Electron. Commun. Probab. **21** (2016), no. 25, 1–10.
DOI: 10.1214/16-ECP4102
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Comparison and converse comparison theorems for backward stochastic differential equations with Markov chain noise*

Zhe Yang† Dimbinirina Ramarimbahoaka‡ Robert J. Elliott§

Abstract

Comparison and converse comparison theorems are important parts of the research on backward stochastic differential equations. In this paper, we obtain comparison results for one dimensional backward stochastic differential equations with Markov chain noise, adapting previous results under simplified hypotheses. We introduce a type of nonlinear expectation, the f-expectation, which is an interpretation of the solution to a BSDE, and use it to establish a converse comparison theorem for the same type of equations as those in the comparison results.

Keywords: BSDEs; comparison theorem; converse comparison; Markov chain.
AMS MSC 2010: 60H15.
Submitted to ECP on August, 2014, final version accepted on February, 12, 2016.

1 Introduction

In 1990 Pardoux and Peng [19] considered general backward stochastic differential equations (BSDEs for short) of the following form:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \qquad t \in [0, T].$$

Here B is a Brownian Motion and g is the driver, or drift, of the above BSDE.

Since then, comparison theorems for BSDEs have attracted extensive attention. El Karoui, Peng and Quenez [12] derived comparison theorems for BSDEs with Lipschitz continuous coefficients. Liu and Ren [18] proved a comparison theorem for BSDEs with linear growth and continuous coefficients. Situ [24] obtained a comparison theorem for BSDEs with jumps. Zhang [27] deduced a comparison theorem for BSDEs with two reflecting barriers. Hu and Peng [15] established a comparison theorem for multidimensional BSDEs. Comparison theorems for BSDEs have received much attention because of their importance in applications. For example, the penalization method for reflected BSDEs is based on a comparison theorem, (see [9], [11], [17] and [22]). Moreover,

*The authors wish to thank SSHRC, NSERC and the ARC for support.

†Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, AB, T2N 1N4, Canada. E-mail: yangzhezhe@gmail.com

‡Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, AB, T2N 1N4, Canada. E-mail: dimbikeli@gmail.com

§Haskayne School of Business, University of Calgary, 2500 University Drive NW, Calgary, AB, T2N 1N4, Canada, School of Mathematical Sciences, University of Adelaide, SA 5005, Australia. E-mail: relliott@ucalgary.ca

research on properties of g-expectations, (see Peng [21]), and the proof of a monotonic limit theorem for BSDEs, (see Peng [20]), both depend on comparison theorems. BSDEs with jumps were also introduced by other authors. We mention [1] and [23]. Crepey and Matoussi [8] considered BSDEs with jumps in a more general framework, where a Brownian motion is incorporated in the model and a general random measure is used to model the jumps, which in [1] is a Poisson random measure.

It is natural to ask whether the converse of the above results holds or not. That is, if we can compare the solutions of two BSDEs with the same terminal conditions, can we compare the driver? Coquet, Hu, Mémin and Peng [7], Briand, Coquet, Mémin and Peng [2], and Jiang [16] derived converse comparison theorems for BSDEs, with no jumps. De Schemaekere [10], derived a converse comparison theorem for a model with jumps.

In 2012, van der Hoek and Elliott [25] introduced a market model where uncertainties are modeled by a finite state Markov chain, instead of Brownian motion or related jump diffusions, which are often used when pricing financial derivatives. The Markov chain has a semimartingale representation involving a vector martingale $M = \{M_t \in \mathbb{R}^N,\ t \geq 0\}$. BSDEs in this framework were introduced by Cohen and Elliott [4] as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s' dM_s, \qquad t \in [0, T].$$

Cohen and Elliott [5] and [6] gave some comparison results for multidimensional BSDEs in the Markov Chain model under conditions involving not only the two drivers but also the two solutions. If we consider two one-dimensional BSDEs driven by the Markov chain, we extend the comparison result to a situation involving conditions only on the two drivers. Consequently our comparison results are easier to use for the one-dimensional case. Moreover, our result in the Markov chain framework needs less conditions on the drivers compared to those in Crepey and Matoussi [8] which are suitable for more general dynamics.

