SPECTRAL GAP FOR STOCHASTIC ENERGY EXCHANGE MODEL WITH NONUNIFORMLY POSITIVE RATE FUNCTION

BY MAKIKO SASADA[1]

Keio University

We give a lower bound on the spectral gap for a class of stochastic energy exchange models. In 2011, Grigo et al. introduced the model and showed that, for a class of stochastic energy exchange models with a uniformly positive rate function, the spectral gap of an N-component system is bounded from below by a function of order N^{-2}. In this paper, we consider the case where the rate function is not uniformly positive. For this case, the spectral gap depends not only on N but also on the averaged energy $\mathcal{E}$, which is the conserved quantity under the dynamics. Under some assumption, we obtain a lower bound of the spectral gap which is of order $C(\mathcal{E})N^{-2}$ where $C(\mathcal{E})$ is a positive constant depending on $\mathcal{E}$. As a corollary of the result, a lower bound of the spectral gap for the mesoscopic energy exchange process of billiard lattice studied by Gaspard and Gilbert [*J. Stat. Mech. Theory Exp.* **2008** (2008) p11021, *J. Stat. Mech. Theory Exp.* **2009** (2009) p08020] and the stick process studied by Feng et al. [*Stochastic Process. Appl.* **66** (1997) 147–182] are obtained.

1. Introduction.

1.1. *Background and model.* Recently, Grigo et al. introduced a class of stochastic energy exchange models, which are pure jump Markov processes with a continuous state space in [11]. The model is a generalization of the mesoscopic energy exchange process of billiard lattice studied in [7] and [8] by Gaspard and Gilbert. Showing the hydrodynamic limit for such a mesoscopic model of mechanical origin is a very important step for a rigorous derivation of a diffusion equation or Fourier law from a system which is purely deterministic.

One of the key estimates required for the hydrodynamical limit is a sharp lower bound on the spectral gap of the finite coordinate process (cf. [12]). What is needed is that the gap, for the process confined to cubes of size N, shrinks at a rate N^{-2}. Up to constants, this is heuristically the best possible lower bound. For a wide class of interacting particle systems or diffusion processes, the desired spectral gap estimates have been obtained (cf. [12, 14]). On the other hand, for pure jump processes with a continuous state space, this type of estimate has been scarcely

shown. To our knowledge, only for the Kac walk and its generalizations (cf. [3, 4, 9]), the sharp estimate of the spectral gap have been shown before the result [11] by Grigo et al. for stochastic energy exchange models. Our goal is, as a first step for proving the hydrodynamic limit, to extend the result in [11] to the class including the mesoscopic energy process of the billiard lattice.

The dynamics of the stochastic energy exchange model introduced by Grigo et al. is described as follows: For each integer $N \geq 2$, denote by Σ_N the one-dimensional cube $\{1, 2, \ldots, N\}$. A configuration of the state space $\mathbb{R}_+^{\Sigma_N} := [0, \infty)^{\Sigma_N}$ is denoted by x, so that x_i indicates the energy at site $i \in \Sigma_N$, which is a positive real number. Fix a nonnegative continuous function $\Lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$, which is called a rate function, and a continuous function $P : \mathbb{R}_+^2 \to \mathcal{P}([0, 1])$ where $\mathcal{P}([0, 1])$ is the set of probability measures on $[0, 1]$. At each nearest neighbor pair of the lattice $(i, i + 1)$, energy exchange independently happens with rate $\Lambda(x_i, x_{i+1})$. When the energy exchange happens between the pair $(i, i + 1)$, a number $0 \leq \alpha \leq 1$ is drawn, independently of everything else, according to a distribution $P(x_i, x_{i+1}, d\alpha)$ and the energy at site i becomes $\alpha(x_i + x_{i+1})$, the energy at site $i + 1$ becomes $(1 - \alpha)(x_i + x_{i+1})$, and all other energies remain unchanged.

More precisely, we consider a continuous time Markov jump process $x(t)$ on $\mathbb{R}_+^N$ by its infinitesimal generator $\mathcal{L}$, acting on measurable bounded functions $f : \mathbb{R}_+^N \to \mathbb{R}$ as

$$(1.1) \quad \mathcal{L}f(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int_0^1 P(x_i, x_{i+1}, d\alpha)\big[f(T_{i,i+1,\alpha}x) - f(x)\big],$$

where

$$(T_{i,i+1,\alpha}x)_k = \begin{cases} x_k & \text{if } k \neq i, i+1, \\ \alpha(x_i + x_{i+1}) & \text{if } k = i, \\ (1-\alpha)(x_i + x_{i+1}) & \text{if } k = i+1. \end{cases}$$

Obviously, the process preserves the total energy $\sum_{i=1}^N x_i$. Therefore, for each $\mathcal{E} > 0$, the set of configurations with mean energy $\mathcal{E}$ per site

$$\mathcal{S}_{\mathcal{E},N} = \left\{ x \in \mathbb{R}_+^N; \frac{1}{N} \sum_{i=1}^N x_i = \mathcal{E} \right\}$$

is invariant for the process. Since $\mathcal{S}_{\mathcal{E},N}$ is compact and invariant, the assumed continuity of Λ and P guarantees the existence of at least one stationary distribution $\pi_{\mathcal{E},N}$ for $x(t)$ on each $\mathcal{S}_{\mathcal{E},N}$. As mentioned, the scaling of the rate of convergence toward the stationary distribution in terms of the lattice size N is of crucial importance in studying the hydrodynamic limit of this model, especially if the system is of nongradient type.

Under certain conditions, Grigo et al. proved that the spectral gap of the generator $\mathcal{L}$ on $\mathcal{S}_{\mathcal{E},N}$ is of order N^{-2} uniformly in the mean energy $\mathcal{E}$ [11]. Since

their proof used the weak convergence in Wasserstein distance, it applies to very general rate functions Λ and transition kernels P. The existence of a lower bound on the rate function Λ and the reversibility of the process are keys of their assumptions. However, as pointed out by themselves, since the mesoscopic energy exchange process of billiard lattice does not satisfy the first assumption, it was desirable to remove the assumption on the existence of a uniform lower bound of the rate function. In this paper, we relax the assumption and study the case where a rate function satisfies $\Lambda(a, b) \geq C(a + b)^m$ for some $C > 0$ and $m \geq 0$ intuitively. We give a precise assumption later, which is satisfied by the mesoscopic energy exchange process of billiard lattice.

To weaken the condition on the rate function Λ, we need a stronger condition on the reversible measure. Precisely, we assume that our process is reversible with respect to a product Gamma-distribution. This condition is satisfied for general mechanical models, and hence mesoscopic energy exchange processes of mechanical origin, such as the mesoscopic energy exchange process of billiard lattice. See also Remark 1.2 below to understand why this condition is natural from a physical point of view.

1.2. *Notation and main result.*   For each $\gamma > 0$, let ν_γ denote a Gamma distribution on $\mathbb{R}_+$ with a scale parameter 1 and a shape parameter γ, that is,

$$\nu^\gamma(dx) = x^{\gamma-1} \frac{e^{-x}}{\Gamma(\gamma)} dx.$$

Let $\nu^{\gamma,N}$ denote the product measure of ν^γ on $\mathbb{R}_+^N$ and $\nu_{\mathcal{E},N}^\gamma := \nu^{\gamma,N}|_{\mathcal{S}_{\mathcal{E},N}}$ denote the conditional probability measure of $\nu^{\gamma,N}$ on $\mathcal{S}_{\mathcal{E},N}$. From now on, we fix an arbitrary $\gamma > 0$ and assume that $\nu_{\mathcal{E},N}^\gamma$ is a reversible measure for $\mathcal{L}$. We also denote $\nu_{\mathcal{E},N}^\gamma$ by $\nu_{\mathcal{E},N}$ when there is no confusion.

Denote by $L^2(\nu_{\mathcal{E},N})$ the Hilbert space of functions f on $\mathcal{S}_{\mathcal{E},N}$ such that $E_{\nu_{\mathcal{E},N}}[f^2] < \infty$. Then the associated Dirichlet form is given by

$$\mathcal{D}(f) = \mathcal{D}_{\mathcal{E},N}(f) := \int_{\mathcal{S}_{\mathcal{E},N}} \nu_{\mathcal{E},N}(dx)[-\mathcal{L}f](x)f(x)$$

$$= \frac{1}{2} \sum_{i=1}^{N-1} \int_{\mathcal{S}_{\mathcal{E},N}} \nu_{\mathcal{E},N}(dx)\Lambda(x_i, x_{i+1})$$

$$\times \int_0^1 P(x_i, x_{i+1}, d\alpha)[f(T_{i,i+1,\alpha}x) - f(x)]^2$$

for all $f \in L^2(\nu_{\mathcal{E},N})$.

We define

(1.2) $\lambda(\mathcal{E}, N) := \inf\left\{ \frac{\mathcal{D}_{\mathcal{E},N}(f)}{E_{\nu_{\mathcal{E},N}}[f^2]} \Big| E_{\nu_{\mathcal{E},N}}[f] = 0, f \in L^2(\nu_{\mathcal{E},N}) \right\}$

and call $\lambda(\mathcal{E}, N)$ the spectral gap of $-\mathcal{L}$ on $\mathcal{S}_{\mathcal{E},N}$ in $L^2(\nu_{\mathcal{E},N})$.

THEOREM 1. *Assume that there exist a positive constant $\tilde{C} > 0$ and a real number $m \geq 0$ such that the following holds*:

(1.3) $$\lambda(\mathcal{E}, 2) \geq \tilde{C}\mathcal{E}^m \qquad \text{for all } \mathcal{E} > 0.$$

Then there exists a positive constant $C > 0$ depending only on m and γ such that

(1.4) $$\lambda(\mathcal{E}, N) \geq C\frac{\tilde{C}\mathcal{E}^m}{N^2}$$

for all $\mathcal{E} > 0$ and $N \geq 2$.

REMARK 1.1. For simplicity, we state the result in one-dimensional setting. However, since our proof relies on the spectral gap estimate for the long-range model and the kind of "moving particle lemma" in the continuous state space, the result is extended to the case in any dimension immediately, with a positive constant $C > 0$ depending on m, γ and d the dimension of the lattice. We refer Section 2 of [15] for the detail of this argument. This is one of the advantage of our proof compared to the preceding study.

Grigo et al. call (Λ, P) is of mechanical form if the rate function Λ and the transition kernel P are of the form

(1.5) $$\Lambda(a, b) = \Lambda_s(a + b)\Lambda_r\left(\frac{a}{a + b}\right), \qquad P(a, b, d\alpha) = P\left(\frac{a}{a + b}, d\alpha\right)$$

Carlen and studied the processes of this form in detail in [11], Section 4. The form naturally occurs in models originating from mechanical systems. Actually, the rate function and the probability kernel of the mesoscopic energy exchange models of billiard lattice satisfies (1.5) and $\Lambda_s(s) = \sqrt{s}$ while Λ_r is a uniformly positive continuous function on $[0, 1]$. See the explicit expression in Section 5.

REMARK 1.2. One of the splendid results of [11] is that if a stochastic energy exchange model of mechanical form admits a reversible product distribution, then this measure must necessarily be a product Gamma-distributions (or a single atom). This is the reason why we concentrate to study the process reversible with respect to a product measure whose marginal is a Gamma distribution.

If the process is of mechanical form, then by the definition, $\lambda(\mathcal{E}, 2) = \Lambda_s(2\mathcal{E})\tilde{C}$ holds where

$$\tilde{C} = \inf\left\{\frac{\int_0^1 \mu(d\beta)\Lambda_r(\beta)\int_0^1 P(\beta, d\alpha)[f(\alpha) - f(\beta)]^2}{2E_\mu[f^2]}\,\Big|\,E_\mu[f] = 0, f \in L^2(\mu)\right\}$$

and $\mu = \mu^\gamma$ is the beta distribution on $[0, 1]$ with parameters (γ, γ). Therefore, if the above $\tilde{C}$ is strictly positive and $\Lambda_s(s) \geq s^m$ for some $m \geq 0$, then (1.3) is satisfied. Moreover, if (Λ, P) is of mechanical form and $\Lambda_s(s) = s^m$ for some

$m \geq 0$, then $\lambda(\mathcal{E}, N) = \mathcal{E}^m \lambda(1, N)$ holds for all $\mathcal{E} > 0$ and $N \in \mathbb{N}$ (cf. Lemma 2.1). Namely, we cannot expect an order N^{-2} bound of the spectral gap to hold uniformly in $\mathcal{E}$. Then it is natural to ask whether such a bound holds if we incorporate the extra factor $\mathcal{E}^m$, and Theorem 1 shows that this is indeed the case.

COROLLARY 1.1. *Assume that* (Λ, P) *is of mechanical form and* $\Lambda_s(s) = s^m$ *for some* $m \geq 0$. *Then, if*

$$\inf\left\{ \frac{\int_0^1 \mu^\gamma(d\beta)\Lambda_r(\beta)\int_0^1 P(\beta, d\alpha)[f(\alpha) - f(\beta)]^2}{E_{\mu^\gamma}[f^2]} \right|$$

$$E_{\mu^\gamma}[f] = 0, f \in L^2(\mu^\gamma)\right\} > 0$$

holds, there exists a positive constant C *independent of* $\mathcal{E}$ *and* N *such that*

$$\lambda(\mathcal{E}, N) \geq C\frac{\mathcal{E}^m}{N^2}$$

for all $\mathcal{E} > 0$ *and* $N \geq 2$.

REMARK 1.3. To give an upper bound of the spectral gap $\lambda(\mathcal{E}, N)$, consider the function $f_{\mathcal{E},N}(x) = \sum_{i=1}^N i(x_i - \mathcal{E})$. Then we have

$\mathcal{D}_{\mathcal{E},N}(f_{\mathcal{E},N})$

$$= \frac{1}{2}\sum_{i=1}^{N-1}\int_{\mathcal{S}_{\mathcal{E},N}} \nu_{\mathcal{E},N}(dx)\Lambda(x_i, x_{i+1})\int_0^1 P(x_i, x_{i+1}, d\alpha)[(1-\alpha)x_i - \alpha x_{i+1}]^2$$

$$= \frac{N-1}{2}\int_{\mathcal{S}_{\mathcal{E},N}} \nu_{\mathcal{E},N}(dx)\Lambda(x_1, x_2)\int_0^1 P(x_1, x_2, d\alpha)[(1-\alpha)x_1 - \alpha x_2]^2$$

$$\leq (N-1)E_{\nu_{\mathcal{E},N}}\big[\Lambda(x_1, x_2)\big(x_1^2 + x_2^2\big)\big].$$

On the other hand, $E_{\nu_{\mathcal{E},N}}[f_{\mathcal{E},N}^2] = \sum_{i,j=1}^N ij E_{\nu_{\mathcal{E},N}}[(x_i - \mathcal{E})(x_j - \mathcal{E})]$ and since $E_{\nu_{\mathcal{E},N}}[(x_i - \mathcal{E})(x_j - \mathcal{E})] = -\frac{1}{N-1}E_{\nu_{\mathcal{E},N}}[(x_i - \mathcal{E})^2]$ for $i \neq j$,

$$E_{\nu_{\mathcal{E},N}}\big[f_{\mathcal{E},N}^2\big] = \frac{N^2(N+1)}{12}E_{\nu_{\mathcal{E},N}}\big[(x_1 - \mathcal{E})^2\big].$$

Then, by the equivalence of ensembles, $E_{\nu_{\mathcal{E},N}}[\Lambda(x_1, x_2)(x_1^2 + x_2^2)]$ and $E_{\nu_{\mathcal{E},N}}[(x_1 - \mathcal{E})^2]$ converge to $E_{\nu_{\mathcal{E}}^2}[\Lambda(x_1, x_2)(x_1^2 + x_2^2)]$ and $E_{\nu_{\mathcal{E}}^2}[(x_1 - \mathcal{E})^2]$ respectively as $N \to \infty$ where $\nu_{\mathcal{E}}$ is the Gamma distribution with a scale parameter $\mathcal{E}$ and a shape parameter γ. Therefore, if $E_{\nu_{\mathcal{E}}^2}[\Lambda(x_1, x_2)(x_1^2 + x_2^2)] < \infty$, then there exist positive constants $A(\mathcal{E})$ and $B(\mathcal{E})$ such that $\mathcal{D}_{\mathcal{E},N}(f_{\mathcal{E},N}) \leq NA(\mathcal{E})$ and $E_{\nu_{\mathcal{E},N}}[f_{\mathcal{E},N}^2] \geq N^3 B(\mathcal{E})$, namely $\lambda(\mathcal{E}, N) \leq \frac{A(\mathcal{E})}{N^2 B(\mathcal{E})}$ for all $N \geq 2$. Therefore, our lower bound (1.4) is optimal up to constant depending on $\mathcal{E}$.

In particular, if $\Lambda(x_1, x_2) \le C(x_1 + x_2)^m$ for some $C > 0$ and $m \ge 0$, then $\lambda(\mathcal{E}, N) \le \frac{C'\mathcal{E}^m}{N^2}$ with some positive constant C'. Therefore, for such rate functions, our lower bound (1.4) is optimal up to constant which is independent of $\mathcal{E}$ and N.

The lack of a uniform lower bound complicates the rigorous analysis of the rate of convergence to equilibrium. A similar problem was found in the zero-range process with constant rate, and it had been an open problem for decades. In 2005, Morris [15] showed that the spectral gap of that model is of order $(1 + \rho)^{-2}N^{-2}$ where ρ is the density of particles and N is the size of the system. In the context of exclusion processes, it has been known that if the jump rates are degenerate, the spectral gap does not have uniform lower bound of order N^{-2}, and instead has a lower bound of order $C(\rho)N^{-2}$ where $C(\rho)$ is a positive constant depending on ρ, the density of particles (cf. [10, 16]). Recently, the spectral gap for the Kac model with hard sphere collisions is studied by Carlen et al. in [5]. By projecting their model to the energy coordinates, we obtain the process called $\mathcal{L}_{LR}^*$ in our paper with parameters $\gamma = \frac{1}{2}$ and $0 \le m \le 1$ (γ in [5] corresponds to m in our paper). $\mathcal{L}_{LR}^*$ is a long-range (or equivalently, mean field) version of our process with specific (Λ, P), which we will denote by (Λ^*, P^*). However, since we study the spectral gap in the whole L^2-space, we cannot apply directly their result, which is for the spectral gap in the symmetric sector. A classical spectral gap for the gradient operator of the product Gamma-distribution was studied by Barthe and Wolff in [1], and they showed that it is of order $\mathcal{E}^{-2}$ for $\gamma \ge 1$.

We also remark that the hydrodynamic limit for a special class of our processes, which are gradient, was studied by Feng et al. in [6]. The process is called the stick process and of mechanical form with $\Lambda_s(s) = s^m$ where $m > 0$ is a fixed model parameter. We show that we can apply the main result of this paper to this class of models in Section 5. As the hydrodynamic equation of the stick process, the porous medium equation

$$\partial_t \mathcal{E}(t, u) = \text{const. } \partial_u\big(\mathcal{E}(t, u)^m \partial_u \mathcal{E}(t, u)\big)$$

was derived. Gaspard and Gilbert conjectured that the hydrodynamic equation of the mesoscopic energy exchange models of billiard lattice is also the porous medium equation with $m = \frac{1}{2}$. By the scaling property of the generator and the reversible measure, the same equation should be derived from the stochastic energy exchange models of mechanical form with $\Lambda_s(s) = s^m$ (under the condition that the process is reversible with respect to a product Gamma-distribution). The same equation is derived also from an exclusion process with degenerate jump rates in [10].

The rest of the article is organized as follows: In Section 2, we give a detailed account of the proof of Theorem 1. Precisely, we reduce the spectral gap estimate of the original process to that of a long-range version of our process with specific

(Λ, P), which we will denote by (Λ^*, P^*). In Section 3, we justify this reduction, and in Section 4 we give an estimate for this specific model. In Section 5, we show that we can apply our result to the mesoscopic energy exchange models of billiard lattice and the stick process. In the Appendix, we give a sharp estimate of the spectral gap of the specific model with $N = 3$. This sharp estimate is the key of our proof.

