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THE SHAPE OF A RANDOM AFFINE WEYL GROUP ELEMENT AND RANDOM CORE PARTITIONS

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Let W be a finite Weyl group and $\hat{W}$ be the corresponding affine Weyl group. We show that a large element in $\hat{W}$, randomly generated by (reduced) multiplication by simple generators, almost surely has one of $|W|$-specific shapes. Equivalently, a reduced random walk in the regions of the affine Coxeter arrangement asymptotically approaches one of $|W|$-many directions. The coordinates of this direction, together with the probabilities of each direction can be calculated via a Markov chain on W.

Our results, applied to type $\tilde{A}_{n-1}$, show that a large random n-core obtained from the natural growth process has a limiting shape which is a piecewise-linear graph. In this case, our random process is a periodic analogue of TASEP, and our limiting shapes can be compared with Rost's theorem on the limiting shape of TASEP.

1. Introduction. Let W denote a finite Weyl group with root system R, and let $\hat{W}$ denote the corresponding affine Weyl group, acting on a real vector space V. They are the most important and classical reflection groups.

1.1. *Random walks in the affine Coxeter arrangement.* The affine Coxeter arrangement of W gives a regular tessellation of V. Define a random walk $X = (X_0, X_1, \ldots)$ in the alcoves, called the *reduced random walk*. We start at the fundamental alcove and at each step we cross one adjacent hyperplane chosen uniformly at random, subject to the condition that we never cross a hyperplane twice. See Figure 1.

This process is a transient Markov chain. More algebraically, it is equivalent to a random infinite reduced word for $\hat{W}$ obtained by multiplying by simple generators one at a time, subject to the condition that the length increases. Nonrandom infinite reduced words in the affine Weyl group have a beautiful structure theory, which we recently studied in relation to factorizations in loop groups [18]. We prove here the following.

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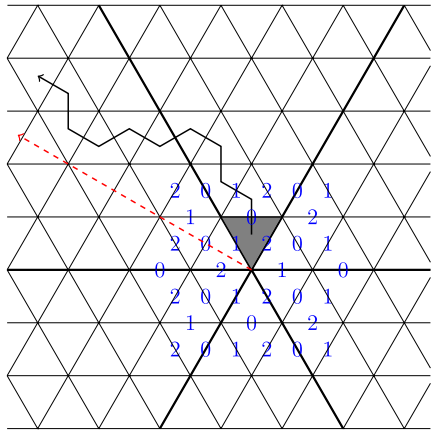

FIG. 1. *A reduced random walk in the alcoves of the $\tilde{A}_2$ arrangement. The shown walk has reduced word $\cdots 1020120210$. The random walk will almost surely be asymptotically parallel to the red dashed line. The thick lines divide V into Weyl chambers.*

THEOREM 1. *Let $(X_0, X_1, \ldots)$ be a reduced random walk in $\hat{W}$. There exists a unit vector $\psi \in V$ so that almost surely we have*

$$\lim_{N\to\infty} v(X_N) \in W \cdot \psi, \tag{1}$$

where $v(X_i)$ denotes the unit vector pointing toward the central point of X_i.

Thus the reduced walk has one of finitely many asymptotic directions. The random walk we study here is different to the walks on hyperplane arrangements that we have seen in the literature; see for example [4, 7].

1.2. *A remarkable Markov chain on W.* In Section 3.1, we define a Markov chain on the finite Weyl group W. Roughly speaking, this Markov chain is obtained by projecting the affine Grassmannian weak order onto W. Unlike the reduced random walk on $\hat{W}$, this Markov chain is irreducible and aperiodic (Proposition 1), and thus has a unique invariant distribution $\{\zeta(w) \mid w \in W\}$.

The vectors $W \cdot \psi$ lie in different Weyl chambers C_w, and we let $X \in C_w$ denote the event that the reduced random walk X eventually stays in C_w. The probabilities $\mathrm{Prob}(X \in C_w)$ vary depending on w: in $\tilde{A}_4$, one Weyl chamber is 96 times more likely than another. The root system notation of the next theorem is reviewed in Section 2.1.

THEOREM 2. *The vector ψ of Theorem 1 is given by*

$$\psi = \frac{1}{Z} \sum_{w\in W\,:\, r_\theta w > w} \zeta(w) w^{-1}(\theta^\vee),$$

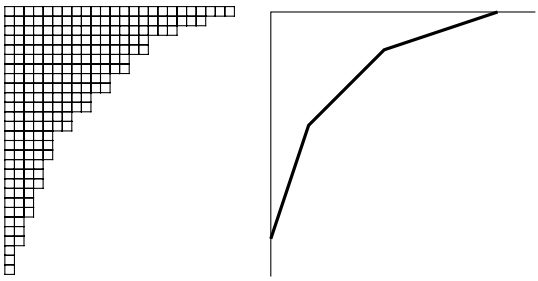

FIG. 2. *A large random* 4*-core, and the piecewise-linear curve* C_4.

where θ *is the highest root of* W *and* Z *is a normalization factor. Furthermore*,

$$\operatorname{Prob}(X \in C_w) = \zeta\left(w^{-1}w_0\right).$$

Thus, the invariant distribution ζ determines two apparently unrelated quantities: the coordinates of the asymptotic directions, and the probabilities of each direction. This surprising duality is ultimately related to the associativity of the Demazure or monoidal product in a Coxeter group. In Section 4.2, we give an alternative formula for $\zeta(w)$, expressed as a calculation involving a sum over the regions of the *Shi arrangement* of W. We also conjecture (Conjecture 2) that in type A the point ψ of Theorem 1 is in the same direction as $\rho^\vee$. In joint work with Williams [20], we conjecture that a multivariate generalization of this Markov chain on the symmetric group has remarkable Schubert positivity properties. Some of these conjectures have been established by Ayyer and Linusson [3] and Linusson and Martin [22].

1.3. *Random* n*-core partitions.* In the case of $W = A_{n-1}$, Theorem 1 applied to a random reduced walk conditioned to remain in the fundamental Weyl chamber can be interpreted in terms of n-core partitions. Recall that a Young diagram is an n-core if no n-ribbon can be removed from it. Grow a random n-core from the empty partition by randomly adding boxes to the Young diagram, subject to the condition that the shape is always an n-core. The notation in the following theorem is explained in Section 5.

THEOREM 3. *For each* n, *there exists a piecewise-linear curve* C_n, *so that for each* $\varepsilon, \delta > 0$, *there exists an* M *such that for every* $N > M$, *we have*

$$\operatorname{Prob}\left(\left|D\left(\lambda^{(N)}\right) - C\right| > \delta\right) < \varepsilon,$$

where $D(\lambda^{(N)})$ *is the diagram of a random* n*-core of degree* N.

Conjecture 2 (verified for $n \leq 6$)[2] gives explicit coordinates for the curve C_n (see Figure 2).

