The Annals of Probability
2015, Vol. 43, No. 1, 405–434
DOI: 10.1214/14-AOP910

BROWNIAN MOTION AND THERMAL CAPACITY[1]

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Let W denote d-dimensional Brownian motion. We find an explicit formula for the essential supremum of Hausdorff dimension of $W(E)\cap F$, where $E\subset(0,\infty)$ and $F\subset\mathbf{R}^d$ are arbitrary nonrandom compact sets. Our formula is related intimately to the thermal capacity of Watson [*Proc. Lond. Math. Soc.* (3) **37** (1978) 342–362]. We prove also that when $d\ge 2$, our formula can be described in terms of the Hausdorff dimension of $E\times F$, where $E\times F$ is viewed as a subspace of space time.

1. Introduction. Let $W:=\{W(t)\}_{t\ge 0}$ denote standard d-dimensional Brownian motion where $d\ge 1$. The principal aim of this paper is to describe the Hausdorff dimension $\dim_{\mathrm{H}}(W(E)\cap F)$ of the random intersection set $W(E)\cap F$, where E and F are compact subsets of $(0,\infty)$ and $\mathbf{R}^d$, respectively. This endeavor solves what appears to be an old problem in the folklore of Brownian motion; see Mörters and Peres [16], page 289.

In general, the Hausdorff dimension of $W(E)\cap F$ is a random variable, and hence we seek only to compute the $L^\infty(\mathrm{P})$-norm of that Hausdorff dimension. The following example—due to Gregory Lawler—highlights the preceding assertion: Consider $d=1$, and set $E:=\{1\}\cup[2,3]$ and $F:=[1,2]$. Also, consider the two events:

$$A_1:=\{1\le W(1)\le 2, W([2,3])\cap[1,2]=\varnothing\},$$
$$A_2:=\{W(1)\notin[1,2], W([2,3])\subset[1,2]\}. \tag{1.1}$$

Evidently, A_1 and A_2 are disjoint; and each has positive probability. However, $\dim_{\mathrm{H}}(W(E)\cap F)=0$ on A_1, whereas $\dim_{\mathrm{H}}(W(E)\cap F)=1$ on A_2. Therefore, $\dim_{\mathrm{H}}(W(E)\cap F)$ is nonconstant, as asserted.

Our first result describes our contribution in the case that $d\ge 2$. In order to describe that contribution, let us define ϱ to be the *parabolic metric* on "space time" $\mathbf{R}_+\times\mathbf{R}^d$, that is,

$$\varrho((s,x);(t,y)):=\max(|t-s|^{1/2},\|x-y\|). \tag{1.2}$$

Received October 2012; revised December 2013.

[1]Supported in part by NSF Grants DMS-10-06903 and DMS-13-07470.

MSC2010 subject classifications. Primary 60J65, 60G17; secondary 28A78, 28A80, 60G15, 60J45.

Key words and phrases. Brownian motion, thermal capacity, Euclidean and space–time Hausdorff dimension.

The metric space $\mathbf{S} := (\mathbf{R}_+ \times \mathbf{R}^d, \varrho)$ is also called *space time*, and Hausdorff dimension of the compact set $E \times F$—viewed as a set in $\mathbf{S}$—is denoted by $\dim_{\mathrm{H}}(E \times F; \varrho)$. That is, $\dim_{\mathrm{H}}(E \times F; \varrho)$ is the infimum of $s \geq 0$ for which

$$\lim_{\varepsilon \to 0} \inf \left(\sum_{j=1}^{\infty} \left| \varrho\text{-diam}(E_j \times F_j) \right|^s \right) < \infty, \tag{1.3}$$

where the infimum is taken over all closed covers $\{E_j \times F_j\}_{j=1}^{\infty}$ of $E \times F$ with $\varrho\text{-diam}(E_j \times F_j) < \varepsilon$, and "$\varrho$-diam($\Lambda$)" denotes the diameter of the space–time set Λ, as measured by the metric ϱ.

THEOREM 1.1. *If $d \geq 2$, then*

$$\left\| \dim_{\mathrm{H}} \big(W(E) \cap F \big) \right\|_{L^\infty(\mathrm{P})} = \dim_{\mathrm{H}}(E \times F; \varrho) - d, \tag{1.4}$$

where "$\dim_{\mathrm{H}} A < 0$" *means* "$A = \varnothing$." *Display* (1.4) *continues to hold for $d = 1$, provided that* "$=$" *is replaced by* "$\leq$."

The following example shows that (1.4) does not always hold for $d = 1$: Consider $E := [0, 1]$ and $F := \{0\}$. Then a computation on the side shows that $\dim_{\mathrm{H}}(W(E) \cap F) = 0$ a.s., whereas $\dim_{\mathrm{H}}(E \times F; \varrho) - d = 1$.

On the other hand, Proposition 1.2 below shows that if $|F| > 0$, where $|\cdot|$ denotes the Lebesgue measure, then $W(E) \cap F$ shares the properties of the image set $W(E)$.

PROPOSITION 1.2. *If $F \subset \mathbf{R}^d$ $(d \geq 1)$ is compact and $|F| > 0$, then*

$$\left\| \dim_{\mathrm{H}} \big(W(E) \cap F \big) \right\|_{L^\infty(\mathrm{P})} = \min\{d, 2 \dim_{\mathrm{H}} E\}. \tag{1.5}$$

If, in addition, $\dim_{\mathrm{H}} E > 1/2$ *and $d = 1$, then* $\mathrm{P}\{|W(E) \cap F| > 0\} > 0$.

When $F \subset \mathbf{R}^d$ satisfies $|F| > 0$, it can be shown that $\dim_{\mathrm{H}}(E \times F; \varrho) = 2 \dim_{\mathrm{H}} E + d$. Hence, (1.5) coincides with (1.4) when $d \geq 2$. Proposition 1.2 is proved by showing that, when $|F| > 0$, there exists an explicit "smooth" random measure on $W(E) \cap F$. Thus, the remaining case, and this is the most interesting case, is when F has Lebesgue measure 0. The following result gives a suitable (though quite complicated) formula that is valid for all dimensions, including $d = 1$.

THEOREM 1.3. *If $F \subset \mathbf{R}^d$ $(d \geq 1)$ is compact and $|F| = 0$, then*

$$\left\| \dim_{\mathrm{H}} \big(W(E) \cap F \big) \right\|_{L^\infty(\mathrm{P})} = \sup\Big\{ \gamma > 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \Big\}, \tag{1.6}$$

where $\mathcal{P}_d(E \times F)$ denotes the collection of all probability measures μ on $E \times F$ that are "diffuse" in the sense that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$\mathcal{E}_\gamma(\mu) := \int\int \frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma} \mu(\mathrm{d}s\,\mathrm{d}x) \mu(\mathrm{d}t\,\mathrm{d}y). \tag{1.7}$$

Theorems 1.1 and 1.3 are the main results of this paper. But it seems natural that we also say a few words about when $W(E) \cap F$ is nonvoid with positive probability, simply because when $\mathrm{P}\{W(E) \cap F = \varnothing\} = 1$ there is no point in computing the Hausdorff dimension of $W(E) \cap F$.

It is a well-known folklore fact that $W(E)$ intersects F with positive probability if and only if $E \times F$ has positive thermal capacity in the sense of Watson [21, 22]. (For a simpler description, see Proposition 1.4 below.) This folklore fact can be proved by combining the results of Doob [2] on parabolic potential theory; specifically, one applies the analytic theory of [2], Chapter XVII, in the context of space–time Brownian motion as in [2], Section 13, pages 700–702. When combined with Theorem 3 of Taylor and Watson [20], this folklore fact tells us the following: If

$$\dim_{\mathrm{H}}(E \times F; \varrho) > d, \tag{1.8}$$

then $W(E) \cap F$ is nonvoid with positive probability; but if $\dim_{\mathrm{H}}(E \times F; \varrho) < d$ then $W(E) \cap F = \varnothing$ almost surely. Kaufman and Wu [9] contain related results. And our Theorem 1.1 states that the essential supremum of the Hausdorff dimension of $W(E) \cap F$ is the slack in the Taylor–Watson condition (1.8) for the nontriviality of $W(E) \cap F$.

The proof of Theorem 1.3 yields a simpler interpretation of the assertion that $E \times F$ has positive thermal capacity, and relates one of the energy forms that appear in Theorem 1.3, namely $\mathcal{E}_0$, to the present context. For the sake of completeness, we state that interpretation next in the form of Proposition 1.4. This proposition provides extra information on the equilibrium measure—in the sense of parabolic potential theory—for the thermal capacity of $E \times F$ when $|F| = 0$. [When $|F| > 0$, there is nothing to worry about, since $\mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0$ for every nonempty Borel set $E \subset (0, \infty)$.]

PROPOSITION 1.4. *Suppose $F \subset \mathbf{R}^d$ $(d \geq 1)$ is compact and has Lebesgue measure 0. Then $\mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0$ if and only if there exists a probability measure $\mu \in \mathcal{P}_d(E \times F)$ such that $\mathcal{E}_0(\mu) < \infty$.*

Theorems 1.1 and 1.3 both proceed by checking to see whether or not $W(E) \cap F$ (and a close variant of it) intersect a sufficiently-thin random set. This so-called "codimension idea" was initiated by S. J. Taylor [19] and has been used in other situations as well [5, 14, 17]. A more detailed account of the history of stochastic codimension can be found in the recent book of Mörters and Peres [16], page 287. The broad utility of this method—using fractal percolation sets as the (thin) testing random sets—was further illustrated by Yuval Peres [18].

Throughout this paper, we adopt the following notation: For all integers $k \geq 1$ and for every $x = (x_1, \dots, x_k) \in \mathbf{R}^k$, $\|x\|$ and $|x|$, respectively, define the ℓ^2 and ℓ^1 norms of x. That is,

$$\|x\| := \left(x_1^2 + \cdots + x_k^2\right)^{1/2} \quad \text{and} \quad |x| := |x_1| + \cdots + |x_k|. \tag{1.9}$$

The rest of the paper is organized as follows. Proposition 1.2 is proved in Section 2. Then, in Sections 5 and 3, Theorems 1.1 and 1.3 are proved in reverse order, since the latter is significantly harder to prove. The main ingredient for proving Theorem 1.3 is Theorem 3.1 whose proof is given in Section 4. Proposition 1.4 is proved in Section 4.5.

2. Proof of Proposition 1.2. The upper bound in (1.5) follows from the well-known fact that $\dim_{\mathrm{H}} W(E) = \min\{d, 2\dim_{\mathrm{H}} E\}$ almost surely. In order to establish the lower bound in (1.5), we first construct a random measure ν on $W(E) \cap F$, and then appeal to a capacity argument. The details follow.

Choose and fix a constant γ such that

$$0 < \gamma < \min\{d, 2\dim_{\mathrm{H}} E\}. \tag{2.1}$$

According to Frostman's theorem, there exists a Borel probability measure σ on E such that

$$\int\int \frac{\sigma(\mathrm{d}s)\sigma(\mathrm{d}t)}{|s-t|^{\gamma/2}} < \infty. \tag{2.2}$$

For every integer $n \geq 1$, we define a random measure μ_n on $E \times F$ via

$$\int f \,\mathrm{d}\mu_n := (2\pi n)^{d/2} \int_{E\times F} f(s,x) \exp\left(-\frac{n\|W(s)-x\|^2}{2}\right)\sigma(\mathrm{d}s)\,\mathrm{d}x \tag{2.3}$$

for every Borel measurable function $f: E \times F \to \mathbf{R}_+$. Equivalently,

$$\int f\,\mathrm{d}\mu_n = \int_{E\times F} \sigma(\mathrm{d}s)\,\mathrm{d}x f(s,x) \int_{\mathbf{R}^d} \mathrm{d}\xi \exp\left(i\langle \xi, W(s)-x\rangle - \frac{\|\xi\|^2}{2n}\right), \tag{2.4}$$

thanks to the characteristic function of a Gaussian vector.

Let ν_n be the image measure of μ_n under the random mapping $g: E \times F \to \mathbf{R}^d$ defined by $g(s,x) := W(s)$. That is, $\int \phi\,\mathrm{d}\nu_n := \int (\phi \circ g)\,\mathrm{d}\mu_n$ for all Borel-measurable functions $\phi: \mathbf{R}^d \to \mathbf{R}_+$. It follows from (2.3) that, if $\{\nu_n\}_{n=1}^\infty$ has a subsequence which converges weakly to ν, then ν is supported on $W(E) \cap F$. This ν will be the desired random measure on $W(E) \cap F$. Thus, we plan to prove that: (i) $\{\nu_n\}_{n=1}^\infty$ indeed has a subsequence which converges weakly; and (ii) use this particular ν to show that $\mathrm{P}\{\dim_{\mathrm{H}}(W(E) \cap F) \geq \gamma\} > 0$. This will demonstrate (1.5).

In order to carry out (i) and (ii), it suffices to verify that there exist positive and finite constants c_1, c_2 and c_3 such that

$$\mathrm{E}\big(\|\nu_n\|\big) \geq c_1, \qquad \mathrm{E}\big(\|\nu_n\|^2\big) \leq c_2 \tag{2.5}$$

and

$$\mathrm{E}\int\int \frac{\nu_n(\mathrm{d}x)\nu_n(\mathrm{d}y)}{\|x-y\|^\gamma} \leq c_3, \tag{2.6}$$

simultaneously for all $n \geq 1$, where $\|\nu_n\| := \nu_n(\mathbf{R})$ denotes the total mass of ν_n. The rest hinges on a well-known capacity argument that is explicitly hashed out in [6], pages 204–206; see also [11], pages 75–76.