2. Proof of Theorem 1. Our basic idea of the proof is to introduce a few suitably chosen reference processes and compare the Dirichlet forms associated with them and that with the original process. First, we introduce a special process given by a generator $\mathcal{L}^*$ with

(2.1)
$$\Lambda^*(x_i, x_{i+1}) = (x_i + x_{i+1})^m,$$

$$P^*(x_i, x_{i+1}, d\alpha) = \frac{\{\alpha(1-\alpha)\}^{\gamma-1}}{B(\gamma, \gamma)} d\alpha = \mu^\gamma(d\alpha),$$

where $B(\gamma, \gamma) = \int_0^1 \{\alpha(1-\alpha)\}^{\gamma-1} d\alpha$ is the normalizing factor and $m \geq 0$ and $\gamma > 0$ are the constants given in the assumption.

We can rewrite the generator $\mathcal{L}^*(= \mathcal{L}^{*,m,\gamma})$ given by (2.1) as

$$\mathcal{L}^* f(x) = \sum_{i=1}^{N-1} (x_i + x_{i+1})^m \{E_{i,i+1} f(x) - f(x)\} = \sum_{i=1}^{N-1} (x_i + x_{i+1})^m D_{i,i+1} f(x),$$

where $E_{i,j} f = E_{\nu_{\mathcal{E},N}^\gamma}[f | \mathcal{F}_{i,j}]$, $D_{i,j} f = E_{i,j} f - f$ and $\mathcal{F}_{i,j}$ is the σ-algebra generated by variables $\{x_k\}_{k \neq i,j}$. Here, we follow the notation used in [3] (see also [18]). Note that $x_i + x_j$ is measurable with respect to $\mathcal{F}_{i,j}$. Using the above expression, we can easily check that $\nu_{\mathcal{E},N}^\gamma = \nu_{\mathcal{E},N}$ is a reversible measure for the process. The associated Dirichlet form is given by

$$\mathcal{D}^*(f) = \mathcal{D}_{\mathcal{E},N}^{*,m}(f) := \int \nu_{\mathcal{E},N}(dx)[-\mathcal{L}^* f](x) f(x)$$

$$= \sum_{i=1}^{N-1} E_{\nu_{\mathcal{E},N}} [(x_i + x_{i+1})^m (E_{i,i+1} f - f)^2]$$

$$= \sum_{i=1}^{N-1} E_{\nu_{\mathcal{E},N}} [(x_i + x_{i+1})^m (D_{i,i+1} f)^2]$$

for all $f \in L^2(\nu_{\mathcal{E},N})$. We use notation $\mathcal{D}^*$ or $\mathcal{D}^{*,m}$ when there is no confusion.

We denote the spectral gap of $\mathcal{L}^*$ in $L^2(\nu_{\mathcal{E},N}^\gamma)$ by $\lambda^{*,m}(\mathcal{E}, N)$. Here, we abbreviate γ. Note that $\lambda^{*,m}(\mathcal{E}, 2) = 2^m \mathcal{E}^m$ since $E_{\nu_{\mathcal{E},2}}[(x_1 + x_2)^m (D_{1,2} f)^2] = (2\mathcal{E})^m E_{\nu_{\mathcal{E},2}}[(D_{1,2} f)^2] = (2\mathcal{E})^m \text{Var}[f^2]$. Namely, this special model satisfies assumption (1.3) of Theorem 1. Moreover, the model is of mechanical form with $\Lambda_s(s) = s^m$, $\Lambda_r(\beta) = 1$ and $P(\beta, d\alpha) = \mu^\gamma(d\alpha)$.

REMARK 2.1. The model defined by the generator $\mathcal{L}^{*,0,1}$ (namely, the above process with parameters $m = 0$ and $\gamma = 1$) was studied by Kipnis et al. in [13] as an exactly solvable model which describes the heat flow.

We consider this special model because of the following guess: Under assumption (1.3), the state of each pair of sites achieves the equilibrium (with respect to the state of this pair of sites) at least with the rate proportional to the mth power of the sum of their energies under the dynamics given by $\mathcal{L}$. Namely, the spectral gap of $\mathcal{L}$ can be bounded from below up to constant by that of the process where the state of any pair of sites achieves the equilibrium exactly with the rate proportional to the mth power of the sum of their energies. The next proposition shows that the guess truly holds.

PROPOSITION 2.1. *Under the assumption of Theorem 1, for any $\mathcal{E} > 0$ and $N \geq 2$,*

$$(2.2) \qquad \lambda(\mathcal{E}, N) \geq \frac{\tilde{C}}{2^m} \lambda^{*,m}(\mathcal{E}, N)$$

holds.

PROOF. Define an operator $\mathcal{L}_0$ on $L^2(\nu_{\mathcal{E},2})$ acting on f as

$$\mathcal{L}_0 f(z_1, z_2) = \Lambda(z_1, z_2) \int P(z_1, z_2, d\alpha)[f(T_{1,2,\alpha}z) - f(z)],$$

where $z = (z_1, z_2) \in \mathbb{R}_+^2$. For $N \geq 3$, $x \in \mathbb{R}_+^N$, $1 \leq i < j \leq N$ and $f : \mathbb{R}_+^N \to \mathbb{R}$, define $f_x^{i,j} : \mathbb{R}_+^2 \to \mathbb{R}$ as

$$f_x^{i,j}(p, q) = f(x_1, x_2, \ldots, x_{i-1}, p, x_{i+1}, \ldots, x_{j-1}, q, x_{j+1}, \ldots, x_N).$$

Note that the function $f_x^{i,j}$ does not depend on x_i nor x_j. Then we can rewrite our generator as follows:

$$\mathcal{L}f(x) = \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} f(x),$$

where $\mathcal{L}_{i,i+1} f(x) = (\mathcal{L}_0 f_x^{i,i+1})(x_i, x_{i+1})$. Then we have

$$E_{\nu_{\mathcal{E},N}}[f(-\mathcal{L}_{i,i+1})f | \mathcal{F}_{i,i+1}]$$
$$= E_{\nu_{\mathcal{E},N}}[f_x^{i,i+1}(x_i, x_{i+1})((-\mathcal{L}_0 f_x^{i,i+1})(x_i, x_{i+1})) | \mathcal{F}_{i,i+1}]$$
$$= E_{\nu_{(x_i+x_{i+1})/2,2}}[f_x^{i,i+1}(-\mathcal{L}_0) f_x^{i,i+1}].$$

Here, we use the *noninterference* property of $\nu_{\mathcal{E},N}$, which is mentioned in [3]. Namely, the conditional distribution with respect to the pair (x_i, x_{i+1}) of $\nu_{\mathcal{E},N}$ on

the configuration space with a fixed value $x_i + x_{i+1}$ is $\nu_{(x_i+x_{i+1})/2,2}$ for any $\mathcal{E} > 0$ and $N \geq 2$.

Now, since $\mathcal{L} = \mathcal{L}_0$ for $N = 2$, by the definition of the spectral gap, we have for any $\mathcal{E} > 0$ and $g \in L^2(\nu_{\mathcal{E},2})$,

$$\lambda(\mathcal{E}, 2) E_{\nu_{\mathcal{E},2}}[(g - E_{\nu_{\mathcal{E},2}}[g])^2] \leq E_{\nu_{\mathcal{E},2}}[g(-\mathcal{L}_0)g].$$

On the other hand, since $E_{i,i+1}$ is the integral operator with respect to $\nu_{(x_i+x_{i+1})/2,2}$ we have

$$E_{\nu_{(x_i+x_{i+1})/2,2}}[(f_x^{i,i+1} - E_{\nu_{(x_i+x_{i+1})/2,2}}[f_x^{i,i+1}])^2] = E_{\nu_{\mathcal{E},N}}[(D_{i,i+1}f)^2|\mathcal{F}_{i,i+1}].$$

Combining the above equations, we have for any $\mathcal{E} > 0$ and $N \geq 2$

$$\lambda\left(\frac{x_i + x_{i+1}}{2}, 2\right) E_{\nu_{\mathcal{E},N}}[(D_{i,i+1}f)^2|\mathcal{F}_{i,i+1}] \leq E_{\nu_{\mathcal{E},N}}[f(-\mathcal{L}_{i,i+1})f|\mathcal{F}_{i,i+1}].$$

Then, by assumption (1.3), we have

$$\frac{\tilde{C}}{2^m} E_{\nu_{\mathcal{E},N}}[(x_i + x_{i+1})^m (D_{i,i+1}f)^2]$$

$$= E_{\nu_{\mathcal{E},N}}\left[\tilde{C}\left(\frac{x_i + x_{i+1}}{2}\right)^m E_{\nu_{\mathcal{E},N}}[(D_{i,i+1}f)^2|\mathcal{F}_{i,i+1}]\right]$$

$$\leq E_{\nu_{\mathcal{E},N}}\left[\lambda\left(\frac{x_i + x_{i+1}}{2}, 2\right) E_{\nu_{\mathcal{E},N}}[(D_{i,i+1}f)^2|\mathcal{F}_{i,i+1}]\right]$$

$$\leq E_{\nu_{\mathcal{E},N}}[f(-\mathcal{L}_{i,i+1})f].$$

Finally, by summing up the terms for $1 \leq i \leq N - 1$, we have $\frac{\tilde{C}}{2^m}\mathcal{D}^*(f) \leq \mathcal{D}(f)$ and complete the proof. $\square$

REMARK 2.2. The similar idea to the proof of Proposition 2.1 was already used in [17, 18].

Hereafter, we only work on the process $\mathcal{L}^*$. The following scaling relation is simple but the key of the rest of the paper.

LEMMA 2.1. *For any $\mathcal{E} > 0$ and $N \geq 2$,*

$$(2.3) \qquad \lambda^{*,m}(\mathcal{E}, N) = \mathcal{E}^m \lambda^{*,m}(1, N).$$

PROOF. Recall that for any $\mathcal{E} > 0$, $\nu_{1,N}$ is the image of $\nu_{\mathcal{E},N}$ under the map $S: x \to \frac{1}{\mathcal{E}}x$, the unitary change of scale from $\mathcal{S}_{\mathcal{E},N}$ to $\mathcal{S}_{1,N}$. Therefore, for any $f \in L^2(\nu_{\mathcal{E},N})$, let $f_{\mathcal{E}}(x) = f(\mathcal{E}x)$, then $f_{\mathcal{E}} \in L^2(\nu_{1,N})$ and

$$(2.4) \qquad E_{\nu_{\mathcal{E},N}}[f(-\mathcal{L}^*)f] = \mathcal{E}^m E_{\nu_{1,N}}[f_{\mathcal{E}}(-\mathcal{L}^*)f_{\mathcal{E}}]$$

holds. Then the statement follows immediately from the definition of the spectral gap. □

To estimate $\lambda^{*,m}(\mathcal{E}, N)$, we introduce a long-range version of the process $\mathcal{L}^*$, which is defined by the generator $\mathcal{L}^*_{\mathrm{LR}}$ $(= \mathcal{L}^{*,m,\gamma}_{\mathrm{LR}})$ as

$$\mathcal{L}^*_{\mathrm{LR}} f(x) = \frac{1}{N} \sum_{i<j} (x_i + x_j)^m \{ E_{i,j} f(x) - f(x) \} = \frac{1}{N} \sum_{i<j} (x_i + x_j)^m D_{i,j} f(x).$$

It is easy to see that $\nu^\gamma_{\mathcal{E},N}$ is a reversible measure of $\mathcal{L}^*_{\mathrm{LR}}$ and the associated Dirichlet form is given by

$$\mathcal{D}^*_{\mathrm{LR}}(f) = \mathcal{D}^{*,m}_{\mathrm{LR},\mathcal{E},N}(f) := \int \nu_{\mathcal{E},N}(dx)[-\mathcal{L}^*_{\mathrm{LR}} f](x) f(x)$$

$$= \frac{1}{N} \sum_{i<j} E_{\nu_{\mathcal{E},N}} [(x_i + x_j)^m (E_{i,j} f - f)^2]$$

$$= \frac{1}{N} \sum_{i<j} E_{\nu_{\mathcal{E},N}} [(x_i + x_j)^m (D_{i,j} f)^2]$$

for all $f \in L^2(\nu_{\mathcal{E},N})$. We use notation $\mathcal{D}^*_{\mathrm{LR}}$ or $\mathcal{D}^{*,m}_{\mathrm{LR}}$ when there is no confusion and denote the spectral gap of $\mathcal{L}^*_{\mathrm{LR}}$ in $L^2(\nu_{\mathcal{E},N})$ by $\lambda^{*,m}_{\mathrm{LR}}(\mathcal{E}, N)$.

Comparison techniques between the spectral gap of a nearest-neighbor interaction process and that of its long-range version are known for general interacting particle systems, or a class of continuous spin systems with uniformly positive rate function (e.g., [16, 18]). However, to apply them to our process, we need to combine their ideas cleverly because of the nonuniformly positive rate function. In fact, unlike general comparison theorems, the spectral gap of $\mathcal{L}^*$ is bounded from below by the spectral gap of $\mathcal{L}^*_{\mathrm{LR}}$ multiplied by N^{-2} and *the spectral gap of 3-site system*. Denote $\kappa_m = \lambda^{*,m}(\frac{1}{3}, 3)$ and $\tilde{\kappa}_m = \lambda^{*,m}_{\mathrm{LR}}(\frac{1}{3}, 3)$. Recall that κ_m and $\tilde{\kappa}_m$ depend also on γ. Our comparison theorem is precisely given in the following way.

THEOREM 2. *For any $m \geq 0$, there exists a positive constant $C = C(m, \gamma)$ such that*

$$\lambda^{*,m}(\mathcal{E}, N) \geq C \kappa_m N^{-2} \lambda^{*,m}_{\mathrm{LR}}(\mathcal{E}, N)$$

for all $\mathcal{E} > 0$ and $N \geq 2$.

We give a proof of this theorem in the next section.

Once we have Theorem 2 and the scaling relation (2.3), then all we have to show is that $\kappa_m > 0$ and $\lambda^{*,m}_{\mathrm{LR}}(1, N)$ is uniformly positive in N. $\kappa_m > 0$ follows from Corollary 2.1 below. The main work of this paper is to give a uniform lower

bound for the spectral gap of $\mathcal{L}_{LR}^*$ in the size of the system with a fixed mean energy $\mathcal{E}$. It is already done for the case $m = 0$ in [9] (also in [18] by a different proof) as follows.

THEOREM 3 ([9, 18]). *For any $\mathcal{E} > 0$ and $N \geq 2$,*

$$(2.5) \qquad \lambda_{LR}^{*,0}(\mathcal{E}, N) = \frac{\gamma N + 1}{N(2\gamma + 1)}.$$

In particular, $\inf_N \lambda_{LR}^{*,0}(\mathcal{E}, N) = \frac{\gamma}{2\gamma+1} > 0$ for any $\mathcal{E} > 0$.

To obtain a uniform bound for positive m, we prove a comparison theorem between the spectral gap of the process with $m \geq 1$ and with $m = 0$. To do this, we use the convexity of the function x^m as a function of $x \in \mathbb{R}_+$, which is true only for $m \geq 1$.

THEOREM 4. *For any $m \geq 1$, $\mathcal{E} > 0$ and $N \geq 2$,*

$$\lambda_{LR}^{*,m}(\mathcal{E}, N) \geq \frac{\mathcal{E}^m \kappa_m}{2} \lambda_{LR}^{*,0}(\mathcal{E}, N).$$

In this estimate, the spectral gap of 3-site system appears again. A proof of the Theorem 4 is given in Section 4.

The analysis of the case $0 < m < 1$ is more difficult and complicated. First, we give a new type of comparison theorem between $\lambda_{LR}^{*,m}(\mathcal{E}, N)$ and $\lambda_{LR}^{*,2m}(\mathcal{E}, N)$:

PROPOSITION 2.2. *For any $m \geq 0$, if $\tilde{\kappa}_m \geq \frac{1}{3}$, then*

$$(2.6) \qquad \lambda_{LR}^{*,m}(\mathcal{E}, N) \geq \sqrt{\left((3\tilde{\kappa}_m - 1)\left(1 - \frac{2}{N}\right) + \frac{1}{N}\right) \lambda_{LR}^{*,2m}(\mathcal{E}, N)}$$

holds for all $\mathcal{E} > 0$ and $N \geq 2$.

We give a proof of this proposition in Section 4. Obviously, to use inequality (2.6), we need the following key lemma and its corollary.

LEMMA 2.2. *For any $\gamma > 0$,*

$$\tilde{\kappa}_1 > \tfrac{1}{3}.$$

Here, we emphasize that $\tilde{\kappa}_m$ depends on γ. A proof of this lemma is given in the Appendix. The next corollary is shown easily from this lemma.

COROLLARY 2.1. *For any $m \geq 0$ and $\gamma > 0$,*

$$3\tilde{\kappa}_m \geq \kappa_m > 0.$$

Moreover, for any $m \leq 1$,

$$\tilde{\kappa}_m > \tfrac{1}{3}.$$

PROOF. Recall that $\kappa_m = \lambda^{*,m}(\frac{1}{3}, 3)$ and $\tilde{\kappa}_m = \lambda_{\mathrm{LR}}^{*,m}(\frac{1}{3}, 3)$. By the explicit expressions,

$$\mathcal{D}^*(f) = \big(E_\nu\big[(x_1 + x_2)^m (D_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)^m (D_{2,3}f)^2\big]\big),$$

$$\mathcal{D}_{\mathrm{LR}}^*(f) = \tfrac{1}{3}\big(E_\nu\big[(x_1 + x_2)^m (D_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)^m (D_{2,3}f)^2\big]$$
$$+ E_\nu\big[(x_1 + x_3)^m (D_{1,3}f)^2\big]\big)$$

for all $f \in L^2(\nu)$ where $\nu = \nu_{1/3,3}^\gamma$, the inequality $3\tilde{\kappa}_m \geq \kappa_m$ follows directly. Note that ν does not depend on m.

Next, we show that $\kappa_1 > 0$. By the definition, for any $f \in L^2(\nu)$ satisfying $E_\nu[f] = 0$, we have

$$3\tilde{\kappa}_1 E_\nu\big[f^2\big]$$
$$\leq E_\nu\big[(x_1 + x_2)(D_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)(D_{2,3}f)^2\big]$$
$$+ E_\nu\big[(x_1 + x_3)(D_{1,3}f)^2\big].$$

Noting $E_\nu[(x_i + x_j)(D_{i,j}f)^2] = E_\nu[(x_i + x_j)f^2] - E_\nu[(x_i + x_j)(E_{i,j}f)^2]$, we have

$$E_\nu\big[(x_1 + x_2)(E_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)(E_{2,3}f)^2\big] + E_\nu\big[(x_1 + x_3)(E_{1,3}f)^2\big]$$
$$\leq (2 - 3\tilde{\kappa}_1)E_\nu\big[f^2\big]$$

since for any $x = (x_1, x_2, x_3) \in \mathcal{S}_{1/3,3}$, $x_1 + x_2 + x_3 = 1$. Therefore,

$$E_\nu\big[(x_1 + x_2)(E_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)(E_{2,3}f)^2\big]$$
$$\leq \big(1 - (3\tilde{\kappa}_1 - 1)\big)E_\nu\big[f^2\big]$$
$$\leq E_\nu\big[(x_1 + x_2)f^2\big] + E_\nu\big[(x_2 + x_3)f^2\big] - (3\tilde{\kappa}_1 - 1)E_\nu\big[f^2\big]$$

which implies $(3\tilde{\kappa}_1 - 1)E_\nu[f^2] \leq E_\nu[(x_1 + x_2)(D_{1,2}f)^2] + E_\nu[(x_2 + x_3)(D_{2,3}f)^2]$ and hence $\kappa_1 \geq 3\tilde{\kappa}_1 - 1$. Then, by Lemma 2.2, $\kappa_1 > 0$ follows.

Now, since $(x_i + x_j)^m$ is decreasing in m for any fixed $x \in \mathcal{S}_{1/3,3}$ and $1 \leq i < j \leq 3$, we have κ_m and $\tilde{\kappa}_m$ are both decreasing in m. Therefore, for any $m \leq 1$, $\kappa_m > 0$ and $\tilde{\kappa}_m > \frac{1}{3}$ holds.

