

ON THE SUBMARTINGALE PROBLEM FOR REFLECTED DIFFUSIONS IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES

BY WEINING KANG AND KAVITA RAMANAN¹

University of Maryland and Brown University

Two frameworks that have been used to characterize reflected diffusions include stochastic differential equations with reflection (SDER) and the so-called submartingale problem. We consider a general formulation of the submartingale problem for (obliquely) reflected diffusions in domains with piecewise C^2 boundaries and piecewise continuous reflection vector fields. Under suitable assumptions, we show that well-posedness of the submartingale problem is equivalent to existence and uniqueness in law of weak solutions to the corresponding SDER. The main step involves showing existence of a weak solution to the SDER given a solution to the submartingale problem. This generalizes the classical construction, due to Stroock and Varadhan, of a weak solution to an (unconstrained) stochastic differential equation, but requires a completely different approach to deal with the geometry of the domain and directions of reflection and properly identify the local time on the boundary, in the presence of multi-valued directions of reflection at nonsmooth parts of the boundary. In particular, our proof entails the construction of classes of test functions that satisfy certain oblique derivative boundary conditions, which may be of independent interest. Other ingredients of the proof that are used to identify the constraining or local time process include certain random linear functionals, suitably constructed exponential martingales and a dual representation of the cone of directions of reflection. As a corollary of our result, under suitable assumptions, we also establish an equivalence between well-posedness of both the SDER and submartingale formulations and well-posedness of the constrained martingale problem, which is another framework for defining (semimartingale) reflected diffusions. Many of our intermediate results are also valid for reflected diffusions that are not necessarily semimartingales, and are used in a companion paper [Equivalence of stochastic equations and the submartingale problem for nonsemimartingale reflected diffusions. Preprint] to extend the equivalence result to a class of nonsemimartingale reflected diffusions.

Received December 2014; revised August 2016.

¹Supported in part by NSF Grants CMMI-1234100 and DMS-14-07504.

MSC2010 subject classifications. Primary 60H10, 60J60, 60J65; secondary 90B15, 90B22.

Key words and phrases. Reflecting diffusions, reflected Brownian motion, submartingale problem, martingale problem, stochastic differential equations with reflection, weak solutions, Skorokhod problem, extended Skorokhod problem, nonsemimartingales, diffusion approximations, queueing networks.

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1. Introduction.

1.1. *Background and motivation.* A reflected diffusion in a nonempty, connected domain G with a vector field $d(\cdot)$ on the boundary ∂G and measurable drift and dispersion coefficients $b : \bar{G} \mapsto \mathbb{R}^J$ and $\sigma : \bar{G} \mapsto \mathbb{R}^{J \times J}$ defined on the closure \bar{G} of the domain is a continuous Markov process that, roughly speaking, behaves like a diffusion with (state-dependent) drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ inside the domain and that is restricted to stay in \bar{G} by a constraining force that is only allowed to act along the directions specified by the vector field on the boundary. For historical reasons, this constrained process is referred to as a reflected diffusion. Two approaches to providing a precise mathematical characterization of this intuitive description are the framework of stochastic differential equations with reflection (SDER), which is used, for example, in [4, 9, 14, 31, 37] and [32], and the submartingale problem formulation introduced by Stroock and Varadhan in [39]. These two approaches are respective generalizations of the stochastic differential equation (SDE) and martingale problem formulations commonly used to analyze diffusions in \mathbb{R}^J . In the case of (unconstrained) diffusions, under fairly

general conditions, there is a well-established equivalence between existence and uniqueness in law of weak solutions to SDEs and well-posedness of the martingale problem (see, e.g., [40] and [27]). Somewhat surprisingly, there appears to be no such general correspondence available in the case of obliquely reflected diffusions in nonsmooth domains. Such reflected diffusions arise in a broad range of applications, including queueing theory, biochemical reaction networks, mathematical finance and the study of interacting particle systems and random matrices.

The goal of the current work and the companion paper [22] is to establish the equivalence between well-posedness of the submartingale problem and well-posedness of the associated SDER formulation for a large class of obliquely reflected diffusions in piecewise smooth domains (see Section 2 for precise definitions). In Theorem 1, we prove this equivalence for semimartingale reflected diffusions with a measurable, locally bounded drift and a continuous and uniformly elliptic diffusion coefficient in piecewise C^2 domains with piecewise continuous reflection that satisfy a certain geometric condition. This condition, which ensures the semimartingale property, is a generalization of the so-called completely- \mathcal{S} condition that is used for RBMs in the orthant [6]. Several intermediate results in the proof hold in greater generality, and are used in [22] to extend the equivalence to a larger class of reflected diffusions that are not necessarily semimartingales, which arise in many situations [7, 12, 15, 20, 33, 34]. In [22], we also provide a counterexample to show that this equivalence can fail to hold for certain nonsemimartingale reflected diffusions outside the class therein, thus underscoring the subtleties involved in the proof of this equivalence. Another approach that can be used to construct reflected diffusions is the so-called constrained martingale problem (CMP) of Kurtz [26], although as formulated in [26], it only applies to semimartingale reflected diffusions. As a corollary of our main result (see Remark 3.2), we also establish equivalence of the well-posedness of the CMP formulation and the SDER formulation for a class of semimartingale reflected diffusions.

Our work unifies and clarifies the connections between these different approaches to constructing reflected diffusions. The submartingale problem was originally formulated only for smooth domains and continuous reflection [39]. Extensions to domains with nonsmooth boundaries had previously been considered only in special cases [12, 29, 30, 42, 44]. With the exception of [44], in each of these cases, the boundaries of the domains considered have only a single point of nonsmoothness. The work [44] considered the class of skew-symmetric reflected Brownian motions (RBMs) in polyhedral domains, which have the special property that they almost surely do not hit the nonsmooth parts of the boundary. For general domains with nonsmooth boundaries and oblique reflection, even the formulation of the submartingale problem is somewhat subtle and a correct formulation in multidimensional nonsmooth domains had been a longstanding open problem [45] [see comment (iii) of Section 4 therein]. In Definition 2.9, we first introduce a general formulation of the submartingale problem in domains with piecewise smooth

boundaries. The equivalence result established here provides additional validation that this is a useful formulation.

Another motivation for this work arises from the fact that whereas some properties of reflected diffusions such as existence and uniqueness in law and large deviation results have been established using the SDER framework in [1, 14, 31, 32], other properties such as boundary properties and characterizations of stationary distributions have been established for reflected diffusions associated with a submartingale problem [21, 43] or (for semimartingale reflected diffusions) the constrained martingale problem [28]. The equivalence result that we establish allows one to transfer results proved in one setting to the other setting. For example, the work [21] provides a characterization of stationary distributions for reflected diffusions in piecewise smooth domains that are defined via a well-posed submartingale problem. When combined with the main result of this paper (Theorem 1), this yields a characterization of stationary distributions of a large class of well-posed SDER that arise in applications. Furthermore, our results show that to establish well-posedness of the submartingale problem one can without loss of generality assume that the drift is zero (see Remark 3.1). In addition, it is sometimes easier to establish existence of reflected diffusions using the submartingale problem formulation, but easier to establish uniqueness in law using the SDER formulation. Our subsidiary result (Theorem 3) shows that uniqueness in law of a solution to the SDER implies uniqueness of the submartingale problem. Thus, our results are potentially also useful for establishing well-posedness of the submartingale problem.

1.2. Discussion of the proof and outline of the paper. Even in cases where the submartingale problem is well formulated, establishing a correspondence between the submartingale problem and weak solutions to SDERs has been considered challenging ([11], page 149). In the case of nonsmooth domains, only very special cases seem to have been previously considered, such as, for example, normal reflection in the d -dimensional nonnegative orthant, which essentially reduces to a one-dimensional problem (see [3], Theorem V.1.1, and [5], Proposition 2.1, for a brief discussion of this case). The proof of Theorem 1 follows from two results, established in Theorem 2 and Theorem 3, respectively. With a view to extending these results to include a class of nonsemimartingale reflected diffusions in [22], these two theorems are established in slightly greater generality, where the geometric (generalized completely- \mathcal{S}) condition is allowed to fail in a certain subset of the boundary of the domain. Theorem 2 shows that any weak solution of the SDER satisfies the submartingale problem. This is an immediate consequence of Itô's formula when the weak solution is a semimartingale, but requires an additional argument when the semimartingale property fails to hold. The crux of this work lies in the proof of the converse, stated in Theorem 3, which entails the construction of a weak solution to an SDER with a specific initial condition from a solution to the corresponding submartingale problem with that same initial condition.

As elaborated below, this construction is considerably complicated by the presence of nonsmooth boundaries and the geometry of the directions of reflection. To describe the main ideas behind our proof, we first briefly recall the classical construction (due to Stroock and Varadhan) of a weak solution to a stochastic differential equation (SDE) from a solution to the martingale problem. Given smooth drift and diffusion coefficients $b : \mathbb{R}^J \mapsto \mathbb{R}^J$ and $\sigma : \mathbb{R}^J \mapsto \mathbb{R}^{J \times N}$, let $a = \sigma \sigma^T$ be the diffusion coefficient and let \mathcal{L} be the usual associated second-order differential operator:

$$(1.1) \quad \mathcal{L}f(x) \doteq \sum_{i=1}^J b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^J a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x),$$

for sufficiently smooth test functions $f : \mathbb{R}^J \mapsto \mathbb{R}$. Let $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$ denote the usual canonical filtered probability space (see Section 2.2 for a precise definition), and recall that a solution to the martingale problem associated with b and σ with initial condition $z \in \mathbb{R}^J$ is a probability measure \mathbb{P}_z on $(\mathcal{C}, \mathcal{M})$ such that for every $z \in \mathbb{R}^J$, $\mathbb{P}_z(\omega(0) = z) = 1$ and for every sufficiently smooth f , the process

$$S^f(t) \doteq f(\omega(t)) - f(\omega(0)) - \int_0^t \mathcal{L}f(\omega(s)) ds, \quad t \geq 0,$$

is a local martingale with respect to the filtration $\{\mathcal{M}_t\}$ under \mathbb{P}_z . To construct a weak solution from the solution \mathbb{P}_z , one first chooses linear test functions of the form $f(x) = x_i$ to conclude that

$$(1.2) \quad S(t) \doteq \omega(t) - \omega(0) - \int_0^t b(\omega(s)) ds, \quad t \geq 0,$$

is a local martingale. Next, using quadratic test functions of the form $f(x) = x_i x_j$, it is easy to show that $[S^{(i)}, S^{(j)}](t) = \int_0^t a_{ij}(\omega(s)) ds$, where $S^{(i)}$ is the i th coordinate process of S and $[S^{(i)}, S^{(j)}]$ represents the covariation of $S^{(i)}$ and $S^{(j)}$. Finally, under a uniform ellipticity condition on a , one can invoke the martingale representation theorem to show that there exists an N -dimensional Brownian motion B (on a possibly extended probability space) such that if S is the process whose i th coordinate is S^i , then $S(t) = \int_0^t \sigma(\omega(s)) dB(s)$. When combined, this proves that under \mathbb{P}_z , the coordinate process $Z(t, \omega) = \omega(t)$ and the Brownian motion B form a weak solution to the SDE with drift b and dispersion σ .

In contrast, in the case of the submartingale problem (see Definition 2.9), one only knows that the process S^f is a submartingale and only for test functions f that satisfy certain oblique derivative boundary conditions. In particular, this typically does not include linear or quadratic test functions. Furthermore, one has to construct not only a Brownian motion, but also a constraining process or local time that pushes in the right directions, as specified by the (possibly multi-valued) reflection vector field. In the case of a smooth domain and smooth reflection vector field, it is possible to first construct the pushing process and then show that the

original process minus the pushing process satisfies a martingale problem [39]. However, this approach appears not to be feasible in the presence of a multi-valued reflection field. Thus, our construction in the case of reflected diffusions requires a completely different approach, which we briefly outline.

For a fixed initial condition z and an associated solution to the submartingale problem \mathbb{Q}_z on $(\mathcal{C}, \mathcal{M})$, we consider the canonical process $Z(t, \omega) = \omega(t)$. In contrast to the unconstrained case, in this case the process S in (1.2) is not a martingale, and may not even be a semimartingale in general. Instead, we construct a nested sequence of domains G_m that increase to G and show that S is a martingale on random time intervals when Z lies within the domain G_m . We then use arguments similar to those used in the unconstrained case and take a suitable limit to construct a candidate driving Brownian motion W in Section 4.2. Then, in Section 4.3 we show that Z is a semimartingale upto a certain stopping time (see Proposition 4.8). This entails an application of the Doob–Meyer decomposition theorem and requires establishing the existence of certain test functions f that satisfy certain oblique derivative boundary conditions (see Lemma 4.3 and Lemma 4.4), and a certain covering argument (Lemma 4.5) to patch together local arguments. The construction of test functions is somewhat involved for piecewise smooth domains, due to both the curvature of the domain and the presence of multiple derivative conditions at the intersections of domains (see Appendix C). Next, in Section 4.4, we show that over intervals during which Z lies in the interior of the domain, the stopped semimartingale S is in fact a martingale, that is equal to the appropriate stochastic integral with respect to the constructed Brownian motion. The most challenging step is the characterization of the behavior of Z on the boundary of the domain, including, in particular identification of the local time or pushing process. This step, which is carried out in Section 4.5, is once again significantly complicated by the fact that multiple directions of reflection are allowed at nonsmooth parts on the boundary and requires several new ingredients in the proof. Specifically, we introduce certain random linear functionals and use functional analytic tools, combined with the Doob–Meyer decomposition theorem and certain exponential martingales, to show that the bounded variation term in the semimartingale decomposition admits an integral representation with respect to a certain random measure (see Sections 4.5.1 and 5). We then use properties of this integral representation along with a certain boundary property of solutions to the submartingale problem established in [21], and a dual representation for the cone of directions of constraint at a point to show that the trace of the local martingale term in the decomposition vanishes on the boundary and that the bounded variation term behaves like a local time, acting only on the boundary (see Section 4.5.2) and “pushing” in the right directions specified by the reflection vector field (see Section 4.5.3).

The outline of the rest of the paper is as follows. In Section 2, we recall some basic definitions related to reflected diffusions. The SDER and submartingale formulations and their properties are introduced in Sections 2.1 and 2.2, respectively,

and a measurability property related to the definition of a weak solution is relegated to Appendix A. Section 2.3 defines the class of domains and reflection directions that we consider. Section 3 contains the main result, Theorem 1, its two auxiliary results Theorem 2 and Theorem 3, and some discussion of the ramifications. The proof of Theorem 2 is relegated to Section 6. The proof of Theorem 3, outlined above, is presented in Section 4. It relies on a certain integral representation, a covering lemma the existence of test functions whose proofs are deferred to Section 5, and Appendices B and C, respectively. First, in the next section we collect some common notation used throughout the paper.

1.3. *Common notation.* Let \mathbb{R} denote the set of real numbers and \mathbb{R}_+ is the set of nonnegative real numbers. Given $a, b \in \mathbb{R}$, $a \wedge b$ ($a \vee b$) denote the minimum (maximum) of a and b . For each $J \in \mathbb{N}$, \mathbb{R}^J is the J -dimensional Euclidean space and $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively, denote the Euclidean norm and the inner product on \mathbb{R}^J . For each set $A \subset \mathbb{R}^J$, A° , ∂A , \bar{A} and A^c denote the interior, boundary, closure and complement of A , respectively. For each $x \in \mathbb{R}^J$ and $A \subset \mathbb{R}^J$, $\text{dist}(x, A)$ is the distance from x to A , that is, $\text{dist}(x, A) = \inf\{y \in A : |y - x|\}$. For each $A \subset \mathbb{R}^J$ and $r > 0$, $B_r(A) = \{y \in \mathbb{R}^J : \text{dist}(y, A) < r\}$, and given $\varepsilon > 0$ let $A^\varepsilon \doteq \{y \in \mathbb{R}^J : \text{dist}(y, A) < \varepsilon\}$ denote the (open) ε -fattening of A . If $A = \{x\}$, we simply denote $B_r(A)$ by $B_r(x)$. With some abuse of notation, we will use 0 to represent both zero and the origin in \mathbb{R}^J , and let $S_1(0)$ denote the unit sphere in \mathbb{R}^J . We also let \mathbb{I}_A denote the indicator function of the set A [i.e., $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise]. Given integers i, j we let δ_{ij} denote the Kronecker delta function, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$, otherwise. Given a set $A \subset \mathbb{R}^J$, let $\overline{\text{co}}[A]$ denote the closure of the convex hull of A , which is defined to be the intersection of all closed convex sets that contain A .

Given a domain E in \mathbb{R}^n , for some $n \in \mathbb{N}$, let $\mathcal{C}(E) = \mathcal{C}^0(E)$ be the space of continuous real-valued functions on E and, for any $m \in \mathbb{Z}_+ \cup \{\infty\}$, let $\mathcal{C}^m(E)$ be the subspace of functions in $\mathcal{C}(E)$ that are m times continuously differentiable on E with continuous partial derivatives of order up to and including m . When E is the closure of a domain, $\mathcal{C}^m(E)$ is to be interpreted as the collection of functions in $\bigcap_{\varepsilon > 0} \mathcal{C}^m(E^\varepsilon)$, where E^ε is an open ε -neighborhood of E , restricted to E . Also, let $\mathcal{C}_b^m(E)$ be the subspace of $\mathcal{C}^m(E)$ consisting of bounded functions whose partial derivatives of order up to and including m are also bounded, let $\mathcal{C}_c^m(E)$ be the subspace of $\mathcal{C}^m(E)$ consisting of functions that vanish outside compact sets. In addition, let $\mathcal{C}_c^m(E) \oplus \mathbb{R}$ be the direct sum of $\mathcal{C}_c^m(E)$ and the space of constant functions, that is, the space of functions that are sums of functions in $\mathcal{C}_c^m(E)$ and constants in \mathbb{R} . The support of a function f is denoted by $\text{supp}(f)$, its gradient of f is denoted by ∇f . For $m \geq 1$ and a sequence of random variables $\{X_n, n \geq 1\}$ defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, we say X_n converges to X in $\mathbb{L}^m(\mathbb{Q})$ as $n \rightarrow \infty$ if $\mathbb{E}^{\mathbb{Q}}[|X_n - X|^m] \rightarrow 0$ as $n \rightarrow \infty$. For $f \in \mathcal{C}^\infty(I)$, with $I \subset \mathbb{R}$ or $f \in \mathcal{C}^2(G)$, we let $\|f\|_\infty$ denote the supremum of f on its domain.

2. Characterizations of reflected diffusions. Throughout, let G be a non-empty connected domain in \mathbb{R}^J and let $d(\cdot)$ be a set-valued mapping defined on the closure \bar{G} of G such that $d(x) = \{0\}$ for $x \in G$, $d(x)$ is a nonempty, closed and convex cone in \mathbb{R}^J with vertex at the origin for every $x \in \partial G$, and the graph of $d(\cdot)$ is closed, that is, the set $\{(x, v) : x \in \bar{G}, v \in d(x)\}$ is a closed subset of \mathbb{R}^{2J} . Let $b : \mathbb{R}^J \mapsto \mathbb{R}^J$ and $\sigma : \mathbb{R}^J \mapsto \mathbb{R}^{J \times N}$ be measurable and locally bounded. Also, let $n(x)$ denote the set of inward normals to G at a point $x \in \partial G$, and let

$$(2.1) \quad \mathcal{V} \doteq \partial G \setminus \mathcal{U},$$

where \mathcal{U} is the subset of the boundary ∂G defined by

$$(2.2) \quad \mathcal{U} \doteq \{x \in \partial G : \exists n \in n(x) \text{ such that } \langle n, d \rangle > 0, \forall d \in d(x) \setminus \{0\}\}.$$

The set \mathcal{V} will play an important role in the analysis. Also, let \mathcal{L} be the usual associated second-order differential operator, as defined in (1.1) for functions $f \in \mathcal{C}_b^2(\bar{G})$, where $\mathcal{C}_b^2(\bar{G})$ is the space of twice continuously differentiable functions on \bar{G} that, along with their first and second partial derivatives, are bounded.

We recall the definition of weak solutions to stochastic differential equations with reflection associated with $(G, d(\cdot))$, b and σ and some of their properties in Section 2.1. We then introduce the formulation of the associated submartingale problem in Section 2.2. Lastly, in Section 2.3 we describe the specific class of piecewise continuous domains $(G, d(\cdot))$ of interest and then state a useful boundary property of reflected diffusions in this class of domains that was established in [21].

2.1. Stochastic differential equations with reflection. The Skorokhod Problem (SP), which was introduced in one dimension by [37] and subsequently extended to higher dimensions by numerous authors [6, 9, 13, 31], and the extended Skorokhod problem (ESP) introduced in [32], are convenient tools for the pathwise construction of reflected diffusions. Roughly speaking, given a continuous path ψ , the ESP associated with $(G, d(\cdot))$ produces a constrained version ϕ of ψ that is restricted to live within \bar{G} by adding to it a “constraining term” η whose increments over any interval lie in the closure of the convex hull of the union of the allowable directions $d(x)$ at the points x visited by ϕ during this interval. Let $\mathcal{C} = \mathcal{C}([0, \infty) : \mathbb{R}^J)$ denote the space of continuous functions from $[0, \infty)$ to \mathbb{R}^J , equipped with the topology of uniform convergence on compact sets. We now rigorously define the ESP.

DEFINITION 2.1 (Extended Skorokhod problem). Suppose $(G, d(\cdot))$ and $\psi \in \mathcal{C}$ with $\psi(0) \in \bar{G}$ are given. Then the pair $(\phi, \eta) \in \mathcal{C} \times \mathcal{C}$ is said to solve the extended Skorokhod Problem (ESP) for ψ if $\phi(0) = \psi(0)$, and if for all $t \in [0, \infty)$, the following properties hold:

1. $\phi(t) = \psi(t) + \eta(t)$;

- 2. $\phi(t) \in \bar{G}$;
- 3. For every $s \in [0, t)$,

$$(2.3) \quad \eta(t) - \eta(s) \in \overline{\text{co}} \left[\bigcup_{u \in [s, t]} d(\phi(u)) \right].$$

If (ϕ, η) is the unique solution to the ESP for ψ , then we write $\phi = \Gamma(\psi)$, and refer to Γ as the extended Skorokhod map (ESM).

The formulation of the ESP in Definition 2.1 appears slightly different from the original one given in [32] since the ESP in [32] was formulated more generally for càdlàg paths. However, as we show below, they coincide for the case of continuous paths, which is all that is required for this work. Indeed, for continuous paths, property 4 of Definition 1.2 of [32] holds automatically, and the following lemma shows that property 3 of Definition 1.2 of [32] is equivalent to property 3 in Definition 2.1.

LEMMA 2.2. *The pair $(\phi, \eta) \in \mathcal{C} \times \mathcal{C}$ satisfies property (2.3) in Definition 2.1 for every $t \geq 0$ if and only if for every $s \in [0, t)$,*

$$(2.4) \quad \eta(t) - \eta(s) \in \overline{\text{co}} \left[\bigcup_{u \in (s, t]} d(\phi(u)) \right].$$

PROOF. Fix $t \in [0, \infty)$. For $s \in [0, t)$, property (2.4) trivially implies property (2.3). To prove the converse, suppose that (2.3) holds for all $s \in [0, t)$. Let $\{s_n, n \in \mathbb{N}\}$ be a sequence of real numbers such that $s < s_n < t$ for each $n \in \mathbb{N}$ and $s_n \rightarrow s$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, by (2.3) we have

$$\eta(t) - \eta(s_n) \in \overline{\text{co}} \left[\bigcup_{u \in [s_n, t]} d(\phi(u)) \right] \subseteq \overline{\text{co}} \left[\bigcup_{u \in (s, t]} d(\phi(u)) \right].$$

Together with the continuity of η and the closedness of the set $\overline{\text{co}}[\bigcup_{u \in (s, t]} d(\phi(u))]$, this implies that (ϕ, η) satisfies property (2.4) holds. \square

REMARK 2.3. Given $(G, d(\cdot))$ and ψ as in Definition 2.1, a pair $(\phi, \eta) \in \mathcal{C} \times \mathcal{C}$ is said to solve the Skorokhod Problem (SP) for ψ if it satisfies properties 1 and 2 of Definition 2.1 and, in addition, η has finite variation on bounded intervals and, in addition, there exists a Borel measurable function $\gamma : [0, \infty) \mapsto S_1(0)$ such that for every $t \in [0, \infty)$,

$$\eta(t) = \int_{[0, t]} \gamma(s) \mathbb{I}_{\{\phi(s) \in \partial G\}} d|\eta|(s),$$

where $\gamma(s) \in d(\phi(s))$ for $d|\eta|$ almost every $s \in [0, \infty)$, and $|\eta|(t)$ represents the total variation of η on the interval $[0, t]$ (see [13]). The ESP is a generalization of

the SP that does not a priori require the constraining term η to have finite variation, and hence allows for the construction of reflected diffusions that are not necessarily semimartingales (see Lemma 2.7 for an elaboration of this point). However, it was shown in Theorem 1.3 of [32] that if the solution (ϕ, η) to the ESP for some ψ is such that η has finite variation on every interval $[0, t]$, then (ϕ, η) is a solution to the SP associated with ψ . Sufficient conditions for the existence of a unique solution to the SP or ESP can be found, for example, in [6, 7, 13, 16, 17, 24, 32].

The ESM can be used to define solutions to stochastic differential equations with reflection (SDERs) associated with a given pair $(G, d(\cdot))$, and drift and dispersion coefficients $b : \bar{G} \mapsto \mathbb{R}^J$ and $\sigma : \bar{G} \mapsto \mathbb{R}^{J \times N}$.

DEFINITION 2.4 (Weak solution). Given $z \in \bar{G}$, a weak solution to the SDER with initial condition z associated with $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ is a triplet $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, \mathbb{P}_z , (Z, W) , where $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ is a filtered space that supports a probability measure \mathbb{P}_z , Z is a continuous, $\{\mathcal{F}_t\}$ -adapted J -dimensional process and W is a continuous, N -dimensional $\{\mathcal{F}_t\}$ -martingale with the following properties:

1. Under \mathbb{P}_z , $\{W_t, \mathcal{F}_t, t \geq 0\}$ is an N -dimensional standard Brownian motion;
2. $\mathbb{P}_z(\int_0^t |b(Z(s))| ds + \int_0^t |\sigma(Z(s))|^2 ds < \infty) = 1, t \in [0, \infty)$;
3. there exists a continuous $\{\mathcal{F}_t\}$ -adapted J -dimensional process Y such that \mathbb{P}_z -almost surely, (Z, Y) solves the ESP associated with $(G, d(\cdot))$ for X , where

$$(2.5) \quad X(t) \doteq z + \int_0^t b(Z(s)) ds + \int_0^t \sigma(Z(s)) dW(s), \quad t \in [0, \infty);$$

4. \mathbb{P}_z -almost surely, the set $\{t : Z(t) \in \partial G\}$ has zero Lebesgue measure. In other words, \mathbb{P}_z -almost surely,

$$(2.6) \quad \int_0^\infty \mathbb{1}_{\partial G}(Z(s)) ds = 0.$$

In addition, we say that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, \mathbb{P}_z , $(Z(\cdot \wedge \tau), W(\cdot \wedge \tau))$ is a weak solution to the τ -stopped SDER associated with $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ and initial condition z if τ is an $\{\mathcal{F}_t\}$ -stopping time, $\{W(t \wedge \tau), \mathcal{F}_t, t \geq 0\}$ is an adapted N -dimensional standard Brownian motion stopped at τ , properties 2–4 hold with (Z, W, X) replaced by $(Z(\cdot \wedge \tau), W(\cdot \wedge \tau), X(\cdot \wedge \tau))$ and (2.6) replaced by $\int_0^\tau \mathbb{1}_{\partial G}(Z(s \wedge \tau)) ds = 0$.

Note that property 3 of Definition 2.4 and the definition of the ESP imply that under \mathbb{P}_z , Z satisfies

$$(2.7) \quad \begin{aligned} Z(t) &= X(t) + Y(t) \\ &= z + \int_0^t b(Z(s)) ds + \int_0^t \sigma(Z(s)) dW(s) + Y(t), \quad t \in [0, \infty). \end{aligned}$$

Also, property 4 can be shown to follow from properties 1–3 in many cases. In particular, it follows from the proof of Theorem 2 and Proposition 2.12 below that property 4 automatically holds when $(G, d(\cdot))$ is piecewise \mathcal{C}^2 with continuous reflection in the sense of Definition 2.11, and $\mathcal{V} = \emptyset$.

The definition of uniqueness in law for the SDER is analogous to the case of an SDE.

DEFINITION 2.5. Uniqueness in law is said to hold for the SDER associated with $(G, d(\cdot)), b(\cdot)$ and $\sigma(\cdot)$ and initial condition z if given any two weak solutions $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}), \mathbb{P}_z, (Z, W)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}), \tilde{\mathbb{P}}_z, (\tilde{Z}, \tilde{W})$, of the SDER with initial condition z , the law of Z under \mathbb{P}_z is the same as the law of \tilde{Z} under $\tilde{\mathbb{P}}_z$. Moreover, uniqueness in law is said to hold for the SDER associated with $(G, d(\cdot)), b(\cdot)$ and $\sigma(\cdot)$ if uniqueness in law holds for each initial condition z .

We now define well-posedness of the SDER.

