ON THE VALIDITY OF RESAMPLING METHODS UNDER LONG MEMORY[1]

BY SHUYANG BAI AND MURAD S. TAQQU

University of Georgia and Boston University

For long-memory time series, inference based on resampling is of crucial importance, since the asymptotic distribution can often be non-Gaussian and is difficult to determine statistically. However, due to the strong dependence, establishing the asymptotic validity of resampling methods is nontrivial. In this paper, we derive an efficient bound for the canonical correlation between two finite blocks of a long-memory time series. We show how this bound can be applied to establish the asymptotic consistency of subsampling procedures for general statistics under long memory. It allows the subsample size b to be $o(n)$, where n is the sample size, irrespective of the strength of the memory. We are then able to improve many results found in the literature. We also consider applications of subsampling procedures under long memory to the sample covariance, M-estimation and empirical processes.

1. Introduction. A stationary time series $\{X_n\}$ is said to have "*long memory*," also called "*long-range dependence*," if the covariance $\mathrm{Cov}[X_n, X_0]$ decays slowly like n^{2d-1} as $n \to \infty$, where $2d - 1 \in (-1, 0)$. The parameter

$$d \in (0, 1/2)$$

is called the *memory parameter*. Time series exhibiting long memory has been found frequently in practice. Statistical problems under such a context have been widely studied. We refer the reader to the recent monographs Doukhan, Oppenheim and Taqqu (2003), Giraitis, Koul and Surgailis (2012), Beran et al. (2013) and Taqqu (1987) for more information.

Long memory creates a challenge for large-sample inference. This is because the distributional scaling limits of some common statistical functionals, for example, sample sum and quadratic forms, may be non-Gaussian distributions due to the so-called *noncentral limit theorems* [see, e.g., Dobrushin (1979), Taqqu (1979) and Terrin and Taqqu (1990)]. This typically leads to the following situation: even though all possible asymptotic distributions of a statistic may have been derived, it is often difficult to determine in practice which the relevant one is based on the observations. In such a situation, it is natural to resort to resampling.

A common strategy for resampling dependent data is the so-called "*moving block bootstrap*." Suppose that a stationary time series $\{X_1, \ldots, X_n\}$ is observed. The moving block bootstrap performs the following procedure:

(1) with $n - b + 1$ consecutive blocks of size b:

$$\{X_1, \ldots, X_b\}, \qquad \{X_2, \ldots, X_{b+1}\}, \qquad \ldots, \qquad \{X_{n-b+1}, \ldots, X_n\},$$

sample randomly about n/b blocks and paste them together to get a bootstrapped copy of $\{X_1, \ldots, X_n\}$;

(2) compute the statistics of interest on this bootstrapped copy;

(3) repeat the preceding two steps many times to get an empirical distribution of the statistics for inference.

Note that the moving block bootstrap involves rearranging the order of the time series and this can destroy the dependence structure.

An alternative resampling scheme, called "*subsampling*," directly computes the statistics on the block subsamples: $\{X_1, \ldots, X_b\}, \{X_2, \ldots, X_{b+1}\}, \ldots, \{X_{n-b+1}, \ldots, X_n\}$, and uses the resulting empirical distribution for inference. It usually involves a proper rescaling since the sample size has been reduced from n to b. Note that in subsampling, the original time order is intact. Subsampling appears to be a more robust procedure when there is strong dependence. Indeed, Lahiri (1993) noticed that the moving block bootstrap procedure may fail in the long-memory case, while the subsampling method is shown to work at least in the special case of sample mean [see Hall, Jing and Lahiri (1998), Nordman and Lahiri, 2005 and Zhang et al., 2013]. For general information on resampling dependent data, see the monographs Politis, Romano and Wolf (1999), Lahiri (2003) and Chapter 10 of Beran et al. (2013).

The asymptotic validity of a subsampling procedure is usually formulated under the following setup: as the sample size $n \to \infty$, the block (subsample) size $b = b_n \to \infty$, while b_n grows more slowly than the sample size:

$$b_n = o(n).$$

Politis and Romano (1994) established the validity of subsampling for general stationary time series by imposing an implicit condition, which we call the "subsampling condition":

$$(1) \qquad \qquad \sum_{k=1}^{n} \alpha_{k,b_n} = o(n) \qquad \text{as } n \to \infty,$$

where $\alpha_{k,b}$ is the between-block mixing coefficient defined by

$$(2) \qquad \alpha_{k,b} = \sup\{|P(A \cap B) - P(A)P(B)|, A \in \mathcal{F}_1^b, B \in \mathcal{F}_{k+1}^{k+b}\},$$

with $\mathcal{F}_p^q$ denoting the sigma field generated by $\{X_p, \ldots, X_q\}$. The condition (1) will be discussed in more detail in Section 4 below. It is, however, not clear whether

the natural block size condition $b_n = o(n)$ implies the subsampling condition (1) in general. Under a typical weak dependence condition, strong mixing [see Bradley (2007)], where it is assumed that, as $k \to \infty$,

$$(3) \qquad \alpha_k := \sup\{|P(A \cap B) - P(A)P(B)|, A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_k^\infty\} \to 0,$$

one can deduce easily that (1) holds if $b_n = o(n)$. See Politis, Romano and Wolf (1999), Theorem 4.2.1. Allowing $b_n = o(n)$ has important statistical implications. See Remark 2.1 below.

In the case where $\{X_n\}$ is given by a long-memory model, the implication $b_n = o(n) \Rightarrow$ (1) has not been established as far as we know. In this paper, for a common class of long-memory models [see (9) below], we shall derive an efficient bound on the between-block mixing coefficient $\alpha_{k,b}$ in (2). Such a bound entails that $b_n = o(n)$ implies (1).

The paper is organized as follows. In Section 2, we introduce the long-memory Gaussian subordination model and discuss its important properties related to $\alpha_{k,b}$ in (2). In Section 3, we state the main results. In Section 4, the implication of the results on subsampling of long-memory Gaussian subordination model is discussed. Many results in the literature are improved. We consider the subsampling of some common statistics in Section 5. Proofs of the main results in Section 3 are given in Section 6.

2. Gaussian subordination and canonical correlation.

In this paper, we focus on a typical class of long memory models, which is obtained by the so-called *Gaussian subordination*. Let $\{Z_n\}$ be stationary Gaussian process with covariance function $\gamma(n) = \text{Cov}[Z_n, Z_0]$. Suppose $\mathbb{E}Z_n = 0$ and $\text{Var}[Z_n] = 1$. Write

$$(4) \qquad \mathbf{Z}_p^q = (Z_p, \ldots, Z_q).$$

Denote the covariance matrix of $\mathbf{Z}_1^b$ by

$$(5) \qquad \Sigma_b = \big(\gamma(i_2 - i_1) = \text{Cov}[Z_{i_1}, Z_{i_2}]\big)_{1 \le i_1, i_2 \le b}.$$

We assume throughout that Σ_b is nonsingular for every $b \in \mathbb{Z}_+$. Consider two vectors

$$\mathbf{Z}_1^b \quad \text{and} \quad \mathbf{Z}_{k+1}^{k+b}$$

distant by $k \in \mathbb{Z}_+$ of the same dimension b. Denote the cross-block covariance matrix between $\mathbf{Z}_1^b$ and $\mathbf{Z}_{k+1}^{k+b}$ by

$$(6) \qquad \Sigma_{k,b} = \big(\gamma(i_2 + k - i_1) = \text{Cov}[Z_{i_1}, Z_{i_2+k}]\big)_{1 \le i_1, i_2 \le b}.$$

The *canonical correlation* $\rho(\cdot, \cdot)$ between two blocks of $\{Z_n\}$ of size b differing by a translation of k units in time is defined as

$$\rho_{k,b} = \rho(\mathbf{Z}_1^b, \mathbf{Z}_{k+1}^{k+b}) := \sup_{\mathbf{u} \in \mathbb{R}^b, \mathbf{v} \in \mathbb{R}^b} \text{Corr}(\langle \mathbf{u}, \mathbf{Z}_1^b \rangle, \langle \mathbf{v}, \mathbf{Z}_{k+1}^{k+b} \rangle)$$

$$(7) \qquad = \sup_{\mathbf{u}, \mathbf{v} \in \mathbb{R}^b} \frac{\mathbf{u}^T \Sigma_{k,b} \mathbf{v}}{\sqrt{\mathbf{u}^T \Sigma_b \mathbf{u}} \sqrt{\mathbf{v}^T \Sigma_b \mathbf{v}}},$$

where $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product. Note that

$$(8) \qquad\qquad 0 \leq \rho_{k,b} \leq 1.$$

Consider now the observed stationary time series $\{X_n\}$. The so-called *Gaussian subordination* model is given by

$$(9) \qquad\qquad X_n = G(Z_n),$$

where $G(\cdot)$ is a measurable function. We then say that $\{X_n\}$ is *subordinated to* the underlying Gaussian $\{Z_n\}$. In fact, our results incorporate also the more general case

$$X_n = G(Z_n, \ldots, Z_{n-l}),$$

where l is a fixed nonnegative integer; see, for example, Bai, Taqqu and Zhang (2016). We focus for simplicity only on the instantaneous case $l = 0$, as is mostly considered in the literature. We note the following important quantity: if $\mathbb{E}G(Z)^2 < \infty$ for a standard Gaussian Z, then the *Hermite rank m* of $G(\cdot)$ is defined as

$$(10) \qquad m = \inf\{p \in \{1, 2, 3, \ldots\} : \mathbb{E}[(G(Z) - \mathbb{E}G(Z))Z^p] \neq 0\}.$$

An equivalent definition with the monomial Z^p replaced by pth order Hermite polynomial is often used as well [see Definition 3.2 in Beran et al. (2013)].

Suppose now that $\{Z_n\}$ has long memory with memory parameter $d \in (0, 1/2)$, that is in our context

$$\text{Cov}[Z_k, Z_0] \sim ck^{2d-1}$$

as $k \to \infty$, where c is a positive constant, so that

$$\sum_{k=-\infty}^{\infty} |\text{Cov}[Z_k, Z_0]| = \infty.$$

Then the covariance $\text{Cov}[X_k, X_0]$ behaves like $k^{(2d-1)m}$ as $k \to \infty$. Hence, if $(2d-1)m > -1$, then $\{X_n\}$ has long memory; if $(2d-1)m < -1$, then $\{X_n\}$ does not have long memory. See Dobrushin and Major (1979) and Taqqu (1979) for more details.

The model (9) has some important mathematical advantages. First, it allows various limit theorems involving short or long memory as well as light tails, namely finite variance [Dobrushin (1979), Taqqu (1979), Breuer and Major (1983)], but also heavy tails [Sly and Heyde (2008)]. The second advantage, which greatly simplifies the analysis of the between-block mixing coefficient $\alpha_{k,b}$ in (2), is the following fact [Theorem 1 of Kolmogorov and Rozanov (1960)]:

For any jointly Gaussian vectors $(\mathbf{Z}_1, \mathbf{Z}_2)$, one has

$$(11) \qquad \sup_{F,G} |\text{Corr}(F_1(\mathbf{Z}_1), F_2(\mathbf{Z}_2))| = \rho(\mathbf{Z}_1, \mathbf{Z}_2),$$

where $\rho(\cdot, \cdot)$ is the canonical correlation as in (7), and the supremum is taken over all functions F_1, F_2 such that $\mathbb{E}F_1(\mathbf{Z}_1)^2 < \infty$, $\mathbb{E}F_2(\mathbf{Z}_2)^2 < \infty$.

Note that (11) reduces nonlinear functions F_1 and F_2 to linear functions. Let $\alpha_{k,b}$ be the between-block mixing coefficient of $\{X_n\}$ defined in (2). In view of (11), one has [see Theorem 2 of Kolmogorov and Rozanov (1960)]

$$(12) \qquad \alpha_{k,b} \leq \rho_{k,b}.$$

Hence, the subsampling condition (1) holds if

$$(13) \qquad \sum_{k=1}^{n} \rho_{k,b_n} = o(n) \qquad \text{as } n \to \infty.$$

In the context of Gaussian subordination, we shall also call (13) the *subsampling condition*.