On the other hand, for $m > 1$, by Hölder's inequality,

$$E_\nu\big[(x_1 + x_2)(D_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)(D_{2,3}f)^2\big]$$

$$\leq E_\nu\big[(x_1 + x_2)^m(D_{1,2}f)^2\big]^{1/m} E_\nu\big[(D_{1,2}f)^2\big]^{1/m'}$$

$$\quad + E_\nu\big[(x_2 + x_3)^m(D_{2,3}f)^2\big]^{1/m} E\big[(D_{2,3}f)^2\big]^{1/m'}$$

$$\leq \big\{E_\nu\big[(x_1 + x_2)^m(D_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)^m(D_{2,3}f)^2\big]\big\}^{1/m}$$

$$\quad \times \big\{E_\nu\big[(D_{1,2}f)^2\big] + E_\nu\big[(D_{2,3}f)^2\big]\big\}^{1/m'}$$

$$\leq \big\{E_\nu\big[(x_1 + x_2)^m(D_{1,2}f)^2\big] + E_\nu\big[(x_2 + x_3)^m(D_{2,3}f)^2\big]\big\}^{1/m}\big\{2E_\nu\big[f^2\big]\big\}^{1/m'},$$

where $\frac{1}{m} + \frac{1}{m'} = 1$. For the second inequality, we use the inequality $a^{1/m}b^{1/m'} + c^{1/m}d^{1/m'} \leq (a + c)^{1/m}(b + d)^{1/m'}$ for any nonnegative numbers a, b, c and d, which is obtained by Hölder's inequality for a two-point space equipped with the counting measure.

Then, combining the above inequality with the fact that $\kappa_1 = \lambda^*(\frac{1}{3}, 3)$, we have

$$\kappa_1 E_\nu\big[f^2\big] \leq 2^{1/m'}\big(\mathcal{D}^{*,m}(f)\big)^{1/m}\big(E_\nu\big[f^2\big]\big)^{1/m'}$$

which implies $\kappa_m \geq \frac{\kappa_1^{m'}}{2} > 0$. $\quad\square$

REMARK 2.3. For the following proof of Theorem 1, the sharp estimate $\tilde{\kappa}_1 > \frac{1}{3}$ is essential for the case $0 < m < 1$, but the weaker condition $\tilde{\kappa}_1 > 0$ is sufficient for the case $m \geq 1$. To prove $\tilde{\kappa}_1 > 0$, we can avoid the complicated argument in the Appendix and instead use a simpler argument, for example, the one used in Section 4.2 of [2].

PROOF OF THEOREM 1. Combining Theorem 3, Theorem 4 and Corollary 2.1, for any $m \geq 1$, there exists a positive constant $C = C(m, \gamma)$ such that

$$(2.7) \qquad\qquad \lambda_{\mathrm{LR}}^{*,m}(\mathcal{E}, N) \geq C\mathcal{E}^m$$

for all $\mathcal{E} > 0$ and $N \geq 2$. Then, applying Proposition 2.2 and Corollary 2.1, for any $\frac{1}{2} \leq m < 1$,

$$\lambda_{\mathrm{LR}}^{*,m}(\mathcal{E}, N) \geq \mathcal{E}^m\sqrt{\Big((3\tilde{\kappa}_m - 1)\Big(1 - \frac{2}{N}\Big) + \frac{1}{N}\Big)C(2m, \gamma)} \geq C(m, \gamma)\mathcal{E}^m,$$

where $C(2m, \gamma)$ is the constant in the inequality (2.7) and

$$C(m, \gamma) = \sqrt{C(2m, \gamma)}\inf_{N \geq 2}\sqrt{\Big((3\tilde{\kappa}_m - 1)\Big(1 - \frac{2}{N}\Big) + \frac{1}{N}\Big)} > 0.$$

Repeating the same argument, we have for any $\frac{1}{2^{k+1}} \leq m < \frac{1}{2^k}$ with some $k \in \mathbb{N}$, there exists a positive constant $C = C(m, \gamma)$ such that

$$\lambda_{\mathrm{LR}}^{*,m}(\mathcal{E}, N) \geq C\mathcal{E}^m$$

for all $\mathcal{E} > 0$ and $N \geq 2$. Therefore, it holds for any $m > 0$ and also for $m = 0$ by Theorem 3.

Now, combining this inequality with Proposition 2.1 and Theorem 2, we complete the proof of Theorem 1. $\square$

REMARK 2.4. We can also consider the process generated by $\mathcal{L}_{\mathrm{LR}}^{*,m}$ with negative m. However, for this case, the statement "$\lambda_{\mathrm{LR}}^{*,m}(\mathcal{E}, N) \geq C\mathcal{E}^m$ for some positive constant C" or the equivalent statement "$\lambda_{\mathrm{LR}}^{*,m}(1, N) \geq C$ for some positive constant C" turns out to be false. Actually, fix $\mathcal{E} = 1$ and consider the function $f_N(x) = \mathbf{1}_{\{x_1 > N/2\}} \in L^2(\nu_{1,N})$, then

$$E\big[(x_i + x_j)^m (D_{i,j} f_N)^2\big] = 0 \qquad \text{for } i, j \neq 1,$$

$$E\big[(x_1 + x_j)^m (D_{1,j} f_N)^2\big] = E\big[(x_1 + x_2)^m (D_{1,2} f_N)^2\big] \qquad \text{for } j \geq 2$$

and

$$
\begin{aligned}
&E\big[(x_1 + x_2)^m (D_{1,2} f_N)^2\big] \\
&\quad = E\big[\mathbf{1}_{\{x_1 + x_2 > N/2\}}(x_1 + x_2)^m (D_{1,2} f_N)^2\big] \\
&\qquad + E\big[\mathbf{1}_{\{x_1 + x_2 \leq N/2\}}(x_1 + x_2)^m (D_{1,2} f_N)^2\big] \\
&\quad = E\big[\mathbf{1}_{\{x_1 + x_2 > N/2\}}(x_1 + x_2)^m (D_{1,2} f_N)^2\big] \\
&\quad \leq \left(\frac{N}{2}\right)^m E\big[(D_{1,2} f_N)^2\big] \leq \left(\frac{N}{2}\right)^m \mathrm{Var}(f_N),
\end{aligned}
$$

where $\mathrm{Var}(f_N)$ is the variance of f_N. Namely, $\lambda_{\mathrm{LR}}^{*,m}(1, N) \leq 2^{-m} N^m$ which means there is no uniform spectral gap for the case where m is negative.

3. Reduction to the long-range model. In this section, we estimate the spectral gap $\lambda^{*,m}(\mathcal{E}, N)$ from below by $\lambda_{\mathrm{LR}}^{*,m}(\mathcal{E}, N)$ by comparing the associated Dirichlet forms.

We first give simple but useful lemmas.

LEMMA 3.1. *For any $x \in \mathbb{R}_+^N$,*

$$\left(\sum_{i=1}^N x_i\right)^m \lambda^{*,m}\left(\frac{1}{N}, N\right) = \lambda^{*,m}\left(\frac{\sum_{i=1}^N x_i}{N}, N\right).$$

In particular, applying the equality for $N = 3$, we have

(3.1) $$\kappa_m(a + b + c)^m = \lambda^{*,m}\left(\frac{a + b + c}{3}, 3\right)$$

for any $a, b, c > 0$.

PROOF. It follows from Lemma 2.1 directly. □

LEMMA 3.2. *For any $m \geq 0$, $\kappa_m \leq \frac{3}{2}$.*

PROOF. Since κ_m is decreasing in m, $\kappa_m \leq \kappa_0$ and by Corollary 2.1 and Theorem 3, $\kappa_0 \leq 3\tilde{\kappa}_0 = \frac{3\gamma+1}{2\gamma+1} \leq \frac{3}{2}$. □

To compare $\mathcal{D}_{LR}^*(f)$ and $\mathcal{D}^*(f)$, we introduce operators $\pi_{i,j} : \mathcal{S}_{\mathcal{E},N} \to \mathcal{S}_{\mathcal{E},N}$ which exchange the energies of sites i and j:

$$(\pi_{i,j}x)_k = \begin{cases} x_k & \text{if } k \neq i, j, \\ x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \end{cases}$$

and $\pi_{i,i}x = x$.

Before going to the main result in this section, we prepare the following key lemma.

LEMMA 3.3. *There exists a universal positive constant C such that for any $m \geq 0$, $\mathcal{E} > 0$, $N \geq 2$ and $1 \leq i < j \leq N$,*

$$(3.2) \quad \begin{aligned} &\kappa_m E_{\nu_{\mathcal{E},N}}\big[x_i^m(f \circ \pi_{i,j} - f)^2\big] \\ &\quad \leq C|j - i| \sum_{k=i}^{j-1} E_{\nu_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m(D_{k,k+1}f)^2\big] \end{aligned}$$

holds for all $f \in L^2(\nu_{\mathcal{E},N})$.

PROOF. Following the strategy used in the study of the spectral gap for multi-species exclusion processes [16], we express the exchange $\pi_{i,j}$ with rate x_i^m by a sequence of neighboring exchange $\pi_{k,k+1}$ with rate x_k^m or x_{k+1}^m, and $\pi_{k,k+2}$ with rate x_k^m or x_{k+2}^m. More precisely, for $i < j$, we denote $K = j - i$ and define a sequence of sites $n_0 = i, n_1, n_2, \dots, n_{4K-3}$ by

$$n_k = \begin{cases} i + k & \text{if } 0 \leq k \leq K, \\ j - 2 - l & \text{if } k = K + 2l + 1, 0 \leq l \leq K - 2, \\ j - l & \text{if } k = K + 2l, 1 \leq l \leq K - 1, \\ i + k - 3K + 3 & \text{if } 3K - 1 \leq k \leq 4K - 3. \end{cases}$$

We define operators $S_k : \mathcal{S}_{\mathcal{E},N} \to \mathcal{S}_{\mathcal{E},N}$ for $0 \leq k \leq 4K - 3$ by $S_0 = Id$ and $S_{k+1} = \pi_{n_k,n_{k+1}} \circ S_k$ for $0 \leq k \leq 4K - 4$. By the construction, $S_{4K-3} = \pi_{i,j}$ and for all $0 \leq k \leq 4K - 3$, $(S_kx)_{n_k} = x_i$ and $|n_k - n_{k+1}| = 1$ or 2. We emphasize that the use of transitions $\pi_{k,k+2}$ is necessary since due to that, we have the crucial relation $(S_kx)_{n_k} = x_i$. Also, it is precisely because of these extra transitions that we are forced to have the factor κ_m in (3.2).

Then, by the Schwarz inequality,

$$E_{\nu_{\mathcal{E},N}}\left[x_i^m\{f(\pi_{i,j}x) - f(x)\}^2\right]$$

$$\leq (4K-3)\sum_{k=0}^{4K-4} E_{\nu_{\mathcal{E},N}}\left[x_i^m\{f(S_{k+1}x) - f(S_k x)\}^2\right]$$

$$= (4K-3)\sum_{k=0}^{4K-4} E_{\nu_{\mathcal{E},N}}\left[x_{n_k}^m\{f(\pi_{n_k,n_{k+1}}x) - f(x)\}^2\right]$$

$$\leq (4K-3)\left\{3\sum_{k=i}^{j-1} E_{\nu_{\mathcal{E},N}}\left[x_k^m\{f(\pi_{k,k+1}x) - f(x)\}^2\right]\right.$$

$$\left. + \sum_{k=i}^{j-2} E_{\nu_{\mathcal{E},N}}\left[x_{k+2}^m\{f(\pi_{k,k+2}x) - f(x)\}^2\right]\right\}.$$

Here, we use the fact that $i \leq n_k \leq j$ for all $0 \leq k \leq 4K-4$ and for each l satisfying $i \leq l < j$, $\sharp\{k; \{n_k, n_{k+1}\} = \{l, l+1\}\} \leq 3$ and $\sharp\{k; \{n_k, n_{k+1}\} = \{l, l+2\}\} \leq 1$ by the construction. We also use the invariance of $\nu_{\mathcal{E},N}$ under the permutation of coordinates, which we will use repeatedly without notice.

Now, since $E_{k,k+1}f(\pi_{k,k+1}x) = E_{k,k+1}f(x)$, we obtain that

$$E_{\nu_{\mathcal{E},N}}\left[x_k^m\{f(\pi_{k,k+1}x) - f(x)\}^2\right]$$

$$= E_{\nu_{\mathcal{E},N}}\left[x_k^m\{f(\pi_{k,k+1}x) - (E_{k,k+1}f)(\pi_{k,k+1}x) + (E_{k,k+1}f)(x) - f(x)\}^2\right]$$

$$\leq 2E_{\nu_{\mathcal{E},N}}\left[x_{k+1}^m\{f(x) - (E_{k,k+1}f)(x)\}^2\right]$$

$$+ 2E_{\nu_{\mathcal{E},N}}\left[x_k^m\{(E_{k,k+1}f)(x) - f(x)\}^2\right]$$

$$\leq 4E_{\nu_{\mathcal{E},N}}\left[(x_k + x_{k+1})^m(D_{k,k+1}f)^2\right].$$

By the same argument,

$$E_{\nu_{\mathcal{E},N}}\left[x_{k+2}^m\{f(\pi_{k,k+2}x) - f(x)\}^2\right] \leq 4E_{\nu_{\mathcal{E},N}}\left[(x_k + x_{k+2})^m(D_{k,k+2}f)^2\right].$$

Then, by the definition of the spectral gap and the scaling relation (3.1), we have

$$\kappa_m E_{\nu_{\mathcal{E},N}}\left[(x_k + x_{k+1} + x_{k+2})^m(D_{k,k+2}f)^2|\mathcal{F}_{k,k+1,k+2}\right]$$

$$= \lambda^{*,m}\left(\frac{x_k + x_{k+1} + x_{k+2}}{3}, 3\right)E_{\nu_{\mathcal{E},N}}\left[(D_{k,k+2}f)^2|\mathcal{F}_{k,k+1,k+2}\right]$$

$$= \lambda^{*,m}\left(\frac{x_k + x_{k+1} + x_{k+2}}{3}, 3\right)E_{\nu_{\mathcal{E},N}}\left[(E_{k,k+2}f - f)^2|\mathcal{F}_{k,k+1,k+2}\right]$$

$$\leq \lambda^{*,m}\left(\frac{x_k + x_{k+1} + x_{k+2}}{3}, 3\right)E_{\nu_{\mathcal{E},N}}\left[(E_{k,k+1,k+2}f - f)^2|\mathcal{F}_{k,k+1,k+2}\right]$$

$$\leq E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1}f)^2 | \mathcal{F}_{k,k+1,k+2}\big]$$
$$+ E_{v_{\mathcal{E},N}}\big[(x_{k+1} + x_{k+2})^m (D_{k+1,k+2}f)^2 | \mathcal{F}_{k,k+1,k+2}\big],$$

where $E_{i,j,k}f = E_v[f|\mathcal{F}_{i,j,k}]$ and $\mathcal{F}_{i,j,k}$ is the σ-algebra generated by variables $\{x_l\}_{l \neq i,j,k}$. At the first inequality, we use the relation that

$$E_{v_{\mathcal{E},N}}\big[(E_{i,j,k}f - f)^2 | \mathcal{F}_{i,j,k}\big]$$
$$= E_{v_{\mathcal{E},N}}\big[(E_{i,j,k}f - E_{i,j}f)^2 | \mathcal{F}_{i,j,k}\big] + E_{v_{\mathcal{E},N}}\big[(E_{i,j}f - f)^2 | \mathcal{F}_{i,j,k}\big].$$

Then, by taking the expectation, we have

(3.3)
$$\kappa_m E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1} + x_{k+2})^m (D_{k,k+2}f)^2\big]$$
$$\leq E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1}f)^2\big]$$
$$+ E_{v_{\mathcal{E},N}}\big[(x_{k+1} + x_{k+2})^m (D_{k+1,k+2}f)^2\big].$$

Then, combing the inequalities, we have

$$\kappa_m E_{v_{\mathcal{E},N}}\big[x_i^m (f \circ \pi_{i,j} - f)^2\big]$$
$$\leq \kappa_m (4K - 3)\bigg\{ 3 \sum_{k=i}^{j-1} E_{v_{\mathcal{E},N}}\big[x_k^m \{f(\pi_{k,k+1}x) - f(x)\}^2\big]$$
$$+ \sum_{k=i}^{j-2} E_{v_{\mathcal{E},N}}\big[x_{k+2}^m \{f(\pi_{k,k+2}x) - f(x)\}^2\big]\bigg\}$$
$$\leq \kappa_m (4K - 3)12 \sum_{k=i}^{j-1} E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1}f)^2\big]$$
$$+ 8(4K - 3) \sum_{k=i}^{j-1} E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1}f)^2\big]$$
$$\leq |j - i|(48\kappa_m + 32) \sum_{k=i}^{j-1} E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1}f)^2\big]$$
$$\leq 104|j - i| \sum_{k=i}^{j-1} E_{v_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1}f)^2\big],$$

where we use $\kappa_m \leq \frac{3}{2}$ in the last inequality. $\quad\square$

REMARK 3.1. The argument used for the estimate (3.3) is exactly the same as the one used in Section 4.2 of [2]. The method is also used in the proof of Theorem 4 in the next section.

Next is the main result in this section, which allows us to compare the Dirichlet forms associated with the nearest neighbor interaction model and the long-range interaction model.

PROPOSITION 3.1.   *There exists a positive constant C depending only on m such that for any $m \geq 0$, $\mathcal{E} > 0$, $N \geq 2$ and $1 \leq i < j \leq N$,*

$$\kappa_m E_{\nu_{\mathcal{E},N}}\big[(x_i + x_j)^m (D_{i,j} f)^2\big] \leq C |j - i| \sum_{k=i}^{j-1} E_{\nu_{\mathcal{E},N}}\big[(x_k + x_{k+1})^m (D_{k,k+1} f)^2\big]$$

holds for all $f \in L^2(\nu_{\mathcal{E},N})$.

PROOF.   Since if $i = j - 1$, taking $C = \frac{3}{2}$, the statement is obvious (with the fact $\kappa_m \leq \frac{3}{2}$), we assume $i < j - 1$. First, we recall that

$$E_{\nu_{\mathcal{E},N}}\big[(x_i + x_j)^m (D_{i,j} f)^2\big]$$

(3.4)
$$= E_{\nu_{\mathcal{E},N}}\left[(x_i + x_j)^m \left\{\int_0^1 P^*(d\alpha) f(T_{i,j,\alpha} x) - f(x)\right\}^2\right]$$

$$= \frac{1}{2} E_{\nu_{\mathcal{E},N}}\left[(x_i + x_j)^m \int_0^1 P^*(d\alpha)\{f(T_{i,j,\alpha} x) - f(x)\}^2\right].$$

Since $(x_i + x_j)^m \leq 2^m (x_i^m + x_j^m)$, the last term of (3.4) is bounded from above by

(3.5)
$$2^{m-1} E_{\nu_{\mathcal{E},N}}\left[x_i^m \int_0^1 P^*(d\alpha)\{f(T_{i,j,\alpha} x) - f(x)\}^2\right]$$

$$+ 2^{m-1} E_{\nu_{\mathcal{E},N}}\left[x_j^m \int_0^1 P^*(d\alpha)\{f(T_{i,j,\alpha} x) - f(x)\}^2\right].$$

Since the second term of (3.5) can be estimated in the same manner as the first term, we only estimate the first term of (3.5). Rewrite the term $f(T_{i,j,\alpha} x) - f(x)$ as

$$f(T_{i,j,\alpha} x) - f(x)$$
$$= \{f(\pi_{i,j-1}(T_{j-1,j,\alpha}(\pi_{i,j-1} x))) - f(T_{j-1,j,\alpha}(\pi_{i,j-1} x))\}$$
$$+ \{f(T_{j-1,j,\alpha}(\pi_{i,j-1} x)) - f(\pi_{i,j-1} x)\} + \{f(\pi_{i,j-1} x) - f(x)\}.$$

Then, using the Schwarz inequality, we can bound the first term of (3.5) from above by

$$E_{\nu_{\mathcal{E},N}}\left[x_i^m \int_0^1 P^*(d\alpha)\{f(\pi_{i,j-1}(T_{j-1,j,\alpha}(\pi_{i,j-1} x))) - f(T_{j-1,j,\alpha}(\pi_{i,j-1} x))\}^2\right]$$

(3.6) $$+ E_{\nu_{\mathcal{E},N}}\left[x_i^m \int_0^1 P^*(d\alpha)\{f(T_{j-1,j,\alpha}(\pi_{i,j-1} x)) - f(\pi_{i,j-1} x)\}^2\right]$$

$$+ E_{\nu_{\mathcal{E},N}}\left[x_i^m \int_0^1 P^*(d\alpha)\{f(\pi_{i,j-1} x) - f(x)\}^2\right]$$

up to constant depending only on m. We estimate three terms of (3.6) separately.