[2]Ayyer and Linusson [2] have reported that they have established Conjecture 2.

There is a growth model on partitions naturally obtained from TASEP on the integer lattice [13, 25], where initially the negative integers are all occupied by balls/particles and the nonnegative integers are all vacant. The particles jump toward the right into adjacent vacant spaces. Our growth process on n-cores corresponds to a periodic analogue of TASEP: now particles that are distance n apart are conditioned to jump together. As explained in Section 5, after appropriate scaling (and assuming Conjecture 2), the limit curve C_n of Theorem 3 approaches, in the limit $n \to \infty$, the degree 2 curve which is the limit shape of TASEP with exponential waiting time [25].

1.4. *(Co)homology of the affine Grassmannian.* In this project, we were initially motivated by the study of families of symmetric functions which represent Schubert classes in the (K)-cohomology of the affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}(n)}$ of $\mathrm{SL}(n)$ [16, 19]. These symmetric functions, called k-Schur functions and affine Stanley symmetric functions, are "affine" analogues of Schur functions, the latter playing a key role in the theory of Schur-measure and Plancherel-measure random partitions. In a similar manner, the symmetric functions mentioned above give rise to Plancherel-like measures on n-cores. These measures are however distinct from the random growth processes studied in this paper.

Instead, our main result may have an interpretation in terms of large products $\xi^N \in K_*(\mathrm{Gr}_{\mathrm{SL}(n)})$ of an element ξ in the K-homology of the affine Grassmannian—it describes the asymptotics of the "spreading out" over the affine Grassmannian of products of this class under the Pontryagin multiplication of a loop group (see Section 5.5).

This connection to the infinite-dimensional geometry of $\mathrm{Gr}_{\mathrm{SL}(n)}$ has concrete probabilistic consequences: in a separate article, we plan to apply this geometry to the calculation of the boundary of the affine Grassmannian weak order.

2. Walks in the affine Coxeter arrangement and reduced words.

2.1. *Affine Weyl groups.* For affine Weyl groups, we use the references [11, 14].

We denote the simple generators of W by $\{s_i \mid i \in I\}$ and by w_0 the longest element of W. Let s_0 be the additional simple generator of $\hat{W}$. The Weyl group acts as linear reflections in a real vector space V, and the affine Weyl group act as affine reflections in V. We let $\ell : W \to \mathbb{Z}$ and $\ell : \hat{W} \to \mathbb{Z}$ denote the length functions.

We let $R \subset V^*$ denote the set of roots of W, and let $R = R^+ \sqcup R^-$ denote the decomposition into positive and negative roots. The set R_{af} of affine roots consists of the elements $\{\alpha + n\delta \mid \alpha \in R \text{ and } n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} - \{0\}\}$. The roots $\hat{\alpha} = \alpha + n\delta$ are the real affine roots, and $\hat{\alpha}$ is positive (resp., negative) if and only if either (a) $\alpha \in R^+$ and $n \geq 0$ (resp., $\alpha \in R^+$ and $n < 0$), or (b) $\alpha \in R^-$ and $n > 0$ (resp., $\alpha \in R^-$ and $n \leq 0$). We denote the positive affine roots by R^+_{af} and the

negative affine roots by R_{af}^-. The simple roots are denoted $\{\alpha_i \mid i \in I \cup \{0\}\}$, and we have $\alpha_0 = \delta - \theta$, where θ is the highest root. We let r_θ denote the reflection in the hyperplane perpendicular to θ.

To each real affine root $\hat{\alpha} = \alpha + k\delta$, we associate the (affine) hyperplane $H_{\hat{\alpha}} = H_\alpha^k = \{v \in V \mid \langle v, \alpha\rangle = -k\}$. The *affine Coxeter arrangement* is the hyperplane arrangement consisting of all such $H_{\hat{\alpha}}$. We also associate to each real affine root $\hat{\alpha}$ a coroot $\hat{\alpha}^\vee$. The connected components of the complement of affine Coxeter arrangement are known as alcoves. The fundamental alcove A° is bounded by the hyperplanes corresponding to the simple roots. There is a bijection $x \mapsto A_x$ between the alcoves and $\hat{W}$, and we shall pick conventions so that $A_{s_i x}$ and A_x are adjacent, separated by the hyperplane corresponding to $x^{-1} \cdot \alpha_i$. The *Weyl chambers* are the connected components of the complement to the finite Coxeter arrangement, where only the H_α's are used for $\alpha \in R$. The *fundamental chamber* is the Weyl chamber containing the fundamental alcove. Affine Weyl group elements corresponding to alcoves inside the fundamental chamber are called *affine Grassmannian*. We shall also need the right action $w : A_x \mapsto A_{xw^{-1}}$ of W on the set of alcoves. The right action of w^{-1} takes the fundamental chamber to the Weyl chamber C_w labeled by w (the one containing the alcove A_w). The elements in C_w are of the form xw, where x is an affine Grassmannian element.

There is an isomorphism $\hat{W} = W \times Q^\vee$, where $Q^\vee$ denotes the coroot lattice of W. If $\lambda \in Q^\vee$, we denote by $t_\lambda \in \hat{W}$ the corresponding element in $\hat{W}$, called a translation element. For $x = wt_\lambda \in \hat{W}$, we have

$$wt_\lambda \cdot (\alpha + n\delta) = w\alpha + \big(n - \langle \lambda, \alpha\rangle\big)\delta. \tag{2}$$

The inversions $\mathrm{Inv}(x) \subset R_{\mathrm{af}}^+$ of x are exactly the real affine roots which are sent to negative roots. Equivalently, $\mathrm{Inv}(x)$ consists of the roots corresponding to hyperplanes separating A_x from A°. Note that with these conventions, A_{t_λ} is obtained from A° by translation by the vector $-\lambda$. The *left weak order* on $\hat{W}$ is given by $x \preceq x'$ if and only if $\mathrm{Inv}(x) \subseteq \mathrm{Inv}(x')$. We shall also write $A \preceq A'$ for the weak order applied to alcoves, and write $A \lessdot A'$ for the cover relations. We say that an alcove A is *of type* w if $A = A_{wt_\lambda}$.

Let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ be the half-sum of positive roots. Recall that $\lambda \in Q^\vee$ is antidominant if $\langle \lambda, \alpha\rangle \le 0$ for $\alpha \in R^+$. The following result is standard [16, 17].

LEMMA 1. *Suppose $x = wt_\lambda$. Then x is affine Grassmannian if and only if λ is antidominant and for every $\alpha \in R^+$ such that $w\alpha \in R^-$ we have $\langle \lambda, \alpha\rangle < 0$. We then have $\ell(x) = -\langle \lambda, 2\rho\rangle - \ell(w)$.*

2.2. *The reduced random walk on alcoves.* We define a random walk on alcoves. The walk begins at $X_0 = A^\circ$. Given $(X_0, X_1, \ldots, X_\ell)$, we pick $X_{\ell+1}$ uniformly at random among the alcoves adjacent to (i.e., sharing a facet with) X_ℓ, with the constraint that the hyperplane separating X_ℓ and $X_{\ell+1}$ has not been crossed

previously. It follows easily from Coxeter group theory that such walks can never "get stuck."