In 2010, Cohen and Elliott [6] also introduced a non-linear expectation, the f-expectation, based on the comparison results in the same paper. Using our comparison results, we shall define f-expectation for one-dimensional BSDEs with Markov chain noise and show its properties as in [6]. Then, we shall provide a converse comparison result for the same model with the use of the f-expectation. f-expectations with strict comparison theorems allow us to describe a type of no-arbitrage as we shall show later in this paper.

The paper is organized as follows. In Section 2, we introduce the model and give some preliminary results. Section 3 shows our comparison result for one-dimensional BSDEs with Markov chain noise. We introduce the f-expectation and give its properties in Section 4. The last section establishes a converse comparison theorem.

2 The model and some preliminary results

Consider a finite state Markov chain. Following [25] and [26] of van der Hoek and Elliott, we assume the finite state Markov chain $X = \{X_t, t \geq 0\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$ and the state space of X is identified with the set of unit vectors $\{e_1, e_2 \cdots, e_N\}$ in $\mathbb{R}^N$, where $e_i = (0, \cdots, 1 \cdots, 0)'$ with 1 in the i-th position. Take $\mathcal{F}_t = \sigma\{X_s; 0 \leq s \leq t\}$ to be the σ-algebra generated by the Markov process $X = \{X_t\}$ and $\{\mathcal{F}_t\}$ to be its completed natural filtration. Since X is a right continuous with left limits (written RCLL) jump-process, the filtration $\{\mathcal{F}_t\}$ is also right-continuous. The Markov chain has the semimartingale representation:

$$X_t = X_0 + \int_0^t A_s X_s ds + M_t.$$

Here, $A = \{A_t, t \geq 0\}$ is the rate matrix of the chain X and M is a vector martingale (See Elliott, Aggoun and Moore [14]). We assume the elements $A_{ij}(t)$ of $A = \{A_t, t \geq 0\}$ are bounded. Then the martingale M is square integrable.

Denote by $[X, X]$ the optional quadratic variation of X, which is a $N \times N$ matrix process and $\langle X, X\rangle$, the unique predictable $N \times N$ matrix process such that $[X, X] - \langle X, X\rangle$ is a matrix valued martingale and write

$$L_t = [X, X]_t - \langle X, X\rangle_t, \quad t \in [0, T].$$

It is shown in [4] that:

$$\langle X, X\rangle_t = \int_0^t \text{diag}(A_s X_s) ds - \int_0^t \text{diag}(X_s) A'_s ds - \int_0^t A_s \text{diag}(X_s) ds. \tag{2.1}$$

For $n \in \mathbb{N}$, denote for $\phi \in \mathbb{R}^n$, the Euclidean norm $|\phi|_n = \sqrt{\phi'\phi}$ and for $\psi \in \mathbb{R}^{n\times n}$, the matrix norm $\|\psi\|_{n\times n} = \sqrt{Tr(\psi'\psi)}$.

Let Ψ be the matrix

$$\Psi_t = \text{diag}(A_t X_{t-}) - \text{diag}(X_{t-}) A'_t - A_t \text{diag}(X_{t-}). \tag{2.2}$$

Then $d\langle X, X\rangle_t = \Psi_t dt$. For any $t > 0$, Cohen and Elliott [4, 6], define the semi-norm $\|.\|_{X_t}$, for $C, D \in \mathbb{R}^{N\times K}$ as:

$$\langle C, D\rangle_{X_t} = Tr(C'\Psi_t D),$$
$$\|C\|^2_{X_t} = \langle C, C\rangle_{X_t}.$$

We only consider the case where $C \in \mathbb{R}^N$, hence we introduce the semi-norm $\|.\|_{X_t}$ as:

$$\langle C, D\rangle_{X_t} = C'\Psi_t D,$$
$$\|C\|^2_{X_t} = \langle C, C\rangle_{X_t}. \tag{2.3}$$