The last term of (3.6) is equal to

$$E_{\nu_{\mathcal{E},N}}\big[x_i^m\{f(\pi_{i,j-1}x) - f(x)\}^2\big]$$

and, therefore, we can apply Lemma 3.3.

By the change of variable, the second term of (3.6) is rewritten as

$$E_{\nu_{\mathcal{E},N}}\left[x_{j-1}^m \int_0^1 P^*(d\alpha)\{f(T_{j-1,j,\alpha}x) - f(x)\}^2\right]$$

which is obviously bounded from above by

$$E_{\nu_{\mathcal{E},N}}\left[(x_{j-1} + x_j)^m \int_0^1 P^*(d\alpha)\{f(T_{j-1,j,\alpha}x) - f(x)\}^2\right]$$
$$= 2E_{\nu_{\mathcal{E},N}}\big[(x_{j-1} + x_j)^m (D_{j-1,j}f)^2\big].$$

Finally, we study the first term of (3.6). By the same way as the second term, the term is rewritten as

$$E_{\nu_{\mathcal{E},N}}\left[x_{j-1}^m \int_0^1 P^*(d\alpha)\big[f(\pi_{i,j-1}(T_{j-1,j,\alpha}x)) - f(T_{j-1,j,\alpha}x)\big]^2\right]$$
$$= E_{\nu_{\mathcal{E},N}}\big[x_{j-1}^m E_{\nu_{\mathcal{E},N}}[(f \circ \pi_{i,j-1} - f)^2 | \mathcal{F}_{j-1,j}]\big]$$

and since $x_{j-1} + x_j$ is measurable with respect to $\mathcal{F}_{j-1,j}$, the last expression is bounded from above by

$$E_{\nu_{\mathcal{E},N}}\big[(x_{j-1} + x_j)^m E_{\nu_{\mathcal{E},N}}[(f \circ \pi_{i,j-1} - f)^2 | \mathcal{F}_{j-1,j}]\big]$$
$$= E_{\nu_{\mathcal{E},N}}\big[(x_{j-1} + x_j)^m (f \circ \pi_{i,j-1} - f)^2\big].$$

Then, using the trivial inequality again, we conclude that the last term is bounded by

$$(3.7) \quad 2^m E_{\nu_{\mathcal{E},N}}\big[x_{j-1}^m (f \circ \pi_{i,j-1} - f)^2\big] + 2^m E_{\nu_{\mathcal{E},N}}\big[x_j^m (f \circ \pi_{i,j-1} - f)^2\big].$$

Though the first term of (3.7) is directly estimated by Lemma 3.3, we need to treat the second term carefully. Precisely, since $\pi_{i,j-1} = \pi_{i,j} \circ \pi_{i,j-1} \circ \pi_{j-1,j}$, we have

$$E_{\nu_{\mathcal{E},N}}\big[x_j^m (f(\pi_{i,j-1}x) - f(x))^2\big] \le 3E_{\nu_{\mathcal{E},N}}\big[x_j^m (f(\pi_{j-1,j}x) - f(x))^2\big]$$
$$+ 3E_{\nu_{\mathcal{E},N}}\big[x_{j-1}^m (f(\pi_{i,j-1}x) - f(x))^2\big]$$
$$+ 3E_{\nu_{\mathcal{E},N}}\big[x_i^m (f(\pi_{i,j}x) - f(x))^2\big].$$

Then we can apply Lemma 3.3 to complete the proof. □

PROOF OF THEOREM 2.   Noting the explicit expressions of $D^*(f)$ and $D^*_{LR}(f)$, by Proposition 3.1, we have

$$\kappa_m D^*_{LR}(f) \leq \frac{C}{N} \sum_{i<j} |j-i| \sum_{k=i}^{j-1} E_{v_{\mathcal{E},N}}[(x_k + x_{k+1})^m (D_{k,k+1}f)^2] \leq C'N^2 D^*(f)$$

for any $f \in L^2(v_{\mathcal{E},N})$ with positive constants C and C' depending only on m. Then, by the definition of the spectral gap, the proof is complete.   □

4. Spectral gap for the long-range model.   In this section, to show that $\inf_N \lambda^{*,m}_{LR}(1, N) > 0$, we give a proof of Theorem 4 and Proposition 2.2.

PROOF OF THEOREM 4.   By the definition of the spectral gap, it is sufficient to show that for all $\mathcal{E} > 0$, $N \geq 2$ and $f \in L^2(v_{\mathcal{E},N})$,

$$\kappa_m \frac{1}{N} \sum_{i<j} E_{v_{\mathcal{E},N}}[(D_{i,j}f)^2] \leq 2\mathcal{E}^{-m} \frac{1}{N} \sum_{i<j} E_{v_{\mathcal{E},N}}[(x_i + x_j)^m (D_{i,j}f)^2]$$

holds.

For $m \geq 1$, since x^m is a convex function, we have $\mathcal{E}^m \leq \frac{1}{N} \sum_{k=1}^N x_k^m$ and, therefore,

$$\frac{1}{N} \sum_{i<j} E_{v_{\mathcal{E},N}}[(D_{i,j}f)^2] \leq \frac{1}{N} \sum_{i<j} \sum_{k=1}^N \frac{\mathcal{E}^{-m}}{N} E_{v_{\mathcal{E},N}}[x_k^m (D_{i,j}f)^2].$$

If $k \neq i, j$, we have

$$E_{v_{\mathcal{E},N}}[x_k^m (D_{i,j}f)^2] = E_{v_{\mathcal{E},N}}[x_k^m (E_{i,j}f - f)^2]$$

$$\leq E_{v_{\mathcal{E},N}}[(x_i + x_j + x_k)^m (E_{i,j}f - f)^2]$$

$$= E_{v_{\mathcal{E},N}}[(x_i + x_j + x_k)^m E_{v_{\mathcal{E},N}}[(E_{i,j}f - f)^2 | \mathcal{F}_{i,j,k}]]$$

$$\leq E_{v_{\mathcal{E},N}}[(x_i + x_j + x_k)^m E_{v_{\mathcal{E},N}}[(E_{i,j,k}f - f)^2 | \mathcal{F}_{i,j,k}]],$$

where we use the relation

$$E_{v_{\mathcal{E},N}}[(E_{i,j,k}f - f)^2 | \mathcal{F}_{i,j,k}] = E_{v_{\mathcal{E},N}}[(E_{i,j,k}f - E_{i,j}f)^2 | \mathcal{F}_{i,j,k}]$$

$$+ E_{v_{\mathcal{E},N}}[(E_{i,j}f - f)^2 | \mathcal{F}_{i,j,k}]$$

again.

Then, by the definition of the spectral gap for 3-site system and the scaling relation (3.1),

$$\kappa_m E_{v_{\mathcal{E},N}}[(x_i + x_j + x_k)^m E[(f - E_{i,j,k}f)^2 | \mathcal{F}_{i,j,k}]]$$

$$\leq E_{v_{\mathcal{E},N}}[(x_i + x_k)^m (D_{i,k}f)^2 + (x_k + x_j)^m (D_{k,j}f)^2].$$

Therefore, noting $x_i^m + x_j^m \leq (x_i + x_j)^m$ for $m \geq 1$ and $\kappa_m \leq \frac{3}{2}$, by summing terms, we obtain

$$\frac{\kappa_m}{N} \sum_{i<j} E_{\nu_{\mathcal{E},N}}[(D_{i,j}f)^2]$$

$$\leq \frac{\mathcal{E}^{-m}}{N} \sum_{i<j} \frac{1}{N}\left(\frac{3}{2} + 2(N-2)\right) E_{\nu_{\mathcal{E},N}}[(x_i + x_j)^m (D_{i,j}f)^2]$$

$$\leq \frac{2\mathcal{E}^{-m}}{N} \sum_{i<j} E_{\nu_{\mathcal{E},N}}[(x_i + x_j)^m (D_{i,j}f)^2]. \qquad \Box$$

PROOF OF PROPOSITION 2.2. Here, we use the idea developed by Caputo in [3] and generalize it. First, recall a well-known equivalent characterization of the spectral gap of a generator $\mathcal{L}$ in $L^2(\nu)$ as the largest constant λ such that the inequality

(4.1) $$E_\nu[(\mathcal{L}f)^2] \geq \lambda E_\nu[f(-\mathcal{L})f]$$

holds for all $f \in L^2(\nu)$. Then, by Schwarz's inequality, we have

$$\lambda = \inf\left\{ \frac{E_\nu[(\mathcal{L}f)^2]}{E_\nu[f(-\mathcal{L})f]} \Big| E_\nu[f] = 0, f \in L^2(\nu) \right\}$$

$$\geq \inf\left\{ \sqrt{\frac{E_\nu[(\mathcal{L}f)^2]}{E_\nu[f^2]}} \Big| E_\nu[f] = 0, f \in L^2(\nu) \right\}.$$

Now, we have

$$E_{\nu_{\mathcal{E},N}}[(\mathcal{L}_{\mathrm{LR}}^{*,m}f)^2] = \frac{1}{N^2} \sum_{b,b'} E_{\nu_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f],$$

where the sum runs over all $\binom{N}{2}$ unordered pairs b and b', and $h_b(x) = (x_i + x_j)^m$ if $b = \{i, j\}$. We write $b \sim b'$ when two unordered pairs have at least one common vertex (including the case $b = b'$). Otherwise, we write $b \not\sim b'$. We observe that if $b \not\sim b'$, then E_b and $E_{b'}$ commute. Moreover, h_b and $h_{b'}$ are both measurable with respect to $\mathcal{F}_b$ and $\mathcal{F}_{b'}$ where $\mathcal{F}_b = \mathcal{F}_{i,j}$ for $b = \{i, j\}$. Therefore, using $D_b^2 = -D_b$ and self-adjointness of D_b and $D_{b'}$, for $b \not\sim b'$

$$E_{\nu_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f] = -E_{\nu_{\mathcal{E},N}}[h_b h_{b'} (D_{b'} D_b f)(D_{b'} f)]$$

$$= E_{\nu_{\mathcal{E},N}}[h_b h_{b'} (D_{b'} D_b f)^2] \geq 0.$$

Therefore, it follows that

$$E_{\nu_{\mathcal{E},N}}[(\mathcal{L}_{\mathrm{LR}}^{*,m}f)^2] \geq \frac{1}{N^2} \sum_{b,b':b\sim b'} E_{\nu_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f].$$

Now, we denote unordered triples $\{i, j, k\}$ of distinct vertices by T (triangles). We say that $b \in T$ if $b = \{i, j\}$ and $i, j \in T$. Clearly, if $b \sim b'$ and $b \neq b'$ there is only one triangle T such that $b, b' \in T$. We may therefore write

$$\sum_{b,b':b\sim b'} E_{v_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f]$$

$$= \sum_{b,b':b\sim b', b\neq b'} E_{v_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f] + \sum_b E_{v_{\mathcal{E},N}}[h_b^2 (D_b f)^2]$$

$$= \sum_T \sum_{b,b' \in T} E_{v_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f] - (N-3) \sum_b E_{v_{\mathcal{E},N}}[h_b^2 (D_b f)^2]$$

since for every b there are exactly $N-2$ triangles T such that $b \in T$.

Let us now apply inequality (4.1) for $\mathcal{L}_{\mathrm{LR}}^{m,*}$ to a fixed triangle T. Let $\mathcal{F}_T$ denote the σ-algebra generated by $\{x_l, l \notin T\}$. Then

$$\frac{1}{3} \sum_{b,b' \in T} E_{v_{\mathcal{E},N}}[h_b h_{b'} D_b f D_{b'} f | \mathcal{F}_T]$$

$$\geq \lambda_{\mathrm{LR}}^{*,m}\left(\frac{\sum_{i \in T} x_i}{3}, 3\right) \sum_b E_{v_{\mathcal{E},N}}[h_b (D_b f)^2 | \mathcal{F}_T]$$

$$= \tilde{\kappa}_m \left(\sum_{i \in T} x_i\right)^m \sum_b E_{v_{\mathcal{E},N}}[h_b (D_b f)^2 | \mathcal{F}_T]$$

$$= \tilde{\kappa}_m \sum_b E_{v_{\mathcal{E},N}}\left[\left(\sum_{i \in T} x_i\right)^m h_b (D_b f)^2 \Big| \mathcal{F}_T\right]$$

$$\geq \tilde{\kappa}_m \sum_b E_{v_{\mathcal{E},N}}[h_b^2 (D_b f)^2 | \mathcal{F}_T],$$

where we use $(\sum_{i \in T} x_i)^m$ is measurable with respect to $\mathcal{F}_T$ and $(\sum_{i \in T} x_i)^m \geq h_b$ for any $b \in T$. Taking $v_{\mathcal{E},N}$-expectation to remove the conditioning on $\mathcal{F}_T$, we obtain that

$$E_{v_{\mathcal{E},N}}[(\mathcal{L}_{\mathrm{LR}}^{*,m} f)^2] \geq \frac{1}{N^2}((3\tilde{\kappa}_m - 1)(N-2) + 1) \sum_b E_{v_{\mathcal{E},N}}[h_b^2 (D_b f)^2].$$

On the other hand, since

$$E_{v_{\mathcal{E},N}}[f(-\mathcal{L}_{\mathrm{LR}}^{2m,*})f] = \frac{1}{N} \sum_b E_{v_{\mathcal{E},N}}[h_b^2 (D_b f)^2],$$

we have

$$\frac{1}{N} \sum_b E_{v_{\mathcal{E},N}}[h_b^2 (D_b f)^2] \geq \lambda_{\mathrm{LR}}^{*,2m}(\mathcal{E}, N) E_{v_{\mathcal{E},N}}[f^2].$$

Therefore, combining the above inequalities, we complete the proof. □

5. Examples. In this section, we present two interesting classes of stochastic energy exchange models for which we can apply Theorem 1.

5.1. *The rarely interacting billiard lattice.* As mentioned in the Introduction, the main motivation of the article [11] was to study the models studied in [7, 8]. Gaspard and Gilbert argued that in the limit of rare collisions, the dynamics of a billiard lattice becomes a Markov jump process. The limiting process is actually in the class considered in this paper. As shown in [11], the process studied in [8] has the generator of the mechanical form with

$$\Lambda_s(s) = s^{1/2}, \qquad \Lambda_r(\beta) = \frac{\sqrt{2\pi}}{6} \frac{1/2 + \beta \vee (1-\beta)}{\sqrt{\beta \vee (1-\beta)}},$$

$$P(\beta, d\alpha) = \frac{3}{2} \frac{1 \wedge \sqrt{(\alpha \wedge (1-\alpha))/(\beta \wedge (1-\beta))}}{1/2 + \beta \vee (1-\beta)} d\alpha.$$

The symbol $\vee$ denotes the maximum and $\wedge$ denotes the minimum. This process is reversible with respect to the product Gamma-distribution with $\gamma = \frac{3}{2}$. Moreover, it is shown in [11] that this measure is also reversible for the process given by the generator corresponding to any other function Λ_s (while keeping Λ_r and P unchanged). Therefore, we consider the generator given by $\Lambda_s(s) = s^m$ for $m \geq 0$, and denote the spectral gap on $\mathcal{S}_{\mathcal{E},N}$ of the process by $\lambda^m_{GG3}(\mathcal{E}, N)$ where 3 represents the dimension of the original mechanical model.

Here, we also consider the process obtained from the two-dimensional billiard lattice studied in [7]. Changing equations (3) and (5) in [7] to our notation yields that

$$\Lambda_s(s) = s^{1/2}, \qquad \Lambda_r(\beta) = \sqrt{\frac{8(\beta \vee (1-\beta))}{\pi^3}} (2E(\beta^*) - (1-\beta^*)K(\beta^*)),$$

$$P(\beta, d\alpha) = \frac{\tilde{P}(\beta, \alpha)}{\Lambda_r(\beta)} d\alpha,$$

where $\beta^* = \frac{\beta}{1-\beta} \wedge \frac{1-\beta}{\beta}$,

$$\tilde{P}(\beta, \alpha) = \sqrt{\frac{2}{\pi^3}} \times \begin{cases} \sqrt{\dfrac{1}{1-\beta}} K\left(\sqrt{\dfrac{\alpha}{1-\beta}}\right) & \text{if } 0 \leq \alpha \leq (\beta \wedge (1-\beta)), \\[2ex] \sqrt{\dfrac{1}{1-\alpha}} K\left(\sqrt{\dfrac{\beta}{1-\alpha}}\right) & \text{if } \beta \leq \alpha \leq (1-\beta), \\[2ex] \sqrt{\dfrac{1}{\alpha}} K\left(\sqrt{\dfrac{1-\beta}{\alpha}}\right) & \text{if } (1-\beta) \leq \alpha \leq \beta, \\[2ex] \sqrt{\dfrac{1}{\beta}} K\left(\sqrt{\dfrac{1-\alpha}{\beta}}\right) & \text{if } (\beta \vee (1-\beta)) \leq \alpha \leq 1, \end{cases}$$

and

$$K(t) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - t^2 \sin^2 \theta}} \, d\theta, \qquad E(t) = \int_0^{\pi/2} \sqrt{1 - t^2 \sin^2 \theta} \, d\theta.$$

Since the underlying mechanical model has a two-dimensional configuration space for each of the constituent particles, this process is reversible with respect to the product Gamma-distribution with $\gamma = 1$. In the same manner as before, this measure is also reversible for the process given by the generator corresponding to any other function Λ_s (while keeping Λ_r and P unchanged). So, we consider the generator given by $\Lambda_s(s) = s^m$ for $m \geq 0$, and denote the spectral gap on $\mathcal{S}_{\mathcal{E},N}$ of the process by $\lambda_{GG2}^m(\mathcal{E}, N)$.

Since these processes are of the mechanical form (1.5), $\lambda_{GG3}^m(\mathcal{E}, 2) = \Lambda_s(2\mathcal{E}) \times \tilde{C}_{GG3} = (2\mathcal{E})^m \tilde{C}_{GG3}$ and $\lambda_{GG2}^m(\mathcal{E}, 2) = \Lambda_s(2\mathcal{E}) \tilde{C}_{GG2} = (2\mathcal{E})^m \tilde{C}_{GG2}$ where $\tilde{C}_{GG3} = \lambda_{GG3}^0(1, 2)$ and $\tilde{C}_{GG2} = \lambda_{GG2}^0(1, 2)$.

LEMMA 5.1.

$$\tilde{C}_{GG3} > 0, \qquad \tilde{C}_{GG2} > 0$$

hold.

PROOF. The fact $\lambda_{GG3}^0(1, 2) > 0$ is shown in [11] since the case $m = 0$ satisfies the condition assumed in Lemma 5.1 of [11]. To show $\tilde{C}_{GG2} > 0$, we write down the explicit Dirichlet form associated to the two-dimensional model:

$$\tilde{C}_{GG2} = \inf\left\{ \frac{\int_0^1 d\beta \int_0^1 \tilde{P}(\beta, \alpha) \, d\alpha [f(\alpha) - f(\beta)]^2}{\int_0^1 d\beta \int_0^1 d\alpha [f(\alpha) - f(\beta)]^2} \,\Big|\, f \in L^2([0, 1]) \right\}.$$

Then, since $\tilde{P}(\beta, \alpha) \geq \sqrt{\frac{2}{\pi^3}} K(0) = \sqrt{\frac{1}{2\pi}}$ for all $0 \leq \alpha, \beta \leq 1$, we have $\tilde{C}_{GG2} \geq \sqrt{\frac{1}{2\pi}}$. $\square$

With this result, we can apply Corollary 1.1 to these models directly and obtain the following corollary.

