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The probability that n random points in a disk are in convex position

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Abstract. Pick n random points $x_1, \dots, x_n$ uniformly and independently in a disk and consider their convex hull C. Let $P_D^{n,m}$ be the probability that exactly m points among the x_i's are on the boundary of the convex hull of $\{x_1, \dots, x_n\}$ (so that $P_D^{n,n}$ is the probability that the x_i's are in a convex position).

In the paper, we provide a formula for $P_D^{n,m}$.

1 Introduction

All the random variables are assumed to be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The expectation is denoted by $\mathbb{E}$. The plane will be sometimes viewed as $\mathbb{R}^2$ or as $\mathbb{C}$ and we will pass from the real notation (e.g., (x, y)) to the complex one ($\rho e^{i\theta}$) without any warning. For a set A in $\mathbb{R}^2$, $|A|$ denotes the Lebesgue measure of A. We denote by ∂B the boundary of a set B. For any $n \geq 1$, any z, notation $z[n]$ stands for the n tuple $(z_1, \dots, z_n)$ and $z\{n\}$ for the set $\{z_1, \dots, z_n\}$. For H a compact convex domain in $\mathbb{R}^2$ with non empty interior and for any $n \geq 0$, $\mathbb{P}_H^n$ denotes the law of n i.i.d. points $z[n]$ taken under the uniform distribution over H. An n-tuple of points $\mathsf{x}[n]$ of the plane is said to be in convex position if the x_i's all belong to $\partial \mathsf{ConvexHull}(\mathsf{x}\{n\})$. Further we define

$$\mathsf{CP}_{n,m} = \{\mathsf{x}[n] : \#\{i : x_i \in \partial \mathsf{ConvexHull}(\mathsf{x}\{n\})\} = m\}$$

the set of n tuples $\mathsf{x}[n]$ for which exactly m are on the boundary of $\mathsf{ConvexHull}(\mathsf{x}\{n\})$. Hence, $\mathsf{CP}_n := \mathsf{CP}_{n,n}$ is the set of n-tuples of points in convex position. Finally, we let

$$P_H^n = \mathbb{P}_H^n(z[n] \in \mathsf{CP}_n), \tag{1}$$

$$P_H^{n,m} = \mathbb{P}_H^n(z[n] \in \mathsf{CP}_{n,m}). \tag{2}$$

The aim of the paper is to establish a formula for P_D^n, the probability that n i.i.d. random points taken under the uniform distribution in a disk D are in convex position; we will also compute $P_D^{n,m}$ the probability that exactly m points among these

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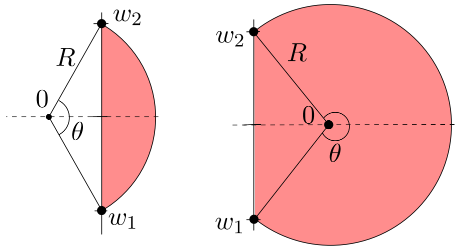

Figure 1 *Representation of typical* $\mathsf{SEG}(\theta, R)$ *for* $0 < \theta < \pi$ *and* $\pi < \theta < 2\pi$.

n points are on $\partial\mathsf{ConvexHull}(\mathsf{z}\{n\})$ in other words the distribution of the number of points on the convex hull boundary. To compute P_D^n we need and obtain a result more general than the disk case only, a result for what we will call *bi-pointed segments* (BSEG). This will play somehow the role of the bi-pointed triangle (see (9)) as studied by Bárány et al. (2000), central also in the approach of Buchta (2009/10) of the computation of $P_T^{n,m}$ and $P_S^{n,m}$ where T stands for triangle, and S for square (see (10)).

For $\theta \in [0, 2\pi]$, $R > 0$, the *arc of circle* $\mathsf{AC}(\theta, R)$ is defined by

$$\mathsf{AC}(\theta, R) = \{Re^{i\nu}, \nu \in [-\theta/2, \theta/2]\}.$$

We denote by $\mathsf{SEG}(\theta, R)$ the *segment* corresponding to the convex hull of $\mathsf{AC}(\theta, R)$ (see Figure 1), which coincides with

$$\mathsf{SEG}(\theta, R) = \{\lambda Re^{i\nu_1} + (1-\lambda)Re^{i\nu_2}, \lambda \in [0,1], \nu_1, \nu_2 \in [-\theta/2, \theta/2]\}.$$

Now consider $w_1(\theta, R) = Re^{-i\theta/2}$ and $w_2(\theta, R) = Re^{i\theta/2}$ the two *extremities* of the *special border* $[w_1(\theta, R), w_2(\theta, R)]$ of $\mathsf{SEG}(\theta, R)$. Let $z_1, \dots, z_n$ be i.i.d. and uniform in $\mathsf{SEG}(\theta, R)$. Set

$$Z[n, \theta, R] = [w_1(\theta, R), w_2(\theta, R), z_1, \dots, z_n],$$

and define the crucial *bi-pointed segment case* (BSEG) function

$$B_{n,m}(\theta) := \mathbb{P}(Z[n, \theta, R] \in \mathsf{CP}_{n+2,m+2}), \qquad \theta \in (0, 2\pi), 1 \le m \le n. \tag{3}$$

The value of R has no importance (since there exists a dilatation sending $\mathsf{SEG}(\theta, R)$ to $\mathsf{SEG}(\theta, R')$, and dilatations conserve convex bodies and uniform distribution) but it will be useful to have the two parameters (θ, R) for subsequent computations. Again, we write B_n instead of $B_{n,n}$ and below L_n instead of $L_{n,n}$. Clearly, for any $\theta \in (0, 2\pi)$, $B_0(\theta) = B_1(\theta) = 1$. Now for any $n \ge 0$, $\theta \in (0, 2\pi)$ define

$$L_{n,m}(\theta) = \frac{B_{n,m}(\theta)(\theta - \sin(\theta))^n \sin(\theta/2)}{n!}. \tag{4}$$

Hence,

$$L_0(\theta) = \sin(\theta/2), \qquad L_1(\theta) = \sin(\theta/2)(\theta - \sin(\theta)). \tag{5}$$

Notice that 0 as well as 2π, which corresponds respectively to the flat case and the disk case, are excluded from definitions (3) and (4). The main contribution of this paper is the following theorem which allows us to compute $P_D^{n,m}$.

Theorem 1.