It follows from equation (2.1) that

$$\int_t^T \|C\|^2_{X_s} ds = \int_t^T C' d\langle X, X\rangle_s C.$$

Denote by $\mathcal{P}$, the σ-field generated by the predictable processes defined on $(\Omega, P, \mathcal{F})$ and with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$. For $t \in [0, \infty)$, consider the following spaces:

$L^2(\mathcal{F}_t) := \{\xi : \xi$ is an $\mathbb{R}$-valued $\mathcal{F}_t$-measurable random variable such that $E[|\xi|^2] < +\infty\}$;

$L^2_{\mathcal{F}}(0, t; \mathbb{R}) := \{\phi : [0, t] \times \Omega \to \mathbb{R};\ \phi$ is an adapted and RCLL process with $E[\int_0^t |\phi(s)|^2 ds] < +\infty\}$;

$P^2_{\mathcal{F}}(0, t; \mathbb{R}^N) := \{\phi : [0, t] \times \Omega \to \mathbb{R}^N;\ \phi$ is a predictable process with $E[\int_0^t \|\phi(s)\|^2_{X_s} ds] < +\infty\}$.

Consider a one-dimensional BSDE with Markov chain noise as follows:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dM_s, \quad t \in [0, T]. \tag{2.4}$$

Here the terminal condition ξ and the coefficient f are known.

Lemma 2.1 (Theorem 6.2 in Cohen and Elliott [4]) gives the existence and uniqueness result of solutions for this kind of equation.

Lemma 2.1. *Assume $\xi \in L^2(\mathcal{F}_T)$, the function $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies a Lipschitz condition, in the sense that there exists two constants $l_1, l_2 > 0$ such that P-a.s. for each $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^N$, $t \in [0,T]$,*

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq l_1|y_1 - y_2| + l_2\|z_1 - z_2\|_{X_t}, \tag{2.5}$$

and for each $(y,z) \in \mathbb{R} \times \mathbb{R}^N$, the process $(f(t,y,z))_{t\in[0,T]}$ is predictable. We also assume f satisfies

$$E[\int_0^T |f(t,0,0)|^2 dt] < \infty. \tag{2.6}$$

Then there exists a solution $(Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times P^2_{\mathcal{F}}(0,T;\mathbb{R}^N)$ to BSDE (2.4). Moreover, this solution is unique among $(Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times P^2_{\mathcal{F}}(0,T;\mathbb{R}^N)$ and up to indistinguishability for Y and equality $d\langle X,X\rangle_t \times \mathbb{P}$-a.s. for Z.

The following lemma is an extension to stopping times of Lemma 2.1 (see Cohen and Elliott [6]).

Lemma 2.2. *Suppose $\tau > 0$ is a stopping time such that there exists a real value T with $P(\tau > T) = 0$, $\xi \in L^2(\mathcal{F}_\tau)$ and f satisfies* (2.5) *and* (2.6)*, with integration from 0 to τ, then the BSDE*

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s,Y_s,Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s' dM_s, \qquad t \geq 0$$

has a solution $(Y,Z) \in L^2_{\mathcal{F}}(0,\tau;\mathbb{R}) \times P^2_{\mathcal{F}}(0,\tau;\mathbb{R}^N)$. Moreover, this solution is unique up to indistinguishability for Y and equality $d\langle X,X\rangle_t \times \mathbb{P}$-a.s. for Z.

See Campbell and Meyer [3] for the following definition:

Definition 2.3 (Moore-Penrose pseudoinverse)**.** *The Moore-Penrose pseudoinverse of a square matrix Q is the matrix $Q^\dagger$ satisfying the properties:*

1) $QQ^\dagger Q = Q$

2) $Q^\dagger QQ^\dagger = Q^\dagger$

3) $(QQ^\dagger)' = QQ^\dagger$

4) $(Q^\dagger Q)' = Q^\dagger Q$.