COROLLARY 5.1. *For any $m \geq 0$, there exists positive constants C and C' independent of $\mathcal{E}$ and N such that*

$$\lambda_{GG3}^m(\mathcal{E}, N) \geq C\mathcal{E}^m \frac{1}{N^2}, \qquad \lambda_{GG2}^m(\mathcal{E}, N) \geq C'\mathcal{E}^m \frac{1}{N^2}.$$

5.2. *Stick processes.* The class of stick processes studied in [6] is another interesting example in the class we considered. The model was first introduced as

the microscopic model which scales to the porous medium equations. The generator of the model is described by the rate function and the probability kernel of the mechanical form as

$$\Lambda_s(s) = s^m, \qquad \Lambda_r(\beta) = \beta^m + (1-\beta)^m, \qquad P(\beta, d\alpha) = \frac{m|\beta - \alpha|^{m-1}}{\Lambda_r(\beta)} \, d\alpha,$$

where m is a positive parameter. $\alpha - 1$ in [6] is associated to m here. The process is reversible with respect to a product Gamma-distribution with $\gamma = 1$.

Denote the spectral gap for the stick process with parameter m on $\mathcal{S}_{\mathcal{E},N}$ by $\lambda_{st}^m(\mathcal{E}, N)$. By the definition,

$$\lambda_{st}^m(1, 2) = \inf_f \left\{ \frac{m \int_0^1 \int_0^1 \{f(t) - f(s)\}^2 |t - s|^{m-1} \, ds \, dt}{\int_0^1 \int_0^1 \{f(t) - f(s)\}^2 \, ds \, dt} \, \middle| \, f \in L^2(\nu_{1,2}) \right\}.$$

Therefore, it is obvious that $\lambda_{st}^1(1, 2) = 1$ and $\lambda_{st}^m(1, 2) > 0$ for $0 < m \leq 1$ as $|t - s|^m \geq |t - s|$ for any $0 \leq t, s \leq 1$ and $0 < m \leq 1$.

On the other hand, for $m > 1$, we need to show $\lambda_{st}^m(1, 2) > 0$ more carefully. Let $k = m - 1 > 0$. For f satisfying $E_{\nu_{1,2}}[f] = \int_0^1 f(t) \, dt = 0$, we have

$$\int_0^1 \int_0^1 \{f(t) - f(s)\}^2 |t - s|^k \, ds \, dt$$

$$= \frac{2}{k+1} \int_0^1 f(t)^2 (t^{k+1} + (1-t)^{k+1}) \, dt - 2 \int_0^1 \int_0^1 f(t) f(s) |t - s|^k \, ds \, dt.$$

Then, for any $a > 0$,

$$\left| \int_0^1 \int_0^1 f(t) f(s) |t - s|^k \, ds \, dt \right|$$

$$= \left| \int_0^1 \int_0^1 f(t) f(s) (|t - s|^k - a^k) \, ds \, dt \right|$$

$$\leq \int_0^1 \int_0^1 |f(t)| |f(s)| \, ||t - s|^k - a^k| \, ds \, dt$$

$$\leq \int_0^1 \int_0^1 \frac{1}{2} (|f(t)|^2 + |f(s)|^2) \, ||t - s|^k - a^k| \, ds \, dt$$

$$= \int_0^1 f(t)^2 \int_0^1 ||t - s|^k - a^k| \, ds \, dt.$$

By simple calculations,

$$\int_0^1 ||t - s|^k - a^k| \, ds$$

$$= \int_0^{1-t} |q^k - a^k| \, dq + \int_0^t |q^k - a^k| \, dq$$

$$= \int_{(1-t)\wedge a}^{1-t} (q^k - a^k)\, dq - \int_0^{(1-t)\wedge a} (q^k - a^k)\, dq$$

$$+ \int_{t\wedge a}^t (q^k - a^k)\, dq - \int_0^{t\wedge a} (q^k - a^k)\, dq$$

$$= \frac{1}{k+1}\big((1-t)^{k+1} + t^{k+1} - 2((1-t)\wedge a)^{k+1} - 2(t\wedge a)^{k+1}\big)$$

$$- a^k\big(1 - 2((1-t)\wedge a) - 2(t\wedge a)\big)$$

$$\leq \frac{1}{k+1}\big((1-t)^{k+1} + t^{k+1}\big) - a^k(1 - 4a).$$

Therefore,

$$\int_0^1 \int_0^1 \{f(t) - f(s)\}^2 |t - s|^k\, ds\, dt$$

$$\geq \frac{2}{k+1} \int_0^1 f(t)^2 \big(t^{k+1} + (1-t)^{k+1}\big)\, dt$$

$$- 2 \int_0^1 f(t)^2 \bigg(\frac{(1-t)^{k+1} + t^{k+1}}{k+1} - a^k(1 - 4a)\bigg)\, dt$$

$$= 2 \int_0^1 f(t)^2 a^k(1 - 4a)\, dt.$$

Namely, for any $0 < a < \frac{1}{4}$, we have $\lambda_{st}^m(1, 2) \geq a^{m-1}(1 - 4a) > 0$.

With this result, we can apply Corollary 1.1 to the stick process directly and obtain the following corollary.

COROLLARY 5.2. *For any $m > 0$, there exists a positive constant C independent of $\mathcal{E}$ and N such that*

$$\lambda_{st}^m(\mathcal{E}, N) \geq C\mathcal{E}^m \frac{1}{N^2}.$$

APPENDIX: SPECTRAL GAP FOR 3-SITE SYSTEM

In this Appendix, we give a proof of Lemma 2.2. From now on, we fix $\nu = \nu_{1/3,3}^\gamma$ and denote by E the integration with respect to ν. For each $n \in \mathbb{N}$, let P_n be the set of polynomials of three variables of degree less than or equal to n and $\tilde{P}_n$ be the set of polynomials of one variable of degree less than or equal to n.

Since P_n is dense in $L^2(\nu)$, we have

$$\tilde{\kappa}_1 = \inf\bigg\{\frac{D_{\mathrm{LR}}^{*,1}(f)}{E[f^2]}; E[f] = 0, f \in L^2(\nu)\bigg\}$$

$$= \inf_{n\in\mathbb{N}} \inf\bigg\{\frac{D_{\mathrm{LR}}^{*,1}(f)}{E[f^2]}; E[f] = 0, f \in P_n\bigg\}.$$

Then, since $D_{LR}^{*,1}(f) = \frac{1}{3} \sum_{i=1}^{3} E[(1 - x_i)(f - E[f|x_i])^2]$ where $E[f|x_i] = E[f|\mathcal{G}_i]$ and $\mathcal{G}_i$ is the σ-algebra generated by x_i,

$$\tilde{\kappa}_1 = \frac{1}{3} \inf_{n \in \mathbb{N}} \inf \left\{ \frac{2E[f^2] - \sum_{i=1}^{3} E[(1 - x_i)E[f|x_i]^2]}{E[f^2]}; E[f] = 0, f \in P_n \right\}.$$

Therefore, to show $\tilde{\kappa}_1 > \frac{1}{3}$, we only need to show that

(A.1) $$\sup_{n \in \mathbb{N}} \sup \left\{ \frac{\sum_{i=1}^{3} E[(1 - x_i)E[f|x_i]^2]}{E[f^2]}; E[f] = 0, f \in P_n \right\} < 1.$$

Now, we construct a set of special functions which generates P_n.

First, for each $n \in \mathbb{N}$, let $J_n \in \tilde{P}_n$ be

(A.2) $$J_n(u) = \frac{\Gamma(n + \gamma)}{n! \Gamma(n + 3\gamma - 1)} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{\Gamma(n + m + 3\gamma - 1)}{\Gamma(m + \gamma)} u^m.$$

$\{J_n\}_{n \in \mathbb{N}}$ are orthogonal polynomials called the Jacobi polynomials with parameters $(\gamma - 1, 2\gamma - 1)$ on the interval $[0, 1]$. We choose the parameter since $\{J_n\}_{n \in \mathbb{N}}$ are orthogonal with respect to the marginal of x_1 under ν, or precisely the beta distribution of parameters $(\gamma, 2\gamma)$. By the construction, for $1 \le i \le 3$,

(A.3) $E[J_n(x_i)] = 0 \qquad (n \in \mathbb{N}), \qquad E[J_n(x_i)J_m(x_i)] = 0 \qquad (n \ne m).$

LEMMA A.1.   *For any $n \in \mathbb{N}$,*

$$E[J_n(x_i)|x_j] = \nu_n J_n(x_j) \qquad for\ i \ne j,$$

where $\nu_n = (-1)^n \frac{\Gamma(2\gamma)\Gamma(n+\gamma)}{\Gamma(\gamma)\Gamma(n+2\gamma)}.$

PROOF.   We first remark that for any $f \in \tilde{P}_n$, $E[f(x_i)|x_j] \in \tilde{P}_n$ as a function of x_j. Moreover, as shown in the proof of Theorem 3.2 in [18], there exists a set of polynomials ψ_n which satisfies $E[\psi_n(x_i)|x_j] = \nu_n \psi_n(x_j)$ for $i \ne j$ and $\psi_n \in \tilde{P}_n$ where $\nu_n = (-1)^n \frac{\Gamma(2\gamma)\Gamma(n+\gamma)}{\Gamma(\gamma)\Gamma(n+2\gamma)}$. Then, since

(A.4) $$\nu_n E[\psi_n(x_1)\psi_m(x_1)] = E[\psi_n(x_2)\psi_m(x_1)] = \nu_m E[\psi_n(x_2)\psi_m(x_2)]$$

and $\nu_n \ne \nu_m$ for $n \ne m$, $\{\psi_n\}_{n \in \mathbb{N}}$ are orthogonal polynomials with respect to the marginal of x_1 under ν, which implies $J_n = c_n \psi_n$ for some $c_n \ne 0$ and $E[J_n(x_i)|x_j] = \nu_n J_n(x_j)$ for $i \ne j$.   □

Next, we consider following polynomials $F_n, G_n, H_n \in P_n$:

(A.5) $F_n(x_1, x_2, x_3) = J_n(x_1) + J_n(x_2) + J_n(x_3),$

(A.6) $G_n(x_1, x_2, x_3) = J_n(x_1) - J_n(x_3),$

(A.7) $H_n(x_1, x_2, x_3) = J_n(x_1) - 2J_n(x_2) + J_n(x_3).$

For any $n \in \mathbb{N}$, let Q_n denote a subspace of P_n generated by $F_0 := 1$ and $\{F_k, G_k, H_k\}_{1 \le k \le n}$ and $Q_n^\perp$ be the orthogonal complement of Q_n of P_n equipped with the inner product induced from $L^2(\nu)$.

PROPOSITION A.1. *For any $n \in \mathbb{N}$ and $f \in Q_n^\perp$,*

(A.8) $$E[f|x_i] = 0, \qquad 1 \le \forall i \le 3.$$

PROOF.   For any $f \in P_n$, by the explicit expression of the integration over two variables, it is not hard to show that $E[f|x_i] \in \tilde{P}_n$. The same property was pointed out in [4]. Therefore, $E[f|x_i] = \sum_{k=1}^n t_k J_k(x_i) + t_0$ with some constants t_k. On the other hand, since $1, J_k(x_i) \in Q_n$ for $1 \le k \le n$ and by the assumption $f \in Q_n^\perp$, we have $t_k = 0$ for $0 \le k \le n$.   □

For any $f \in P_n$, we can write $f = \sum_{i=0}^n a_i F_i + \sum_{i=1}^n b_i G_i + \sum_{i=1}^n c_i H_i + K$ with some $K \in Q_n^\perp$ and constants a_i, b_i and c_i. In particular, if $E[f] = 0$, then $a_0 = 0$. Moreover, since $F_1 = 0$ on $\mathcal{S}_{1/3,3}$, we take $a_1 = 0$. Then, for any $f \in P_n$ satisfying $E[f] = 0$,

$$\sum_{i=1}^3 E\big[(1-x_i)E[f|x_i]^2\big]$$

$$= \sum_{i=1}^3 E\left[(1-x_i)\left(E\left[\sum_{k=2}^n a_k F_k + \sum_{k=1}^n b_k G_k + \sum_{k=1}^n c_k H_k|x_i\right]\right)^2\right]$$

$$= E\left[(1-x_1)\left(\sum_{k=2}^n a_k(1+2v_k)J_k(x_1) + \sum_{k=1}^n (b_k + c_k)(1-v_k)J_k(x_1)\right)^2\right]$$

$$\quad + E\left[(1-x_2)\left(\sum_{k=2}^n a_k(1+2v_k)J_k(x_2) - \sum_{k=1}^n 2c_k(1-v_k)J_k(x_2)\right)^2\right]$$

$$\quad + E\left[(1-x_3)\left(\sum_{k=2}^n a_k(1+2v_k)J_k(x_3) + \sum_{k=1}^n (-b_k + c_k)(1-v_k)J_k(x_3)\right)^2\right]$$

$$= 3E\left[(1-x_1)\left(\sum_{k=2}^n a_k(1+2v_k)J_k(x_1)\right)^2\right]$$

$$\quad + 2E\left[(1-x_1)\left(\sum_{k=1}^n b_k(1-v_k)J_k(x_1)\right)^2\right]$$

$$\quad + 6E\left[(1-x_1)\left(\sum_{k=1}^n c_k(1-v_k)J_k(x_1)\right)^2\right].$$

On the other hand,

$$E[f^2] = E\left[\left(\sum_{k=2}^n a_k F_k + \sum_{k=1}^n b_k G_k + \sum_{k=1}^n c_k H_k + K\right)^2\right]$$

$$= 3\sum_{k=2}^n a_k^2(1+2\nu_k)E[J_k(x_1)^2] + 2\sum_{k=1}^n b_k^2(1-\nu_k)E[J_k(x_1)^2]$$

$$+ 6\sum_{k=1}^n c_k^2(1-\nu_k)E[J_k(x_1)^2] + E[K^2].$$

Therefore, to show (A.1) we only need to show that

(A.9) $$\sup_{n\in\mathbb{N}}\sup_{a=(a_k)}\left\{\frac{E_\mu[(1-u)(\sum_{k=2}^n a_k(1+2\nu_k)J_k(u))^2]}{\sum_{k=2}^n a_k^2(1+2\nu_k)E_\mu[J_k(u)^2]}\right\} < 1$$

and

(A.10) $$\sup_{n\in\mathbb{N}}\sup_{b=(b_k)}\left\{\frac{E_\mu[(1-u)(\sum_{k=1}^n b_k(1-\nu_k)J_k(u))^2]}{\sum_{k=1}^n b_k^2(1-\nu_k)E_\mu[J_k(u)^2]}\right\} < 1,$$

where μ is the beta distribution with parameters $(\gamma, 2\gamma)$.

Since $\{J_n\}$ is a series of orthogonal polynomials, we have

$$E_\mu\left[(1-u)\left(\sum_{k=2}^n a_k(1+2\nu_k)J_k(u)\right)^2\right]$$

$$= \sum_{k=2}^n a_k^2(1+2\nu_k)^2 E_\mu[(1-u)J_k(u)^2]$$

$$- 2\sum_{k=2}^{n-1} a_k a_{k+1}(1+2\nu_k)(1+2\nu_{k+1})E_\mu[u J_k(u) J_{k+1}(u)].$$

Define $J_{n,m} \in \mathbb{R}$ for $n \in \mathbb{N}$ and $1 \le m \le n$ as

$$J_n(u) = \sum_{m=0}^n J_{n,m} u^m.$$

Then we have

$$E_\mu[u J_k(u)^2] = E_\mu[u^{k+1} J_{k,k} J_k(u)] + E_\mu[u^k J_{k,k-1} J_k(u)]$$

$$= \frac{J_{k,k}}{J_{k+1,k+1}} E_\mu[(J_{k+1}(u) - J_{k+1,k}u^k)J_k(u)] + \frac{J_{k,k-1}}{J_{k,k}} E_\mu[J_k(u)^2]$$

$$= -\frac{J_{k+1,k}}{J_{k+1,k+1}} E_\mu[J_k(u)^2] + \frac{J_{k,k-1}}{J_{k,k}} E_\mu[J_k(u)^2]$$

and

$$E_\mu\big[u J_k(u) J_{k+1}(u)\big] = E_\mu\big[u^{k+1} J_{k,k} J_{k+1}(u)\big] = \frac{J_{k,k}}{J_{k+1,k+1}} E_\mu\big[J_{k+1}(u)^2\big].$$

Therefore, we have

$$E_\mu\left[(1-u)\left(\sum_{k=2}^n a_k(1+2v_k) J_k(u)\right)^2\right]$$

$$= \sum_{k=2}^n a_k^2(1+2v_k)^2\left(1 + \frac{J_{k+1,k}}{J_{k+1,k+1}} - \frac{J_{k,k-1}}{J_{k,k}}\right) E_\mu\big[J_k(u)^2\big]$$

$$- 2\sum_{k=2}^{n-1} a_k a_{k+1}(1+2v_k)(1+2v_{k+1})\frac{J_{k,k}}{J_{k+1,k+1}} E_\mu\big[J_{k+1}(u)^2\big].$$

In the same manner, we have

$$E_\mu\left[(1-u)\left(\sum_{k=1}^n b_k(1-v_k) J_k(u)\right)^2\right]$$

$$= \sum_{k=1}^n b_k^2(1-v_k)^2\left(1 + \frac{J_{k+1,k}}{J_{k+1,k+1}} - \frac{J_{k,k-1}}{J_{k,k}}\right) E_\mu\big[J_k(u)^2\big]$$

$$- 2\sum_{k=1}^{n-1} b_k b_{k+1}(1-v_k)(1-v_{k+1})\frac{J_{k,k}}{J_{k+1,k+1}} E_\mu\big[J_{k+1}(u)^2\big].$$

Now, we change variables as $\tilde{a}_k = a_k\sqrt{(1+2v_k)E_\mu[J_k(u)^2]}$ and $\tilde{b}_k = b_k\sqrt{(1-v_k)E_\mu[J_k(u)^2]}$. Note that $1+2v_k > 0$ for $k \geq 2$ and $1 - v_k > 0$ for $k \geq 1$. Then conditions (A.9) and (A.10) can be rewritten as

$$
\text{(A.11)} \quad \sup_{n\in\mathbb{N}} \sup_{\tilde{a}=(\tilde{a}_k)} \left\{\left(\sum_{k=2}^n \tilde{a}_k^2(1+2v_k)p_k \right.\right.
$$

$$
\left.\left. - 2\sum_{k=2}^{n-1} \tilde{a}_k\tilde{a}_{k+1}\sqrt{(1+2v_k)(1+2v_{k+1})}q_k\right)\bigg/\sum_{k=2}^n \tilde{a}_k^2\right\} < 1
$$

and

$$
\text{(A.12)} \quad \sup_{n\in\mathbb{N}} \sup_{\tilde{b}=(\tilde{b}_k)} \left\{\left(\sum_{k=1}^n \tilde{b}_k^2(1-v_k)p_k \right.\right.
$$

$$
\left.\left. - 2\sum_{k=1}^{n-1} \tilde{b}_k\tilde{b}_{k+1}\sqrt{(1-v_k)(1-v_{k+1})}q_k\right)\bigg/\sum_{k=1}^n \tilde{b}_k^2\right\} < 1,
$$

where $p_k = 1 + \frac{J_{k+1,k}}{J_{k+1,k+1}} - \frac{J_{k,k-1}}{J_{k,k}}$ and $q_k = \frac{J_{k,k}}{J_{k+1,k+1}} \sqrt{\frac{E_\mu[J_{k+1}(u)^2]}{E_\mu[J_k(u)^2]}}$. Note that $|q_k| = -q_k$ for all $k \in \mathbb{N}$.