There has been some recent progress on deriving (13) from some growth conditions more stringent than $b_n = o(n)$. Bai, Taqqu and Zhang (2016) proved the following bound on the canonical correlation $\rho_{k,b}$ [see Lemma 3.4 of Bai, Taqqu and Zhang (2016)].

PROPOSITION 2.1. *Let*

$$(14) \qquad M_\gamma(k) = \max_{n > k} |\gamma(n)|,$$

and let

$$(15) \qquad \lambda_b = \text{the minimum eigenvalue of } \Sigma_b.$$

Then

$$(16) \qquad \rho_{k,b} \leq b \frac{M_\gamma(k-b)}{\lambda_b}.$$

The bound (16) is obtained by optimizing the denominator and numerator in (7) separately. Note that the minimum eigenvalue λ_b is positive since Σ_b is assumed to be positive definite.

How useful is the bound (16)? As shown in Bai, Taqqu and Zhang (2016), this bound is satisfactory in cases where $\{Z_n\}$ has weak dependence. For example, suppose that $\gamma(n)$, $n \geq 1$, is bounded by $cn^{-\beta}$ with $\beta > 1$ and $c > 0$, and that the spectral density is bounded away from zero. Then

$$\sum_{k=1}^{\infty} M_\gamma(k) < \infty, \quad \text{and} \quad \lambda_{\min} := \inf_m \lambda_m > 0$$

[Brockwell and Davis (1991), Proposition 4.5.3]. Consequently, if $b_n = o(n)$, we have

$$\sum_{k=1}^{n} \rho_{k,b_n} \leq b_n \lambda_{\min}^{-1} \sum_{k=1}^{\infty} M_\gamma(k) = o(n).$$

Thus, $b_n = o(n)$ implies (13).

However, in the case where $\{Z_n\}$ has long memory with a memory parameter $d \in (0, 1/2)$, the crude bound (16) is not satisfactory as it requires a block size

$$(17) \qquad\qquad b_n = o(n^{1-2d})$$

to obtain the subsampling condition (13). See the relation (57) in Bai, Taqqu and Zhang (2016). The block size condition (17) depends on d and becomes quite restrictive when d is close to $1/2$.

Based on the result of Adenstedt (1974), Betken and Wendler (2015) have obtained recently a bound which improves (16) in the long memory situation. Under the assumptions given in the Appendix, Betken and Wendler (2015) derived the following bound for $\rho_{k,b}$.

PROPOSITION 2.2. *Let $\rho_{k,b}$ be as in* (7). *Suppose that the assumptions* BW1 *and* BW2 *in the Appendix hold, where the covariance is*

$$\gamma(n) = n^{2d-1} L_\gamma(n)$$

for some slowly varying L_γ, $d \in (0, 1/2)$. Then when $k > b$, we have for some constants $C_1, C_2 > 0$ that

$$(18) \quad \rho_{k,b} \le C_1 \left(\frac{b}{k-b} \right)^{1-2d} L_\gamma(k-b) + C_2 b^2 (k-b)^{2d-2} \max\{L_\gamma(k-b), 1\}.$$

In order to ensure (13) using (18), it is imposed in Betken and Wendler (2015) that

$$(19) \qquad\qquad b_n = o(n^{1-d-\varepsilon})$$

for some arbitrarily small $\varepsilon > 0$. Note that the requirement (19), although it depends on d, is less restrictive than (17), and the order $b_n = O(n^{1/2})$ is always allowed since $d < 1/2$. The right-hand side of (18) involves two terms. If it were not for the presence of the second term, one would be able to impose only $b_n = o(n)$.

REMARK 2.1. Allowing the nonrestrictive condition, $b_n = o(n)$ has an important practical implication: the valid range of block size choice does not depend on the memory parameter d of $\{Z_n\}$, which is typically unknown. This is particularly desirable for resampling procedures designed to avoid treating the nuisance parameter d under long memory [see, e.g., Jach, McElroy and Politis (2012) and Bai, Taqqu and Zhang (2016)].

In Section 3 below, we shall bound $\rho_{k,b}$ in such a way that only the first term in (18) is effectively present for a wide class of long memory models. This will allow the nonrestrictive condition $b_n = o(n)$ to imply the subsampling condition (13) (see Theorem 3.3 below). We obtain our result by establishing a tight bound in the special case of a FARIMA$(0, d, 0)$ time series, where there is explicit information

on the model, and then extending the result to a more general setup using Fourier analysis.

We mention that our results can be easily generalized to obtain a bound on the canonical correlation $\rho(\mathbf{Z}_1^a, \mathbf{Z}_{k+1}^{k+b})$, where possibly different block sizes a and b are involved. Since the main application, subsampling, involves only $a = b$, the proofs are given only for this case. Nevertheless, the statements concerning the case $a \neq b$ are given in Remark 3.6 below.

3. Main results. In this section, we state the main result. Let $\{Z_n\}$ be a second-order stationary process with covariance function $\gamma(n)$ and spectral density $f(\lambda)$ so that

$$\gamma(n) = \int_{-\pi}^{\pi} f(\lambda) e^{in\lambda} \, d\lambda.$$

Let $\rho_{k,b}$ be as in (7). Consider the spectral density of a FARIMA$(0, d, 0)$ time series [see, e.g., Section 13.2 of Brockwell and Davis (1991)]:

$$(20) \qquad f_d(\lambda) := \frac{1}{2\pi} |1 - e^{i\lambda}|^{-2d} = \frac{1}{2\pi} [\sin(\lambda/2)^2]^{-d}, \qquad 0 < d < 1/2.$$

First, we state a result in the special case of FARIMA$(0, d, 0)$ time series.

THEOREM 3.1. *Suppose that $f(\lambda) = \sigma^2 f_d(\lambda)$, for some constant $\sigma^2 > 0$, is the spectral spectral density of $\{Z_n\}$. Then when $1 \leq b < k$, there exists a constant $c > 0$, such that*

$$(21) \qquad \rho_{k,b} \leq c \left(\frac{b}{k-b} \right)^{1-2d}.$$

This theorem is proved in Section 6.2.

REMARK 3.1. We note that the bound (21) is sharp when $b \ll k$. Indeed, suppose that $\gamma_d(n)$ is the covariance of the FARIMA$(0, d, 0)$ [see (60) below]. $\gamma_d(n)$ is decreasing in $n \in \mathbb{Z}_+$ following the asymptotic order n^{2d-1}. Then it is well known that [see, e.g., Corollary 1.2 of Beran et al. (2013)]

$$V_d(b) := \text{Var}[Z_1 + \cdots + Z_b] \sim cb^{2d+1}$$

for some constant $c > 0$. Now take in (7) $\mathbf{u} = \mathbf{v} = (1, \ldots, 1)^T$. One has, for $k > b \geq 1$,

$$\rho_{k,b} \geq \text{Corr}(Z_1 + \cdots + Z_b, Z_{k+1} + \cdots + Z_{k+b})$$

$$= V_d(b)^{-1} \sum_{i,j=1}^{b} \gamma_d(i - j + k)$$

$$(22) \qquad \geq c_1 b^{-2d-1} \sum_{i,j=1}^{b} \gamma_d(b-1+k)$$

$$\geq c_2 b^{-2d-1} b^2 (k+b-1)^{2d-1}$$

$$= c_2 \left(\frac{b}{k+b-1} \right)^{1-2d},$$

for some constants $c_1, c_2 > 0$, where the first inequality in (22) follows from the fact that $\gamma_d(n)$ is decreasing in n, and $\gamma_d(i - j + k)$ attains its minimum at $i = b$, $j = 1$. When $b \ll k$ (e.g., when $k > 2b$), the bounds (21) and (22) are both following the order $(b/k)^{1-2d}$.

REMARK 3.2.   It is instructive to relate the bound (21) to strong mixing, a weak dependence notion in the case where $\{Z_n\}$ is a stationary Gaussian sequence. By Kolmogorov and Rozanov (1960), the Gaussian $\{Z_n\}$ is strong mixing[2] if and only if

$$(23) \qquad \lim_{k \to \infty} \sup_{b \in \mathbb{Z}_+} \rho_{k+b,b} = 0.$$

Since the long memory $\{Z_n\}$ with a spectral density in (20) is not strong mixing, we must have

$$(24) \qquad \lim_{k \to \infty} \sup_{b \in \mathbb{Z}_+} \rho_{k+b,b} > 0.$$

Let us then rewrite (21) as

$$(25) \qquad \rho_{k+b,b} \leq c \left(\frac{b}{k} \right)^{1-2d}.$$

The bound (25) shows that with a fixed b, we have

$$\rho_{k+b,b} = O(k^{2d-1}),$$

and hence

$$\lim_{k \to \infty} \rho_{k+b,b} = 0.$$

This relation is weaker than (23), and hence in accord with the fact that FARIMA$(0, d, 0)$ with $0 < d < 1/2$ is not strong mixing.

We also note that it follows from Lemma 6.6 that a reversed version of (23),

$$\lim_{b_n \to \infty} \sup_{[\varepsilon n] \leq k \leq n} \rho_{k,b_n} = 0, \qquad b_n = o(n),$$

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[2]For stationary Gaussian processes, the "complete regularity coefficient" $\rho_{k,b}$ is expressed as (7). It is equivalent to $\alpha_{k,b}$ in (2), that is, $\alpha_{k,b} \leq \rho_{k,b} \leq 2\pi \alpha_{k,b}$. Hence, a stationary Gaussian process is strong mixing if and only if (23) holds. See Kolmogorov and Rozanov (1960), Theorems 1 and 2 or Ibragimov and Rozanov (1978), Section IV.1. See also Bai, Taqqu and Zhang (2016).

where $\varepsilon > 0$ is arbitrarily small, is a sufficient condition for the subsampling condition (13).[3]

We now turn to the general case.

THEOREM 3.2. *Suppose that the spectral density of $\{Z_n\}$ is given by*

(26)
$$f(\lambda) = f_d(\lambda) f_0(\lambda),$$

where $f_0(\lambda)$ is the spectral density which corresponds to a covariance function (or Fourier coefficient)

$$\gamma_0(n) = \int_{-\pi}^{\pi} f_0(\lambda) e^{in\lambda} \, d\lambda.$$

Assume that the following hold:

(a) $c_0 := \inf_{\lambda \in (-\pi, \pi]} f_0(\lambda) > 0$;
(b) $\sum_{n=-\infty}^{\infty} |\gamma_0(n)| < \infty$.

Then, depending on the rate of decay of $\gamma_0(n)$, we have the following:

- *Suppose $\gamma_0(n) = O(n^{-\alpha})$ for some $\alpha > 0$:*
  Then for any fixed small $\varepsilon > 0$, there exist constants $c_1, c_2 > 0$, such that for all b, k', k satisfying

(27)
$$1 \le b < k' \le (1 - \varepsilon)k,$$

  we have

(28)
$$\rho_{k,b} \le c_1 \left(\frac{b}{k' - b} \right)^{1-2d} + c_2 k' k^{-\alpha}.$$

- *Suppose $\gamma_0(n) = o(n^{-\alpha})$ for some $\alpha > 0$:*
  Then the second term on the right-hand side of (28) can be replaced by $c_2 k' o(k^{-\alpha})$.
- *Suppose $\gamma_0(n) = O(e^{-cn})$ for some $c > 0$:*
  Then the second term on the right-hand side of (28) can be replaced by $c_2 e^{-c_3 k}$ for some $c_3 > 0$.

This theorem is proved in Section 6.2.

REMARK 3.3. The conditions (a) and (b) together state that the spectral density $f_0(\lambda)$ corresponds to a short-memory time series. The condition (b) implies that $f_0(\lambda)$ is bounded and continuous. Note that $\gamma_0(n) = O(n^{-\alpha})$ with $\alpha > 1$ implies the statement (b). In view of the linear filter theory [Theorem 4.10.1 of

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[3]The remark is suggested by an anonymous referee.