DEFINITION 2.6 (Well-posedness of the SDER). The SDER associated with $(G, d(\cdot)),$ drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ is said to be well posed if for every $z \in \bar{G}$, there exists a weak solution to the SDER with initial condition z and uniqueness in law holds for the SDER.

Well-posedness has been established for SDER associated with many classes of polyhedral and piecewise smooth domains. In general, solutions to the SDER defined via the ESP need not be semimartingales when $\mathcal{V} \neq \emptyset$ (see [7, 18, 20, 32–34] for examples where $\mathcal{V} \neq \emptyset$ and such nonsemimartingales arise). However, we now make some observations on the link between weak solutions and the semimartingale property, which we will use in our subsequent analysis.

LEMMA 2.7. Given $z \in \bar{G}$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}), \mathbb{P}_z, (Z, W)$, be a weak solution of the SDER with initial condition z , and let X, Y be as in Definition 2.4. Let θ_1, θ_2 be two $\{\mathcal{F}_t\}$ -stopping times such that θ_1 is \mathbb{P}_z -almost finite and $\theta_2 \geq \theta_1$, and consider the shifted and stopped processes

$$(2.8) \quad \tilde{Y}(\omega, u) \doteq Y(\omega, (\theta_1(\omega) + u) \wedge \theta_2(\omega)) - Y(\omega, \theta_1(\omega)),$$

$$\omega \in \Omega, u \in [0, \infty)$$

and

$$(2.9) \quad \tilde{Z}(\omega, u) \doteq Z(\omega, (\theta_1(\omega) + u) \wedge \theta_2(\omega)), \quad \omega \in \Omega, u \in [0, \infty).$$

If $\tilde{Z}(\omega, t) \notin \mathcal{V}$ for all $t \in [0, \theta_2(\omega) - \theta_1(\omega)]$ (which should be interpreted as $t \in [0, \infty)$ when $\theta_2(\omega) = \infty$), then the total variation of \tilde{Y} is \mathbb{P}_z -almost finite on

every bounded interval, and there exists a measurable function $\tilde{\gamma} : (\Omega \times \mathbb{R}_+, \mathcal{F} \times \mathcal{B}(\mathbb{R}_+)) \mapsto (\mathbb{R}^J, \mathcal{B}(\mathbb{R}^J))$ such that for each $\omega \in \Omega$, and $0 \leq s \leq t < \infty$,

$$(2.10) \quad \tilde{Y}(\omega, t) - \tilde{Y}(\omega, s) = \int_{[s,t]} \tilde{\gamma}(\omega, u) d|\tilde{Y}|(\omega, u),$$

and for \mathbb{P}_z -almost every ω , $\tilde{\gamma}(\omega, u) \in d(\tilde{Z}(\omega, u))$ for $d|\tilde{Y}|(\omega)$ -almost every $u \in [0, \infty)$.

As observed in Remark 2.3, a deterministic analog of this property was established in [32]. The main new content of this lemma is the claim that the process γ can be chosen to be jointly measurable in ω and t . The proof of the lemma is relegated to Appendix A.

We close this section with an observation that will allow us to assume without loss of generality that the Brownian motion driving the weak solution is J -dimensional. Recall that the diffusion coefficient $a(\cdot) = \sigma(\cdot)\sigma^T(\cdot)$ is uniformly elliptic if there exists $\bar{a} > 0$ such that

$$(2.11) \quad v^T a(x)v \geq \bar{a}|v|^2 \quad \text{for all } v \in \mathbb{R}^J, x \in \bar{G}.$$

REMARK 2.8. Under the uniform ellipticity condition (2.11), standard arguments can be used to show that existence of a weak solution with an $\mathbb{R}^{J \times N}$ -valued dispersion coefficient $\sigma(\cdot)$ is equivalent to existence of a weak solution with the $\mathbb{R}^{J \times J}$ dispersion coefficient $a^{1/2}(\cdot)$ (for one direction, see Proposition 4.6 in Chapter 5 of [23]).

2.2. *The submartingale problem.* Let $(G, d(\cdot))$, $b(\cdot)$, $\sigma(\cdot)$, \mathcal{V} and \mathcal{L} be as defined at the beginning of Section 2. We first introduce a class of test functions that arises in the formulation of the submartingale problem. Recall that $\mathcal{C}_c^2(\bar{G}) \oplus \mathbb{R}$ is the space of functions that are sums of functions in $\mathcal{C}_c^2(\bar{G})$ and constants in \mathbb{R} . Define

$$\mathcal{H} \doteq \left\{ f \in \mathcal{C}_c^2(\bar{G}) \oplus \mathbb{R} : \begin{array}{l} f \text{ is constant in a neighborhood of } \mathcal{V}, \\ \langle d, \nabla f(y) \rangle \geq 0 \text{ for } d \in d(y) \text{ and } y \in \partial G \end{array} \right\},$$

where for each function f defined on \mathbb{R}^J , we say f is constant in a neighborhood of \mathcal{V} if for each $x \in \mathcal{V}$, f is constant in some open neighborhood of x . When $\mathcal{V} = \emptyset$, the condition that f be constant in a neighborhood of \mathcal{V} is understood to be void.

We now define the submartingale problem associated with the data $(G, d(\cdot))$, \mathcal{V} , $b(\cdot)$ and $\sigma(\cdot)$. Recall that $\mathcal{C} = \mathcal{C}([0, \infty) : \mathbb{R}^J)$ denote the space of continuous functions from $[0, \infty)$ to \mathbb{R}^J , equipped with the topology of uniform convergence on compact sets. Let \mathcal{M} be the associated Borel σ -algebra, which is generated by sets of the form $\{\omega \in \mathcal{C} : \omega(t) \in A\}$ for $t \in [0, \infty)$ and $A \in \mathcal{B}(\mathbb{R}^J)$. We equip the measurable space $(\mathcal{C}, \mathcal{M})$ with the filtration $\{\mathcal{M}_t\}$, where for $t \in [0, \infty)$, \mathcal{M}_t is the smallest σ -algebra with respect to which the map $\omega \in \mathcal{C} \mapsto \omega(s) \in \mathbb{R}^J$ is measurable for every $s \in [0, t]$.

DEFINITION 2.9 (Submartingale problem). Given $z \in \bar{G}$, a probability measure \mathbb{Q}_z on the measurable space $(\mathcal{C}, \mathcal{M})$ is a solution to the submartingale problem starting from z associated with $(G, d(\cdot))$, \mathcal{V} , drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ if the following four properties hold:

1. $\mathbb{Q}_z(\omega(0) = z) = 1$;
2. $\mathbb{Q}_z(\omega(t) \in \bar{G} \text{ for every } t \in [0, \infty)) = 1$;
3. For every $f \in \mathcal{H}$, the process

$$f(\omega(t)) - f(\omega(0)) - \int_0^t \mathcal{L}f(\omega(u)) du, \quad t \geq 0,$$

is a \mathbb{Q}_z -submartingale on $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$;

4. \mathbb{Q}_z -almost surely, $\int_0^\infty \mathbb{I}_{\mathcal{V}}(\omega(u)) du = 0$.

A family $\{\mathbb{Q}_z, z \in \bar{G}\}$ of probability measures on $(\mathcal{C}, \mathcal{M})$ is a solution to the submartingale problem if for each $z \in \bar{G}$, \mathbb{Q}_z is a solution to the submartingale problem starting from z .

DEFINITION 2.10 (Well-posedness of the submartingale problem). The submartingale problem associated with $(G, d(\cdot))$, \mathcal{V} , drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ is said to be well posed if there exists exactly one solution $\{\mathbb{Q}_z, z \in \bar{G}\}$ to the submartingale problem.

Definition 2.9 differs slightly from past formulations of the submartingale problem in domains with nonsmooth boundaries. As mentioned in the Introduction, essentially all these works [12, 29, 30, 42, 44] consider domains that have only a single point of nonsmoothness on the boundary and the formulation they use is Definition 2.9, but with \mathcal{V} replaced by the set of nonsmooth points on the boundary. In either formulation, since the test functions in property 3 are required to be constant in a neighborhood of some subset of the boundary, property 3 provides no information on the behavior of the processes in a neighborhood of \mathcal{V} . Thus, an additional property (property 4) needs to be imposed to ensure that the reflected diffusion spends zero Lebesgue time on the boundary. For the class of domains, we consider in Section 2.3, it is shown in Proposition 2.12 that any solution to the submartingale problem formulated as in Definition 2.9 spends zero Lebesgue time on the boundary. On the other hand, since \mathcal{V} is typically a subset of the nonsmooth part of the domain, property 3 in our formulation has to be satisfied by a larger class of test functions, and hence, it is a priori easier to establish uniqueness and harder to establish existence of solutions. However, in the cases studied previously, it seems not much harder to establish existence of solutions for our formulation of the submartingale problem. For example, for the two-dimensional wedge considered in [42], the two formulations coincide when $\mathcal{V} = \{0\}$, which is precisely the case when the parameter α in [42] satisfies $\alpha \geq 1$. When $\alpha < 1$, $\mathcal{V} = \emptyset$, the formulations are different. Hence, existence of a solution to the submartingale problem in

Definition 2.9 does not follow directly from the results in [42], but it can nevertheless be deduced using similar arguments or, alternatively, by applying Theorem 2 in conjunction with the results of [41]. We believe our formulation is more convenient for obtaining results in domains with piecewise smooth boundaries (that potentially have more than one nonsmooth point). In particular, this formulation was used to obtain a characterization of stationary distributions of a large class of reflected diffusions in domains with piecewise smooth boundaries in [21]. As discussed in [22], the correct formulation of the submartingale problem is even more subtle for multidimensional domains whose \mathcal{V} sets have more complex geometries.

2.3. *A class of domains with piecewise smooth boundary.* We now introduce the general class of domains and reflection directions $(G, d(\cdot))$ covered by our results.

DEFINITION 2.11 (Piecewise \mathcal{C}^2 with continuous reflection). The pair $(G, d(\cdot))$ is said to be piecewise \mathcal{C}^2 with continuous reflection if it satisfies the following properties:

1. G is a nonempty domain in \mathbb{R}^J with representation

$$G = \bigcap_{i \in \mathcal{I}} G^i,$$

where \mathcal{I} is a finite index set and for each $i \in \mathcal{I}$, G^i is a nonempty domain with \mathcal{C}^2 boundary in the sense that for each $x \in \partial G$, there exist a neighborhood \mathcal{N}_x of x , and functions $\varphi_x^i \in \mathcal{C}^2(\mathbb{R}^J)$, $i \in \mathcal{I}(x) \doteq \{i \in \mathcal{I} : x \in \partial G^i\}$, such that

$$\mathcal{N}_x \cap G^i = \{z \in \mathcal{N}_x : \varphi_x^i(z) > 0\}, \quad \mathcal{N}_x \cap \partial G^i = \{z \in \mathcal{N}_x : \varphi_x^i(z) = 0\},$$

and $\nabla \varphi_x^i \neq 0$ on \mathcal{N}_x . For each $x \in \partial G^i$ and $i \in \mathcal{I}(x)$, let

$$n^i(x) \doteq \frac{\nabla \varphi_x^i(x)}{|\nabla \varphi_x^i(x)|}$$

denote the unit inward normal vector to ∂G^i at x .

2. The (set-valued) direction “vector field” $d(\cdot) : \bar{G} \mapsto \mathbb{R}^J$ is given by $d(x) = \{0\}$ if $x \in G$ and

$$d(x) = \left\{ \sum_{i \in \mathcal{I}(x)} s_i d^i(x) : s_i \geq 0, i \in \mathcal{I}(x) \right\}, \quad x \in \partial G,$$

where, for each $i \in \mathcal{I}$, $d^i(\cdot)$ is a continuous unit vector field defined on ∂G^i that satisfies $\|d^i(x)\| = 1$ and

$$\langle n^i(x), d^i(x) \rangle > 0 \quad \text{for each } x \in \partial G_i.$$

If $d^i(\cdot)$ is constant for every $i \in \mathcal{I}$, then the pair $(G, d(\cdot))$ is said to be piecewise \mathcal{C}^2 with constant reflection. If, in addition, $n^i(\cdot)$ is constant for every $i \in \mathcal{I}$, then the pair $(G, d(\cdot))$ is said to be polyhedral with piecewise constant reflection.

Note that, with the definition given above, the set of inward normal vectors to G takes the form

$$n(x) = \left\{ \sum_{i \in \mathcal{I}(x)} s_i n^i(x) : s_i \geq 0, i \in \mathcal{I}(x) \right\}, \quad x \in \partial G.$$

Since $(G, d(\cdot))$ is piecewise \mathcal{C}^2 with continuous reflection, it can readily be verified that \mathcal{U} is relatively open to ∂G , and hence \mathcal{V} is a closed set. We now state a boundary property, which extends results established in [35] for RBMs in polyhedral domains. A version of this boundary property was established in Proposition 6.1 of [21]. (Note that in [21] the set \mathcal{V} in the submartingale problem was allowed to be any arbitrary subset of ∂G and Proposition 6.1 of [21] was established under the condition that $\partial G \setminus \mathcal{U} \subseteq \mathcal{V}$, which is in particular satisfied when we set $\mathcal{V} = \partial G \setminus \mathcal{U}$, as specified here in (2.1).) The statement of Proposition 6.1 of [21] assumes well-posedness of the submartingale problem associated with $(G, d(\cdot))$, \mathcal{V} , $b(\cdot)$ and $\sigma(\cdot)$ for every initial condition, whereas in the version below we only require existence of a solution to the submartingale problem for a fixed z . The version we need can be obtained by slightly modifying the proof given in [21] to use stopping times and a covering argument (in a manner analogous to the proof of Theorem 3 in Section 4) instead of regular conditional probability distributions. The complete proof is relegated to Appendix D.

PROPOSITION 2.12 (Boundary property). *Suppose that $(G, d(\cdot))$ is a piecewise \mathcal{C}^2 domain with continuous reflection, $b(\cdot)$, $\sigma(\cdot)$ are measurable and locally bounded, $a = \sigma \sigma^T$ is uniformly elliptic. If, for some $z \in \bar{G}$, \mathbb{Q}_z is a solution to the submartingale problem associated with $(G, d(\cdot))$, \mathcal{V} , $b(\cdot)$ and $\sigma(\cdot)$ with initial condition z , then we have*

$$(2.12) \quad \int_0^\infty \mathbb{I}_{\partial G}(w(u)) du = 0, \quad \mathbb{Q}_z\text{-almost surely.}$$

3. Main results. We now state our main results. We will assume throughout, without always stating this explicitly, that the drift and dispersion coefficients are measurable and locally bounded, and that the diffusion coefficient is continuous and uniformly elliptic, that is, (2.11) holds for some $\bar{a} > 0$.

THEOREM 1. *Suppose $(G, d(\cdot))$ is piecewise \mathcal{C}^2 with continuous reflection and $\mathcal{V} = \emptyset$. Then the SDER associated with $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ is well posed if and only if the corresponding submartingale problem is well posed.*

Suppose $G = \mathbb{R}_+^J$ and $d(x)$ is equal to the vector d^j when x is in the relative interior of the face $\bar{G} \cap \{x : x_j = 0\}$. Then the condition $\mathcal{V} = \emptyset$, with \mathcal{V} defined by (2.1), is equivalent to the condition that the so-called reflection matrix $[d_j^i]_{i,j \in \{1, \dots, J\}}$ is completely- \mathcal{S} (see [6]). Given constant drift and dispersion coefficients b and σ , for different classes of polyhedral domains with piecewise constant reflection $(G, d(\cdot))$, it was shown in [10, 35, 41] that the condition $\mathcal{U} = \partial G$ is sufficient for well-posedness of the associated SDER, and is also necessary for existence of a weak solution that is a semimartingale. For more general G and $d(\cdot)$, the condition $\mathcal{V} = \emptyset$ imposed in Theorem 1 can be viewed as a generalized completely- \mathcal{S} condition, and it follows from Lemma 2.7 that in this case the reflected diffusion is a semimartingale.

Theorem 1 is a direct consequence of Theorems 2 and 3, which prove slightly more general results that do not assume that $\mathcal{V} = \emptyset$.

THEOREM 2. *Given $(G, d(\cdot))$, \mathcal{V} , $b(\cdot)$, $\sigma(\cdot)$, suppose that for some $z \in \bar{G}$, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, \mathbb{P}_z , (Z, W) is a weak solution to the associated SDER with initial condition z and let $\mathbb{Q}_z = \mathbb{P}_z \circ Z^{-1}$ denote the law of Z on $(\mathcal{C}, \mathcal{M})$ under \mathbb{P}_z . If \mathcal{V} is the union of finitely many closed connected sets, then \mathbb{Q}_z is a solution to the corresponding submartingale problem starting from z . Consequently, if the submartingale problem has at most one solution with initial condition z , then uniqueness in law holds for the associated SDER with initial condition z .*

The proof of Theorem 2 is relegated to Section 6. It is essentially a consequence of Itô’s formula; however, since we also allow weak solutions that are not necessarily semimartingales, the proof requires some additional approximation arguments, which use the results on the ESP from [32] summarized in Lemma 2.7. The more substantial result is its (partial) converse, Theorem 3 below.

THEOREM 3. *Suppose $(G, d(\cdot))$ is piecewise \mathcal{C}^2 with continuous reflection, \mathcal{V} is the union of finitely many closed connected sets, and, for some $z \in \bar{G}$, the submartingale problem associated with $(G, d(\cdot))$, \mathcal{V} , $b(\cdot)$ and $\sigma(\cdot)$ has a solution \mathbb{Q}_z starting from z . Let*

$$(3.1) \quad Z(\omega, t) = \omega(t), \quad t \geq 0, \omega \in \mathcal{C},$$

and consider the $\{\mathcal{M}_t\}$ -stopping time given by

$$(3.2) \quad \tau_{\mathcal{V}} = \inf\{t \geq 0 : \omega(t) \in \mathcal{V}\}.$$

Then there exists a process W defined on $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$ such that $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$, \mathbb{Q}_z , $(Z(\cdot \wedge \tau_{\mathcal{V}}), W(\cdot \wedge \tau_{\mathcal{V}}))$ is a weak solution to the associated $\tau_{\mathcal{V}}$ -stopped SDER with initial condition z . Consequently, if $\mathcal{V} = \emptyset$ and there is uniqueness in law for the SDER with initial condition z , then there is a unique solution \mathbb{Q}_z to the submartingale problem with initial condition z .

Section 4 is devoted to the proof of Theorem 3. A broad outline of the proof, broken down into several steps, is first provided in Section 4.1, and details of the various steps are presented in Sections 4.2–4.5.

Theorem 1 allows us to transfer results that have been established for solutions to well-posed submartingale problems to reflected diffusions characterized as solutions to well-posed SDER. We end this section by discussing some additional consequences of Theorem 1.

REMARK 3.1. If $b(\cdot)$ and $\sigma(\cdot)$ satisfy suitable conditions, then Girsanov's theorem can be used to show that well-posedness of the SDER associated with $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ is equivalent to well-posedness of the SDER associated with $(G, d(\cdot))$, $b \equiv 0$ and $\sigma(\cdot)$. In other words, under suitable conditions, to show well-posedness of an SDER one can assume without loss of generality that $b \equiv 0$. Due to Theorem 1, under the same conditions on $b(\cdot)$ and $\sigma(\cdot)$, when establishing well-posedness of a submartingale problem, one can also without loss of generality assume $b \equiv 0$. This can be a very convenient simplification. While it is natural to expect such an equivalence, in the generality we are considering, it does not seem to be straightforward to establish this result directly for the submartingale problem without invoking Theorem 1 and the corresponding result for weak solutions to SDER. For example, for the case of skew-symmetric diffusions considered in [44], this was established by invoking the corresponding result for smooth domains and then using an approximation argument and the fact that the RBMs almost surely do not hit the nonsmooth parts of the boundary.

REMARK 3.2. As mentioned in the [Introduction](#), a third approach to the construction of reflected diffusions is the controlled or constrained martingale problem (CMP) of Kurtz [25, 26, 28]. In contrast to the submartingale formulation, the formulation of the CMP given in [26] can only be used when $\mathcal{V} = \emptyset$ and the reflected diffusions are semimartingales. When $\mathcal{V} = \emptyset$, it is trivial to see (by a simple application of Itô's formula), that any weak solution to the SDER solves the CMP (see, e.g., [28], Example 1.4), and that any solution of the CMP is also a solution to the submartingale problem (see [26], Section 3). However, the converse, namely existence of a weak solution to the SDER given a solution to the CMP, was not known. Under a certain assumption on the existence of a test function, it was shown in [26], Theorem 3.1, that a solution to the submartingale problem solves the CMP. Our results in particular verify the existence of this test function for the class of data $(G, d(\cdot))$ considered here when $\mathcal{V} = \emptyset$ and, therefore establish, in this setting, the equivalence of well-posedness of the submartingale problem and well-posedness of the CMP. Together with our main result, Theorem 1, this also shows, in the case $\mathcal{V} = \emptyset$, the equivalence of all three formulations for the class of data considered here.

4. Proof of Theorem 3. The broad outline of the proof of Theorem 3 is given in Section 4.1, and the details are provided in Sections 4.2–4.5. For the rest of the paper, we consider $(G, d(\cdot))$ that is piecewise C^2 with continuous reflection and \mathcal{V} is the union of finitely many closed connected sets. Also, in light of Remark 2.8, we can (and will) assume that $\sigma = a^{1/2}$, where $a = \sigma \sigma^T$.

4.1. *Common notation and broad outline of the proof.* It is clear that the second assertion of Theorem 3 follows immediately from the first. Therefore, we focus on the proof of the first assertion. When $z \in \mathcal{V}$, the conclusion of the first assertion of Theorem 3 holds trivially. Thus, we fix $z \in \bar{G} \setminus \mathcal{V}$, and let \mathbb{Q}_z be a solution to the associated submartingale problem associated with the data $(G, d(\cdot))$, \mathcal{V} , b and σ , and let Z be the canonical process on $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$, defined by (3.1). Throughout this proof, unless mentioned otherwise, all martingales, submartingales, semimartingales and stopping times will be with respect to the probability measure \mathbb{Q}_z and the filtration $\{\mathcal{M}_t\}$, and this will typically not be stated explicitly, except on occasion for emphasis. For conciseness, we also use the following notation. Let $\{S^f(t), t \geq 0\}$ be the process given by

$$(4.1) \quad S^f(t) \doteq f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{L}f(Z(u)) du, \quad t \geq 0,$$

for functions f for which the process is well defined. In particular, this is well defined for all $f \in C_c^2(\bar{G})$. Also, let χ denote the identity function on \mathbb{R}^J : $\chi(x) = x$, and let $\chi_i(x) = x_i$. Let S denote the process whose i th component is S^{χ_i} , so that

$$(4.2) \quad S(t) \doteq Z(t) - Z(0) - \int_0^t b(Z(u)) du, \quad t \geq 0.$$

Note that the measurability and local boundedness of b ensures that the right-hand side of (4.2) is well defined.

The proof of the first assertion of Theorem 3 consists of three main steps. First, in Section 4.2, for each $z \in \bar{G}$, we construct a continuous adapted stochastic process W on the canonical filtered probability space $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$, and show that under \mathbb{Q}_z , $\{W(t), \mathcal{M}_t, t \geq 0\}$ is a J -dimensional standard Brownian motion. Next, given the $\{\mathcal{M}_t\}$ -adapted process Z as in (3.1), for any $K < \infty$, we let

$$(4.3) \quad G^K \doteq \{x \in \bar{G} : \text{dist}(x, \mathcal{V}) > 1/K \text{ and } |x| < K\},$$

and define the stopping time

$$(4.4) \quad \theta^K \doteq \inf\{t \geq 0 : Z(t) \notin G^K\}.$$

Since $\theta^K \rightarrow \tau_{\mathcal{V}}$ as $K \rightarrow \infty$, to prove the first assertion of Theorem 3, it clearly suffices to show that for every $0 < K < \infty$, $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$, \mathbb{Q}_z , $(Z(\cdot \wedge \theta^K), W(\cdot \wedge \theta^K))$ is a weak solution to the θ^K -stopped SDER with initial condition z .

For this, we define X in terms of Z and W via (2.5), and let

$$(4.5) \quad Y(t) \doteq Z(t) - X(t), \quad t \geq 0.$$

Then X represents the candidate unconstrained process and Y the corresponding candidate “pushing” or local time process in the definition of a weak solution. Now, note that from the definitions of S and X in (4.2) and (2.5), respectively, we have

$$(4.6) \quad S(t) = Y(t) + \int_0^t a^{1/2}(Z(u)) dW(u), \quad t \geq 0.$$

In the second step of the proof (see Proposition 4.8 of Section 4.3), we show that $S(\cdot \wedge \theta^K)$ is a continuous semimartingale, and represent the local martingale component by $M(\cdot \wedge \theta^K)$ and bounded variation component by $A(\cdot \wedge \theta^K)$.

Now, observe that $W(\cdot \wedge \theta^K)$ satisfies property 1 of the weak solution to the θ^K -stopped SDER in Definition 2.4, property 2 follows from the continuity of $Z(\cdot \wedge \theta^K)$ and the local boundedness of b and σ , and property 4 is a consequence of the boundary property established in Proposition 2.12. Thus, to show that $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\}, \mathbb{Q}_z, (Z(\cdot \wedge \theta^K), W(\cdot \wedge \theta^K)))$ is a weak solution to the θ^K -stopped SDER with initial condition z , it only remains to verify the third property, namely, to show that almost surely, $(Z(\cdot \wedge \theta^K), Y(\cdot \wedge \theta^K))$ solves the ESP for $X(\cdot \wedge \theta^K)$. However, property 1 of the ESP holds trivially by the definition of Y in (4.5), and property 2 of the ESP is a direct consequence of property 2 of the submartingale problem and the definition of Z in (2.7). Thus, the proof of Theorem 3 is reduced to verifying that $Y(\cdot \wedge \theta^K)$ and $Z(\cdot \wedge \theta^K)$ satisfy the following “reflection property” embodied in property 3 of the ESP: almost surely,

$$(4.7) \quad Y(t \wedge \theta^K) - Y(s \wedge \theta^K) \in \overline{\text{co}} \left[\bigcup_{u \in [s \wedge \theta^K, t \wedge \theta^K]} d(Z(u)) \right] \quad \text{for every } 0 \leq s < t < \infty.$$

In turn, in view of the semimartingale decomposition for S obtained in Section 4.3, this is equivalent to showing that $M(\cdot \wedge \theta^K)$ coincides with the stochastic integral term on the right-hand side of (4.6) and that the bounded variation term $A(\cdot \wedge \theta^K) = Y(\cdot \wedge \theta^K)$ satisfies the reflection property specified in (4.7). The third step of the proof establishes this latter property. On the intervals when Z is in the interior of the domain, this property is established in Section 4.4. The proof of (4.7) for intervals in which Z also hits the boundary is given in Section 4.5. This proof is rather involved, and requires a careful analysis of the behavior of Z at the boundary of the domain, which relies on a certain integral representation that is proved in Section 5.

4.2. *Construction of a Brownian motion.* In Lemma 4.1, we show that S is a martingale on certain random time intervals during which the process lies strictly inside the domain G . This allows us to construct, a sequence $\{W^m\}_{m \in \mathbb{N}}$ of martingales, which is shown in Lemma 4.2 to converge (along a subsequence) to a standard Brownian motion.

