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Optimal transport between random measures

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Abstract. We analyze the optimal transport problem between two equivariant random measures and derive sufficient conditions for the existence of a unique Monge solution. Moreover, we show that equivariance naturally appears in this context by proving that classical optimal couplings on bounded sets converge to the optimal coupling on the whole space. Finally, we derive sufficient conditions for the L^p cost to be finite by introducing a suitable metric.

Résumé. Nous analysons le problème du transport optimal entre deux mesures aléatoires et équivariantes et démontrons des conditions qui garantissent l'existence d'une solution de type Monge. En outre, nous démontrons que l'équivariance apparaît naturellement dans ce contexte en prouvant que les couplages optimaux classiques dans des ensembles bornés convergent vers le couplage optimal dans tout l'espace. Finalement nous démontrons des conditions suffisantes pour que le coût au sens L^p soit fini en introduisant une métrique appropriée.

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1. Introduction and statement of main results

In this article, we investigate the optimal transportation problem between two *equivariant random measures* $\lambda^\bullet$ and $\mu^\bullet$ (the $\bullet$ is to remind us on the randomness of the two measures). In particular, we are interested in conditions ensuring the existence of a (unique) Monge solution, i.e. a map $T : \Omega \times M \to M$ such that the optimal coupling is concentrated on the graph of T. It turns out that we can derive similar existence and uniqueness results as in the classical theory.

More than two hundred years ago, Monge [41] posed his famous transportation problem: Given two probability measures λ and μ on some Polish metric measure space (M, d, m) and a *cost function* $c : M \times M \to \mathbb{R}$ consider

$$\min_{T:T_*\lambda=\mu} \int c\big(x, T(x)\big)\lambda(\mathrm{d}x), \tag{1.1}$$

where $T_*\lambda$ denotes the push forward of the measure λ under the map T. This problem is very hard to solve. For example it is not clear in general if the set over which is minimized is nonempty. One hundred fifty years later Kantorovich [29] proposed a relaxation of this problem. He considered

$$\min_{q\in\Pi(\lambda,\mu)} \int c(x, y)q(\mathrm{d}x, \mathrm{d}y), \tag{1.2}$$

where $\Pi(\lambda, \mu)$ denotes the set of *all couplings* between λ and μ. The set $\Pi(\lambda, \mu)$ is always nonempty and compact. If the cost function c is lower semicontinuous this directly yields the existence of minimizers (see Section 4 in [51]). In general these two minimization problems lead to different solutions. It is a question of ongoing research to find sufficient conditions for the two minimization problems to coincide (e.g. see [8,11,20,21,40]). The first satisfactory result was achieved by Brenier and Rachev and Rüschendorf [10,46]. It was this discovery together with the seminal Ph.D. thesis of McCann [39] which prompted the fascinating development the optimal transport theory has undergone in the last twenty years with applications ranging from mathematical physics and PDEs to geometry and economics, e.g. [45,50,51] and references therein.

So far optimal transport problems have mainly been studied between finite measures. In [26], Sturm and the author, are the first who analyze an optimal transportation problem between two infinite measures, the Lebesgue measure and an equivariant simple point process. In this article we obtain sufficient conditions for the existence of a Monge solution for the optimal transport problem between two random measures. Furthermore, we show that the optimal coupling is the limit of classical optimal couplings on bounded sets. Let us start by describing our results in more detail.

Let (M, d, m) be a geodesic Polish metric measure space. We assume that there is a (countable, discrete) group G of isometries of M acting properly discontinuously, cocompactly and freely on M (see Definition 3.1) leaving the measure m invariant. A random measure $\lambda^\bullet$ on M is a measure valued random variable modeled on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. It is called *invariant* if $\lambda^\bullet(B) \stackrel{d}{=} \lambda^\bullet(gB)$ for any $g \in G$ and Borel set $B \subset M$. It turns out that for the sequel a more subtle concept is required. We assume that the probability space admits a measurable flow $\theta = (\theta_g)_{g\in G}$ which we interpret as the action of G on the support of a realization λ^ω. A random measure $\lambda^\bullet$ is called *equivariant* if

$$\lambda^{\theta_g \omega}(g\cdot) = \lambda^\omega(\cdot) \quad \text{for all } \omega \in \Omega, g \in G.$$

We will assume that $\mathbb{P}$ is stationary, that is $\mathbb{P}$ is invariant under the flow θ. In particular, this implies that equivariant random measures are invariant. All random measures will be defined on the *same* probability space (see also Section 3.3).