(i) *For any* $n \geq 1$,

$$P_D^n = \lim_{t \to 2\pi^-} B_{n-1}(t).$$

(i′) *For any* $n \geq 2$,

$$P_D^n = \frac{(n-2)!}{2^{n-2}\pi^{n-1}} \int_0^{2\pi} \sum_{k=0}^{n-2} L_k(\phi) L_{n-2-k}(2\pi - \phi)\, d\phi.$$

(ii) *For any* $\theta \in (0, 2\pi)$ *and any* $n \geq 1$,

$$\frac{L_n(\theta)}{2} = \int_0^\theta \frac{\sin(\theta/2)^{2n+1}}{\sin(\phi/2)^{2n+1}} \int_0^\phi \sum_{k=0}^{n-1} L_k(\eta) L_{n-1-k}(\phi - \eta)\, d\eta\, d\phi. \tag{6}$$

Analogous results can be obtained for $P_D^{n,m}$:

(iii) *For any* $\theta \in (0, 2\pi)$ *any* k, *and any* $l \geq k+1$, $L_{k,l}(\theta) = 0$. *For any* $\theta \in (0, 2\pi)$, *any* $n \geq 1$ *and any* $1 \leq m \leq n$

$$\begin{aligned}\frac{L_{n,m}(\theta)}{2} = \int_0^\theta \int_0^\phi & \frac{\sin(\theta/2)^{2n+1}}{\sin(\phi/2)^{2n+1}} \\ & \times \sum_{\substack{n_1+n_2+n_3=n-1 \\ m_1+m_2=m-1}} \frac{(\sin(\eta) + \sin(\phi-\eta) - \sin(\phi))^{n_3}}{n_3!} \\ & \times L_{n_1,m_1}(\eta) L_{n_2,m_2}(\phi - \eta)\, d\eta\, d\phi.\end{aligned}$$

An alternative form can be given using

$$\sin(\eta) + \sin(\phi - \eta) - \sin(\phi) = 4 \sin\left(\frac{\phi - \eta}{2}\right) \sin(\phi/2) \sin(\eta/2).$$

(iii′) *For any* $n \geq 2$ *and any* $1 \leq m \leq n$

$$P_D^{n,m} = \frac{(n-2)!}{2^{n-2}\pi^{n-1}} \int_0^{2\pi} \sum_{\substack{n_1+n_2=n-1 \\ m_1+m_2=m-1}} L_{n_1,m_1}(\phi) L_{n_2,m_2}(2\pi - \phi)\, d\phi.$$

(iv) *For any* $n \geq 1$ *and any* $1 \leq m \leq n$,

$$P_D^{n,m} = \lim_{t \to 2\pi^-} B_{n-1,m-1}(t).$$

From (ii), one can compute successively the $L_j(\theta)$'s, and by (4), this allows one to compute the $B_j(\theta)$'s. By (i) it suffices then to take the limit when $\theta \to 2\pi^-$.

Despite great effort we were not able to find a simpler formula for B_n than that presented in the theorem. Nevertheless, explicit computation can be done but closed formula for the first L_j given below shows a rapid growth in complexity (L_{10} would need one page to be written down). The effective computation of the first L_n is complex and very few can be computed by hand. In particular, the singularity apparent in (6) is difficult to handle since the terms in the sum need to be combined to compensate the singularity.

In Section 3, we present an algorithm allowing one to compute the L_j's. With this algorithm we have computed the first 32 values of L_n, before running out of computer memory, which allows the computation of $(P_D^n, 1 \le n \le 33)$. They can be found at Marckert (2015). This is just a matter of power of computer/computer algebra system, or code optimization, to go further. L_0 and L_1 have been given in (5); writing for short S and C instead of $\sin(\theta/2)$ and $\cos(\theta/2)$ respectively, one founds

$$\begin{aligned}
L_2(\theta) &= -\frac{2}{3}S^5 - 2S^3 + \frac{1}{2}S\theta^2,\\
L_3(\theta) &= \frac{2CS^6}{27} + \frac{7S^4C}{27} + \frac{35CS^2}{9} + \frac{S^3\theta}{2} + \frac{S\theta^3}{6} - \frac{35S\theta}{18},\\
L_4(\theta) &= -\frac{10CS^2\theta}{9} + \frac{S^9}{270} + \frac{S^7}{81} + \frac{S^5}{216} + \frac{S\theta^4}{24} + \frac{155S^3}{24} - \frac{305S\theta^2}{288},\\
L_5(\theta) &= -\frac{CS^{10}}{10{,}125} - \frac{17S^8C}{40{,}500} - \frac{73CS^6}{81{,}000} + \frac{4427S^4C}{64{,}800} + \frac{CS^2\theta^2}{16} - \frac{473{,}473CS^2}{43{,}200}\\
&\quad + \frac{S^5\theta}{108} + \frac{S\theta^5}{120} - \frac{305S^3\theta}{144} - \frac{61S\theta^3}{144} + \frac{473{,}473S\theta}{86{,}400}.
\end{aligned}$$

We can also compute $L_{m,n}(\theta)$ for small values of m, n (they are available at Marckert (2015), for all $n \le 12$). For any $n \ge 2$, $\sum_{k=1}^{n} B_{n,k}(\theta) = 1$. Since $B_{2,2} = B_2$ is known, so do $B_{2,1}$. The next ones are

$$\begin{aligned}
L_{3,1}(\theta) &= \frac{2}{3}CS^6 - 5S^4C - 2S^5\theta + \frac{5}{2}S^3\theta,\\
L_{3,2}(\theta) &= \frac{i}{54}(32S^6 + 54iS^2\theta + 168S^4 - 54S^2\theta^2 - 105i\theta - 302S^2\\
&\quad + 27\theta^2 + 105)S + \frac{S(16S^4 + 92S^2 - 27\theta^2 - 105)}{108iS^2 + 108SC - 54i},\\
L_{4,1}(\theta) &= \frac{4}{3}CS^6\theta - \frac{7}{3}S^4\theta C + \frac{4S^9}{15} - \frac{38S^7}{9} + \frac{14}{3}S^5.
\end{aligned}$$

The next ones are too large to be written here. Using these formulae, one finds the following explicit values for P_D^n, given in Table 1 and below.