It follows from (2.4) and Fubini's theorem that

$$\begin{aligned} \mathrm{E}(\|\nu_n\|) &= \int_{E\times F} \sigma(\mathrm{d}s)\,\mathrm{d}x \int_{\mathbf{R}^d} \mathrm{d}\xi\, \mathrm{E}(\mathrm{e}^{i\langle \xi, W(s)-x\rangle})\mathrm{e}^{-\|\xi\|^2/(2n)} \\ &= \int_{E\times F} \left(\frac{2\pi}{s+n^{-1}}\right)^{d/2} \exp\left(-\frac{\|x\|^2}{2(s+n^{-1})}\right)\sigma(\mathrm{d}s)\,\mathrm{d}x. \end{aligned} \tag{2.7}$$

Since $E \subset (0,\infty)$ is compact, we have $\inf E \geq \delta$ for some constant $\delta > 0$. Hence, (2.7) implies that $\inf_{n\geq 1} \mathrm{E}(\|\nu_n\|) \geq c_1$ for some constant $c_1 > 0$, and this verifies the first inequality in (2.5). For the second inequality, we use (2.3) to see that

$$\|\nu_n\| = \|\mu_n\| = (2\pi n)^{d/2} \int_{E\times F} \exp\left(-\frac{n\|W(s)-x\|^2}{2}\right)\sigma(\mathrm{d}s)\,\mathrm{d}x. \tag{2.8}$$

We may replace F by all of $\mathbf{R}^d$ in order to find that $\|\nu_n\| \leq (2\pi)^d$ a.s.; whence follows the second inequality in (2.5). Similarly, we prove (2.6) by writing

$$\begin{aligned} &\int\int \frac{\nu_n(\mathrm{d}x)\nu_n(\mathrm{d}y)}{\|x-y\|^\gamma} \\ &\quad = \int_{(E\times F)^2} \frac{\sigma(\mathrm{d}s)\sigma(\mathrm{d}t)\,\mathrm{d}x\,\mathrm{d}y}{\|W(t)-W(s)\|^\gamma} \\ &\qquad\qquad \times (2\pi n)^d \exp\left(-\frac{n\|W(s)-x\|^2 - n\|W(t)-y\|^2}{2}\right). \end{aligned}$$

We may replace F by $\mathbf{R}^d$, use the scaling property of W and the fact that $\gamma < d$ in order to see that

$$\mathrm{E}\int\int \frac{\nu_n(\mathrm{d}x)\nu_n(\mathrm{d}y)}{\|x-y\|^\gamma} \leq c\int\int \frac{\sigma(\mathrm{d}s)\sigma(\mathrm{d}t)}{|s-t|^{\gamma/2}} \qquad \text{a.s.}$$

Therefore, (2.6) follows from (2.2).

Finally, we prove the last statement in Proposition 1.2. Since $\dim_{\mathrm{H}} E > \frac{1}{2}$, Frostman's theorem assures us that there exists a Borel probability measure σ on E such that (2.2) holds with $\gamma = 1$. We construct a sequence of random measures $\{\nu_n\}_{n=1}^\infty$ as before, and extract a subsequence that converges weakly to a random Borel measure ν on $W(E)\cap F$ such that $\mathrm{P}\{\|\nu\| > 0\} > 0$.

Let $\widehat{\nu}$ denote the Fourier transform of ν. In accord with Plancherel's theorem, a sufficient condition for $\mathrm{P}\{|W(E)\cap F| > 0\} > 0$ is that $\widehat{\nu} \in L^2(\mathbf{R})$. We apply Fatou's lemma to reduce our problem to the following:

$$\sup_{n\geq 1} \mathrm{E}\int_{-\infty}^{\infty} \left|\widehat{\nu}_n(\theta)\right|^2 \mathrm{d}\theta < \infty. \tag{2.9}$$

By (2.4) and Fubini's theorem,

$$
\begin{aligned}
\mathrm{E}\int_{-\infty}^{\infty}\left|\widehat{\nu}_n(\theta)\right|^2 \mathrm{d}\theta
&= \int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{E}\int_{\mathbf{R}^2} \mu_n(\mathrm{d}s\,\mathrm{d}x)\mu_n(\mathrm{d}t\,\mathrm{d}y)\mathrm{e}^{i\theta(W(s)-W(t))} \\
&= \int_{-\infty}^{\infty} \mathrm{d}\theta \int_{(E\times F)^2} \sigma(\mathrm{d}s)\sigma(\mathrm{d}t)\,\mathrm{d}x\,\mathrm{d}y \int_{\mathbf{R}^2} \mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad \times \exp\left(-i(\xi x+\eta y) - \frac{\xi^2+\eta^2}{2n}\right)\mathrm{E}\left(\mathrm{e}^{i[(\xi+\theta)W(s)+(\eta-\theta)W(t)]}\right).
\end{aligned} \tag{2.10}
$$

When $0 < s < t$, this last expectation can be written as

$$
\exp\left(-\frac{s}{2}(\xi+\eta)^2 - \frac{t-s}{2}(\eta-\theta)^2\right). \tag{2.11}
$$

By plugging this into (2.10), we can write the triple integral in $[\mathrm{d}\theta\,\mathrm{d}\xi\,\mathrm{d}\eta]$ of (2.10) as

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \mathrm{e}^{-i(\xi x+\eta y)} \exp\left(-\frac{\xi^2+\eta^2}{2n} - \frac{s}{2}(\xi+\eta)^2\right)\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{t-s}{2}(\eta-\theta)^2\right)\mathrm{d}\theta \\
&= p(x,y)\sqrt{\frac{2\pi}{t-s}},
\end{aligned} \tag{2.12}
$$

where $p(x, y)$ denotes the joint density function of a bivariate normal distribution with mean vector 0 and covariance matrix Γ^{-1}, where

$$
\Gamma := \begin{pmatrix} s+n^{-1} & s \\ s & s+n^{-1} \end{pmatrix}. \tag{2.13}
$$

We plug (2.12) into (2.10), replace F by $\mathbf{R}^d$ to integrate $[\mathrm{d}x\,\mathrm{d}y]$ in order to find that

$$
\sup_{n\geq 1} \mathrm{E}\int_{-\infty}^{\infty}\left|\widehat{\nu}_n(\theta)\right|^2 \mathrm{d}\theta \leq \text{const}\cdot \int\int \frac{\sigma(\mathrm{d}s)\sigma(\mathrm{d}t)}{|s-t|^{1/2}} < \infty. \tag{2.14}
$$

This yields (2.9) and completes the proof of Proposition 1.2.

3. Proof of Theorem 1.3.

Here and throughout,

$$
B_x(\epsilon) := \left\{y \in \mathbf{R}^d : \|x-y\| \leq \epsilon\right\} \tag{3.1}
$$

denotes the radius-ϵ ball about $x \in \mathbf{R}^d$. Also, define ν_d to be the volume of $B_0(1)$; that is,

$$
\nu_d := \frac{2\cdot\pi^{d/2}}{d\Gamma(d/2)}. \tag{3.2}
$$

Recall that $\{W(t)\}_{t\geq 0}$ denotes a standard Brownian motion in $\mathbf{R}^d$, and consider the following "parabolic Green function": For all $t>0$ and $x \in \mathbf{R}^d$,

$$p_t(x) := \frac{e^{-\|x\|^2/(2t)}}{(2\pi t)^{d/2}} \mathbf{1}_{(0,\infty)}(t). \tag{3.3}$$

The seemingly-innocuous indicator function plays an important role in the sequel; this form of the heat kernel appears earlier in Watson [21, 22] and Doob [2], (4.1), page 266.

As indicated in the Introduction, our proof of Theorem 1.3 is based on the codimension argument to check whether or not $W(E) \cap F$ intersect a sufficiently-thin "testing" random set. One example of such testing sets could be the range of a stable Lévy process $X = \{X(t)\}_{t\geq 0}$ in $\mathbf{R}^d$ with index $\alpha \in (0, 2]$. However, this choice of testing set will only work for $d \leq 3$, because the range $X((0,\infty))$ will not be able to intersect $W(E) \cap F$ if $d \geq 4$ due to the fact that $X((0,\infty)) \cap G = \varnothing$ a.s. for any Borel set $G \subset \mathbf{R}^d$ with $\dim_{\mathrm{H}} G < d - \alpha$.

To avoid this restriction and for future applications, we will use the range of an N-parameter additive stable Lévy process with index α as the testing set for proving Theorem 1.3.

Let $X^{(1)}, \ldots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are totally independent from one another, as well as from the process W, and all take their values in $\mathbf{R}^d$. We assume also that $X^{(1)}, \ldots, X^{(N)}$ have right-continuous sample paths with left-limits. This assumption can be—and will be—made without incurring any real loss in generality. Finally, our normalization of the processes $X^{(1)}, \ldots, X^{(N)}$ is described as follows:

$$\mathrm{E}\big[\exp\big(i\langle \xi, X^{(k)}(1)\rangle\big)\big] = e^{-\|\xi\|^\alpha/2} \qquad \text{for all } 1 \leq k \leq N \text{ and } \xi \in \mathbf{R}^d. \tag{3.4}$$

Define the corresponding *additive stable process* $X_\alpha := \{X_\alpha(\mathbf{t})\}_{\mathbf{t}\in\mathbf{R}^N_+}$ as

$$X_\alpha(\mathbf{t}) := \sum_{k=1}^N X^{(k)}(t_k) \qquad \text{for all } \mathbf{t} := (t_1, \ldots, t_N) \in \mathbf{R}^N_+. \tag{3.5}$$

Also, define $\mathcal{C}_\gamma$ to be the capacity corresponding to the energy form (1.7). That is, for all compact sets $U \subset \mathbf{R}_+ \times \mathbf{R}^d$ and $\gamma \geq 0$,

$$\mathcal{C}_\gamma(U) := \Big[\inf_{\mu\in\mathcal{P}_d(U)} \mathcal{E}_\gamma(\mu)\Big]^{-1}. \tag{3.6}$$

THEOREM 3.1. *If $d > \alpha N$ and $F \subset \mathbf{R}^d$ has Lebesgue measure* 0, *then*

$$\mathrm{P}\big\{W(E) \cap X_\alpha\big(\mathbf{R}^N_+\big) \cap F \neq \varnothing\big\} > 0 \quad \Longleftrightarrow \quad \mathcal{C}_{d-\alpha N}(E \times F) > 0. \tag{3.7}$$

We can now apply Theorem 3.1 to prove Theorem 1.3. Theorem 3.1 will be established subsequently.

PROOF OF THEOREM 1.3. Suppose $\alpha \in (0,2]$ and $N \in \mathbf{Z}_+$ are chosen such that $d - \alpha N \in (0,2)$. If X_α denotes an N-parameter additive stable process $\mathbf{R}^d$ whose index is $\alpha \in (0,2]$, then [12], Theorem 4.4, implies that

$$\operatorname{codim} X_\alpha\big(\mathbf{R}_+^N\big) = d - \alpha N. \tag{3.8}$$

This means that $X_\alpha(\mathbf{R}_+^N)$ will intersect any nonrandom Borel set $G \subset \mathbf{R}^d \setminus \{0\}$ with $\dim_{\mathrm{H}}(G) > d - \alpha N$, with positive probability; whereas $X_\alpha(\mathbf{R}_+^N)$ does not intersect any $G \subset \mathbf{R}^d \setminus \{0\}$ with $\dim_{\mathrm{H}}(G) < d - \alpha N$, almost surely.

Define

$$\Delta := \sup\Big\{\gamma > 0 : \inf_{\mu \in \mathcal{P}_d(E\times F)} \mathcal{E}_\gamma(\mu) < \infty\Big\} \tag{3.9}$$

with the convention that $\sup \varnothing = 0$.

If $\Delta > 0$ and $d - \alpha N < \Delta$, then $\mathcal{C}_{d-\alpha N}(E \times F) > 0$. It follows from Theorem 3.1 and (3.8) that

$$\mathrm{P}\big\{\dim_{\mathrm{H}}\big(W(E) \cap F\big) \geq d - \alpha N\big\} > 0. \tag{3.10}$$

Because $d - \alpha N \in (0, \Delta)$ is arbitrary, we have $\| \dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathrm{P})} \geq \Delta$.

Similarly, Theorem 3.1 and (3.8) imply that

$$d - \alpha N > \Delta \quad \Longrightarrow \quad \dim_{\mathrm{H}}\big(W(E) \cap F\big) \leq d - \alpha N \qquad \text{almost surely.} \tag{3.11}$$

Hence, $\| \dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathrm{P})} \leq \Delta$ whenever $\Delta \geq 0$. This proves the theorem. □

4. Proof of Theorem 3.1.

Our proof of Theorem 3.1 is divided into separate parts. We begin by developing a requisite result in harmonic analysis. Then we develop some facts about additive Lévy processes. After that, we prove Theorem 3.1 in two separate parts.