Based on the definition, somewhat surprisingly we get:

LEMMA 2. *The process* $(X_0, X_1, \ldots)$ *is a Markov chain.*

PROOF. The hyperplanes that have been crossed during the first ℓ steps of the walk $(X_0, X_1, \ldots, X_\ell)$ are exactly the hyperplanes separating X_ℓ from $X_0 = A^\circ$. □

We call this process the *random walk in* $\hat{W}$ (or sometimes the *reduced random walk in* $\hat{W}$), starting at the fundamental alcove. We shall also consider the process $(Y_0, Y_1, \ldots)$ where the random walk is constrained to stay within the fundamental Weyl chamber. We call this the *reduced affine Grassmannian random walk* in $\hat{W}$.

2.3. *Reformulation in terms of infinite reduced words.* An infinite reduced word $\mathbf{i} = \cdots i_3 i_2 i_1$ is an infinite word such that $i_r i_{r-1} \cdots i_1$ is a reduced word for $\hat{W}$, for any r. The Coxeter-equivalence of reduced words can be extended to *braid limits* of infinite reduced words. It is known that any infinite reduced word $\mathbf{i}$ of $\hat{W}$ is braid equivalent to an infinite reduced word of the form $\cdots \tau\tau\tau u$, where τ is the reduced word of a translation element, and u is a finite reduced word for $\hat{W}$ (see [12, 18]).

Sequences $(X_0, X_1, \ldots)$ of alcoves as considered in Section 2.2 are tautologically in bijection with infinite reduced words. Thus, Theorem 1 says that a random infinite reduced word $\mathbf{i}$ is not only almost surely braid equivalent to τ^∞ for one of $|W|$-many τ's, but indeed that almost surely $\mathbf{i}$ and τ^∞ asymptotically converge to the same point of the boundary of the Tits cone (cf. [18], Remark 4.5).

3. Projection to the finite Weyl group.

3.1. *A Markov chain on* W. We define a Markov chain with finite state space W, which appears to be of independent combinatorial interest. Let $r = |I| + 1$ be the rank of $\hat{W}$. The transition probability from w to v is given by

$$p_{w,v} = \begin{cases} 1/r, & \text{if } v = s_i w \text{ and } \ell(v) < \ell(w), \\ 1/r, & \text{if } v = r_\theta w \text{ and } \ell(v) > \ell(w), \\ k/r, & \text{if } v = w, \\ 0, & \text{otherwise,} \end{cases}$$

where k is chosen so that $\sum_{v \in W} p_{w,v} = 1$. Let $P = (p_{w,v})$ denote the transition matrix. Let Θ_W denote the directed graph on W with edges given by the nonzero transitions (see Figure 3). Let $Z_0, Z_1, \ldots$ be the Markov chain on Θ_W with transition matrix P.

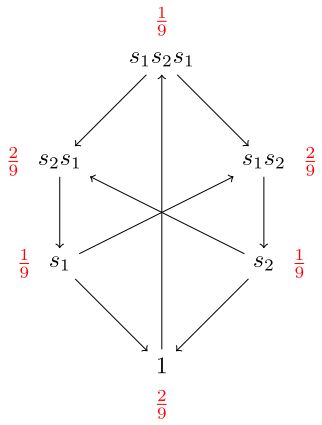

FIG. 3. *The graph* Θ_{S_3} *(with the transitions from a vertex to itself removed) and the stationary distribution* ζ_{S_3}.

PROPOSITION 1. *The Markov chain* $(Z_0, Z_1, \ldots)$ *is irreducible and aperiodic.*

PROOF. Aperiodicity is clear from the definition. Strong connectedness follows from [10], Theorem 4.2. □

It follows that $(Z_0, Z_1, \ldots)$ has a unique limit stationary distribution.

PROBLEM 1. *Explicitly describe the stationary distribution* $\zeta = \zeta_W$ *of* $(Z_0, Z_1, \ldots)$ *for each* W.

This distribution appears to have remarkable enumerative properties, especially for the symmetric group [20].

CONJECTURE 1. *Let* $W = S_n$. *Then* $\zeta(w)/\zeta(w_0)$ *is an integer for all* $w \in W$, *and* $\zeta(1)/\zeta(w_0) = \prod_{k=0}^{n-1} \binom{n}{k} = \max_{w \in W}(\zeta(w)/\zeta(w_0))$.[3]

REMARK 1. The integrality part of Conjecture 1 fails for other types. For example, it is false for W of type B_3. However, the weighted version of Θ_W, as described in Remark 5 and Section 5.5, still appears to retain these properties.

REMARK 2. Let μ_N be the probability measure on length N elements of $\hat{W}$, where $\mu_N(x)$ is proportional to the number of reduced words of x. Define P' by

[3]After this paper was written, Svante Linusson pointed out to us that the integrality part of Conjecture 1 follows from the work of Ferrari and Martin [8] on multitype TASEP. Aas [1] has announced a proof of the product expression for $\zeta(1)/\zeta(w_0)$.

setting the diagonal entries of P to 0. The matrix P' is a sub-stochastic matrix, which nevertheless calculates the projected measures $\pi(\mu_N)$ after scaling. (The matrix P' weights each path equally regardless of the valency of the vertices that it passes through.)

After scaling, and conjugation by a suitable diagonal matrix D, one does obtain a Markov chain with transition matrix given by $Q = rD^{-1}P'D$. The methods in this section will still prove Corollary 1 for the measures μ_N (but with a different limit ψ).

3.2. *Projection.* Let $(Y_0, Y_1, \ldots)$ denote the affine Grassmannian random walk of 2.2. We let $(\tilde{Y}_0, \tilde{Y}_1, \ldots)$ denote the delayed random walk, where $\tilde{Y}_{i+1}$ has probability k/r of being equal to $\tilde{Y}$, where $r = |I| + 1$ is the rank of the affine Weyl group, and k is the number of facets of $\tilde{Y}_i$ which separate $\tilde{Y}_i$ from A°. Each of the transitions in the original random walk now have probability $1/r$. Similarly, define $\tilde{X}$.

Let $\pi : \hat{W} \to W$ be the projection given by $wt_\lambda \mapsto w$. The following proposition is a key observation of the paper.

PROPOSITION 2. *The projection $\pi(\tilde{Y}_0, \tilde{Y}_1, \ldots)$ of the delayed affine Grassmannian random walk is the Markov chain $(Z_0, Z_1, \ldots)$, with initial condition $Z_0 = \mathrm{id}$.*

The result follows from Lemmas 1 and 4.