Recall the matrix Ψ given by (2.2). We adapt Lemma 3.5 in Cohen and Elliott [6] to our one-dimensional framework as follows:

Lemma 2.4. *For any driver satisfying* (2.5) *and* (2.6)*, for any Y and Z*

$$P(f(t,Y_{t-},Z_t) = f(t,Y_{t-},\Psi_t\Psi_t^\dagger Z_t), \text{ for all } t \in [0,+\infty]) = 1$$

and

$$\int_0^t Z_s' dM_s = \int_0^t (\Psi_s\Psi_s^\dagger Z_s)' dM_s.$$

Therefore, without any loss of generality, assume $Z = \Psi\Psi^\dagger Z$.

The following lemma which gives the duality between the solutions to linear BSDEs and linear SDEs is Theorem 2 in [5], adapted for our one-dimensional case with Markov chain noise:

Lemma 2.5. *(Linear BSDEs) Let (η,μ) be a $du \times P - a.s.$ bounded $(\mathbb{R}^{1\times N},\mathbb{R})$ valued predictable process, $g \in P^2_{\mathcal{F}}(0,T,\mathbb{R})$ and $\xi \in L^2(\mathcal{F}_T)$. Then the linear BSDE given by*

$$Y_t = \xi + \int_t^T (\mu_s Y_s + \eta_s Z_s + g_s)ds - \int_t^T Z_s' dM_s, \quad t \in [0,T]$$

has a unique solution $(Y,Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times P^2_{\mathcal{F}}(0,T;\mathbb{R}^N)$*, (up to appropriate sets of measure zero). Furthermore, if for all* $s \in [t,T]$

$$1 + \eta_s \Psi_s^{\dagger}(e_j - X_{s-}) \tag{2.7}$$

is non-zero (invertible for the multi-dimensional case) and non-negative for all j *such that* $e_j' A_s X_{s-} > 0$*, except possibly on some evanescent set, then* Y *is given by the explicit formula*

$$Y_t = E[\xi U_T + \int_t^T g_s U_s ds | \mathcal{F}_t] \tag{2.8}$$

up to indistinguishability. Here U *is the solution to the one-dimensional SDE:*

$$\begin{cases} dU_s = U_s \mu_s ds + U_{s-} \eta_s (\Psi_s^{\dagger})' dM_s, & s \in [t,T]; \\ U_t = 1. \end{cases} \tag{2.9}$$

3 A comparison theorem for one-dimensional BSDEs with Markov chain noise

We need the following assumption for our comparison results below.

Assumption 3.1. *Assume the Lipschitz constant* l_2 *of the driver* f *given in* (2.5) *satisfies*

$$l_2 \|\Psi_t^{\dagger}\|_{N\times N} \sqrt{6m} < 1, \quad \textit{for any } t \in [0,T],$$

where Ψ *is given in* (2.2) *and* $m > 0$ *is the bound of* $\|A_t\|_{N\times N}$*, for any* $t \in [0,T]$*.*

For $i = 1,2$, suppose $(Y^{(i)}, Z^{(i)})$ is the solution of a one-dimensional BSDE with Markov chain noise:

$$Y_t^{(i)} = \xi_i + \int_t^T f_i(s, Y_s^{(i)}, Z_s^{(i)}) ds - \int_t^T (Z_s^{(i)})' dM_s, \quad t \in [0,T].$$

Theorem 3.2. *Assume* $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$ *and* $f_1, f_2 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy some conditions such that the above two BSDEs have unique solutions. Moreover assume* f_1 *satisfies* (2.5) *and Assumption 3.1. If* $\xi_1 \le \xi_2$*, a.s. and* $f_1(t, Y_t^{(2)}, Z_t^{(2)}) \le f_2(t, Y_t^{(2)}, Z_t^{(2)})$*, a.e., a.s., then*

$$P(Y_t^{(1)} \le Y_t^{(2)}, \quad \textit{for any } t \in [0,T]) = 1.$$

Moreover,

$$Y_0^{(1)} = Y_0^{(2)} \Longleftrightarrow \begin{cases} f_1(t, Y_t^{(2)}, Z_t^{(2)}) = f_2(t, Y_t^{(2)}, Z_t^{(2)}), \ \textit{a.e., a.s.;} \\ \xi_1 = \xi_2, \ \textit{a.s.} \end{cases}$$