LEMMA 4.1. *Let $\mathcal{O}_i, i = 1, 2$, be connected bounded open subsets of G such that $\bar{\mathcal{O}}_1 \subset \mathcal{O}_2$ and $\bar{\mathcal{O}}_2 \subset G$. Given any $\{\mathcal{M}_t\}$ -stopping time ϱ , define the two stopping times*

$$(4.8) \quad \zeta \doteq \inf\{t > \varrho : Z(t) \in \bar{\mathcal{O}}_1\},$$

$$(4.9) \quad \tau \doteq \inf\{t > \zeta : Z(t) \notin \mathcal{O}_2\}.$$

Then $\{S(t \wedge \tau) - S(t \wedge \zeta), \mathcal{M}_t, t \geq 0\}$ is a continuous martingale and for $t \geq 0, i, j = 1, \dots, J$,

$$(4.10) \quad [S_i, S_j](t \wedge \tau) - [S_i, S_j](t \wedge \zeta) = \int_{t \wedge \zeta}^{t \wedge \tau} a_{ij}(Z(u)) du.$$

PROOF. Since $\bar{\mathcal{O}}_2 \cap \partial G = \emptyset$, for each $i = 1, \dots, J$, there exists $f^{(i)} \in \mathcal{C}_c^2(G)$ such that $f^{(i)}(x) = x_i$ for $x \in \bar{\mathcal{O}}_2$ and $f^{(i)}(x) = 0$ for each x in a neighborhood of ∂G . Then, for every $i, j = 1, \dots, J$, the functions $f^{(i)}, -f^{(i)}, f^{(i)}f^{(j)}$ and $-f^{(i)}f^{(j)}$ clearly lie in \mathcal{H} . Let $S^{(i)}$ and $S^{(i,j)}$ be equal to S^f , as defined in (4.1), when $f = f^{(i)}$ and $f = f^{(i)}f^{(j)}$, respectively. Then by property 3 of the submartingale problem and the optional sampling theorem, $S^{(i)}, S^{(i,j)}, S^{(i)}(\cdot \wedge \tau) - S^{(i)}(\cdot \wedge \zeta)$ and $S^{(i,j)}(\cdot \wedge \tau) - S^{(i,j)}(\cdot \wedge \zeta)$ are all continuous martingales. Since $Z(s) \in \bar{\mathcal{O}}_2$ for $s \in [t \wedge \zeta, t \wedge \tau]$, and $f^{(i)}(x) = x_i$ and $\mathcal{L}f^{(i)}(x) = b_i(x)$ for $x \in \bar{\mathcal{O}}_2$, it follows that $S_i(\cdot \wedge \tau) - S_i(\cdot \wedge \zeta)$ is equal to $S^{(i)}(\cdot \wedge \tau) - S^{(i)}(\cdot \wedge \zeta)$, and hence, is a continuous martingale. In addition, observing that $\mathcal{L}(f^{(i)}f^{(j)})(x) = x_j b_i(x) + x_i b_j(x) + \frac{1}{2}a_{ij}(x)$, a standard argument [e.g., see (4.10)–(4.12) on page 315 of [23], where $M^{(i)}$ there plays the role of S_i here] can be used to show that $S_i S_j(t \wedge \tau) - S_i S_j(t \wedge \zeta) - \int_{t \wedge \zeta}^{t \wedge \tau} a_{ij}(Z(u)) du$ is a continuous martingale. This establishes (4.10). \square

Let $\{G_m, m \in \mathbb{N}\}$ be a sequence of bounded domains in G such that $\bar{G}_m \subset G_{m+1}$ for each $m \in \mathbb{N}$ and $\bigcup_{m \in \mathbb{N}} G_m = G$. Also, for each $m \in \mathbb{N}$, let $\tau_0^m \doteq 0$ and let $\{\zeta_k^m : k \in \mathbb{N}\}$ and $\{\tau_k^m : k \in \mathbb{N}\}$ be nested sequences of stopping times defined by

$$(4.11) \quad \zeta_k^m \doteq \inf\{t > \tau_{k-1}^m : Z(t) \in \bar{G}_m\},$$

$$(4.12) \quad \tau_k^m \doteq \inf\{t > \zeta_k^m : Z(t) \notin G_{2m}\}.$$

For each $m \in \mathbb{N}$, since \bar{G}_m is compact and $\bar{G}_m \subset G_{2m}$, then the distance between \bar{G}_m and ∂G_{2m} is strictly positive. By the continuity of Z , $\zeta_k^m \rightarrow \infty$ as $k \rightarrow \infty$.

For each $k \in \mathbb{N}$, applying Lemma 4.1 with $\mathcal{O}_1 = G_m, \mathcal{O}_2 = G_{2m}, \tau_0 = \tau_{k-1}^m, \zeta = \zeta_k^m$ and $\tau = \tau_k^m$, it follows that $\{S(t \wedge \tau_k^m) - S(t \wedge \zeta_k^m), t \geq 0\}$ is a continuous martingale with covariation processes

$$(4.13) \quad [S_i, S_j](t \wedge \tau_k^m) - [S_i, S_j](t \wedge \zeta_k^m) = \int_{t \wedge \zeta_k^m}^{t \wedge \tau_k^m} a_{ij}(Z(u)) du, \quad t \geq 0,$$

for $i, j = 1, \dots, J$. Now, for $m \in \mathbb{N}$, define

$$(4.14) \quad W^m(t) \doteq \sum_{k=1}^{\infty} \int_{t \wedge \zeta_k^m}^{t \wedge \tau_k^m} \mathbb{I}_{G_m}(Z(u)) a^{-1/2}(Z(u)) dS(u), \quad t \geq 0.$$

LEMMA 4.2. *For each $m \in \mathbb{N}$, the process $\{W^m(t), t \geq 0\}$ is a continuous martingale with covariation processes given by*

$$(4.15) \quad [W_i^m, W_j^m](t) = \delta_{ij} \int_0^t \mathbb{I}_{G_m}(Z(u)) du, \quad t \geq 0, 1 \leq i, j \leq J,$$

where δ_{ij} represents the Kronecker delta: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. Moreover, there exist a process $\{W(t), \mathcal{M}_t\}$ that is a J -dimensional standard Brownian motion (under \mathbb{Q}_z), and a subsequence $\{W^{m_n}, n \in \mathbb{N}\}$ such that, as $n \rightarrow \infty$, W^{m_n} almost surely converges uniformly on bounded intervals to W .

PROOF. The uniform ellipticity condition (2.11) implies that $a^{-1/2}(\cdot)$ is uniformly bounded on \bar{G} . Therefore, $\{\mathbb{I}_{G_m}(Z(t))a^{-1/2}(Z(t)), t \geq 0\}$ is a bounded $\{\mathcal{M}_t\}$ -adapted process. Moreover, for each $m, k \in \mathbb{N}$, by (4.13) the covariation of the continuous martingale $S(\cdot \wedge \tau_k^m) - S(\cdot \wedge \zeta_k^m)$ is absolutely continuous with respect to Lebesgue measure. Hence, the stochastic integral

$$H^{k,m}(t) \doteq \int_{t \wedge \zeta_k^m}^{t \wedge \tau_k^m} \mathbb{I}_{G_m}(Z(u)) a^{-1/2}(Z(u)) dS(u), \quad t \geq 0,$$

is a continuous martingale, with covariation, for $i, j, k, k' \in \mathbb{N}$, given by

$$\begin{aligned} & [H_i^{k,m}, H_j^{k',m}] \\ &= \delta_{kk'} \sum_{\ell, \ell'=1}^J \int_{t \wedge \zeta_k^m}^{t \wedge \tau_k^m} \mathbb{I}_{G_m}(Z(u)) (a^{-1/2})_{i\ell}(Z(u)) (a^{-1/2})_{j\ell'}(Z(u)) d[S_\ell, S_{\ell'}](u) \\ &= \delta_{kk'} \sum_{\ell, \ell'=1}^J \int_{t \wedge \zeta_k^m}^{t \wedge \tau_k^m} \mathbb{I}_{G_m}(Z(u)) (a^{-1/2})_{i\ell}(Z(u)) (a^{-1/2})_{j\ell'} \\ & (4.16) \quad \times (Z(u)) a_{\ell\ell'}(Z(u)) du \\ &= \delta_{ij} \delta_{kk'} \int_{t \wedge \zeta_k^m}^{t \wedge \tau_k^m} \mathbb{I}_{G_m}(Z(u)) du \\ &= \delta_{ij} \delta_{kk'} \int_{t \wedge \zeta_k^m}^{t \wedge \zeta_{k+1}^m} \mathbb{I}_{G_m}(Z(u)) du, \end{aligned}$$

where the last equality uses the fact that $Z(u) \notin G_m$ for $u \in [\tau_k^m, \zeta_{k+1}^m]$.

We next show that for a fixed $t > 0$ and $1 \leq i \leq J$, $\sum_{k=1}^n H_i^{k,m}(t)$ converges in $\mathbb{L}^2(\mathbb{Q}_z)$ to $W_i^m(t)$ as $n \rightarrow \infty$. Applying Fatou’s lemma, (4.16), the monotone convergence theorem and the fact that $\zeta_k^m \rightarrow \infty$ as $k \rightarrow \infty$, we obtain for $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_z} \left[\left| W_i^m(t) - \sum_{k=1}^n H_i^{k,m}(t) \right|^2 \right] &\leq \sum_{k=n+1}^{\infty} \mathbb{E}^{\mathbb{Q}_z} [(H_i^{k,m}(t))^2] \\ &= \mathbb{E}^{\mathbb{Q}_z} \left[\int_{t \wedge \zeta_{n+1}^m}^t \mathbb{I}_{G_m}(Z(u)) du \right]. \end{aligned}$$

Also, sending $n \rightarrow \infty$, since $\zeta_{n+1}^m \rightarrow \infty$, the right-hand side above converges to zero by the dominated convergence theorem, which proves that $\sum_{k=1}^n H_i^{k,m}(t)$ converges in $\mathbb{L}^2(\mathbb{Q}_z)$ to $W_i^m(t)$. Moreover, since for each i , $\sum_{k=1}^n H_i^{k,m}$ is a continuous martingale, by [8], Proposition 1.3, $W^m(t) = \sum_{k=1}^n H^{k,m}(t)$ is also a continuous martingale. Moreover, using (4.16), the fact that $\zeta_{n+1}^m \rightarrow \infty$ as $n \rightarrow \infty$, and the dominated convergence theorem, it follows that

$$[W_i^m, W_j^m](t) = \delta_{ij} \sum_{k=1}^{\infty} \int_{t \wedge \zeta_k^m}^{t \wedge \zeta_{k+1}^m} \mathbb{I}_{G_m}(Z(u)) du = \delta_{ij} \int_0^t \mathbb{I}_{G_m}(Z(u)) du,$$

which proves (4.15).

We now extract a convergent subsequence of $\{W^m, m \in \mathbb{N}\}$. Let $\tilde{m} > m, \tilde{m}, m \in \mathbb{N}$. Since $\tilde{G}_m \subset G_{\tilde{m}}$, for any $k \in \mathbb{N}$, there exists $\tilde{k} \in \mathbb{N}$ such that $[\zeta_k^m, \tau_k^m] \subset [\zeta_{\tilde{k}}^{\tilde{m}}, \tau_{\tilde{k}}^{\tilde{m}}]$. Moreover, for any $\tilde{k} \in \mathbb{N}$, if $Z(u) \in G_m$ for some $u \in [\zeta_{\tilde{k}}^{\tilde{m}}, \tau_{\tilde{k}}^{\tilde{m}}]$ then it is easy to see that $u \in [\zeta_k^m, \tau_k^m]$ for some k . Together, this implies that

$$W^{\tilde{m}}(t) - W^m(t) = \sum_{\tilde{k}=1}^{\infty} \int_{t \wedge \zeta_{\tilde{k}}^{\tilde{m}}}^{t \wedge \tau_{\tilde{k}}^{\tilde{m}}} \mathbb{I}_{G_{\tilde{m}} \setminus G_m}(Z(u)) a^{-1/2}(Z(u)) dS(u).$$

The argument used to establish (4.15) can then be used to show that

$$[W_i^{\tilde{m}} - W_i^m](t) = \int_0^t \mathbb{I}_{G_{\tilde{m}} \setminus G_m}(Z(u)) du, \quad i = 1, \dots, J.$$

For any $k \in \mathbb{N}, T < \infty$, and $i = 1, \dots, J$, by Doob’s maximal inequality we have

$$\begin{aligned} \mathbb{Q}_z \left(\sup_{t \in [0, T]} |W_i^{\tilde{m}}(t) - W_i^m(t)| \geq \frac{1}{2^k} \right) &\leq 2^{2k} \mathbb{E}^{\mathbb{Q}_z} [|W_i^{\tilde{m}}(T) - W_i^m(T)|^2] \\ &= 2^{2k} \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^T \mathbb{I}_{G_{\tilde{m}} \setminus G_m}(Z(u)) du \right]. \end{aligned}$$

Since $\bigcup_{m \in \mathbb{N}} G_m = G$, taking first $\tilde{m} \rightarrow \infty$ and then $m \rightarrow \infty$, the expectation on the right-hand side converges to zero by the bounded convergence theorem. Hence,

there exists a subsequence $\{W^{m_k}, k \in \mathbb{N}\}$ such that for $i = 1, \dots, J$,

$$\begin{aligned} \mathbb{Q}_z \left(\sup_{t \in [0, T]} |W_i^{m_{k+1}}(t) - W_i^{m_k}(t)| \geq \frac{1}{2^k} \right) &\leq 2^{2k} \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^T \mathbb{I}_{G_{m_{k+1}} \setminus G_{m_k}}(Z(s)) ds \right] \\ &\leq \frac{1}{2^k}. \end{aligned}$$

By the Borel–Cantelli lemma there exist $\mathcal{C}^0 \subset \mathcal{C}$ with $\mathbb{Q}_z(\mathcal{C}^0) = 1$ and a continuous process $W = \{W(\omega, t), t \geq 0\}$ such that for $\omega \in \mathcal{C}^0$, as $k \rightarrow \infty$, the sequence of continuous functions $\{W^{m_k}(\omega, t), t \geq 0\}$ converges uniformly on bounded intervals to $\{W(\omega, t), t \geq 0\}$. Furthermore, for $\omega \in \mathcal{C}^0$, by (4.15) and the fact that $\bigcup_{m \in \mathbb{N}} G_m = G$, we have for $i, j = 1, \dots, J$,

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \left| [W_i^{m_k}, W_j^{m_k}](t, \omega) - \delta_{ij} \int_0^t \mathbb{I}_G(Z(u, \omega)) du \right| = 0.$$

The boundary property (2.12) shows that almost surely, $\int_0^t \mathbb{I}_G(Z(u)) du = t$ and thus, that $(W^m, [W^m, W^m])$ converge jointly to $(W, I_J t)$, where I_J is the $J \times J$ identity matrix. Thus, by [19], Theorem 2.4; page 528, W is a continuous \mathbb{Q}_z -martingale with $[W_i, W_j](t) = \delta_{ij}t$, thus proving that W is a J -dimensional standard Brownian motion under \mathbb{Q}_z . \square

4.3. The semimartingale property.

4.3.1. *Preliminary results.* We first establish certain geometric properties of the directions of reflection and the existence of suitable test functions.

LEMMA 4.3. *Suppose $(G, d(\cdot))$ is a piecewise \mathcal{C}^2 domain with continuous reflection. Then, for each $y \in \mathcal{U}$, there exist $\alpha_y > 0$ and $0 < R_y < \text{dist}(y, \mathcal{V})$ such that:*

1. $\mathcal{I}(x) \subseteq \mathcal{I}(y)$ for all $x \in B_{R_y}(y) \cap \partial G$;
2. $\sup_{n \in \mathbb{N}(y): |n|=1} \inf_{x \in B_{R_y}(y) \cap \partial G} \inf_{d \in d(x): |d|=1} \langle n, d \rangle \geq \alpha_y$;
3. *There exist $r_y < R_y$, an increasing, continuous function $\kappa_y : (0, \infty) \mapsto (0, \infty)$ that satisfies $\kappa_y(r) < r$ if $r \leq r_y$ and $\kappa_y(r) = \kappa_y(r_y)$ if $r > r_y$ and a collection of functions $\{f^{y,r}, r \in (0, r_y]\}$ on \mathbb{R}^J such that:*
 - (a) $-f^{y,r} \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$;
 - (b) $\text{supp}[f^{y,r}] \cap \bar{G} \subset B_r(y) \cap \bar{G}$;
 - (c) $0 \leq f^{y,r}(x) \leq 1$ for all $x \in \bar{G}$;
 - (d) $f^{y,r}(x) = 1$ for all $x \in B_{\kappa_y(r)}(y) \cap \bar{G}$.

PROOF. The proof is deferred to Appendix C. \square

For $0 < r < s < \infty$, let

$$\mathcal{U}_{r,s} \doteq \{x \in \partial G : |x| \leq r, d(x, \mathcal{V}) \geq s\}.$$

LEMMA 4.4. *For every $0 < K < \infty$, $s \geq 1/K$ and $r \leq K$, there exists a function $f_{r,s} \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$ such that $\langle v, \nabla f_{r,s}(x) \rangle \geq 1$ for each $x \in \mathcal{U}_{r,s}$, $v \in d(x)$ and $|v| = 1$.*

PROOF. The existence of $f_{r,s}$ follows from [21], Theorem 2. Note that the proof of the existence of $f_{r,s}$ in [21], Theorem 2 (which is the verification of part 2 of Assumption 1 of [21]) does not require Assumption 2 therein to hold. \square

We now prove a covering lemma that is used in certain localization arguments in the proof of Theorem 3 and is also used to verify that certain random times constructed in the sequel are stopping times. The proof of the lemma is deferred to Appendix B.

LEMMA 4.5 (Covering lemma). *Suppose we are given a compact subset \hat{G} of $\bar{G} \setminus \mathcal{V}$, and a collection of open sets $\{\mathcal{O}_y, y \in \hat{G}\}$ such that $y \in \mathcal{O}_y$ for every $y \in \hat{G}$, $\mathcal{O}_y \cap \mathcal{V} = \emptyset$ if $y \in \hat{G} \cap \mathcal{U}$ and $\mathcal{O}_y \subset G$ if $y \in \hat{G} \cap G$. Then there exists a finite set of points $\hat{F} \subset \hat{G}$ such that*

$$\hat{G} \subseteq \bigcup_{y \in \hat{F}} \mathcal{O}_y,$$

and there exists a measurable mapping $\hat{\lambda}$ from \hat{G} onto \hat{F} such that $y \in \mathcal{O}_{\hat{\lambda}(y)}$ and $\hat{\lambda}(y) \in \partial G$ if $y \in \hat{G} \cap \partial G$.

4.3.2. *Proof of the semimartingale property.* We start with two preliminary results. The first result considers the behavior of S in the interior of the domain G .

LEMMA 4.6. *Given two stopping times ς and τ such that almost surely $\varsigma \leq \tau$ and $Z(t) \in G$ for each $t \in [\varsigma, \tau)$, the process $S(\cdot \wedge \tau) - S(\cdot \wedge \varsigma)$ is a martingale that satisfies*

$$(4.17) \quad S(t \wedge \tau) - S(t \wedge \varsigma) = \int_{t \wedge \varsigma}^{t \wedge \tau} a^{1/2}(Z(u)) dW(u), \quad t \geq 0.$$

PROOF. Let $G_m, m \in \mathbb{N}$, be the sequence of nested domains introduced in Section 4.2, let $\tau_0 \doteq \varsigma$, and let ς_1^m and $\tau_1^m, m \in \mathbb{N}$, be defined as in (4.11) and (4.12), respectively, with $k = 1$. Since $Z(\varsigma) \in G$ by assumption and $\bigcup_m G_m = G$, for each ω there exists $m_0(\omega) < \infty$ such that for all $m \geq m_0(\omega)$, $Z(\varsigma(\omega), \omega) \in G_m$ and $\varsigma_1^m(\omega) = \varsigma(\omega)$. Let $\tilde{\tau}_1^m \doteq \inf\{t \geq \varsigma : Z(t) \notin G_m\}$. It follows that for each ω , $\tilde{\tau}_1^m(\omega) \leq \tau_1^m(\omega)$ for $m \geq m_0(\omega)$ and

$$(4.18) \quad \tau_1(\omega) = \lim_{m \rightarrow \infty} \tau_1^m(\omega) = \lim_{m \rightarrow \infty} \tilde{\tau}_1^m(\omega) \geq \tau(\omega),$$

where

$$(4.19) \quad \tau_1(\omega) \doteq \inf\{t \geq \varsigma(\omega) : Z(t, \omega) \in \partial G\}.$$

From the discussion preceding (4.13), it follows that $S(\cdot \wedge \tau_1^m) - S(\cdot \wedge \zeta_1^m)$, $m \in \mathbb{N}$, is a sequence of continuous martingales with covariation processes

$$(4.20) \quad [S_i, S_j](t \wedge \tau_1^m) - [S_i, S_j](t \wedge \zeta_1^m) = \int_{t \wedge \zeta_1^m}^{t \wedge \tau_1^m} a_{ij}(Z(u)) du, \quad t \geq 0,$$

for $i, j = 1, \dots, J$. By (4.18), this sequence converges uniformly on compact intervals to $S(\cdot \wedge \tau_1) - S(\cdot \wedge \zeta)$. Since the dispersion matrix $\sigma(\cdot)$ is locally bounded, then $a_{ij}(\cdot)$ is also locally bounded, and hence, $a_{ij}(Z(\cdot))$ is locally integrable. Together with (4.20) and (4.18), this shows that almost surely, for every $T < \infty$,

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \left| [S_i, S_j](t \wedge \tau_1^m) - [S_i, S_j](t \wedge \zeta_1^m) - \int_{t \wedge \zeta}^{t \wedge \tau_1} a_{ij}(Z(u)) du \right| = 0.$$

Thus, by [19], Theorem 2.4, page 528, the optional stopping theorem and the fact that $\zeta \leq \tau \leq \tau_1$, $S(\cdot \wedge \tau) - S(\cdot \wedge \zeta)$ is a continuous martingale with

$$(4.21) \quad [S_i, S_j](t \wedge \tau) - [S_i, S_j](t \wedge \zeta) = \int_{t \wedge \zeta}^{t \wedge \tau} a_{ij}(Z(u)) du, \quad t \geq 0.$$

Next, note that by (4.14),

$$W^m(t \wedge \tau_1^m) - W^m(t \wedge \zeta_1^m) = \int_{t \wedge \zeta_1^m}^{t \wedge \tau_1^m} \mathbb{I}_{G_m}(Z(u)) a^{-1/2}(Z(u)) dS(u).$$

Since $Z(u) \in G_m$ for $u \in [\zeta_1^m, \tilde{\tau}_1^m)$, we have

$$\begin{aligned} W^m(t \wedge \tilde{\tau}_1^m) - W^m(t \wedge \zeta_1^m) &= \int_{t \wedge \zeta_1^m}^{t \wedge \tilde{\tau}_1^m} \mathbb{I}_{G_m}(Z(u)) a^{-1/2}(Z(u)) dS(u) \\ &= \int_{t \wedge \zeta_1^m}^{t \wedge \tilde{\tau}_1^m} a^{-1/2}(Z(u)) dS(u). \end{aligned}$$

Now, recall that $\zeta_1^m(\omega) = \zeta(\omega)$ for all large enough $m \geq m_0(\omega)$ and, from (4.18), that $\tilde{\tau}_1^m \rightarrow \tau_1$. Therefore, for any $t > 0$, using (4.21), the right-hand side of the last display converges in $\mathbb{L}^2(\mathbb{Q}_z)$ to $\int_{t \wedge \zeta}^{t \wedge \tau_1} a^{-1/2}(Z(u)) dS(u)$, and by Lemma 4.2 the left-hand side converges almost surely to $W(t \wedge \tau_1) - W(t \wedge \zeta)$ along a subsequence. Since $\tau \leq \tau_1$, this proves

$$W(t \wedge \tau) - W(t \wedge \zeta) = \int_{t \wedge \zeta}^{t \wedge \tau} a^{-1/2}(Z(u)) dS(u).$$

Taking the stochastic integral of the martingales on both sides of the last equation with respect to $a^{1/2}(Z(\cdot))$, we obtain (4.17). \square

Recall the definition of S^f given in (4.1). We now show that, for a suitable class of functions g , certain localized versions of the process S^g are submartingales.

LEMMA 4.7. *Let ϱ be an $\{\mathcal{M}_t\}$ -stopping time, let $r_x, \kappa_x, x \in \bar{G}$, be the constants in Lemma 4.3, and on the set $\{\varrho < \infty\}$, define*

$$(4.22) \quad \theta^x \doteq \inf\{t > \varrho : Z(t) \notin B_{\kappa_x(r_x)}(x)\}.$$

Then, for any $x \in \mathcal{U}$ and $g \in \mathcal{C}^2(\mathbb{R}^J)$ such that $\langle \nabla g(y), d \rangle \geq 0$ for all $d \in d(y)$ and $y \in B_{r_x}(x) \cap \partial G$, there exists a function $h \in \mathcal{H}$ such that $h(y) - h(x) = g(y) - g(x)$ for $y \in B_{\kappa_x(r_x)}(x)$, and the process $\mathbb{I}_{\{\varrho < t\}}[S^g(t \wedge \theta^x) - S^g(\varrho)], t \geq 0$, is a continuous submartingale.

PROOF. Fix $x \in \mathcal{U}$, an $\{\mathcal{M}_t\}$ -stopping time ϱ , and let $f = f^{x,r_x}$ be a function that satisfies property 3 of Lemma 4.3. Given g as in the statement of the lemma, define

$$h(y) \doteq \left(g(y) - \sup_{l \in B_{r_x}(x)} g(l) \right) f(y), \quad y \in \mathbb{R}^J.$$

For $y \in B_{\kappa_x(r_x)}(x)$, by property 3(d) of Lemma 4.3, $f(y) = 1$, and hence, $h(y) - h(x) = g(y) - g(x)$ and $\mathcal{L}h(y) = \mathcal{L}g(y)$. Note that on the set $\{\varrho < \infty\}$, $Z(t \wedge \theta^x) \in B_{\kappa_x(r_x)}(x)$ for each $t \in (\varrho, \theta^x)$. Therefore, on the set $\{\varrho < \infty\}$, it is clear from (4.1) that

$$(4.23) \quad S^g(\cdot \wedge \theta^x) - S^g(\varrho) = S^h(\cdot \wedge \theta^x) - S^h(\varrho).$$

In addition, clearly $h \in \mathcal{C}^2(\bar{G})$, with $\text{supp}[h] \subseteq \text{supp}[f] \subset B_{r_x}(x)$, and

$$\nabla h(y) = f(y)\nabla g(y) + \left(g(y) - \sup_{l \in B_{r_x}(x)} g(l) \right) \nabla f(y).$$

The assumed properties of g , together with the fact that $f \geq 0$, $\text{supp}[\nabla f] \subset B_{r_x}(x)$ and $-f \in \mathcal{H}$, imply that $h \in \mathcal{H}$. By property 3 of the submartingale problem, the fact that $\varrho \leq \theta^x$, and the optional stopping theorem, it follows that $S^h(t \wedge \theta^x) - S^h(t \wedge \varrho) = \mathbb{I}_{\{\varrho < t\}}[S^h(t \wedge \theta^x) - S^h(\varrho)]$ is a continuous submartingale. By (4.23), this implies that $\mathbb{I}_{\{\varrho < t\}}[S^g(t \wedge \theta^x) - S^g(\varrho)], t \geq 0$, is also a continuous submartingale. \square

We now show that $S(\cdot \wedge \theta^K)$ and $Z(\cdot \wedge \theta^K)$ are semimartingales. We will establish this locally and then extend using the covering lemma and suitable stopping times, which we introduce below. This notation, and a similar extension argument, is also used in the proof of Lemma 4.11. For each $0 < K < \infty$, let G^K be the set in (4.3) and let \bar{G}^K be the closure of G^K . Also, recall that θ^K is the stopping time defined in (4.4). For $y \in \bar{G}^K \cap \mathcal{U}$, let the constant r_y and function $\kappa_y : (0, \infty) \mapsto (0, \infty)$ be as in property 3 of Lemma 4.3, and for $y \in \bar{G}^K \setminus \mathcal{U}$, let

$$r_y \doteq \frac{1}{2} \sup\{r > 0 : B_r(y) \subset G\},$$

and let κ_y be the linear function $\kappa_y(r) = r, r \geq 0$. Applying Lemma 4.5 with $\hat{G} = \bar{G}^K$ and $\mathcal{O}_y = B_{\kappa_y(r_y)/2}(y), y \in \bar{G}^K$ (this collection of sets is easily seen to satisfy the assumptions of the lemma), there exist a finite set $\hat{F}^K \subset \bar{G}^K$ and a measurable map $\hat{\lambda}^K : \bar{G}^K \mapsto \hat{F}^K$ such that $\bar{G}^K \subseteq \bigcup_{y \in \hat{F}^K} \mathcal{O}_y$ and $\hat{\lambda}^K(y) \in \partial G$ if $y \in \bar{G}^K \cap \partial G$. Now, let $\{\varrho_k, k \in \mathbb{N} \cup \{0\}\}$ be an increasing sequence of stopping times defined by $\varrho_0 = 0$ and let

$$(4.24) \quad \varrho_{k+1} \doteq \inf\{t \geq \varrho_k : Z(t) \notin B_{\ell_k}(\bar{y}_k)\} \wedge \theta^K, \quad k \in \mathbb{N} \cup \{0\},$$

where $\bar{y}_k \doteq \hat{\lambda}^K(Z(\varrho_k))$ and $\ell_k \doteq \kappa_{\bar{y}_k}(r_{\bar{y}_k})$. We claim that almost surely, $\varrho_k \rightarrow \theta^K$ as $k \rightarrow \infty$. To see why the claim is true, for $\omega \in \mathcal{C}$, let $\varrho^*(\omega)$ denote the limit of the nondecreasing sequence $\{\varrho_k(\omega)\}$. Now, let $\tilde{\mathcal{C}} \subset \mathcal{C}$ be the set of measure one on which Z is continuous, and suppose that there exists $\omega \in \tilde{\mathcal{C}}$ such that $\varrho^*(\omega) < \theta^K(\omega)$. Then $Z(\varrho_k(\omega)) \rightarrow Z(\varrho^*(\omega)) \in \bar{G}^K$ as $k \rightarrow \infty$. Note that for each $k \geq 1$,

$$\begin{aligned} |Z(\varrho_{k+1}(\omega)) - Z(\varrho_k(\omega))| &\geq |Z(\varrho_{k+1}(\omega)) - \bar{y}_k(\omega)| - |\bar{y}_k(\omega) - Z(\varrho_k(\omega))| \\ &= \ell_k(\omega) - |\bar{y}_k(\omega) - Z(\varrho_k(\omega))|. \end{aligned}$$

Now, for each $k \geq 1$, since $\bar{y}_k \doteq \hat{\lambda}^K(Z(\varrho_k))$ the property of $\hat{\lambda}$ stated in Lemma 4.5 shows that $Z(\varrho_k(\omega)) \in \mathcal{O}_{\bar{y}_k(\omega)} = B_{\ell_k(\omega)/2}(\bar{y}_k(\omega))$, which implies $|\bar{y}_k(\omega) - Z(\varrho_k(\omega))| \leq \ell_k(\omega)/2$. Hence, for each $k \geq 1$,

$$|Z(\varrho_{k+1}(\omega)) - Z(\varrho_k(\omega))| \geq \frac{\ell_k(\omega)}{2} \geq \frac{1}{2} \min_{y \in \hat{F}^K} \kappa_y(r_y) > 0,$$

which contradicts the convergence of $\{Z(\varrho_k(\omega)), k \in \mathbb{N}\}$. Thus, $\varrho^*(\omega) = \theta^K(\omega)$, and the claim holds.