Since for any sequence of positive numbers $\{\alpha_k\}_{k\geq 2}$

$$\sum_{k=2}^{n-1} \left(\tilde{a}_k \sqrt{\frac{(1+2v_k)|q_k|}{\alpha_k}} - \tilde{a}_{k+1}\sqrt{(1+2v_{k+1})|q_k|\alpha_k} \right)^2 \geq 0$$

we have

$$-2\sum_{k=2}^{n-1} \tilde{a}_k \tilde{a}_{k+1} \sqrt{(1+2v_k)(1+2v_{k+1})}q_k$$

$$= 2\sum_{k=2}^{n-1} \tilde{a}_k \tilde{a}_{k+1} \sqrt{(1+2v_k)(1+2v_{k+1})}|q_k|$$

$$\leq \sum_{k=2}^{n-1} \left(\tilde{a}_k^2 \frac{(1+2v_k)|q_k|}{\alpha_k} + \tilde{a}_{k+1}^2 (1+2v_{k+1})|q_k|\alpha_k \right).$$

Namely,

$$\sup_{n\geq 2} \sup_{\tilde{a}} \left\{ \frac{\sum_{k=2}^{n} \tilde{a}_k^2 (1+2v_k)p_k - 2\sum_{k=2}^{n-1} \tilde{a}_k \tilde{a}_{k+1} \sqrt{(1+2v_k)(1+2v_{k+1})}q_k}{\sum_{k=2}^{n} \tilde{a}_k^2} \right\}$$

$$\leq \sup_{n\geq 2} \sup_{\tilde{a}} \left\{ \frac{\sum_{k=2}^{n} \tilde{a}_k^2 (1+2v_k)(p_k + |q_k|/\alpha_k + |q_{k-1}|\alpha_{k-1})}{\sum_{k=2}^{n} \tilde{a}_k^2} \right\},$$

where $\alpha_1 = 0$ for convention. Therefore, we can conclude (A.11) if we succeed to show the following proposition.

PROPOSITION A.2. *There exists a sequence of positive numbers $\{\alpha_n\}_{n=2}^{\infty}$ which satisfies*

$$\sup_{n\geq 2} \left\{ (1+2v_n)\left(p_n + \frac{|q_n|}{\alpha_n} + |q_{n-1}|\alpha_{n-1} \right) \right\} < 1,$$

where $\alpha_1 = 0$ for convention.

In the same manner, to show (A.12), we only need to show the following proposition.

PROPOSITION A.3. *There exists a sequence of positive numbers $\{\beta_n\}_{n=1}^{\infty}$ which satisfies*

$$\sup_{n\geq 1} \left\{ (1-v_n)\left(p_n + \frac{|q_n|}{\beta_n} + |q_{n-1}|\beta_{n-1} \right) \right\} < 1,$$

where $\beta_0 = 0$ for convention.

A.1. Some properties of constants. To prove the desired propositions, we first study some properties of constants v_n, p_n and q_n. Hereafter, to emphasize the fact that v_n, p_n and q_n depend not only on n but also γ, we denote them by $v_n(\gamma)$, $p_n(\gamma)$ and $q_n(\gamma)$.

LEMMA A.2.   *For each fixed $\gamma > 0$, $|v_n(\gamma)|$ is decreasing as a function of n for $n \geq 1$. Moreover, for each fixed $n \in \mathbb{N}$, $|v_n(\gamma)|$ is decreasing as a function of γ for $\gamma > 0$.*

PROOF.   Since $|v_n(\gamma)| = \prod_{k=0}^{n-1} \frac{\gamma+k}{2\gamma+k}$, it is obvious.   □

LEMMA A.3. $p_n(\gamma) > 0$ *for any $n \in \mathbb{N}$ and $\gamma > 0$. Moreover, for each fixed $\gamma < \frac{2}{3}$, $p_n(\gamma)$ is increasing as a function of n for $n \geq 1$ and $p_n(\gamma) < \frac{1}{2}$ for any $n \in \mathbb{N}$. If $\gamma = \frac{2}{3}$, then $p_n(\gamma) = \frac{1}{2}$ for all $n \in \mathbb{N}$. For each fixed $\gamma > \frac{2}{3}$, $p_n(\gamma)$ is decreasing as a function of n for $n \geq 1$.*

PROOF.   By the definition,

$$p_n(\gamma) = 1 + \frac{-(n+1)(n+\gamma)}{2n+3\gamma} - \frac{-n(n+\gamma-1)}{2n+3\gamma-2}$$

$$= \frac{2n(n-1) + (6n-4)\gamma + 6\gamma^2}{(2n+3\gamma)(2n+3\gamma-2)} = \frac{1}{2} + \frac{-\gamma + 3/2\gamma^2}{(2n+3\gamma)(2n+3\gamma-2)}. \quad □$$

LEMMA A.4.   *For each fixed $n \in \mathbb{N}$, $p_n(\gamma)$ is increasing as a function of γ for $\gamma \geq \frac{1}{3}$.*

PROOF.   By the definition,

$$\frac{d}{d\gamma} p_n(\gamma) = \frac{d}{d\gamma} \left(\frac{6\gamma^2 + (6n-4)\gamma + 2n(n-1)}{(2n+3\gamma)(2n+3\gamma-2)} \right)$$

and the numerator of the derivative is

$$(12\gamma + 6n - 4)(2n+3\gamma)(2n+3\gamma-2)$$
$$- 3(6\gamma^2 + (6n-4)\gamma + 2n(n-1))(4n+6\gamma-2)$$
$$= 2n(9\gamma^2 + 6\gamma(n-1) - 2(n-1)) > 0$$

for $\gamma \geq \frac{1}{3}$.   □

LEMMA A.5.   *For each fixed $\frac{2}{3} \leq \gamma \leq 2$, $|q_n(\gamma)|$ is decreasing as a function of n for $n \geq 2$. For each fixed $2 < \gamma \leq \frac{7}{3}$, $|q_n(\gamma)|$ is decreasing as a function of n for $n \geq 3$. For each fixed $\gamma < \frac{2}{3}$, $|q_n(\gamma)|$ is decreasing as a function of n for $n \geq 2$.*

PROOF. First, note that by the definition

$$q_n(\gamma) = -\frac{\Gamma(2n+3\gamma-1)}{n!\Gamma(n+3\gamma-1)}\frac{(n+1)!\Gamma(n+3\gamma)}{\Gamma(2n+3\gamma+1)}$$

$$\times \sqrt{\frac{\Gamma(n+1+\gamma)(2n+3\gamma-1)n!\Gamma(n+3\gamma-1)}{(2n+3\gamma+1)(n+1)!\Gamma(n+3\gamma)\Gamma(n+\gamma)}}$$

$$= -\frac{(n+1)(n+3\gamma-1)}{(2n+3\gamma)(2n+3\gamma-1)}\sqrt{\frac{(n+\gamma)(2n+3\gamma-1)}{(2n+3\gamma+1)(n+1)(n+3\gamma-1)}}$$

$$= -\frac{1}{(2n+3\gamma)}\sqrt{\frac{(n+3\gamma-1)(n+1)(n+\gamma)}{(2n+3\gamma+1)(2n+3\gamma-1)}}.$$

For each $n \in \mathbb{N}$, we have

$$\frac{|q_{n+1}(\gamma)|^2}{|q_n(\gamma)|^2} = \frac{(2n+3\gamma)^2(2n+3\gamma-1)}{(2n+3\gamma+2)^2(2n+3\gamma+3)}\frac{(n+2)(n+3\gamma)(n+\gamma+1)}{(n+1)(n+3\gamma-1)(n+\gamma)}.$$

For any $\gamma > 0$, $(2n+3\gamma+3)(n+\gamma) - (2n+3\gamma)(n+\gamma+1) = n$, so

$$\frac{(2n+3\gamma)}{(2n+3\gamma+3)}\frac{(n+\gamma+1)}{(n+\gamma)} < 1.$$

On the other hand, for $2/3 \le \gamma$, since $(2n+3\gamma+2)(n+3\gamma-1) - (2n+3\gamma)(n+3\gamma) = 3\gamma - 2$,

$$\frac{(2n+3\gamma)}{(2n+3\gamma+2)}\frac{(n+3\gamma)}{(n+3\gamma-1)} \le 1.$$

In the same manner, $(2n+3\gamma+2)(n+1) - (2n+3\gamma-1)(n+2) = n-3\gamma+4$, then if $\gamma \le \frac{n+4}{3}$, then

$$\frac{(2n+3\gamma-1)}{(2n+3\gamma+2)}\frac{(n+2)}{(n+1)} \le 1.$$

Next, we assume that $\gamma < 2/3$. As in the same way, since $(2n+3\gamma+2)(n+3\gamma-1) - (2n+3\gamma-1)(n+3\gamma) = n+6\gamma-2$, for any $n \ge 2$,

$$\frac{(2n+3\gamma-1)}{(2n+3\gamma+2)}\frac{(n+3\gamma)}{(n+3\gamma-1)} < 1$$

and $(2n+3\gamma+2)(n+1) - (2n+3\gamma)(n+2) = 2-3\gamma$,

$$\frac{(2n+3\gamma)}{(2n+3\gamma+2)}\frac{(n+2)}{(n+1)} < 1. \qquad \square$$

LEMMA A.6. *For each fixed $n \ge 3$, $|q_n(\gamma)|$ is decreasing as a function of γ for $\gamma > 0$ and $|q_2(\gamma)|$ is decreasing as a function of γ for $\gamma \ge \frac{1}{10}$.*

PROOF.　Instead of $|q_n(\gamma)|$ itself, we will consider the derivative of $|q_n(\gamma)|^2$. By the definition,

$$\frac{d}{d\gamma}|q_n(\gamma)|^2 = \sqrt{n+1}\frac{d}{d\gamma}\left(\frac{(n+\gamma)(n+3\gamma-1)}{(2n+3\gamma)^4-(2n+3\gamma)^2}\right).$$

The numerator of the derivative is

$$(4n+6\gamma-1)((2n+3\gamma)^4-(2n+3\gamma)^2)$$
$$-(n+\gamma)(n+3\gamma-1)(12(2n+3\gamma)^3-6(2n+3\gamma))$$
$$=-(2n+3\gamma)((18\gamma^2+(24n-9)\gamma+(4n^2-10n))(2n+3\gamma)^2$$
$$+3\gamma+2n^2+4n)$$

which is negative for any $\gamma>0$ if $n\geq 3$, and at least for $\gamma\geq\frac{1}{10}$ if $n=2$.　□

LEMMA A.7.　*For any $\gamma\geq\frac{1}{5}$ and $n\in\mathbb{N}$, $|q_n(\gamma)|\leq\frac{1}{4\sqrt{n+\gamma}}$.*

PROOF.　For any positive numbers a,b, $\sqrt{ab}\leq\frac{1}{2}(a+b)$. Therefore,

$$|q_n(\gamma)|=\frac{\sqrt{n+\gamma}}{(2n+3\gamma)}\frac{\sqrt{(n+1)(n+3\gamma-1)}}{\sqrt{(2n+3\gamma)^2-1}}\leq\frac{\sqrt{n+\gamma}}{2\sqrt{(2n+3\gamma)^2-1}}.$$

Now, for any $\gamma\geq\frac{1}{5}$,

$$(2n+3\gamma)^2-1=4n^2+12n\gamma+9\gamma^2-1\geq 4(n+\gamma)^2+(5\gamma-1)(\gamma+1)$$
$$\geq 4(n+\gamma)^2.\qquad\qquad\qquad\square$$

A.2. Proof of Proposition A.2 for $\gamma\geq\frac{2}{3}$.　Here, we give a proof of Proposition A.2 for the case $\gamma\geq\frac{2}{3}$.

LEMMA A.8.　*For any fixed $\gamma\geq 2$, $\frac{1}{4\sqrt{4+\gamma}}+\frac{1}{4\sqrt{3+\gamma}}<\frac{1}{1+2v_4(\gamma)}-p_4(\gamma)$.*

PROOF.　By the definition, for any $\gamma\geq 2$,

$$\frac{1}{1+2v_4(\gamma)}-p_4(\gamma)$$
$$=\frac{2(2\gamma+1)(2\gamma+3)}{2(2\gamma+1)(2\gamma+3)+(\gamma+2)(\gamma+3)}-\frac{6\gamma^2+20\gamma+24}{(3\gamma+8)(3\gamma+6)}$$
$$=\frac{2\gamma(46+67\gamma+29\gamma^2+3\gamma^3)}{3(\gamma+2)(3\gamma+8)(3\gamma+4)(\gamma+1)}$$
$$=\frac{2}{9}+\frac{2(-64-30\gamma+43\gamma^2+24\gamma^3)}{9(\gamma+2)(3\gamma+8)(3\gamma+4)(\gamma+1)}>\frac{2}{9}.$$

On the other hand, for any $\gamma \geq 2$,

$$\frac{1}{4\sqrt{4+\gamma}} + \frac{1}{4\sqrt{3+\gamma}} < \frac{1}{4\sqrt{6}} + \frac{1}{4\sqrt{5}} < \frac{2}{9},$$

and the lemma follows. $\square$

LEMMA A.9. *For any fixed $\frac{2}{3} \leq \gamma \leq 2$, $|q_6(\gamma)| + |q_5(\gamma)| < \frac{1}{1+2v_6(\gamma)} - p_6(\gamma)$.*

PROOF. By Lemmas A.2, A.4 and A.6,

$$\sup_{2/3 \leq \gamma \leq 2} (|q_6(\gamma)| + |q_5(\gamma)|) = |q_6|\left(\frac{2}{3}\right) + |q_5|\left(\frac{2}{3}\right)$$

and

$$\inf_{2/3 \leq \gamma \leq 2} \left(\frac{1}{1+2v_6(\gamma)} - p_6(\gamma)\right) \geq \frac{1}{1+2v_6(2/3)} - p_6(2).$$

Then the exact calculation shows

$$\left|q_6\left(\frac{2}{3}\right)\right| + \left|q_5\left(\frac{2}{3}\right)\right| < \frac{1}{1+2v_6(2/3)} - p_6(2). \qquad \square$$

LEMMA A.10. *For $\frac{2}{3} \leq \gamma \leq 2$, let $\alpha_2(\gamma) = |q_2(\gamma)|(\frac{1}{1+2v_2(\gamma)} - p_2(\gamma))^{-1}$, $\alpha_3(\gamma) = \frac{1/(1+2v_4(\gamma)) - p_4(\gamma)}{2|q_3(\gamma)|}, \alpha_4(\gamma) = 2|q_4(\gamma)|(\frac{1}{1+2v_4(\gamma)} - p_4(\gamma))^{-1}$ and $\alpha_5(\gamma) = 1$. Then*

$$(A.13) \qquad (1 + 2v_2(\gamma))\left(p_2(\gamma) + \frac{|q_2(\gamma)|}{\alpha_2(\gamma)}\right) = 1,$$

$$(A.14) \qquad (1 + 2v_4(\gamma))\left(p_4(\gamma) + \frac{|q_4(\gamma)|}{\alpha_4(\gamma)} + |q_3(\gamma)|\alpha_3(\gamma)\right) = 1$$

and

$$(A.15) \quad \max_{n=3,5}\left\{(1 + 2v_n(\gamma))\left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\alpha_n(\gamma)} + |q_{n-1}(\gamma)|\alpha_{n-1}(\gamma)\right)\right\} < 1$$

hold.

PROOF. Equations (A.13) and (A.14) hold by the choice of $\{\alpha_n(\gamma)\}_{n=2}^4$. Therefore, we only need to show (A.15). For $n = 3$,

$$(1 + 2v_3(\gamma))\left(p_3(\gamma) + \frac{|q_3(\gamma)|}{\alpha_3(\gamma)} + |q_2(\gamma)|\alpha_2(\gamma)\right)$$

$$= \frac{3\gamma}{4\gamma + 2}\left(p_3(\gamma) + 2|q_3(\gamma)|^2\left(\frac{1}{1+2v_4(\gamma)} - p_4(\gamma)\right)^{-1}\right.$$

$$\left. + |q_2(\gamma)|^2\left(\frac{1}{1+2v_2(\gamma)} - p_2(\gamma)\right)^{-1}\right),$$

$$\frac{3\gamma}{4\gamma+2}p_3(\gamma) = \frac{3\gamma}{4\gamma+2}\frac{6\gamma^2+14\gamma+12}{(3\gamma+6)(3\gamma+4)} = \frac{\gamma(3\gamma^2+7\gamma+6)}{6\gamma^3+23\gamma^2+26\gamma+8} < \frac{1}{2},$$

$$\frac{6\gamma}{4\gamma+2}|q_3(\gamma)|^2\left(\frac{1}{1+2\nu_4(\gamma)}-p_4(\gamma)\right)^{-1}$$

$$\leq \frac{3\gamma}{2\gamma+1}\frac{3(\gamma+2)(3\gamma+8)(3\gamma+4)(\gamma+1)}{2\gamma(46+67\gamma+29\gamma^2+3\gamma^3)}\left|q_3\left(\frac{2}{3}\right)\right|^2$$

$$= \left(\frac{27}{4}+\frac{351}{124(1+2\gamma)}-\frac{9(1052+1388\gamma+471\gamma^2)}{62(46+67\gamma+29\gamma^2+3\gamma^3)}\right)\frac{11}{756}$$

$$< \left(\frac{27}{4}+\frac{351}{124}\right)\frac{11}{756} = \frac{121}{868},$$

$$\frac{3\gamma}{4\gamma+2}|q_2(\gamma)|^2\left(\frac{1}{1+2\nu_2(\gamma)}-p_2(\gamma)\right)^{-1}$$

$$\leq \frac{3\gamma}{4\gamma+2}\frac{(3\gamma+4)(3\gamma+2)}{3\gamma}\left|q_2\left(\frac{2}{3}\right)\right|^2$$

$$= \left(\frac{27}{8}+\frac{9\gamma}{4}+\frac{5}{8(1+2\gamma)}\right)\frac{2}{105} < \frac{17}{105}$$

for the last inequality we use the fact that $\gamma \leq 2$. For $n = 5$,

$$(1+2\nu_5(\gamma))\left(p_5(\gamma)+\frac{|q_5(\gamma)|}{\alpha_5(\gamma)}+|q_4(\gamma)|\alpha_4(\gamma)\right)$$

$$= (1+2\nu_5(\gamma))\left(p_5(\gamma)+|q_5(\gamma)|+2|q_4(\gamma)|^2\left(\frac{1}{1+2\nu_4(\gamma)}-p_4(\gamma)\right)^{-1}\right).$$

Here,

$$(1+2\nu_5(\gamma))p_5(\gamma) \leq \left(1+2\nu_5\left(\frac{2}{3}\right)\right)p_5(2) = \frac{15}{26}\frac{29}{56} < \frac{1}{3},$$

$$(1+2\nu_5(\gamma))|q_5(\gamma)| \leq |q_5(\gamma)| \leq \left|q_5\left(\frac{2}{3}\right)\right| = \sqrt{\frac{17}{1716}} < \frac{1}{10},$$

$$(1+2\nu_5(\gamma))2|q_4(\gamma)|^2\left(\frac{1}{1+2\nu_4(\gamma)}-p_4(\gamma)\right)^{-1}$$

$$\leq \frac{25}{18}2(1+2\nu_3(\gamma))|q_3(\gamma)|^2\left(\frac{1}{1+2\nu_4(\gamma)}-p_4(\gamma)\right)^{-1} \leq \frac{25}{18}\frac{121}{868} < \frac{2}{9}. \qquad \square$$

LEMMA A.11. *For $\gamma \geq 2$, let $\alpha_2(\gamma) = |q_2(\gamma)|(\frac{1}{1+2\nu_2(\gamma)}-p_2(\gamma))^{-1}$, and $\alpha_3(\gamma) = 1$. Then*

$$(A.16) \qquad\qquad (1+2\nu_2(\gamma))\left(p_2(\gamma)+\frac{|q_2(\gamma)|}{\alpha_2(\gamma)}\right) = 1$$

and

(A.17) $$\left\{ (1 + 2v_3(\gamma)) \left(p_3(\gamma) + \frac{|q_3(\gamma)|}{\alpha_3(\gamma)} + |q_2(\gamma)| \alpha_2(\gamma) \right) \right\} < 1$$

hold.