Proof. Set $Y_t = Y_t^{(2)} - Y_t^{(1)}$, $Z_t = Z_t^{(2)} - Z_t^{(1)}$, $\xi = \xi_2 - \xi_1$, $f_s = f_2(s, Y_s^{(2)}, Z_s^{(2)}) - f_1(s, Y_s^{(2)}, Z_s^{(2)})$, and define

$$a_s = \begin{cases} \dfrac{f_1(s, Y_s^{(2)}, Z_s^{(2)}) - f_1(s, Y_s^{(1)}, Z_s^{(2)})}{Y_s}, & \text{if } \ Y_s \neq 0; \\ 0, & \text{if } \ Y_s = 0 \end{cases}$$

and

$$b_s = \begin{cases} \dfrac{f_1(s, Y_s^{(1)}, Z_s^{(2)}) - f_1(s, Y_s^{(1)}, Z_s^{(1)})}{|Z_s|_N^2} Z_s', & \text{if } \ Z_s \neq 0; \\ 0, & \text{if } \ Z_s = 0. \end{cases}$$

Then, we have:

$$Y_t = \xi + \int_t^T (a_s Y_s + b_s Z_s + f_s) ds - \int_t^T Z_s' dM_s, \quad t \in [0,T]. \tag{3.1}$$

□

Lemma 3.3. *For any $C \in \mathbb{R}^N$,*

$$\|C\|_{X_t} \leq \sqrt{3m}|C|_N, \quad \text{for any } t \in [0,T],$$

where $m > 0$ is the bound of $\|A_t\|_{N\times N}$, for any $t \in [0,T]$.

Proof. Since the elements $A_{ij}(t)$ of $A = \{A_t, t \geq 0\}$ are bounded, there exists a constant $m > 0$ such that $\|A_t\|_{N\times N} \leq m$, for any $t \in [0,T]$. From the definition in (2.3), we have:

$$\begin{aligned}
\|C\|^2_{X_t} &\leq |C|^2_N \cdot \|\text{diag}(A_t X_t) - \text{diag}(X_t)A'_t - A_t\text{diag}(X_t)\|_{N\times N} \\
&\leq |C|^2_N \cdot (\|\text{diag}(A_t X_t)\|_{N\times N} + \|\text{diag}(X_t)A'_t\|_{N\times N} + \|A_t\text{diag}(X_t)\|_{N\times N}) \\
&\leq |C|^2_N \cdot (|A_t X_t|_N + |X_t|_N \cdot \|A_t\|_{N\times N} + \|A_t\|_{N\times N} \cdot |X_t|_N) \\
&\leq |C|^2_N \cdot (\|A_t\|_{N\times N} \cdot |X_t|_N + |X_t|_N \cdot \|A_t\|_{N\times N} + \|A_t\|_{N\times N} \cdot |X_t|_N) \\
&\leq 3|C|^2_N \cdot \|A_t\|_{N\times N} \leq 3m|C|^2_N \ .
\end{aligned}$$

□

We return to the proof of Theorem 3.2. Denote

$$dV_s = a_s ds + b_{s-}(\Psi^\dagger_s)' dM_s, \ \ s \in [0,T].$$

The solution to SDE (2.9) is given by the Doléan-Dade exponential (See [13]):

$$U_s = \exp(V_s - \frac{1}{2}\langle V^c, V^c\rangle_s) \prod_{0\leq u\leq s}(1+\Delta V_u)e^{-\Delta V_u}, \ \ s \in [0,T],$$

where

$$\Delta V_u = b_{u-}(\Psi^\dagger_u)'\Delta M_u = b_{u-}(\Psi^\dagger_u)'\Delta X_u.$$

If f_1 satisfies Assumption 3.1, we deduce

$$\begin{aligned}
|\Delta V_u| &\leq |b_{u-}|_N \cdot \|(\Psi^\dagger_u)'\|_{N\times N} \cdot |\Delta X_u|_N \\
&< l_2 \frac{\|Z_u\|_{X_u}}{|Z_u|_N} \frac{1}{\sqrt{6m}l_2}\sqrt{2} \\
&< \sqrt{3m}l_2 \frac{1}{\sqrt{6m}l_2}\sqrt{2} \\
&= 1.
\end{aligned}$$