PROPOSITION 4.8. *For each $0 < K < \infty$, let θ^K be the stopping time defined in (4.4). Then $S(\cdot \wedge \theta^K)$ is a continuous semimartingale, that is, there exist a continuous local martingale M with $M(0) = 0$ and a continuous process A with $A(0) = 0$ that is of locally bounded variation such that*

$$(4.25) \quad S(t \wedge \theta^K) = M(t \wedge \theta^K) + A(t \wedge \theta^K), \quad t \geq 0.$$

Furthermore, $Z(\cdot \wedge \theta^K)$ is also a continuous semimartingale with local martingale component $z + M(\cdot \wedge \theta^K)$ and locally finite variation component $A(\cdot \wedge \theta^K) + \int_0^{\cdot \wedge \theta^K} b(Z(u)) du$.

PROOF. For each $0 < K < \infty$, let \bar{G}^K be the closure of the set G^K defined in (4.3). Clearly, \bar{G}^K is compact. Fix K large enough such that $z \in \bar{G}^K$, and let $\{\varrho_k\}$ be the sequence of stopping times introduced above. We now prove by induction that for each $k \in \mathbb{N} \cup \{0\}$, $S(\cdot \wedge \varrho_k)$ is a continuous semimartingale. When $k = 0$, $S(\cdot \wedge \varrho_0) = S(0)$ is clearly a continuous semimartingale. Now, suppose that $S(\cdot \wedge$

ϱ_k) is a continuous semimartingale for some $k \in \mathbb{N} \cup \{0\}$. We show that $S(\cdot \wedge \varrho_{k+1})$ is also a continuous semimartingale. Observe that for $t \geq 0$,

$$\begin{aligned} S(t \wedge \varrho_{k+1}) - S(t \wedge \varrho_k) &= \mathbb{I}_{\{t \wedge \theta^K > \varrho_k\}} [S(t \wedge \varrho_{k+1}) - S(\varrho_k)] \\ &= \mathbb{I}_{\{t \wedge \theta^K > \varrho_k\}} \sum_{x \in \hat{F}^K} \mathbb{I}_{\{Z(\varrho_k) \in (\hat{\lambda}^K)^{-1}(x)\}} [S(t \wedge \theta_k^x \wedge \theta^K) - S(\varrho_k)], \end{aligned}$$

where, for each $x \in \hat{F}^K$ and $k \in \mathbb{N} \cup \{0\}$, we define θ_k^x as in (4.22), but with ϱ replaced by ϱ_k . Then, since $S(\cdot \wedge \varrho_k)$ is a semimartingale by the induction assumption, to show that $S(\cdot \wedge \varrho_{k+1})$ is a semimartingale, it suffices to establish the claim that for each $x \in \hat{F}^K$, the process

$$D_k^x(t) \doteq \mathbb{I}_{\{\varrho_k < t \wedge \theta^K\}} \mathbb{I}_{\{Z(\varrho_k) \in (\hat{\lambda}^K)^{-1}(x)\}} [S(t \wedge \theta_k^x \wedge \theta^K) - S(\varrho_k)], \quad t \geq 0,$$

is a semimartingale.

To prove the claim, we consider the cases $x \in \hat{F}^K \cap G$ and $x \in \hat{F}^K \cap \mathcal{U}$ separately. For $x \in \hat{F}^K \cap G$, note that $B_{\kappa_x(r_x)}(x) = B_{r_x}(x)$ and the closure of $B_{r_x}(x)$ lies in G by the definition of r_x . Thus, when $\varrho_k < \theta^K$ and $Z(\varrho_k) \in (\hat{\lambda}^K)^{-1}(x)$, $Z(s) \in G$ for every $s \in [\varrho_k, \theta_k^x \wedge \theta^K]$. Applying Lemma 4.6, with $\varsigma = \varrho_k$ and $\tau = \theta_k^x \wedge \theta^K$, $S(\cdot \wedge \theta_k^x \wedge \theta^K) - S(\cdot \wedge \varrho_k)$ and, therefore, D_k^x , is a continuous semimartingale. Next, for $x \in \hat{F}^K \cap \mathcal{U}$, by properties 1 and 2 of Lemma 4.3, there exist $\alpha_x > 0$ and $n^x \in n(x)$ such that $\langle n^x, d \rangle \geq \alpha_x$ for all $d \in d(y)$ with $|d| = 1$ and $y \in B_{r_x}(x) \cap \partial G$. Let $\{\vec{e}^\ell, \ell = 1, \dots, J\}$ be an orthonormal basis for \mathbb{R}^J and for $\tilde{\varepsilon}_x > 0$, let $v_x^\ell \doteq n^x + \tilde{\varepsilon}_x \vec{e}^\ell$. Choose $\tilde{\varepsilon}_x > 0$ small enough such that for each $\ell = 1, \dots, J$, $\langle v_x^\ell, d \rangle > 0$ for all nonzero $d \in d(y)$ and $y \in B_{r_x}(x) \cap \partial G$. For each $\ell = 1, \dots, J$, define g_x^ℓ via $g_x^\ell(y) = \langle y, v_x^\ell \rangle$ for $y \in \mathbb{R}^J$. Then, clearly $g_x^\ell \in \mathcal{C}^2(\mathbb{R}^J)$ and $\langle \nabla g_x^\ell(y), d \rangle \geq 0$ for $d \in d(y)$, $y \in B_{r_x}(x) \cap \partial G$. Now, applying Lemma 4.7 with $\varrho = \varrho_k$ and $g = g_x^\ell$, together with the optional stopping theorem, it follows that $\{\mathbb{I}_{\{\varrho_k < t \wedge \theta^K\}} [S^{g_x^\ell}(t \wedge \theta_k^x \wedge \theta^K) - S^{g_x^\ell}(\varrho_k)], t \geq 0\}$ is a submartingale. Since, on the set $\varrho_k < \infty$, $S^{g_x^\ell}(t \wedge \theta_k^x \wedge \theta^K) - S^{g_x^\ell}(\varrho_k)$ is equal to $\langle S(t \wedge \theta_k^x \wedge \theta^K) - S(\varrho_k), v_x^\ell \rangle$ and the drift b is locally bounded, this shows that $\mathbb{I}_{\{\varrho_k < t \wedge \theta^K\}} \langle S(t \wedge \theta_k^x \wedge \theta^K) - S(\varrho_k), v_x^\ell \rangle$ is a bounded submartingale. By the Doob–Meyer decomposition theorem, and the linear independence of the vectors $v^\ell, \ell = 1, \dots, J$, it follows that D_k^x is a semimartingale. This completes the proof of the claim, and thus shows that $S(\cdot \wedge \varrho_{k+1})$ is a semimartingale.

By induction, $S(\cdot \wedge \varrho_k)$ is a semimartingale for every k . Sending $k \rightarrow \infty$ and using the fact that $\varrho_k \rightarrow \theta^K$ almost surely, it follows that $S(\cdot \wedge \theta^K)$ is a semimartingale. Let M and A denote the continuous local martingale and continuous, locally bounded variation components in the semimartingale decomposition of $S(\cdot \wedge \theta^K)$. Then (4.25) holds. Finally, since (4.2) shows that $Z = Z(0) + S + \int_0^\cdot b(Z(u)) du$, the second assertion of the lemma follows directly from the first. \square

4.4. *Behavior of the semimartingale in the interior of the domain.* We now characterize the behavior of the components $A(\cdot \wedge \theta^K)$ and $M(\cdot \wedge \theta^K)$ of the semimartingale decomposition (4.25) of $S(\cdot \wedge \theta^K)$ in G . As in Section 4.2, let $\{G_m, m \in \mathbb{N}\}$ be a sequence of bounded domains with $\bar{G}_m \subset G_{\tilde{m}}$ for $m < \tilde{m}$ and $\bigcup_{m \in \mathbb{N}} G_m = G$. For each $\omega \in \mathcal{C}[0, \infty)$ and fixed $m \in \mathbb{N}$, set $\tau_0^m(\omega) \doteq 0$ and for $k \in \mathbb{N}$, recursively define

$$(4.26) \quad \zeta_k^m = \zeta_k^m(\omega) \doteq \inf\{t \geq \tau_{k-1}^m : Z(t) \in \bar{G}_m\},$$

$$(4.27) \quad \tau_k^m = \tau_k^m(\omega) \doteq \inf\{t \geq \zeta_k^m : Z(t) \in \partial G\}.$$

Note that for each m , almost surely, since Z is continuous and the distance between $\bar{G}_m \cap \bar{G}^K$ and $\partial G \cap \bar{G}^K$ is strictly positive, $\tau_k^m \wedge \theta^K$ and $\zeta_k^m \wedge \theta^K$ converge to θ^K as $k \rightarrow \infty$.

LEMMA 4.9. *Let $M(\cdot \wedge \theta^K)$ and $A(\cdot \wedge \theta^K)$, respectively, be the continuous local martingale and continuous bounded variation processes that arise in the local semimartingale decomposition of $S(\cdot \wedge \theta^K)$ given in (4.25). Then almost surely, for every $k, m \in \mathbb{N}$,*

$$(4.28) \quad M(t \wedge \tau_k^m \wedge \theta^K) - M(t \wedge \zeta_k^m \wedge \theta^K) = \int_{t \wedge \zeta_k^m \wedge \theta^K}^{t \wedge \tau_k^m \wedge \theta^K} a^{1/2}(Z(u)) dW(u),$$

for $t \geq 0$. Moreover, almost surely, for every $t \geq 0$,

$$(4.29) \quad \int_0^{t \wedge \theta^K} \mathbb{I}_G(Z(u)) d|A|(u) = 0,$$

and

$$(4.30) \quad \int_0^{t \wedge \theta^K} \mathbb{I}_G(Z(u)) d[M_i, M_j](u) = \int_0^{t \wedge \theta^K} \mathbb{I}_G(Z(u)) a_{ij}(Z(u)) du.$$

PROOF. For $k, m \in \mathbb{N}$, since $Z(t) \in G$ for $t \in [\zeta_k^m \wedge \theta^K, \tau_k^m \wedge \theta^K)$, by Lemma 4.6,

$$S(t \wedge \tau_k^m \wedge \theta^K) - S(t \wedge \zeta_k^m \wedge \theta^K) = \int_{t \wedge \zeta_k^m \wedge \theta^K}^{t \wedge \tau_k^m \wedge \theta^K} a^{1/2}(Z(u)) dW(u)$$

for each $t \geq 0$. Thus, the process Y defined in (4.5)–(4.6) satisfies $Y(\cdot \wedge \tau_k^m \wedge \theta^K) - Y(\cdot \wedge \zeta_k^m \wedge \theta^K) = 0$. Comparing this with (4.25), we have (4.28) and $A(t \wedge \theta^K) - A(\zeta_k^m \wedge \theta^K) = 0$ for $t \in [\zeta_k^m, \tau_k^m]$. Because the latter equality holds for all $k \in \mathbb{N}$, $\tau_k^m \wedge \theta^K \rightarrow \theta^K$, $\zeta_k^m \wedge \theta^K \rightarrow \theta^K$ as $k \rightarrow \infty$ and for $u \in [0, \theta^K]$, $Z(u) \in G_m$ implies $u \in \bigcup_k [\zeta_k^m \wedge \theta^K, \tau_k^m \wedge \theta^K]$, it follows that almost surely

$$\int_0^{\theta^K} \mathbb{I}_{G_m}(Z(u)) d|A|(u) = 0$$

and

$$\int_0^{\theta^K} \mathbb{I}_{G_m}(Z(u)) d[M_i, M_j](u) = \int_0^{\theta^K} \mathbb{I}_{G_m}(Z(u)) a_{ij}(Z(u)) du.$$

Taking limits as $m \rightarrow \infty$, recalling that $\bigcup_m G_m = G$ and applying the dominated convergence theorem, we obtain (4.29) and (4.30). \square

4.5. *Boundary behavior of the semimartingale.* To complete the proof of Theorem 3, it only remains to show the following generalization of Lemma 4.9:

$$(4.31) \quad M(\cdot \wedge \theta^K) = \int_0^{\cdot \wedge \theta^K} a^{1/2}(Z(u)) dW(u),$$

$$(4.32) \quad A(\cdot \wedge \theta^K) = Y(\cdot \wedge \theta^K),$$

and Y satisfies the reflection property specified in (4.7). We establish relations (4.31) and (4.32) in Section 4.5.2 (see Corollary 4.15 therein) by showing that the trace of the quadratic variation of the martingale M vanishes on the boundary. We then establish the local reflection property in Section 4.5.3. Both results use certain properties of the semimartingale decomposition of S^f that are first established in Section 4.5.1.

4.5.1. *A random measure and an integral representation.* Given the semimartingale decomposition for $Z(\cdot \wedge \theta^K)$ established in Proposition 4.8, a simple application of Itô’s formula shows that for $f \in \mathcal{C}^2(\mathbb{R}^J)$, the semimartingale $S^f(\cdot \wedge \theta^K)$ admits the decomposition

$$(4.33) \quad S^f(\cdot \wedge \theta^K) = M^f(\cdot \wedge \theta^K) + A^f(\cdot \wedge \theta^K),$$

where

$$(4.34) \quad M^f(t \wedge \theta^K) \doteq \int_0^{t \wedge \theta^K} \langle \nabla f(Z(u)), dM(u) \rangle,$$

and

$$(4.35) \quad \begin{aligned} A^f(t \wedge \theta^K) &\doteq \int_0^{t \wedge \theta^K} \langle \nabla f(Z(u)), dA(u) \rangle \\ &+ \frac{1}{2} \sum_{i,j=1}^J \int_0^{t \wedge \theta^K} \frac{\partial^2 f}{\partial x_i \partial x_j}(Z(u)) d[M_i, M_j](u) \\ &- \frac{1}{2} \sum_{i,j=1}^J \int_0^{t \wedge \theta^K} a_{ij}(Z(u)) \frac{\partial^2 f}{\partial x_i \partial x_j}(Z(u)) du, \end{aligned}$$

for $t \geq 0$. Since ∇f is continuous and $M(\cdot \wedge \theta^K)$ is a local martingale, the stochastic integral $M^f(\cdot \wedge \theta^K)$ is a local martingale, and $A^f(\cdot \wedge \theta^K)$ is the locally

bounded variation component. In Lemma 4.11, we show that in fact $M^f(\cdot \wedge \theta^K)$ is a martingale. Note that we want to show (4.31) and (4.32), which implies that $A^f(\cdot \wedge \theta^K) = \int_0^{\cdot \wedge \theta^K} \langle \nabla f(Z(u)), dY(u) \rangle$. With that in mind, in Proposition 4.12 below, we establish an integral representation for $A^f(\cdot \wedge \theta^K)$.

We first introduce some localizing stopping times. For each $0 < c < \infty$, let

$$(4.36) \quad \zeta_c \doteq \left\{ t \geq 0 : |A|(t) \geq c \text{ or } \sum_{i,j=1}^J |[M_i, M_j]|(t) \geq c \right\},$$

where recall that $|A|$ and $|[M_i, M_j]|$ denote the total variation processes associated with A and $[M_i, M_j]$, respectively. It is clear that

$$\mathbb{E}^{\mathbb{Q}_z}[|A|(t \wedge \zeta_c)] \leq c \quad \text{and} \quad \sum_{i,j=1}^J \mathbb{E}^{\mathbb{Q}_z}[|[M_i, M_j]|(t \wedge \zeta_c)] \leq c.$$

REMARK 4.10. The following separability property of \mathcal{H} , which is used in Lemma 4.11, can be proved in a manner similar to the proof of [21], Lemma 5.2, under the assumption that \mathcal{V} is the union of finitely many closed connected sets: \mathcal{H} has a countable subset \mathcal{H}_0 with the property that for each $f \in \mathcal{H}$ and each $N \in \mathbb{N}$, there exists a sequence $\{g_k : k \in \mathbb{N}\} \subset \mathcal{H}_0$ such that

$$(4.37) \quad \lim_{k \rightarrow \infty} \sup_{y \in \bar{G} \cap B_N(0)} \max_{i,j=1}^J |f(y) - g_k(y)| \vee \left| \frac{\partial f(y)}{\partial x_i} - \frac{\partial g_k(y)}{\partial x_i} \right| \vee \left| \frac{\partial^2 f(y)}{\partial x_i \partial x_j} - \frac{\partial^2 g_k(y)}{\partial x_i \partial x_j} \right| = 0.$$

LEMMA 4.11. Fix $f \in \mathcal{C}^2(\bar{G})$ and $0 < c < \infty$. Then almost surely,

$$(4.38) \quad \int_0^{t \wedge \theta^K} \mathbb{I}_G(Z(u)) dA^f(u) = 0, \quad \forall t \geq 0,$$

and $M^f(\cdot \wedge \theta^K \wedge \zeta_c)$ is bounded on every finite interval of $[0, \infty)$. Moreover, $M^f(\cdot \wedge \theta^K)$ is a continuous martingale. Furthermore, almost surely, for all $f \in \mathcal{C}^2(\bar{G})$ such that $\langle \nabla f(x), d \rangle \geq 0$ for all $d \in d(x)$ and $x \in \bar{G}^{2K} \cap \partial G$, the process $A^f(\cdot \wedge \theta^K)$ is adapted, continuous and increasing.

PROOF. Fix $f \in \mathcal{C}^2(\bar{G})$. Then, from (4.4) and (4.33)–(4.36), it is clear that for each $0 < c, T < \infty$, $\sup_{t \in [0, T]} |M^f(t \wedge \theta^K \wedge \zeta_c)|$ is bounded by

$$(4.39) \quad 2 \sup_{x \in \bar{G}^K} |f(x)| + T \sup_{x \in \bar{G}^K} |\mathcal{L}f(x)| + c \sup_{x \in \bar{G}^K} |\nabla f(x)| + \frac{c}{2} \sup_{x \in \bar{G}^K} \left| \sum_{i,j=1}^J \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right| + \frac{T}{2} \sup_{x \in \bar{G}^K} \left| \sum_{i,j=1}^J a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right|.$$

Also, the relation (4.38) follows from (4.29), (4.30) and (4.35).

For the proof of the martingale property for $M^f(\cdot \wedge \theta^K)$, first consider the case when f additionally satisfies $\langle \nabla f(x), d \rangle \geq 0$ for all $d \in d(x)$ and $x \in \bar{G}^{2K} \cap \partial G$. Then (4.1) and the boundedness of f and $\mathcal{L}f$ on \bar{G}^K implies that $S^f(\cdot \wedge \theta^K)$ is bounded on every finite interval. Together with property 3 of the submartingale problem and the optional stopping theorem, this shows that $S^f(\cdot \wedge \theta^K)$ is a continuous submartingale of class DL. Thus, by the Doob–Meyer decomposition theorem, and the semimartingale decomposition in (4.33), it follows that $A^f(\cdot \wedge \theta^K)$ is a continuous increasing process and $M^f(\cdot \wedge \theta^K)$ is a continuous martingale. Now, consider $f \in \mathcal{C}_c^2(\bar{G})$ that is constant in a neighborhood of \mathcal{V} . For each such f , there exist constants $0 < s < u < \infty$ satisfying $\text{supp}(\nabla f) \cup \bar{G}^K \subseteq \mathcal{U}_{u,s} = \{x \in \partial G : |x| \leq u, d(x, \mathcal{V}) \geq s\}$ and a constant $C > 0$ such that $f + Cf_{u,s} \in \mathcal{H}$, where $f_{u,s}$ is the function in Lemma 4.4. Since $M^g(\cdot \wedge \theta^K \wedge \zeta_c)$ is a continuous martingale for both $g = f_{u,s}$ and $g = f + Cf_{u,s}$, it also a continuous martingale for $g = f \in \mathcal{C}_c^2(\bar{G})$. Finally, for $f \in \mathcal{C}^2(\bar{G})$, since the process $Z(\cdot \wedge \theta^K)$ lives in \bar{G}^K , there exists a function $g \in \mathcal{C}_c^2(\bar{G})$ such that g is constant in a neighborhood of \mathcal{V} and $f = g$ on \bar{G}^K . Thus, $M^f(\cdot \wedge \theta^K) = M^g(\cdot \wedge \theta^K)$ is a continuous martingale for each $f \in \mathcal{C}^2(\bar{G})$.

It only remains to establish the last assertion of the lemma. In the last paragraph, we proved that A^f is increasing almost surely for a fixed f with the stated properties. We now show that it is almost surely true simultaneously for all $f \in \mathcal{H}$. Let \mathcal{H}_0 be the countable dense subset of \mathcal{H} mentioned in Remark 4.10. Since \mathcal{H}_0 is countable, the continuity and the monotonicity of $A^f(\cdot \wedge \theta^K)$ hold almost surely (simultaneously) for all $f \in \mathcal{H}_0$. Now, note that for each $T > 0$, $A^{f_n} - A^f = A^{f-f_n}$ and $\sup_{t \in [0, T]} |A^{f-f_n}(t \wedge \theta^K \wedge \zeta_c)|$, is bounded above by (the sum of the last three terms on the) right-hand side of (4.39) with $f_n - f$ in place of f . Since any $f \in \mathcal{H}$ can be approximated by a sequence $\{f_n\}$ in \mathcal{H}_0 in the strong sense made precise in (4.37), this implies that as $n \rightarrow \infty$, $\sup_{t \in [0, T]} |A^{f_n}(t \wedge \theta^K \wedge \zeta_c) - A^f(t \wedge \theta^K \wedge \zeta_c)|$ converges to 0 both pointwise (i.e., for each $\omega \in \mathcal{C}$) and in $L^1(\mathbb{Q}_z)$. It follows that $A^f(\cdot \wedge \theta^K \wedge \zeta_c)$, and hence $A^f(\cdot \wedge \theta^K)$ is continuous and increasing on $[0, T]$. Since T is arbitrary, the desired property holds almost surely for all $f \in \mathcal{H}$.

We now consider the larger class of functions $f \in \mathcal{C}^2(\mathbb{R}^J)$ such that $\langle \nabla f(x), d \rangle \geq 0$ for all $d \in d(x)$ and $x \in \bar{G}^{2K} \cap \partial G$. Let the quantities $\hat{F}^K \subset \bar{G}^K$, $r_x, \kappa_x(\cdot), x \in \hat{F}^K$, be as introduced prior to Proposition 4.8, and let $\{\varrho_k, k \in \mathbb{N} \cup \{0\}\}$ be the sequence of stopping times introduced in (4.24). We prove by induction that for each $k \in \mathbb{N} \cup \{0\}$, $A^f(\cdot \wedge \varrho_k)$ is continuous and increasing. When $k = 0$, $\varrho_0 = 0$, and the conclusion holds trivially. Now, suppose that $A^f(\cdot \wedge \varrho_k)$ is continuous and increasing for some $k \in \mathbb{N} \cup \{0\}$. We show below that $A^f(\cdot \wedge \varrho_{k+1})$ is also continuous and increasing. Due to the induction assumption, it clearly suffices to show that $D_k \doteq A^f(\cdot \wedge \varrho_{k+1}) - A^f(\cdot \wedge \varrho_k)$ is continuous and increasing. Now, note that

$$D_k(t) = \mathbb{I}_{\{\varrho_k < t \wedge \theta^K\}} \sum_{x \in \hat{F}^K} \mathbb{I}_{\{Z(\varrho_k) \in (\hat{\lambda}^K)^{-1}(x)\}} D_k^x(t),$$

where, for $x \in \hat{F}^K$, $D_k^x(t) = A^f(t \wedge \theta_k^x \wedge \theta^K) - A^f(\varrho_k)$, and θ_k^x is defined as in (4.22), but with ϱ_k in place of ϱ . Now, fix $x \in \hat{F}^K$. Then, by Lemma 4.7, there exists a function $h_x \in \mathcal{H}$ such that $h_x(y) - h_x(x) = f(y) - f(x)$ for each $y \in B_{\kappa_x(r_x)}(x)$. Hence, on the set $\{\varrho_k < \theta^K\}$, $D_k^x(t) = A^{h_x}(t \wedge \theta_k^x \wedge \theta^K) - A^{h_x}(\varrho_k)$. But, this is an increasing process since h_x lies in \mathcal{H} . In turn, this implies D_k^x is nondecreasing, thus proving that $A^f(\cdot \wedge \varrho_{k+1})$ is increasing. By induction, this proves the last assertion of the lemma. \square

The last result will allow us to define a random measure and establish a convenient integral representation for the process A^f .

PROPOSITION 4.12. *For each $\omega \in \mathcal{C}$, there exists a σ -finite measure $\tilde{\mu}(\omega, \cdot)$ on $(\mathbb{R}_+ \times S_1(0), \mathcal{B}(\mathbb{R}_+ \times S_1(0)))$, and there exists a subset $\Omega_0 \subset \mathcal{C}$ with $\mathbb{Q}_z(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, for all $f \in \mathcal{H}$ and $t \geq 0$,*

$$(4.40) \quad A^f(\omega, t \wedge \theta^K(\omega)) = \int_{\mathcal{R}_t(\omega)} \langle v, \nabla f(Z(\omega, u)) \rangle \tilde{\mu}(\omega, du, dv),$$

where, for $t \geq 0$, $\mathcal{R}_t(\omega)$ is the random Borel subset of $[0, t] \times S_1(0)$ given by

$$(4.41) \quad \mathcal{R}_t(\omega) \doteq \{(u, v) \in [0, t] \times S_1(0) : Z(\omega, u) \in \partial G \setminus \mathcal{V}, v \in d(Z(\omega, u))\}.$$

Moreover, for any continuous function $g : \mathbb{R}^J \rightarrow \mathbb{R}^J$, $\int_{\mathcal{R}_t} \langle v, g(Z(u)) \rangle \tilde{\mu}(\cdot, du, dv)$, $t \geq 0$, is a continuous adapted stochastic process starting from 0.

The proof of Proposition 4.12 is long and functional analytic in nature, involving an application of the Hahn–Banach and Riesz representation theorems for random linear functionals. So as not to interrupt the flow of the main construction, we defer the proof to Section 5.

4.5.2. A boundary property of the martingale component. We start in Lemma 4.13 by identifying a family of exponential martingales associated with the random measure $\tilde{\mu}$ from Proposition 4.12, which is used to show that the trace of the quadratic variation of the martingale M vanishes on the boundary in Proposition 4.14, and establish the desired identities (4.31) and (4.32) in Lemma 4.15.

Define $\alpha(0) = 0$ and set

$$(4.42) \quad \alpha(t) \doteq S(t \wedge \theta^K) - \int_{\mathcal{R}_t} v \tilde{\mu}(\cdot, du, dv), \quad t > 0.$$

Applying the last assertion of Proposition 4.12 with $g(x) = \bar{e}^\ell$ for $\ell = 1, \dots, J$, it follows that α is a well-defined continuous adapted stochastic process. Moreover, since $S(\cdot \wedge \theta^K)$ is a semimartingale, (4.42) shows that $\alpha(\cdot)$ is also a continuous semimartingale. We now identify some exponential martingales associated with the process α . Recall the family of stopping times $\{\zeta_c, c > 0\}$ defined by (4.36).

LEMMA 4.13. For each $0 < c < \infty$ and every bounded, $\{\mathcal{M}_t\}$ -adapted process $\{\vartheta(t), t \geq 0\}$,

$$(4.43) \quad \exp\left\{\int_0^{t \wedge \theta^K \wedge \zeta_c} \langle \vartheta(u), d\alpha(u) \rangle - \frac{1}{2} \int_0^{t \wedge \theta^K \wedge \zeta_c} \langle \vartheta(u), a(Z(u))\vartheta(u) \rangle du\right\},$$

$t \geq 0,$

is a continuous martingale.

PROOF. Fix $0 < c < \infty$. We first reduce the proof of the lemma to showing the result for constant $\vartheta(\cdot)$, namely, to showing that for all $\vartheta \in \mathbb{R}^J$,

$$(4.44) \quad \exp\left\{\langle \vartheta, \alpha(t \wedge \theta^K \wedge \zeta_c) \rangle - \frac{1}{2} \int_0^{t \wedge \theta^K \wedge \zeta_c} \langle \vartheta, a(Z(u))\vartheta \rangle du\right\}$$

is a continuous martingale. Indeed, suppose the result holds for constant $v \in \mathbb{R}^J$. Given the local boundedness of a , the nondegeneracy condition (2.11), and the continuity of $\alpha(\cdot \wedge \theta^K)$, it follows that the conditions of Theorem 3.1 (and therefore Theorem 3.2) of [38] are fulfilled with P, ξ and s therein replaced by $\mathbb{Q}_z, \alpha(\cdot \wedge \theta^K \wedge \zeta_c)$ and 0, respectively. Therefore, we can apply part (v) of Theorem 3.2 of [38], with $\xi = \alpha$ and $\theta = \vartheta$, to conclude that for every bounded adapted process $\vartheta(\cdot)$, the process in (4.13) is a continuous martingale.