Take two equivariant random measures $\lambda^\bullet$ and $\mu^\bullet$ of equal intensity, i.e. $\mathbb{E}[\lambda^\bullet(B_0)] = \mathbb{E}[\mu^\bullet(B_0)]$ for a fundamental region B_0 (cf. Definitions 3.11 and 3.2), and a cost function c. We are interested in couplings $q^\bullet$ of $\lambda^\bullet$ and $\mu^\bullet$, i.e. measure valued random variables $\omega \mapsto q^\omega$ such that for any $\omega \in \Omega$ the measure q^ω on $M \times M$ is a coupling of λ^ω and μ^ω. Due to the almost sure infinite mass of λ^ω and μ^ω the usual notion of optimality, being a minimizer of the total transport cost (1.2), is not meaningful. However, the cost per volume is a reasonable quantity to consider. Hence, we look for minimizers of the *mean transportation cost*

$$\mathfrak{C}(q^\bullet) := \sup_{B\in \mathrm{Adm}(M)} \frac{1}{m(B)} \mathbb{E}\left[\int_{M\times B} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right], \tag{1.3}$$

where $\mathrm{Adm}(M)$ is the set of all bounded Borel sets that can be written as the finite union of "G-translates" of fundamental regions (see Section 3.7). For example for $M = \mathbb{R}^d$, $G = \mathbb{Z}^d$ acting by translation, a typical set would be a finite union of unit cubes (see also Section 1.1). We always consider cost functions of the form $c(x, y) = \vartheta(d(x, y))$ for some continuous strictly increasing function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\vartheta(0) = 0$ and $\lim_{r\to\infty} \vartheta(r) = \infty$ (see also Remark 3.12). As we are interested in Monge solutions and the restriction of optimal couplings $q^\bullet$ to bounded sets should be optimal in the classical sense, we additionally assume that the classical Monge problem (1.1) between two compactly supported probability measures λ and μ with $\lambda \ll m$ has a unique solution.

The supremum in (1.3) is a supremum over big sets (see Lemma 3.13). Hence, the existence of one minimizer of (1.3) implies the existence of many minimizers due to the normalizing factor $1/m(B)$. Therefore, opposed to the classical setting, we require the minimizer to have an additional property, namely equivariance. Thus, a coupling $q^\bullet$ of $\lambda^\bullet$ and $\mu^\bullet$ is called *optimal* if it is equivariant and minimizes the mean transportation cost (1.3) *among all equivariant couplings*. The set of all equivariant couplings between $\lambda^\bullet$ and $\mu^\bullet$ will be denoted by $\Pi_e(\lambda^\bullet, \mu^\bullet)$. Last and Thorisson resp. Last derive in [34] resp. [32] necessary and sufficient conditions for $\Pi_e(\lambda^\bullet, \mu^\bullet)$ to be nonempty for M being an Abelian resp. locally compact second countable group. We will show that there always is at least one optimal coupling as soon as the optimal/minimal mean transportation cost is finite, i.e. we show the existence of a Kantorovich solution. In many cases this solution happens to be a Monge solution:

Theorem 1.1. *Let $(\lambda^\bullet, \mu^\bullet)$ be two equivariant random measures of unit intensity on M. Let the cost function be given by $c(x, y) = \vartheta(d(x, y))$ for some strictly increasing continuous function ϑ with $\lim_{r\to\infty} \vartheta(r) = \infty$ such that the* *classical Monge problem* (1.1) *between two probability measures* $\nu_0 \ll m$ *and* ν_1 *has a unique solution. If the optimal mean transportation cost is finite*

$$\mathfrak{c}_{e,\infty} = \inf_{q^\bullet \in \Pi_e(\lambda^\bullet,\mu^\bullet)} \mathfrak{C}(q^\bullet) < \infty$$

and if λ^ω *is absolutely continuous to* m *for almost all* ω, *then there is a unique optimal coupling* $q^\bullet$ *between* $\lambda^\bullet$ *and* $\mu^\bullet$. *It can be represented as* $q^\omega = (id, T^\omega)_* \lambda^\omega$ *for some measurable map* $T^\omega : \mathrm{supp}(\lambda^\omega) \to \mathrm{supp}(\mu^\omega)$, *measurably only dependent on the* σ*-algebra generated by* $(\lambda^\bullet, \mu^\bullet)$.

At first sight the requirement that the optimal coupling needs to be equivariant might look unsatisfactory. However, this condition naturally appears. As mentioned above, any coupling $q^\bullet$ which claims to be optimal should have the property that $1_{A\times B} q^\bullet$ is optimal between its marginals in the classical sense for any two bounded Borel sets A and B. Reversing this argument one should expect that the optimal coupling $q^\bullet$, if it is unique, can be obtained as the limit of classical optimal couplings between $\lambda^\bullet$ and $\mu^\bullet$ restricted to bounded sets. This is indeed the case. In particular, the limit will be equivariant. However, to be able to prove this, we need to additionally assume that the group G is amenable (see Definition 3.5).