PROOF. Equation (A.16) holds by the choice of $\alpha_2(\gamma)$. Therefore, we only need to show (A.17). Note that

$$(1 + 2v_3(\gamma)) \left(p_3(\gamma) + \frac{|q_3(\gamma)|}{\alpha_3(\gamma)} + |q_2(\gamma)| \alpha_2(\gamma) \right)$$

$$= \frac{3\gamma}{4\gamma + 2} \left(p_3(\gamma) + |q_3(\gamma)| + |q_2(\gamma)|^2 \left(\frac{1}{1 + 2v_2(\gamma)} - p_2(\gamma) \right)^{-1} \right).$$

Then, since

$$\frac{3\gamma}{4\gamma + 2} p_3(\gamma) = \frac{3\gamma}{4\gamma + 2} \frac{6\gamma^2 + 14\gamma + 12}{(3\gamma + 6)(3\gamma + 4)} = \frac{\gamma(3\gamma^2 + 7\gamma + 6)}{6\gamma^3 + 23\gamma^2 + 26\gamma + 8} < \frac{1}{2},$$

$$\frac{3\gamma}{4\gamma + 2} |q_3(\gamma)| \le \frac{3}{4} |q_3(2)| = \frac{1}{3} \sqrt{\frac{10}{143}} < \frac{1}{9},$$

$$\frac{3\gamma}{4\gamma + 2} |q_2(\gamma)|^2 \left(\frac{1}{1 + 2v_2(\gamma)} - p_2(\gamma) \right)^{-1}$$

$$\le \frac{3\gamma}{4\gamma + 2} \frac{(3\gamma + 4)(3\gamma + 2)}{3\gamma} |q_2(\gamma)|^2 = \frac{(3\gamma + 1)(3\gamma + 2)(3\gamma + 6)}{(4\gamma + 2)(3\gamma + 3)(3\gamma + 4)(3\gamma + 5)}$$

$$< \frac{1}{4\gamma + 2} \le \frac{1}{10},$$

the lemma follows. □

PROOF OF PROPOSITION A.2 FOR $\gamma \ge \frac{2}{3}$. First, assume that $\gamma \ge 2$. Take $\alpha_2(\gamma) = \frac{|q_2(\gamma)|}{1/(1 + 2v_2(\gamma)) - p_2(\gamma)} + \varepsilon(\gamma)$ where $\varepsilon(\gamma) > 0$ will be specified later, and $\alpha_n(\gamma) = 1$ for $n \ge 3$. By Lemmas A.2, A.3, A.7 and A.8,

$$\sup_{n \ge 4} \left\{ (1 + 2v_n(\gamma)) \left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\alpha_n(\gamma)} + |q_{n-1}(\gamma)| \alpha_{n-1}(\gamma) \right) \right\}$$

$$= \sup_{n \ge 4} \{ (1 + 2v_n(\gamma))(p_n(\gamma) + |q_n(\gamma)| + |q_{n-1}(\gamma)|) \}$$

$$\le \sup_{n \ge 4} \left\{ (1 + 2|v_n(\gamma)|) \left(p_n + \frac{1}{4\sqrt{n + \gamma}} + \frac{1}{4\sqrt{n - 1 + \gamma}} \right) \right\}$$

$$= (1 + 2v_4(\gamma)) \left(p_4(\gamma) + \frac{1}{4\sqrt{4 + \gamma}} + \frac{1}{4\sqrt{3 + \gamma}} \right) < 1$$

holds. On the other hand, by Lemma A.11, for sufficiently small $\varepsilon(\gamma) > 0$

$$(1 + 2v_2(\gamma))\left(p_2(\gamma) + \frac{|q_2(\gamma)|}{\alpha_2(\gamma)}\right) < 1$$

and

$$(1 + 2v_3(\gamma))\left(p_3(\gamma) + \frac{|q_3(\gamma)|}{\alpha_3(\gamma)} + |q_2(\gamma)|\alpha_2(\gamma)\right) < 1$$

hold. Therefore, the proof is complete. The same argument works for the case $\frac{2}{3} \le \gamma \le 2$ with Lemmas A.10 and A.8. $\square$

A.3. Proof of Proposition A.3 for $\gamma \ge \frac{2}{3}$. Here, we give a proof of Proposition A.3 for the case $\gamma \ge \frac{2}{3}$.

LEMMA A.12. *For any fixed $\gamma \ge 2$, $\frac{1}{4\sqrt{3+\gamma}} + \frac{1}{4\sqrt{2+\gamma}} < \frac{1}{1-v_3(\gamma)} - p_3(\gamma)$.*

PROOF. By the definition,

$$\frac{1}{1 - v_3(\gamma)} - p_3(\gamma) = \frac{8 + 40\gamma + 38\gamma^2 + 6\gamma^3}{48 + 132\gamma + 108\gamma^2 + 27\gamma^3}$$

$$= \frac{2}{9} + \frac{2(-4 + 16\gamma + 21\gamma^2)}{9(16 + 44\gamma + 36\gamma^2 + 9\gamma^3)}.$$

Therefore, for any $\gamma \ge 3$,

$$\frac{1}{4\sqrt{3 + \gamma}} + \frac{1}{4\sqrt{2 + \gamma}} < \frac{2}{9} < \frac{1}{1 - v_3(\gamma)} - p_3(\gamma).$$

On the other hand,

$$\frac{1}{1 - v_3(\gamma)} - p_3(\gamma) = \frac{1}{4} + \frac{-16 + 28\gamma + 44\gamma^2 - 3\gamma^3}{12(16 + 44\gamma + 36\gamma^2 + 9\gamma^3)}$$

and since $-16 + 28\gamma + 44\gamma^2 - 3\gamma^3 > 0$ for any $2 \le \gamma \le 3$, we have

$$\frac{1}{4\sqrt{3 + \gamma}} + \frac{1}{4\sqrt{2 + \gamma}} < \frac{1}{4} < \frac{1}{1 - v_3(\gamma)} - p_3(\gamma)$$

for $2 \le \gamma \le 3$. $\square$

LEMMA A.13. *For any fixed $\frac{2}{3} \le \gamma \le 2$, $|q_3(\gamma)| + |q_2(\gamma)| < \frac{1}{1-v_3(\gamma)} - p_3(\gamma)$.*

PROOF. By Lemmas A.2, A.4 and A.6,

$$\sup_{2/3 \le \gamma \le 2} \left(|q_3(\gamma)| + |q_2(\gamma)|\right) = |q_3|\left(\frac{2}{3}\right) + |q_2|\left(\frac{2}{3}\right) = \frac{1}{6}\sqrt{\frac{11}{21}} + \frac{1}{3}\sqrt{\frac{6}{35}} < \frac{13}{50}$$

and

$$\frac{1}{1-v_3(\gamma)} - p_3(\gamma) = \frac{13}{50} + \frac{-224 + 284\gamma + 496\gamma^2 - 51\gamma^3}{150(16 + 44\gamma + 36\gamma^2 + 9\gamma^3)}.$$

Then, since $-224 + 284\gamma + 496\gamma^2 - 51\gamma^3 > 0$ for $\frac{2}{3} \leq \gamma \leq 2$,

$$|q_3(\gamma)| + |q_2(\gamma)| < \frac{1}{1-v_3(\gamma)} - p_3(\gamma). \qquad \square$$

LEMMA A.14. *Let* $0 < \varepsilon(\gamma) < \frac{2}{1+3\gamma}$ *and* $\beta_1(\gamma) = \frac{|q_1(\gamma)|3(3\gamma+2)}{2-\varepsilon(\gamma)}$ *and* $\beta_2(\gamma) = 1$. *Then*

$$(1 - v_1(\gamma))\left(p_1(\gamma) + \frac{|q_1(\gamma)|}{\beta_1(\gamma)}\right) < 1$$

and

(A.18) $$(1 - v_2(\gamma))\left(p_2(\gamma) + \frac{|q_2(\gamma)|}{\beta_2(\gamma)} + |q_1(\gamma)|\beta_1(\gamma)\right) < 1$$

hold.

PROOF. By the definition,

$$(1 - v_1(\gamma))\left(p_1(\gamma) + \frac{|q_1(\gamma)|}{\beta_1(\gamma)}\right) = \frac{3}{2}\left(\frac{2(3\gamma + 1)}{3(3\gamma + 2)} + \frac{2 - \varepsilon(\gamma)}{3(3\gamma + 2)}\right)$$

$$= \frac{(3\gamma + 1)}{(3\gamma + 2)} + \frac{2 - \varepsilon(\gamma)}{2(2 + 3\gamma)} < 1.$$

On the other hand,

$$1 - v_2(\gamma) = \frac{3\gamma + 1}{2(2\gamma + 1)},$$

$$p_2(\gamma) = \frac{2(3\gamma^2 + 4\gamma + 2)}{(4 + 3\gamma)(2 + 3\gamma)} = \frac{2\gamma}{2 + 3\gamma} + \frac{4}{(4 + 3\gamma)(2 + 3\gamma)},$$

$$|q_1(\gamma)|\beta_1(\gamma) = \frac{|q_1(\gamma)|^2 3(3\gamma + 2)}{2 - \varepsilon(\gamma)} = \frac{18\gamma(\gamma + 1)(3\gamma + 2)}{(3 + 3\gamma)(2 + 3\gamma)^2(1 + 3\gamma)(2 - \varepsilon(\gamma))}$$

$$= \frac{6\gamma}{(2 + 3\gamma)(1 + 3\gamma)(2 - \varepsilon(\gamma))} < \frac{1}{2 + 3\gamma}$$

and

$$|q_2(\gamma)| = \frac{\sqrt{3(\gamma + 2)(3\gamma + 1)}}{\sqrt{(3 + 3\gamma)(4 + 3\gamma)^2(5 + 3\gamma)}} < \frac{\sqrt{3}(3 + 4\gamma)}{2(4 + 3\gamma)(2 + 3\gamma)}.$$

Therefore,

$$(1 - v_2(\gamma))\left(p_2(\gamma) + \frac{|q_2(\gamma)|}{\beta_2(\gamma)} + |q_1(\gamma)|\beta_1(\gamma)\right)$$

$$< \frac{3\gamma + 1}{2(2\gamma + 1)}\left(\frac{2\gamma + 1}{2 + 3\gamma} + \frac{4}{(4 + 3\gamma)(2 + 3\gamma)} + \frac{\sqrt{3}(3 + 4\gamma)}{2(4 + 3\gamma)(2 + 3\gamma)}\right)$$

$$= \frac{3\gamma + 1}{2(2 + 3\gamma)} + \frac{3\gamma + 1}{2(2\gamma + 1)}\frac{8 + \sqrt{3}(3 + 4\gamma)}{2(4 + 3\gamma)(2 + 3\gamma)}.$$

Now, to show (A.18), we only need to show that

$$\frac{(8 + \sqrt{3}(3 + 4\gamma))(3\gamma + 1)}{4(2\gamma + 1)(4 + 3\gamma)(2 + 3\gamma)} \leq 1 - \frac{3\gamma + 1}{2(2 + 3\gamma)} = \frac{3\gamma + 3}{2(2 + 3\gamma)}$$

which is equivalent to

$$(8 + 3\sqrt{3} + 4\sqrt{3}\gamma)(3\gamma + 1) \leq 2(3\gamma + 3)(2\gamma + 1)(4 + 3\gamma).$$

Then, by comparing coefficients of both sides, we complete the proof. □

PROOF OF PROPOSITION A.3 FOR $\gamma \geq \frac{2}{3}$.   First, assume that $\gamma \geq 2$. Take $\beta_1(\gamma)$ as in Lemma A.14 and $\beta_n(\gamma) = 1$ for $n \geq 2$. By Lemmas A.2, A.3, A.7 and A.12,

$$\sup_{n \geq 3}\left\{(1 - v_n(\gamma))\left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma)\right)\right\}$$

$$= \sup_{n \geq 3}\{(1 - v_n(\gamma))(p_n(\gamma) + |q_n(\gamma)| + |q_{n-1}(\gamma)|)\}$$

$$\leq \sup_{n \geq 3}\left\{(1 + |v_n(\gamma)|)\left(p_n(\gamma) + \frac{1}{4\sqrt{n + \gamma}} + \frac{1}{4\sqrt{n - 1 + \gamma}}\right)\right\}$$

$$= (1 - v_3(\gamma))\left(p_3(\gamma) + \frac{1}{4\sqrt{3 + \gamma}} + \frac{1}{4\sqrt{2 + \gamma}}\right) < 1$$

holds. Therefore, with Lemma A.14, we have

$$\sup_{n \geq 1}\left\{(1 - v_n(\gamma))\left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma)\right)\right\} < 1.$$

The same argument works for the case $\frac{2}{3} \leq \gamma \leq 2$ with Lemmas A.5, A.13 and A.14. □

A.4. Proof of Propositions A.2 and A.3 for $\gamma < \frac{2}{3}$.   Here, we give a proof of Propositions A.2 and A.3 for the case $\gamma < \frac{2}{3}$.

For any $\gamma > 0$, $(\frac{1}{1+2|v_n(\gamma)|} - \frac{1}{2})$ is positive for any $n \geq 2$ and increasing for $n \geq 2$. Therefore, by Lemma A.5, $|q_n(\gamma)|(\frac{1}{1+2|v_n(\gamma)|} - \frac{1}{2})^{-1}$ is decreasing for $n \geq 2$ and

$$\lim_{n \to \infty} |q_n(\gamma)| \left(\frac{1}{1+2|v_n(\gamma)|} - \frac{1}{2} \right)^{-1} = 0$$

holds. Therefore, there exists $n_0 = n_0(\gamma) \in \mathbb{N}$ satisfying for any $n \geq n_0$

$$|q_n(\gamma)| \left(\frac{1}{1+2|v_n(\gamma)|} - \frac{1}{2} \right)^{-1} < \frac{1}{2}.$$

Then it is obvious that

$$\sup_{n \geq n_0+1} \left\{ (1 + 2v_n(\gamma)) \left(\frac{1}{2} + |q_n(\gamma)| + |q_{n-1}(\gamma)| \right) \right\} < 1$$

and

$$\sup_{n \geq n_0+1} \left\{ (1 - v_n) \left(\frac{1}{2} + |q_n(\gamma)| + |q_{n-1}(\gamma)| \right) \right\} < 1.$$

Define $\alpha_n(\gamma)$ as follows:

$$(A.19) \quad \alpha_n(\gamma) = \begin{cases} \max \left\{ |q_2(\gamma)| \left(\frac{1}{1+2v_2(\gamma)} - \frac{1}{2} \right)^{-1}, 1 \right\} \\ \quad \text{if } n = 2, \\ \max \left\{ 2|q_n(\gamma)| \left(\frac{1}{1+2v_n(\gamma)} - \frac{1}{2} \right)^{-1}, 1 \right\} \\ \quad \text{if } n \geq 4 \text{ and } n \text{ is even,} \\ \left(\max \left\{ 2|q_n(\gamma)| \left(\frac{1}{1+2v_{n+1}(\gamma)} - \frac{1}{2} \right)^{-1}, 1 \right\} \right)^{-1} \\ \quad \text{if } n \geq 3 \text{ and } n \text{ is odd} \end{cases}$$

and $\alpha_1 = 0$. Obviously, $\alpha_n(\gamma) = 1$ for any $n \geq n_0 = n_0(\gamma)$.

LEMMA A.15. *For any $n \geq 2$,*

$$(1 + 2v_n(\gamma)) \left(\frac{1}{2} + \frac{|q_n(\gamma)|}{\alpha_n(\gamma)} + |q_{n-1}(\gamma)| \alpha_{n-1}(\gamma) \right) \leq 1.$$

PROOF. In this proof, we denote $v_n(\gamma)$, $p_n(\gamma)$ and $q_n(\gamma)$ by v_n, p_n and q_n when there is no confusion.

By the definition,

$$(1 + 2v_2) \left(\frac{1}{2} + \frac{|q_2|}{\alpha_2} \right) \leq (1 + 2v_2) \left(\frac{1}{2} + \left(\frac{1}{1+2v_2} - \frac{1}{2} \right) |q_2|/|q_2| \right) = 1$$

and for any even number $n \geq 4$,

$$(1 + 2v_n)\left(\frac{1}{2} + \frac{|q_n|}{\alpha_n} + |q_{n-1}|\alpha_{n-1}\right)$$

$$\leq (1 + 2v_n)\left(\frac{1}{2} + \left(\frac{1}{1 + 2v_n} - \frac{1}{2}\right)|q_n|/(2|q_n|)\right.$$

$$\left. + \left(\frac{1}{1 + 2v_n} - \frac{1}{2}\right)|q_{n-1}|/(2|q_{n-1}|)\right) = 1.$$

Next, for $n = 3$,

$$(1 + 2v_3)\left(\frac{1}{2} + \frac{|q_3|}{\alpha_3} + |q_2|\alpha_2\right)$$

$$\leq (1 + 2v_3)\left(\frac{1}{2} + \max\left\{2|q_3|^2\left(\frac{1}{1 + 2v_4} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_4} - \frac{1}{2}\right)\right\}\right.$$

$$\left. + \max\left\{|q_2|^2\left(\frac{1}{1 + 2v_2} - \frac{1}{2}\right)^{-1}, \left(\frac{1}{1 + 2v_2} - \frac{1}{2}\right)\right\}\right)$$

and for any odd number $n \geq 5$,

$$(1 + 2v_n)\left(\frac{1}{2} + \frac{|q_n|}{\alpha_n} + |q_{n-1}|\alpha_{n-1}\right)$$

$$\leq (1 + 2v_n)\left(\frac{1}{2} + \max\left\{2|q_n|^2\left(\frac{1}{1 + 2v_{n+1}} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_{n+1}} - \frac{1}{2}\right)\right\}\right.$$

$$\left. + \max\left\{2|q_{n-1}|^2\left(\frac{1}{1 + 2v_{n-1}} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_{n-1}} - \frac{1}{2}\right)\right\}\right).$$

To conclude the proof, we will show that

$$\max\left\{2|q_3|^2\left(\frac{1}{1 + 2v_4} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_4} - \frac{1}{2}\right)\right\} \leq \frac{1}{2}\left(\frac{1}{1 + 2v_3} - \frac{1}{2}\right),$$

$$\max\left\{|q_2|^2\left(\frac{1}{1 + 2v_2} - \frac{1}{2}\right)^{-1}, \left(\frac{1}{1 + 2v_2} - \frac{1}{2}\right)\right\} \leq \frac{1}{2}\left(\frac{1}{1 + 2v_3} - \frac{1}{2}\right)$$

and for any odd number $n \geq 5$,

$$\max\left\{2|q_n|^2\left(\frac{1}{1 + 2v_{n+1}} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_{n+1}} - \frac{1}{2}\right)\right\} \leq \frac{1}{2}\left(\frac{1}{1 + 2v_n} - \frac{1}{2}\right),$$

$$\max\left\{2|q_{n-1}|^2\left(\frac{1}{1 + 2v_{n-1}} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_{n-1}} - \frac{1}{2}\right)\right\} \leq \frac{1}{2}\left(\frac{1}{1 + 2v_n} - \frac{1}{2}\right).$$

Note that for any odd number $n \geq 3$, $\frac{1}{1+2\nu_n} - \frac{1}{2} > \frac{1}{2}$ and even number $n \geq 4$, $\frac{1}{1+2\nu_n} - \frac{1}{2} < \frac{1}{2}$ and $\frac{1}{1+2\nu_2} - \frac{1}{2} = \frac{3\gamma}{6\gamma+4} < \frac{1}{4}$. Namely, we only need to show that

$$4|q_3|^2 \left(\frac{1}{1+2\nu_4} - \frac{1}{2} \right)^{-1} \leq \left(\frac{1}{1+2\nu_3} - \frac{1}{2} \right),$$

$$2|q_2|^2 \left(\frac{1}{1+2\nu_2} - \frac{1}{2} \right)^{-1} \leq \left(\frac{1}{1+2\nu_3} - \frac{1}{2} \right),$$

$$4|q_n|^2 \left(\frac{1}{1+2\nu_{n+1}} - \frac{1}{2} \right)^{-1} \leq \left(\frac{1}{1+2\nu_n} - \frac{1}{2} \right),$$

$$4|q_{n-1}|^2 \left(\frac{1}{1+2\nu_{n-1}} - \frac{1}{2} \right)^{-1} \leq \left(\frac{1}{1+2\nu_n} - \frac{1}{2} \right).$$

We can rewrite these inequalities as

$$16|q_3|^2 \frac{1+2|\nu_4|}{1-2|\nu_4|} \leq \frac{1+2|\nu_3|}{1-2|\nu_3|}, \qquad 8|q_2|^2 \frac{1+2|\nu_2|}{1-2|\nu_2|} \leq \frac{1+2|\nu_3|}{1-2|\nu_3|},$$

$$16|q_n|^2 \frac{1+2|\nu_{n+1}|}{1-2|\nu_{n+1}|} \leq \frac{1+2|\nu_n|}{1-2|\nu_n|}, \qquad 16|q_{n-1}|^2 \frac{1+2|\nu_{n-1}|}{1-2|\nu_{n-1}|} \leq \frac{1+2|\nu_n|}{1-2|\nu_n|}.$$

Combing the fact that $|q_n|^2$ is decreasing in $n \geq 2$ and $|\nu_n|$ is also decreasing in n, we only need to prove that

$$16|q_3|^2 \leq 1, \qquad 8|q_2|^2 \frac{1+2|\nu_2|}{1-2|\nu_2|} \leq \frac{1+2|\nu_3|}{1-2|\nu_3|}$$

and for any odd number $n \geq 5$,

$$16|q_4|^2 \frac{1+2|\nu_{n-1}|}{1-2|\nu_{n-1}|} \leq \frac{1+2|\nu_n|}{1-2|\nu_n|}.$$

Since $|q_3(\gamma)|^2 < |q_3(0)|^2 = \frac{2}{105}$, the first inequality holds for all $\gamma > 0$. The second inequality is rewritten as

$$|q_2(\gamma)|^2 = \frac{(3\gamma+1)(3\gamma+6)}{(3\gamma+3)(3\gamma+4)^2(3\gamma+5)} \leq \frac{5\gamma+4}{24(3\gamma+2)}$$

and since the coefficients of the polynomial

$$(5\gamma+4)(3\gamma+3)(3\gamma+4)^2(3\gamma+5) - 24(3\gamma+2)(3\gamma+1)(3\gamma+6)$$

are all positive, it is satisfied for any $\gamma > 0$.

