb{R}_+\backslash\{0\}$ arbitrary. By integrating both sides of the identity in (4.6) in Lemma 4.3 over $\mathbb{R}^2$ against the measure $\mathbf{I}_{\mathbb{R}_+\times\mathbb{R}_+}(a,b) a\, b\, \exp(-ua - vb)\mathrm{d}a\mathrm{d}b$ we have (4.19) by noting that the integral of the $o(1)$ term in (4.6) tends to zero by the dominated convergence theorem (as it is bounded by one).
(ii) The proof is a modification of the argument in part (i). Note first that (4.9) in Lemma 4.3 also holds with C replaced by its complement $C^c = \mathbb{R}\backslash C$. For given $a, b, c \in \mathbb{R}_+$ it follows from (4.9) in Lemma 4.3 (taking $C = (-\infty, c]$ and $C = (-\infty, 0]$ and substracting)

$$\begin{aligned} & P(Y(t) \le a, Z(x) \le b, m(t) \in (0, c]) \\ & \quad = P(Y(t) \le a) P(Z(x) \le b) P(m(t) \in (0, c]) + o(1), \qquad \min(t,x) \to \infty, \quad (4.22) \end{aligned}$$

and similarly (taking $C = (-\infty, -c]^c = (-c, \infty)$ and $C = (-\infty, 0]^c = (0, \infty]$ and substracting)

$$\begin{aligned} & P(Y(t) \le a, Z(x) \le b, -m(t) \in [0, c)) \\ & \quad = P(Y(t) \le a) P(Z(x) \le b) P(m(t) \in (-c, 0]) + o(1), \qquad \min(t,x) \to \infty. \quad (4.23) \end{aligned}$$

Let next $u, v, w \in \mathbb{R}_+\backslash\{0\}$ be arbitrary. Integrating both sides of the identity in (4.22) over $\mathbb{R}^3$ against the measure

$$\mu(\mathrm{d}a, \mathrm{d}b, \mathrm{d}c) = \mathbf{I}_{\mathbb{R}_+^2 \times \mathbb{R}_+}(a,b,c) a\, b\, c\, \exp(-ua - vb - wc)\mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c$$

and applying the dominated convergence theorem shows that also the integral of the $o(1)$-term tends to zero, which yields the "$-$"-version of (4.20). The "$+$"-version of follows similarly by integrating both sides of the identity in (4.23) against μ. As (4.21) follows as direct consequence of (4.20), the proof is complete. □

5 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. As $Y(t)$ and $Z(x)$ each admit a weak limit Y_∞, Z_∞ as $t, x \to \infty$, given in (2.8) and in Proposition 3.3, the joint Laplace transform of (Y_∞, Z_∞) follows from (4.19) in Lemma 4.4(i). Finally, the factorisation of the exponential distribution is obtained by setting $u = v$ in (1.2). □

Proof of Theorem 1.3. The asymptotic independence of $Y(t), Z(x)$ and $m(t)$ follows from (4.8) in Lemma 4.3. The joint Fourier-Laplace transform then follows from a direct calculation using (4.21) in Lemma 4.4(ii) and the laws of Y_∞, Z_∞ and m_∞ given in (2.8) and Propositions 3.1 and 2.5, respectively. □

References

[1] Søren Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability. MR-1978607

[2] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. MR-1406564

[3] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21(5):363–365, 1994. MR-1325211

[4] J. Bertoin, K. van Harn, and F. W. Steutel. Renewal theory and level passage by subordinators. *Statist. Probab. Lett.*, 45(1):65–69, 1999. MR-1718352

[5] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication. MR-1700749

[6] Robert M. Blumenthal. *Excursions of Markov processes*. Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. MR-1138461

[7] R. A. Doney and R. A. Maller. Cramér's estimate for a reflected Lévy process. *Ann. Appl. Probab.*, 15(2):1445–1450, 2005. MR-2134110

[8] Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1997. For insurance and finance. MR-1458613

[9] Priscilla Greenwood and Jim Pitman. Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. in Appl. Probab.*, 12(4):893–902, 1980. MR-0588409

[10] Philip S. Griffin, Ross A. Maller, and Kees van Schaik. Asymptotic distributions of the overshoot and undershoots for the Lévy insurance risk process in the Cramér and convolution equivalent cases. *Insurance Math. Econom.*, 51(2):382–392, 2012. MR-2949454

[11] Olympia Hadjiliadis and Jan Večeř. Drawdowns preceding rallies in the Brownian motion model. *Quant. Finance*, 6(5):403–409, 2006. MR-2261219

[12] Donald L. Iglehart. Extreme values in the $GI/G/1$ queue. *Ann. Math. Statist.*, 43:627–635, 1972. MR-0305498

[13] Kiyosi Itô. Poisson point processes attached to Markov processes. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory*, pages 225–239. Univ. California Press, Berkeley, Calif., 1972. MR-0402949

[14] Samuel Karlin and Amir Dembo. Limit distributions of maximal segmental score among Markov-dependent partial sums. *Adv. in Appl. Probab.*, 24(1):113–140, 1992. MR-1146522

[15] J. F. C. Kingman. *Poisson processes*. Oxford, 1993.

[16] A.E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy processes with Application*. Springer, Berlin, 2006.

[17] Aleksandar Mijatović and Martijn R. Pistorius. On the drawdown of completely asymmetric Lévy processes. *Stochastic Process. Appl.*, 122(11):3812–3836, 2012. MR-2965927

[18] Aleksandar Mijatović and Martijn Pistorius. Buffer-overflows: joint limit laws of undershoots and overshoots of reflected processes. *Stochastic Process. Appl.*, 125(8):2937–2954, 2015. MR-3343283

[19] George V. Moustakides. Optimality of the CUSUM procedure in continuous time. *Ann. Statist.*, 32(1):302–315, 2004. MR-2051009

[20] N. U. Prabhu. *Stochastic storage processes*, volume 15 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998. Queues, insurance risk, dams, and data communication. MR-1492990

[21] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author. MR-1739520

[22] A. N. Shiryaev. Minimax optimality of the method of cumulative sums (CUSUM) in the continuous time case. *Uspekhi Mat. Nauk*, 51(4(310)):173–174, 1996. MR-1422244

Acknowledgments. AM was supported by the Humboldt Foundation Research Fellowship GRO/1151787 STP. MP’s research was supported in part by the NWO-STAR cluster. We thank the anonymous referees for their careful reading and useful suggestions.