4.1. *Isoperimetry.* Recall that a function $\kappa : \mathbf{R}^n \to \overline{\mathbf{R}}_+ := [0, \infty]$ is *tempered* if it is measurable and

$$\int_{\mathbf{R}^n} \frac{\kappa(x)}{(1 + \|x\|)^m}\, \mathrm{d}x < \infty \qquad \text{for some } m \geq 0. \tag{4.1}$$

A function $\kappa : \mathbf{R}^n \to \overline{\mathbf{R}}_+$ is said to be *positive definite* if it is tempered and for all rapidly-decreasing test functions $\phi : \mathbf{R}^n \to \mathbf{R}$,

$$\int_{\mathbf{R}^n} \mathrm{d}x \int_{\mathbf{R}^n} \mathrm{d}y \phi(x)\kappa(x-y)\phi(y) \geq 0. \tag{4.2}$$

Let $\widehat{g}$ denote the Fourier transform of a function (or a measure) g. We use the following normalization: $\widehat{g}(\xi) = \int_{\mathbf{R}^n} \exp(i\xi \cdot z) g(z)\, \mathrm{d}z$ when $g \in L^1(\mathbf{R}^n)$. We will make heavy use of the following result.

LEMMA 4.1. *If $\kappa:\mathbf{R}^n \to \overline{\mathbf{R}}_+$ is positive definite and lower semicontinuous, then for all finite Borel measures μ on $\mathbf{R}^n$,*

$$\int\int \kappa(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \widehat{\kappa}(\xi)\left|\widehat{\mu}(\xi)\right|^2 \mathrm{d}\xi. \tag{4.3}$$

If κ is in addition bounded, then in fact for all finite Borel measures μ and ν on $\mathbf{R}^n$,

$$\int\int \kappa(x-y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \widehat{\kappa}(\xi)\widehat{\mu}(\xi)\overline{\widehat{\nu}(\xi)}\,\mathrm{d}\xi. \tag{4.4}$$

PROOF. Equation (4.3) is proved in Foondun and Khoshnevisan [3], Corollary 3.4; for a weaker version see [13], Theorem 5.2. We can derive (4.4) from (4.3) in a standard way ("polarization"): Apply (4.3) with $\mu+\nu$ in place of μ to see that

$$\int\int \kappa(x-y)(\mu+\nu)(\mathrm{d}x)(\mu+\nu)(\mathrm{d}y) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \widehat{\kappa}(\xi)\left|(\widehat{\mu}+\widehat{\nu})(\xi)\right|^2 \mathrm{d}\xi. \tag{4.5}$$

Develop both sides, and match the quadratic terms, using (4.3), to finish. □

Lemma 4.1 implies two "isoperimetric inequalities" that are stated below as Propositions 4.2 and 4.4. Recall that a finite Borel measure ν on $\mathbf{R}^d$ is said to be *positive definite* if $\widehat{\nu}(\xi) \ge 0$ for all $\xi \in \mathbf{R}^d$.

PROPOSITION 4.2. *Suppose $\kappa:\mathbf{R}^d \to \overline{\mathbf{R}}_+$ is a lower semicontinuous positive-definite function such that $\kappa(x)=\infty$ iff $x=0$. Suppose ν and σ are two positive definite probability measures on $\mathbf{R}^d$ that satisfy the following:*

1. *κ and $\kappa*\nu$ are uniformly continuous on every compact subset of $\mathbf{R}^d\setminus\{0\}$; and*
2. *$(\tau,x)\mapsto(p_\tau*\sigma)(x)$ is uniformly continuous on every compact subset of $(0,\infty)\times(\mathbf{R}^d\setminus\{0\})$.*

Then, for all finite Borel measures μ on $\mathbf{R}_+\times\mathbf{R}^d$,

$$\begin{aligned}&\int\int (p_{|t-s|}*\sigma)(x-y)(\kappa*\nu)(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\ &\quad\le \int\int p_{|t-s|}(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.6}$$

REMARK 4.3. The very same proof shows the following slight enhancement: *Suppose κ and ν are the same as in Proposition 4.2. If σ_1 and σ_2 share the properties of σ in Proposition 4.2 and $\widehat{\sigma}_1(\xi)\le\widehat{\sigma}_2(\xi)$ for all $\xi\in\mathbf{R}^d$, then for all finite Borel measures μ on $\mathbf{R}_+\times\mathbf{R}^d$,*

$$\begin{aligned}&\int\int (p_{|t-s|}*\sigma_1)(x-y)(\kappa*\nu)(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\ &\quad\le \int\int (p_{|t-s|}*\sigma_2)(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.7}$$

Proposition 4.2 is this in the case that $\sigma_2 := \delta_0$. An analogous result holds for positive definite probability measures ν_1 and ν_2 which satisfy $\widehat{\nu}_1(\xi) \le \widehat{\nu}_2(\xi)$ for all $\xi \in \mathbf{R}^d$.

PROOF. Throughout this proof, we choose and fix $\epsilon > 0$.

Without loss of generality, we may and will assume that

$$\int\int p_{|t-s|}(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y) < \infty; \tag{4.8}$$

for there is nothing to prove, otherwise.

Because $p_{|t-s|}$ is positive definite for every nonnegative $t \neq s$, so are $p_{|t-s|} * \sigma$ and $\kappa * \nu$. Because $p_{|t-s|}$ is bounded and continuous when $s \neq t$, it follows from the Bochner–Minlos–Schwartz theorem that $p_{|t-s|} \times (\kappa * \nu)$ is positive definite. Therefore, for fixed $t > s$, Lemma 4.1 applies, and tells us that for all Borel probability measures ρ on $\mathbf{R}^d$, and for all nonnegative $t \neq s$,

$$\begin{aligned}\int\int &(p_{|t-s|} * \sigma)(x-y)(\kappa * \nu)(x-y)\rho(\mathrm{d}x)\rho(\mathrm{d}y)\\ &= \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\mathrm{d}\xi\int_{\mathbf{R}^d}\mathrm{d}\zeta\,\mathrm{e}^{-(t-s)\|\xi\|^2/2}\widehat{\sigma}(\xi)\widehat{\kappa}(\zeta)\widehat{\nu}(\xi)\big|\widehat{\rho}(\xi-\zeta)\big|^2.\end{aligned} \tag{4.9}$$

Because the preceding is valid also when $\sigma = \nu = \delta_0$, and since $0 \le \widehat{\sigma}(\xi), \widehat{\nu}(\xi) \le 1$ for all $\xi \in \mathbf{R}^d$, it follows that for all nonnegative $t \neq s$,

$$\begin{aligned}\int\int &(p_{|t-s|} * \sigma)(x-y)(\kappa * \nu)(x-y)\rho(\mathrm{d}x)\rho(\mathrm{d}y)\\ &\le \int\int p_{|t-s|}(x-y)\kappa(x-y)\rho(\mathrm{d}x)\rho(\mathrm{d}y).\end{aligned} \tag{4.10}$$

This inequality continues to holds when ρ is a finite Borel measure on $\mathbf{R}^d$, by scaling. Thus, thanks to Tonelli's theorem, the proposition is valid whenever $\mu(\mathrm{d}t\,\mathrm{d}x) = \lambda(\mathrm{d}t)\rho(\mathrm{d}x)$ for two finite Borel measures λ and ρ, respectively defined on $\mathbf{R}_+$ and $\mathbf{R}^d$.

Now let us consider a compactly-supported finite measure μ on $\mathbf{R}_+ \times \mathbf{R}^d$. For all $\eta > 0$, define

$$\mathcal{G}(\eta) := \big\{(t,s,x,y) \in (\mathbf{R}_+)^2 \times \big(\mathbf{R}^d\big)^2 : |t-s| \wedge \|x-y\| \ge \eta\big\}. \tag{4.11}$$

It suffices to prove that for all $\eta > 0$,

$$\begin{aligned}\int\int_{\mathcal{G}(\eta)} &(p_{|t-s|} * \sigma)(x-y)(\kappa * \nu)(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\ &\le \int\int_{\mathcal{G}(\eta)} p_{|t-s|}(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.12}$$

This is so, because $\kappa(0)=\infty$ and (4.8) readily tell us that the product measure $\mu\otimes\mu$ does not charge

$$\{(t,s,x,y)\in(\mathbf{R}_+)^2\times(\mathbf{R}^d)^2 : x=y\}; \tag{4.13}$$

and, therefore,

$$\begin{aligned}\lim_{\eta\downarrow 0}\int\int_{\mathcal{G}(\eta)}&(p_{|t-s|}*\sigma)(x-y)(\kappa*\nu)(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\ &=\int\int_{\substack{s\neq t\\ x\neq y}}(p_{|t-s|}*\sigma)(x-y)(\kappa*\nu)(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\ &=\int\int(p_{|t-s|}*\sigma)(x-y)(\kappa*\nu)(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.14}$$

And similarly,

$$\begin{aligned}\lim_{\eta\downarrow 0}\int\int_{\mathcal{G}(\eta)}&p_{|t-s|}(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\ &=\int\int p_{|t-s|}(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.15}$$

And the proposition follows, subject to (4.12).

Next, we verify (4.12) to finish the proof. One can check directly that $\mathcal{G}(\eta)\cap\operatorname{supp}(\mu\otimes\mu)$ is compact, and both mappings $(t,s,x,y)\mapsto(p_{|t-s|}*\sigma)(x-y)\times(\kappa*\nu)(x-y)$ and $(t,s,x,y)\mapsto p_{|t-s|}(x-y)\kappa(x-y)$ are uniformly continuous on $\mathcal{G}(\eta)\cap\operatorname{supp}(\mu\otimes\mu)$.

By discretization, we can find finite Borel measures $\{\lambda_j\}_{j=1}^\infty$—on $\mathbf{R}_+$—and $\{\rho_j\}_{j=1}^\infty$—on $\mathbf{R}^d$—such that μ is the weak limit of $\mu_N:=\sum_{j=1}^N(\lambda_j\otimes\rho_j)$ as $N\to\infty$. It follows from (4.4) and an argument similar to (4.9) that for all $\eta>0$ and $N\geq 1$,

$$\begin{aligned}\int\int_{\mathcal{G}(\eta)}&(p_{|t-s|}*\sigma)(x-y)(\kappa*\nu)(x-y)\mu_N(\mathrm{d}t\,\mathrm{d}x)\mu_N(\mathrm{d}s\,\mathrm{d}y)\\ &\leq\int\int_{\mathcal{G}(\eta)}p_{|t-s|}(x-y)\kappa(x-y)\mu_N(\mathrm{d}t\,\mathrm{d}x)\mu_N(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.16}$$

Let $N\uparrow\infty$ to deduce (4.12), and hence the proposition. □

PROPOSITION 4.4. *Suppose $\kappa:\mathbf{R}\to\overline{\mathbf{R}}_+$ is a lower semicontinuous positive-definite function such that $\kappa(x)=\infty$ iff $x=0$. Suppose ν and σ are two positive definite probability measures, respectively on $\mathbf{R}$ and $\mathbf{R}^d$, that satisfy the following*:

1. *κ and $\kappa*\nu$ are uniformly continuous on every compact subset of $\mathbf{R}\setminus\{0\}$; and*
2. *$(\tau,x)\mapsto(p_\tau*\sigma)(x)$ is uniformly continuous on every compact subset of $(0,\infty)\times(\mathbf{R}^d\setminus\{0\})$.*

Then, for all finite Borel measures μ *on* $\mathbf{R}_+ \times \mathbf{R}^d$,

$$\begin{aligned}&\int\int (p_{|t-s|} * \sigma)(x-y)(\kappa * \nu)(s-t)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\&\qquad \le \int\int p_{|t-s|}(x-y)\kappa(s-t)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y).\end{aligned} \tag{4.17}$$

PROOF. It suffices to prove the proposition in the case that

$$\mu(\mathrm{d}s\,\mathrm{d}x) = \lambda(\mathrm{d}s)\rho(\mathrm{d}x), \tag{4.18}$$

for finite Borel measures λ and ρ, respectively on $\mathbf{R}_+$ and $\mathbf{R}^d$. See, for instance, the argument beginning with (4.11) in the proof of Proposition 4.2. We shall extend the definition λ so that it is a finite Borel measure on all of $\mathbf{R}$ in the usual way: If $A \subset \mathbf{R}$ is Borel measurable, then $\lambda(A) := \lambda(A \cap \mathbf{R}_+)$. This slight abuse in notation should not cause any confusion in the sequel.