Brockwell and Davis (1991)], the process $\{Z_n\}$ whose spectral density is given as in (26) can be interpreted as a FARIMA$(0, d, 0)$ model with dependent noise:

$$(1 - B)^d Z_n = \xi_n,$$

where $\{\xi_n\}$ is short-memory with spectral density $f_0(\lambda)$ and covariance function $\gamma_0(n)$, B is the back-shift operator, and $(1 - B)^d$ is interpreted using binomial series. If $f_0(\lambda)$ is chosen so that $f(\lambda)\lambda^{2d}$, $\lambda > 0$, satisfies the so-called *quasi monotonicity* condition near $\lambda = 0$ [see Definition 1.29 of Soulier (2009), which holds if $f(\lambda)\lambda^{2d}$, $\lambda > 0$, is monotone in a neighborhood of $\lambda = 0$], one can derive the time-domain long memory condition: the covariance $\gamma(k) \sim ck^{2d-1}$ as $k \to \infty$ for some constant $c > 0$ [see Theorem 1.37 of Soulier (2009)].

REMARK 3.4. The decay rate of $\gamma_0(n)$ depends on the smoothness of $f_0(\lambda)$. The following facts are well known:

- If $f_0(\lambda)$ is α-Hölder continuous with $\alpha \in (0, 1]$, then

$$\gamma_0(n) = O(n^{-\alpha})$$

[Zygmund (2002), Theorem 4.7],
- and if further $\alpha > 1/2$, then the absolute convergence in (b) holds [Zygmund (2002), Theorem 3.1];
- if $f_0(\lambda)$ is of bounded variation, then

$$\gamma_0(n) = O(n^{-1})$$

[Zygmund (2002), Theorem 4.12];
- if f_0 is r times differentiable, then

$$\gamma_0(n) = o(n^{-r})$$

(integration by parts and the Riemann–Lebesgue lemma);
- if $f_0(\lambda)$ has an analytic extension $f_0(z)$ to the open complex strip $\{z : \text{Im}(z) < c\}$, $c > 0$, then

$$\gamma_0(n) = O(e^{-c'n})$$

for any $c' \in (0, c)$ [Timan (1963), Section 3.12, Formula (19)].

REMARK 3.5. Let us now focus on the second term in (28). Since it involves the exponent α in $\gamma_0(n)$, and since $\gamma_0(n)$ satisfies (b), that second term may be viewed as resulting from the short-memory part. We will only need to consider the case $\alpha \geq 1$. If $\alpha > 1$ or if $\alpha = 1$ and $k' = o(k)$, then the second term always tends to zero as $k \to \infty$. It resembles a strong-mixing-type bound, which depends only on the gap k separating the infinite past from the infinite future [see (3)].

REMARK 3.6. By straightforward modifications of the proofs (including some statements in Section 6.1), Theorems 3.1 and 3.2 can be generalized for the canonical correlation $\rho(\mathbf{Z}_1^a, \mathbf{Z}_{k+1}^{k+b})$ with possibly different a and b. Under the assumptions of Theorem 3.1, one can obtain the bound

$$\rho(\mathbf{Z}_1^a, \mathbf{Z}_{k+1}^{k+b}) \leq c(k - \max(a, b))^{2d-1} a^{1/2-d} b^{1/2-d}.$$

Moreover, under the assumptions of Theorem 3.2 with $a, b < k'$, one can obtain, for example, in the case where $\gamma_0(n) = O(n^{-\alpha})$, $\alpha > 0$, the following bound:

$$(29) \qquad \rho(\mathbf{Z}_1^a, \mathbf{Z}_{k+1}^{k+b}) \leq c_1(k' - \max(a, b))^{2d-1} a^{1/2-d} b^{1/2-d} + c_2 k' k^{-\alpha}.$$

The bounds for the other two cases are obtained by replacing the second term on the right-hand side of (29) as in Theorem 3.2.

The following theorem is the key to establish the consistency of subsampling procedures for a typical class of long-memory time series models [see (9) below]. Let b_n be a sequence which will stand for the block size in the context of subsampling.

THEOREM 3.3. *If the spectral density $f(\lambda)$ satisfies the assumptions in Theorem 3.2 with*

$$\gamma_0(n) = O(n^{-1}),$$

then any

$$b_n = o(n)$$

implies the subsampling condition (13), namely

$$\sum_{k=1}^{n} \rho_{k,b_n} = o(n) \qquad \text{as } n \to \infty.$$

This theorem is proved in Section 6.2. The conditions in Theorem 3.2 include some typical long-memory models.

EXAMPLE 3.1 [FARIMA(p, d, q)]. A FARIMA(p, d, q), $0 < d < 1/2$, time series $\{Z_n\}$ is the solution of the following stochastic difference equation:

$$\Phi(B)(1 - B)^d Z_n = \Theta(B)\xi_n,$$

where $\{\xi_n\}$ is a white noise sequence with variance σ^2, $\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ is a polynomial with no zeros on the unit complex circle, and $\Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$. See Definition 13.2.2 of Brockwell and Davis (1991). It is known

that the spectral density of $\{Z_n\}$ is given by [Theorem 13.2.2 of Brockwell and Davis (1991)]

$$(30) \qquad f(\lambda) = \frac{\sigma^2}{2\pi} |1 - e^{i\lambda}|^{-2d} \frac{|\Theta(e^{i\lambda})|^2}{|\Phi(e^{i\lambda})|^2}.$$

The function $f_0(\lambda) := \sigma^2 |\Theta(e^{i\lambda})|^2 / |\Phi(e^{i\lambda})|^2$ can be extended to an analytic $f(z)$ in a strip $\{z, \operatorname{Im}(z) < c\}$, $c > 0$, since all zeros of $\Phi(z)$ are outside an open annulus containing the unit circle. By Remark 3.4, $\gamma_0(n)$ decays exponentially. Hence, the assumptions of Theorem 3.2 are satisfied and the bound (28) holds with the second term decaying exponentially. Thus, by Theorem 3.3, the block size $b_n = o(n)$ implies the subsampling condition (13).

EXAMPLE 3.2 (Fractional Gaussian noise with $H > 1/2$).   Fractional Gaussian noise with Hurst index $H \in (0, 1)$, which is the increments of fractional Brownian motion [see, e.g., Taqqu (2003)], is a centered stationary Gaussian sequence $\{Z_n\}$ whose covariance is given by

$$\gamma(n) = \frac{\sigma^2}{2} (|n + 1|^{2H} - |n|^{2H} + |n - 1|^{2H}), \qquad \sigma^2 > 0.$$

Note that $\gamma(n) \sim cn^{2H-2}$ as $n \to \infty$ for some $c > 0$. The spectral density of $\{Z_n\}$ is derived by Sinai (1976), namely, for some $c_1 > 0$, we have

$$f(\lambda) = c_1 |1 - e^{i\lambda}|^2 \sum_{k=-\infty}^{\infty} \frac{1}{|\lambda + 2\pi k|^{2H+1}}.$$

We are interested in the case where

$$d := H - 1/2 > 0,$$

the long-memory regime. We mention that in the so-called antipersistent regime $H < 1/2$, one can show using (16) that the condition (13) holds under $b_n = o(n^{1-\varepsilon})$ for an $\varepsilon > 0$ arbitrarily small. See Bai, Taqqu and Zhang (2016). Write the spectral density as

$$f(\lambda) = f_d(\lambda) f_0(\lambda),$$

where f_d is as in (20), and

$$
\begin{aligned}
(31) \quad f_0(\lambda) &= c_1 |1 - e^{i\lambda}|^{2d+2} \sum_{k=-\infty}^{\infty} \frac{1}{|\lambda + 2\pi k|^{2d+2}} \\
&= c_1 \left[\left(\frac{4\sin(\lambda/2)^2}{\lambda^2} \right)^{d+1} + (4\sin(\lambda/2)^2)^{d+1} \sum_{k \neq 0} \frac{1}{|\lambda + 2\pi k|^{2d+2}} \right].
\end{aligned}
$$

In view of the term $(4\sin(\lambda/2)^2 / \lambda^2)^{d+1}$, we see that $f_0(\lambda)$ is bounded below away from 0 when $\lambda \in (-\pi, \pi]$. Furthermore, note that since $d \in (0, 1/2)$, the series

$\sum_{k \neq 0} |\lambda + 2\pi k|^{-2d-2}$ and its term-by-term derivatives (with respect to λ) converge uniformly for all $\lambda \in (-\pi, \pi]$. By Theorem 7.17 of Rudin (1976), the series and hence $f_0(\lambda)$ is infinitely differentiable. Then the assumptions of Theorem 3.2 hold with arbitrarily large $\alpha > 0$ in Theorem 3.2 (see Remark 3.4), and hence so does the bound (28). Thus, by Theorem 3.3, the block size $b_n = o(n)$ implies the subsampling condition (13).

3.1. *Multivariate extension.* Theorem 3.3 can be directly extended to the case of a multivariate second-order stationary process

$$\{\mathbf{Z}_n = (Z_{n,1}, \ldots, Z_{n,J})\}$$

with *uncorrelated* components. One can define similarly the canonical correlation $\rho_{k,b}$ between two blocks $(\mathbf{Z}_1, \ldots, \mathbf{Z}_b)$ and $(\mathbf{Z}_{k+1}, \ldots, \mathbf{Z}_{k+b})$. Let $\rho_{k,b,j}$ be the canonical correlation between $(Z_{1,j}, \ldots, Z_{b,j})$ and $(Z_{k+1,j}, \ldots, Z_{k+b,j})$ involving the jth component of the vectors, $j = 1, \ldots, J$. Since the covariance between Z_{n_1,j_1} and Z_{n_2,j_2} vanishes if $j_1 \neq j_2$, the covariance matrices Σ_b in (5) and $\Sigma_{k,b}$ in (6) corresponding to $\{\mathbf{Z}_n\}$ are block-diagonal, and hence so are the $U_{k,b}$ and $V_{k,b}$ matrices defined in (57) and (58) below. Because the eigenvalues of a block-diagonal matrix consists of eigenvalues of each block, one has in view of Lemma 6.1 below that

$$(32) \qquad \rho_{k,b} = \max_{1 \leq j \leq J} \rho_{k,b,j}.$$

Hence, to establish the subsampling condition (13) for $\rho_{k,b}$ of $\{\mathbf{Z}_n\}$ which has uncorrelated components, one only needs to establish the corresponding ones for each component $\{Z_{n,j}\}$. See Example 4.8 below for an application of such a multivariate case to a nonlinear time series.

4. Validity of subsampling. In this section, we discuss the role that the subsampling condition (1) plays in ensuring the consistency of a subsampling procedure.

4.1. *Subsampling of time series.* We start with a brief introduction to subsampling procedures for time series. Let $\{X_n\}$ be a stationary time series. One is interested in using the quantity

$$T_n(\mathbf{X}_1^n; \theta) = T_n(X_1, \ldots, X_n; \theta)$$

for inference, where θ is an unknown parameter, which may not be present in some cases. Suppose that

$$(33) \qquad T_n(\mathbf{X}_1^n; \theta) \xrightarrow{d} T$$

as $n \to \infty$, where T is some random variable with distribution function $F_T(x)$. In general, the limit T depends on θ. But we suppress this in the notation for

simplicity. Whenever we mention convergence of distribution functions, we mean convergence at continuity points of the limit. The convergence (33) is established on a case-by-case basis.