Table 1 *First values of* P_D^n

n	4	5	6	7	8
$1-P_D^n$	$\frac{35}{12\pi^2}$	$\frac{305}{48\pi^2}$	$\frac{305}{24\pi^2}-\frac{473{,}473}{11{,}520\pi^4}$	$\frac{2135}{96\pi^2}-\frac{2{,}900{,}611}{23{,}040\pi^4}$	$\frac{427}{12\pi^2}-\frac{185{,}227}{480\pi^4}+\frac{62{,}664{,}108{,}221}{48{,}384{,}000\pi^6}$

$$
\begin{aligned}
1-P_D^9 &= \frac{427}{8\pi^2}-\frac{1{,}826{,}293}{1920\pi^4}+\frac{221{,}424{,}913{,}259}{43{,}008{,}000\pi^6},\\
1-P_D^{10} &= \frac{305}{4\pi^2}-\frac{7{,}956{,}347}{3840\pi^4}+\frac{275{,}822{,}571{,}959}{12{,}902{,}400\pi^6}\\
&\quad -\frac{11{,}959{,}334{,}618{,}379{,}662{,}657}{163{,}870{,}801{,}920{,}000\pi^8},\\
1-P_D^{11} &= \frac{3355}{32\pi^2}-\frac{15{,}780{,}457}{3840\pi^4}+\frac{10{,}435{,}892{,}451{,}347}{154{,}828{,}800\pi^6}\\
&\quad -\frac{116{,}756{,}045{,}890{,}280{,}952{,}727}{327{,}741{,}603{,}840{,}000\pi^8},\\
1-P_D^{12} &= \frac{3355}{24\pi^2}-\frac{14{,}549{,}381}{1920\pi^4}+\frac{35{,}864{,}761{,}139{,}141}{193{,}536{,}000\pi^6}\\
&\quad -\frac{153{,}063{,}833{,}227{,}904{,}154{,}127}{81{,}935{,}400{,}960{,}000\pi^8}\\
&\quad +\frac{24{,}568{,}177{,}984{,}436{,}193{,}008{,}990{,}903{,}477}{3{,}815{,}698{,}848{,}546{,}816{,}000{,}000\pi^{10}}.
\end{aligned}
$$

By Theorem 1, we can also compute the first values of $P_D^{n,m}$ presented in Table 2. We have computed $P_D^{n,m}$ for all (n,m) such that $n \le 13$ (they are available at Marckert (2015)).

Some explicit results for bi-pointed half disk are in Table 3. Again, the method we have provide all the results till $n = 33$.

The value $P_D^{4,4} = 1 - 35/(12\pi^2)$ is due to Woolhouse in 1867.

The[1] values $P_D^{5,3}, P_D^{5,4}$ and $P_D^{5,5}$ as well as the values $P_D^{n,3}$ for arbitrary n are due to Miles (1971). Buchta (1984) computed the expected area V_n of the convex hull of n uniform and independent points in a disk with unit area, and found

$$V_5 = \frac{175}{72\pi^2} - \frac{23{,}023}{6912\pi^4}.$$

[1]This paragraph is due to one of the referees of the paper. We thank her/him for these precisions.

Table 2 *First values of* $P_D^{n,m}$

$n \setminus m$	3	4	5	6	7	8
4	$\frac{35}{12\pi^2}$	P_D^4	0	0	0	0
5	$\frac{15}{16\pi^2}$	$\frac{65}{12\pi^2}$	P_D^5	0	0	0
6	$\frac{1001}{320\pi^4}$	$\frac{15}{8\pi^2}+\frac{19{,}019}{1280\pi^4}$	$\frac{65}{6\pi^2}-\frac{17{,}017}{288\pi^4}$	P_D^6	0	0
7	$\frac{35}{32\pi^4}$	$\frac{777}{40\pi^4}$	$\frac{105}{32\pi^2}+\frac{106{,}099}{7680\pi^4}$	$\frac{455}{24\pi^2}-\frac{184{,}583}{1152\pi^4}$	P_D^7	0
8	$\frac{138{,}567}{35{,}840\pi^6}$	$\frac{875}{192\pi^4}+\frac{4{,}110{,}821}{64{,}512\pi^6}$	$\frac{7413}{160\pi^4}-\frac{203{,}739{,}679}{2{,}688{,}000\pi^6}$	$\frac{21}{4\pi^2}+\frac{803{,}747}{8640\pi^4}-\frac{22{,}301{,}758{,}193}{20{,}736{,}000\pi^6}$	$\frac{91}{3\pi^2}-\frac{457{,}751}{864\pi^4}+\frac{1{,}233{,}200{,}111}{518{,}400\pi^6}$	P_D^8

Table 3 *First values for the bi-pointed half disk case*

n	2	3	4	5	6
$1 - B_n(\pi)$	$\frac{16}{3\pi^2}$	$\frac{26}{3\pi^2}$	$\frac{305}{12\pi^2} - \frac{20{,}992}{135\pi^4}$	$\frac{305}{6\pi^2} - \frac{97{,}091}{240\pi^4}$	$\frac{2135}{24\pi^2} - \frac{3{,}102{,}211}{1440\pi^4} + \frac{960{,}925{,}696}{70{,}875\pi^6}$

By Efron (1965),

$$3P_D^{6,3} + 4P_D^{6,4} + 5P_D^{6,5} + 6P_D^{6,6} = 6(1 - V_5) = 6 - \frac{175}{12\pi^2} + \frac{23{,}023}{6912\pi^4}$$

and since $P_D^{6,3} + P_D^{6,4} + P_D^{6,5} + P_D^{6,6} = 1$ and Miles' result $P_{6,3} = \frac{1001}{320\pi^4}$ the following relations hold

$$P_D^{6,4} = P_D^{6,6} - 1 + \frac{175}{12\pi^2} - \frac{151{,}151}{5760\pi^4}$$

and

$$P_D^{6,5} = 2 - \frac{175}{12\pi^2} + \frac{133{,}133}{5760\pi^4} - 2P_D^{6,6}.$$

Of course, the result Table 2 we obtained are compatible with these relations.

Besides these results and those exposed in Theorem 1, the only explicit results in the literature concern triangles and parallelograms (we here discuss only results known for any n, in 2D). Valtr (1995) showed that if S is a square (or a non flat parallelogram) then, for $n \geq 1$,

$$P_S^n = \left(\frac{\binom{2n-2}{n-1}}{n!} \right)^2, \tag{7}$$

and in a second paper, Valtr (1996), he proved that if T is a (non flat) triangle then, for $n \geq 1$,

$$P_T^n = \frac{2^n (3n-3)!}{(n-1)!^3 (2n)!}. \tag{8}$$

Buchta (2009/10) goes further and gives an expression for $P_S^{n,m}$ and $P_T^{n,m}$ as a finite sum of explicit terms.