Tonelli's theorem and Lemma 4.1 together imply that in the case that (4.18) holds:

$$\begin{aligned}&\int\int (p_{|t-s|} * \sigma)(x-y)(\kappa * \nu)(s-t)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\&\quad = \int\int \lambda(\mathrm{d}t)\lambda(\mathrm{d}s)(\kappa*\nu)(s-t)\int\int \rho(\mathrm{d}x)\rho(\mathrm{d}y)(p_{|t-s|}*\sigma)(x-y)\\&\quad = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \widehat{\sigma}(\xi)\big|\widehat{\rho}(\xi)\big|^2\,\mathrm{d}\xi \int\int \lambda(\mathrm{d}t)\lambda(\mathrm{d}s)(\kappa*\nu)(s-t)\mathrm{e}^{-|t-s|\cdot\|\xi\|^2/2}\\&\quad \le \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \big|\widehat{\rho}(\xi)\big|^2\,\mathrm{d}\xi \int\int \lambda(\mathrm{d}t)\lambda(\mathrm{d}s)(\kappa*\nu)(s-t)\mathrm{e}^{-|t-s|\cdot\|\xi\|^2/2}.\end{aligned} \tag{4.19}$$

The map $\tau \mapsto \exp\{-|\tau|\cdot\|\xi\|^2/2\}$ is positive definite on $\mathbf{R}$ for every fixed $\xi \in \mathbf{R}^d$; in fact, its inverse Fourier transform is a (scaled) Cauchy density function, which we refer to as ϑ_ξ. Therefore, in accord with Lemma 4.1,

$$\begin{aligned}&\int\int (\kappa*\nu)(s-t)\mathrm{e}^{-|t-s|\cdot\|\xi\|^2/2}\lambda(\mathrm{d}t)\lambda(\mathrm{d}s)\\&\quad = \frac{1}{2\pi}\int_{\mathbf{R}}\big|\widehat{\lambda}(\tau)\big|^2(\widehat{\kappa}\widehat{\nu}*\vartheta_\xi)(\tau)\,\mathrm{d}\tau \le \frac{1}{2\pi}\int_{\mathbf{R}}\big|\widehat{\lambda}(\tau)\big|^2(\widehat{\kappa}*\vartheta_\xi)(\tau)\,\mathrm{d}\tau\\&\quad = \int\int \kappa(s-t)\mathrm{e}^{-|t-s|\cdot\|\xi\|^2/2}\lambda(\mathrm{d}t)\lambda(\mathrm{d}s).\end{aligned} \tag{4.20}$$

The last line follows from the first identity, since we can consider $\nu = \delta_0$ as a possibility. Therefore, it follows from (4.19) and (4.20) that

$$\begin{aligned}&\int\int (p_{|t-s|}*\sigma)(x-y)(\kappa*\nu)(s-t)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\\&\quad \le \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\big|\widehat{\rho}(\xi)\big|^2\,\mathrm{d}\xi\int\int\lambda(\mathrm{d}t)\lambda(\mathrm{d}s)\kappa(s-t)\mathrm{e}^{-|t-s|\cdot\|\xi\|^2/2}\\&\quad = \int\int\lambda(\mathrm{d}t)\lambda(\mathrm{d}s)\kappa(s-t)\int\int\rho(\mathrm{d}x)\rho(\mathrm{d}y)p_{|t-s|}(x-y);\end{aligned}$$

the last line follows from the first identity in (4.19) by considering the special case that $\nu = \delta_0$ and $\sigma = \delta_0$. This proves the proposition in the case that μ has the form (4.18), and the result follows. □

4.2. *Additive stable processes.* In this subsection, we develop a "resolvent density" estimate for the additive stable process X_α.

First of all, note that the characteristic function $\xi \mapsto \mathrm{E}\exp(i\langle \xi, X_\alpha(\mathbf{t})\rangle)$ of $X_\alpha(\mathbf{t})$ is absolutely integrable for every $\mathbf{t} \in \mathbf{R}_+^N \setminus \{\mathbf{0}\}$. Consequently, the inversion formula applies and tells us that we can always choose the following as the probability density function of $X_\alpha(\mathbf{t})$:

$$g_{\mathbf{t}}(x) := g_{\mathbf{t}}(\alpha; x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \mathrm{e}^{-i\langle x, \xi\rangle - |\mathbf{t}|\cdot\|\xi\|^\alpha/2}\, \mathrm{d}\xi. \tag{4.21}$$

LEMMA 4.5. *Choose and fix some* $\mathbf{a}, \mathbf{b} \in (0, \infty)^N$ *such that* $a_j \le b_j$ *for all* $1 \le j \le N$. *Define*

$$[\mathbf{a}, \mathbf{b}] := \{\mathbf{s} \in \mathbf{R}_+^N : a_j \le s_j \le b_j \text{ for all } 1 \le j \le N\}. \tag{4.22}$$

Then, for all $M > 0$ *there exists a constant* $A_0 \in (1, \infty)$*—depending only on the parameters* d, N, M, α, $\min_{1\le j\le N} a_j$, *and* $\max_{1\le j\le N} b_j$*—such that for all* $x \in [-M, M]^d$,

$$A_0^{-1} \le \int_{[\mathbf{a},\mathbf{b}]} g_{\mathbf{t}}(x)\, \mathrm{d}\mathbf{t} \le A_0. \tag{4.23}$$

PROOF. Let $\vec{1} := (1, \ldots, 1) \in \mathbf{R}^N$. Then we may also observe the scaling relation,

$$g_{\mathbf{t}}(x) = |\mathbf{t}|^{-d/\alpha} g_{\vec{1}}\left(\frac{x}{|\mathbf{t}|^{1/\alpha}}\right), \tag{4.24}$$

together with the fact $g_{\vec{1}}$ is an isotropic stable-α density function on $\mathbf{R}^d$. The upper bound in (4.23) follows from (4.24) and the boundedness of $g_{\vec{1}}(z)$.

On the other hand, since $\mathbf{a} \in (0, \infty)^N$, the lower bound in (4.23) follows from (4.24) and the well-known fact that $g_{\vec{1}}(z)$ is continuous and strictly positive everywhere. □

PROPOSITION 4.6. *Choose and fix some* $\mathbf{b} \in (0, \infty)^N$ *and define* $[\mathbf{0}, \mathbf{b}]$ *as in Lemma* 4.5, *and assume* $d > \alpha N$. *Then, for all* $M > 0$ *there exists a constant* $A_1 \in (1, \infty)$*—depending only on* d, N, M, α, $\min_{1\le j\le N} b_j$, *and* $\max_{1\le j\le N} b_j$*—such that for all* $x \in [-M, M]^d$,

$$\frac{1}{A_1\|x\|^{d-\alpha N}} \le \int_{[\mathbf{0},\mathbf{b}]} g_{\mathbf{t}}(x)\, \mathrm{d}\mathbf{t} \le \frac{A_1}{\|x\|^{d-\alpha N}}. \tag{4.25}$$

PROOF. Recall the following standard estimate: For all $R > 0$, there exists $C(R) \in (1, \infty)$ and $c(R) \in (0, 1)$ such that

$$\frac{c(R)}{\|z\|^{d+\alpha}} \le g_{\vec{1}}(z) \le \frac{C(R)}{\|z\|^{d+\alpha}} \qquad \text{for all } z \in \mathbf{R}^d \text{ with } \|z\| \ge R. \tag{4.26}$$

See [10], Proposition 3.3.1, page 380, where this is proved for $R = 2$. The slightly more general case where $R > 0$ is arbitrary is proved in exactly the same manner.

Since

$$\int_{[\mathbf{0},\mathbf{b}]} g_{\mathbf{t}}(x)\, d\mathbf{t} \le e^{|\mathbf{b}|} \int_{\mathbf{R}_+^N} e^{-|\mathbf{t}|} g_{\mathbf{t}}(x)\, d\mathbf{t}, \tag{4.27}$$

the proof of Proposition 4.1.1 of [10], page 420, shows that the upper bound in (4.25) holds for all $x \in \mathbf{R}^d$.

For the lower bound, we first recall the notation $\vec{1} := (1, \dots, 1) \in \mathbf{R}^N$, and then apply (4.24) and (4.26) in order to find that

$$\begin{aligned}\int_{[\mathbf{0},\mathbf{b}]} g_{\mathbf{t}}(x)\, d\mathbf{t} &= \int_{[\mathbf{0},\mathbf{b}]} |\mathbf{t}|^{-d/\alpha} g_{\vec{1}}\left(\frac{x}{|\mathbf{t}|^{1/\alpha}}\right) d\mathbf{t} \\ &\ge \frac{c(1)}{\|x\|^{d+\alpha}} \cdot \int_{\substack{\mathbf{t}\in[\mathbf{0},\mathbf{b}]:\\ |\mathbf{t}|^{1/\alpha}\le\|x\|}} |\mathbf{t}|\, d\mathbf{t}.\end{aligned} \tag{4.28}$$

Clearly, there exists $R_0 > 0$ sufficiently small such that whenever $\|x\| \le R_0$,

$$\int_{\substack{\mathbf{t}\in[\mathbf{0},\mathbf{b}]:\\ |\mathbf{t}|^{1/\alpha}\le\|x\|}} |\mathbf{t}|\, d\mathbf{t} \ge \text{const} \cdot \|x\|^{\alpha(N+1)}, \tag{4.29}$$

and the result follows. On the other hand, if $\|x\| > R_0$, then the preceding display still holds uniformly for all $x \in [-M, M]^d$. This proves the proposition. □

We mention also the following; it is an immediate consequence of Proposition 4.6 and the scaling relation (4.24).

LEMMA 4.7. *Choose and fix some* $\mathbf{b} \in (0, \infty)^N$ *and define* $[\mathbf{0}, \mathbf{b}]$ *as in Lemma* 4.5. *Then there exists a constant* $A_2 \in (1, \infty)$*—depending only on* d, N, α, $\min_{1\le j\le N} b_j$, *and* $\max_{1\le j\le N} b_j$*—such that for all* $x \in \mathbf{R}^d$,

$$\int_{[\mathbf{0},2\mathbf{b}]} g_{\mathbf{t}}(x)\, d\mathbf{t} \le A_2 \int_{[\mathbf{0},\mathbf{b}]} g_{\mathbf{t}}(x)\, d\mathbf{t}. \tag{4.30}$$

PROOF. Let $M > 1$ be a constant. If $x \in [-M, M]^d$, then (4.30) follows from Proposition 4.6. And if $\|x\| \ge M$, then (4.30) holds because of (4.24) and (4.26), together with the well-known fact that g_1 is continuous and strictly positive everywhere; compare with the first line in (4.28). □

4.3. *First part of the proof.* Our goal, in this first half, is to prove the following:

$$\text{(4.31)}\quad \mathcal{C}_{d-\alpha N}(E\times F)>0 \quad\Longrightarrow\quad \mathrm{P}\{W(E)\cap X_\alpha(\mathbf{R}_+^N)\cap F\neq\varnothing\}>0.$$

By Lemma 4.1 in [12], it is equivalent to prove

$$\text{(4.32)}\quad \mathcal{C}_{d-\alpha N}(E\times F)>0 \quad\Longrightarrow\quad \mathrm{E}\{\lambda_d((W(E)\cap F)\ominus X_\alpha(\mathbf{R}_+^N))\}>0,$$

where λ_d is the Lebesgue measure in $\mathbf{R}^d$ and $A\ominus B:=\{a-b:a\in A,b\in B\}$.

First, let us make some reductions. Because $E\subset(0,\infty)$ and $F\subset\mathbf{R}^d$ are assumed to be compact, there exists a $q\in(1,\infty)$ such that

$$\text{(4.33)}\qquad E\subseteq[q^{-1},q]\quad\text{and}\quad F\subseteq[-q,q]^d.$$

We will use q for this purpose unwaiveringly. Notice that if, either

$$\mathrm{E}\{\lambda_d(X_\alpha(\mathbf{R}_+^N))\}>0$$

or there exist some $n\le N-1$ and distinct $i_1,\dots,i_n\in\{1,\dots,N\}$ such that

$$\mathrm{E}\{\lambda_d((W(E)\cap F)\ominus X_{i_1,\dots,i_n}(\mathbf{R}_+^n))\}>0,$$

then (4.32) holds trivially. In the above, similarly to (3.5), $X_{i_1,\dots,i_n}$ is defined by

$$X_{i_1,\dots,i_n}(\mathbf{t})=\sum_{k=1}^n X^{(i_k)}(t_{i_k})\qquad\text{for all }\mathbf{t}:=(t_{i_1},\dots,t_{i_n})\in\mathbf{R}_+^n.$$

Hence, without loss of generality, we can and will assume that $\mathrm{E}(X_\alpha(\mathbf{R}_+^N))=0$ and $\mathrm{E}\{\lambda_d((W(E)\cap F)\ominus X_{i_1,\dots,i_n}(\mathbf{R}_+^n))\}=0$ for all $n\le N-1$ and all distinct $i_1,\dots,i_n\in\{1,\dots,N\}$. Since each Lévy process X_j has only countable number of jumps, this assumption implies

$$\text{(4.34)}\qquad \lambda_d\{(W(E)\cap F)\ominus(\overline{X_\alpha(\mathbf{R}_+^N)}\setminus X_\alpha(\mathbf{R}_+^N))\}=0\qquad \text{P-a.s.}$$

Now we provide some preliminary result for proving (4.32). Define

$$\text{(4.35)}\qquad f_\epsilon(x):=\frac{1}{\nu_d\epsilon^d}\mathbf{1}_{B_0(\epsilon)}(x)\quad\text{and}\quad \phi_\epsilon(x):=(f_\epsilon*f_\epsilon)(x).$$

For every $\mu\in\mathcal{P}_d(E\times F)$ and $\epsilon>0$ we define a random variable $Z_\epsilon(\mu)$ by

$$\text{(4.36)}\quad Z_\epsilon(\mu):=\int_{[1,2]^N}d\mathbf{u}\int_{E\times F}\mu(ds\,dx)\phi_\epsilon(W(s)-x)\phi_\epsilon(X_\alpha(\mathbf{u})-x).$$