To show that the process in (4.44) is a continuous martingale, we establish a slightly more general result. Suppose $f \in \mathcal{C}^2(\bar{G})$ is positive. Then for $t \geq 0$ and $\omega \in \mathcal{C}$, define $V^f(t) = V^f(\omega, t)$ to be

$$V^f(t) \doteq \exp\left\{-\int_0^t \frac{\mathcal{L}f(Z(\omega, u))}{f(Z(\omega, u))} du - \int_{\mathcal{R}_t(\omega)} \left\langle v, \frac{\nabla f(Z(\omega, u))}{f(Z(\omega, u))} \right\rangle \tilde{\mu}(\omega, du, dv)\right\},$$

and

$$(4.45) \quad H^f(t) \doteq f(Z(t \wedge \theta^K \wedge \zeta_c))V^f(t \wedge \theta^K \wedge \zeta_c).$$

Applying the last assertion of Proposition 4.12 with $g(x) = \nabla f(x)/f(x)$, it follows that V^f and therefore H^f are well-defined, continuous adapted stochastic processes. We now claim that for any positive $f \in \mathcal{C}^2(\bar{G})$, H^f is a positive continuous martingale starting from $f(z)$. Suppose the claim were true. Then for fixed $\vartheta \in \mathbb{R}^J$, define $f(x) = f^\vartheta(x) = \exp\{\langle \vartheta, x \rangle\}$, $x \in \mathbb{R}^J$. Then f is clearly positive, lies in $\mathcal{C}^2(\bar{G})$ and satisfies

$$\frac{\mathcal{L}f(x)}{f(x)} = \langle \vartheta, b(x) \rangle + \frac{1}{2} \langle \vartheta, a(x)\vartheta \rangle \quad \text{and} \quad \frac{\nabla f(x)}{f(x)} = \vartheta.$$

Substituting this into the definitions of V^f and H^f and recalling the definition of α in (4.42) and of S in (4.2), it is easy to verify that the process in (4.44) is equal to $e^{-\langle \vartheta, Z(0) \rangle} H^f(t)$, $t \geq 0$, and hence, is a continuous martingale.

Thus, it only remains to establish the claim. Fix $f \in \mathcal{C}^2(\bar{G})$ that is positive. To prove the claim, we will first establish the relation

$$(4.46) \quad H^f(t) - f(Z(0)) = N^f(t \wedge \theta^K \wedge \zeta_c), \quad t \geq 0,$$

where

$$(4.47) \quad N^f(t) \doteq M^f(t)V^f(t) - \int_0^t M^f(u) dV^f(u), \quad t \geq 0$$

and then show that $N^f(\cdot \wedge \theta^K \wedge \zeta_c)$ is a continuous martingale. Using the relation (4.33) for M^f in the first and last lines below, the definition of V^f , the representation for A^f in (4.40), and integration-by-parts, we obtain

$$\begin{aligned} & \int_0^t M^f(u \wedge \theta^K \wedge \zeta_c) dV^f(u \wedge \theta^K \wedge \zeta_c) \\ &= \int_0^{t \wedge \theta^K \wedge \zeta_c} \left(f(Z(u)) - f(Z(0)) - \int_0^u \mathcal{L}f(Z(s)) ds - A^f(u) \right) dV^f(u) \\ &= - \int_0^{t \wedge \theta^K \wedge \zeta_c} V^f(u) d \left(\int_0^u \mathcal{L}f(Z(s)) ds + A^f(u) \right) \\ &\quad - \int_0^{t \wedge \theta^K \wedge \zeta_c} \left(f(Z(0)) + \int_0^u \mathcal{L}f(Z(s)) ds + A^f(u) \right) dV^f(u) \\ &= -V^f(t \wedge \theta^K \wedge \zeta_c) \left(\int_0^{t \wedge \theta^K \wedge \zeta_c} \mathcal{L}f(Z(u)) du + A^f(t \wedge \theta^K \wedge \zeta_c) \right) \\ &\quad + \int_0^{t \wedge \theta^K \wedge \zeta_c} \left(\int_0^u \mathcal{L}f(Z(s)) ds + A^f(u) \right) dV^f(u) \\ &\quad - \int_0^{t \wedge \theta^K \wedge \zeta_c} \left(f(Z(0)) + \int_0^u \mathcal{L}f(Z(s)) ds + A^f(u) \right) dV^f(u) \\ &= f(Z(0)) \\ &\quad - V^f(t \wedge \theta^K \wedge \zeta_c) \left(f(Z(0)) + \int_0^{t \wedge \theta^K \wedge \zeta_c} \mathcal{L}f(Z(u)) du \right. \\ &\quad \left. + A^f(t \wedge \theta^K \wedge \zeta_c) \right) \\ &= f(Z(0)) - V^f(t \wedge \theta^K \wedge \zeta_c) (-M^f(t \wedge \theta^K \wedge \zeta_c) + f(Z(t \wedge \theta^K \wedge \zeta_c))). \end{aligned}$$

Together with the definitions of N^f and H^f in (4.47) and (4.45), respectively, this proves (4.46).

We now show that $N^f(\cdot \wedge \theta^K \wedge \zeta_c)$ is a martingale. By Lemma 4.11, $M^f(\cdot \wedge \theta^K \wedge \zeta_c)$ is a continuous martingale that is bounded on every finite interval and $V^f(\cdot \wedge \theta^K \wedge \zeta_c)$ is a continuous finite variation process. If we can show that

$\mathbb{E}[|V^f|(t \wedge \theta^K \wedge \zeta_c)] < \infty$ for every $t > 0$, where recall that $|V^f|(t)$ denotes the total variation of V^f on $[0, t]$, then the desired result will follow from a standard application of Itô's formula [e.g., applying Lemma 2.1 of [39] with $\phi = M^f(\cdot \wedge \theta^K \wedge \zeta_c)$ and $\psi = V^f(\cdot \wedge \theta^K \wedge \zeta_c)$]. Let $C_f < \infty$ be the maximum of the supremum of f and the suprema of its first and second partial derivatives over \bar{G}^K and let $c_f > 0$ with $\inf_{x \in \bar{G}^K} f(x) > c_f$. Choose $f_{r,s}$ from Lemma 4.4, and note that then $\langle v, f_{r,s} \rangle \geq 1$ for and $v \in d(x) \cap S_1(0)$, $x \in \mathcal{U}_{r,s}$. It follows from (4.40) and (4.35) that

$$\begin{aligned} \tilde{\mu}(\omega, \mathcal{R}_{t \wedge \theta^K(\omega) \wedge \zeta_c(\omega)}(\omega)) &\leq A^{f_{r,s}}(\omega, t \wedge \theta^K(\omega) \wedge \zeta_c(\omega)) \\ &\leq \left(cJ + \frac{cJ^2}{2} + \frac{tJ^2}{2} \sum_{i,j=1}^J \sup_{x \in \bar{G}^K} |a_{ij}|(x) \right) C_{f_{r,s}}. \end{aligned}$$

In turn, this implies that for all $t > 0$,

$$\begin{aligned} &\mathbb{E}[|V^f|(t \wedge \theta^K \wedge \zeta_c)] \\ &\leq \exp \left\{ \frac{tJ \sup_{x \in \bar{G}^K} |\mathcal{L}f(x)|}{c_f} \right. \\ &\quad \left. + \frac{JC_f}{c_f} \left(\frac{3cJ^2}{2} + \frac{tJ^2}{2} \sum_{i,j=1}^J \sup_{x \in \bar{G}^K} |a_{ij}|(x) \right) C_{f_{r,s}} \right\} \\ &< \infty, \end{aligned}$$

as desired. This completes the proof of the lemma. \square

PROPOSITION 4.14. *The continuous local martingale $M(\cdot \wedge \theta^K)$ in the decomposition (4.25) for $S(\cdot \wedge \theta^K)$ satisfies almost surely,*

$$(4.48) \quad \int_0^{\theta^K} \mathbb{I}_{\partial G}(Z(u)) d[M_i, M_i](u) = 0, \quad i = 1, \dots, J.$$

PROOF. For each $\vartheta \in \mathbb{R}^J$, choosing $\vartheta(\cdot) = \vartheta \mathbb{I}_{\partial G}(Z(\cdot))$ in Lemma 4.13, we see that for each $c > 0$,

$$\exp \left\{ \left\langle \vartheta, \int_0^{t \wedge \theta^K \wedge \zeta_c} \mathbb{I}_{\partial G}(Z(u)) d\alpha(u) \right\rangle - \frac{1}{2} \int_0^{t \wedge \theta^K \wedge \zeta_c} \mathbb{I}_{\partial G}(Z(u)) \langle \vartheta, a(Z(u)) \vartheta \rangle du \right\}$$

is a continuous martingale. Since almost surely Z spends zero Lebesgue time on the boundary by Proposition 2.12, this implies that for each $\vartheta \in \mathbb{R}^J$ and $t \geq 0$,

$$\mathbb{E}^{\mathbb{Q}_z} \left[\exp \left\{ \left\langle \vartheta, \int_0^{t \wedge \theta^K \wedge \zeta_c} \mathbb{I}_{\partial G}(Z(u)) d\alpha(u) \right\rangle \right\} \right] = 1.$$

Hence, for each $t \geq 0$, we have almost surely, $\int_0^{t \wedge \theta^K \wedge \zeta_c} \mathbb{I}_{\partial G}(Z(u)) d\alpha(u) = 0$, which in turn implies that for each $i = 1, \dots, J$, $\int_0^{\theta^K \wedge \zeta_c} \mathbb{I}_{\partial G}(Z(u)) d[\alpha_i, \alpha_i](u) = 0$. By letting $c \rightarrow \infty$, we have

$$(4.49) \quad \int_0^{\theta^K} \mathbb{I}_{\partial G}(Z(u)) d[\alpha_i, \alpha_i](u) = 0, \quad i = 1, \dots, J.$$

From (4.25) and (4.42), we know that

$$\alpha(t \wedge \theta^K) = M(t \wedge \theta^K) + A(t \wedge \theta^K) - \int_{\mathcal{R}_t} v \tilde{\mu}(du, dv).$$

Since $A(t \wedge \theta^K) - \int_{\mathcal{R}_t} v \tilde{\mu}(du, dv)$ is an adapted process with locally bounded variation, it follows that $[\alpha_i, \alpha_i] = [M_i, M_i]$, and (4.48) follows directly from (4.49). This completes the proof of Proposition 4.14. \square

LEMMA 4.15. *We have almost surely, that (4.31) and (4.32) hold for all $t \geq 0$.*

PROOF. From (4.6) and (4.25), it is clear that (4.32) follows from (4.31). To establish (4.31), let the sequences of stopping times $\zeta_k^m \uparrow \infty, m \in \mathbb{N}$, and $\tau_k^m \uparrow \infty, m \in \mathbb{N}$, be defined by (4.26) and (4.27). We use the fact that almost surely $\zeta_k^m \uparrow \infty$ and $\tau_k^m \uparrow \infty$ as $k \rightarrow \infty$, to conclude that for any $t \geq 0$,

$$(4.50) \quad M(t \wedge \theta^K) = M^{1,m}(t) + M^{2,m}(t), \quad m \in \mathbb{N},$$

where for $m \in \mathbb{N}$,

$$M^{1,m}(t) \doteq \sum_{k \in \mathbb{N}} [M(t \wedge \tau_k^m \wedge \theta^K) - M(t \wedge \zeta_k^m \wedge \theta^K)],$$

$$M^{2,m}(t) \doteq \sum_{k \in \mathbb{N}} [M(t \wedge \zeta_k^m \wedge \theta^K) - M(t \wedge \tau_{k-1}^m \wedge \theta^K)].$$

Now, by (4.28) we have for any $m \in \mathbb{N}$ and $t \geq 0$,

$$M^{1,m}(t) - \int_0^{t \wedge \theta^K} a^{1/2}(Z(u)) dW(u) = - \sum_{k \in \mathbb{N}} \int_{t \wedge \tau_{k-1}^m \wedge \theta^K}^{t \wedge \zeta_k^m \wedge \theta^K} a^{1/2}(Z(u)) dW(u).$$

The last term is a square integrable martingale, with covariation

$$(4.51) \quad \int_0^{t \wedge \theta^K} \mathbb{I}_{\cup_{k \in \mathbb{N}} [\tau_{k-1}^m, \zeta_k^m]}(u) \mathbb{I}_G(Z(u)) a_{ij}(Z(u)) du, \quad i, j = 1, \dots, J.$$

Each integral in (4.51) converges almost surely to zero as $m \rightarrow \infty$ because

$$(4.52) \quad \lim_{m \rightarrow \infty} \mathbb{I}_{\cup_{k \in \mathbb{N}} [\tau_{k-1}^m, \zeta_k^m]}(u) = \mathbb{I}_{\partial G}(Z(u)).$$

Thus, we have shown that for any $t > 0$, as $m \rightarrow \infty$, $M^{1,m}(t)$ converges in $\mathbb{L}^2(\mathbb{Q}_z)$ to $\int_0^{t \wedge \theta^K} a^{1/2}(Z(u)) dW(u)$. In view of (4.50), to complete the proof of (4.31), it

suffices to show that $M^{2,m}(t)$ converges to zero in $\mathbb{L}^2(\mathbb{Q}_z)$, as $m \rightarrow \infty$. Now, for each $i = 1, \dots, J$, by (4.52) and the bounded convergence theorem,

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}_z} [|M^{2,m}(t)|^2] \\ &= \sum_{i=1}^J \lim_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^{t \wedge \theta^K} \mathbb{I}_{\cup_{k \in \mathbb{N}} [\tau_{k-1}^m, \zeta_k^m]}(u) d[M_i, M_i](u) \right] \\ &= \sum_{i=1}^J \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^{t \wedge \theta^K} \mathbb{I}_{\partial G}(Z(u)) d[M_i, M_i](u) \right], \end{aligned}$$

which is identically zero due to Proposition 4.14. This proves the lemma. \square

4.5.3. *Proof of the reflection property.* In this section, we establish the reflection property (4.7). Roughly speaking, this requires establishing that the constraining term pushes in the right directions, as dictated by the reflection vector field. The proof relies on the following simple geometric property.

LEMMA 4.16. *Let Θ be a convex cone with vertex at 0 and let*

$$(4.53) \quad \Lambda \doteq \{v \in \mathbb{R}^J : \langle v, b \rangle \geq 0 \text{ for each } b \in \Theta\}.$$

If there exists $\Upsilon \in \mathbb{R}^J$ such that $\langle v, \Upsilon \rangle \geq 0$ for all $v \in \Lambda$, then $\Upsilon \in \Theta$.

PROOF. We use an argument by contradiction to establish the lemma. Suppose that there exists $\Upsilon \in \mathbb{R}^J \setminus \Theta$ such that $\langle v, \Upsilon \rangle \geq 0$ for all $v \in \Lambda$. Let $P_\Theta : \mathbb{R}^J \rightarrow \Theta$ be the metric projection onto the cone Θ (which assigns to each point $x \in \mathbb{R}^J$ the point on Θ that is closest to x). Since $P_\Theta(\Upsilon) - \Upsilon$ is the inward normal to Θ at $P_\Theta(\Upsilon)$, and Θ is convex and has vertex at the origin, we have

$$\langle P_\Theta(\Upsilon) - \Upsilon, b \rangle \geq 0 \quad \text{for each } b \in \Theta.$$

This implies that $P_\Theta(\Upsilon) - \Upsilon \in \Lambda$, and hence, by the assumed property of Υ , $\langle P_\Theta(\Upsilon) - \Upsilon, \Upsilon \rangle \geq 0$. On the other hand, since P_Θ is nonexpansive and $P_\Theta(\Upsilon) \neq \Upsilon$ because $\Upsilon \notin \Theta$, it follows that $\langle P_\Theta(\Upsilon), \Upsilon \rangle < \langle \Upsilon, \Upsilon \rangle$, which yields a contradiction. \square

We now use this to establish the reflection property.

LEMMA 4.17.

$$\mathbb{Q}_z \left(Y(t \wedge \theta^K) - Y(s \wedge \theta^K) \in \overline{\text{co}} \left[\bigcup_{u \in [s \wedge \theta^K, t \wedge \theta^K]} d(Z(u)) \right] \right) = 1.$$

PROOF. For each $\varepsilon > 0$ and $y \in \mathcal{U}$, by Lemma 4.3 there exists $R_y < \varepsilon$ such that properties 1 and 2 of the lemma hold and further, we can choose constants $r = r_{y,\varepsilon} < R_y < \varepsilon$ and maps $\kappa = \kappa_y$ such that property 3 is also satisfied. By applying Lemma 4.5 with $\hat{G} = \bar{G}^K \cap \mathcal{U}$ and $\mathcal{O}_y = B_{\kappa_y(r_{y,\varepsilon})/2}(y)$, $y \in \bar{G}^K \cap \mathcal{U}$, there exist a finite set $\hat{F}^\varepsilon \subset \bar{G}^K \cap \mathcal{U}$ and a measurable map $\hat{\lambda}^\varepsilon$ from $\bar{G}^K \cap \mathcal{U}$ onto \hat{F}^ε such that Lemma 4.5 holds. Let $\iota_0^\varepsilon = s \wedge \theta^K$ and for each $k \in \mathbb{N}$, recursively define the following two nested sequences of stopping times:

$$\begin{aligned} \varrho_k^\varepsilon &\doteq \inf\{t \geq \iota_{k-1}^\varepsilon : Z(t) \in \partial G\} \wedge \theta^K, \\ \iota_k^\varepsilon &\doteq \inf\{t \geq \varrho_k^\varepsilon : Z(t) \notin B_{\ell_k^\varepsilon}(\bar{y}_k^\varepsilon)\} \wedge \theta^K, \end{aligned}$$

where $\bar{y}_k^\varepsilon \doteq \hat{\lambda}^\varepsilon(Z(\varrho_k^\varepsilon))$ and $\ell_k^\varepsilon = \kappa_{\bar{y}_k^\varepsilon}(r_{\bar{y}_k^\varepsilon,\varepsilon})$. Using an argument exactly analogous to that used prior to Proposition 4.8, it is possible to show that almost surely, $\varrho_k^\varepsilon \rightarrow \theta^K$, $\iota_k^\varepsilon \rightarrow \theta^K$ as $k \rightarrow \infty$.

First, observe that the relations $\int_0^{\theta^K} \mathbb{I}_G(Z(u)) d|A|(u) = 0$ and $A(t \wedge \theta^K) = Y(t \wedge \theta^K)$ established in (4.29) and (4.32), respectively, along with the fact that $Z(t) \in G$ for $t \in \bigcup_{k \in \mathbb{N}} [\iota_{k-1}^\varepsilon, \varrho_k^\varepsilon]$ imply that for every $t \geq 0$,

$$(4.54) \quad Y(t \wedge \varrho_k^\varepsilon) - Y(t \wedge \iota_{k-1}^\varepsilon) = 0, \quad k \in \mathbb{N}.$$

Next, for each $k \in \mathbb{N}$, if $\varrho_k^\varepsilon < \infty$, let

$$\Theta_k^\varepsilon \doteq \overline{\text{co}} \left[\bigcup_{y \in B_{\ell_k^\varepsilon}(\bar{y}_k^\varepsilon)} d(y) \right],$$

with \bar{y}_k^ε and ℓ_k^ε as defined above, and otherwise, let $\Theta_k^\varepsilon \doteq \{0\}$. We now show that for every $\varepsilon > 0$, and $k \in \mathbb{N}$,

$$(4.55) \quad \mathbb{Q}_z(Y(t \wedge \iota_k^\varepsilon) - Y(t \wedge \varrho_k^\varepsilon) \in \Theta_k^\varepsilon) = 1.$$

Observe that

$$\begin{aligned} &\{Y(t \wedge \iota_k^\varepsilon) - Y(t \wedge \varrho_k^\varepsilon) \in \Theta_k^\varepsilon\} \\ (4.56) \quad &= \{\varrho_k^\varepsilon \geq t \wedge \theta^K, 0 \in \Theta_k^\varepsilon\} \\ &\cup \left[\{\varrho_k^\varepsilon < t \wedge \theta^K\} \cap \bigcup_{x \in \hat{F}^\varepsilon} [\{Z(\varrho_k^\varepsilon) \in (\hat{\lambda}^\varepsilon)^{-1}(x)\} \cap \mathcal{D}_k^{\varepsilon,x}] \right], \end{aligned}$$

where, on the set $\{\varrho_k^\varepsilon < \infty\}$, we define

$$\mathcal{D}_k^{\varepsilon,x} \doteq \left\{ Y(t \wedge \iota_k^\varepsilon) - Y(\varrho_k^\varepsilon) \in \overline{\text{co}} \left[\bigcup_{y \in B_{\kappa_x(r_{x,\varepsilon})}(x)} d(y) \right] \right\}.$$

For $x \in \hat{F}^\varepsilon$, let Λ_x^ε be the set defined in (4.53) with $\overline{\text{co}}[\bigcup_{y \in B_{\kappa_x}(r_{x,\varepsilon})(x)} d(y)]$ in place of Θ . For each $v \in \Lambda_x^\varepsilon$, define $g^v(y) \doteq \langle v, y \rangle$, $y \in \mathbb{R}^J$. Then by Lemma 4.16,

$$\mathcal{D}_k^{\varepsilon,x} = \bigcap_{v \in \Lambda_x^\varepsilon} \{g^v(Y(t \wedge \iota_k^\varepsilon)) - g^v(Y(\varrho_k^\varepsilon)) \geq 0\}.$$

Since $g^v \in \mathcal{C}^2(\mathbb{R}^J)$ and $\langle \nabla g^v(y), d \rangle = \langle v, d \rangle \geq 0$ for $d \in d(y)$, $y \in B_{\kappa_x}(r_{x,\varepsilon})(x)$, Lemma 4.7 and Lemma 4.15 together imply that almost surely, for every $v \in \Lambda_x^\varepsilon$, $A^{g^v}(\cdot \wedge \iota_k^\varepsilon) - A^{g^v}(\cdot \wedge \varrho_k^\varepsilon)$ is a continuous increasing process on the set $\{Z(\varrho_k^\varepsilon) \in (\hat{\lambda}^\varepsilon)^{-1}(x)\}$. However, $A^{g^v}(\cdot \wedge \theta^K) = \langle A(\cdot \wedge \theta^K), v \rangle$ by (4.35) and $A(\cdot \wedge \theta^K) = Y(\cdot \wedge \theta^K)$ on $[0, \theta^K]$ by (4.32). Therefore, almost surely, for every $v \in \Lambda_x^\varepsilon$, $\langle v, Y(\cdot \wedge \iota_k^\varepsilon) - Y(\cdot \wedge \varrho_k^\varepsilon) \rangle$ is a continuous increasing process on the set $\{Z(\varrho_k^\varepsilon) \in (\hat{\lambda}^\varepsilon)^{-1}(x)\}$, which shows that $\mathbb{Q}_z(\mathcal{D}_k^{\varepsilon,x} \cap \{Z(\varrho_k^\varepsilon) \in (\hat{\lambda}^\varepsilon)^{-1}(x)\}) = 1$ for $x \in \hat{F}^\varepsilon$. Since \hat{F}^ε is finite, together with (4.56), this implies (4.55).

Now, (4.54) and (4.55) together imply that for every $\varepsilon > 0$,

$$(4.57) \quad \mathbb{Q}_z\left(Y(t \wedge \theta^K) - Y(s \wedge \theta^K) \in \overline{\text{co}}\left[\bigcup_{k \in \mathbb{N}: \varrho_k^\varepsilon < t \wedge \theta^K} \Theta_k^\varepsilon\right]\right) = 1.$$

For each $k \in \mathbb{N}$, $\varepsilon > 0$ and $\omega \in \mathcal{C}$, define

$$\mathcal{E}_k^\varepsilon(\omega) \doteq \bigcup_{u \in [\varrho_k^\varepsilon(\omega), \varrho_{k+1}^\varepsilon(\omega)]} \{y \in \bar{G} : |y - Z(\omega, u)| \leq 4\varepsilon\},$$

and, recalling that $r_{y,\varepsilon} < \varepsilon$, note that

$$\Theta_k^\varepsilon(\omega) \subseteq \overline{\text{co}}\left[\bigcup_{y \in \mathcal{E}_k^\varepsilon(\omega)} d(y)\right].$$

Since the graph $d(\cdot)$ is closed, we have almost surely,

$$\bigcap_{\varepsilon > 0} \bigcup_{k \in \mathbb{N}: \varrho_k^\varepsilon < t \wedge \theta^K} \overline{\text{co}}\left[\bigcup_{y \in \mathcal{E}_k^\varepsilon} d(y)\right] = \overline{\text{co}}\left[\bigcup_{u \in [0, t \wedge \theta^K]} d(Z(u))\right].$$

Thus, sending $\varepsilon \downarrow 0$ on both sides of (4.57), we obtain (4.7). \square

5. Proof of integral representations. This section is devoted to the proof of Proposition 4.12. First, in Section 5.1, we introduce a random positive linear functional, which we use in Section 5.2 to establish a preliminary integral representation for A^f (see Lemma 5.3). The proof of Proposition 4.12 is then given in Section 5.3.

5.1. *A random positive linear functional.* In what follows, let

$$\mathcal{K} \doteq \{(x, v) \in \mathbb{R}^{2J} : x \in \partial G \setminus \mathcal{V}, v \in d(x), |v| = 1\}.$$

For each $f \in \mathcal{H}$, let $h_f : \mathcal{K} \mapsto \mathbb{R}$ be the function given by

$$h_f(x, v) \doteq \langle v, \nabla f(x) \rangle, \quad (x, v) \in \mathcal{K}.$$

Clearly, $h_f \in C_c^1(\mathcal{K})$ for each $f \in \mathcal{H}$. Note that $C_c(\mathbb{R}_+ \times \mathcal{K})$, equipped with the uniform norm, is a separable linear space. Let \mathcal{T}_0 be the linear subspace of $C_c(\mathbb{R}_+ \times \mathcal{K})$ given by

$$\mathcal{T}_0 \doteq \left\{ g \in C_c(\mathbb{R}_+ \times \mathcal{K}) : \begin{aligned} g(u, x, v) &= \sum_{i=1}^n \ell_i(u) h_{f_i}(x, v), \\ n \in \mathbb{N}, f_i \in \mathcal{H}, \ell_i \in C_c(\mathbb{R}_+), i &= 1, \dots, n \end{aligned} \right\}.$$

Now, let $A^f, f \in \mathcal{H}$, be the family of processes defined in (4.35), and recall from Lemma 4.11 that there exists a set $\Omega_0 \in \mathcal{M}$ with $\mathbb{Q}_z(\Omega_0) = 1$ such that for all $\omega \in \Omega_0, f \in \mathcal{H}, t \mapsto A^f(\omega, t \wedge \theta^K(\omega))$ is increasing and the map $f \mapsto A^f(\omega, \cdot \wedge \theta^K(\omega))$ is linear. For each $g \in \mathcal{T}_0$ that has a representation of the form $g(u, x, v) = \sum_{i=1}^n \ell_i(u) h_{f_i}(x, v)$, with $\ell_i \in C_c(\mathbb{R}_+)$ and $f_i \in \mathcal{H}, i = 1, \dots, n$, define

$$(5.1) \quad \Lambda(\omega, g) \doteq \begin{cases} \sum_{i=1}^n \int_0^\infty \ell_i(u) dA^{f_i}(\omega, u \wedge \theta^K(\omega)), & \text{if } \omega \in \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

We will sometimes suppress the dependence of Λ and A^f on ω and simply write $\Lambda(g)$ and $A^f(t \wedge \theta^K)$, respectively.

We will show that Λ is a random positive linear functional on \mathcal{T}_0 , in a sense made precise below.

DEFINITION 5.1. Let X be a topological linear space. A map $\Psi : \Omega \times X \rightarrow \mathbb{R}$ is a random linear functional on X if it satisfies the following two properties:

- (i) $\Psi(\cdot, x)$ is a random variable for each $x \in X$;
- (ii) $\Psi(\omega, \cdot)$ is a linear functional on X for each $\omega \in \Omega$.

The positivity of Λ will be shown with respect to a suitable positive cone. Define

$$\mathcal{P} \doteq \{g \in C_c(\mathbb{R}_+ \times \mathcal{K}) : 0 \leq g(u, x, v) \leq h_f(x, v), (x, v) \in \mathcal{K} \text{ for some } f \in \mathcal{H}\}.$$

Consider the partial order \leq on $C_c(\mathbb{R}_+ \times \mathcal{K})$ defined by $h \leq g$ if $g - h \in \mathcal{P}$.

LEMMA 5.1. *The set \mathcal{P} is a positive cone in $C_c(\mathbb{R}_+ \times \mathcal{K})$. Moreover, for each $g \in C_c(\mathbb{R}_+ \times \mathcal{K})$, there exists $\hat{g} \in \mathcal{T}_0$ such that $g \leq \hat{g}$. Furthermore, if $g \in C_c(\mathbb{R}_+ \times \mathcal{K})$ is nonnegative, then $g \in \mathcal{P}$ and $0 \leq g$.*

PROOF. Note that if $g, \tilde{g} \in \mathcal{P}$, there exist $f, \tilde{f} \in \mathcal{H}$ such that for $(x, v) \in \mathcal{K}$, $0 \leq g(u, x, v) \leq h_f(x, v)$ and $0 \leq \tilde{g}(u, x, v) \leq h_{\tilde{f}}(x, v)$. Hence, by the linearity of the mapping $f \mapsto h_f$, $0 \leq g(u, x, v) + \tilde{g}(u, x, v) \leq h_f(x, v) + h_{\tilde{f}}(x, v) = h_{f+\tilde{f}}(x, v)$, and for $a > 0$, $0 \leq ag(u, x, v) \leq ah_f(x, v) = h_{af}(x, v)$. Thus, $g + \tilde{g} \in \mathcal{P}$ and $ag \in \mathcal{P}$, showing that \mathcal{P} is a positive cone in $\mathcal{C}_c(\mathbb{R}_+ \times \mathcal{K})$.