For the bi-pointed triangle, Bárány et al. (2000) have shown the following. Let $T = (A, B, C)$ be a (non-flat) triangle, and let $(z_1, \ldots, z_n)$ be $\mathbb{P}_T^n$ distributed, and let $\overline{\mathsf{z}[n]} = (A, B, z_1, \ldots, z_n)$ be the $n+2$ tuple obtained by adding A, B to $\mathsf{z}[n]$. For any $n \geq 0$,

$$\mathbb{P}_T^n\left(\overline{\mathsf{z}[n]} \in \mathsf{CP}_{n+2}\right) = \frac{2^n}{n!(n+1)!}. \tag{9}$$

These results are at the start of several works concerning limit shape for convex bodies in a domain (Bárány et al. (2000), Bárány (1999)) and for the evaluation of the probability that n points chosen in a convex domain H are in convex position (see Bárány (1999)).

Buchta (2006) proved the following fact: For any $n \geq 1$, any $1 \leq m \leq n$,

$$\mathbb{P}_T^n\big(\overline{\mathsf{z}[n]} \in \mathsf{CP}_{n+2,m+2}\big) = \sum_{C \in \mathsf{Comp}(n,m)} 2^m \prod_{i=1}^{m} \frac{C_i}{SC_i(1+SC_i)}, \tag{10}$$

where $SC_i = C_1 + \cdots + C_i$ and $\mathsf{Comp}(n,m)$ is the set of compositions of n in m non-empty parts (Examples: $\mathsf{Comp}(2,3) = \varnothing$, $\mathsf{Comp}(4,2) = \{(1,3),(3,1),(2,2)\}$).

Additional references

The literature concerning the question of the number of points on the convex hull for i.i.d. random points taken in a convex domain is huge. We won't make a survey here but refer to Reitzner (2010), Hug (2013) and to the papers cited in the present paper. We will focus on what concerns the disk.

Blaschke (1917) proves that for the 4 points problem (the so-called *problem of Sylvester*), for any convex K,

$$\frac{2}{3} \leq P_T^4 \leq P_K^4 \leq P_D^4 = 1 - \frac{35}{12\pi^2}.$$

Bárány (1999) has shown that

$$\lim_{n \to +\infty} n^2 \big(P_K^n\big)^{1/n} = e^2 A^3(K)/4, \tag{11}$$

where $A^3(K)$ is the supremum of the *affine perimeter* of all convex sets $S \subset K$. For the disk one gets

$$\log\big(P_D^n\big) = -2n\log n + n\log(2\pi^2 e^2) - 2\varepsilon_0\big(3\pi^4 n\big)^{1/5} + \cdots, \tag{12}$$

where the last term, not really proved in the mathematical sense, has been obtained by Hilhorst, Calka and Schehr (2008). Central limit theorems exists also for the number of points on $\partial\mathsf{ConvexHull}(\mathsf{x}\{n\})$ under $\mathbb{P}_D^n$ (and for more general domain, under the uniform or Poisson distribution), see Groeneboom (2012), Buchta (2013), Pardon (2012), Bárány and Reitzner (2010).

2 Proof of Theorem 1

Beyond the appearances, the proof of Theorem 1 is quite simple and it relies on a paradigm of combinatorics that can be stated as follows: *always try to decompose the structure you are studying!* But how can we decompose P_D^n? The two main ideas of the paper are the following:

– $B_n(\theta)$ can be decomposed,
– with $B_n(\theta)$ one can compute P_D^n.

2.1 Proof of (i)

Throughout this section, $n \geq 1$ is fixed. Take a closed disk $\overline{B} = \overline{B}((0,0), R_c)$, with center $(0,0)$ and radius $R_c = 1/\sqrt{\pi}$, that is with area 1, and pick n i.i.d. uniform points $U_1, \dots, U_n$ in $\overline{B}$. Now consider the smallest disk $\overline{B}((0,0), R_n)$ that contains all the U_i's. Clearly

$$R_n = \inf\{r : \#(\overline{B}(0,r) \cap \{U_1, \dots, U_n\}) = n\}.$$

Proposition 2. *Conditionally on $R_n = r$, there is a.s. exactly one index $J \in \{1, \dots, n\}$ such that U_J belongs to the circle $\partial B((0,0), r)$. Conditionally on $\{J = j, R_n = r\}$, U_j and $(U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_n)$ are independent, U_j has the uniform law on the circle $\partial B((0,0), r)$, and $U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_n$ are uniform in $\overline{B}((0,0), r)$.*

Proof. A.s. the points $U_1, \dots, U_n$ belong to different circles with center $(0,0)$, and by symmetry conditionally on $R_n = r$ and $J = j$, U_j is uniform on $B((0,0), r)$. Now, conditionally on $R_n = r$ and $J = j$, each variable U_ℓ (for $\ell \neq j$) are just conditioned to satisfy $\|U_\ell\|_2 \leq r$, and this conditioning conserves the uniform distribution. □

Proof of Theorem 1(i). Theorem 1(i) is—or should be—intuitively obvious, taking into account Proposition 2. But of course, a formal argument is needed. Consider the three following models:

(a) n points i.i.d. uniform in a disk $B((0,0), R)$,
(b) one point uniform on the circle $\partial B((0,0), R)$ and, independently, $n-1$ i.i.d. uniform inside the disk $B((0,0), R)$,
(c) one point is placed at $(-R, 0)$ and $n-1$ are taken uniformly and independently in $B((0,0), R)$.

We claim that these three models are equivalent with respect to the probability to be in convex position.

The equivalence between (a) and (b) follows Proposition 2. Indeed, dilatation conserves uniform distribution and convexity. Therefore, conditionally on $R_n = r$, and $J = j$, since U_j is uniform on $\partial B((0,0), r)$ and the other U_i's are i.i.d. uniform inside $B((0,0), r)$, by a dilatation, the probability that these n points are in a convex position is the same as in the case where U_j is taken on $\partial B((0,0), R)$, and the other ones taken independently and uniformly inside $B((0,0), R)$ and this is true for any fixed R and j.

Now (b) and (c) are equivalent for the following reason: Consider the rotation ψ with center $(0,0)$ which sends U_j on $(-R, 0)$. This rotation conserves convexity, and the uniform distribution on $B((0,0), R)$. Hence, the random variables $\psi(U_i)$'s for $i \neq j$ are independent and uniform on $B((0,0), R)$.

Hence, we have establish that the probability that n points are in a convex position is the same in the model (a), and in the model (c). We will then work on this third model.

Now if we come back to the BSEG considerations, when $\theta \to 2\pi$, the points $w_1(R_c, \theta)$ and $w_2(R_c, \theta)$ become closer and closer, and the line passing by these points lets all the other points in one of the half plane it defines. It is intuitively clear that replacing $w_1(R_c, \theta)$ and $w_2(R_c, \theta)$ by a single point *close* to them (for example, at position $(-R_c, 0)$) will not dramatically change the model nor the probability to be in convex position. This is the essence of Theorem 1(i).