LEMMA 4.8. *There exists a constant $a\in(0,\infty)$ such that*

$$\text{(4.37)}\qquad \inf_{\mu\in\mathcal{P}_d(E\times F)}\inf_{\epsilon\in(0,1)}\mathrm{E}[Z_\epsilon(\mu)]\ge a.$$

PROOF. Thanks to the triangle inequality, whenever $u \in B_0(\epsilon/2)$ and $v \in B_0(\epsilon/2)$, we have $u - v \in B_0(\epsilon)$ and $v \in B_0(\epsilon)$. Therefore, for all $u \in \mathbf{R}^d$ and $\epsilon > 0$,

$$\begin{aligned} \phi_\epsilon(u) &= \frac{1}{\nu_d^2 \epsilon^{2d}} \int_{\mathbf{R}^d} \mathbf{1}_{B_0(\epsilon)}(u - v) \mathbf{1}_{B_0(\epsilon)}(v)\,\mathrm{d}v \\ &\geq \frac{1}{\nu_d^2 \epsilon^{2d}} \mathbf{1}_{B_0(\epsilon/2)}(u) \int_{\mathbf{R}^d} \mathbf{1}_{B_0(\epsilon/2)}(v)\,\mathrm{d}v \geq 2^{-d} f_{\epsilon/2}(u). \end{aligned} \tag{4.38}$$

Because $f_{\epsilon/2}$ is a probability density, and since $\epsilon \in (0, 1)$, the preceding implies that for all $\mathbf{u} \in [1, 2]^N$ and $x \in \mathbf{R}^d$,

$$\begin{aligned} (\phi_\epsilon * g_{\mathbf{u}})(x) &= \int_{\mathbf{R}^d} \phi_\epsilon(u) g_{\mathbf{u}}(x - u)\,\mathrm{d}u \\ &\geq 2^{-d} \int_{\mathbf{R}^d} f_{\epsilon/2}(u) g_{\mathbf{u}}(x - u)\,\mathrm{d}u \geq 2^{-d} \inf_{\|z - x\| \leq 1/2} g_{\mathbf{u}}(z). \end{aligned} \tag{4.39}$$

Since $F \subset [-q, q]^d$, (4.39) and (4.24) in the Lemma 4.5 tell us that

$$a_0 := \inf_{\mathbf{u} \in [1,2]^N} \inf_{x \in F} \inf_{\epsilon \in (0,1)} (\phi_\epsilon * g_{\mathbf{u}})(x) > 0. \tag{4.40}$$

And, therefore, for all $\epsilon > 0$ and $\mu \in \mathcal{P}_d(E \times F)$,

$$\begin{aligned} \mathrm{E}[Z_\epsilon(\mu)] &= \int_{E \times F} \mu(\mathrm{d}s\,\mathrm{d}x)(\phi_\epsilon * p_s)(x) \int_{[1,2]^N} \mathrm{d}\mathbf{u} (\phi_\epsilon * g_{\mathbf{u}})(x) \\ &\geq a_0 \int_{E \times F} (\phi_\epsilon * p_s)(x) \mu(\mathrm{d}s\,\mathrm{d}x) \\ &\geq a_0 \inf_{s \in [1/q, q]} \inf_{x \in F} \inf_{\epsilon \in (0,1)} (\phi_\epsilon * p_s)(x), \end{aligned} \tag{4.41}$$

which is clearly positive. □

PROPOSITION 4.9. *There exists a constant $b \in (0, \infty)$ such that the following inequality holds simultaneously for all $\mu \in \mathcal{P}_d(E \times F)$:*

$$\sup_{\epsilon > 0} \mathrm{E}(|Z_\epsilon(\mu)|^2) \leq b \mathcal{E}_{d - \alpha N}(\mu). \tag{4.42}$$

PROOF. First of all, let us note the following complement to (4.38):

$$\phi_\epsilon(z) \leq 2^d f_{2\epsilon}(z) \qquad \text{for all } \epsilon > 0 \text{ and } z \in \mathbf{R}^d. \tag{4.43}$$

Define, for the sake of notational simplicity,

$$\mathcal{Q}_\epsilon(t, x; s, y) := \phi_\epsilon(W(t) - x) \phi_\epsilon(W(s) - y). \tag{4.44}$$

Next, we apply the Markov property to find that for all (t, x) and (s, y) in $E \times F$ such that $s < t$, and all $\epsilon > 0$,

$$\mathrm{E}[\mathcal{Q}_\epsilon(t, x; s, y)] = \mathrm{E}[\phi_\epsilon(W(s) - y) \phi_\epsilon(\tilde{W}(t - s) + W(s) - x)], \tag{4.45}$$

where $\tilde{W}$ is a Brownian motion independent of W. An application of (4.43) yields

$$\begin{aligned} \mathrm{E}\big[\mathcal{Q}_\epsilon(t,x;s,y)\big] &\le 4^d\mathrm{E}\big[f_{2\epsilon}\big(W(s)-y\big)f_{2\epsilon}\big(\tilde{W}(t-s)+W(s)-x\big)\big] \\ &\le 8^d\mathrm{E}\big[f_{2\epsilon}\big(W(s)-y\big)f_{4\epsilon}\big(\tilde{W}(t-s)-x+y\big)\big], \end{aligned} \tag{4.46}$$

thanks to the triangle inequality. Consequently, we may apply independence and (4.38) to find that

$$\begin{aligned} \mathrm{E}\big[\mathcal{Q}_\epsilon(t,x;s,y)\big] &\le 8^d\mathrm{E}\big[f_{2\epsilon}\big(W(s)-y\big)\big]\cdot\mathrm{E}\big[f_{4\epsilon}\big(W(t-s)-x+y\big)\big] \\ &\le 32^d\mathrm{E}\big[\phi_{4\epsilon}\big(W(s)-y\big)\big]\cdot\mathrm{E}\big[\phi_{8\epsilon}\big(W(t-s)-x+y\big)\big] \\ &= 32^d(\phi_{4\epsilon}*p_s)(y)\cdot(\phi_{8\epsilon}*p_{t-s})(x-y). \end{aligned} \tag{4.47}$$

Since $s\in E$, it follows that $s\ge 1/q$, and hence $\sup_{z\in\mathbf{R}^d}p_s(z)\le p_{1/q}(0)$. Thus,

$$\mathrm{E}\big[\phi_\epsilon\big(W(t)-x\big)\phi_\epsilon\big(W(s)-y\big)\big]\le 32^d p_{1/q}(0)\cdot(\phi_{8\epsilon}*p_{t-s})(x-y). \tag{4.48}$$

By symmetry, the following holds for all $(t,x),(s,y)\in E\times F$ and $\epsilon>0$:

$$\mathrm{E}\big[\phi_\epsilon\big(W(t)-x\big)\phi_\epsilon\big(W(s)-y\big)\big]\le 32^d p_{1/q}(0)\cdot(\phi_{8\epsilon}*p_{|t-s|})(x-y). \tag{4.49}$$

Similarly, we can show that for all $(\mathbf{u},x),(\mathbf{v},y)\in[1,2]^N\times F$ and $\epsilon>0$:

$$\mathrm{E}\big[\phi_\epsilon\big(X_\alpha(\mathbf{u})-x\big)\phi_\epsilon\big(X_\alpha(\mathbf{v})-y\big)\big]\le 16^d K\cdot(\phi_{8\epsilon}*g_{\mathbf{u}-\mathbf{v}})(x-y), \tag{4.50}$$

where $K:=g_{(1/q,\dots,1/q)}(0)<\infty$ by (4.21), and the definition of $g_{\mathbf{t}}(z)$ has been extended to all $\mathbf{t}\in\mathbf{R}^N\setminus\{0\}$ by symmetry, namely,

$$g_{\mathbf{t}}(z):=|\mathbf{t}|^{-d/\alpha}g_{\vec{1}}\left(\frac{z}{|\mathbf{t}|^{1/\alpha}}\right)\qquad\text{for all } z\in\mathbf{R}^d \text{ and } \mathbf{t}\in\mathbf{R}^N\setminus\{0\}, \tag{4.51}$$

where we recall $\vec{1}:=(1,\dots,1)\in\mathbf{R}^N$.

To verify (4.50), we define $Z_1=X_\alpha(\mathbf{u})-X_\alpha(\mathbf{u}\curlywedge\mathbf{v})$ and $Z_2=X_\alpha(\mathbf{v})-X_\alpha(\mathbf{u}\curlywedge\mathbf{v})$, where $\mathbf{u}\curlywedge\mathbf{v}=(u_1\wedge v_1,\dots,u_N\wedge v_N)$. Then the random variables Z_1, Z_2 and $X_\alpha(\mathbf{u}\curlywedge\mathbf{v})$ are independent. Similarly to (4.46) and (4.47), the left-hand side of (4.50) is bounded from above by

$$8^d\mathrm{E}\big[f_{2\epsilon}\big(Z_1+X_\alpha(\mathbf{u}\curlywedge\mathbf{v})-x\big)f_{4\epsilon}(Z_2-Z_1+x-y)\big]. \tag{4.52}$$

By conditional on Z_1 and Z_2 and applying the unmorality of $X_\alpha(\mathbf{u}\curlywedge\mathbf{v})$ (see Remark 2.3 in [11]), we see that (4.52) is at most

$$\begin{aligned} &\frac{8^d}{\nu_d(2\epsilon)^d}\mathrm{P}\big[\big|X_\alpha(\mathbf{u}\curlywedge\mathbf{v})\big|\le 2\epsilon\big]\mathrm{E}\big[f_{4\epsilon}(Z_2-Z_1+x-y)\big] \\ &\qquad\le 16^d g_{(1/q,\dots,1/q)}(0)\cdot(\phi_{8\epsilon}*g_{\mathbf{u}-\mathbf{v}})(x-y), \end{aligned} \tag{4.53}$$

where we have also use the fact that Z_2-Z_1 has density function $g_{\mathbf{u}-\mathbf{v}}$. This proves (4.50).

It follows easily from (4.49) and (4.50) that $\mathrm{E}(|Z_\epsilon(\mu)|^2)$ is bounded from above by a constant multiple of

$$\int\int (\phi_{8\epsilon} * p_{|t-s|})(x-y)\biggl(\int_{[1,2]^{2N}} (\phi_{8\epsilon} * g_{\mathbf{u}-\mathbf{v}})(x-y)\,\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}\biggr) \times \mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y), \tag{4.54}$$

uniformly for all $\epsilon > 0$. Define

$$\kappa(z) := \int_{[0,1]^N} g_{\mathbf{u}}(z)\,\mathrm{d}\mathbf{u} \qquad \text{for all } z \in \mathbf{R}^d. \tag{4.55}$$

Then we have shown that, uniformly for every $\epsilon > 0$,

$$\mathrm{E}\bigl(|Z_\epsilon(\mu)|^2\bigr) \le \mathrm{const}\cdot \int\int (\phi_{8\epsilon} * p_{|t-s|})(x-y)(\phi_{8\epsilon} * \kappa)(x-y) \times \mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y). \tag{4.56}$$

It follows easily from (4.21) that the conditions of Proposition 4.2 are met for $\sigma(\mathrm{d}x) := \nu(\mathrm{d}x) := \phi_{8\epsilon}(x)\,\mathrm{d}x$ and, therefore, that proposition yields the following bound: Uniformly for all $\epsilon > 0$,

$$\mathrm{E}\bigl(|Z_\epsilon(\mu)|^2\bigr) \le \mathrm{const}\cdot \int\int p_{|t-s|}(x-y)\kappa(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y). \tag{4.57}$$

According to Proposition 4.6, $\kappa(z) \le \mathrm{const}/\|z\|^{d-\alpha N}$ uniformly for all $z \in \{x - y : x, y \in F\}$, and the proof is thus completed. □

Now we establish (4.31).

PROOF OF THEOREM 3.1 (First half). If $\mathcal{C}_{d-\alpha N}(E \times F) > 0$, then there exists $\mu_0 \in \mathcal{P}_d(E \times F)$ such that $\mathcal{E}_{d-\alpha N}(\mu_0) < \infty$, by definition. We apply the Paley–Zygmund inequality [10], page 72, to Lemma 4.8 and Proposition 4.9, with μ replaced by μ_0, to find that for all $\epsilon > 0$,

$$\mathrm{P}\bigl\{Z_\epsilon(\mu_0) > 0\bigr\} \ge \frac{|\mathrm{E}Z_\epsilon(\mu_0)|^2}{\mathrm{E}(|Z_\epsilon(\mu_0)|^2)} \ge \frac{a^2/b}{\mathcal{E}_{d-\alpha N}(\mu_0)}. \tag{4.58}$$

If $Z_\epsilon(\mu_0)(\omega) > 0$ for some ω in the underlying sample space, then it follows from (4.36) and (4.38) that

$$\inf_{s\in E}\inf_{x\in F}\inf_{\mathbf{u}\in[1,2]^N} \max\bigl(\|W(s)-x\|, \|X_\alpha(\mathbf{u})-x\|\bigr)(\omega) \le \epsilon \tag{4.59}$$

for the very same ω. Letting $\epsilon \to 0$ in (4.58) we see that, as the right-most term in (4.58) is independent of $\epsilon > 0$, the preceding establishes

$$\mathrm{P}\bigl\{W(E) \cap \overline{X_\alpha\bigl([a,b]^N\bigr)} \cap F \neq \varnothing\bigr\} > 0.$$

From the proof of Lemma 4.1 in [12], we see that the above implies

$$\mathrm{E}\big\{\lambda_d\big(\big(W(E)\cap F\big)\ominus \overline{X_\alpha\big([a,b]^N\big)}\big)\big\}>0. \tag{4.60}$$

Because of (4.34), we obtain (4.32). This proves the first half of the proof of Theorem 3.1. □

4.4. *Second part of the proof.* For the second half of our proof, we aim to prove that

$$\mathrm{P}\big\{W(E)\cap X_\alpha\big([a,b]^N\big)\cap F\neq\varnothing\big\}>0 \quad\Longrightarrow\quad \mathcal{C}_{d-\alpha N}(E\times F)>0 \tag{4.61}$$

for all positive real numbers $a<b$. This would complete our derivation of Theorem 3.1. In order to simplify the exposition, we make some reductions. Since F has Lebesgue measure 0, we may and will assume that E has no isolated points. Furthermore, we will take $[a,b]^N=[1,3/2]^N$.