LEMMA 3. *Let $\alpha \in R^+ - \{\theta\}$. Then $\langle \theta^\vee, \alpha \rangle \in \{0, 1\}$.*

PROOF. The sum $\alpha - k\theta$ can be a root only if $k \in \{0, 1\}$. □

LEMMA 4. *Suppose $x = wt_\lambda \in W_{\mathrm{af}}$ is affine Grassmannian. Then $\ell(r_\theta w) > \ell(w)$ in W if and only if $s_0 x$ is affine Grassmannian and $s_0 x \succ x$.*

PROOF. Suppose that $\ell(r_\theta w) > \ell(w)$. Let $\alpha = w^{-1}\theta \in R^+$. To show that $s_0 x \succ x$, we compute

$$x^{-1}\alpha_0 = t_{-\lambda} w^{-1}(\delta - \theta) = \delta - t_{-\lambda}\alpha = \big(1 - \langle \lambda, \alpha \rangle\big)\delta - \alpha \in R^+_{\mathrm{af}}$$

since λ is antidominant by Lemma 1. To show that $s_0 x$ is affine Grassmannian, we calculate for $\beta \in R^+$

$$\begin{aligned} r_\theta t_{-\theta^\vee} x(\beta) &= r_\theta t_{-\theta^\vee}\big(w\beta - \langle \lambda, \beta \rangle \delta\big) \\ &= (r_\theta w)(\beta) + \big(\langle \theta^\vee, w\beta \rangle - \langle \lambda, \beta \rangle\big)\delta. \end{aligned}$$

We need to show that the root $(r_\theta w)(\beta) + (\langle \theta^\vee, w\beta \rangle - \langle \lambda, \beta \rangle)\delta$ is positive.

First suppose that $\langle \lambda, \beta \rangle = 0$. Then by Lemma 1, we have $w\beta \in R^+$, so since θ is the highest root we must have $\langle \theta^\vee, w\beta \rangle \geq 0$ by Lemma 3. If $\langle \theta^\vee, w\beta \rangle > 0$, we are done. If $\langle \theta^\vee, w\beta \rangle = 0$, we must show that $(r_\theta w)\beta \in R^+$. We calculate that $(r_\theta w)\beta = w r_\alpha \beta$. But $\langle \alpha^\vee, \beta \rangle = \langle \theta^\vee, w\beta \rangle = 0$, so that $w r_\alpha \beta = w\beta \in R^+$.

Now suppose $\langle \lambda, \beta \rangle < 0$. If $w\beta \in R^+$ then by Lemma 3 we have $\langle \theta^\vee, w\beta \rangle \geq 0$, so we would be done. If $w\beta \in R^-$, we note that $w\beta \neq -\theta$ so by Lemma 3 it suffices to assume that $\langle \theta^\vee, w\beta \rangle = -1$ and show that $r_\theta w\beta \in R^+$. But $r_\theta w\beta = w r_\alpha \beta = w(\beta + \alpha) = w\beta + \theta \in R^+$.

For the converse, let us suppose that $\ell(r_\theta w) < \ell(w)$. Let $\alpha = -w^{-1}\theta \in R^+$. We have

$$x^{-1}\alpha_0 = t_{-\lambda} w^{-1}(\delta - \theta) = \alpha + \big(1 + \langle \lambda, \alpha \rangle\big)\delta.$$

But $w\alpha = -\theta \in R^-$, so by Lemma 1, we have $\langle \lambda, \alpha \rangle < 0$. If $\langle \lambda, \alpha \rangle < -1$, then $x^{-1}\alpha_0$ is a negative root, so that $s_0 x \prec x$. Otherwise, we have $\langle \lambda, \alpha \rangle = -1$. In this case, we calculate that

$$(s_0 x)\alpha = (r_\theta t_{-\theta^\vee} w t_\lambda)\alpha = (r_\theta w t_{\lambda + \alpha^\vee})\alpha = r_\theta w\alpha - \big\langle \lambda + \alpha^\vee, \alpha \big\rangle \delta.$$

But $\langle \alpha^\vee, \alpha \rangle = 2$, so $(s_0 x)\alpha \in R^-_{\mathrm{af}}$, and thus $s_0 x$ is not affine Grassmannian. □

3.3. *Proof of Theorem* 1. Let $Z = (Z_0, Z_1, \ldots)$ be a random walk on Θ_W with transition matrix P, and $e = (w \to u)$ an edge in Θ_W. Write $\kappa_{e,N}(Z)$ for the number of times the edge e is used in $(Z_0, Z_1, \ldots, Z_N)$.

LEMMA 5. *We have*

$$\lim_{N \to \infty} \frac{1}{N} \kappa_{e,N}(Z) = \zeta(w)/r,$$

almost surely.

PROOF. This follows from the ergodic theorem for Markov chains; see for example [6], Corollary 4.1. □

PROOF OF THEOREM 1 AND FIRST STATEMENT OF THEOREM 2. We first establish the statement for the delayed affine Grassmannian random walk $(\tilde{Y}_0, \tilde{Y}_1, \ldots)$. Outside a set of measure 0 (those $\tilde{Y}$ that eventually stop), $\tilde{Y}$ naturally maps (by removing repeats) to the random walk Y defined in Section 2.2.

Let the projection of $\tilde{Y}$ to W be $\pi(\tilde{Y}) = Z$, which is a Markov chain on Θ_W by Proposition 2. Write $\tilde{Y}_i = A_{x_i}$, where $x_i = w_i t_{\lambda^{(i)}}$. The translation element $\lambda^{(i)}$ only changes from i to $i+1$ if $x_{i+1} = s_0 x_i$. By Lemma 4, this corresponds to transitions $(w_i \to r_\theta w_i)$ in Z, which changes $\lambda^{(i)}$ by $w_i^{-1}(-\theta^\vee)$ (using $s_0 = r_\theta t_{-\theta^\vee}$).

For two edges e, e', by Lemma 5, the ratio $\frac{\kappa_{e,N}(Z)}{\kappa_{e',N}(Z)}$ converges almost surely to $\zeta(w)/\zeta(w')$. It follows that

$$\lim_{N \to \infty} \operatorname{span}\big(\lambda^{(N)}\big) \to \operatorname{span}\Bigg(\sum_{w \in W \,:\, \ell(r_\theta w) > \ell(w)} \zeta(w) w^{-1}\big(-\theta^\vee\big) \Bigg) \tag{3}$$

almost surely. The alcove $A_{w_i t_{\lambda^{(i)}}}$ shares a vertex with the alcove $A_{t_{\lambda^{(i)}}}$, and so $-\lambda^{(i)}$ points in almost the same direction as $v(\tilde{Y}_i)$. We thus obtain Theorem 1 and the first statement of Theorem 2 for the reduced affine Grassmannian random walk Y.