We are interested in the distribution function

$$(34) \qquad\qquad F_{T_n}(x) = P\big(T_n(\mathbf{X}_1^n; \theta) \leq x\big),$$

which is in general difficult to obtain. Suppose that the limit distribution $F_T(x)$ is not available either, due for example, to the presence of nuisance parameters. So we resort to some resampling procedure. Consider the statistic defined on a block of length b starting at i, namely,

$$T_b(\mathbf{X}_i^{i+b-1}; \theta).$$

One expects that the distribution of $T_b(\mathbf{X}_i^{i+b-1}; \theta)$ is close to that of $T_n(\mathbf{X}_1^n; \theta)$ since, in view of (33), both are close to F_T when b and n are reasonably large. By varying i while keeping b fixed, one obtains many subsamples. One then wants to use the empirical distribution

$$(35) \qquad\qquad \widehat{F}_{n,b}^*(x) = \frac{1}{n-b+1} \sum_{i=1}^{n-b+1} \mathrm{I}\{T_b(\mathbf{X}_i^{i+b-1}; \theta) \leq x\}$$

as an approximation of F_{T_n} for inference. But $\widehat{F}_{n,b}^*(x)$ involves the unknown parameter θ. We thus replace θ by a consistent estimate $\hat{\theta}_n$ which depends on the whole sample $\{X_1, \ldots, X_n\}$. This leads to

$$(36) \qquad\qquad \widehat{F}_{n,b}(x) = \frac{1}{n-b+1} \sum_{i=1}^{n-b+1} \mathrm{I}\{T_b(\mathbf{X}_i^{i+b-1}; \hat{\theta}_n) \leq x\}.$$

Note that each term $\widehat{F}_{n,b_n}(x)$ depends on the whole sample $\{X_1, \ldots, X_n\}$ while each term of $\widehat{F}_{n,b}^*(x)$ depends only on the an individual block $\{X_i, \ldots, X_{i+b-1}\}$. On the other hand $\widehat{F}_{n,b_n}(x)$ is computable from data since it does not involve the unknown parameter θ.

EXAMPLE 4.1. As a typical case, suppose [see Chapter 3 of Politis, Romano and Wolf (1999)] that θ is indeed the parameter on which we want to carry out the inference. Let $\hat{\theta}_{n,b,i}$ be an estimator of θ computed using the block $\{X_i, \ldots, X_{i+b-1}\}$. To get an approximation of the distribution of $\hat{\theta}_n - \theta$ for inference, one then proposes to use the empirical distribution:

$$(37) \qquad\qquad \widehat{F}_{n,b}(x) = \frac{1}{n-b+1} \sum_{i=1}^{n-b+1} \mathrm{I}\{\tau_b(\hat{\theta}_{n,b,i} - \hat{\theta}_n) \leq x\},$$

where τ_b is an appropriate deterministic normalizer which ensures that $\tau_n(\hat{\theta}_n - \theta)$ converges in distribution as $n \to \infty$. In this case, we have

$$T_b(\mathbf{X}_i^{i+b-1}; \theta) = \tau_b(\hat{\theta}_{n,b,i} - \theta), \qquad T_b(\mathbf{X}_i^{i+b-1}; \hat{\theta}_n) = \tau_b(\hat{\theta}_{n,b,i} - \hat{\theta}_n).$$

If the convergence rate τ_n is unknown, it needs to be consistently estimated by some $\hat{\tau}_n$ in the sense that $\hat{\tau}_n / \tau_n$ converges in probability to a positive constant [see Politis, Romano and Wolf (1999), Chapter 8]. In the cases of heavy tails or long memory, often τ_b needs to be replaced by a random normalizer $\hat{\tau}_{n,b,i}$ computed using the block $\{X_i, \ldots, X_{i+b-1}\}$, and $\hat{\tau}_{n,b,i} / \tau_b$ would typically only converge in distribution as $b \to \infty$. This is done, for example, in Romano and Wolf (1999), Jach, McElroy and Politis (2012) and Bai, Taqqu and Zhang (2016). We mention that even without a proper scale estimate $\hat{\tau}_n$, the shape of the unscaled empirical distribution

$$\frac{1}{n-b+1} \sum_{i=1}^{n-b+1} I\{(\hat{\theta}_{n,b,i} - \hat{\theta}_n) \le x\}$$

can be useful for diagnostic purposes [see Sherman and Carlstein (1996)].

EXAMPLE 4.2. There are cases where no unknown θ is involved. For example, in change-point problems using the Wilcoxon test statistics, which involves $\sum_{i=1}^{k} R_i - k/n \sum_{i=1}^{n} R_i$ where R_i's are ranks. In this case, one can suppress the θ in (35) and the $\hat{\theta}$ in (36), and hence the distinction between $\widehat{F}^*_{n,b}(x)$ and $\widehat{F}_{n,b}(x)$. See Betken and Wendler (2015).

4.2. *Asymptotic consistency of subsampling procedures.* To obtain convergence results, we let the block size $b = b_n$ depend explicitly on the sample size n, which tends to infinity as $n \to \infty$.

DEFINITION 4.1. Let $\widehat{F}_{n,b_n}(x)$ be as in (36) and let $F_{T_n}(x)$ be as in (34). We say that the subsampling procedure is consistent, if

$$(38) \qquad \left| \widehat{F}_{n,b_n}(x) - F_{T_n}(x) \right| \overset{p}{\to} 0$$

as $n \to \infty$ for x at the continuity point of the limit distribution T.

By some standard argument using Polya's theorem [see the proofs of Theorem 3.1 and Corollary 3.1 in Bai, Taqqu and Zhang (2016)], if $F_T(x)$ is continuous, one can have a stronger form of consistency, namely,

$$\sup_x \left| \widehat{F}_{n,b_n}(x) - F_{T_n}(x) \right| \overset{p}{\to} 0.$$

When proving the consistency of the empirical distribution of subsampling, for example, of $\widehat{F}_{n,b_n}$ in (37), a common strategy is to first replace $T_b(\mathbf{X}_i^{i+b-1}; \hat{\theta}_n)$, if necessary, by $T_b(\mathbf{X}_i^{i+b-1}; \theta)$, so that it depends only on the block $\{X_i, \ldots, X_{i+b-1}\}$. One needs to show that this modification is asymptotically negligible

[see Politis, Romano and Wolf (1999), the proofs of Theorem 3.2.1 and Theorem 11.3.1]. After this reduction, we are basically working with $\widehat{F}^*_{n,b_n}(x)$ in (35). To establish the consistency (38), we then only need to show that

$$\widehat{F}^*_{n,b_n}(x) \xrightarrow{p} F_T(x), \tag{39}$$

since $F_{T_n}(x) \to F_T(x)$ by the assumption (33). To do so, we write the bias-variance decomposition of the mean-square error

$$\mathbb{E}\big|\widehat{F}^*_{n,b_n}(x) - F_T(x)\big|^2 = \big[\mathbb{E}\widehat{F}^*_{n,b_n}(x) - F_T(x)\big]^2$$
$$+ \big[\mathbb{E}\big(\widehat{F}^*_{n,b_n}(x)^2\big) - \big(\mathbb{E}\widehat{F}^*_{n,b_n}(x)\big)^2\big] \tag{40}$$
$$= \big|F^*_{T_{b_n}}(x) - F_T(x)\big|^2 + \mathrm{Var}\big[\widehat{F}^*_{n,b_n}(x)\big]$$

since

$$\mathbb{E}\widehat{F}^*_{n,b_n}(x) = F^*_{T_{b_n}}(x)$$

in view of (35) and (34). The first term in (40) is the bias term, which converges to zero due to the assumption $b_n \to \infty$ and (33). The key is to bound the second variance term. By a standard argument [see the proof of Theorem 3.1 of Bai, Taqqu and Zhang (2016)],

$$\mathrm{Var}\big[\widehat{F}^*_{n,b_n}(x)\big]$$
$$\leq \frac{2}{n - b_n + 1} \sum_{k=0}^{n-b_n+1} \big|\mathrm{Cov}[\mathrm{I}\{T_{b_n}(\mathbf{X}^{b_n}_1; \theta) \leq x\}, \mathrm{I}\{T_{b_n}(\mathbf{X}^{k+b_n}_{k+1}; \theta) \leq x\}]\big|. \tag{41}$$

Note that $\mathrm{I}\{T_{b_n}(\mathbf{X}^{k+b_n}_{k+1}; \theta) \leq x\}$ is a function of $\{X_{k+1}, \ldots, X_{k+b_n}\}$ which is measurable with respect to the sigma field $\mathcal{F}^{k+b_n}_{k+1}$. Because $\mathrm{Cov}[\mathrm{I}_A, \mathrm{I}_B] = P(A)P(B) - P(A \cap B)$, we have

$$\big|\mathrm{Cov}[\mathrm{I}\{T_{b_n}(\mathbf{X}^{b_n}_1; \theta) \leq x\}, \mathrm{I}\{T_{b_n}(\mathbf{X}^{k+b_n}_{k+1}; \theta) \leq x\}]\big| \leq \alpha_{k,b_n},$$

where α_{k,b_n} is the between-block mixing coefficient defined in (2). Hence, from the variance bound (41), one has by the subsampling condition (1) that

$$\mathrm{Var}\big[\widehat{F}^*_{n,b_n}(x)\big] \leq \frac{2}{n - b_n + 1} \sum_{k=0}^{n-b_n+1} \alpha_{k,b_n} \to 0 \tag{42}$$

when $n \to \infty$ and $b_n = o(n)$. For convenience, we formulate below a corresponding result in the context of Gaussian subordination (9).

PROPOSITION 4.1. *Suppose that $\{X_n\}$ follows the Gaussian subordination model* (9) *and let $\widehat{F}^*_{n,b}(x)$ be defined as in* (35). *If the subsampling condition* (13) *holds and $b_n = o(n)$, we have*

$$\lim_{n\to\infty} \mathrm{Var}\big[\widehat{F}^*_{n,b_n}(x)\big] = 0.$$

PROOF. In view of (12), the condition (13) implies the condition (1). Then apply (42). □

This proposition shows why the subsampling condition (13) is useful. Consequently, we have

THEOREM 4.1. *Assume that the following hold*:

1. *$\{X_n\}$ is given by the Gaussian subordination model* (9), *where the underlying Gaussian $\{Z_n\}$ satisfies the assumptions of Theorem* 3.3.
2. *As $n \to \infty$, the convergence* (33) *holds.*
3. *Let x be any continuity point of $F_T(x) = P(T \le x)$. Assume that for any $\varepsilon > 0$ and $\delta > 0$, with probability tending to 1 as $n \to \infty$, one has the following*:

$$\widehat{F}^*_{n,b_n}(x - \varepsilon) - \delta \le \widehat{F}_{n,b_n}(x) \le \widehat{F}^*_{n,b_n}(x + \varepsilon) + \delta.$$

Then if $b_n \to \infty$ and $b_n = o(n)$ as $n \to \infty$, the consistency of subsampling in the sense of Definition 4.1 *holds, namely,*

$$\left| F_{T_n}(x) - \widehat{F}_{n,b_n}(x) \right| \xrightarrow{p} 0$$

for x at any continuity point of F_T.

PROOF. Assumption 1 yields Proposition 4.1. By Assumption 2, we have $F_{T_n}(x) \to F_T(x)$ as $n \to \infty$. Combining Proposition 4.1 and Assumption 2, we get

$$\widehat{F}^*_{n,b_n}(x) \xrightarrow{p} F_T(x)$$

by (40). Combining this further with Assumption 3, and using the arguments in the proof of Theorem 3.1 of Bai, Taqqu and Zhang (2016) yields

$$\widehat{F}_{n,b_n}(x) \xrightarrow{p} F_T(x).$$

The conclusion then follows from the triangle inequality:

$$\left| F_{T_n}(x) - \widehat{F}_{n,b_n}(x) \right| \le \left| F_{T_n}(x) - F_T(x) \right| + \left| \widehat{F}_{n,b_n}(x) - F_T(x) \right|. \qquad \square$$

REMARK 4.1. Assumption 3 needs to be checked in a case-by-case basis. See, for example, the proofs of Theorem 3.1 of Bai, Taqqu and Zhang (2016) and Theorem 11.3.1 of Politis, Romano and Wolf (1999). It may take different forms depending on the specific problem at hand.