Henceforth, we assume that the displayed probability in (4.61) is positive. Let ∂ be a point that is not in $\mathbf{R}_+\times\mathbf{R}_+^N$, and we define an $E\times[1,3/2]^N\cup\{\partial\}$-valued random variable $T=(S,\mathbf{U})$ as follows:

1. If there is no $(s,\mathbf{u})\in E\times[1,3/2]^N$ such that $W(s)=X_\alpha(\mathbf{u})\in F$, then $T=(S,\mathbf{U}):=\partial$.
2. If there exists $(s,\mathbf{u})\in E\times[1,3/2]^N$ such that $W(s)=X_\alpha(\mathbf{u})\in F$, then we define $T=(S,\mathbf{U})$ inductively. Let S denote the first time in E when W hits $X_\alpha([1,3/2]^N)\cap F$, namely,

$$S:=\inf\big\{s\in E: W(s)\in X_\alpha\big([1,3/2]^N\big)\cap F\big\}. \tag{4.62}$$

It follows from (4.62) that there is a sequence $(s^n,\mathbf{u}^n)\in E\times[1,3/2]^N$ such that $s^n\downarrow S$ and $W(s^n)=X_\alpha(\mathbf{u}^n)\in F$ for all $n\geq 1$. Notice that for any subsequence of $\{\mathbf{u}^n\}$, say $\{\mathbf{u}^{n_k}\}$, which converges to some $\mathbf{u}=(u_1,\dots,u_N)\in[1,3/2]^N$, we have $\lim_{k\to\infty}X_\alpha(\mathbf{u}^{n_k})=W(S)$. The limit on the left-hand side can be expressed as the sum of left or right limits of the Lévy processes $X^{(j)}$ at u_j ($j=1,\dots,N$). For simplicity of notation, we denote this limit by $\overline{X}_\alpha(u_1,\dots,u_N)$. Then we can define inductively,

$$\begin{aligned}
U_1 &:= \inf\big\{u_1\in[1,3/2]:\overline{X}_\alpha(u_1,u_2,\dots,u_N)=W(S)\\
&\qquad\qquad \text{for some } u_2,\dots,u_N\in[1,3/2]\big\},\\
U_2 &:= \inf\big\{u_2\in[1,3/2]:\overline{X}_\alpha(U_1,u_2,\dots,u_N)=W(S)\\
&\qquad\qquad \text{for some } u_3,\dots,u_N\in[1,3/2]\big\},\\
&\ \ \vdots\\
U_N &:= \inf\big\{u_N\in[1,3/2]:\overline{X}_\alpha(U_1,\dots,U_{N-1},u_N)=W(S)\big\}.
\end{aligned} \tag{4.63}$$

Note that $\mathbf{U} = (U_1, \ldots, U_N) \in [1, 3/2]^N$ and $\overline{X}_\alpha(\mathbf{U}) = W(S) \in F$ on the event $\{(S, \mathbf{U}) \neq \partial\}$.

Now for every two Borel sets $G_1 \subseteq E$ and $G_2 \subseteq F$ we define

$$\mu(G_1 \times G_2) := \mathrm{P}\big\{S \in G_1, \overline{X}_\alpha(\mathbf{U}) \in G_2 | T \neq \partial\big\}. \tag{4.64}$$

Since $\mathrm{P}\{T \neq \partial\} > 0$, it follows that μ is a bona fide probability measure on $E \times F$. Moreover, $\mu \in \mathcal{P}_d(E \times F)$, since for every $t > 0$,

$$\mu\big(\{t\} \times F\big) = \mathrm{P}\big\{S = t, \overline{X}_\alpha(\mathbf{U}) \in F | T \neq \partial\big\} \leq \frac{\mathrm{P}\{W(t) \in F\}}{\mathrm{P}\{T \neq \partial\}} = 0, \tag{4.65}$$

because F has Lebesgue measure 0.

For every $\epsilon > 0$, we define $Z_\epsilon(\mu)$ by (4.36), but insist on one (important) change. Namely, now, we use the Gaussian mollifier,

$$\phi_\epsilon(z) := \frac{1}{(2\pi\epsilon^2)^{d/2}} \exp\left(-\frac{\|z\|^2}{2\epsilon^2}\right), \tag{4.66}$$

in place of $f_\epsilon * f_\epsilon$. (The change in the notation is used only in this portion of the present proof.)

Thanks to the proof of Lemma 4.8,

$$\inf_{\epsilon \in (0,1)} \mathrm{E}\big[Z_\epsilon(\mu)\big] > 0. \tag{4.67}$$

We can argue, as we did in the proof of (4.56) [e.g., up to a constant factor, the inequalities (4.49) and (4.50) still hold], to find that

$$\begin{aligned} \sup_{\epsilon \in (0,1)} \mathrm{E}\big(\big|Z_\epsilon(\mu)\big|^2\big) \leq \mathrm{const} \cdot \int\int (\phi_{8\epsilon} * p_{|t-s|})(x-y)(\phi_{8\epsilon} * \kappa)(x-y) \\ \times \mu(\mathrm{d}s\,\mathrm{d}x)\mu(\mathrm{d}t\,\mathrm{d}y), \end{aligned} \tag{4.68}$$

where κ is defined by (4.55). Define

$$\tilde{\kappa}(z) := \int_{[0,1/2]^N} g_{\mathbf{t}}(z)\,\mathrm{d}\mathbf{t} \qquad \text{for all } z \in \mathbf{R}^d. \tag{4.69}$$

Thanks to Lemma 4.7,

$$\begin{aligned} \sup_{\epsilon \in (0,1)} \mathrm{E}\big(\big|Z_\epsilon(\mu)\big|^2\big) \leq \mathrm{const} \cdot \int\int (\phi_{8\epsilon} * p_{|t-s|})(x-y)(\phi_{8\epsilon} * \tilde{\kappa})(x-y) \\ \times \mu(\mathrm{d}s\,\mathrm{d}x)\mu(\mathrm{d}t\,\mathrm{d}y). \end{aligned} \tag{4.70}$$

Now we are ready to explain why we had to change the definition of ϕ_ϵ from $f_\epsilon * f_\epsilon$ to the present Gaussian ones: In the present Gaussian case, both subscripts of "8ϵ" can be replaced by "ϵ" at no extra cost; see (4.71) below. Here is the reason why:

First of all, note that ϕ_ϵ is still positive definite; in fact, $\widehat{\phi}_\epsilon(\xi) = \mathrm{e}^{-\epsilon^2\|\xi\|^2/2} > 0$ for all $\xi \in \mathbf{R}^d$. Next—and this is important—we can observe that $\widehat{\phi}_\epsilon \le \widehat{\phi}_\delta$ whenever $0 < \delta < \epsilon$. And hence, the following holds, thanks to Remark 4.3:

$$\sup_{\epsilon\in(0,1)} \mathrm{E}\big(|Z_\epsilon(\mu)|^2\big) \le \mathrm{const}\cdot \int\int (\phi_\epsilon * p_{|t-s|})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y) \times \mu(\mathrm{d}s\,\mathrm{d}x)\mu(\mathrm{d}t\,\mathrm{d}y). \tag{4.71}$$

This proves the assertion that "8ϵ can be replaced by ϵ."

Now define a partial order $\prec$ on $\mathbf{R}^N$ as follows: $\mathbf{u} \prec \mathbf{v}$ if and only if $u_i \le v_i$ for all $i = 1,\dots,N$. Let $\mathcal{X}_\mathbf{v}$ denote the σ-algebra generated by the collection $\{X_\alpha(\mathbf{u})\}_{\mathbf{u}\prec\mathbf{v}}$. Also define $\mathcal{G} := \{\mathcal{G}_t\}_{t\ge 0}$ to be the usual augmented filtration of the Brownian motion W.

According to Theorem 2.3.1 of [10], page 405, $\{\mathcal{X}_\mathbf{v}\}$ is a commuting N-parameter filtration [10], page 233. Hence, so is the $(N+1)$-parameter filtration

$$\mathcal{F} := \{\mathcal{F}_{s,\mathbf{u}};\, s \ge 0, \mathbf{u} \in \mathbf{R}_+^N\}, \tag{4.72}$$

where $\mathcal{F}_{s,\mathbf{u}} := \mathcal{G}_s \times \mathcal{X}_\mathbf{u}$ is the product σ-algebra.

Now, for any fixed $(s,\mathbf{u}) \in E \times [1,3/2]^N$,

$$\mathrm{E}\big[Z_\epsilon(\mu)\,|\,\mathcal{F}_{s,\mathbf{u}}\big] \ge \int_{V(\mathbf{u})} \mathrm{d}\mathbf{v} \int_{\substack{E\times F\\ t\ge s}} \mu(\mathrm{d}t\,\mathrm{d}x)\mathcal{T}_\epsilon(t,x;\mathbf{v}), \tag{4.73}$$

where

$$V(\mathbf{u}) := \{\mathbf{v} \in [1,2]^N : u_j \le v_j \text{ for all } 1 \le j \le N\} \tag{4.74}$$

and

$$\mathcal{T}_\epsilon(t,x;\mathbf{v}) := \mathrm{E}\big[\phi_\epsilon\big(W(t)-x\big)\phi_\epsilon\big(X_\alpha(\mathbf{v})-x\big)\,|\,\mathcal{F}_{s,\mathbf{u}}\big]. \tag{4.75}$$

Thanks to independence, and the respective Markov properties of the processes $W, X^{(1)},\dots,X^{(N)}$,

$$\begin{aligned}\mathcal{T}_\epsilon(t,x;\mathbf{v}) &= \mathrm{E}\big[\phi_\epsilon\big(W(t)-x\big)\,|\,\mathcal{G}_s\big]\cdot \mathrm{E}\big[\phi_\epsilon\big(X_\alpha(\mathbf{v})-x\big)\,|\,\mathcal{X}_\mathbf{u}\big]\\ &= (\phi_\epsilon * p_{t-s})\big(x-W(s)\big)\cdot(\phi_\epsilon * g_{\mathbf{v}-\mathbf{u}})\big(x-X_\alpha(\mathbf{u})\big).\end{aligned} \tag{4.76}$$

Therefore, the definition (4.69) of $\tilde{\kappa}$ and the triangle inequality together reveal that with probability one,

$$\begin{aligned}&\mathrm{E}\big[Z_\epsilon(\mu)\,|\,\mathcal{F}_{s,\mathbf{u}}\big]\\ &\quad\ge \mathbf{1}_{\{(S,\mathbf{U})\ne\partial\}}(\omega)\\ &\qquad\times \int_{\substack{E\times F\\ t>s}} (\phi_\epsilon * p_{t-s})\big(x-W(s)\big)(\phi_\epsilon * \tilde{\kappa})\big(x-X_\alpha(\mathbf{u})\big)\mu(\mathrm{d}t\,\mathrm{d}x).\end{aligned} \tag{4.77}$$

This inequality is valid almost surely, simultaneously for all s in a dense countable subset of E (which will be assumed as a subset of $\mathbf{Q}_+$ for simplicity of notation) and all $\mathbf{u} \in [1, 3/2]^N \cap \mathbf{Q}_+^N$.