Now, the random walk $X = (X_0, X_1, \ldots)$ will eventually stay in some Weyl chamber, since each Weyl chamber is separated from the fundamental alcove by some hyperplanes which can be crossed at most once, and there are finitely many Weyl chambers.

The asymptotic direction of Y does not depend on initial point of the random walk, but only the constraint that the walk remains inside the fundamental chamber and heads away from A°. Thus, if we know that $X \in C_w$, we can apply the right action of W to the part of X lying inside C_w to get a random walk in the fundamental chamber which almost surely has asymptotic direction ψ, completing the proof. □

The almost sure convergence of Theorem 1 implies convergence in probability. Pick a norm on V.

COROLLARY 1. *For each $\varepsilon > 0$ and $\delta > 0$, there is a $M = M(\varepsilon, \delta)$ so that*

$$\mathrm{Prob}\big(|v(Y_N) - \psi| \geq \varepsilon\big) < \delta$$

for $N > M$.

REMARK 3. It follows from the proof of Theorem 1 that the point ψ has rational coordinates, when written in terms of simple coroots. This implies that there is a translation element of $\hat{W}$ which points in the same direction as ψ.

REMARK 4. In Theorem 1 and Corollary 1, only the limiting direction is discussed. The formula in Lemma 1 for the length $\ell(t_\lambda)$ of a translation element allows us to calculate the speed that the random walk is traveling from the fundamental alcove.

We give an explicit conjecture for ψ when $W = S_n$. In the next result we treat ρ as a point in V by identifying V and V^* in the usual way.

CONJECTURE 2. *For $W = S_n$, we have $\psi = \gamma\rho$ for some $\gamma > 0$.*

REMARK 5. Conjecture 2 does not hold as stated for other types. Define $\{a_i \mid i \in I\}$ by $\theta = \sum_i a_i \alpha_i$, and set $a_0 = 1$. Now, weight the transitions corresponding to left multiplication by s_i by a factor of a_i. Then our computations suggest that Conjecture 2 still holds for type B_n, and that it is close to holding in other types. The coefficients a_i here are connected via affine Dynkin diagram duality to the coefficients $a_i^\vee$ that we expected to see for reasons related to the topology of the affine Grassmannian; see Section 5.5. The duality may be an artifact of our choice of $Q^\vee$ instead of Q for the definition of an affine Weyl group.

4. The probability of eventually staying in a Weyl chamber.

4.1. *Global reversal of the random walk on* $\hat{W}$. Let $X = (X_0, X_1, \ldots)$ be the reduced random walk in $\hat{W}$. Write $X \in C_w$ for the event that X eventually stays in the Weyl chamber C_w. Write $X_N \in C_w^v$ if $X_N \in C_w$ and the type of X_N is v. We use the same notation for the delayed random walk $\tilde{X}$.

The reverse of the random walks X or $\tilde{X}$ is a very different process to the original process. For example, X can go in many directions, at least at the beginning of the walk, but reversing X gives a walk which heads toward the fundamental chamber. Thus, the next result is very surprising. It relies on a very special feature of Coxeter groups, namely the associativity of the Demazure product.

Let K denote the affine 0-Hecke algebra of $\hat{W}$ (see [19]), with generators $\{T_i \mid i \in I \cup \{0\}\}$, a $\mathbb{Z}$-basis $\{T_x \mid x \in \hat{W}\}$ where $T_{\mathrm{id}} = 1$, satisfying the multiplication formulae

$$T_i T_x = \begin{cases} T_{s_i x}, & \text{if } \ell(s_i x) > \ell(x), \\ T_x, & \text{otherwise,} \end{cases}$$

and also

$$T_x T_i = \begin{cases} T_{x s_i}, & \text{if } \ell(x s_i) > \ell(x), \\ T_x, & \text{otherwise.} \end{cases}$$

In the following, we will freely identify alcoves with elements of $\hat{W}$.

LEMMA 6. *For each* $x \in \hat{W}$, *we have* $\mathrm{Prob}(\tilde{X}_N = x) = \mathrm{Prob}(\tilde{X}_N = x^{-1})$, *and* $\mathrm{Prob}(X_N = x) = \mathrm{Prob}(X_N = x^{-1})$.

PROOF. Let $\xi = \frac{1}{|I|+1}(\sum_{i \in I \cup \{0\}} T_i) \in K$. Then $\mathrm{Prob}(\tilde{X}_N = x) = [T_x](\xi)^N$ where $[T_x]$ denotes the coefficient of T_x when an element of K is written in the basis $\{T_y \mid y \in \hat{W}\}$. But the element ξ of K is invariant under the algebra antimorphism $T_x \mapsto T_{x^{-1}}$ of K. It follows that the coefficient of T_x and $T_{x^{-1}}$ in the product ξ^N coincides. Restricting to elements with length N gives the second statement. □

We call $x = w t_\lambda \in \hat{W}$ regular if $\lambda \in Q^\vee$ is regular, that is, the stabilizer subgroup of W acting on λ is trivial.

LEMMA 7. *Suppose* $x \in C_w^v$ *is regular. Then* $x^{-1} \in C_{w_0 w v^{-1}}^{v^{-1}}$.

PROOF. If $x \in C_w^v$ is regular, then $x = v t_{w^{-1}\mu}$, where μ is a regular and antidominant. Then $x^{-1} = w^{-1} t_{-\mu} w v^{-1} = w^{-1} w_0 t_{w_0(-\mu)} w_0 w v^{-1}$, and $w_0(-\mu)$ is antidominant. □

PROOF OF SECOND STATEMENT OF THEOREM 2. It is clear that $\mathrm{Prob}(X \in C_w) = \mathrm{Prob}(\tilde{X} \in C_w)$, so we shall focus on the delayed walk. Let $\eta(w) = \mathrm{Prob}(\tilde{X} \in C_w)$. In the proof of Theorem 1, we considered the delayed affine Grassmannian walk $\tilde{Y}$, or equivalently, a walk conditioned to lie in C_{id}. It follows from Proposition 2 that for such a walk $\mathrm{Prob}(\tilde{Y} \in C_{\mathrm{id}}^v) = \zeta(v)$. This same argument can be applied to a walk conditioned to lie in any of the cones C_w, and we obtain

$$\lim_{N\to\infty} \mathrm{Prob}\big(\tilde{X}_N \in C_w^v\big) = \eta(w)\zeta\big(vw^{-1}\big).$$

It follows from Theorem 1 that $\mathrm{Prob}(\tilde{X}_N \text{is regular}) \to 1$ as $N \to \infty$. Thus, using Lemmas 6 and 7, for each ε we can find N sufficiently large so that

$$\big|\mathrm{Prob}\big(\tilde{X}_N \in C_w^v\big) - \mathrm{Prob}\big(\tilde{X}_N \in C_{w_0 w v^{-1}}^{v^{-1}}\big)\big| < \varepsilon.$$

It follows that $\eta(w)\zeta(vw^{-1}) = \eta(w_0 w v^{-1})\zeta(w^{-1}w_0)$ for every $v, w \in W$. We note that setting $\eta(w) = \zeta(w^{-1}w_0)$ solves this equation, and since η is a probability measure on W this must be the solution. $\square$

4.2. *The Shi arrangement*. The ideas here are related to the language of reduced words in affine Coxeter groups; see, for example, [5, 9]. The *Shi arrangement* is the hyperplane arrangement consisting of the hyperplanes $\{H_\alpha^0, H_\alpha^1 \mid \alpha \in R^+\}$. One of the regions (connected components of the complement) of the Shi arrangement is exactly the fundamental alcove A°.