Hence we have $U_s > 0$ for any $s \in [0,T]$. Moreover, condition (2.7) is satisfied. Hence, by Lemma 2.5, we know for any $t \in [0,T]$,

$$Y_t = E[\xi U_T + \int_t^T f_s U_s ds|\mathcal{F}_t], \ a.s.$$

As $\xi \geq 0$, a.s., and $f_s \geq 0$, a.e., a.s., it follows that for any $t \in [0,T]$, $Y_t \geq 0$, a.s. Since $Y_.$ and $E[\xi U_T + \int_.^T f_s U_s ds|\mathcal{F}_.]$ are both RCLL, by Lemma 2.21 in [13],

$$P(Y_t \geq 0, \text{ for any } t \in [0,T]) = 1.$$

Moreover, for any $s \in [0,T]$,

$$Y_0 = 0 \Longleftrightarrow \xi = 0, \ a.s., \text{ and } f_t = 0, \ a.e., \ a.s.$$

4 f-expectation

Now we introduce the nonlinear expectation, the f-expectation. The f-expectation, for a fixed driver f, is an interpretation of the solution to a BSDE as a type of nonlinear expectation. Here, we give the one-dimensional case of the definitions and properties in Cohen and Elliott [6], under the assumptions of our comparison theorems.

Assumption 4.1. *Suppose $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies* (2.5) *and* (2.6) *such that*

(I) For all $(t,y) \in \mathbb{R} \times \mathbb{R}$, $f(t,y,0) = 0$, a.s.;

(II) For all $(y,z) \in \mathbb{R} \times \mathbb{R}^N$, $t \to f(t,y,z)$ is continuous.

Before introducing the f-expectation, we shall give the following definition:

Definition 4.2. *For a fixed driver f, given $t \in [0,T]$ and $\xi \in L^2(\mathcal{F}_t)$, define for each $s \in [0,t]$,*

$$\mathcal{E}^f_{s,t}(\xi) = Y_s,$$

where (Y,Z) is the solution of

$$Y_s = \xi + \int_s^t f(u, Y_u, Z_u) du - \int_s^t Z'_u dM_u, \quad s \in [0,t].$$

Proposition 4.3. *Suppose the driver f satisfies Assumption 3.1 and Assumption 4.1. $\mathcal{E}^f_{s,t}(\cdot)$ defined above satisfies:*
(a) For any $\xi \in L^2(\mathcal{F}_s)$, $\mathcal{E}^f_{s,t}(\xi) = \xi$, a.s.
(b) If for any $\xi_1, \xi_2 \in L^2(\mathcal{F}_t)$, $\xi_1 \geq \xi_2$, a.s., then $\mathcal{E}^f_{s,t}(\xi_1) \geq \mathcal{E}^f_{s,t}(\xi_2)$. Moreover,

$$\mathcal{E}^f_{s,t}(\xi_1) = \mathcal{E}^f_{s,t}(\xi_2) \Longleftrightarrow \xi_1 = \xi_2, \text{ a.s.}$$

(c) For any $r \leq s \leq t$, $\mathcal{E}^f_{r,s}(\mathcal{E}^f_{s,t}(\xi)) = \mathcal{E}^f_{r,t}(\xi)$, a.s.
(d) For any $A \in \mathcal{F}_s$, $\mathbb{I}_A \mathcal{E}^f_{s,t}(\xi) = \mathbb{I}_A \mathcal{E}^f_{s,t}(\mathbb{I}_A \xi)$, a.s.

The proof of Proposition 4.3 is as in [6].

Definition 4.4. *Define, for $\xi \in L^2(\mathcal{F}_T)$ and a driver f,*

$$\mathcal{E}_f(\xi) := \mathcal{E}^f_{0,T}(\xi), \text{ and } \mathcal{E}_f(\xi|\mathcal{F}_t) := \mathcal{E}^f_{t,T}(\xi).$$

$\mathcal{E}_f(\xi)$ is called f-expectation and $\mathcal{E}_f(\xi|\mathcal{F}_t)$ is called conditional f-expectation.