REMARK 4.2. Theoretical understanding of the optimal choice of block size b in subsampling has been in general a difficult problem [see Chapter 9 of Politis, Romano and Wolf (1999)]. Our results have some heuristic implication for the choice of b. Consider the bias-variance decomposition in (40) for the empirical

distribution $\widehat{F}_{n,b_n}^*(x)$ (which involves the unknown θ) instead of that of $\widehat{F}_{n,b_n}(x)$. For simplicity, let $\{X_n = Z_n\}$ be a FARIMA$(0, d, 0)$ Gaussian process. From (12), (42) and Theorem 3.1, one can derive that when $b_n \ll n$, the variance term satisfies

(43)
$$
\begin{aligned}
\mathrm{Var}\big[\widehat{F}_{n,b}^*(x)\big] &\le \frac{2}{n - b_n + 1} \sum_{k=0}^{n-b_n+1} \rho_{k,b_n} \\
&\le \frac{c}{n} \left[\sum_{k=0}^{b_n} \rho_{k,b_n} + \sum_{k=b_n+1}^{n} \rho_{k,b_n} \right] \\
&\le c_1 \frac{b_n}{n} + \frac{c_2}{n} \left(\sum_{k=1}^{n} k^{2d-1} \right) b_n^{1-2d} \\
&\le c_1 \frac{b_n}{n} + c_2 \left(\frac{b_n}{n} \right)^{1-2d} \le c_3 \left(\frac{b_n}{n} \right)^{1-2d}.
\end{aligned}
$$

On the other hand, one also needs the rate at which the bias term $[F_{T_b}^*(x) - F_T(x)]^2$ in (40) tends to zero. It is in general difficult to assess such a rate, unless T is some special random variable, for example a Gaussian. Suppose such a rate is available: $[F_{T_b}^*(x) - F_T(x)]^2 \asymp b^{-2\gamma}$ for some $\gamma > 0$. Suppose also that the behavior of the variance is captured exactly by the bound in (43). Then by solving $(b/n)^{1-2d} \asymp b^{-2\gamma}$, one gets the optimal order:

$$ b \asymp n^{(1-2d)/(1-2d+2\gamma)}. $$

Under weak dependence, the usual Berry–Esseen rate of Gaussian approximation is $\gamma = 1/2$. If one supposes that the rate is given by $\gamma = 1/2 - d$ under long memory, then an optimal order of b would be

$$ b \asymp n^{1/2}, $$

which has been empirically observed to perform well in the context of subsampling under long memory [see, e.g., Hall, Jing and Lahiri (1998), Betken and Wendler (2015) and Zhang et al. (2013)]. Note that the preceding argument is only heuristic and is not necessarily correct. It does not apply for example to the case of the subsampling estimation of the scale parameter $\sigma_{n,d} := n^{-1-2d} \mathrm{Var}[\sum_{i=1}^n Z_i]$ considered in Kim and Nordman (2011), where the optimal order can be $b_n \asymp n^a$ for a ranging to the whole interval $(0, 1)$, depending on d.

4.3. *Application to cases considered in the literature.* Based on Theorem 3.3, some results on subsampling procedures under the Gaussian subordination model (9) considered in the literature can be established or improved as indicated below.

EXAMPLE 4.3. Hall, Jing and Lahiri (1998) considered subsampling for the sample mean. Their assumptions, however, allow only the case where the underlying Gaussian $\{Z_n\}$ in (9) is completely regular, which is equivalent to strong

mixing [see Ibragimov and Rozanov (1978), Section IV.1], but not the long memory regime considered here. Replacing their assumptions on $\{Z_n\}$ by the ones described in Theorem 3.3, the condition $b_n = o(n)$ will yield consistency of their subsampling procedure. See also Lahiri (2003), Section 10.4.

EXAMPLE 4.4. Conti et al. (2008) considered subsampling for an estimator of the long memory parameter. They assumed as in Hall, Jing and Lahiri (1998) that $\{Z_n\}$ is completely regular. Replacing their assumptions on $\{Z_n\}$ by the ones given in Theorem 3.3, the condition $b_n = o(n)$ will yield consistency of their subsampling procedure.

EXAMPLE 4.5. Psaradakis (2010) studied subsampling for the one-sample sign statistic under the same framework of Hall, Jing and Lahiri (1998). Replacing the assumptions on $\{Z_n\}$ in Theorem 3.3, the block size condition $b_n = o(n)$ will yield consistency of the subsampling procedure.

EXAMPLE 4.6. Betken and Wendler (2015) considered subsampling for a general statistic as in (35) under the model (9), and also discussed a robust change-point test as an example. If their assumptions on $\{Z_n\}$ are replaced by those of Theorem 3.2(a), their block size condition $b_n = o(n^{1-d-\varepsilon})$ with arbitrarily small $\varepsilon > 0$ can be relaxed to $b_n = o(n)$ and consistency still holds for a general statistic.

EXAMPLE 4.7. Bai, Taqqu and Zhang (2016) studied subsampling for sample mean with the self-normalization considered in Shao (2010), which avoids dealing with various nuisance parameters. They adopted the assumptions in Theorem 3.3, so that $b_n = o(n)$ yields consistency. See Bai, Taqqu and Zhang (2016), Corollary 3.1.

EXAMPLE 4.8. Suppose that $\{X_n\}$ is subordinated to a multivariate stationary Gaussian process

$$\{\mathbf{Z}_n = (Z_{n,1}, \ldots, Z_{n,J})\}$$

with independent components. As noted in Section 3.1, the analog of $\rho_{k,b}$ in (7) for $\{\mathbf{Z}_n\}$ is the maximum of the $\rho_{k,b,j}$'s computed from the individual $\{Z_{n,j}\}$'s. This fact is useful when one considers some nonlinear time series models.

For example, let

$$\mathbf{Z}_n = (\eta_n, \xi_n)$$

be a bivariate stationary Gaussian process, where $\{\eta_n\}$ satisfies the long memory conditions as in Theorem 3.3, and $\{\xi_n\}$ is i.i.d. standard normal and is independent of $\{\eta_n\}$. Consider the following stochastic volatility model [see, e.g., Beran et al. (2013), Section 2.1.3.8]:

$$(44) \qquad\qquad X_n = G(\eta_n)F(\xi_n),$$

where $Z_{n,1} = \eta_n$ and $Z_{n,2} = \xi_n$, and $G(\cdot)$ is a positive function and $F(\cdot)$ is chosen so that $\mathbb{E}F(\xi_n) = 0$. The flexibility of choosing $G(\cdot)$ and $F(\cdot)$ allows a variety of marginal distributions for $\{X_n\}$. The sequence $\{X_n\}$ is uncorrelated, but the volatility $G(\eta_n)$ of $\{X_n\}$ may display long memory if $G(\cdot)$ is chosen appropriately. See also Deo, Hsieh and Hurvich (2010) and Jach, McElroy and Politis (2012) for variants of (44).

Note that in (32), the canonical correlation $\rho_{k,b,1}$ for $\{\eta_n\}$ dominates since the canonical correlation $\rho_{k,b,2}$ for $\{\xi_n\}$ vanishes when $k > b$ due to independence. Hence, if $b_n = o(n)$, the subsampling condition (13) holds for the underlying $\{\mathbf{Z}_n\}$ because it holds for $\{\eta_n\}$ by Theorem 3.3.

REMARK 4.3. In the special case of subsampling for the sample mean with a *deterministic* normalization, where $\{X_n\}$ is a long-memory linear process which is not necessarily Gaussian, Nordman and Lahiri (2005) showed that a block size condition $b_n = o(n^{1-\varepsilon})$ for any $\varepsilon > 0$ suffices for consistency. In the same setup but with $\{X_n\}$ replaced by a nonlinear function of a long-memory linear process, Zhang et al. (2013) obtained consistency with the same block size condition $b_n = o(n^{1-\varepsilon})$ for arbitrarily small $\varepsilon > 0$. These do not fall within the Gaussian subordination framework (9). Establishing (1) for long memory models beyond Gaussian subordination under a nonrestrictive condition close to $b_n = o(n)$ may be an interesting open problem.

5. Subsampling of common statistics. In the following sections, we discuss subsampling of some common types of statistics under long memory. The case of the sample mean is discussed in Examples 4.3, 4.7 and Remark 4.3. See also Chapter 4 of Lahiri (2003) for expositions about block bootstrap methods under weak dependence.

5.1. *Sample autocovariance.* Let $\{Z_n\}$ be a stationary long-memory Gaussian process satisfying the assumptions in Theorem 3.3 and $\gamma(k) = \text{Cov}[Z_k, Z_0] \sim ck^{2d-1}$ for some $c > 0$ as $k \to \infty$. Let the memory parameter be $d \in (0, 1/2)$. For simplicity, we focus on the case where

$$X_n = Z_n.$$

We consider the estimation of $\gamma(m)$ by the sample autocovariance

$$\hat{\gamma}(m) = \frac{1}{n} \sum_{i=1}^{n-b} (X_i - \bar{X}_n)(X_{i+m} - \bar{X}_n),$$

where $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$. We treat for simplicity here only the subsampling inference for the $\gamma(m)$ at a single lag m, while the joint inference at different lags can also be considered in the same way. Even under the assumption of $\{X_n\}$ being

Gaussian, the asymptotic behavior of $\hat{\gamma}_n(m)$ is intricate. Indeed, we have [see, e.g., Dehling and Taqqu (1991) and Hosking (1996)] as $n \to \infty$ that

$$(45) \qquad \tau_n(\hat{\gamma}_n(m) - \gamma(m)) \overset{d}{\to} \begin{cases} N(0, \sigma_1^2) & d \in (0, 1/4), \tau_n = n^{1/2}; \\ \sigma_2 R_d & d \in (1/4, 1/2), \tau_n = n^{1-2d}, \end{cases}$$

where σ_1 and σ_2 are positive scale constants, and R_d is a non-Gaussian distribution termed "modified Rosenblatt distribution" by Hosking (1996). Using (45) directly for asymptotic inference involves some difficulties: (1) the dichotomy of asymptotic distributions as the memory parameter d varies, (2) τ_n takes a nonstandard rate if $d > 1/4$ which is unknown. This forces one to deal with several nuisance parameters.

Now consider using subsampling for inference. We need to account for the nonstandard scaling τ_n. Let $\hat{\tau}_n = \hat{\tau}_n(\mathbf{X}_1^n)$ be a random normalizer computed from the data so that

$$(46) \qquad T_n := \hat{\tau}_n(\hat{\gamma}_n(m) - \gamma(m)) \overset{d}{\to} T$$

as $n \to \infty$ for some nondegenerate random variable T. It is practically attractive to have just one unified form of the normalizer $\hat{\tau}_n$ which works across the different cases in (45). Such $\hat{\tau}_n$ can be in principle achieved by the "self-normalization" considered in Lobato (2001), Shao (2010), although some functional limit theorems involved are yet to be established. See also McElroy and Jach (2012) for a different self-normalization. We require that the self-normalization $\hat{\tau}_n$ captures the scaling τ_n in the sense of that as $n \to \infty$,

$$(47) \qquad \tau_n/\hat{\tau}_n \overset{d}{\to} W$$

for some random variable W satisfying $P(W \neq 0) = 1$. Now the conditions (45), (46) and (47) are in line with Assumption 11.3.1 of Politis, Romano and Wolf (1999).