We now turn to the proof of the second assertion of the lemma. Fix $g \in \mathcal{C}_c(\mathbb{R}_+ \times \mathcal{K})$. Then there exists a compact set $K \subset \mathcal{K}$, an interval $[t_1, t_2) \subset \mathbb{R}_+$ and a constant $0 < C < \infty$ such that $|g(u, x, v)| \leq C\mathbb{I}_{[t_1, t_2)}(u)\mathbb{I}_K(x, v)$ for each $(u, x, v) \in \mathbb{R}_+ \times \mathcal{K}$. Since $\mathcal{K} \cap \mathcal{V} \times \mathbb{R}^J = \emptyset$ and \mathcal{V} is closed, there exist $r, s > 0$ such that

$$(5.2) \quad \{x \in \mathbb{R}^J : (x, v) \in K\} \subseteq \mathcal{U}_{r,s} = \{x \in \partial G : |x| \leq r, d(x, \mathcal{V}) \geq s\}.$$

Now, choose $f_{r,s}$ from Lemma 4.4. Then $f \doteq Cf_{r,s}$ satisfies $|g(u, x, v)| \leq C\mathbb{I}_K(x, v) \leq h_f(x, v) = \langle v, \nabla f(x) \rangle$ for each $(u, x, v) \in \mathbb{R}_+ \times \mathcal{K}$. Let $\ell \in \mathcal{C}_c(\mathbb{R}_+)$ be a function such that $\mathbb{I}_{[t_1, t_2)}(u) \leq \ell(u) \leq 1$ for each $u \in \mathbb{R}_+$, and choose $\hat{g}(u, x, v) = \ell(u)h_f(x, v)$. Then $\hat{g} \in \mathcal{T}_0$ and $0 \leq \hat{g} - g \leq 2\hat{g} \leq 2h_f = h_{2f}$ on \mathcal{K} . Since $2f \in \mathcal{H}$, this shows that $\hat{g} - g \in \mathcal{P}$, and hence, that $g \leq \hat{g}$. Lastly, if $g \in \mathcal{C}_c(\mathbb{R}_+ \times \mathcal{K})$ and $g \geq 0$, the last argument shows that $0 \leq g(u, x, v) \leq C\mathbb{I}_{[t_1, t_2)}(u)\mathbb{I}_K(x, v) \leq \ell(u)h_f(x, v)$ for each $(x, v) \in \mathcal{K}$. This shows that $g \in \mathcal{P}$ and $0 \leq g$. \square

LEMMA 5.2. *The map $\Lambda : \Omega \times \mathcal{T}_0 \rightarrow \mathbb{R}$ in (5.1) defines a random linear functional on \mathcal{T}_0 . Moreover, Λ is positive in the sense that $\Lambda(g) \geq 0$ whenever $g \geq 0$.*

PROOF. For $\omega \notin \Omega_0$, $\Lambda(\omega, \cdot)$ is trivially well defined, and is positive and linear on \mathcal{T}_0 . So, fix $\omega \in \Omega_0$. To show that $\Lambda(\omega, \cdot)$ is well defined, we need to show that if $g \in \mathcal{T}_0$ admits two representations

$$(5.3) \quad \begin{aligned} g(u, x, v) &= \sum_{i=1}^n \ell_i(u)h_{f_i}(x, v) \\ &= \sum_{j=1}^m \tilde{\ell}_j(u)h_{\tilde{f}_j}(x, v), \quad (u, x, v) \in \mathbb{R}_+ \times \mathcal{K}, \end{aligned}$$

with $\ell_i, \tilde{\ell}_j \in \mathcal{C}_c(\mathbb{R}_+)$, $f_i, \tilde{f}_j \in \mathcal{H}$, $i = 1, \dots, n$, $j = 1, \dots, m$, $m, n \in \mathbb{N}$, then

$$(5.4) \quad \sum_{i=1}^n \int_0^\infty \ell_i(u) dA^{f_i}(\omega, u \wedge \theta^K(\omega)) = \sum_{j=1}^m \int_0^\infty \tilde{\ell}_j(u) dA^{\tilde{f}_j}(\omega, u \wedge \theta^K(\omega)).$$

First, note that since $A^{f_i}(\omega, \cdot \wedge \theta^K(\omega))$ and $A^{\tilde{f}_j}(\omega, \cdot \wedge \theta^K(\omega))$ are increasing functions, and ℓ_i and $\tilde{\ell}_j$ are continuous with compact support, each of the integrals in (5.1) is well defined as a Riemann–Stieltjes integral. In fact, since the functions ℓ_i lie in $\mathcal{C}_c(\mathbb{R}_+)$, $i = 1, \dots, n$, they are uniformly continuous and so for each

$\varepsilon > 0$, there exists $h > 0$ such that $|\ell_i(u) - \ell_i(v)| < \varepsilon$ whenever $|u - v| \leq h$, $i = 1, \dots, n, j = 1, \dots, m$. Thus, for T large enough such that $[0, T]$ contains the supports of every $\ell_i, \tilde{\ell}_j, i = 1, \dots, n, j = 1, \dots, m$, we see that the quantity

$$(5.5) \quad \left| \sum_{i=1}^n \int_0^\infty \ell_i(u) dA^{f_i}(u \wedge \theta^K) - \sum_{i=1}^n \sum_{k=0}^\infty \ell_i(kh) (A^{f_i}((kh + h) \wedge \theta^K) - A^{f_i}(kh \wedge \theta^K)) \right|$$

is bounded by $\varepsilon \sum_{i=1}^n A^{f_i}(T \wedge \theta^K)$, and, likewise, the quantity

$$\left| \sum_{j=1}^m \int_0^\infty \tilde{\ell}_j(u) dA^{\tilde{f}_j}(u \wedge \theta^K) - \sum_{j=1}^m \sum_{k=0}^\infty \tilde{\ell}_j(kh) (A^{\tilde{f}_j}((kh + h) \wedge \theta^K) - A^{\tilde{f}_j}(kh \wedge \theta^K)) \right|$$

is bounded above by $\varepsilon \sum_{j=1}^m A^{\tilde{f}_j}(T \wedge \theta^K)$. We now claim that

$$(5.6) \quad \sum_{i=1}^n \ell_i(u) A^{f_i}(\omega, t \wedge \theta^K(\omega)) = \sum_{j=1}^m \tilde{\ell}_j(u) A^{\tilde{f}_j}(\omega, t \wedge \theta^K(\omega)), \quad u, t \geq 0.$$

If the claim (5.6) holds, then in particular,

$$\begin{aligned} & \sum_{i=1}^n \sum_{k=0}^\infty \ell_i(kh) (A^{f_i}(\omega, (kh + h) \wedge \theta^K(\omega)) - A^{f_i}(\omega, kh \wedge \theta^K(\omega))) \\ &= \sum_{j=1}^m \sum_{k=0}^\infty \tilde{\ell}_j(kh) (A^{\tilde{f}_j}(\omega, (kh + h) \wedge \theta^K(\omega)) - A^{\tilde{f}_j}(\omega, kh \wedge \theta^K(\omega))), \end{aligned}$$

which, when combined with the two previous bounds, implies

$$\begin{aligned} & \left| \sum_{i=1}^n \int_0^\infty \ell_i(u) dA^{f_i}(\omega, u \wedge \theta^K(\omega)) - \sum_{j=1}^m \int_0^\infty \tilde{\ell}_j(u) dA^{\tilde{f}_j}(\omega, u \wedge \theta^K(\omega)) \right| \\ & \leq \varepsilon \left(\sum_{i=1}^n A^{f_i}(\omega, T \wedge \theta^K(\omega)) + \sum_{j=1}^m A^{\tilde{f}_j}(\omega, T \wedge \theta^K(\omega)) \right). \end{aligned}$$

Sending $\varepsilon \downarrow 0$, we obtain (5.4).

Thus, to prove (5.4), it suffices to establish (5.6). Define

$$\Delta^u(x) \doteq \sum_{i=1}^n \ell_i(u) f_i(x) - \sum_{j=1}^m \tilde{\ell}_j(u) \tilde{f}_j(x), \quad (u, x) \in \mathbb{R}_+ \times \bar{G}.$$

Due to the linearity of the space \mathcal{H} and of the map $f \mapsto h_f$, (5.3) implies that for each $u \geq 0$, Δ^u lies in \mathcal{H} and $h_{\Delta^u}(x, v) = \langle \nabla \Delta^u(x), v \rangle = 0$ for every $(x, v) \in \mathcal{K}$. In turn, this implies that $A^{\Delta^u}(t \wedge \theta^K)$ and $-A^{\Delta^u}(t \wedge \theta^K) = A^{-\Delta^u}(t \wedge \theta^K)$ are both increasing, and hence $A^{\Delta^u}(t \wedge \theta^K) = 0$ for every $t \geq 0$. By linearity of the mapping $f \mapsto A^f(\cdot \wedge \theta^K)$, this is equivalent to (5.6). Thus, we have shown that $\Lambda(\omega, \cdot)$ is a well-defined functional on \mathcal{T}_0 . The fact that $g \mapsto \Lambda(\omega, g)$ is linear is an immediate consequence of the definition of Λ in (5.1), and the fact that the sum of representations of two functions g, \tilde{g} in \mathcal{T}_0 is a representation for the sum $g + \tilde{g}$. Furthermore, for any $g \in \mathcal{T}_0$, given any representation for g of the form (5.3), each stochastic Riemann–Stieltjes integral $\int_0^\infty \ell_i(u) dA^{f_i}(u \wedge \theta^K)$ is a random variable, and so is its sum. Since Ω_0 is a measurable set, it follows immediately from (5.1) that $\Lambda(\cdot, g)$ is a random variable. Thus, Λ satisfies both properties of Definition 5.1 and is a random linear functional on \mathcal{T}_0 .

We now establish the positivity of Λ . Let $g \in \mathcal{T}_0$ be such that $g \geq 0$. Since $\mathcal{T}_0 \subset \mathcal{C}_c(\mathbb{R}_+ \times \mathcal{K})$, $g \in \mathcal{P}$ by the last assertion of Lemma 5.1. Now, since $g \in \mathcal{T}_0$, it also admits a representation of the form $g(u, x, v) = \sum_{i=1}^n \ell_i(u) h_{f_i}(x, v)$ for $\ell_i \in \mathcal{C}_c(\mathbb{R}_+)$ and $f_i \in \mathcal{H}, i = 1, \dots, n$. For $\omega \notin \Omega_0$, $\Lambda(\omega, \cdot) \equiv 0$. On the other hand, for $\omega \in \Omega_0$ and each $u \geq 0$, $\langle v, \nabla(\sum_{i=1}^n \ell_i(u) f_i)(x) \rangle = \sum_{i=1}^n \ell_i(u) h_{f_i}(x, v) \geq 0$ for each $x \in \partial G \setminus \mathcal{V}, v \in d(x)$ and $|v| = 1$. So $\sum_{i=1}^n \ell_i(u) f_i \in \mathcal{H}$, and hence

$$\sum_{i=1}^n \ell_i(u) (A^{f_i}(\omega, (u + h) \wedge \theta^K(\omega)) - A^{f_i}(\omega, u \wedge \theta^K(\omega))) \geq 0.$$

Together with the approximation (5.5) to the Riemann–Stieltjes integral, this implies that for any $\varepsilon > 0$, $\sum_{i=1}^n \int_0^\infty \ell_i(u) dA^{f_i}(u \wedge \theta^K) \geq -\varepsilon \sum_{i=1}^n A^{f_i}(T \wedge \theta^K)$. Sending ε down to 0, we conclude that for all $\omega \in \Omega_0$,

$$\Lambda(\omega, g) = \sum_{i=1}^n \int_0^\infty \ell_i(u) dA^{f_i}(\omega, u \wedge \theta^K(\omega)) \geq 0.$$

This shows that Λ is positive, and completes the proof of the lemma. \square

5.2. *An integral representation.* We now use the random positive linear functional Λ to show that $A^f(\cdot \wedge \theta^K)$ admits a suitable integrable representation.

LEMMA 5.3. *There exists a unique positive regular Borel measure $\mu(\omega, \cdot)$ on $\mathbb{R}_+ \times \mathcal{K}$ such that for each $f \in \mathcal{H}$ and $t \geq 0$,*

$$(5.7) \quad A^f(\omega, t \wedge \theta^K(\omega)) = \int_{[0, t] \times \mathcal{K}} \langle v, \nabla f(x) \rangle \mu(\omega, du, dx, dv).$$

PROOF. Fix $\omega \in \mathcal{C}$. By Lemma 5.2, $\Lambda(\omega, \cdot)$ is a positive linear functional on \mathcal{T}_0 . Thus, by the positive cone version of the Hahn–Banach theorem for positive linear functionals (see Theorem 2.1 of [2]), $\Lambda(\omega, \cdot)$ can be extended to a positive

linear functional on $\mathcal{C}_c(\mathbb{R}_+ \times \mathcal{K})$, which we denote by $\bar{\Lambda}(\omega, \cdot)$. In turn, an application of the Riesz–Markov–Kakutani representation theorem for positive linear functionals (see Theorem 2.14 of [36]) shows that there exists a unique positive regular Borel measure $\mu(\omega, \cdot)$ on $\mathbb{R}_+ \times \mathcal{K}$ such that

$$(5.8) \quad \bar{\Lambda}(\omega, g) = \int_{\mathbb{R}_+ \times \mathcal{K}} g(u, x, v) \mu(\omega, du, dx, dv)$$

for each $g \in \mathcal{C}_c(\mathbb{R}_+ \times \mathcal{K})$.

Now, for each $t > 0$, let $\{\ell^n, n \in \mathbb{N}\}$ be a sequence of nonnegative functions in $\mathcal{C}_c(\mathbb{R}_+)$ such that $\ell^n \uparrow \mathbb{I}_{[0,t]}$ as $n \rightarrow \infty$. For each $f \in \mathcal{H}$, substituting $g^n(u, x, v) = \ell^n(u)h_f(x, v) \in \mathcal{T}_0$ into both the definition (5.1) and the representation (5.8) of $\bar{\Lambda}$, taking limits as $n \rightarrow \infty$ and invoking the monotone convergence theorem, we obtain (5.7). \square

We now establish some additional properties of the measure $\mu(\omega, \cdot)$. For each $\omega \in \mathcal{C}$, consider the set

$$\bar{\mathcal{K}}(\omega) \doteq \{(u, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2J} : x = Z(\omega, u) \in \partial G \setminus \mathcal{V}, v \in d(x), |v| = 1\},$$

where we have written $Z(\omega, u)$ instead of $Z(u)$ to make clear the dependence of the right-hand side on ω .

LEMMA 5.4. *There exists $\Omega_0 \in \mathcal{M}$ with $\mathbb{Q}_z(\Omega_0) = 1$ such that*

$$(5.9) \quad \mu(\omega, [\mathbb{R}_+ \times \mathcal{K}] \setminus \bar{\mathcal{K}}(\omega)) = 0, \quad \text{for } \omega \in \Omega_0.$$

PROOF. Let $\Omega_0 \in \mathcal{M}$ be the set of full \mathbb{Q}_z -measure such that the relation (4.38) holds. For each $\omega \in \Omega_0$ and $\varepsilon > 0$, let $\zeta_0^\varepsilon(\omega) \doteq 0$ and for each $k \in \mathbb{N}$, let

$$\begin{aligned} \tau_k^\varepsilon(\omega) &\doteq \inf\{t \geq \zeta_{k-1}^\varepsilon(\omega) : Z(\omega, t) \in \partial G\}, \\ \zeta_k^\varepsilon(\omega) &\doteq \inf\{t \geq \tau_k^\varepsilon(\omega) : \text{dist}(Z(\omega, t), Z(\omega, \tau_k^\varepsilon(\omega))) \geq \varepsilon\}. \end{aligned}$$

Note that for some k and ω , we could have $\tau_{k+1}^\varepsilon(\omega) = \zeta_k^\varepsilon(\omega)$. For $\omega \in \Omega_0$, $\varepsilon > 0$ and $k \in \mathbb{N}$, define the sets

$$\mathcal{A}_k^\varepsilon(\omega) \doteq \{(u, x, v) \in [\tau_k^\varepsilon(\omega), \zeta_k^\varepsilon(\omega)] \times \mathcal{K} : \text{dist}(x, Z(\omega, \tau_k^\varepsilon(\omega))) \leq \varepsilon\}$$

and

$$\mathcal{E}_k^\varepsilon(\omega) \doteq \{(u, x, v) \in [\tau_k^\varepsilon(\omega), \zeta_k^\varepsilon(\omega)] \times \mathcal{K} : \text{dist}(x, Z(\omega, \tau_k^\varepsilon(\omega))) > \varepsilon\}.$$

Note that $\bar{\mathcal{K}}(\omega) = \bigcap_{\varepsilon > 0} \bigcup_{k \geq 1} \mathcal{A}_k^\varepsilon(\omega)$. In fact, it is easy to see that $\bar{\mathcal{K}}(\omega) \subset \bigcap_{\varepsilon > 0} \bigcup_{k \geq 1} \mathcal{A}_k^\varepsilon(\omega)$. For the converse, let $(u, x, v) \in \bigcap_{\varepsilon > 0} \bigcup_{k \geq 1} \mathcal{A}_k^\varepsilon(\omega)$. Then for each $\varepsilon > 0$, there exists $k(\varepsilon) \geq 1$ such that $(u, x, v) \in \mathcal{A}_{k(\varepsilon)}^\varepsilon(\omega)$. Thus, $u \in [\tau_{k(\varepsilon)}^\varepsilon(\omega), \zeta_{k(\varepsilon)}^\varepsilon(\omega)]$ and $(x, v) \in \mathcal{K}$ such that $\text{dist}(x, Z(\omega, \tau_{k(\varepsilon)}^\varepsilon(\omega))) \leq \varepsilon$. It follows that $\text{dist}(Z(\omega, u), Z(\omega, \tau_{k(\varepsilon)}^\varepsilon(\omega))) \leq \varepsilon$. and hence $\text{dist}(x, Z(\omega, u)) \leq 2\varepsilon$. Since

$Z(\omega, \tau_{k(\varepsilon)}^\varepsilon(\omega)) \in \partial G$, we have $\text{dist}(Z(\omega, u), \partial G) \leq \varepsilon$. By letting $\varepsilon \downarrow 0$, it follows that $x = Z(\omega, u) \in \partial G$. Since $(x, v) \in \mathcal{K}$, we have $x = Z(\omega, u) \in \partial G \setminus \mathcal{V}$. Thus, $(u, x, v) \in \bar{\mathcal{K}}(\omega)$. This completes the proof of $\bar{\mathcal{K}}(\omega) = \bigcap_{\varepsilon > 0} \bigcup_{k \geq 1} \mathcal{A}_k^\varepsilon(\omega)$.

Observe that for every $\varepsilon > 0$,

$$[\mathbb{R}_+ \times \mathcal{K}] \setminus \left[\bigcup_{k \geq 1} \mathcal{A}_k^\varepsilon(\omega) \right] = \left[\bigcup_{k \geq 1} \mathcal{E}_k^\varepsilon(\omega) \right] \cup \left[\bigcup_{k \geq 0} (\zeta_k^\varepsilon(\omega), \tau_{k+1}^\varepsilon(\omega)) \times \mathcal{K} \right].$$

Thus, to show (5.9), it suffices to show that for each $\omega \in \Omega_0$ and $\varepsilon > 0$,

$$(5.10) \quad \mu(\omega, \mathcal{E}_k^\varepsilon(\omega)) = 0, \quad k \in \mathbb{N},$$

and

$$(5.11) \quad \mu(\omega, (\zeta_k^\varepsilon(\omega), \tau_{k+1}^\varepsilon(\omega)) \times \mathcal{K}) = 0, \quad k \in \mathbb{N}.$$

To establish (5.10) and (5.11), choose $\bar{r}, s > 0$ such that $\bar{G}^K \subseteq \mathcal{U}_{\bar{r}, s} = \{x \in \partial G : |x| \leq \bar{r}, d(x, \mathcal{V}) \geq s\}$ and recall the definition of θ^K in (4.4). Let $f_{\bar{r}, s}$ be the function in Lemma 4.4. Together with (5.7), (4.38) and the continuity of $A^{f_{\bar{r}, s}}(\cdot \wedge \theta^K)$, this shows that

$$\begin{aligned} 0 &= A^{f_{\bar{r}, s}}(\omega, \tau_{k+1}^\varepsilon(\omega) \wedge \theta^K(\omega)) - A^{f_{\bar{r}, s}}(\omega, \zeta_k^\varepsilon(\omega) \wedge \theta^K(\omega)) \\ &= \int_{(\zeta_k^\varepsilon(\omega), \tau_{k+1}^\varepsilon(\omega)) \times \mathcal{K}} \langle v, \nabla f_{\bar{r}, s}(x) \rangle \mu(\omega, du, dx, dv) \\ &\geq \mu(\omega, (\zeta_k^\varepsilon(\omega), \tau_{k+1}^\varepsilon(\omega)) \times \mathcal{K}_{\bar{r}, s}), \end{aligned}$$

where $\mathcal{K}_{\bar{r}, s} = \{(x, v) \in \mathbb{R}^{2J} : x \in \mathcal{U}_{\bar{r}, s}, v \in d(x), |v| = 1\}$. Then we see that $\mu(\omega, (\zeta_k^\varepsilon(\omega), \tau_{k+1}^\varepsilon(\omega)) \times \mathcal{K}_{\bar{r}, s}) = 0$ for each such $\bar{r}, s > 0$. Note that $\mathcal{K}_{\bar{r}, s} \uparrow \mathcal{K}$ as $\bar{r} \rightarrow \infty$ and $s \rightarrow 0$. This proves (5.11). To show (5.10), by (5.7) again, for each $f \in \mathcal{H}$ we have

$$(5.12) \quad \begin{aligned} &A^f(\omega, \zeta_k^\varepsilon(\omega) \wedge \theta^K(\omega)) - A^f(\omega, \tau_k^\varepsilon(\omega) \wedge \theta^K(\omega)) \\ &= \int_{[\tau_k^\varepsilon(\omega), \zeta_k^\varepsilon(\omega)] \times \mathcal{K}} \langle v, \nabla f(x) \rangle \mu(\omega, du, dx, dv). \end{aligned}$$

Note that for each $f \in \mathcal{H}$ with support outside $B_\varepsilon(Z(\omega, \tau_k^\varepsilon(\omega)))$, by the definition of A^f in (4.35), we have

$$A^f(\omega, \zeta_k^\varepsilon(\omega) \wedge \theta^K(\omega)) - A^f(\omega, \tau_k^\varepsilon(\omega) \wedge \theta^K(\omega)) = 0.$$

Thus, by running over all $f \in \mathcal{H}$ with support outside $B_\varepsilon(Z(\omega, \tau_k^\varepsilon(\omega)))$, we have that (5.10) holds. This completes the proof of the lemma. \square

5.3. *Proof of Proposition 4.12.* We now present the proof of Proposition 4.12. Note that as a consequence of (5.7) and Lemma 5.4, for every $\omega \in \Omega_0$, we have

$$A^f(\omega, t \wedge \theta^K(\omega)) = \int_{[0,t] \times \mathcal{K} \cap \bar{\mathcal{K}}(\omega)} \langle v, \nabla f(x) \rangle \mu(\omega, du, dx, dv), \quad f \in \mathcal{H}.$$

Define a random measure $\tilde{\mu} : \mathcal{C} \times \mathbb{R}_+ \times \mathcal{K} \mapsto \mathbb{R}$ as follows:

$$\tilde{\mu}(\omega, du, dv) \doteq \begin{cases} \mathbb{I}_{\{x=Z(\omega,u) \in \partial G \setminus \mathcal{V}\}} \mu(\omega, du, dx, dv), & \text{if } \omega \in \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $\omega \in \Omega_0$, clearly (4.40) holds. Since $\mu(\omega, \cdot)$, and hence $\tilde{\mu}(\omega, \cdot)$ is a Borel measure by Lemma 5.3, it follows that for each compact set $K \subset \mathbb{R}_+ \times \mathcal{K}$, $\tilde{\mu}(\omega, K) < \infty$ and, therefore, $\tilde{\mu}(\omega, \cdot)$ is σ -finite. Since $\mathbb{Q}_z(\Omega_0) = 1$, this proves the first part of Proposition 4.12.

Next, note that $Z(\omega, \cdot \wedge \theta^K(\omega))$ lives in \bar{G}^K . For each $f \in \mathcal{C}_c^2(\bar{G})$ that is constant in a neighborhood of \mathcal{V} , there exists a constant $C > 0$ and, by Lemma 4.4, a function $f_{\bar{r},s} \in \mathcal{H}$ such that $\langle v, \nabla f_{\bar{r},s}(x) \rangle \geq 1$ for each $x \in \partial G \cap \bar{G}^K$, $v \in d(x)$ and $|v| = 1$ such that $f + Cf_{\bar{r},s} \in \mathcal{H}$. Since (4.40) holds for both $f_{\bar{r},s}$ and $f + Cf_{\bar{r},s}$, then (4.40) holds for f . For any function $f \in \mathcal{C}^2(\bar{G})$, there exists a function $g \in \mathcal{C}_c^2(\bar{G})$ such that g is constant in a neighborhood of \mathcal{V} and $f = g$ on \bar{G}^K . Since $Z(\omega, \cdot \wedge \theta^K(\omega))$ lives in \bar{G}^K , (4.40) also holds for each $f \in \mathcal{C}^2(\bar{G})$. It follows that for each $t \geq 0$ and $f \in \mathcal{C}^2(\bar{G})$ that is uniformly positive,

$$\begin{aligned} & \int_0^t \int_{d(Z(\omega,u)) \cap \mathcal{S}_1(0)} \mathbb{I}_{\{Z(\omega,u) \in \partial G \setminus \mathcal{V}\}} \left\langle v, \frac{\nabla f(Z(\omega,u))}{f(Z(\omega,u))} \right\rangle \tilde{\mu}(\omega, du, dv) \\ &= \begin{cases} \int_0^t \frac{1}{f(Z(\omega,u))} dA^f(\omega, u \wedge \theta^K(\omega)), & \text{if } \omega \in \Omega_0, \\ 0, & \text{if } \omega \notin \Omega_0 \end{cases} \end{aligned}$$

as a function of ω is a random variable. Moreover, since it follows from the definition given in (4.35) that A^f is an adapted continuous process of bounded variation, it follows that the integral on the left-hand side of the above equality is also an adapted continuous process.

Let $\vartheta \in \mathbb{R}^J$. Choose $f(x) = \exp\{\langle \vartheta, x \rangle\}$. Then $f \in \mathcal{C}^2(\bar{G})$ is uniformly positive. Simple calculations yield that $\frac{\nabla f(x)}{f(x)} = \vartheta$, then by substituting $f(x) = \exp\{\langle \vartheta, x \rangle\}$ into the previous display and recalling the definition of \mathcal{R}_t from (4.41), we have $\langle \vartheta, \int_{\mathcal{R}_t} v \tilde{\mu}(\cdot, du, dv) \rangle$, $t \geq 0$, is a one-dimensional continuous adapted stochastic process starting from 0. Since ϑ is arbitrary, then we have $R^v(t) \doteq \int_{\mathcal{R}_t} v \tilde{\mu}(\cdot, du, dv)$, $t \geq 0$, is a J -dimensional continuous adapted stochastic process starting from 0 and the i th component of $R^v(t)$, denoted by $R_i^v(t)$, is $\int_{\mathcal{R}_t} v_i \tilde{\mu}(\cdot, du, dv)$. Let $g : \mathbb{R}^J \rightarrow \mathbb{R}^J$ be a continuous function and let g_i denote its

i th component. It follows that

$$\begin{aligned} \int_{\mathcal{R}_t} \langle v, g(Z(u)) \rangle \tilde{\mu}(\cdot, du, dv) &= \sum_{i=1}^J \int_{\mathcal{R}_t} v_i g_i(Z(u)) \tilde{\mu}(\cdot, du, dv) \\ &= \sum_{i=1}^J \int_0^t g_i(Z(u)) dR_i^v(u). \end{aligned}$$

Hence, $\int_{\mathcal{R}_t} \langle v, g(Z(u)) \rangle \tilde{\mu}(\cdot, du, dv)$, $t \geq 0$, is also a continuous adapted process starting from 0. This completes the proof of Proposition 4.12.

6. Proof of Theorem 2. Given $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$, suppose the associated set \mathcal{V} is the union of finitely many closed connected sets, and suppose that for each $z \in \bar{G}$, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, \mathbb{P}_z , (Z, W) satisfies properties 1–3 of Definition 2.4 for the associated SDER with initial condition z , and let \mathbb{Q}_z be the law of Z induced by \mathbb{P}_z on the canonical filtered probability space $(\mathcal{C}, \mathcal{M}, \{\mathcal{M}_t\})$. Fix $z \in \bar{G}$. Then the definition of the ESP implies that \mathbb{Q}_z satisfies properties 1 and 2 of the submartingale problem associated with $(G, d(\cdot))$, b and σ . We now show that \mathbb{Q}_z also satisfies property 3 of the submartingale problem. Fix $f \in \mathcal{H}$, and for some $\bar{L} \in \mathbb{N}$, let $\mathcal{V} = \bigcup_{i=1}^{\bar{L}} \mathcal{V}_i$ be the unique decomposition of \mathcal{V} into a finite union of its connected components, each of which is closed. Any $f \in \mathcal{H}$ is the sum of a constant and a function \tilde{f} , where \tilde{f} has compact support and is constant in a neighborhood of every point in \mathcal{V} . The set $\mathcal{V}_i \cap \text{supp}[\tilde{f}]$ is compact for every $i = 1, \dots, \bar{L}$, and hence, a standard covering argument shows that there exists $\varepsilon > 0$ such that for each $i = 1, \dots, \bar{L}$, f is constant on $B_\varepsilon(\mathcal{V}_i) \cap \bar{G}$. We assume without loss of generality that ε is smaller than the minimum distance between any two closed sets \mathcal{V}_i and \mathcal{V}_j , $i, j = 1, \dots, \bar{L}$, $i \neq j$.