The following properties follow directly from Definition 4.4, Proposition 4.3 and Lemma 2.2.

Proposition 4.5. *Suppose the driver f satisfies Assumption 3.1 and Assumption 4.1. Let $s, t \leq T$, be two stopping times.*
(a') For $\xi \in L^2(\mathcal{F}_t)$, $\mathcal{E}_f(\xi|\mathcal{F}_t) = \xi$, a.s.
(b') If for any $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$, $\xi_1 \geq \xi_2$, a.s., then $\mathcal{E}_f(\xi_1|\mathcal{F}_t) \geq \mathcal{E}_f(\xi_2|\mathcal{F}_t)$. Moreover, $\mathcal{E}_f(\xi_1) = \mathcal{E}_f(\xi_2) \Longleftrightarrow \xi_1 = \xi_2$, a.s.
(c') For any $s \leq t$, $\mathcal{E}_f(\mathcal{E}_f(\xi|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}_f(\xi|\mathcal{F}_s)$, a.s. Moreover, $\mathcal{E}_f(\mathcal{E}_f(\xi|\mathcal{F}_s)) = \mathcal{E}_f(\xi)$.
(d') For any $A \in \mathcal{F}_t$, $\mathbb{I}_A \mathcal{E}_f(\xi|\mathcal{F}_t) = \mathbb{I}_A \mathcal{E}_f(\mathbb{I}_A \xi|\mathcal{F}_t)$, a.s.

Remark 4.6. *In finance, f-expectations define a non-linear pricing system where $\mathcal{E}_f$ represents the initial price of an asset which is to be traded at time T. In Proposition 4.5, statement (b$'$) ensures No-arbitrage opportunity taking ξ_2 to be zero. Hence, the strict comparison theorem guarantees No-arbitrage pricing in a market where all agents agree that the pricing system is non-linear.*

5 A converse comparison theorem, for one-dimensional BSDE with Markov chain noise

Our converse comparison theorem uses the theory of an f-expectation in the previous section. The arguments in this section are adapted from [7]. For $i = 1, 2$, consider two BSDEs with the same terminal condition ξ:

$$Y_t^{(i)} = \xi + \int_t^T f_i(s, Y_s^{(i)}, Z_s^{(i)})ds - \int_t^T (Z_s^{(i)})' dM_s, \quad t \in [0, T].$$

Theorem 5.1. *Suppose f_1 satisfies Assumption 3.1, Assumption 4.1 and f_2 satisfies Assumption 4.1. Then the following are equivalent:*
i) For any $\xi \in L^2(\mathcal{F}_T)$, $\mathcal{E}_{f_1}(\xi) \le \mathcal{E}_{f_2}(\xi)$;
ii) $P(f_1(t, y, z) \le f_2(t, y, z),$ for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^N) = 1$.

Proof. ii) $\Rightarrow$ i) is given by Theorem 3.2.

Let us prove i) $\Rightarrow$ ii). For each $\delta > 0$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^N$, introduce the stopping time:

$$\tau_\delta = \tau_\delta(y, z) = \inf\{t \ge 0;\ f_2(t, y, z) \le f_1(t, y, z) - \delta\} \wedge T.$$

Suppose ii) does not hold, then there exists $\delta > 0$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^N$ such that $P(\tau_\delta(y, z) < T) > 0$. For (δ, y, z) such that $P(\tau_\delta(y, z) < T) > 0$, consider for $i = 1, 2$, the following SDE

$$\begin{cases} dY^i(t) = -f_i(t, Y^i(t), z)dt + z dM_t, & t \in [\tau_\delta, T], \\ Y^i(\tau_\delta) = y. \end{cases}$$

For $i = 1, 2$, the above equation admits a unique solution $Y^{(i)}$ (See Elliott[13], Chapter 14). Define:

$$\tau'_\delta = \inf\{t \ge \tau_\delta;\ f_2(t, Y^{(2)}(t), z) \ge f_1(t, Y^{(1)}(t), z) - \frac{\delta}{2}\} \wedge T,$$

with $\tau'_\delta = T$ if $\tau_\delta = T$. We know $\Omega = \{\tau_\delta \le \tau'_\delta\} = \{\tau_\delta < \tau'_\delta\} \cup \{\tau_\delta = \tau'_\delta\}$, which is a disjoint union, and $\{\tau_\delta = \tau'_\delta\} = \{\tau_\delta = T\}$. Hence, $\{\tau_\delta < \tau'_\delta\} = \{\tau_\delta = T\}^c = \{\tau_\delta < T\}$. It follows that $P(\tau_\delta < \tau'_\delta) > 0$.