For sake of completeness, let us give a formal proof. Take $R > 0$ and consider the two sets $S(\varepsilon) = \mathsf{SEG}(2\pi - \varepsilon, R)$ and $S = \mathsf{SEG}(2\pi, R) = \overline{B}((0,0), R)$. These two sets are closed for the Hausdorff topology when ε is small. We always have $S(\varepsilon) \subset S$, and $|S \setminus S(\varepsilon)|$ goes to 0. This property implies that if we fix $\varepsilon' > 0$, for ε small enough, for $z_1, \dots, z_n$ chosen uniformly and independently under $\mathbb{P}_S$,

$$\mathbb{P}\big(\{z_1, \dots, z_n\} \subset S(\varepsilon)\big) \geq 1 - \varepsilon'. \tag{13}$$

Conditionally on the event $\Lambda_\varepsilon := \{\{z_1, \dots, z_n\} \subset S(\varepsilon)\}$, the z_i's are i.i.d. uniform in $S(\varepsilon)$. Let $w_1(\varepsilon) = w_1(R_c, 2\pi - \varepsilon)$, $w_2(\varepsilon) = w_2(R_c, 2\pi - \varepsilon)$, $w = -R$.

We want to show that $\mathbb{P}((z_1, \dots, z_n, w_1^\varepsilon, w_2^\varepsilon) \in \mathsf{CP}_{n+2} | \Lambda_\varepsilon) \to \mathbb{P}((z_1, \dots, z_n, w) \in \mathsf{CP}_{n+1})$. Consider the following sets (subsets of S^n):

$$E_1(\varepsilon) := \big\{(t_1, \dots, t_n) \in S(\varepsilon) : \big(t_1, \dots, t_n, w_1(\varepsilon), w_2(\varepsilon)\big) \in \mathsf{CP}_{n+2}\big\},$$
$$E_2 := \big\{(t_1, \dots, t_n) \in S : (t_1, \dots, t_n, w) \in \mathsf{CP}_{n+1}\big\}.$$

It suffices to prove that $|E_1(\varepsilon)| \underset{\varepsilon \to 0}{\to} |E_2|$. First $E_1(\varepsilon) \subset E_2$ since if $(t_1, \dots, t_n, w_1(\varepsilon), w_2(\varepsilon))$ belongs to CP_{n+2} and since the segments $[w_1(\varepsilon), w]$ and $[w_2(\varepsilon), w]$ are chords, then $(z_1, \dots, z_n, w_1(\varepsilon), w_2(\varepsilon), w)$ is in CP_{n+3} from what we deduce that E_2 is in CP_{n+1}.

To end the proof, take $(t_1, \dots, t_n) \in E_2$. We show that when ε is small enough, it is in $E_1(\varepsilon)$. More precisely, we will see that it is not the case only if the t_i belongs to a null set (for Lebesgue measure). We assume that $n \geq 2$ since for $n = 1$ the result is clear.

First, for $\varepsilon > 0$ small enough, if the t_i's are different and different to $-R$, all the t_i belongs to $S(\varepsilon)$. Since $(t_1, \dots, t_n, w) \in \mathsf{CP}_{n+1}$ draw the convex polygon p passing by these points, and relabel the t_i's as $t_1^\star, \dots, t_n^\star$ clockwise around p so that the neighbours of w are $t_1^\star$ and $t_n^\star$. Again, up to null set, the angles $(w, t_1^\star, t_2^\star)$ and $(t_{n-1}^\star, t_n^\star, w)$ are not 0, and it appears clearly that for ε small enough, $(t_1, \dots, t_n, w_1(\varepsilon), w_2(\varepsilon)) \in \mathsf{CP}_{n+2}$. We then have $E_2 = \bigcup_\varepsilon E_1(\varepsilon)$ and the $E_1(x) \cup E_1(x')$ if $x' < x$, so $|E_1(\varepsilon)| \to |E_2|$ when ε goes to 0. $\square$

2.2 Proof of (ii)

For any $\theta \in [0, 2\pi]$, $R > 0$,

$$\big|\mathsf{SEG}(\theta, R)\big| := \frac{R^2}{2}\big(\theta - \sin(\theta)\big) \tag{14}$$

and then for

$$R_\theta = \sqrt{\frac{2}{\theta - \sin(\theta)}}, \tag{15}$$

the area $|\mathsf{SEG}(\theta, R_\theta)| = 1$. Fix θ and denote for abbreviation by SEG_θ the segment $\mathsf{SEG}(\theta, R_\theta)$ with unit area. The size L_θ of the special border $[w_1(\theta, R_\theta), w_2(\theta, R_\theta)]$ for this segment is

$$L_\theta = 2R_\theta \sin(\theta/2). \tag{16}$$

In this section, we fix $\theta \in (0, 2\pi)$ and search to express $B_n(\theta)$ with some combinations of $B_j(\nu)$, for $\nu < \theta$ and $j < n$. To get the decomposition, we will "push the arc of circle" $\mathsf{AC}(\theta, R)$ inside $\mathsf{SEG}(\theta, R_\theta)$ till it touches one of the z_i's doing something similar to the Buchta's method (for the computation of P_S^n and P_T^n). Here it is a bit more complex: we need the arc of circle to stay an arc of circle during the operation in order to get a nice decomposition, and also we need to keep the bi-pointed elements. The arc angle and radius will change during the operation. This will lead to a quadratic formula for B_n. Almost all quantities appearing in this section should be indexed by θ. In order to avoid heavy notation, we won't do this. Draw SEG_θ in the plane. We consider the family of segments

$$\mathcal{F}_\theta := \big(\mathsf{SEG}[\phi], 0 \le \phi \le \theta\big)$$

having as special border the special border of SEG_θ, that is $[w_1(\theta, R_\theta), w_2(\theta, R_\theta)]$, and lying at its right, such that the angle of $\mathsf{SEG}[\phi]$ is ϕ (see Figure 2).