The assumptions on the group action imply that G is finitely generated. Let $(F_r)_{r\in\mathbb{N}}$ be an exhausting Følner sequence (see Proposition 3.6) of G. Let B_0 be a fundamental region (see Definition 3.2) and $B_r = F_r B_0$. Let Q_{B_r} be the unique optimal *semicoupling* between $\lambda^\bullet$ and $1_{B_r}\mu^\bullet$, that is the unique optimal coupling between $\rho \cdot \lambda^\bullet$ and $1_{B_r}\mu^\bullet$ for some optimal choice of density ρ (see also Sections 3.2 and 3.4). Put

$$\tilde{Q}_g^r := \frac{1}{|F_r|} \sum_{h\in gF_r^{-1}} Q_{hB_r}. \tag{1.4}$$

Theorem 1.2. *Let* $(\lambda^\bullet, \mu^\bullet)$ *be two equivariant random measures on* M, *such that the optimal mean transportation cost is finite*, $\mathfrak{c}_{e,\infty} < \infty$. *Assume, that* G *is amenable and* λ^ω *is absolutely continuous to* m *for almost all* ω. *Then, for every* $g \in G$

$$\tilde{Q}_g^r \to Q^\infty \quad \textit{vaguely}$$

in $\mathcal{M}(M \times M \times \Omega)$, *where* Q^∞ *denotes the unique optimal coupling.*

For the proof of this theorem the assumption of absolute continuity is only needed to ensure uniqueness of Q_{gB_r} and Q^∞. If we do not have absolute continuity but uniqueness of Q_{gB_r} and Q^∞ the same theorem with the same proof holds.

In the case of absolute continuity we can even say a bit more and get rid of the symmetrization procedure in (1.4). By Theorem 1.1 the unique optimal coupling Q^∞ is given by a Monge solution, that is

$$Q^\infty = (id, T)_* \lambda^\bullet.$$

Moreover, the optimal semicoupling Q_{gB_r} will be shown to be given by (see Theorem 4.7)

$$Q_{gB_r} = (id, T_{g,r})_* (\rho_{g,r} \lambda^\bullet),$$

for some measurable map $T_{g,r}$ and some density $\rho_{g,r}$. Then, we have

Theorem 1.3. *For every* $g \in G$

$$T_{g,r} \to T \quad \textit{locally in } \lambda^\bullet \otimes \mathbb{P} \textit{ measure.}$$

Analogous results will be obtained in the more general case of optimal semicouplings between $\lambda^\bullet$ and $\mu^\bullet$ where $\lambda^\bullet$ has intensity one and $\mu^\bullet$ has intensity $\beta \in (0, \infty)$ (see Theorem 5.9, Theorem 6.3, Proposition 6.7 and Section 7). In the case $\beta \le 1$, $\lambda^\bullet$ is allowed to not transport all of its mass. There will be some areas from which nothing is transported and the $\mu^\bullet$ mass can choose its favorite $\lambda^\bullet$ mass. In the case $\beta \geq 1$ the situation is the opposite. There is too much $\mu^\bullet$ mass. Hence, $\lambda^\bullet$ can choose its favorite $\mu^\bullet$ mass and some part of the $\mu^\bullet$ mass will not be satisfied, that is they will not get enough or even any of the $\lambda^\bullet$ mass.

The necessity of the amenability assumption in the last two theorems is closely related to the question whether

$$\mathfrak{c}_{e,\infty} = \mathfrak{c}_\infty := \inf_{q^\bullet \in \Pi(\lambda^\bullet, \mu^\bullet)} \mathfrak{C}(q^\bullet),$$

where $\Pi(\lambda^\bullet, \mu^\bullet)$ denotes the set of *all (semi)couplings* between $\lambda^\bullet$ and $\mu^\bullet$. In the case that G is amenable this is true (see Corollary 6.5). For the nonamenable case we do not know the answer. In particular, if $\mathfrak{c}_\infty < \mathfrak{c}_{e,\infty}$ there will be no approximation results in the spirit of Theorems 1.2 and 1.3 due to Lemma 3.13.