Let B and B' be two regions of the Shi arrangement. We say that B is less than or equal to B', and write $B \trianglelefteq B'$ if the set of hyperplanes of the Shi arrangement separating B' from the fundamental alcove, contains the same set for B.

Let Γ denote the set of pairs (B, w), where B is a region of the Shi arrangement, and $w \in W$ is such that B contains an alcove of type w. We make Γ into a directed graph by defining edges $(B, w) \to (B', w')$ whenever $B \trianglelefteq B'$, and an alcove A of type w in B is adjacent (shares a facet) with an alcove A' of type w' in B', satisfying $A \lessdot A'$.

LEMMA 8. *If $(B, w) \to (B', w')$ then every alcove A of type w in B shares a facet with an alcove A' of type w' in B', and we have $A \lessdot A'$.*

PROOF. Suppose A and $\tilde{A}$ are both of type w inside B. Set $\tilde{A} = A + \lambda$. Let H be a hyperplane (not necessarily belonging to the Shi arrangement) cutting out a facet of (the closure of) A, and suppose A' is on the other side of H, adjacent to A and satisfying $A \lessdot A'$. Similarly, define $\tilde{A}'$ adjacent to $\tilde{A}$, on the other side of $\tilde{H} := H + \lambda$. Clearly, $\tilde{A}' = A' + \lambda$.

Since A and $\tilde{A}$ belong to the same region of the Shi arrangement, the line segment joining the center of A to the center of $\tilde{A}$ does not intersect the Shi arrangement. But one can go from A' to $\tilde{A}'$ by crossing H, traveling from A to $\tilde{A}$ and

crossing $\tilde{H}$. Thus, the only hyperplanes of the Shi arrangement that could separate A' from $\tilde{A}'$ are the parallel hyperplanes H and $\tilde{H}$.

Suppose first that H belongs to the Shi arrangement. If at least one of H or $\tilde{H}$ separates A' from $\tilde{A}'$, then since A and $\tilde{A}$ are on the same side of H, it follows that H separates A' from $\tilde{A}'$. We have that λ cannot be parallel to H (otherwise $H = \tilde{H}$). Let H be orthogonal to the root α, so that we must have $\langle \lambda, \alpha \rangle \neq 0$. But from (2) it is easy to see that one of the hyperplanes H_α^k was crossed going from A to $\tilde{A}$. It follows that the region B is not bounded in the α direction. The hyperplane H must thus be H_α^0 or H_α^1. In either case, it separates A from A°, contradicting the assumption $A \lessdot A'$.

So if H belongs to the Shi arrangement, we conclude that $H = \tilde{H}$, and that $\tilde{A}'$ and A' belong to the same region B' of the Shi arrangement. Since H separates A' from A°, and we also have $\tilde{A} \lessdot \tilde{A}'$.

Finally, suppose that H does not belong to the Shi arrangment. Then A, A', $\tilde{A}$, $\tilde{A}'$ all belong to the same region B, and are all separated from A° by some H_α^0 or H_α^1 parallel to H. In this case, the claim is clear. $\square$

Denote by B_w the unique region of the Shi arrangement that is a translation of the Weyl chamber C_w. Let Γ' be the graph obtained from Γ by removing $\{(B_v, u) \mid v, u \in W\}$. Let M be the transition matrix of Γ and let M' be its restriction to Γ'. Let $\mathbf{p}^w$ be the vector with components labeled by vertices of Γ', given by $\mathbf{p}^w_{(B,v)} = \sum_{u \in W} \mathrm{Prob}((B, v) \to (B_w, u))$. Note that for each (B, v), there is at most one $u \in W$ for which the probability $\mathrm{Prob}((B, v) \to (B_w, u))$ is nonzero.

Let $\varepsilon_{(B,w)}$ denote the unit vector corresponding to a vertex of Γ, and $\langle \cdot, \cdot \rangle$ denote the natural inner product on the vertex space spanned by vertices of Γ.

THEOREM 4. *For each $w \in W$,*

$$\zeta\left(w^{-1} w_0\right) = \mathrm{Prob}(X \in C_w) = \left\langle \left(I - M'\right)^{-1} \cdot \varepsilon_{(A^\circ, 1)}, \mathbf{p}^w \right\rangle.$$

PROOF. Lemma 8 guarantees that the Markov chain $X = (X_0, X_1, \ldots)$ projects to a Markov chain on Γ via $x = v t_\lambda \mapsto (B, v)$ where the alcove A_x lies in the region B. Thus, the probability $\mathrm{Prob}(X \in C_w)$ we desire is equal to the probability that a random walk in Γ starting from $(A^\circ, 1)$, with transition matrix M, eventually ends up at one of the vertices (B_w, v). This immediately gives the stated formula, assuming that $(I - M')^{-1}$ is invertible, and is equal to $I + M' + (M')^2 + \cdots$.

Let B be a region of the Shi arrangement which lies between two parallel hyperplanes H_α^0 and H_α^1. Then for each $A \in B$, there is some $A' \succ A$ outside of B. It follows that the random walk $(X_0, X_1, \ldots)$ has probability 0 of staying in a region of the Shi arrangement other than one of the B_w's. Thus, $I - M'$ must be invertible, M' must be strictly substochastic, and $I + M' + (M')^2 + \cdots = I - M'$. $\square$

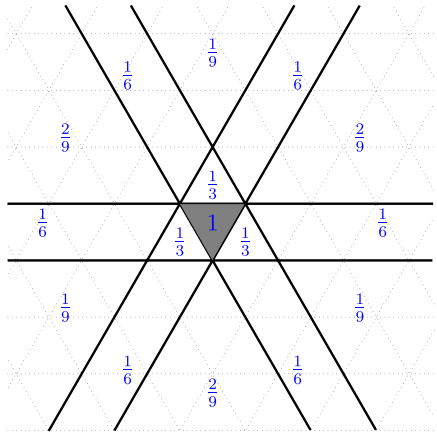

FIG. 4. *Probabilities that X passes through each region of the Shi arrangement of $\tilde{A}_2$. The probabilities of the (translated) Weyl chambers should be compared with Figure 3, illustrating Theorems 2 and 4.*

Theorem 4 is illustrated in Figure 4.