When ϕ goes from θ to 0, the center $O[\phi]$ of (the circle which defines) $\mathsf{SEG}[\phi]$ moves on the x-axis from $O[\theta] = 0$ to $(-\infty, 0)$. Comparing the distance from $O[\phi]$ to the special border, we can compute the coordinate of $O[\phi]$:

$$O[\phi] = \frac{L_\theta}{2}\left(\cot\left(\frac{\theta}{2}\right) - \cot\left(\frac{\phi}{2}\right)\right) \tag{17}$$

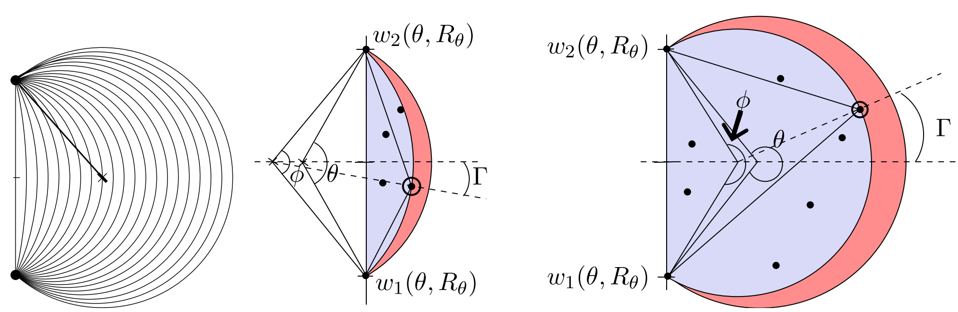

Figure 2 *Representation of the family $\mathcal{F}_\theta$. The angle $\phi < \theta$ and $\mathsf{SEG}[\phi] \le \mathsf{SEG}[\theta]$. The angles are taken at the center of the circle that defines the segments.*

and the radius of $\mathsf{SEG}[\phi]$,

$$R[\phi] = R_\theta \frac{\sin(\theta/2)}{\sin(\phi/2)}. \tag{18}$$

Since the special border of all the $\mathsf{SEG}[\phi]$ is the same one sees that if $\phi < \phi'$ then $\mathsf{SEG}[\phi] \subset \mathsf{SEG}[\phi']$. When ϕ goes to 0, $\mathsf{SEG}[\phi]$ goes to $[w_1(\theta, R_\theta), w_2(\theta, R_\theta)]$ (for the Hausdorff topology). One also sees that $\mathsf{SEG}[\theta] = \mathsf{SEG}_\theta$, and for $\phi < \theta$, by (14) and (15),

$$\left|\mathsf{SEG}[\phi]\right| = \left(\frac{\sin(\theta/2)}{\sin(\phi/2)}\right)^2 \frac{\phi - \sin(\phi)}{\theta - \sin(\theta)} \tag{19}$$

and then the other segments of the family $\mathcal{F}_\theta$ have area smaller than 1 (see Figure 2).

Again θ is fixed. Let $z_1, \dots, z_n$ be $n \geq 1$ i.i.d. uniform random points in SEG_θ. Denote by

$$\Phi = \min\{\phi : \#(\{z_1, \dots, z_n\} \cap \mathsf{SEG}[\phi]\}) = n\},$$

and let J the (a.s. unique) index of the variable z_j on $\partial\mathsf{SEG}[\phi]$. Finally, let Γ be the (signed) angle $((+\infty, 0), O[\Phi], z_J)$ formed by the x-axis and the line $(0[\Phi], z_J)$ (see Figure 2). We have the following proposition.

Proposition 3. *The distribution of* (Φ, Γ) *admits the following density* $f_{(\Phi,\Gamma)}$ *with respect to the Lebesgue measure*

$$f_{(\Phi,\Gamma)}(\phi, \gamma) = n \frac{\sin(\theta/2)^{2n}}{(\theta - \sin(\theta))^n} \frac{(\phi - \sin(\phi))^{n-1}}{\sin(\phi/2)^{2n+1}} \\ \times \left(\cos(\gamma) - \cos(\phi/2)\right) 1_{0 \leq \phi \leq \theta} 1_{|\gamma| \leq \phi/2}.$$

Proof. First, the density of $z_J = (x, y)$ with respect to the Lebesgue measure on $|\mathsf{SEG}_\theta|$ is $n\,dx\,dy|\mathsf{SEG}_{x,y}|^{n-1}$ where $|\mathsf{SEG}_{x,y}|^{n-1}$ is the area of the unique element of the family $\mathcal{F}_\theta$ whose border contains (x, y) (indeed $z_J = (x, y)$ if (a.s.) all the points $z_1, \dots, z_n$ are inside $\mathsf{SEG}_{x,y}$ except one of them, which lies exactly at (x, y)). We then just have to make a change of variables in this formula.

We search the unique pair (ϕ, γ) such that

$$x + iy = R[\phi]e^{i\gamma} + O[\phi].$$

Since by (19) and (17) everything is explicit, we can compute the Jacobian

$$\left| \det \begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \gamma} \end{pmatrix} \right| = \frac{\sin(\theta/2)^2}{\sin(\phi/2)^3} \frac{(\cos(\gamma) - \cos(\phi/2))}{(\theta - \sin(\theta))}.$$

From what we deduce the wanted formula, using (19). □

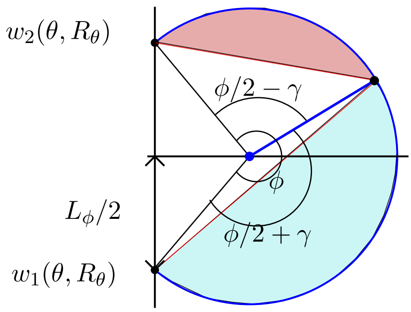

Figure 3 *Decomposition of the computation of $B_n(\theta)$, and definition of the two sub-segments appearing in the decomposition.*

Now, it remains to end the decomposition of our problem. Conditionally on $(\Phi, \Gamma, J) = (\phi, \gamma, j)$, the points $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$ are i.i.d. uniform in $\mathsf{SEG}[\phi]$.

The triangle $T := (w_1(\theta, R_\theta), w_2(\theta, R_\theta), z_j)$ is inscribed in $\mathsf{SEG}[\phi]$ and $\mathsf{SEG}[\phi] \setminus T$ produces two segments S_1 and S_2. Since we may rescale $\mathsf{SEG}[\phi]$ to be SEG_ϕ (to get area 1), the question now is that of the area of the two rescaled segments. After rescaling, S_1 and S_2 appear to be $\mathsf{SEG}[\phi/2+\gamma, R_\phi]$ and $\mathsf{SEG}[\phi/2-\gamma, R_\phi]$ by identification of the angles. Using (14)

$$\left|\mathsf{SEG}[\alpha, R_\phi]\right| = \frac{\alpha - \sin(\alpha)}{\phi - \sin(\phi)}. \tag{20}$$

We keep temporarily notation S_1 and S_2 instead of $\mathsf{SEG}[\phi/2+\gamma, R_\phi]$ and $\mathsf{SEG}[\phi/2-\gamma, R_\phi]$ for short. The following proposition is a consequence of the fact that uniform distribution is preserved by conditioning. It is the "combinatorial decomposition" of the computation of $B_n(\theta)$, illustrated on Figure 3.