Define now analogously the sample autocovariance computed on the block $\mathbf{X}_{k+1}^{k+b}$:

$$\hat{\gamma}_b(m; \mathbf{X}_{k+1}^{k+b}) = \frac{1}{b} \sum_{i=k+1}^{k+b-m} (X_i - \bar{X}_{k+1}^{k+b})(X_{i+m} - \bar{X}_{k+1}^{k+b}),$$

where

$$\bar{X}_{k+1}^{k+b} = b^{-1}(X_{k+1} + \cdots + X_{k+b}), \qquad k = 0, 1, \ldots, n-b, 0 \leq m \leq b.$$

Set the statistics on the block as

$$(48) \qquad T_b(\mathbf{X}_{k+1}^{k+b}; \hat{\gamma}(m)) = \hat{\tau}_b(\mathbf{X}_{k+1}^{k+b})(\hat{\gamma}_b(m; \mathbf{X}_{k+1}^{k+b}) - \hat{\gamma}_n(m)),$$

where $\hat{\gamma}_b(m; \mathbf{X}_{k+1}^{k+b})$ satisfies (47) and is based on the block $\mathbf{X}_{k+1}^{k+b}$.

We formulate a result as follows, which is within the framework of Theorem 4.1. The proof is similar to that of Theorem 11.3.1 of Politis, Romano and Wolf (1999). See also the proof of Theorem 3.1 of Bai, Taqqu and Zhang (2016).

THEOREM 5.1.   *Let $\{X_n = Z_n\}$ be a long-memory Gaussian process satisfying the assumptions in Theorem* 3.3. *Assume that* (45), (46) *and* (47) *hold. Let*

$$\widehat{F}_{n,b}(x) = (n - b + 1)^{-1} \sum_{i=1}^{n-b+1} \mathrm{I}\{T_b(\mathbf{X}_i^{i+b-1}; \hat{\gamma}(m)) \le x\},$$

where $T_b(\mathbf{X}_i^{i+b-1}; \hat{\gamma}_n(m))$ is as in (48). *Let $F_{T_n}(x)$ be the distribution function of T_n in* (46). *Then the consistency of the subsampling procedure holds: if $n \to \infty$, $b_n \to \infty$ and $b_n = o(n)$, we have*

$$\left| F_{T_n}(x) - \widehat{F}_{n,b_n}(x) \right| \overset{p}{\to} 0$$

for x at the continuity point of the distribution of T in (46).

Sample autocovariance falls within the category called "smooth function of mean" [see, e.g., Example 4.4.2 of Politis, Romano and Wolf (1999) and Section 4.2 of Lahiri (2003)]. Theorem 5.1 may be extended to this general category, given that asymptotic results analogous to (45), (46) and (47) are established specifically.

5.2. *M-estimation.*   Beran (1991) considered the M-estimation for the following location model:

$$(49) \qquad\qquad X_i = \mu + Q(Z_i),$$

where $\{Z_i\}$ is a *standardized* long-memory Gaussian process satisfying the assumptions in Theorem 3.3, and $\gamma(k) = \mathrm{Cov}[Z_k, Z_0] \sim ck^{2d-1}$ for some $c > 0$ as $k \to \infty$. The function $Q(\cdot)$ satisfies $\mathbb{E}Q(Z_i) = 0$ and $\sigma^2 := \mathbb{E}Q(Z_i)^2 < \infty$. Assume for simplicity that $\sigma^2 = 1$, while in general σ^2 can be consistently estimated by the sample variance and this does not affect the asymptotic results. The estimating equation is given by

$$(50) \qquad\qquad \sum_{i=1}^{n} \psi(X_i - x) = 0,$$

where ψ is some deterministic function such that $\mathbb{E}\psi(X_i - x) = 0$ if and only if $x = \mu$. Let $\hat{\mu}_n$ be the resulting M-estimator of μ [the solution to (50)]. Theorem 1 of Beran (1991) states that

$$(51) \qquad\qquad n^{1/2-d}(\hat{\mu}_n - \mu) \overset{d}{\to} N(0, \sigma_M^2)$$

for some scale constant $\sigma_M > 0$. Theorem 1 of Beran (1991) has five assumptions: the assumptions 1-4 are standard regularity conditions in the M-estimation context, while the 5th one is imposed to restrict to the Gaussian asymptotics in (51). If the

5th assumption is dropped, depending on the Hermite rank m [see (10)] of the composite transform $\psi \circ Q$, one may have

$$(52) \qquad \tau_n(\hat{\mu}_n - \mu) \xrightarrow{d} \begin{cases} N(0, \sigma_1^2) & (2d-1)m < -1, \tau_n = n^{1/2}; \\ \sigma_2 Z_{m,d} & (2d-1)m > -1, \tau_n = n^{(d-1/2)m+1}, \end{cases}$$

where σ_1 and σ_2 are positive scale constants, and $Z_{m,d}$ is the so-called *Hermite distribution* [see Dobrushin and Major (1979)] which is non-Gaussian if the Hermite rank $m \geq 2$. Using directly (52) for inference is again difficult due to the dichotomy and the nuisance parameters.

However, as in Section 5.1, we can consider a subsampling procedure with a proper self-normalization $\hat{\tau}_n$ [see, e.g., Shao (2010)]. We omit the formal statement of the result, which is similar to Theorem 5.1. Note that asymptotic results similar to (45), (46) and (47) need to be established.

5.3. *Empirical process.* Let $\{Z_n\}$ be long-memory Gaussian with memory parameter $d \in (0, 1/2)$ that satisfies the assumptions in Theorem 3.3 as well as $\mathrm{Cov}[Z_k, Z_0] \sim ck^{2d-1}$ for some constant $c > 0$. Consider the setup in Dehling and Taqqu (1989): let the data $\{X_n\}$ be given by the Gaussian subordination model (9), where G is any measurable function. We consider a case where the parameter is an infinite-dimensional object: the distribution function F. Assume that F is continuous. Consider the empirical distribution

$$(53) \qquad \widehat{F}_n(x) = \widehat{F}_n(x; \mathbf{X}_1^n) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{I}\{X_i \leq x\}$$

as an estimate of the distribution function $F(x) = P(X_n \leq x)$. Let m be the Hermite rank of the class of functions:

$$\mathcal{E} := \{\mathrm{I}\{G(\cdot) \leq x\} - F(x), x \in \mathbb{R}\}$$

[see Definition before Theorem 1.1 of Dehling and Taqqu (1989)]. Theorem 1.1 of Dehling and Taqqu (1989) established the following functional limit theorem: if $(2d-1)m > -1$, then we have the weak convergence in the Skorohod space $D(-\infty, \infty)$:

$$(54) \qquad \tau_n(\widehat{F}_n(x) - F(x)) \Rightarrow J_m(x) Z_{m,d}, \qquad \tau_n = n^{(d-1/2)m+1},$$

where $Z_{m,d}$ is the Hermite distribution as in (52), and $J_m(x)$ is a nonrandom function determined by the class $\mathcal{E}$. The corresponding result for the short-memory regime $(2d-1)m < -1$ has not been established up to our knowledge [there may be technical issues with tightness, see Chambers and Slud (1989)], but a weak convergence to a Gaussian process with the rate $\tau_n = n^{1/2}$ is expected [see, e.g., Dehling and Philipp (2002)].

Now consider the block version of (53):

$$(55) \qquad \widehat{F}_{b,k}(x) = \widehat{F}_b(x; \mathbf{X}_k^{k+b-1}) = \frac{1}{b} \sum_{i=k}^{k+b-1} \mathrm{I}\{X_i \le x\}.$$

Again as in the previous sections, we can consider applying subsampling with a proper estimation of scale $\hat{\tau}_n$ so that

$$\hat{\tau}_n / \tau_n \xrightarrow{d} W$$

for some nonzero random variable W and

$$\hat{\tau}_n (\widehat{F}_n(x) - F(x)) \Rightarrow T(x)$$

for some nondegenerate random function $T(x)$. Then one can use the empirical "observations"

$$\hat{\tau}_b (\widehat{F}_{b,k}(x) - \widehat{F}_n(x)), \qquad k = 1, \dots, n - b + 1$$

for inference. For example, to construct a uniform confidence band for F, consider

$$S_k = \hat{\tau}_b \sup_x |\widehat{F}_{b,k}(x) - \widehat{F}_n(x)|, \qquad k = 1, \dots, n - b + 1.$$

Then use the empirical quantile of $\{S_k\}$ to find the cutoff s_α for the confidence band

$$(56) \qquad [\max\{\widehat{F}_n(x) - s_\alpha, 0\}, \min\{\widehat{F}_n(x) + s_\alpha, 1\}], \qquad x \in \mathbb{R}.$$

For more information on subsampling empirical processes, see Section 7.4 of Politis, Romano and Wolf (1999). One needs again to establish results similar to (45), (46) and (47). The asymptotic consistency of the confidence band (56) will then follow under the block size condition $b = b_n \to \infty$ and $b_n = o(n)$ as $n \to \infty$.

6. Proofs of the main results. We now give the proofs of the main results stated in Section 3. In all the proofs below, the letters $c, c_1, c_2, \dots$ will denote positive constants whose values can change from line to line.

6.1. *Preliminary lemmas.* We give a number of lemmas which will be used in the proofs of the main results. Note that the covariance matrix of the joint vector $(\mathbf{Z}_1^b, \mathbf{Z}_{k+1}^{k+b})$ is

$$\begin{pmatrix} \Sigma_b & \Sigma_{k,b} \\ \Sigma_{k,b}^T & \Sigma_b \end{pmatrix},$$

where Σ_b is the in-block covariance matrix (5), and $\Sigma_{k,b}$ is the cross-block covariance matrix (6). The following lemma states a well-known relation between the maximization problem (7) and the matrices Σ_b and $\Sigma_{k,b}$.

LEMMA 6.1. *The supremum in* (7) *is attained when* $\mathbf{u} = \mathbf{u}^*$ *and* $\mathbf{v} = \mathbf{v}^*$, *where* $\mathbf{u}^*$ *is an eigenvector of the matrix*:

$$(57) \qquad U_{k,b} = \Sigma_b^{-1} \Sigma_{k,b} \Sigma_b^{-1} \Sigma_{k,b}^T$$

corresponding to its maximum eigenvalue λ_U, *and where* $\mathbf{v}^*$ *is an eigenvector of the matrix*

$$(58) \qquad V_{k,b} = \Sigma_b^{-1} \Sigma_{k,b}^T \Sigma_b^{-1} \Sigma_{k,b}$$

corresponding to its maximum eigenvalue λ_V. *In addition, the canonical correlation equals*

$$\rho_{k,b} = \sqrt{\lambda_U} = \sqrt{\lambda_V},$$

and $\mathbf{u}^*$ *and* $\mathbf{v}^*$ *are related through*

$$(59) \qquad \mathbf{u}^* = \rho_{k,b}^{-1} \Sigma_b^{-1} \Sigma_{k,b} \mathbf{v}^*, \qquad \mathbf{v}^* = \rho_{k,b}^{-1} \Sigma_b^{-1} \Sigma_{k,b}^T \mathbf{u}^*.$$

PROOF. See Hotelling (1936) and also Section 21.5.3 of Seber (2008). □

The following facts about the FARIMA$(0, d, 0)$ model can be found in Brockwell and Davis (1991), Section 13.2. See also Hosking (1981). The FARIMA$(0, d, 0)$ time series $\{Z_n\}$ with spectral density $f_d(\lambda)$ in (20) has covariance function

$$(60) \qquad \gamma_d(n) = \gamma_d(0) \prod_{k=1}^n \frac{k-1+d}{k-d} \sim c_d n^{2d-1} \qquad \text{as } n \to \infty,$$

$$\gamma_d(0) = \frac{\Gamma(1-2d)}{\Gamma(1-d)^2},$$

where $c_d > 0$ is a constant, and $\Gamma(\cdot)$ denotes the gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \qquad \text{if } x > 0;$$

$$\Gamma(x) = +\infty \qquad \text{if } x = 0;$$

$$\Gamma(x) = x^{-1} \Gamma(1+x) \qquad \text{if } x < 0.$$

Notice that $\gamma_d(n) > 0$ and is decreasing as $n > 0$ grows. It is also known that the mean-square best linear predictor of Z_{b+1} in terms of $Z_1, \ldots, Z_b$ is given as

$$(61) \qquad \widehat{Z}_{b+1} = P_{[1,b]} Z_{b+1} = \sum_{j=1}^b \phi_{bj} Z_{b-j+1},$$

where $P_{[1,b]}$ denotes the $L^2(\Omega)$ projection onto the closed linear span sp$\{Z_1, \ldots, Z_b\}$, and the coefficients are given by

$$(62) \qquad \phi_{bj} = -\Gamma(-d)^{-1} \binom{b}{j} \frac{\Gamma(j-d)\Gamma(b-d-j+1)}{\Gamma(b-d+1)}, \qquad j = 1, \ldots, b.$$

Note also that each $\phi_{bj} > 0$ since $\Gamma(x) > 0$ if $x > 0$ and $\Gamma(x) < 0$ if $-1 < x < 0$. One can then easily deduce the following fact.