5. *n*-cores, periodic TASEP and the connection to symmetric functions.

5.1. *n-cores and affine Grassmannian permutations.* In this section, we suppose $W = S_n$ is the symmetric group. We assume basic familiarity with Young diagrams. Recall that a skew Young diagram λ/μ is a ribbon if it is edge-connected and does not contain any 2×2 square. A Young diagram λ is called an *n-core* if no ribbons of size n can be removed from it (and still leaving a Young diagram).

The set of n-cores can be built from the empty partition by the following procedure. Take an n-core λ, and suppose b is an addable-corner of λ on diagonal d. Then the Young diagram obtained from λ by adding all addable-corners on diagonals d' satisfying $d' \equiv d \mod n$, is also an n-core, and recursively one obtains every n-core in this way. Figure 5 shows the start of the 3-core graph, where the edges denote the above box adding operation. The 3-core graph is the one-skeleton of a hexagonal planar tiling. The following result is well known; see [17].

PROPOSITION 3. *There is a natural bijection between n-cores and the affine Grassmannian elements of $\tilde{S}_n$. The edges of the n-core graph correspond to left-multiplication by simple generators.*

In the following, we use the standard coordinates for $Q^\vee$, so that $\alpha_i^\vee = e_i - e_{i+1}$.

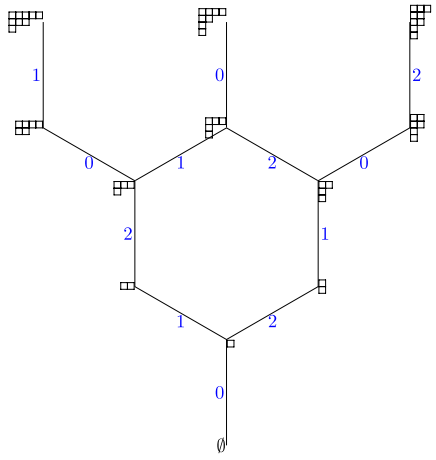

FIG. 5. *The graph of* 3*-cores, with edges labeled by the corresponding simple generator. Note that* 3*-cores on the same level do not have the same number of boxes.*

LEMMA 9. *Let* $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in Q^\vee$ *be an antidominant element of the coroot lattice. Then the* n*-core of the translation element* $t_{(\mu_1,\mu_2,\ldots,\mu_n)}$ *has slope* $(n-i)/i$ *between diagonals* $n\mu_i + i - 2$ *and* $n\mu_{i+1} + i - 2$*, for* $i = 1, 2, \ldots, n-1$.[4]

PROOF. Follows from [17], Proposition 8.10. □

The 4-core in Figure 2 corresponds to $(-7, -2, 3, 6) \in Q^\vee$.

5.2. *The shape of a random* n*-core.* By a random n-core we will mean an n-core generated by applying the bijection in Proposition 3 to the Markov chain Y described in Section 2.2. If λ is a n-core, then we let $D(\lambda)$ denote the curve drawing out the lower-right boundary of λ, scaled by the degree $\deg(\lambda)$ in both directions. Here, the degree is the length of the corresponding affine Grassmannian element from Proposition 3, or equivalently, the distance from the empty partition in the n-core graph. By convention, $D(\lambda)$ includes a vertical ray going to $-\infty$ along the y-axis, and a horizontal ray going to $+\infty$ along the x-axis. Given two curves D, D' of this form, we write $|D - D'|$ to denote the supremum of the distance between D and D', measured along the diagonals $y = -x + k$. With this notation, Corollary 1 combined with Lemma 9 translates to Theorem 3.

Let us use Conjecture 2 to predict the piecewise-linear curve C_n of Theorem 3. Let μ be an antidominant element of $Q^\vee$ satisfying $\mu_2 - \mu_1 = \mu_3 - \mu_2 = \cdots = \mu_n - \mu_{n-1} = A$ (i.e., μ is in the same direction as ρ). To calculate the correct

[4]The slope should be calculated between the points of intersection of the boundary of the core, and the diagonals, but for our asymptotic purposes this is not important.

scaling we use Lemma 1 which says that $\ell(t_\mu) = \sum_{1\le i<j\le n} \mu_j - \mu_i = A/\alpha$, where $\alpha = \frac{6}{(n-1)n(n+1)}$.

Now consider the piecewise-linear curve C_ρ which successively connects the points

$$(0,-\infty),\qquad \left(0,-\frac{n(n-1)}{2}\alpha\right),\qquad (\alpha,-(1+2+\cdots+n-2)\alpha),$$
$$((1+2)\alpha,-(1+2+\cdots+n-3)\alpha),\qquad \ldots,$$
$$((1+2+\cdots+n-2)\alpha,-\alpha),\qquad \left(\frac{n(n-1)}{2}\alpha,0\right),\qquad (\infty,0).$$

Using Lemma 9, one calculates that the core λ corresponding to t_μ has diagram $D(\lambda)$ extremely close to C_n: namely, it passes through the specified points but may not be linear in between those points. Thus, we have the following proposition.

PROPOSITION 4. *Assuming Conjecture 2, the curve C_n of Theorem 3 is C_ρ.*

This proposition allows us to make some predictions, for example, of the length of the first row of a random n-core. This might be compared to corresponding results for random partitions (see, e.g., [23, 27]).

COROLLARY 2. *Assuming Conjecture 2, the expected length of the first row of a random n-core of degree d is asymptotic to $\frac{3d}{n+1}$.*

The area between C_ρ and the axes is equal to

$$\operatorname{area}(C_\rho) = \frac{1}{2}\alpha^2\big((n-1)^2+2(n-2)^2+\cdots+(n-1)1^2\big) = \frac{n^2(n^2-1)\alpha^2}{24}.$$

If we scale the limit shape so that this area is normalized to 1, then the x-intercept of C_ρ would become $\frac{\sqrt{6}(n-1)}{\sqrt{n^2-1}}$.

COROLLARY 3. *Assuming Conjecture 2, the first row of a large random n-core is asymptotic to $\frac{\sqrt{6}(n-1)}{\sqrt{n^2-1}}\sqrt{N}$, where N is the number of boxes in the n-core.*

5.3. *Periodic TASEP.* There is a well-known correspondence between growth models on Young diagrams, and the totally asymmetric exclusion process (TASEP). The random growth model on n-cores we have described gives rise to a periodic analogue of TASEP that we now describe.