Proposition 4.

(i) *Conditionally on* $(\Phi, \Gamma, J) = (\phi, \gamma, j)$, *the respective number* (N_1, N_2, N_3) *of points of* $z\{n\} \setminus \{z_j\}$ *in* S_1, S_2 *and* $\mathsf{SEG}_\phi - (S_1 \cup S_2)$ *is*

$$\mathsf{Multinomial}\big(n-1, |S_1|, |S_2|, 1-|S_1|-|S_2|\big).$$

(ii) *Conditionally on* $(\Phi, \Gamma, J) = (\phi, \gamma, j)$ *and* $(N_1, N_2, N_3) = (k_1, k_2, k_3)$ *the points* $z_1, \dots, z_n$ *are in convex position with probability* $1_{k_3=0, k_1+k_2=n-1} \times B_{k_1}(\phi/2+\gamma) B_{k_2}(\phi/2-\gamma)$.

Putting everything together, we have obtained

$$B_n(\theta) = \int_0^\theta \int_{-\frac{\phi}{2}}^{\frac{\phi}{2}} f_{(\Phi,\Gamma)}(\phi, \gamma) \sum_{k=0}^{n-1} \binom{n-1}{k} |S_1|^k |S_2|^{n-1-k}$$
$$\times B_k(\phi/2+\gamma) B_{n-1-k}(\phi/2-\gamma)\, d\gamma\, d\phi$$

Set $\eta = \phi/2 + \gamma$, $d\eta = d\gamma$, η goes from 0 to ϕ (and $\phi/2 - \gamma = \phi - \eta$), giving

$$B_n(\theta) = \int_0^\theta \int_0^\phi f_{(\Phi,\Gamma)}(\phi, \eta - \phi/2) \sum_{k=0}^{n-1} \binom{n-1}{k} \tag{21}$$

$$\begin{aligned} &\times \left|\mathsf{SEG}[\eta, R_\phi]\right|^k \left|\mathsf{SEG}[\phi - \eta, R_\phi]\right|^{n-1-k} \\ &\times B_k(\eta) B_{n-1-k}(\phi - \eta)\, d\eta\, d\phi \end{aligned} \tag{22}$$

from which we get

$$B_n(\theta) = \int_0^\theta \int_0^\phi n \frac{\sin(\theta/2)^{2n}}{(\theta - \sin(\theta))^n} \frac{\cos(\eta - \phi/2) - \cos(\phi/2)}{\sin(\phi/2)^{2n+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \tag{23}$$

$$\begin{aligned} &\times \big(\eta - \sin(\eta)\big)^k B_k(\eta) \big((\phi - \eta) - \sin(\phi - \eta)\big)^{n-1-k} \\ &\times B_{n-1-k}(\phi - \eta)\, d\eta\, d\phi. \end{aligned} \tag{24}$$

Now, $\cos(\eta - \phi/2) - \cos(\phi/2) = 2\sin(\eta/2)\sin((\phi - \eta)/2)$. Finally setting $L_n(\theta)$ as done in (4), we obtain Theorem 1(ii).

2.3 Proof (i′)

Recall Proposition 2. To compute P_n^D we can work under the model where $n-1$ points $z_1, \ldots, z_{n-1}$ are picked independently and uniformly inside the disk $B((0,0), R_c)$ (with $R_c = \pi^{-1/2}$) and one point on the boundary. We place this last point at position $-R_c$ which is allowed since rotation keeps convex bodies and the uniform distribution.

Now take a family of circles $\mathcal{G} = \{B[r], 0 \le r \le R_c\}$ such that $B[r]$ as radius r, its center at position $-R_c + r$, implying that $-R_c$ belongs to all these circles (see Figure 4).

If $r' < r$, $B[r'] \subset B[r]$. Let $r^\star$ be the largest circle such that exists $1 \le k \le n-1$, $z_k \in \partial B[r^\star]$. Denote then by ϕ the angle such that $z_k = (-R_c + r) + re^{i(-\pi+\phi)}$. If

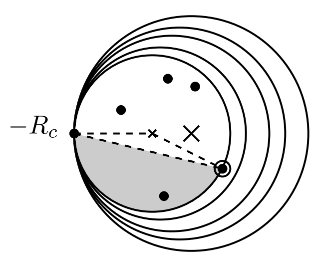

Figure 4 *Decomposition of the computation of P_n^D. The big cross is the center of the initial circle, the small one, the center of the smallest circle containing all the points.*

we denote by (X, Y) the (Euclidean) position of z_k, the density of the distribution of (x, y) is

$$(n-1)1_{(x,y)\in B((0,0),R_c)}\left|B[r]\right|^{n-2}dx\,dy,$$

where $B[r]$ is the unique circle in the family $\mathcal{G}$ which passes by (x, y). We can then compute the Jacobian and find the distribution of (r, ϕ) to be with density $1_{0\le r\le R_c,0\le\phi\le 2\pi}r(1-\cos(\phi))(\pi r^2)^{n-2}\,dr\,d\phi$. Once z_k is given, we can once again normalise the problem, and come back on a circle of area R_c. We then get, using $1+\cos(\phi)=2\sin^2(\phi/2)$

$$\begin{aligned}P_D^n = (n-1)\int_0^{R_c}\int_0^{2\pi}\sum_{k=0}^{n-2}\binom{n-2}{k}2\sin^2(\phi/2)r\left(\pi r^2\right)^{n-2}\\ \times B_k(\phi)B_{n-2-k}(2\pi-\phi)\left|\mathsf{SEG}(\phi, R_c)\right|^k\left|\mathsf{SEG}(2\pi-\phi, R_c)\right|^{n-2-k}d\phi\,dr.\end{aligned}$$

The integration with respect to dr gives

$$\begin{aligned}P_D^n = \frac{1}{\pi}\int_0^{2\pi}\sum_{k=0}^{n-2}\binom{n-2}{k}\sin^2(\phi/2)B_k(\phi)B_{n-2-k}(2\pi-\phi)\\ \times\left(\frac{\phi-\sin(\phi)}{2\pi}\right)^k\left(\frac{2\pi-\phi+\sin(\phi)}{2\pi}\right)^{n-2-k}d\phi\end{aligned}$$

since once ϕ is known, the convexity follows that on the pair of bi-pointed segments with angles ϕ and $2\pi-\phi$, and the number of elements in these segments is $\mathsf{binomial}(n-2, |\mathsf{SEG}(\phi, R_c)|)$.