Now, define $\iota_0 \doteq 0$ and for $k \in \mathbb{N}$, let

$$\begin{aligned} \varrho_k &\doteq \inf\{t > \iota_{k-1} : Z(t) \in \bar{B}_{\varepsilon/2}(\mathcal{V})\}, \\ \iota_k &\doteq \inf\{t > \varrho_k : Z(t) \notin B_\varepsilon(\mathcal{V})\}, \end{aligned}$$

where, by convention, the infimum over any empty set is taken to be infinity. Since $\bar{B}_{\varepsilon/2}(\mathcal{V})$ and $(B_\varepsilon(\mathcal{V}))^c$ are closed sets, ι_k and ϱ_k are $\{\mathcal{F}_t\}$ -stopping times. Since the process Z is continuous, almost surely, $\iota_k, \varrho_k \rightarrow \infty$ as $k \rightarrow \infty$. For $t \in [0, \infty)$,

$$\begin{aligned} (6.1) \quad f(Z(t)) - f(Z(0)) &= \sum_{k=1}^{\infty} [\mathbb{I}_{\{\iota_{k-1} \leq t\}} (f(Z(t \wedge \varrho_k)) - f(Z(\iota_{k-1}))) \\ &\quad + \mathbb{I}_{\{\varrho_k \leq t\}} (f(Z(t \wedge \iota_k)) - f(Z(\varrho_k)))] \\ &= \sum_{k=1}^{\infty} [\mathbb{I}_{\{\iota_{k-1} \leq t\}} (f(Z(t \wedge \varrho_k)) - f(Z(\iota_{k-1})))], \end{aligned}$$

where the last equality holds because f is constant on each $B_\varepsilon(\mathcal{V}_i)$ and the continuity of Z implies that almost surely, Z lies in the ε -neighborhood of exactly one connected component of \mathcal{V} during each interval $[\varrho_k, \iota_k)$. Fix $k \in \mathbb{N}$. Now, (Z, Y) is a solution to the ESP for X , where X is defined by (2.5), and $Z(t) \notin \mathcal{V}$ for $t \in [\iota_{k-1}, \varrho_k]$. Therefore, on the set $\{\varrho_k < t\}$, applying Lemma 2.7 with $\theta_1 = \iota_{k-1}$, $\theta_2 = \varrho_k$, $s = 0$ and t replaced by $t - \iota_{k-1}$, and defining $\tilde{Y}^k(u) \doteq Y((\iota_{k-1} + u) \wedge \varrho_k) - Y(\iota_{k-1})$ for $u \in [0, \infty)$, it follows that there exists a measurable function $\gamma^k : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^J$ such that

$$\begin{aligned} Y(t \wedge \varrho_k) - Y(\iota_{k-1}) &= \tilde{Y}^k(t - \iota_{k-1}) - \tilde{Y}^k(0) \\ &= \int_0^{(t-\iota_{k-1}) \wedge (\varrho_k - \iota_{k-1})} \gamma^k(u) d|\tilde{Y}^k|(u), \end{aligned}$$

where $\gamma^k(u) \in d(Z(\iota_{k-1} + u))$ for $d|\tilde{Y}^k|$ almost every u [where we have replaced $d(Z((\iota_{k-1} + u) \wedge \varrho_k))$ by $d(Z(\iota_{k-1} + u))$ because $d|\tilde{Y}^k|(u) = 0$ for $u > \varrho_k - \iota_{k-1}$]. In turn, this implies that the process $Z(\cdot \wedge \varrho_k) - Z(\iota_{k-1})$ admits the following semimartingale decomposition: for $t \geq \iota_{k-1}$,

$$\begin{aligned} Z(t \wedge \varrho_k) - Z(\iota_{k-1}) &= \int_{\iota_{k-1}}^{t \wedge \varrho_k} b(Z(u)) du + \int_{\iota_{k-1}}^{t \wedge \varrho_k} \sigma(Z(u)) dW(u) \\ &\quad + \int_0^{(t-\iota_{k-1}) \wedge (\varrho_k - \iota_{k-1})} \gamma^k(u) d|\tilde{Y}^k|(u), \end{aligned}$$

and by Itô’s formula, on the set $\{\iota_{k-1} \leq t\}$ we have

$$\begin{aligned} f(Z(t \wedge \varrho_k)) - f(Z(\iota_{k-1})) &= \int_{\iota_{k-1}}^{t \wedge \varrho_k} \mathcal{L}f(Z(u)) du + \int_{\iota_{k-1}}^{t \wedge \varrho_k} \langle \nabla f(Z(u)), \sigma(Z(u)) dW(u) \rangle \\ &\quad + \int_0^{(t-\iota_{k-1}) \wedge (\varrho_k - \iota_{k-1})} \langle \nabla f(Z(\iota_{k-1} + u)), \gamma^k(u) d|\tilde{Y}^k|(u) \rangle. \end{aligned}$$

Multiplying both sides of the last display by $\mathbb{I}_{\{\iota_{k-1} \leq t\}}$, summing over $k \in \mathbb{N}$ and observing that ∇f and $\mathcal{L}f$ are identically zero on $B_\varepsilon(\mathcal{V})$ because f is constant on each connected component of \mathcal{V} , we have the equalities

$$\begin{aligned} \sum_{k=1}^\infty \mathbb{I}_{\{\iota_{k-1} \leq t\}} \int_{\iota_{k-1}}^{t \wedge \varrho_k} \langle \nabla f(Z(u)), \sigma(Z(u)) dW(u) \rangle \\ = \int_0^t \langle \nabla f(Z(u)), \sigma(Z(u)) dW(u) \rangle \end{aligned}$$

and, likewise,

$$\sum_{k=1}^\infty \mathbb{I}_{\{\iota_{k-1} \leq t\}} \int_{\iota_{k-1}}^{t \wedge \varrho_k} \mathcal{L}f(Z(u)) du = \int_0^t \mathcal{L}f(Z(u)) du.$$

Combining the last three displays with (6.1), we conclude that \mathbb{P}_z -almost surely, for every $t \geq 0$, $S^f(t) \doteq f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{L}f(Z(u)) du$ is equal to

$$\int_0^t \langle \nabla f(Z(u)), \sigma(Z(u)) dW(u) \rangle + \sum_{k=1}^{\infty} \mathbb{I}_{\{\tau_{k-1} \leq t\}} \int_0^{(t-\tau_{k-1}) \wedge (\varrho_k - \tau_{k-1})} \langle \nabla f(Z(\tau_{k-1} + u)), \gamma^k(u) d|\tilde{Y}^k|(u) \rangle.$$

Since $f \in \mathcal{H}$, $\gamma^k(u) \in d(Z(\tau_{k-1} + u))$ for $d|\tilde{Y}^k|$ almost every u , the second term on the right-hand side is almost surely nondecreasing, whereas the local boundedness of σ and the fact that f has compact support shows that the first term on the right-hand side is a martingale. This implies that the process described by the right-hand side and, therefore, the left-hand side, is a submartingale, and hence, shows that \mathbb{Q}_z satisfies property 3 of the submartingale problem. Next, if the triplet $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, $\mathbb{P}_z, (Z, W)$ is actually a weak solution (so that property 4 of Definition 2.4 also holds) then \mathbb{Q}_z clearly also satisfies property 4 of Definition 2.9, and so \mathbb{Q}_z a solution to the submartingale problem. This completes the proof of the first assertion of Theorem 2. The second assertion follows immediately from the first.

APPENDIX A: PROOF OF A MEASURABILITY PROPERTY FOR THE ESP

We now establish Lemma 2.7. Let the processes X, Y, Z , and the stopping times θ_1, θ_2 be as in Lemma 2.7, let the stopped shifted processes \tilde{Y} and \tilde{Z} be defined as in (2.8) and (2.9), respectively, and also define the corresponding process $\tilde{X}(u) = Z(\theta_1) + X((u + \theta_1) \wedge \theta_2) - X(\theta_1), u \in [0, \infty)$. Also, given any \mathbb{R}^J -valued process H , recall that $|H|(u)$ represents the total variation of H on $[0, u]$. It follows from Lemma 2.3 of [32] that \mathbb{P}_z -almost surely on the set $A = \{\theta_1 < \theta_2\}$, (\tilde{Z}, \tilde{Y}) satisfies the ESP for \tilde{X} . On the other hand, Theorem 2.9 of [32] shows that for $\omega \in A$ such that $\tilde{Z}(\omega, s) \notin \mathcal{V}$ for all $s \in [0, \theta_2(\omega) - \theta_1(\omega)]$ (which should be interpreted as $s \in [0, \infty)$ when $\theta_2(\omega) = \infty$), the total variation $|\tilde{Y}|$ of \tilde{Y} is finite on every bounded interval $[0, t], t < \infty$. It then follows from property 2 of Theorem 1.3 of [32] (which is restated in Remark 2.3 of this paper) that for each ω , one can find a Borel measurable function $\gamma(\omega, \cdot)$ on $[0, \infty)$, with the desired properties stated in (2.10). However, to prove the assertion in Lemma 2.7, we need to show the existence of a version of γ that is jointly measurable in $\Omega \times [0, \infty)$.

To show this, for each $N \in \mathbb{N}$, we define the stopping time

$$\tau^N \doteq \inf\{t \geq 0 : |\tilde{Y}|(t) \geq N\},$$

let $\theta^N \doteq \tau^N \wedge \theta_2$, and let $\tilde{Y}^{\theta_2}(\cdot)$ and $\tilde{Y}^{\theta^N}(\cdot)$, respectively, be the stopped processes $\tilde{Y}(\cdot \wedge \theta_2)$ and $\tilde{Y}(\cdot \wedge \theta^N) = \tilde{Y}(\cdot \wedge \theta_2 \wedge \tau^N)$.

Consider the measure $\bar{\mu}^N$ on $(\Omega \times \mathbb{R}_+, \mathcal{F} \times \mathcal{B}(\mathbb{R}_+))$ defined by

$$\begin{aligned} \bar{\mu}^N(A \times (s, t]) &\doteq \mathbb{E}^{\mathbb{P}_z}[\mathbb{I}_A(|\tilde{Y}|(t \wedge \theta^N) - |\tilde{Y}|(s \wedge \theta^N))] \\ \text{(A.1)} \qquad &= \mathbb{E}^{\mathbb{P}_z} \left[\mathbb{I}_A \int_{(s,t]} d|\tilde{Y}^{\theta^N}|(u) \right] \end{aligned}$$

for $A \in \mathcal{F}$ and $0 \leq s < t < \infty$. Since $|\tilde{Y}|$ is almost surely nondecreasing, the definition of τ^N implies $\bar{\mu}^N(\Omega \times \mathbb{R}_+) \leq N$, and thus, each $\bar{\mu}^N$ is a finite measure. In an analogous fashion, for each $i = 1, \dots, J$, define μ_i^N to be the finite signed measure on $(\Omega \times \mathbb{R}_+, \mathcal{F} \times \mathcal{B}(\mathbb{R}_+))$ that satisfies

$$(A.2) \quad \begin{aligned} \mu_i^N(A \times (s, t]) &\doteq \mathbb{E}^{\mathbb{P}_z}[\mathbb{I}_A(\tilde{Y}_i(t \wedge \theta^N) - \tilde{Y}_i(s \wedge \theta^N))], \\ A \in \mathcal{F}, 0 \leq s < t < \infty. \end{aligned}$$

From (A.1) and (A.2), it is clear that $\mu_i^N \ll \bar{\mu}^N$ for $i = 1, \dots, J$. Let μ^N denote the J -dimensional vector of finite signed measures whose i th entry is μ_i^N . By the Radon–Nikodým theorem, there exists a measurable function $\gamma^N : (\Omega \times \mathbb{R}_+, \mathcal{F} \times \mathcal{B}(\mathbb{R}_+)) \mapsto (\mathbb{R}^J, \mathcal{B}(\mathbb{R}^J))$ such that

$$(A.3) \quad \begin{aligned} \mu^N(A \times (s, t]) &= \int_{A \times (s, t]} \gamma^N(\omega, u) d\bar{\mu}^N(\omega, u), \\ A \in \mathcal{F}, 0 \leq s < t < \infty. \end{aligned}$$

Moreover, it is also clear that $\gamma^N(\omega, u) = \gamma^N(\omega, u \wedge \tau^N(\omega))$ for each $\omega \in \Omega, 0 \leq u < \infty$.

Now, from (A.2), (A.3) and (A.1), it follows that for each random variable ξ and measurable function h defined on \mathbb{R}_+ :

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_z} \left[\xi \int_{[0, \infty)} h(u) d\tilde{Y}^{\theta^N}(u) \right] &= \int_{\Omega \times \mathbb{R}_+} \xi(\omega) h(u) d\mu^N(\omega, u) \\ &= \int_{\Omega \times \mathbb{R}_+} \xi(\omega) h(u) \gamma^N(\omega, u) d\bar{\mu}^N(\omega, u) \\ &= \mathbb{E}^{\mathbb{P}_z} \left[\xi(\cdot) \int_{[0, \infty)} h(u) \gamma^N(\cdot, u) d|\tilde{Y}^{\theta^N}|(u) \right]. \end{aligned}$$

Hence, for each $0 \leq s < t < \infty$, since the above display holds for each ξ , by choosing $h(u) = \mathbb{I}_{(s, t]}(u)$, we see that \mathbb{P}_z -almost surely,

$$\tilde{Y}(t \wedge \theta^N) - \tilde{Y}(s \wedge \theta^N) = \int_{(s, t]} \gamma^N(\cdot, u) d|\tilde{Y}^{\theta^N}|(u),$$

and the continuity of \tilde{Y} implies that \mathbb{P}_z -almost surely,

$$(A.4) \quad \tilde{Y}(t \wedge \theta^N) - \tilde{Y}(s \wedge \theta^N) = \int_{[s, t]} \gamma^N(\cdot, u) d|\tilde{Y}^{\theta^N}|(u), \quad 0 \leq s < t < \infty.$$

In turn, this shows that for \mathbb{P}_z -almost every ω , $\gamma^N(\omega, \cdot)$ is a version of the Radon–Nikodým derivative of $d\tilde{Y}^{\theta^N}(\omega, \cdot)$ with respect to $d|\tilde{Y}^{\theta^N}|(\omega, \cdot)$.

We now show that the sequence $\gamma^N, N \in \mathbb{N}$, is consistent in the sense that for $N \in \mathbb{N}$,

$$(A.5) \quad \gamma^{N+1}(\cdot, u \wedge \tau^N) = \gamma^N(\cdot, u) \quad \text{for } d|\tilde{Y}^{\theta_2}| \text{-a.e. } u \in [0, \infty).$$

Indeed, first note that for each $N \in \mathbb{N}$ and $0 \leq s < t < \infty$, (A.4), with N replaced by $N + 1$, t replaced by $t \wedge \tau^N$ and s replaced by $s \wedge \tau^N$, yields

$$\tilde{Y}(t \wedge \tau^N \wedge \theta^{N+1}) - \tilde{Y}(s \wedge \tau^N \wedge \theta^{N+1}) = \int_{[s \wedge \tau^N, t \wedge \tau^N]} \gamma^{N+1}(\cdot, u) d|\tilde{Y}^{\theta^{N+1}}|(u),$$

which can be equivalently rewritten as

$$\tilde{Y}(t \wedge \theta^N) - \tilde{Y}(s \wedge \theta^N) = \int_{[s, t]} \gamma^{N+1}(\cdot, u \wedge \tau^N) d|\tilde{Y}^{\theta^N}|(u).$$

A comparison with (A.4) shows that

$$\gamma^{N+1}(\cdot, u \wedge \tau^N) = \gamma^N(\cdot, u) \quad \text{for } d|\tilde{Y}^{\theta^N}| \text{-a.e. } u \in [0, \tau^N],$$

which is equivalent to (A.5). Next, define

$$\tilde{\gamma}(\omega, u) = \limsup_{N \rightarrow \infty} \gamma^N(\omega, u), \quad (\omega, u) \in \Omega \times [0, \infty).$$

It follows that $\tilde{\gamma}$ is a measurable function from $(\Omega \times \mathbb{R}_+, \mathcal{F} \times \mathcal{B}(\mathbb{R}_+))$ to $(\mathbb{R}^J, \mathcal{B}(\mathbb{R}^J))$, and \mathbb{P}_z -almost surely, $\tilde{\gamma}(\cdot, u \wedge \tau^N) = \gamma^N(\cdot, u)$ for $d|\tilde{Y}^{\theta^2}|$ -a.e. $u \in [0, \infty)$ and for each $0 \leq s < t < \infty$,

$$\begin{aligned} \tilde{Y}(t \wedge \theta_2 \wedge \tau^N) - \tilde{Y}(s \wedge \theta_2 \wedge \tau^N) &= \int_{[s, t]} \tilde{\gamma}(\cdot, u \wedge \tau^N) d|\tilde{Y}^{\theta_2 \wedge \tau^N}|(u) \\ &= \int_{[s \wedge \tau^N, t \wedge \tau^N]} \tilde{\gamma}(\cdot, u) d|\tilde{Y}^{\theta_2}|(u). \end{aligned}$$

Sending $N \rightarrow \infty$ and using the continuity of \tilde{Y} and the fact that $\tau^N \rightarrow \infty$ because \tilde{Y} has finite variation on every bounded interval, we have \mathbb{P}_z -almost surely, for each $0 \leq s < t < \infty$,

$$\begin{aligned} \tilde{Y}(t) - \tilde{Y}(s) &= \tilde{Y}(t \wedge \theta_2) - \tilde{Y}(s \wedge \theta_2) \\ &= \int_{[s, t]} \tilde{\gamma}(\cdot, u) d|\tilde{Y}^{\theta_2}|(u) = \int_{[s, t]} \tilde{\gamma}(\cdot, u) d|\tilde{Y}|(u). \end{aligned}$$

Finally, since the pair (Z, Y) solves the ESP for X , it follows from Remark 2.3 that \mathbb{P}_z -almost surely, $\tilde{\gamma}(u) \in d(\tilde{Z}(u))$ for $d|\tilde{Y}|$ -a.e. $u \in [0, \infty)$. This completes the proof.

APPENDIX B: PROOF OF THE COVERING LEMMA

In this section, we prove Lemma 4.5. Fix a compact subset \hat{G} of $\bar{G} \setminus \mathcal{V}$ and a collection of open sets $\{\mathcal{O}_y, y \in \hat{G}\}$ such that $y \in \mathcal{O}_y$ for all $y \in \hat{G}$, $\mathcal{O}_y \cap \mathcal{V} = \emptyset$ if $y \in \hat{G} \cap \mathcal{U}$ and $\mathcal{O}_y \subset G$ if $y \in \hat{G} \cap G$. Since $\hat{G} \cap \mathcal{U} = \hat{G} \cap \partial G$ is compact, and $\{\mathcal{O}_y, y \in \hat{G} \cap \mathcal{U}\}$ is an open cover of $\hat{G} \cap \mathcal{U}$, there exists a finite set $F \subset \hat{G} \cap \mathcal{U}$ such that $\hat{G} \cap \mathcal{U} \subset \bigcup_{y \in F_1} \mathcal{O}_y$. Enumerating the elements of F as $F = \{y_i, i = 0, \dots, N\}$

for some $N \in \mathbb{N}$, we let $D_0 = \mathcal{O}_0$ and $D_k = \mathcal{O}_{y_k} \setminus (\bigcup_{i=0}^{k-1} \mathcal{O}_{y_i})$ for $k = 1, \dots, N$. Then $\{D_k \cap \hat{G} \cap \mathcal{U}, 1 \leq k \leq N\}$ is a partition of $\hat{G} \cap \mathcal{U}$, and for each $x \in \bigcup_{y \in F} \mathcal{O}_y$ there is a unique index $i = i(x)$ such that $x \in D_i \subseteq \mathcal{O}_{y_i}$. Define $\lambda(x) \doteq y_{i(x)}$ for $x \in \bigcup_{y \in F} \mathcal{O}_y$. Then, since each D_i is Borel measurable, the mapping λ from $\bigcup_{y \in F} \mathcal{O}_y$ onto F is clearly measurable. Moreover, since $\hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y$ is compact, contained in G and $\{\mathcal{O}_y, y \in \hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y\}$ is an open cover of $\hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y$, there exists a finite set $F' \subset \hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y$ such that $\hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y \subset \bigcup_{y \in F'} \mathcal{O}_y$. By a similar construction as for λ , there exists a measurable mapping λ' from $\hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y$ onto F' such that $y \in \mathcal{O}_{\lambda'(y)}$. The mapping $\hat{\lambda}$ from \hat{G} to $\hat{F} \doteq F \cup F'$ that is equal to λ on $\hat{G} \cap \bigcup_{y \in F} \mathcal{O}_y$ and is equal to λ' on $\hat{G} \setminus \bigcup_{y \in F} \mathcal{O}_y$ is clearly measurable and has the desired properties.

APPENDIX C: CONSTRUCTION OF TEST FUNCTIONS

This section is devoted to the proof of Lemma 4.3. The first property of the lemma follows from the upper-semicontinuity of the set function $\mathcal{I}(y) = \{i \in \mathcal{I} : y \in \partial G_i\}$ introduced in Definition 2.11. Since $y \in \mathcal{U}$, property 2 follows from property 1, the definition of \mathcal{U} in (2.2) and the continuity of $n^i(\cdot)$ and $\gamma^i(\cdot)$, which holds by Definition 2.11.

We devote the rest of this section to the proof of property 3. For each $y \in \mathcal{U}$, the family of functions $\{f^{y,r}\}$ will be constructed as suitably smoothed and localized versions of the distance function to a certain cone. The construction is similar in spirit to (although more complicated than) that carried out in [32], Section 6.1, for polyhedral domains. We start by establishing two preliminary results, the first of which paraphrases a result from [32].

LEMMA C.1. *Let Θ be a closed convex cone with vertex at the origin and a boundary that is C^∞ , except possibly at the vertex. Given any closed, convex, compact subset \mathcal{K} of the interior of Θ , constants $0 < \eta < \lambda < \infty$ and $\varepsilon > 0$, there exist $\nu > 0$ and a C^∞ function ℓ on the set*

$$\Lambda \doteq \{x \in \mathbb{R}^J : \eta < \text{dist}(x, \Theta) < \lambda\}$$

that satisfy the following two properties:

1. $\sup_{x \in \Lambda} (|\ell(x) - \text{dist}(x, \Theta)| \vee ||\nabla \ell(x)| - 1|) \leq \varepsilon;$
2. for $p \in \mathcal{K}$ and $x \in \Lambda$, we have $\langle \nabla \ell(x), p \rangle \leq -\nu.$

Moreover, if Θ is a half-space, given any subset \mathcal{K} of Θ , the function $\ell(x) \doteq \text{dist}(x, \Theta)$, $x \in \Lambda$, is a C^2 function on Λ that satisfies property 1 above, and also satisfies property 2 with $\nu = 0$.

PROOF. The function ℓ with the properties stated above can be constructed as a suitable mollification of the distance function to the cone Θ . Indeed, Lemma C.1

can be deduced from the proof of [32], Lemma 6.2, with $g_C, L_{C,\delta_C}, K_C^{\delta_C/3}, \tilde{\eta}_C, \tilde{\lambda}_C$ and $\tilde{\varepsilon}_C$ therein replaced by $\ell(\cdot), \Theta, \mathcal{K}, \eta, \lambda$ and ε , respectively. \square

Fix $y \in \mathcal{U}$. We now construct a certain cone associated with the directions of reflection at y , which will serve as the analog to the cone Θ from Lemma C.1. Define

$$(C.1) \quad K_y \doteq \left\{ - \sum_{i \in \mathcal{I}(y)} a_i d^i(y) : a_i \geq 0, i \in \mathcal{I}(y), \sum_{i \in \mathcal{I}(y)} a_i = 1 \right\},$$

where recall the definitions of $d^i(y)$ and $\mathcal{I}(y)$ given in Definition 2.11. Note that K_y is a convex, compact subset of \mathbb{R}^J . Therefore, there exist $\delta_y > 0$ and a compact, convex set K_{y,δ_y} such that K_{y,δ_y} has C^∞ boundary and satisfies

$$(C.2) \quad K_y^{\delta_y/2} \subset (K_{y,\delta_y})^\circ \subset K_{y,\delta_y} \subset K_y^{\delta_y},$$

where $K_y^\varepsilon \doteq \{x \in \mathbb{R}^J : \text{dist}(x, K_y) \leq \varepsilon\}$ for every $\varepsilon > 0$. Since $y \in \mathcal{U}$, by the definition of \mathcal{U} in (2.2), it is easy to see that $0 \notin K_y$ and

$$\min_{i \in \mathcal{I}(y)} \langle n^i(y), d \rangle < 0 \quad \text{for every } d \in K_y.$$

Therefore, $\delta_y > 0$ can be chosen such that $0 \notin K_{y,\delta_y}$ and

$$(C.3) \quad \min_{i \in \mathcal{I}(y)} \langle n^i(y), d \rangle < 0 \quad \text{for every } d \in K_y^{\delta_y}.$$

For each $i \in \mathcal{I}(y)$, since ∂G_i is C^1 near $y \in \partial G$, the hyperplane $\{x \in \mathbb{R}^J : \langle n^i(y), x - y \rangle = 0\}$ is the tangent plane to ∂G_i at y . Let

$$\mathcal{S}_y \doteq \bigcap_{i \in \mathcal{I}(y)} \{x \in \mathbb{R}^J : \langle n^i(y), x - y \rangle \geq 0\}.$$

Then \bar{G} can be locally approximated near y by the polyhedral cone \mathcal{S}_y in the sense that for each $N < \infty$,

$$(C.4) \quad \left\{ y + \frac{(x - y)}{r} \in \mathbb{R}^J : x \in \bar{G}, |x - y| \leq Nr \right\} \rightarrow \mathcal{S}_y \cap B_N(y) \quad \text{as } r \rightarrow 0,$$

where the convergence is with respect to the Hausdorff distance. In view of (C.10), it follows that there exist

$$(C.5) \quad 0 < r_y < \text{dist}\left(y, \mathcal{V} \cup \bigcup_{i \notin \mathcal{I}(y)} (\partial G \cap \partial G_i)\right)$$

and $\lambda_y \in (0, 1)$ small enough (not depending on r_y) such that for each $r \in (0, r_y)$,

$$(C.6) \quad \left\{ x \in \mathbb{R}^J : \text{dist}\left(x, y + \bigcup_{t \in [0, \bar{R}_y]} t K_{y,\delta_y}\right) \leq 3\lambda_y r \right\} \cap \partial G \subset B_r(y) \cap \partial G$$

and

$$(C.7) \quad \left\{ x \in \mathbb{R}^J : \text{dist}\left(x, y + \bigcup_{t \in [0, \bar{R}_y]} tK_{y, \delta_y}\right) \leq 3\lambda_y r \right\} \cap \bar{G} \cap \partial B_r(y) = \emptyset.$$

Choose $\eta_y \in (0, \lambda_y)$, and define

$$(C.8) \quad \Pi_y \doteq \{x \in \mathbb{R}^J : \eta_y < \text{dist}(x, \Theta) \leq 2\lambda_y\}.$$

Then, it follows from Lemma C.1, with $\Theta = \bigcup_{t \geq 0} tK_{y, \delta_y}$, $\mathcal{K} = K_y^{\delta_y/3}$, $\lambda = 2\lambda_y$, $\eta = \eta_y \in (0, \lambda_y)$, $\Lambda = \Pi_y$, and $\varepsilon_y = \lambda_y/12 \wedge \eta_y/2$, that there exists a constant $\nu_y > 0$ (that does not depend on r_y) and a function $\ell_y : \Pi_y \rightarrow \mathbb{R}$ that satisfy the properties stated in Lemma C.1.

We will construct the family of functions $\{f^{y,r}\}$ as suitably scaled and truncated versions of ℓ_y , whose supports have the desired properties. The construction uses certain properties of K_{y, δ_y} summarized in Lemma C.2.

LEMMA C.2. *For $y \in \mathcal{U}$, there exist $\bar{R}_y \in (0, 1)$ and $\beta_y > 0$ such that*

$$(C.9) \quad \min_{i \in \mathcal{I}(y)} \langle n^i(y), d \rangle < -2\beta_y |d| \quad \text{for every } d \in \bigcup_{t \in [0, \bar{R}_y]} tK_{y, \delta_y}$$

and

$$(C.10) \quad \left(y + \bigcup_{t \in [0, \bar{R}_y]} tK_{y, \delta_y}\right) \cap \bar{G} = \{y\}.$$

PROOF. Fix $y \in \mathcal{U}$. We first use an argument by contradiction to prove that

$$(C.11) \quad \sup_{d \in K_{y, \delta_y}} \min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{d}{|d|} \right\rangle < 0.$$

Since K_{y, δ_y} is compact and $d \mapsto \min_{i \in \mathcal{I}(y)} \langle n^i(y), d/|d| \rangle$ is continuous, the supremum can be replaced by a maximum in (C.11). Thus, if (C.11) does not hold, then there exists $d \in K_{y, \delta_y}$ such that $\min_{i \in \mathcal{I}(y)} \langle n^i(y), d/|d| \rangle \geq 0$. But this contradicts (C.3). Thus, (C.11) holds, which, in particular, implies that there exists $\beta_y > 0$ such that (C.9) holds for any $\bar{R}_y > 0$.