Set $\tilde{Y} = Y^{(1)} - Y^{(2)}$, then

$$d\tilde{Y}(t) = (f_2(t, Y^{(2)}(t), z) - f_1(t, Y^{(1)}(t), z))dt.$$

Hence, by taking the integral of the above from τ_δ to τ'_δ and $\tilde{Y}(\tau_\delta) = 0$, we have

$$\tilde{Y}(\tau'_\delta) = Y^{(1)}(\tau'_\delta) - Y^{(2)}(\tau'_\delta) \le -\frac{\delta}{2}(\tau'_\delta - \tau_\delta) \le 0. \tag{5.1}$$

So we deduce

$$\tilde{Y}(\tau'_\delta) \le 0, \text{ with strict inequality on } \{\tau_\delta < \tau'_\delta\}. \tag{5.2}$$

Note, $(Y^{(i)}, z)$, $i = 1, 2$ are solutions of BSDEs with coefficients $(f_i, Y^{(i)}(T))$. It follows from Proposition 4.5 (c'), that

$$\mathcal{E}_{f_1}(Y^{(1)}(\tau'_\delta)|\mathcal{F}_{\tau_\delta}) = \mathcal{E}_{f_1}(\mathcal{E}_{f_1}(Y^{(1)}(T)|\mathcal{F}_{\tau'_\delta})|\mathcal{F}_{\tau_\delta}) = \mathcal{E}_{f_1}(Y^{(1)}(T)|\mathcal{F}_{\tau_\delta}) = y,$$

and similarly

$$\mathcal{E}_{f_2}(Y^{(2)}(\tau'_\delta)|\mathcal{F}_{\tau_\delta}) = \mathcal{E}_{f_2}(Y^{(2)}(T)|\mathcal{F}_{\tau_\delta}) = y.$$

Moreover, again from Proposition 4.5 (c'),

$$\mathcal{E}_{f_1}(Y^{(1)}(\tau'_\delta)) = \mathcal{E}_{f_2}(Y^{(2)}(\tau'_\delta)) = y.$$

On the other hands, by (5.1) and (5.2), we know

$$Y^{(1)}(\tau'_\delta) \le Y^{(2)}(\tau'_\delta)$$

and

$$P(Y^{(1)}(\tau'_\delta) < Y^{(2)}(\tau'_\delta)) > 0.$$

It then follows from Definition 4.4 and Proposition 4.5 (b') that

$$y = \mathcal{E}_{f_1}(Y^{(1)}(\tau'_\delta)) < \mathcal{E}_{f_1}(Y^{(2)}(\tau'_\delta)),$$

but from i), we have

$$\mathcal{E}_{f_1}(Y^{(2)}(\tau'_\delta)) \le \mathcal{E}_{f_2}(Y^{(2)}(\tau'_\delta)) = y,$$

which is a contradiction. So we conclude ii) holds. □

The above results may be used for market models using Markov chain noise and f-expectations. With the terminal condition fixed the converse theorem compares the f-expectations, then the drivers and vice-versa. If different pricing systems, arising from different drivers, can be chosen from the markets the converse theorem ensures the existence of a minimal pricing system because as long as the drivers are comparable, the f-expectations are.

6 Conclusion

We have considered BSDEs driven by Markov chain noise. A comparison theorem has been established using more simple conditions only on the drivers. This simplifies applications of the result. A non-linear f-expectation is defined for one dimensional BSDEs with Markov chain noise and finally a converse comparison theorem is proved using the f-expectation.

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