2.4 Proof of (iii)

The proof is the same as that of (ii) except that in Proposition 4 we need to follow the number of points falling in the triangle. We then get

$$\begin{aligned}B_{n,m}(\theta) = \int_0^{\theta}\int_0^{\phi} f_{(\Phi,\Gamma)}(\phi, \eta-\phi/2)\sum_{\substack{n_1+n_2+n_3=n-1\\ m_1+m_2=m-1}}\binom{n-1}{n_1, n_2, n_3}\\ \times\left|\mathsf{SEG}[\eta, R_\phi]\right|^{n_1}\left|\mathsf{SEG}[\phi-\eta, R_\phi]\right|^{n_2}\\ \times\left(1-|\mathsf{SEG}[\eta, R_\phi]|-|\mathsf{SEG}[\phi-\eta, R_\phi]|\right)^{n_3}\\ \times B_{n_1,m_1}(\eta)B_{n_2,m_2}(\phi-\eta)\,d\eta\,d\phi.\end{aligned}$$

Using the notation introduced in (4), we get (iii).

2.5 Proof of (iii′)

Copy the arguments in Section 2.3. In the same way, for $n \geq 2$, $1 \leq m \leq n$

$$P_D^{n,m} = \frac{1}{\pi} \int_0^{2\pi} \sum_{k=0}^{n-2} \sum_{1 \leq m_1 \leq n-2} \binom{n-2}{k} \sin^2(\phi/2) \\ \times B_{k,m_1}(\phi) B_{n-2-k,m-m_1-2}(2\pi - \phi) \\ \times \left(\frac{\phi - \sin(\phi)}{2\pi}\right)^k \left(\frac{2\pi - \phi + \sin(\phi)}{2\pi}\right)^{n-2-k} d\phi$$

with the condition that $B_{k,k+l} = 0$. (iii′) follows.

2.6 Proof of (iv)

The same proof of (i) does the job.

3 Effective computation of L_n

We explain in this part how to effectively compute the sequences L_n and $L_{n,m}$. There exist maybe some “simple close formulae” for these functions that can be proved by recurrence, but even with the 30 first L_n in hand, we were not able to find one. So, the method we propose allows one to make the successive computations of the L_j’s with a computer algebra system as Maple, Mathematica or Sage: additionally to standard polynomial computations, the needed operations are: Laplace transforms, inverse Laplace transforms, and integration. We wrote a program for Maple helped by Salvy (2013) (the code of the program is available at Marckert (2015)). Here are the main lines of the algorithm.

Instead of computing $L_n(\theta)$ we compute $M_n(\theta) = L_n(2\theta)$ which satisfies a simpler recurrence:

$$M_n(t) = 8 \int_0^t \frac{\sin(t)^{2n+1}}{\sin(\phi)^{2n+1}} \int_0^\phi \sum_{k=0}^{n-1} M_k(\eta) M_{n-1-k}(\phi - \eta)\, d\eta\, d\phi.$$

Since $B_1(\theta) = B_0(\theta) = 1$, $L_0(\theta)$ and $L_1(\theta)$ are known by (4), and thus $M_0(\theta)$ and $M_1(\theta)$ too.

Denote by $J_n(t) = \int_0^t \sum_{k=0}^{n-1} M_k(u) M_{n-1-k}(t-u)\, du$, and by TJ_n, TM_n the Laplace transform of J_n and M_n. We have

$$TJ_n(s) = \sum_{k=0}^{n-1} TM_k(s) TM_{n-k-1}(s).$$

A computation of J_n by integration seems difficult to the computer algebra system, but TJ_n can be computed easily, and the computation of the Laplace inversion of

TJ_n which gives J_n works without any harm with Maple. Maple provides $J_n(t)$ under the form of a polynomial of $\cos(kt)$, $\sin(kt)$, t (for a fixed n, several k are involved). We linearize $\cos(kt)$ and $\sin(kt)$ (replacing them by some polynomials in $\cos(t), \sin(t)$). Using $\cos(t)^2 + \sin(t)^2 = 1$, it is possible to rewrite J_n as polynomial of degree at most 1 in $\cos(t)$ (for this, we take the rest of $J_n(t)$ by the division by $\cos(t)^2 + \sin(t)^2 - 1$). This reduction step is important as it provides much shorter formulae for J_n, and allows one to compute L_n for larger n.

It remains to compute $M_n(t)/\sin(t)^{2n+1}$ which is equal to $8\int_0^t J_n(v)/\sin(v)^{2n+1}\,dv$. Again, Maple is not able to make this integration directly, and needs some help. We then observe that M_n is solution to the following ordinary differential equation:

$$\sin(t)M_n'(t) - (2n+1)\cos(t) - 8\sin(t)J_n(t) = 0, \qquad \lim_{t\to 0}\frac{M_n(t)}{\sin(t)^{2n+1}} = 0, \quad (25)$$

the last equation following from (4). Now, the form of M_n can be guessed: this is a polynomial in $\sin(t), \cos(t), t$. The degree in $\cos(t)$ can be taken equal to 1, and some bounds on the degrees of $\sin(t)$, and t can be guessed (by trial and error, for example). Plugging $M_n(t) = \sum_{k_1,k_2,k_3} a_{k_1,k_2,k_3}\sin(t)^{k_1}\cos(t)^{k_2}t^{k_3}$ into (25), replacing $\cos(t)$ by C, $\sin(t)$ by S, and again taking the rest by the division by $C^2 + S^2 - 1$, we get the nullity of a polynomial in C, S, t. This provides a linear system on the coefficients a_{k_1,k_2,k_3}, easy to solve. This provides a close form for M_n.

The method is a bit demanding in computer resources mainly because M_n becomes more and more complex as n grows, which implies that Laplace transform and inverse Laplace transform devours the memory resources of the computer.

A similar change of variable provides a formula for $M_{n,m}(\theta) = L_{n,m}(2\theta)$. The computation of $M_{n,m}$ is possible using the same algorithm, except that some complications arise since some polylogarithm terms (of the type $\mathsf{polylog}(n, e^{i\theta}) + \mathsf{polylog}(n, -e^{i\theta})$) appears in some intermediate computations. We are able to compute $L_{n,m}$ for $n \le 12$ (which provides the values of $P_D^{n+1,m+1}$ for the pairs (n, m) such that $n \le 13$). Here, the computation are slow, and we renounced to go further for a matter of time (several hours are needed to compute the case $n = 13$).

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