Finally, by Lemma A.16 below, to show the last inequality we only need to show that

$$16|q_4(\gamma)|^2 \leq \frac{1+2|\nu_3(\gamma)|}{1-2|\nu_3(\gamma)|} \frac{1-2|\nu_2(\gamma)|}{1+2|\nu_2(\gamma)|}$$

and it follows from the fact

$$16\big|q_4(\gamma)\big|^2 \le 16\big|q_4(0)\big|^2 = \frac{5}{21} < \frac{1}{3} < \frac{5\gamma + 4}{3(3\gamma + 2)}$$

$$= \frac{1 + 2|v_3(\gamma)|}{1 - 2|v_3(\gamma)|}\frac{1 - 2|v_2(\gamma)|}{1 + 2|v_2(\gamma)|}. \qquad \square$$

LEMMA A.16. *For any $n \ge 2$,*

$$\frac{1 + 2|v_{n+1}(\gamma)|}{1 - 2|v_{n+1}(\gamma)|}\frac{1 - 2|v_n(\gamma)|}{1 + 2|v_n(\gamma)|} \ge \frac{1 + 2|v_3(\gamma)|}{1 - 2|v_3(\gamma)|}\frac{1 - 2|v_2(\gamma)|}{1 + 2|v_2(\gamma)|}.$$

PROOF. Consider a function $f(x, a) = \frac{(1+ax)(1-a)}{(1-ax)(1+a)}$ for $0 < a < 1$ and $0 < x < 1$. Then it is easy to see that $\partial_x f(x, a) > 0$ and $\partial_a f(x, a) < 0$ for all $0 < x < 1$ and $0 < a < 1$. Therefore, for any $n \ge 2$,

$$\frac{1 + 2|v_{n+1}(\gamma)|}{1 - 2|v_{n+1}(\gamma)|}\frac{1 - 2|v_n(\gamma)|}{1 + 2|v_n(\gamma)|} = f\left(\frac{n + \gamma}{n + 2\gamma}, 2|v_n(\gamma)|\right)$$

$$\ge f\left(\frac{2 + \gamma}{2 + 2\gamma}, 2|v_n(\gamma)|\right) \ge f\left(\frac{2 + \gamma}{2 + 2\gamma}, 2|v_2(\gamma)|\right)$$

$$= \frac{1 + 2|v_3(\gamma)|}{1 - 2|v_3(\gamma)|}\frac{1 - 2|v_2(\gamma)|}{1 + 2|v_2(\gamma)|}. \qquad \square$$

PROOF OF PROPOSITION A.2 FOR $\gamma < \frac{2}{3}$. Define $\{\alpha_n(\gamma)\}$ as (A.19). Then, for sufficiently large $n \in \mathbb{N}$, $\alpha_n(\gamma) = 1$. Therefore,

$$\limsup_{n\to\infty}\left((1 + 2v_n(\gamma))\left(p_n + \frac{|q_n(\gamma)|}{\alpha_n(\gamma)} + |q_{n-1}(\gamma)|\alpha_{n-1}(\gamma)\right)\right)$$

$$\le \limsup_{n\to\infty}\left((1 + 2v_n(\gamma))\left(\frac{1}{2} + |q_n(\gamma)| + |q_{n-1}(\gamma)|\right)\right) = \frac{1}{2} < 1$$

holds. Then, to show Proposition A.2, we only need to show that with this $\alpha_n(\gamma)$,

$$(1 + 2v_n(\gamma))\left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\alpha_n(\gamma)} + |q_{n-1}(\gamma)|\alpha_{n-1}(\gamma)\right) < 1$$

for all $n \ge 2$. Then, since for any $\gamma < \frac{2}{3}$ and $n \ge 2$, $p_n(\gamma) < \frac{1}{2}$, the proof is complete with Lemma A.15. $\square$

Define β_n as follows:

$$
\text{(A.20)} \qquad \beta_n = \begin{cases}
\max\left\{ |q_1(\gamma)| \left(\dfrac{1}{1 - v_1(\gamma)} - \dfrac{1}{2} \right)^{-1}, 1 \right\} \\
\qquad \text{if } n = 1, \\[6pt]
\max\left\{ 2|q_n(\gamma)| \left(\dfrac{1}{1 - v_n(\gamma)} - \dfrac{1}{2} \right)^{-1}, 1 \right\} \\
\qquad \text{if } n \geq 3 \text{ and } n \text{ is odd}, \\[6pt]
\left(\max\left\{ 2|q_n(\gamma)| \left(\dfrac{1}{1 - v_{n+1}(\gamma)} - \dfrac{1}{2} \right)^{-1}, 1 \right\} \right)^{-1} \\
\qquad \text{if } n \geq 2 \text{ and } n \text{ is even}
\end{cases}
$$

and $\beta_0 = 0$. Obviously, $\beta_n = 1$ for any $n \geq n_0$.

LEMMA A.17. *For any $n \geq 1$*

$$
(1 - v_n(\gamma)) \left(\frac{1}{2} + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma) \right) \leq 1.
$$

PROOF. By the definition,

$$
(1 - v_1(\gamma)) \left(\frac{1}{2} + \frac{|q_1(\gamma)|}{\beta_1(\gamma)} \right) \leq 1
$$

and for any odd number $n \geq 3$,

$$
(1 - v_n(\gamma)) \left(\frac{1}{2} + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma) \right)
$$

$$
\leq (1 - v_n(\gamma)) \left(\frac{1}{2} + \left(\frac{1}{1 - v_n(\gamma)} - \frac{1}{2} \right) |q_n(\gamma)| / (2|q_n(\gamma)|) \right.
$$

$$
\left. + \left(\frac{1}{1 - v_n(\gamma)} - \frac{1}{2} \right) |q_{n-1}(\gamma)| / (2|q_{n-1}(\gamma)|) \right) = 1.
$$

Next, for $n = 2$,

$$
(1 - v_2(\gamma)) \left(\frac{1}{2} + \frac{|q_2(\gamma)|}{\beta_2(\gamma)} + |q_1(\gamma)|\beta_1(\gamma) \right)
$$

$$
\leq (1 - v_2(\gamma)) \left(\frac{1}{2} + \max\left\{ 2|q_2(\gamma)|^2 \left(\frac{1}{1 - v_3(\gamma)} - \frac{1}{2} \right)^{-1}, \right. \right.
$$

$$
\left. \frac{1}{2} \left(\frac{1}{1 - v_3(\gamma)} - \frac{1}{2} \right) \right\}
$$

$$
+ \max\left\{ |q_1(\gamma)|^2 \left(\frac{1}{1 - v_1(\gamma)} - \frac{1}{2} \right)^{-1}, \left(\frac{1}{1 - v_1(\gamma)} - \frac{1}{2} \right) \right\} \right)
$$

and for any even number $n \geq 4$,

$$(1 - v_n(\gamma))\left(\frac{1}{2} + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma)\right)$$

$$\leq (1 - v_n(\gamma))\left(\frac{1}{2} + \max\left\{2|q_n(\gamma)|^2\left(\frac{1}{1 - v_{n+1}(\gamma)} - \frac{1}{2}\right)^{-1},\right.\right.$$

$$\left.\frac{1}{2}\left(\frac{1}{1 - v_{n+1}(\gamma)} - \frac{1}{2}\right)\right\}$$

$$+ \max\left\{2|q_{n-1}(\gamma)|^2\left(\frac{1}{1 - v_{n-1}(\gamma)} - \frac{1}{2}\right)^{-1},\right.$$

$$\left.\left.\frac{1}{2}\left(\frac{1}{1 + 2v_{n-1}(\gamma)} - \frac{1}{2}\right)\right\}\right).$$

To complete the proof, we will show that

$$\max\left\{2|q_2(\gamma)|^2\left(\frac{1}{1 - v_3(\gamma)} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 - v_3(\gamma)} - \frac{1}{2}\right)\right\}$$

$$\leq \frac{1}{2}\left(\frac{1}{1 - v_2(\gamma)} - \frac{1}{2}\right),$$

$$\max\left\{|q_1(\gamma)|^2\left(\frac{1}{1 - v_1(\gamma)} - \frac{1}{2}\right)^{-1}, \left(\frac{1}{1 - v_1(\gamma)} - \frac{1}{2}\right)\right\} \leq \frac{1}{2}\left(\frac{1}{1 - v_2(\gamma)} - \frac{1}{2}\right)$$

and for any even number $n \geq 4$,

$$\max\left\{2|q_n(\gamma)|^2\left(\frac{1}{1 - v_{n+1}(\gamma)} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 - v_{n+1}(\gamma)} - \frac{1}{2}\right)\right\}$$

$$\leq \frac{1}{2}\left(\frac{1}{1 - v_n(\gamma)} - \frac{1}{2}\right),$$

$$\max\left\{2|q_{n-1}(\gamma)|^2\left(\frac{1}{1 - v_{n-1}(\gamma)} - \frac{1}{2}\right)^{-1}, \frac{1}{2}\left(\frac{1}{1 + 2v_{n-1}(\gamma)} - \frac{1}{2}\right)\right\}$$

$$\leq \frac{1}{2}\left(\frac{1}{1 - v_n(\gamma)} - \frac{1}{2}\right).$$

Note that for any even number $n \geq 2$, $\frac{1}{1 - v_n(\gamma)} - \frac{1}{2} > \frac{1}{2}$ and odd number $n \geq 3$, $\frac{1}{1 - v_n(\gamma)} - \frac{1}{2} < \frac{1}{2}$ and $\frac{1}{1 - v_1(\gamma)} - \frac{1}{2} = \frac{1}{6} < \frac{1}{4}$. Namely, we only need to show that

$$4|q_2(\gamma)|^2\left(\frac{1}{1 - v_3(\gamma)} - \frac{1}{2}\right)^{-1} \leq \left(\frac{1}{1 - v_2(\gamma)} - \frac{1}{2}\right),$$

$$2|q_1(\gamma)|^2\left(\frac{1}{1 - v_1(\gamma)} - \frac{1}{2}\right)^{-1} \leq \left(\frac{1}{1 - v_2(\gamma)} - \frac{1}{2}\right),$$

$$4|q_n(\gamma)|^2 \left(\frac{1}{1 - v_{n+1}(\gamma)} - \frac{1}{2} \right)^{-1} \leq \left(\frac{1}{1 - v_n(\gamma)} - \frac{1}{2} \right),$$

$$4|q_{n-1}(\gamma)|^2 \left(\frac{1}{1 - v_{n-1}(\gamma)} - \frac{1}{2} \right)^{-1} \leq \left(\frac{1}{1 - v_n(\gamma)} - \frac{1}{2} \right)$$

for any even number $n \geq 4$. We can rewrite these inequalities as

$$16|q_2(\gamma)|^2 \frac{1 + |v_3(\gamma)|}{1 - |v_3(\gamma)|} \leq \frac{1 + |v_2(\gamma)|}{1 - |v_2(\gamma)|},$$

$$8|q_1(\gamma)|^2 \frac{1 + |v_1(\gamma)|}{1 - |v_1(\gamma)|} \leq \frac{1 + |v_2(\gamma)|}{1 - |v_2(\gamma)|},$$

$$16|q_n(\gamma)|^2 \frac{1 + |v_{n+1}(\gamma)|}{1 - |v_{n+1}(\gamma)|} \leq \frac{1 + |v_n(\gamma)|}{1 - |v_n(\gamma)|},$$

$$16|q_{n-1}(\gamma)|^2 \frac{1 + |v_{n-1}(\gamma)|}{1 - |v_{n-1}(\gamma)|} \leq \frac{1 + |v_n(\gamma)|}{1 - |v_n(\gamma)|}.$$

Combing the fact that $|q_n(\gamma)|^2$ is decreasing in $n \geq 2$ and $|v_n(\gamma)|$ is also decreasing in n, we only need to prove that

$$16|q_2(\gamma)|^2 \leq 1, \qquad 8|q_1(\gamma)|^2 \frac{1 + |v_1(\gamma)|}{1 - |v_1(\gamma)|} \leq \frac{1 + |v_2(\gamma)|}{1 - |v_2(\gamma)|}$$

and for any even number $n \geq 4$,

$$16|q_3(\gamma)|^2 \frac{1 + |v_{n-1}(\gamma)|}{1 - |v_{n-1}(\gamma)|} \leq \frac{1 + |v_n(\gamma)|}{1 - |v_n(\gamma)|}.$$

Since $|q_2(\gamma)|^2 = \frac{(3\gamma+1)(2+\gamma)}{(3\gamma+5)(3\gamma+4)^2(\gamma+1)} < \frac{(3\gamma+1)(2+\gamma)}{16(3\gamma+5)(\gamma+1)} = \frac{3\gamma^2+7\gamma+2}{16(3\gamma^2+8\gamma+5)}$, the first inequality holds. The second inequality is rewritten as

$$|q_1(\gamma)|^2 = \frac{6\gamma(\gamma+1)}{(3\gamma+1)(3\gamma+2)^2(3\gamma+3)} = \frac{2\gamma}{(3\gamma+1)(3\gamma+2)^2} < \frac{5\gamma+3}{24(3\gamma+1)}$$

and since the coefficients of the polynomial

$$(5\gamma + 3)(3\gamma + 2)^2 - 48\gamma$$

are all positive, it is satisfied for any $\gamma > 0$.

Finally, by the same argument of the proof of Lemma A.16, to prove the last inequality we only need to show that

$$16|q_3(\gamma)|^2 \leq \frac{1 + |v_2(\gamma)|}{1 - |v_2(\gamma)|} \frac{1 - |v_1(\gamma)|}{1 + |v_1(\gamma)|}$$

and it follows from the fact

$$16|q_3(\gamma)|^2 \leq 16|q_3(0)|^2 = \frac{32}{105} < \frac{1}{3} < \frac{5\gamma + 3}{3(3\gamma + 1)}$$

$$= \frac{1 + |\nu_2(\gamma)|}{1 - |\nu_2(\gamma)|} \frac{1 - |\nu_1(\gamma)|}{1 + |\nu_1(\gamma)|}. \qquad \square$$

PROOF OF PROPOSITION A.3 FOR $\gamma < \frac{2}{3}$.   Define $\{\beta_n(\gamma)\}$ as (A.20). Then, for sufficiently large $n \in \mathbb{N}$, $\beta_n(\gamma) = 1$. Therefore,

$$\limsup_{n \to \infty} \left((1 - \nu_n(\gamma)) \left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma) \right) \right)$$

$$\leq \limsup_{n \to \infty} \left((1 - \nu_n(\gamma)) \left(\frac{1}{2} + |q_n(\gamma)| + |q_{n-1}(\gamma)| \right) \right) = \frac{1}{2} < 1$$

holds. Then, to show Proposition A.3, we only need to show that with this $\beta_n(\gamma)$,

$$(1 - \nu_n(\gamma)) \left(p_n(\gamma) + \frac{|q_n(\gamma)|}{\beta_n(\gamma)} + |q_{n-1}(\gamma)|\beta_{n-1}(\gamma) \right) < 1$$

for all $n \geq 1$. Then, since for any $\gamma < \frac{2}{3}$ and $n \geq 1$, $p_n(\gamma) < \frac{1}{2}$, the proof is complete with Lemma A.17. $\square$

REFERENCES

[1] BARTHE, F. and WOLFF, P. (2009). Remarks on non-interacting conservative spin systems: The case of gamma distributions. *Stochastic Process. Appl.* **119** 2711–2723. MR2532220

[2] BHATNAGAR, N., CAPUTO, P., TETALI, P. and VIGODA, E. (2007). Analysis of top-swap shuffling for genome rearrangements. *Ann. Appl. Probab.* **17** 1424–1445. MR2344312

[3] CAPUTO, P. (2008). On the spectral gap of the Kac walk and other binary collision processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 205–222. MR2429910

[4] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2003). Determination of the spectral gap for Kac's master equation and related stochastic evolution. *Acta Math.* **191** 1–54. MR2020418

[5] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2014). Spectral gap for the Kac model with hard sphere collisions. *J. Funct. Anal.* **266** 1787–1832. MR3146836

[6] FENG, S., ISCOE, I. and SEPPÄLÄINEN, T. (1997). A microscopic mechanism for the porous medium equation. *Stochastic Process. Appl.* **66** 147–182. MR1440397

[7] GASPARD, P. and GILBERT, T. (2008). On the derivation of Fourier's law in stochastic energy exchange systems. *J. Stat. Mech. Theory Exp.* **2008** p11021.

[8] GASPARD, P. and GILBERT, T. (2009). Heat transport in stochastic energy exchange models of locally confined hard spheres. *J. Stat. Mech. Theory Exp.* **2009** p08020.

[9] GIROUX, G. and FERLAND, R. (2008). Global spectral gap for Dirichlet–Kac random motions. *J. Stat. Phys.* **132** 561–567. MR2415119

[10] GONÇALVES, P., LANDIM, C. and TONINELLI, C. (2009). Hydrodynamic limit for a particle system with degenerate rates. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **45** 887–909. MR2572156

[11] GRIGO, A., KHANIN, K. and SZÁSZ, D. (2012). Mixing rates of particle systems with energy exchange. *Nonlinearity* **25** 2349–2376. MR2956578

[12] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314

[13] KIPNIS, C., MARCHIORO, C. and PRESUTTI, E. (1982). Heat flow in an exactly solvable model. *J*. *Stat*. *Phys*. **27** 65–74. MR0656869

[14] LANDIM, C., SETHURAMAN, S. and VARADHAN, S. (1996). Spectral gap for zero-range dynamics. *Ann*. *Probab*. **24** 1871–1902. MR1415232

[15] MORRIS, B. (2006). Spectral gap for the zero range process with constant rate. *Ann*. *Probab*. **34** 1645–1664. MR2271475

[16] NAGAHATA, Y. and SASADA, M. (2011). Spectral gap for multi-species exclusion processes. *J*. *Stat*. *Phys*. **143** 381–398. MR2793170

[17] OLLA, S. and SASADA, M. (2013). Macroscopic energy diffusion for a chain of anharmonic oscillators. *Probab*. *Theory Related Fields* **157** 721–775. MR3129802

[18] SASADA, M. (2013). On the spectral gap of the Kac walk and other binary collision processes on d-dimensional lattice. In *Symmetries*, *Integrable Systems and Representations* (K. Iohara et al., eds.). *Springer Proc*. *Math*. *Stat*. **40** 543–560. Springer, Heidelberg. MR3077700

DEPARTMENT OF MATHEMATICS
KEIO UNIVERSITY
3-14-1, HIYOSHI, KOHOKU-KU
YOKOHAMA 223-8522
JAPAN
E-MAIL: sasada@math.keio.ac.jp