LEMMA 6.2.   *Let $\{Z_n\}$ be the* FARIMA$(0, d, 0)$ *process with spectral density f_d given in* (20). *Let ϕ_{bj}^n, $j = 1, \ldots, b$, $n \geq b + 1$, be the coefficient[4] of the best linear predictor of Z_n in terms of $Z_1, \ldots, Z_b$, namely,*

$$P_{[1,b]}Z_n = \sum_{j=1}^{b} \phi_{bj}^n Z_{b-j+1}.$$

Then

$$\phi_{bj}^n > 0$$

for $j = 1, \ldots, b$ and $n \geq b + 1$.

PROOF.   Note that

$$P_{[1,b]}Z_n = P_{[1,b]} \cdots P_{[1,n-2]}P_{[1,n-1]}Z_n.$$

Then apply (61), (62) and use the positiveness of ϕ_{bj}'s recursively.   □

The next result plays a key role in the proof of Theorem 3.1.

LEMMA 6.3.   *If $\gamma(n)$ is the covariance function of a* FARIMA$(0, d, 0)$ *time series whose spectral density f_d is given in* (20), *then the matrices $U_{b,k}$ and $V_{b,k}$ in* (57) *and* (58), *respectively, have all the entries positive, and the extremal eigenvectors $\mathbf{u}^*$ and $\mathbf{v}^*$ in Lemma 6.1 can be chosen[5] to have all positive entries.*

PROOF.   To show that the matrices $U_{k,b}$ and $V_{k,b}$ have positive entries, it is enough to show that $\Sigma_b^{-1}\Sigma_{k,b}$ and $\Sigma_b^{-1}\Sigma_{k,b}^T$ have positive entries. Note that a column of $\Sigma_{k,b}$ is of the form

$$\boldsymbol{\gamma}_{n-1}^{n-b} := \big(\gamma(n-1), \ldots, \gamma(n-b)\big)^T$$

for some $n > b$. The corresponding column of $\Sigma_b^{-1}\Sigma_{k,b}$ is then $\Sigma_b^{-1}\boldsymbol{\gamma}_{n-1}^{n-b} = (\phi_{bb}^n, \ldots, \phi_{b1}^n)^T$ by the Yule–Walker equation [see, e.g., (5.1.9) of Brockwell and Davis (1991)]. Similarly, a column of $\Sigma_b^{-1}\Sigma_{k,b}^T$ is of the form $(\phi_{b1}^n, \ldots, \phi_{bb}^n)^T$. Hence by Lemma 6.2, all entries of $\Sigma_b^{-1}\Sigma_{k,b}$ and $\Sigma_b^{-1}\Sigma_{k,b}^T$ are positive.

The Perron–Frobenius theorem [see Item 9.16 of Seber (2008)] states that the eigenvector corresponding to the maximum eigenvalue of a matrix with positive entries can be chosen to have all components positive. Since the matrix $U_{k,b}$ has

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[4]The superscript n in ϕ_{bj}^n is an index.

[5]Eigenvectors are determined up to a multiplicative constant.

positive entries, we deduce that the extremal eigenvector $\mathbf{u}^*$ can be chosen to have all positive entries. In view of (59), this also makes $\mathbf{v}^*$ positive. $\quad\square$

The following simple fact will be useful.

LEMMA 6.4. *Let $\{Z_n\}$ be a* FARIMA$(0, d, 0)$ *process with spectral density f_d given in* (20). *Let $c > 0$ be a fixed constant. Then for all nonnegative u_j's such that*

$$(63) \qquad \mathrm{Var}\left[\sum_{j=1}^b u_j Z_j\right] \le c,$$

there exists a constant $c_1 > 0$ which does not depend on b, such that

$$(64) \qquad \sum_{j=1}^b u_j \le c_1 b^{1/2-d}.$$

PROOF. Using the fact that $u_j \ge 0$, as well as the positiveness, monotonicity and the asymptotics of the covariance function $\gamma_d(n)$ in (60), we have

$$c \ge \mathrm{Var}\left[\sum_{j=1}^b u_j Z_j\right] = \sum_{i,j=1}^b u_i u_j \gamma_d(i - j)$$

$$\ge \gamma_d(b-1) \sum_{i,j=1}^b u_i u_j \ge c_2 b^{2d-1}\left(\sum_{j=1}^b u_j\right)^2$$

for some constant $c_2 > 0$ not depending on b, which yields (64). $\quad\square$

REMARK 6.1. By making use of the result of Adenstedt (1974), one can strengthen (64) to $|\sum_{j=1}^b u_j| \le cb^{1/2-d}$ with u_j's not necessarily nonnegative. See Lemma 2 of Betken and Wendler (2015).

We recall here the time and frequency domain isomorphism (Kolmogorov isomorphism).

LEMMA 6.5 [Brockwell and Davis (1991), Theorem 4.8.1]. *Suppose f is the spectral density of $\{Z_n\}$. Let $\mathcal{H} = \overline{\mathrm{sp}}\{Z_n, n \in \mathbb{Z}\}$ be the Hilbert space spanned by $\{Z_n\}$ in $L^2(\Omega)$. Let $\mathfrak{H} = \overline{\mathrm{sp}}\{e^{in\cdot}, n \in \mathbb{Z}\}$ be the Hilbert space spanned by $\{e^{in\cdot}\}$ in $L^2((-\pi, \pi], \mathbb{C}; f)$ (f-weighted complex-valued L^2 space on $(-\pi, \pi]$). Then there is a unique Hilbert space isomorphism*

$$T : \mathcal{H} \longrightarrow \mathfrak{H}, \qquad Z_n \longrightarrow e^{in\cdot}.$$

We note that the translation in the time domain of k units by the isomorphism acts as multiplication by $e^{ik\lambda}$ in the frequency domain.

Let us return to the maximization problem (7). First, note that

$$\sup_{\mathbf{u}\in\mathbb{R}^b, \mathbf{v}\in\mathbb{R}^b} \text{Corr}(\langle\mathbf{u}, \mathbf{Z}_1^b\rangle, \langle\mathbf{v}, \mathbf{Z}_{k+1}^{k+b}\rangle) = \sup_{\mathbf{u}\in\mathbb{R}^b, \mathbf{v}\in\mathbb{R}^b} |\text{Corr}(\langle\mathbf{u}, \mathbf{Z}_1^b\rangle, \langle\mathbf{v}, \mathbf{Z}_{k+1}^{k+b}\rangle)|.$$

Note also that the preceding maximization can be stated in an equivalent constrained optimization form

$$\rho_{k,b} = \sup_{\mathbf{u}\in\mathbb{R}^b, \mathbf{v}\in\mathbb{R}^b} |\text{Cov}(\langle\mathbf{u}, \mathbf{Z}_1^b\rangle, \langle\mathbf{v}, \mathbf{Z}_{k+1}^{k+b}\rangle)|$$

$$= \sup_{\mathbf{u}\in\mathbb{R}^b, \mathbf{v}\in\mathbb{R}^b} \left| \sum_{i,j=1}^{b} u_i v_j \gamma(k+j-i) \right|,$$

(65)
$$\text{subject to: } \text{Var}[\langle\mathbf{u}, \mathbf{Z}_1^b\rangle] = \sum_{i,j=1}^{b} u_i u_j \gamma(i-j) \le 1,$$

$$\text{Var}[\langle\mathbf{v}, \mathbf{Z}_{k+1}^{k+b}\rangle] = \sum_{i,j=1}^{b} v_i v_j \gamma(i-j) \le 1,$$

that is, the conditions

$$\text{Var}[\langle\mathbf{v}, \mathbf{Z}_1^b\rangle] = 1 \quad \text{and} \quad \text{Var}[\langle\mathbf{v}, \mathbf{Z}_{k+1}^{k+b}\rangle] = 1$$

can be replaced by

$$\text{Var}[\langle\mathbf{v}, \mathbf{Z}_1^b\rangle] \le 1 \quad \text{and} \quad \text{Var}[\langle\mathbf{v}, \mathbf{Z}_{k+1}^{k+b}\rangle] \le 1.$$

This is because the maximum will be attained at the boundaries where the variances are equal to 1 by scaling. In view of Lemma 6.5, the preceding constrained optimization can be expressed in the frequency domain as

$$\rho_{k,b} = \sup_{U_b, V_b} \left| \int_{-\pi}^{\pi} e^{ik\lambda} U_b(e^{i\lambda}) \overline{V_b(e^{i\lambda})} f(\lambda) \, d\lambda \right|$$

(66)
$$= \sup_{U_b, V_b} \left| \int_{-\pi}^{\pi} e^{-ik\lambda} U_b(e^{i\lambda}) \overline{V_b(e^{i\lambda})} f(\lambda) \, d\lambda \right|,$$

$$\text{subject to: } \int_{-\pi}^{\pi} |U_b(e^{i\lambda})|^2 f(\lambda) \, d\lambda \le 1,$$

(67)
$$\int_{-\pi}^{\pi} |V_b(e^{i\lambda})|^2 f(\lambda) \, d\lambda \le 1,$$

where the supremum is taken over all polynomials $U_b(z) = \sum_{j=0}^{b-1} u_{j+1} z^j$, $V_b(z) = \sum_{j=0}^{b-1} v_{j+1} z^j$.

REMARK 6.2. If one replaces the constraints "$\cdots \leq 1$" in (65) or (67) by "$\cdots \leq c$" for some constant $c > 0$, then the supremum obtained in (65) and (67) becomes $c\rho_{k,b}$.

6.2. Proof of the main theorems.

PROOF OF THEOREM 3.1. We first use the maximizer $\mathbf{u}^* = (u_1^*, \ldots, u_b^*)^T$ and $\mathbf{v}^* = (v_1^*, \ldots, v_b^*)^T$ of (65) to get

$$\rho_{k,b} = \sum_{i,j=1}^{b} u_i^* v_j^* \gamma_d(k + j - i).$$

By Lemma 6.3, the components u_j^* and v_j^*, $j = 1, \ldots, b$, can be chosen to be all positive. In view of (65), we can suppose (63) with $c = 1$. By the positiveness and the monotone decreasing property of $\gamma_d(n)$ in (60), as well as the relation (64) in Lemma 6.4, we have for $k > b$,

$$\rho_{k,b} \leq \left(\sum_{j=1}^{b} u_j^*\right)\left(\sum_{j=1}^{b} v_j^*\right)\gamma_d(k + 1 - b)$$

$$\leq cb^{1/2-d}b^{1/2-d}(k - b)^{2d-1} = cb^{1-2d}(k - b)^{2d-1}. \qquad \square$$

The following corollary will be used in the proof of Theorem 3.2. It is an immediate consequence of Theorem 3.1, the frequency domain characterization (67) and Remark 6.2.