Select points with rational coordinates that converge, coordinatewise from the above and below, to $(S(\omega), \mathbf{U}(\omega))$. In this way, we find that

$$\begin{aligned} \sup_{\substack{s\in E, \mathbf{u}\in[1,3/2]^N \\ \text{all rational coords}}} & \mathrm{E}\big[Z_\epsilon(\mu) | \mathcal{F}_{s,\mathbf{u}}\big] \\ & \geq \mathbf{1}_{\{(S,\mathbf{U})\neq\partial\}}(\omega) \\ & \quad \times \int_{\substack{E\times F \\ t>S}} (\phi_\epsilon * p_{t-S})\big(x - W(S)\big)(\phi_\epsilon * \tilde{\kappa})\big(x - \overline{X}_\alpha(\mathbf{U})\big)\mu(\mathrm{d}t\,\mathrm{d}x). \end{aligned} \tag{4.78}$$

This is valid ω by ω. We square both sides of (4.78) and then apply expectations to both sides in order to obtain the following:

$$\begin{aligned} \mathrm{E}\Big\{\Big(& \sup_{(s,\mathbf{u})\in\mathbf{Q}_+^{N+1}} \mathrm{E}\big[Z_\epsilon(\mu)|\mathcal{F}_{s,\mathbf{u}}\big]\Big)^2\Big\} \\ & \geq \mathrm{P}\big\{(S,\mathbf{U}) \neq \partial\big\} \\ & \quad \times \mathrm{E}\bigg[\bigg(\int_{\substack{E\times F \\ t>S}} \Psi_\epsilon(t,x)\mu(\mathrm{d}t\,\mathrm{d}x)\bigg)^2 \Big| (S,\mathbf{U}) \neq \partial\bigg], \end{aligned} \tag{4.79}$$

where

$$\Psi_\epsilon(t,x) := (\phi_\epsilon * p_{t-S})\big(x - W(S)\big)(\phi_\epsilon * \tilde{\kappa})\big(x - \overline{X}_\alpha(\mathbf{U})\big).$$

According to (4.64), and because $W(S) = \overline{X}_\alpha(\mathbf{U})$ on $\{(S,\mathbf{U}) \neq \partial\}$, the conditional expectation in (4.79) is equal to the following:

$$\int\bigg(\int_{\substack{E\times F \\ t>s}} (\phi_\epsilon * p_{t-s})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\bigg)^2 \mu(\mathrm{d}s\,\mathrm{d}y). \tag{4.80}$$

In view of the Cauchy–Schwarz inequality, the quantity in (4.80) is at least

$$\bigg(\int\int_{\substack{E\times F \\ t>s}} (\phi_\epsilon * p_{t-s})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\bigg)^2,$$

which is, in turn, greater than or equal to

$$\frac{1}{4}\bigg(\int\int (\phi_\epsilon * p_{|t-s|})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y)\mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\bigg)^2, \tag{4.81}$$

by symmetry.

The preceding estimates from below the conditional expectation in (4.79). And this yields a bound on the right-hand side of (4.79). We can also obtain a good estimate for the left-hand side of (4.79). Indeed, the $(N+1)$-parameter filtration $\mathcal{F}$ is commuting; therefore, according to Cairoli's strong $(2,2)$ inequality [10], Theorem 2.3.2, page 235,

$$\mathrm{E}\Big\{\Big(\sup_{(s,\mathbf{u})\in\mathbf{Q}_+^{N+1}} \mathrm{E}\big[Z_\epsilon(\mu)\big|\mathcal{F}_{s,\mathbf{u}}\big]\Big)^2\Big\} \le 4^{N+1}\mathrm{E}\big(|Z_\epsilon(\mu)|^2\big), \tag{4.82}$$

and this is in turn at most a constant times the final quantity in (4.81); compare with (4.71). In this way, we are led to the following bound:

$$\begin{aligned}\mathrm{P}\big\{(S,\mathbf{U})\neq\partial\big\} \le \mathrm{const}\cdot\Big[\int\!\!\int (\phi_\epsilon * p_{|t-s|})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y) \\ \times \mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\Big]^{-1}.\end{aligned} \tag{4.83}$$

Since the implied constant is independent of ϵ, we can let $\epsilon \downarrow 0$. As the integrand is lower semicontinuous, we obtain the following from simple real-variables considerations:

$$\begin{aligned}\mathrm{P}\big\{(S,\mathbf{U})\neq\partial\big\} \le \mathrm{const}\cdot\Big[\int\!\!\int p_{|t-s|}(x-y)\tilde{\kappa}(x-y) \\ \times \mu(\mathrm{d}t\,\mathrm{d}x)\mu(\mathrm{d}s\,\mathrm{d}y)\Big]^{-1}.\end{aligned} \tag{4.84}$$

By Proposition 4.6, the term in the reciprocated brackets is equivalent to the energy $\mathcal{E}_{d-\alpha N}(\mu)$ of μ, and because μ is a probability measure on $E\times F$, we obtain the following:

$$\mathrm{P}\big\{(S,\mathbf{U})\neq\partial\big\} \le \mathrm{const}\cdot\mathcal{C}_{d-\alpha N}(E\times F). \tag{4.85}$$

This yields (4.61), and hence Theorem 3.1.

4.5. *Proof of Proposition* 1.4. The method for proving Theorem 3.1 can be modified to prove Proposition 1.4.

PROOF OF PROPOSITION 1.4 (Sketch). The proof for the sufficiency follows a similar line as in Section 4.3; we merely exclude all appearances of $X_\alpha(\mathbf{u})$, and keep careful track of the incurred changes. This argument is based on a second-moment argument and is standard. Hence, we only give a brief sketch for the proof of the more interesting necessity.

Assume that $\mathrm{P}\{W(E)\cap F\neq\varnothing\}>0$ and let Δ be a point that is not in $\mathbf{R}_+$. Define $\tau := \inf\{s\in E : W(s)\in F\}$ on $\{W(E)\cap F\neq\varnothing\}$, where $\inf\varnothing := \Delta$ (in this instance).

Let μ be the probability measure on $E\times F$ defined by

$$\mu(G_1\times G_2) := \mathrm{P}\big\{\tau\in G_1, W(\tau)\in G_2\big|\tau\neq\Delta\big\}. \tag{4.86}$$

Since F has Lebesgue measure 0, we have $\mu \in \mathcal{P}_d(E \times F)$. The rest of the proof is similar to the argument of Section 4.4, but is considerably simpler. Therefore, we omit the many remaining details. $\square$

5. Proof of Theorem 1.1. Let us recall Kaufman's uniform dimension result for Brownian motion [7]: *If $d \geq 2$, then outside a single null set* $\dim_{\mathrm{H}} W(G) = 2\dim_{\mathrm{H}} G$ *for all analytic sets $G \subset \mathbf{R}_+$*. Note that the set G can be random; that is, G can depend on the Brownian path itself. By considering the random set $G := W^{-1}(F)$, we can reduce the proof of Theorem 1.1 to one about determining a formula for $\| \dim_{\mathrm{H}}(E \cap W^{-1}(F))\|_{L^\infty(\mathrm{P})}$;[2] see the paragraph that precedes (5.22).

For this purpose, we choose and fix an $\alpha \in (0,1)$, and let X_α to be a symmetric stable Lévy process in $\mathbf{R}$ with index α. As before, we denote the transition probabilities of X_α by

$$g_t(x) := \frac{\mathrm{P}\{X_\alpha(t) \in \mathrm{d}x\}}{\mathrm{d}x} = \frac{1}{\pi}\int_0^\infty \cos(\xi|x|)\mathrm{e}^{-t\xi^\alpha/2}\,\mathrm{d}\xi. \tag{5.1}$$

We define υ to be the corresponding *1-potential density*. That is,

$$\upsilon(x) := \int_0^\infty g_t(x)\mathrm{e}^{-t}\,\mathrm{d}t. \tag{5.2}$$

It is known that for all $m > 0$ there exists $c_m = c_{m,\alpha} > 1$ such that

$$c_m^{-1}|x|^{\alpha-1} \leq \upsilon(x) \leq c_m|x|^{\alpha-1} \qquad \text{if } |x| \leq m; \tag{5.3}$$

see [10], Lemma 3.4.1, page 383. Since $\alpha \in (0,1)$, the preceding remains valid even when $x = 0$, as long as we recall that $1/0 := \infty$.

For any $\mu \in \mathcal{P}(E \times F)$, the collections of all probability measures on $E \times F$, and $\beta > 0$, define

$$\mathcal{I}_\beta(\mu) := \int\int \frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{\beta/2}}\mathbf{1}_{\{s\neq t\}}\mu(\mathrm{d}s\,\mathrm{d}x)\mu(\mathrm{d}t\,\mathrm{d}y). \tag{5.4}$$

The following forms the first step toward our proof of Theorem 1.1.

LEMMA 5.1. *Suppose there exists a $\mu \in \mathcal{P}(E \times F)$ such that $\mathcal{I}_{d+2(1-\alpha)}(\mu)$ is finite. Then, the random set $E \cap W^{-1}(F)$ intersects the closure of $X_\alpha(\mathbf{R}_+)$ with positive probability.*

REMARK 5.2. It is possible, but significantly harder, to prove that the sufficient condition of Lemma 5.1 is also necessary. We will omit the proof of that theorem, since we will not need it.

[2] Here is where we study the case $d \geq 2$ separately from the case $d = 1$. Kaufman's theorem fails to hold for one-dimensional Brownian motion. The standard example is the random set $G := W^{-1}\{0\}$. For this set, $\dim_{\mathrm{H}} W(G) = \dim_{\mathrm{H}}\{0\} = 0$. And this quantity is clearly different from $2\dim_{\mathrm{H}} G$, which is 1 thanks to a well-known theorem of Paul Lévy.

PROOF OF LEMMA 5.1. The proof is similar in spirit to that of Proposition 1.2. For all fixed $\epsilon > 0$ and probability measures μ on $(0,\infty) \times \mathbf{R}^d$, we define the following parabolic version of (4.36), using the same notation for $\phi_\epsilon := f_\epsilon * f_\epsilon$, etc.:

$$Y_\epsilon(\mu) := \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t \int \mu(\mathrm{d}s\,\mathrm{d}x)\phi_\epsilon\big(W(s)-x\big)\phi_\epsilon\big(X_\alpha(t)-s\big). \tag{5.5}$$

Just as we did in Lemma 4.8, we can find a constant $c \in (0,\infty)$—depending only on the geometry of E and F—such that uniformly for all $\mu \in \mathcal{P}(E \times F)$ and $\epsilon \in (0,1)$,

$$\mathrm{E}\big[Y_\epsilon(\mu)\big] = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t \int \mu(\mathrm{d}s\,\mathrm{d}x)(\phi_\epsilon * p_s)(x)(\phi_\epsilon * g_t)(s) \geq c; \tag{5.6}$$

but now we apply (5.3) in place of Lemma 4.5.

And we proceed, just as we did in Proposition 4.9, and prove that

$$\mathrm{E}\big(\big|Y_\epsilon(\mu)\big|^2\big) \leq \text{ const } \cdot \mathcal{I}_{d+2(1-\alpha)}(\mu). \tag{5.7}$$

The only differences between the proof of (5.7) and that of Proposition 4.9 are the following:

- Here we appeal to Proposition 4.4, whereas in Proposition 4.9 we made use of Proposition 4.2; and
- We apply (5.3) in place of both Proposition 4.6 and Lemma 4.7. Otherwise, the details of the two computations are essentially the same.

Lemma 5.1 follows from another application of the Paley–Zygmund lemma [10], page 72, to (5.6) and (5.7); the Paley–Zygmund lemma is used in a similar way as in the proof of the first half of Theorem 3.1. We omit the details, since this is a standard second-moment computation. □

Next, we present measure-theoretic conditions that are respectively sufficient and necessary for $\mathcal{I}_{d+2(1-\alpha)}(\mu)$ to be finite for some Borel space–time probability measure μ on $E \times F$.

LEMMA 5.3. *We always have*

$$\dim_{\mathrm{H}}(E \times F; \varrho) \leq \sup\Big\{\beta > 0 : \inf_{\mu \in \mathcal{P}(E\times F)} \mathcal{I}_\beta(\mu) < \infty\Big\}. \tag{5.8}$$

PROOF. For all space–time probability measures μ, and $\tau > 0$ define the *space–time τ-dimensional Bessel–Riesz energy* of μ as

$$\Upsilon_\tau(\mu;\varrho) := \int\int \frac{\mu(\mathrm{d}s\,\mathrm{d}x)\mu(\mathrm{d}t\,\mathrm{d}y)}{[\varrho((s,x);(t,y))]^\tau}. \tag{5.9}$$

A suitable formulation of Frostman's theorem [20] implies that

$$\dim_{\mathrm{H}}(E \times F; \varrho) = \sup\{\tau > 0 : \Upsilon_\tau(\mu;\varrho) < \infty\}. \tag{5.10}$$

We can consider separately the cases that $\|x-t\|^2 \le |s-t|$ and $\|x-y\|^2 > |s-t|$, and hence deduce that

$$\frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|s-t|^\beta} \le \min\left(\frac{c}{\|x-y\|^{2\beta}}, \frac{1}{|s-t|^\beta}\right), \tag{5.11}$$

where $c := \sup_{z>1} z^{2\beta}\mathrm{e}^{-z/2}$ is finite. Consequently, $\mathcal{I}_{2\beta}(\mu) \le c'\Upsilon_{2\beta}(\mu;\varrho)$, with $c' := \max(c, 1)$, and (5.8) follows from (5.10). $\square$

LEMMA 5.4. *With probability one*,

$$\dim_{\mathrm{H}}\big(E \cap W^{-1}(F)\big) \le \frac{\dim_{\mathrm{H}}(E \times F;\varrho) - d}{2}. \tag{5.12}$$

PROOF. Choose and fix some $r > 0$. Let $\mathcal{T}(r)$ denote the collection of all intervals of the form $[t-r^2, t+r^2]$ that are in $[1/q, q]$. Also, let $\mathcal{S}(r)$ denote the collection of all closed Euclidean $[\ell^2]$ balls of radius r that are contained in $[-q,q]^d$. Recall that X_α is a symmetric stable process of index $\alpha \in (0,1)$ that is independent of W. It is well known that uniformly for all $r \in (0,1)$,

$$\sup_{I\in\mathcal{T}(r)} \mathrm{P}\big\{X_\alpha\big([0,1]\big) \cap I \ne \varnothing\big\} \le \mathrm{const}\cdot r^{2(1-\alpha)}; \tag{5.13}$$

see [10], Lemma 1.4.3, page 355, for example. It is just as simple to prove that the following holds uniformly for all $r \in (0,1)$:

$$\sup_{I\in\mathcal{T}(r)} \sup_{J\in\mathcal{S}(r)} \mathrm{P}\big\{W(I) \cap J \ne \varnothing\big\} \le \mathrm{const}\cdot r^d. \tag{5.14}$$

[Indeed, conditional on $\{W(I) \cap J \ne \varnothing\}$, the random variable $W(t)$ comes to within r of J with a minimum positive probability, where t denotes the smallest point in I.] Because $W(I) \cap J \ne \varnothing$ if and only if $W^{-1}(J) \cap I \ne \varnothing$, it follows that uniformly for all $r \in (0,1)$,

$$\sup_{I\in\mathcal{T}(r)} \sup_{J\in\mathcal{S}(r)} \mathrm{P}\big\{W^{-1}(J) \cap I \cap X_\alpha\big([0,1]\big) \ne \varnothing\big\} \le \mathrm{const}\cdot r^{d+2(1-\alpha)}. \tag{5.15}$$

Define

$$\mathcal{R} := \bigcup_{r\in(0,1)} \big\{I \times J : I \in \mathcal{T}(r) \text{ and } J \in \mathcal{S}(r)\big\}. \tag{5.16}$$

Thus, $\mathcal{R}$ denotes the collection of all "space–time parabolic rectangles" whose ϱ-diameter lies in the interval $(0,1)$.