Let $\sigma = (\sigma_i \in \{0,1\} \mid i \in \mathbb{Z})$ be a doubly infinite sequence of 0-s and 1-s, labeled by the integers. The sequence σ is to be thought of as a sequence of balls and empty spaces: $\sigma_i = 0$ mean that position i is empty, and $\sigma_i = 1$ means that position i is

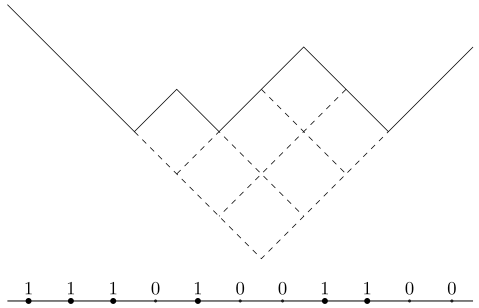

FIG. 6. *The calculation of* $\sigma((3,2,2))$ *is illustrated here*: *we first rotate the Young diagram* 135 *degrees counterclockwise*, *then we draw the outline curve* (*illustrated in solid lines*). *Downward steps in the outline curve corresponds to* 1-*s*, *and upward steps correspond to* 0-*s*.

occupied. There is a natural map $\lambda \mapsto \sigma(\lambda)$, illustrated in Figure 6. The indexing is normalized so that $\sigma(\varnothing)$ is the step-function satisfying $\sigma_i = 1$ for $i < 0$ and $\sigma_i = 0$ for $i \geq 0$. It is clear that adding a box to λ corresponds to moving a ball to an empty space immediately to its right.

Suppose λ is an n-core. Then $\sigma(\lambda)$ satisfies:

(1) $\sigma_i = 1$ for $i \ll 0$,
(2) if $\sigma_i = 1$ then $\sigma_{i-n} = 1$ and
(3) if $d_1, d_2, \ldots, d_n \in \mathbb{Z}$ are such that $\sigma_{d_j} = 1$ and $\sigma_{d_j+n} = 0$, then we have $\sum_{j=1}^{n} d_j = -\binom{n+1}{2}$,

and these conditions characterize the sequences that arise from n-cores. Periodic TASEP is a random process on these sequences, given by the rules:

(1) At time $t = 0$, we have $\sigma(0)$ is the step-function.
(2) At each time t, an element $\bar{i} \in \mathbb{Z}/n\mathbb{Z}$ is chosen uniformly at random, subject to the condition that there exists $i_0 \equiv \bar{i} \bmod n$ satisfying $\sigma_{i_0} = 1$ and $\sigma_{i_0+1} = 0$. We then define $\sigma(t+1)$ by moving all balls at positions $i \equiv i \bmod n$ one step to the right, if possible.

The conditioning implies that at each time step finitely many, but nonzero number of balls are moved.

PROPOSITION 5. *The random* n*-core process is transformed under* $\lambda \mapsto \sigma(\lambda)$ *to the periodic TASEP process*.

When $n = \infty$, periodic TASEP becomes one of the standard discrete time versions of the TASEP process. Namely, at each time t, one of the balls that can be moved is chosen uniformly at random, and moved one step to the right. The asymptotic behavior of TASEP is a very well-studied problem. In particular, Rost [25] (see also Johansson [13]) has described the asymptotic shape of the result.

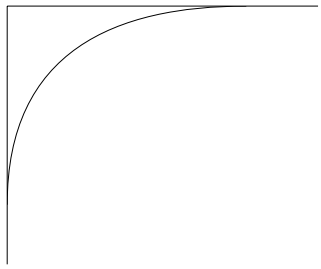

FIG. 7. *The limiting curve C for TASEP.*

We describe their result in terms of Young diagrams, and also rotated so that Young diagrams are upper-left justified. As $t \to \infty$, the Young diagram of this growth process, after suitable scaling, approaches the limiting curve (see Figure 7)

$$C = \{(x,0) \mid x \in [1,\infty)\} \cup \{(x,y) \in [0,1] \times [-1,0] \mid \sqrt{x} + \sqrt{-y} = 1\} \cup \{(y,0) \mid y \in [-1,-\infty)\}.$$

It is not hard to see that after a suitable scaling, the piecewise-linear curves C_ρ of Section 5.2 approaches C pointwise, as $n \to \infty$.

5.4. *Plancherel measure for n-cores.* This work was motivated by the connections to a family $\tilde{F}_x(X)$ of symmetric functions labeled by $x \in \tilde{S}_n$, known as *affine Stanley symmetric functions* [15] (and also a closely related family $\tilde{G}_x(X)$ called the affine stable Grothendieck polynomials [19]). The coefficient $[m_{1^{\ell(x)}}]\tilde{F}_x(X)$ of the square-free monomial in $\tilde{F}_x$ is equal to the number of reduced words of x. Whereas Stanley's seminal work [26] studies *exact* formulae for the number of reduced words, our approach looks for *asymptotic* formulae. The symmetric functions $\tilde{F}_x$ plays the same role for affine permutations, namely, a generating function for "semi-standard" objects, as the Schur functions s_λ play for Grassmannian permutations. Schur functions play a crucial role in the study of random partitions; see, for example, [24].

The measure we obtain on the set $\{x \in \tilde{S}_n \mid \ell(x) = N\}$ of affine permutations of length N from our random walk is not the same measure as the one obtained by letting $\mathrm{Prob}(x)$ be proportional to the number of reduced words of x. Nevertheless, Corollary 1 and Theorem 3 still apply (see Remark 2).

In [17], we proved an enumerative identity

$$m! = \sum_\lambda \#\{\text{weak tableaux of shape } \lambda\} \cdot \#\{\text{strong tableaux of shape } \lambda\}, \tag{4}$$

where the sum is over n-cores of degree m. *Weak tableaux* count paths in the n-core graph. *Strong tableaux* are defined in terms of the strong (Bruhat) order. The terms on the right-hand side of (4) would give the natural analogue of the Plancherel

measure for partitions. In [17], a symmetric function generalization of (4) is also given, and involves affine Stanley symmetric functions and *k-Schur functions*. The identity (4) is generalized to the Kac–Moody case in [21].

5.5. *K-homology of the affine Grassmannian.* Recall from the proof of Lemma 6 in Section 4.1 that the probabilities $\mathrm{Prob}(X_N = x)$ were given by the coefficients $[T_x]\xi^N$ for an element $\xi \in K$. In the case $W = S_n$, by [19], Corollary 7.5, the element ξ can be interpreted (up to a factor) as the divisor Schubert class in the K-homology $K^*(\mathrm{Gr}_{\mathrm{SL}(n)})$ of the affine Grassmannian of $\mathrm{SL}(n)$. The affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}(n)}$ is an infinite-dimensional space of central importance in representation theory. In the case of a complex simple algebraic group with Weyl group W, the natural element to consider from the point of view of the geometry of Gr_G is

$$\xi' = \sum_{i=I\cup\{0\}} a_i^\vee T_i,$$

where the definition of the weights $a_i^\vee$ can be found in [14]; see [21], Proposition 2.17, for an explanation of these weights (the argument in [21] is for the homology case, but easily extends to K-homology). Probabilistically, this amounts to considering random walks where the allowable transitions are not taken uniformly at random, but left multiplication by s_i is weighted by the $a_i^\vee$. Note that Theorem 1 and its proof still remain valid in this situation. See also Remark 5.

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