COROLLARY 6.1. *Define* $g_b(\lambda) = U_b(e^{i\lambda})\overline{V_b(e^{i\lambda})}f_d(\lambda)$, *where* f_d *is as in* (20), *and let*

$$\hat{g}_b(k) = \int_{-\pi}^{\pi} e^{ik\lambda} g_b(\lambda)\,d\lambda, \qquad k \in \mathbb{Z},$$

be its Fourier coefficient. Then under the constraints

$$\int_{-\pi}^{\pi} |U_b(e^{i\lambda})|^2 f_d(\lambda)\,d\lambda \leq c_1 \quad \text{and} \quad \int_{-\pi}^{\pi} |V_b(e^{i\lambda})|^2 f_d(\lambda)\,d\lambda \leq c_1,$$

where $c_1 > 0$ *is a constant, we have for some constant* $c > 0$ *and* $1 \leq b < k$ *that*

$$\sup_{U_b, V_b} |\hat{g}_b(k)| = c_1\rho_{k,b} \leq cb^{1-2d}(k - b)^{2d-1},$$

where $\rho_{k,b}$ *is the canonical correlation corresponding to the spectral density* $f_d(\lambda)$.

We are now ready to prove Theorem 3.2.

PROOF OF THEOREM 3.2. Because $f = f_d f_0$ and $f_0 \geq c_0 > 0$ by assumption, the constraint in (67) implies the following constraint:

$$(68) \qquad \int_{-\pi}^{\pi} |U_b(e^{i\lambda})|^2 f_d(\lambda) \, d\lambda \leq c_0^{-1} \quad \text{and} \quad \int_{-\pi}^{\pi} |V_b(e^{i\lambda})|^2 f_d(\lambda) \, d\lambda \leq c_0^{-1}.$$

Using the notation in (67), set as in Corollary 6.1

$$(69) \qquad g_b(\lambda) = U_b(e^{i\lambda}) \overline{V_b(e^{i\lambda})} f_d(\lambda),$$

which involves the spectral density f_d. The first integral in (66) becomes

$$\int_{-\pi}^{\pi} e^{ik\lambda} g_b(\lambda) f_0(\lambda) \, d\lambda.$$

We have [see Corollary 4.3.2 of Brockwell and Davis (1991)]

$$f_0(\lambda) = \frac{1}{2\pi} \sum_n e^{-in\lambda} \gamma_0(n),$$

and $f_0(\lambda) \leq (2\pi)^{-1} \sum_n |\gamma_0(n)| < \infty$. Note also that $\int_{-\pi}^{\pi} |g_b(\lambda)| \, d\lambda < \infty$ since U_b and V_b are polynomials which are bounded on $(-\pi, \pi]$. So one gets by Fubini's theorem that

$$
\begin{aligned}
(70) \qquad \int_{-\pi}^{\pi} e^{ik\lambda} g_b(\lambda) f_0(\lambda) \, d\lambda &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda} g_b(\lambda) \left(\sum_n \gamma_0(n) e^{-in\lambda} \right) d\lambda \\
&= \sum_n \hat{g}_b(k-n) \gamma_0(n) = \sum_n \hat{g}_b(n) \gamma_0(k-n).
\end{aligned}
$$

In view of (66) and (70), one has

$$(71) \qquad \rho_{k,b} = \sup_{U_b, V_b} \left| \int_{-\pi}^{\pi} e^{ik\lambda} g_b(\lambda) f_0(\lambda) \, d\lambda \right| = \sup_{U_b, V_b} \left| \sum_n \hat{g}_b(n) \gamma_0(k-n) \right|,$$

where $\rho_{k,b}$ is the canonical correlation corresponding to the spectral density $f = f_d f_0$. We will now split the last expression in (71) into two terms, one involving $|n| > k'$ and other $|n| \leq k'$, where $b < k' \leq k(1 - \varepsilon)$ as in (27). Hence,

$$(72) \quad \rho_{k,b} \leq \sup_{U_b, V_b} \sum_{|n| > k'} |\hat{g}_b(n) \gamma_0(k-n)| + \sup_{U_b, V_b} \sum_{|n| \leq k'} |\hat{g}_b(n) \gamma_0(k-n)| =: T_1 + T_2.$$

The first term can be bounded as

$$T_1 \leq \left(\sum_{s=-\infty}^{\infty} |\gamma_0(s)| \right) \max_{|n| > k'} \sup_{U_b, V_b} |\hat{g}_b(n)|,$$

where

$$\sup_{U_b, V_b} |\hat{g}_b(n)| = \sup_{U_b, V_b} \left| \int_{-\pi}^{\pi} e^{in\lambda} U_b(e^{i\lambda}) \overline{V_b(e^{i\lambda})} f_d(\lambda) \, d\lambda \right|$$

$$= \sup_{U_b, V_b} \left| \int_{-\pi}^{\pi} e^{-in\lambda} U_b(e^{i\lambda}) \overline{V_b(e^{i\lambda})} f_d(\lambda) \, d\lambda \right|.$$

In view of (68), Corollary 6.1 with $c_1 = c_0^{-1}$ yields that

$$\sup_{U_b, V_b} |\hat{g}_b(n)| \le c(n-b)^{2d-1} b^{1-2d}.$$

Hence,

$$(73) \qquad T_1 \le \left(\sum_{s=-\infty}^{\infty} |\gamma_0(s)| \right) c \max_{n>k'} (n-b)^{2d-1} b^{1-2d} \le c_1 (k'-b)^{2d-1} b^{1-2d}.$$

We now deal with T_2. First, note that by (69), the Cauchy–Schwarz inequality and (68), one has

$$(74) \qquad |\hat{g}_b(n)| \le \int_{-\pi}^{\pi} |U_b(e^{i\lambda}) \overline{V_b(e^{i\lambda})}| f_d(\lambda) \, d\lambda \le c_0^{-1}.$$

- If the assumption $\gamma_0(n) = O(n^{-\alpha})$ holds, then (74) and the restriction $k' \le k(1-\varepsilon)$ imply that

$$(75) \qquad T_2 \le c_0^{-1} \sum_{|n| \le k'} |\gamma_0(k-n)| \le c \sum_{|n| \le k'} (k-n)^{-\alpha}$$

$$\le c \sum_{|n| \le k'} (k-k')^{-\alpha} \le c_1 k' (k-k')^{-\alpha} \le c_1 \varepsilon^{-\alpha} k' k^{-\alpha}.$$

- If instead $\gamma_0(n) = o(n^{-\alpha})$ is assumed, then

$$(76) \qquad T_2 \le c_0^{-1} \sum_{|n| \le k'} |\gamma_0(k-n)| \le ck' o((k-k')^{-\alpha}) \le c\varepsilon^{-\alpha} k' o(k^{-\alpha}).$$

- If alternatively $\gamma_n(n) = O(e^{-cn})$ is assumed, then

$$(77) \qquad T_2 \le c_0^{-1} \sum_{n=-\infty}^{k'} |\gamma_0(k-n)| \le c_1 e^{-c(k-k')} \le c_1 e^{-\varepsilon ck}.$$

Combining (72), (73), (75), (76) and (77) yields the desired bounds. □

LEMMA 6.6. *Suppose that for any $\varepsilon \in (0, 1)$, as $n \to \infty$ and $b_n = o(n)$, we have*

$$(78) \qquad \max_{[\varepsilon n] \le k \le n} \rho_{k, b_n} \to 0.$$

Then $\sum_{k=1}^{n} \rho_{k, b_n} = o(n)$ as $n \to \infty$. Furthermore, if $\rho_{k, b}$ is nonincreasing in k for each b, then the converse holds.[6]

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[6]This argument and the one in the following theorem were suggested by an anonymous referee.

PROOF.   For an arbitrarily small $\varepsilon > 0$. Indeed, if (78) holds, then using $\rho_{k,b} \leq 1$, one has

$$\sum_{k=1}^{n} \rho_{k,b_n} = \sum_{k<[\varepsilon n]} \rho_{k,b_n} + \sum_{[\varepsilon n] \leq k \leq n} \rho_{k,b_n} \leq \varepsilon n + \sum_{[\varepsilon n] \leq k \leq n} \max_{[\varepsilon n] \leq k \leq n} \rho_{k,b_n}.$$

By (78),

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \rho_{k,b_n} \leq \varepsilon.$$

By the arbitrariness of ε, we get $\sum_{k=1}^{n} \rho_{k,b_n} = o(n)$.

Now suppose that $\rho_{k,b}$ is nonincreasing in k. Assume $\frac{1}{n} \sum_{k=1}^{n} \rho_{k,b_n} \to 0$ as $n \to \infty$. Then for any $\varepsilon \in (0, 1)$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \rho_{k,b_n} \geq \frac{1}{n} \sum_{k=1}^{[\varepsilon n]} \rho_{k,b_n} \geq \frac{1}{n} \sum_{k=1}^{[\varepsilon n]} \rho_{[\varepsilon n],b_n} = \frac{[n\varepsilon]}{n} \rho_{[\varepsilon n],b_n}.$$

Taking $n \to \infty$ on both sides, we get $\rho_{[\varepsilon n],b_n} = \max_{[\varepsilon n] \leq k \leq n} \rho_{k,b_n} \to 0$. □

PROOF OF THEOREM 3.3.   By Lemma 6.6, we need to prove (78). First, fix an integer $m \geq 1$ and choose

$$(79) \qquad\qquad k' = b_n(m+1).$$

Even though m can be quite large, $k' = b_n(m+1) = o(n)$ because $b_n = o(n)$. In fact, for n large enough, we have the following chain of inequalities:

$$1 \leq b_n < k' = b_n(m+1) \leq k/2 \qquad \text{for } [\varepsilon n] \leq k \leq n.$$

The inequalities $1 \leq b_n < k' \leq k/2$ allow the application of Theorem 3.2(a), yielding

$$(80) \qquad\qquad \rho_{k,b_n} \leq c_1 \left(\frac{b_n}{k' - b_n} \right)^{1-2d} + c_2 \frac{k'}{k}.$$

Then by $k' = b_n(m+1)$, (80) and $b_n = o(n)$,

$$\limsup_{n \to \infty} \max_{[n\varepsilon] \leq k \leq n} \rho_{k,b_n} \leq c_1 m^{2d-1} + c_2(m+1) \limsup_{n \to \infty} b_n [\varepsilon n]^{-1} = c_1 m^{2d-1}.$$

Since m can be chosen arbitrarily large and $2d - 1 < 0$, we get (78). □

APPENDIX

To obtain Proposition 2.2 and get a bound on $\rho_{k,b}$ in (7), Betken and Wendler (2015) imposed the following assumptions on the covariance function $\gamma(n)$ and the spectral density $f(\lambda)$ of $\{Z_n\}$:

*BW*1. The covariance function satisfies $\gamma(n) = n^{2d-1}L_\gamma(n)$, where $d \in (0, 1/2)$ and $L_\gamma(n)$ is a function slowly varying as $n \to \infty$, which satisfies $\max_{n+1 \leq k \leq n+2m-1} |L_\gamma(k) - L_\gamma(n)| \leq C(m/n) \min\{L_\gamma(n), 1\}$ for some constant $C > 0$ and all $n, m \in \mathbb{Z}_+$.

*BW*2. The spectral density $f(\lambda) = |\lambda|^{-2d} L_f(\lambda)$, where $L_f(\lambda)$ is a slowly varying function as $\lambda \to 0+$ and satisfies $L_f(\lambda) \geq c$ for some $c > 0$ and the limit of $L_f(\lambda)$ exists (can be $+\infty$) as $\lambda \to 0$.[7]

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[7]In Betken and Wendler (2015), the existence of the limit $L_f(\lambda)$ as $\lambda \to 0$ is not assumed. However, this seems necessary, since the result in Adenstedt (1974) applied by Betken and Wendler (2015) requires continuity at $\lambda = 0$ of $1/L(\lambda)$ [see the proof of Lemma 2 in Betken and Wendler (2015), which quoted Lemma 4.4 of Adenstedt (1974)].

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DEPARTMENT OF STATISTICS
UNIVERSITY OF GEORGIA
101 CEDAR STREET
ATHENS, GEORGIA 30606
USA
E-MAIL: bsy9142@uga.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS
BOSTON UNIVERSITY
111 CUMMINTON MALL
BOSTON, MASSACHUSETTS 02215
USA
E-MAIL: murad@bu.edu