Suppose $d + 2(1-\alpha) > \dim_{\mathrm{H}}(E \times F;\varrho)$. By the definition of Hausdorff dimension, and a Vitali-type covering argument (see Mattila [15], Theorem 2.8, page 34) for all $\epsilon > 0$, we can find a countable collection $\{E_j \times F_j\}_{j=1}^\infty$ of elements of $\mathcal{R}$ such that: (i) $\bigcup_{j=1}^\infty (E_j \times F_j)$ contains $E \times F$; (ii) the ϱ-diameter of $E_j \times F_j$ is positive and less than one (strictly) for all $j \geq 1$; and (iii) $\sum_{j=1}^{\infty} |\varrho\text{-diam}(E_j \times F_j)|^{d+2(1-\alpha)} \leq \epsilon$. Thanks to (5.15),

$$\begin{aligned}
&\mathrm{P}\{W^{-1}(F) \cap E \cap X_\alpha([0,1]) \neq \varnothing\} \\
&\quad \leq \sum_{j=1}^{\infty} \mathrm{P}\{W^{-1}(F_j) \cap E_j \cap X_\alpha([0,1]) \neq \varnothing\} \qquad (5.17)\\
&\quad \leq \mathrm{const} \cdot \sum_{j=1}^{\infty} \bigl|\varrho\text{-diam}(E_j \times F_j)\bigr|^{d+2(1-\alpha)} \leq \mathrm{const} \cdot \epsilon.
\end{aligned}$$

Since neither the implied constant nor the left-most term depend on the value of ϵ, the preceding shows that $W^{-1}(F) \cap E \cap X_\alpha([0,1])$ is empty almost surely.

Now let us recall half of McKean's theorem [10], Example 2, page 436: *If* $\dim_{\mathrm{H}}(A) > 1-\alpha$, *then* $X_\alpha([0,1]) \cap A$ *is nonvoid with positive probability.* We apply McKean's theorem, conditionally, with $A := W^{-1}(F) \cap E$ to find that if $d + 2(1-\alpha) > \dim_{\mathrm{H}}(E \times F; \varrho)$, then

$$\dim_{\mathrm{H}}\bigl(W^{-1}(F) \cap E\bigr) \leq 1-\alpha \qquad \text{almost surely.} \tag{5.18}$$

The preceding is valid almost surely, simultaneously for all rational values of $1-\alpha$ that are strictly between one and $\frac{1}{2}(\dim_{\mathrm{H}}(E \times F; \varrho) - d)$. Thus, the result follows. □

PROOF OF THEOREM 1.1. By the modulus of continuity of Brownian motion, there exists a null set off which $\dim_{\mathrm{H}} W(A) \leq 2 \dim_{\mathrm{H}} A$, simultaneously for all Borel sets $A \subseteq \mathbf{R}_+$ that might—or might not—depend on the Brownian path itself. Since $W(E \cap W^{-1}(F)) = W(E) \cap F$, Lemma 5.4 implies that

$$\dim_{\mathrm{H}}\bigl(W(E) \cap F\bigr) \leq \dim_{\mathrm{H}}(E \times F; \varrho) - d \qquad \text{almost surely.} \tag{5.19}$$

For the remainder of the proof, we assume that $d \geq 2$, and propose to prove that

$$\bigl\|\dim_{\mathrm{H}}\bigl(W(E) \cap F\bigr)\bigr\|_{L^\infty(\mathrm{P})} \geq \dim_{\mathrm{H}}(E \times F; \varrho) - d. \tag{5.20}$$

Henceforth, we assume without loss of generality that

$$\dim_{\mathrm{H}}(E \times F; \varrho) > d; \tag{5.21}$$

for there is nothing left to prove otherwise. In accord with the theory of Taylor and Watson [20], (5.21) implies that $\mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0$.

According to Kaufman's uniform-dimension theorem [7], the Hausdorff dimension of $W(E) \cap F$ is almost surely equal to twice the Hausdorff dimension of $E \cap W^{-1}(F)$. Therefore, it suffices to prove the following in the case that $d \geq 2$:

$$\bigl\|\dim_{\mathrm{H}}\bigl(E \cap W^{-1}(F)\bigr)\bigr\|_{L^\infty(\mathrm{P})} \geq \frac{\dim_{\mathrm{H}}(E \times F; \varrho) - d}{2}, \tag{5.22}$$

as long as the right-hand side is positive. If $\alpha \in (0, 1)$ satisfies

$$1 - \alpha < \frac{\dim_{\mathrm{H}}(E \times F; \varrho) - d}{2}, \tag{5.23}$$

than Lemma 5.3 implies that $\mathcal{I}_{d+2(1-\alpha)}(\mu) < \infty$ for some $\mu \in \mathcal{P}(E \times F)$. Thanks to Lemma 5.1, $E \cap W^{-1}(F) \cap \overline{X_\alpha([0, 1])} \neq \varnothing$ with positive probability. Consequently,

$$\mathrm{P}\{\dim_{\mathrm{H}}(E \cap W^{-1}(F)) \geq 1 - \alpha\} > 0, \tag{5.24}$$

because the second half of McKean's theorem implies that *if* $\dim_{\mathrm{H}}(A) < 1 - \alpha$, *then* $\overline{X_\alpha(\mathbf{R}_+)} \cap A = \varnothing$ *almost surely*. Since (5.24) holds for all $\alpha \in (0, 1)$ that satisfy (5.23), (5.22) follows. This completes the proof. □

REMARK 5.5. Let us mention the following byproduct of our proof of Theorem 1.1: For every $d \geq 1$,

$$\left\| \dim_{\mathrm{H}}(E \cap W^{-1}(F)) \right\|_{L^\infty(\mathrm{P})} = \frac{\dim_{\mathrm{H}}(E \times F; \varrho) - d}{2}. \tag{5.25}$$

When $d = 1$, this was found first by Kaufman [8], who used other arguments (for the harder half). See Hawkes [4] for similar results in case W is replaced by a stable subordinator of index $\alpha \in (0, 1)$.

We conclude this paper with some problems that continue to elude us.

OPEN PROBLEMS. Theorems 1.1 and 1.3 together imply that when $d \geq 2$ and $F \subset \mathbf{R}^d$ has Lebesgue measure 0,

$$\sup\left\{\gamma > 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty\right\} = \dim_{\mathrm{H}}(E \times F; \rho) - d. \tag{5.26}$$

The preceding is a kind of "parabolic Frostman theorem." And we saw in the Introduction that (5.26) is in general false when $d = 1$. We would like to better understand why the one-dimensional case is so different from the case $d \geq 2$. Thus, we are led naturally to a number of questions, three of which we state below:

P1. Equation (5.26) is, by itself, a theorem of geometric measure theory. Therefore, we ask, "*Is there a direct proof of* (5.26) *that does not involve random processes*, *broadly speaking*, *and Kaufman's uniform-dimension theorem* [7], *in particular*"?

P2. When $d \geq 2$, (5.26) gives an interpretation of the capacity form on the left-hand side of (5.26) in terms of the geometric object on the right-hand side. Can we understand the left-hand side of (5.26) geometrically in the case that $d = 1$?

P3. The following interesting question is due to an anonymous referee: Are there quantitative relationships between a rough hitting-type probability of the form $\mathrm{P}\{\dim_{\mathrm{H}}(W(E) \cap F) > \gamma\}$ and the new capacity form of Benjamini et al. [1] (see also [16], Theorem 8.24)? We suspect the answer is "yes," but do not have a proof.

Acknowledgments. Many hearty thanks are due to Professors Gregory Lawler and Yuval Peres. The former showed us the counterexample in the Introduction, and the latter introduced us to the problem that is being considered here.

We thank the anonymous referee for pointing out several mistakes in an earlier formulation of Theorem 1.3 in a previous draft of this manuscript.

REFERENCES

[1] BENJAMINI, I., PEMANTLE, R. and PERES, Y. (1995). Martin capacity for Markov chains. *Ann. Probab.* **23** 1332–1346. MR1349175

[2] DOOB, J. L. (1984). *Classical Potential Theory and Its Probabilistic Counterpart*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **262**. Springer, New York. MR0731258

[3] FOONDUN, M. and KHOSHNEVISAN, D. (2013). On the stochastic heat equation with spatially-colored random forcing. *Trans. Amer. Math. Soc.* **365** 409–458. MR2984063

[4] HAWKES, J. (1978). Measures of Hausdorff type and stable processes. *Mathematika* **25** 202–212. MR0533127

[5] HAWKES, J. (1981). Trees generated by a simple branching process. *J. Lond. Math. Soc.* (2) **24** 373–384. MR0631950

[6] KAHANE, J.-P. (1985). *Some Random Series of Functions*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **5**. Cambridge Univ. Press, Cambridge. MR0833073

[7] KAUFMAN, R. (1969). Une propriété métrique du mouvement brownien. *C. R. Acad. Sci. Paris Sér. A–B* **268** A727–A728. MR0240874

[8] KAUFMAN, R. (1972). Measures of Hausdorff-type, and Brownian motion. *Mathematika* **19** 115–119. MR0319280

[9] KAUFMAN, R. and WU, J. M. (1982). Parabolic potential theory. *J. Differential Equations* **43** 204–234. MR0647063

[10] KHOSHNEVISAN, D. (2002). *Multiparameter Processes*: *An Introduction to Random Fields*. Springer, New York. MR1914748

[11] KHOSHNEVISAN, D. and XIAO, Y. (2002). Level sets of additive Lévy processes. *Ann. Probab.* **30** 62–100. MR1894101

[12] KHOSHNEVISAN, D. and XIAO, Y. (2005). Lévy processes: Capacity and Hausdorff dimension. *Ann. Probab.* **33** 841–878. MR2135306

[13] KHOSHNEVISAN, D. and XIAO, Y. (2009). Harmonic analysis of additive Lévy processes. *Probab. Theory Related Fields* **145** 459–515. MR2529437

[14] LYONS, R. (1990). Random walks and percolation on trees. *Ann. Probab.* **18** 931–958. MR1062053

[15] MATTILA, P. (1995). *Geometry of Sets and Measures in Euclidean Spaces*: *Fractals and Rectifiability*. *Cambridge Studies in Advanced Mathematics* **44**. Cambridge Univ. Press, Cambridge. MR1333890

[16] MÖRTERS, P. and PERES, Y. (2010). *Brownian Motion*. Cambridge Univ. Press, Cambridge. MR2604525

[17] PERES, Y. (1996). Remarks on intersection-equivalence and capacity-equivalence. *Ann. Inst. H. Poincaré Phys. Théor.* **64** 339–347. MR1400296

[18] PERES, Y. (1999). Probability on trees: An introductory climb. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1997). *Lecture Notes in Mathematics* **1717** 193–280. Springer, Berlin. MR1746302

[19] TAYLOR, S. J. (1966). Multiple points for the sample paths of the symmetric stable process. *Z. Wahrsch. Verw. Gebiete* **5** 247–264. MR0202193

[20] TAYLOR, S. J. and WATSON, N. A. (1985). A Hausdorff measure classification of polar sets for the heat equation. *Math. Proc. Cambridge Philos. Soc.* **97** 325–344. MR0771826

[21] WATSON, N. A. (1976). Green functions, potentials, and the Dirichlet problem for the heat equation. *Proc. Lond. Math. Soc.* (3) **33** 251–298. [Corrigendum: *Proc. Lond. Math. Soc.* (3) **37** (1978) 32–34. MR0499239] MR0425145

[22] WATSON, N. A. (1978). Thermal capacity. *Proc. Lond. Math. Soc.* (3) **37** 342–362. MR0507610

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