In addition, since for each $i \in \mathcal{I}(y)$, ∂G_i is C^1 near y , it follows that

$$\lim_{\delta \rightarrow 0} \inf_{x \in \bar{G}: |x-y| \leq \delta} \min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{x-y}{|x-y|} \right\rangle \geq 0.$$

Together with (C.11) this shows that there exists $\bar{R}_y \in (0, 1)$ such that

$$(C.12) \quad \inf_{x \in \bar{G}: |x-y| \leq \bar{R}_y (\sum_{i \in \mathcal{I}(y)} |d^i(y)| + \delta_y)} \min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{x-y}{|x-y|} \right\rangle > \sup_{d \in K_{y, \delta_y}} \min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{d}{|d|} \right\rangle.$$

We use this to prove (C.10) by contradiction. Suppose that (C.10) does not hold. Then there exists $d \in \bigcup_{t \in [0, \bar{R}_y]} tK_{y, \delta_y}$ such that $d \neq 0$ and $y + d \in \bar{G}$. We can write $d = t^*d^*$ for some $t^* \in [0, \bar{R}_y]$ and $d^* \in K_{y, \delta_y}$. Also, by the definition of K_{y, δ_y} in (C.1) and (C.2), $|d| \leq \bar{R}_y(\sum_{i \in \mathcal{I}(y)} |d^i(y)| + \delta_y)$. Hence, (C.12) implies that

$$\min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{d^*}{|d^*|} \right\rangle = \min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{d}{|d|} \right\rangle > \sup_{d \in K_{y, \delta_y}} \min_{i \in \mathcal{I}(y)} \left\langle n^i(y), \frac{d}{|d|} \right\rangle,$$

which contradicts the fact that $d^* \in K_{y, \delta_y}$. Thus, (C.10) holds for the chosen $\bar{R}_y \in (0, 1)$. \square

We now introduce some more geometric objects that will allow us to identify a family of sets $O(y, r)$, $r > 0$, where $O(y, r)$ will contain the support of the function $\nabla f^{y, r}$ that we want to construct. Let L_{y, δ_y} be a truncated (half) cone with vertex at the origin defined by

$$L_{y, \delta_y} \doteq \bigcup_{t \in [0, \frac{\bar{R}_y}{2}]} tK_{y, \delta_y}.$$

Then (C.10) and the fact that $0 \in L_{y, \delta_y}$ imply

$$(y + L_{y, \delta_y}) \cap \bar{G} = \{y\}.$$

Due to the fact that $y \in \mathcal{U}$, there exists a vector in the set K_y defined in (C.1) such that $-q_y$ points into G from y and $|q_y| \leq 1$ [here we have used the fact that $d^i(y)$ is a unit vector for every i and y]. For each $r \in (0, 1)$, define

$$(C.13) \quad M(y, r) \doteq y - \lambda_y \frac{\bar{R}_y}{2} r q_y + rL_{y, \delta_y},$$

and observe that, since a δ_y neighborhood of the vector $-\bar{R}_y q_y / 2$ lies in L_{y, δ_y} , y lies in the interior of $M(y, r)$. For each $\varepsilon \geq 0$, let

$$M^\varepsilon(y, r) \doteq \{x \in \mathbb{R}^J : \text{dist}(x, M(y, r)) \leq \varepsilon\}.$$

Since $\bar{R}_y < 1$ and $|q_y| \leq 1$, it is clear that for each $x \in M^{2\lambda_y r}(y, r)$,

$$\text{dist}\left(x, y + \bigcup_{t \in [0, \bar{R}_y]} tK_{y, \delta_y}\right) \leq 2\lambda_y r + \left| \lambda_y \frac{\bar{R}_y}{2} r q_y \right| < 3\lambda_y r.$$

Thus, we have

$$M^{2\lambda_y r}(y, r) \subseteq \left\{x \in \mathbb{R}^J : \text{dist}\left(x, y + \bigcup_{t \in [0, \bar{R}_y]} tK_{y, \delta_y}\right) < 3\lambda_y r\right\}$$

and hence, by (C.6)–(C.7) we have

$$(C.14) \quad M^{2\lambda_y r}(y, r) \cap \bar{G} \cap \partial B_r(y) = \emptyset \quad \text{and} \quad M^{2\lambda_y r}(y, r) \cap \bar{G} \subset B_r(y) \cap \bar{G}.$$

Let

$$(C.15) \quad O(y, r) \doteq \bar{G} \cap (M^{2\lambda_y r}(y, r) \setminus M^{\eta_y r}(y, r)) \subset B_r(y) \cap \bar{G}.$$

For each $x \in O(y, r)$, it is clear that $x \in \bar{G}$ and from (C.13) and the fact that $\eta_y < \lambda_y$, it follows that

$$(C.16) \quad \eta_y < \text{dist}\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y, L_{y, \delta_y}\right) \leq 2\lambda_y.$$

Since \bar{G} can be locally approximated at y by \mathcal{S}_y as in (C.4), by choosing r_y and λ_y sufficiently small, we can ensure that for each $r \in (0, r_y)$ and $x \in O(y, r)$, the projection of $(x-y)/r + \lambda_y (R_y/2) q_y$ to L_{y, δ_y} coincides with the projection of $(x-y)/r + \lambda_y (R_y/2) q_y$ to $\bigcup_{t \geq 0} tK_{y, \delta_y}$ since L_{y, δ_y} is the portion of $\bigcup_{t \geq 0} tK_{y, \delta_y}$ truncated near its vertex. Hence, for each $x \in O(y, r)$, we have

$$\text{dist}\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y, L_{y, \delta_y}\right) = \text{dist}\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y, \bigcup_{t \geq 0} tK_{y, \delta_y}\right).$$

Together with (C.16), this shows that for each $x \in O(y, r)$, $\frac{x-y}{r} + \lambda_y \frac{\bar{R}_y}{2} q_y$ lies in the set Π_y specified in (C.8).

Now, let $k_{y,r}$ be the function on $O(y, r)$ given by

$$k_{y,r}(x) \doteq \ell_y\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y\right), \quad x \in O(y, r).$$

Then the properties of ℓ_y and v_y stated in Lemma C.1 and the definition of $O(y, r)$ in (C.15) imply that $k_{y,r} \in C^\infty(O(y, r))$,

$$(C.17) \quad \sup_{x \in O(y,r)} \left(\left| k_{y,r}(x) - \text{dist}\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y, L_{y, \delta_y}\right) \right| \vee (r|\nabla k_{y,r}(x)| - 1) \right) \leq \frac{\lambda_y}{12},$$

and $\langle r \nabla k_{y,r}(x), p \rangle \leq -v_y$ for each $p \in K_y^{\delta_y/3}$ and $x \in O(y, r)$. From the second property in Lemma C.1, it also follows that

$$\langle r \nabla k_{y,r}(x), d^i(y) \rangle \geq v_y \quad \text{for } i \in \mathcal{I}(y) \text{ and } x \in O(y, r).$$

Since $d^i(\cdot)$ is continuous for each $i \in \mathcal{I}$, r_y satisfies (C.5) and \mathcal{I} is upper semicontinuous, by possibly making r_y yet smaller and using the first property of $k_{y,r}$ from Lemma C.1, we have for each $r \in (0, r_y)$,

$$(C.18) \quad \langle r \nabla k_{y,r}(x), d^i(x) \rangle \geq \frac{v_y}{2} \quad \text{for } i \in \mathcal{I}(x) \text{ and } x \in O(y, r).$$

Now, choose $\bar{h}_y \in C^\infty(\mathbb{R})$ to be a decreasing function such that

$$(C.19) \quad \bar{h}_y(s) = \begin{cases} 1, & \text{if } s \in (-\infty, 5\lambda_y/4], \\ \text{strictly decreasing,} & \text{if } s \in (5\lambda_y/4, 23\lambda_y/12], \\ 0, & \text{if } s \in (23\lambda_y/12, \infty), \end{cases}$$

and define $f^{y,r} : \mathbb{R}^J \rightarrow \mathbb{R}_+$ as follows:

$$(C.20) \quad f^{y,r}(x) \doteq \begin{cases} \bar{h}_y(k_{y,r}(x)), & \text{if } x \in O(y,r), \\ 1, & \text{if } x \in \bar{G} \cap M^{\eta_y r}(y,r), \\ 0, & \text{otherwise.} \end{cases}$$

When combined with the definitions of $M(y,r)$ and $O(y,r)$ given in (C.13) and (C.15), respectively, and properties (C.17) and (C.14), we infer that

$$(C.21) \quad \begin{aligned} & \text{supp}[f^{y,r}] \cap \bar{G} \\ & \subset \left\{ x \in \bar{G} : k_{y,r}(x) \leq \frac{23\lambda_y}{12} \right\} \\ & \subset \left\{ x \in \bar{G} : \text{dist}\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y, L_{y,\delta_y}\right) \leq \frac{23\lambda_y}{12} + \frac{\lambda_y}{12} \right\} \\ & = \left\{ x \in \bar{G} : \text{dist}\left(x-y + \frac{\lambda_y \bar{R}_y}{2} r q_y, r L_{y,\delta_y}\right) \leq 2\lambda_y r \right\} \\ & = M^{2\lambda_y r}(y,r) \cap \bar{G} \\ & \subset B_r(y) \cap \bar{G}, \end{aligned}$$

which establishes property 3(b) of Lemma 4.3. In addition, we also have

$$(C.22) \quad \begin{aligned} & \{x \in \bar{G} : k_{y,r}(x) \geq 5\lambda_y/4\} \\ & \subset \left\{ x \in \bar{G} : \text{dist}\left(\frac{x-y}{r} + \frac{\lambda_y \bar{R}_y}{2} q_y, L_{y,\delta_y}\right) \geq \frac{5\lambda_y}{4} - \frac{\lambda_y}{12} \right\} \\ & \subset \left\{ x \in \bar{G} : \text{dist}\left(x-y + \frac{\lambda_y \bar{R}_y}{2} r q_y, r L_{y,\delta_y}\right) \geq \lambda_y r \right\} \\ & \subset (M^{\eta_y r}(y,r))^c \cap \bar{G}, \end{aligned}$$

where the last inclusion uses the fact that $\eta_y < \lambda_y$. Relations (C.19)–(C.22), together with (C.15), show that the set on which $f^{y,r}$ is neither 0 nor 1 is a strict subset of $O(y,r)$. Combining this with (C.21) and the fact that $\bar{h}_y \in C^\infty(\mathbb{R})$ and $k_{y,r} \in C^\infty(O(y,r))$, it follows that

$$(C.23) \quad \bar{h}'_y(k_{y,r}(x)) < 0, \quad \text{if } x \in O(y,r)$$

and $f^{y,r} \in C^\infty(\bar{G})$.

By the definition of \bar{h}_y in (C.19), $f^{y,r}$ clearly satisfies property 3(c) of Lemma 4.3. Moreover, since y is an interior point of $M(y, r)$, there exists $\kappa_y(r) \in (0, r)$ such that $B_{\kappa_y(r)}(y) \subset M(y, r)$. For each $x \in B_{\kappa_y(r)}(y) \cap \bar{G}$, the definition of $f^{y,r}$ in (C.20) implies that $f^{y,r}(x) = 1$. Thus, $f^{y,r}$ satisfies property 3(d) of Lemma 4.3. Finally, for each $x \in O(y, r)$, since $\nabla f^{y,r}(x) = \bar{h}'_y(k_{y,r}(x)) \nabla k_{y,r}(x)$, (C.18), (C.23) and the fact that $\mathcal{I}(x) \subset \mathcal{I}(y)$ for all x sufficiently close to y imply that

$$\langle \nabla f^{y,r}(x), d^i(x) \rangle \leq 0 \quad \text{for } i \in \mathcal{I}(x) \text{ and } x \in O(y, r),$$

which proves that $-f^{y,r} \in \mathcal{H}$. Since $f^{y,r}$ has compact support on \bar{G} by property 3(b), this implies property 3(a) of Lemma 4.3. This completes the proof of Lemma 4.3.

APPENDIX D: PROOF OF PROPOSITION 2.12

Fix $z \in \bar{G}$. Due to property 4 of the submartingale problem and the fact that $\mathcal{V} = \partial G \setminus \mathcal{U}$ by (2.1), to show (2.12) it suffices to show that

$$(D.1) \quad \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^\infty \mathbb{I}_{\mathcal{U}}(\omega(s)) ds \right] = 0.$$

Recall the definition of $\mathcal{I}(\cdot)$ given in Definition 2.11 and for each $\delta > 0$, let

$$(D.2) \quad \mathcal{U}_\delta \doteq \left\{ x \in \mathcal{U} : \begin{array}{l} \mathcal{I}(y) \subseteq \mathcal{I}(x) \text{ for all } y \in B_\delta(x) \cap \partial G \text{ and } \exists n \in n(x) \\ \text{such that } n = \sum_{i \in \mathcal{I}(x)} \theta_i n^i(x), \text{ where } \theta_i \geq 0, i \in \mathcal{I}(x), \\ \sum_{i \in \mathcal{I}(x)} \theta_i = 1, \text{ and } \langle n, d \rangle \geq \delta |d| \text{ for all } d \in d(x) \end{array} \right\},$$

and for each $\mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset$, let

$$(D.3) \quad \mathcal{U}_\delta^\mathcal{J} \doteq \{x \in \mathcal{U}_\delta : \mathcal{I}(x) = \mathcal{J}\}.$$

It is immediate from the definition that any two elements in $\{\mathcal{U}_\delta^\mathcal{J}, \mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset\}$ are disjoint, and

$$(D.4) \quad \mathcal{U}_\delta = \bigcup_{\mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset} \mathcal{U}_\delta^\mathcal{J}, \quad \mathcal{U} = \bigcup_{\delta > 0} \mathcal{U}_\delta.$$

In light of (D.4), to prove (D.1), and hence Proposition 2.12, it is clearly sufficient to show that for every $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset$, such that $\mathcal{U}_\delta^\mathcal{J} \neq \emptyset$,

$$(D.5) \quad \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^\infty \mathbb{I}_{\mathcal{U}_\delta^\mathcal{J}}(\omega(s)) ds \right] = 0.$$

Indeed, taking first the sum in (D.5) over $\mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset$, next the limit as $\delta \rightarrow 0$ in (D.5) and then applying Fatou's lemma, we obtain (D.1).

We now state three results from [21] that will be used in the proof of Proposition 2.12.

LEMMA D.1 (Lemma 6.2 of [21]). *For each $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, $\mathcal{U}_\delta^{\mathcal{J}}$ is closed.*

The next lemma states the existence of a family of test functions that lie in the set \mathcal{H} . From Definition 2.11, for each $y \in \partial G$ there exist a neighborhood \mathcal{N}_y , and functions $\varphi_y^i \in \mathcal{C}^2(\mathbb{R}^J)$, $i \in \mathcal{I}(y)$, that characterize the domain G in the neighborhood \mathcal{N}_y of y . For each $y \in \mathcal{U}$, let $\theta_i(y) > 0$, $i \in \mathcal{I}(y)$, be constants such that for each $j \in \mathcal{I}(y)$,

$$(D.6) \quad \left\langle \sum_{i \in \mathcal{I}(y)} \theta_i(y) \frac{\nabla \varphi_y^i(y)}{|\nabla \varphi_y^i(y)|}, \gamma^j(y) \right\rangle > 0.$$

Such constants exist by the definition of n^i and \mathcal{U} in property 1 of Definition 2.11 and (2.2), respectively. Then, for $y \in \mathcal{U}$, define

$$(D.7) \quad g^y(x) \doteq \sum_{i \in \mathcal{I}(y)} \frac{\theta_i(y)}{|\nabla \varphi_y^i(y)|} \varphi_y^i(x), \quad x \in \mathbb{R}^J.$$

From property 1 of Definition 2.11 observe that

$$(D.8) \quad \begin{aligned} \mathcal{N}_y \cap \left(\bigcap_{j \in \mathcal{I}(y)} \partial G^j \right) &= \{x \in \mathcal{N}_y : \varphi_y^j(x) = 0, j \in \mathcal{I}(y)\} \\ &= \{x \in \mathcal{N}_y : g^y(x) = 0\}. \end{aligned}$$

LEMMA D.2. *There exists a function $\kappa' : (0, 1) \mapsto (0, 1/2)$ with $\kappa'(\varepsilon) < \varepsilon/2$ for every $\varepsilon \in (0, 1)$ such that for each $y \in \mathcal{U}$, there exist constants $0 < r'_y < r_y < \text{dist}(y, \mathcal{V})$ such that $B_{r'_y}(y) \subset \mathcal{N}_y$, $0 < c_y < \infty$, $\beta_y > 0$, and a family of functions $\{q_{\varepsilon,y} \in \mathcal{H} : \varepsilon \in (0, 1)\}$ that has the following properties:*

1. $\text{supp}[q_{\varepsilon,y}] \cap \bar{G} \subset \bar{G} \cap B_{r_y}(y)$;
2. $-\varepsilon^2 - \varepsilon^{3/2} \leq q_{\varepsilon,y} \leq 0$;
3. $|\nabla q_{\varepsilon,y}| \leq c_y \varepsilon$;
4. for every $x \in \bar{G} \cap B_{r'_y}(y)$,

$$\sum_{i,j=1}^J a_{ij}(x) \frac{\partial^2 q_{\varepsilon,y}}{\partial y_i \partial y_j}(x) \geq \begin{cases} 2\alpha\beta_y - c_y\varepsilon, & \text{if } 0 \leq g^y(x) \leq \varepsilon/2, \\ -c_x\varepsilon, & \text{if } \varepsilon/2 < g^y(x) < \varepsilon - \kappa'(\varepsilon), \end{cases}$$

and

$$\left| \sum_{i,j=1}^J a_{ij}(x) \frac{\partial^2 q_{\varepsilon,y}}{\partial y_i \partial y_j}(x) \right| \leq c_y \sqrt{\varepsilon} \quad \text{if } g^y(x) \geq \varepsilon - \kappa'(\varepsilon).$$

PROOF. This is a restatement of Lemma 6.3 of [21]. Its proof, which is given in [21], Appendix B, relies on the existence of a certain family of test functions for each $x \in \mathcal{U}$, described in Proposition 7.1 of [21]. Although, the proof of Proposition 7.1 in [21] for $x \in \mathcal{U}$ has a gap, the existence of the required family of test functions follows alternatively from Lemma 4.3 of this paper. Specifically, properties (1)–(3) of Proposition 7.1 are satisfied with $g_{x,r} = f^{x,r}$, $\kappa_x, r_x, \alpha_x(r) = r$, where $f^{x,r}, \kappa_x, r_x$ are the quantities in Lemma 4.3. \square

For each $\delta > 0$, $\mathcal{J} \subseteq \mathcal{I}$ with $\mathcal{J} \neq \emptyset$ and $x \in \mathcal{U}_\delta^\mathcal{J}$, let r'_x be the constant from Lemma D.2. The neighborhoods $\{B_{r'_x/2}(y) : y \in \mathcal{U}_\delta^\mathcal{J}\}$ form an open cover of the closed set $\mathcal{U}_\delta^\mathcal{J}$. The next lemma states that we can choose a countable open subcover that has certain properties.

LEMMA D.3. *For each $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, there exists a countable set of points $\hat{F}_\delta^\mathcal{J} \subset \mathcal{U}_\delta^\mathcal{J}$ such that*

$$(D.9) \quad \mathcal{U}_\delta^\mathcal{J} \subseteq \bigcup_{x \in \hat{F}_\delta^\mathcal{J}} B_{r'_x/2}(x),$$

and there exists a measurable mapping $\lambda_\delta^\mathcal{J}$ from $\mathcal{U}_\delta^\mathcal{J}$ onto $\hat{F}_\delta^\mathcal{J}$ such that $x \in B_{l_{\bar{x}}/2}(\bar{x})$, where $\bar{x} = \lambda_\delta^\mathcal{J}(x)$ and $l_{\bar{x}} = r'_{\bar{x}}$ and $\mathcal{I}(x) = \mathcal{I}(\bar{x})$ for each $x \in \mathcal{U}_\delta^\mathcal{J}$.

PROOF. This is essentially Lemma 6.4 of [21], with the slight difference that r_x in [21] is replaced with $r'_x/2$ in the inclusion (D.9); however, the proof of the version above is exactly analogous to the proof of Lemma 6.4 given in [21]. \square

We now use the preliminary results above to establish Proposition 2.12.

PROOF OF PROPOSITION 2.12. Fix $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, such that $\mathcal{U}_\delta^\mathcal{J} \neq \emptyset$. We first introduce a sequence of stopping times. Let $\hat{F}_\delta^\mathcal{J}, \{B_{r'_x}(x) : x \in \hat{F}_\delta^\mathcal{J}\}$ and the measurable mapping $\lambda_\delta^\mathcal{J}$ be as in Lemma D.3. Now, set $\sigma_0 \doteq 0$ and for $n \in \mathbb{N}$, recursively define the stopping times

$$(D.10) \quad \tau_n \doteq \inf\{t \geq \sigma_{n-1} : \omega(t) \in \mathcal{U}_\delta^\mathcal{J}\},$$

$$(D.11) \quad \sigma_n \doteq \inf\{t \geq \tau_n : \omega(t) \notin B_{l_{\bar{x}_n}}(\bar{x}_n)\},$$

where $\bar{x}_n = \lambda_\delta^\mathcal{J}(\omega(\tau_n))$ and $l_{\bar{x}_n} = r'_{\bar{x}_n}$. Note that when $\sigma_n < \infty$, $|\omega(\sigma_n) - \omega(\tau_n)| \geq |\omega(\sigma_n) - \bar{x}_n| - |\bar{x}_n - \omega(\tau_n)| \geq l_{\bar{x}_n}/2 > 0$. Thus, by the continuity of $\omega(\cdot)$ and a simple contradiction argument, $\tau_n \rightarrow \infty$ and $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$.

Now, we establish (D.5) using a proof by induction. Note that for $n = 1$, $\sigma_{n-1} = 0$ and so we trivially have

$$(D.12) \quad \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^{\sigma_{n-1}} \mathbb{1}_{\mathcal{U}_\delta^\mathcal{J}}(\omega(s)) ds \right] = 0.$$

Now, suppose that (D.12) holds for some $n \in \mathbb{N}$. We will show that then (D.12) also holds with n replaced by $n + 1$. Since under \mathbb{Q}_z , $\omega(t) \notin \mathcal{U}_\delta^{\mathcal{J}}$ for $t \in [\sigma_{n-1}, \tau_n)$, it is clear that

$$(D.13) \quad \mathbb{E}^{\mathbb{Q}_z} \left[\int_{\sigma_{n-1}}^{\tau_n} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s)) ds \right] = 0.$$

Due to (D.12) and (D.13), it follows that

$$(D.14) \quad \mathbb{E}^{\mathbb{Q}_z} \left[\int_0^{\sigma_n} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s)) ds \right] = \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s)) ds \right].$$

Next, for each $y \in \hat{F}_\delta^{\mathcal{J}}$, let the constant $c_y \in (0, \infty)$ and the family of test functions $q_{\varepsilon,y}$, $\varepsilon \in (0, 1)$, be as specified in Lemma D.2. For each $y \in \hat{F}_\delta^{\mathcal{J}}$ and $\varepsilon \in (0, 1)$, since $q_{\varepsilon,y} \in \mathcal{H}$, by Lemma 4.7 with ϱ, θ^x, g replaced by $\tau_n, \theta^y = \inf\{t \geq \tau_n : \omega(t) \notin B_{l_y}(y)\}$, $q_{\varepsilon,y}$, respectively,

$$(D.15) \quad \mathbb{I}_{\{\tau_n < t\}} \left[q_{\varepsilon,y}(\omega(t \wedge \theta^y)) - q_{\varepsilon,y}(\omega(\tau_n)) - \int_{\tau_n}^{t \wedge \theta^y} \mathcal{L}q_{\varepsilon,y}(\omega(u)) du \right]$$

is a \mathbb{Q}_z -submartingale. Since, on the event $\{\tau_n < \infty, \bar{x}_n = y\}$, $\sigma_n = \theta^y$ and $\omega(s) \in \bar{G} \cap B_{r'_n}(\bar{x}_n)$ for every $s \in [\tau_n, \theta^y)$, it follows from the submartingale property of (D.15) and property (2) of $q_{\varepsilon,y}$ in Lemma D.2 that

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} \int_{\tau_n}^{t \wedge \sigma_n} \mathcal{L}q_{\varepsilon,y}(\omega(u)) du \right] \\ & \leq \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} [q_{\varepsilon,y}(\omega(t \wedge \sigma_n)) - q_{\varepsilon,y}(\omega(\tau_n))] \right] \\ & \leq \varepsilon^2 + \varepsilon^{3/2}. \end{aligned}$$

On the other hand, note that by the definition of \mathcal{L} in (1.1),

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} \int_{\tau_n}^{t \wedge \sigma_n} \mathcal{L}q_{\varepsilon,y}(\omega(u)) du \right] \\ & = \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} \int_{\tau_n}^{t \wedge \sigma_n} \frac{1}{2} \sum_{i,j=1}^J a_{ij}(\omega(u)) \frac{\partial^2 q_{\varepsilon,y}(\omega(u))}{\partial x_i \partial x_j} du \right] \\ & \quad + \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} \int_{\tau_n}^{t \wedge \sigma_n} \sum_{j=1}^J b_j(\omega(u)) \frac{\partial q_{\varepsilon,y}(\omega(u))}{\partial x_j} du \right]. \end{aligned}$$

Combining the last two displays with property (3) of $q_{\varepsilon,y}$ in Lemma D.2, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} \int_{\tau_n}^{t \wedge \sigma_n} \sum_{i,j=1}^J a_{ij}(\omega(u)) \frac{\partial^2 q_{\varepsilon,y}(\omega(u))}{\partial x_i \partial x_j} du \right] \\ & \leq 2\varepsilon^2 + 2\varepsilon^{3/2} + 2c_y t \sup_{z \in \bar{G} \cap B_{r'_n}(y)} |b(z)| \varepsilon. \end{aligned}$$

Together with property (4) of $q_{\varepsilon, y}$ in Lemma D.2, this implies that

$$\begin{aligned} & (2\alpha\beta_y - c_y\varepsilon)\mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < t, \bar{x}_n = y\}} \int_{\tau_n}^{t \wedge \sigma_n} \mathbb{I}_{\{0 \leq g^y(\omega(u)) \leq \varepsilon/2\}} du \right] \\ & \leq c_y t \sqrt{\varepsilon} + c_y t \varepsilon + 2\varepsilon^2 + 2\varepsilon^{3/2} + 2c_y t \sup_{z \in \bar{G} \cap B_{r'_y}(y)} |b(z)| \varepsilon. \end{aligned}$$

Letting first $\varepsilon \downarrow 0$ and then $t \rightarrow \infty$, we obtain

$$\mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty, \bar{x}_n = y\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\{g^y(\omega(u)) = 0\}} du \right] = 0.$$

Recall that $\mathcal{I}(\omega(\tau_n)) = \mathcal{I}(\bar{x}_n) = \mathcal{J}$ by Lemma D.3. From the definition of σ_n and g^y given in (D.11) and (D.7), respectively, the fact that $B_{l_y}(y) \subset \mathcal{N}_y$ and (D.8), we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty, \bar{x}_n = y\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\cap_{j \in \mathcal{J}} \partial G_j}(\omega(u)) du \right] \\ & = \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty, \bar{x}_n = y\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\cap_{j \in \mathcal{I}(y)} \partial G_j}(\omega(u)) du \right] \\ & = \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty, \bar{x}_n = y\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\{g^y(\omega(u)) = 0\}} du \right] = 0. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(u)) du \right] \\ & \leq \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\cap_{j \in \mathcal{J}} \partial G_j}(\omega(u)) du \right] \\ & = \sum_{y \in \hat{F}_\delta^{\mathcal{J}}} \mathbb{E}^{\mathbb{Q}_z} \left[\mathbb{I}_{\{\tau_n < \infty, \bar{x}_n = y\}} \int_{\tau_n}^{\sigma_n} \mathbb{I}_{\cap_{j \in \mathcal{J}} \partial G_j}(\omega(u)) du \right] = 0. \end{aligned}$$

When combined with (D.14), this shows that (D.12) holds with n replaced by $n + 1$. Since $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$, the proposition follows by induction. \square

Acknowledgment. The authors would like to thank Ruth Williams for insightful comments on a previous version of the paper, which led us to remove a measurability condition in the definition of the submartingale problem and also establish a slightly stronger version of Theorem 3.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
 UNIVERSITY OF MARYLAND, BALTIMORE COUNTY
 1000 HILLTOP CIRCLE
 BALTIMORE, MARYLAND 21250
 USA
 E-MAIL: wkang@umbc.edu

DIVISION OF APPLIED MATHEMATICS
 BOX F
 BROWN UNIVERSITY
 182 GEORGE STREET
 PROVIDENCE, RHODE ISLAND 02912
 USA
 E-MAIL: Kavita_Ramanan@brown.edu