A very important condition in all the results is the finiteness of the minimal mean transportation cost $\mathfrak{c}_{e,\infty}$. In general, this is very difficult to check but in the case of the cost functions $c(x, y) = d(x, y)^p$ we can derive a sufficient condition. We write the optimal mean transportation cost between $\lambda^\bullet$ and $\mu^\bullet$ as $\mathbb{W}_p^p(\lambda^\bullet, \mu^\bullet)$, i.e.

$$\mathbb{W}_p^p(\lambda^\bullet, \mu^\bullet) = \inf_{q^\bullet \in \Pi_e(\lambda^\bullet, \mu^\bullet)} \mathfrak{C}(q^\bullet).$$

Denote the set of all equivariant random measures $\mu^\bullet$ on M with unit intensity s.t. $\mathbb{W}_p(m, \mu^\bullet) < \infty$ by $\mathcal{P}_p$. Then $\mathbb{W}_p$ defines a metric on $\mathcal{P}_p$ which implies the vague convergence of the Campbell measures (see Propositions 8.1 and 8.3). In particular, if $\lambda^\bullet, \mu^\bullet \in \mathcal{P}_p$ they have finite $\mathbb{W}_p$ distance. Sturm and the author derive in [26] a rather general technique giving sufficient conditions for a random measure $\eta^\bullet$ to lie in $\mathcal{P}_p$. As a byproduct of these results we can show stability of optimal couplings under vague convergence of the Campbell measures of their marginals (see Proposition 8.5).

1.1. *Examples*

To illustrate the abstract concepts and theorems we give some examples. We start with examples of spaces satisfying our assumptions.

(i) $M = \mathbb{R}^d$ with group action translation by $\mathbb{Z}^d$, the Euclidean distance $|\cdot|$ and a $\mathbb{Z}^d$-invariant Borel measure m. A fundamental region is the unit cube $[0, 1)^d$. If ϑ is strictly convex and C^2 and m does not charge sets of dimension less or equal to $d - 1$, then there is always a unique Monge solution (see Example 10.35 in [51]). In particular this is the case, if m is the Lebesgue measure, but also other choices are possible. $\mathbb{Z}^d$ is amenable so that all the results are applicable.

(ii) Hyperbolic space, e.g. $M = \mathbb{H}^2$ the two dimensional hyperbolic space with a Fuchsian group G acting cocompactly and freely, the hyperbolic distance d and a G invariant Borel measure, e.g. the volume measure (other constructions are possible, see Remark 3.4). A fundamental region is a suitable subset of a hyperbolic polygon (see [30]). If ϑ is strictly convex, there is a unique Monge solution (see Theorem 10.28 [51]). Note that G is not amenable. In particular, this means that only Theorem 1.1 and Proposition 3.18 apply, the existence and uniqueness of the Monge solutions. It is not clear whether Theorems 1.2 and 1.3 hold.

(iii) Heisenberg groups, e.g. $M = H^3(\mathbb{R})$ the three dimensional Heisenberg group with Carnot–Caratheodory metric d and Lebesgue measure m and G the standard lattice acting on $H^3(\mathbb{R})$. A fundamental region is $[0, 1)^3$ (see [17]). For the existence of Monge solutions it is again sufficient to require ϑ to be a strictly convex function (see [11]). As for $\mathbb{R}^d$ all our results apply.

Our results open the door for the construction of optimal couplings between various random measures. Let us mention a few to fix ideas some of which have been studied in the literature (see next section).

(i) Allocations to a point process: $\lambda^\bullet$ being the volume measure on Euclidean or hyperbolic space and $\mu^\bullet$ a point process, e.g. the Poisson point process or the zeros of the Gaussian entire function. This is probably the most studied example in the literature.

(ii) Power diagrams in particular Laguerre tessellations (see [35]): These are generalizations of Voronoi tessellations which naturally appear in this setting, see Sections 5.1 and 7. However, with the tools developed in this article we cannot only prescribe the volume of the different cells we can also allow a nonhomogeneous background by considering couplings between a random $\lambda^\bullet$ and a point process with respect to the cost functions $d(x, y)^p$. Especially concerning applications as modeling cellular structures this might be interesting.

(iii) Couplings of a diffuse random measure not charging sets of dimension $\le d-1$ with another random measure. A special case of this problem are couplings of mixtures of local times of one dimensional Brownian motion (see [33]).

Finally, there is at least one other class of very natural candidates from which one could expect the existence of Monge solutions but where our results do not apply: optimal couplings between two independent simple point processes. An infinite dimensional variant of Birkhoff's Theorem together with the existence of Kantorovich solutions (see Proposition 3.18) yield the existence of at least one Monge solution. However, even in the case of two independent Poisson processes we do not know if there is a unique optimal (Monge) coupling. Note that in the case of two independent Poisson processes on $\mathbb{R}^d$ the semicouplings on bounded sets, Q_{B_r}, are unique. In particular, along the lines of the proof of Theorem 1.2 we get convergence along a subsequence to an optimizer.

Question 1.4. *Suppose we are given two independent Poisson point processes on $\mathbb{R}^d$ such that the optimal mean transportation cost is finite. Is the optimal coupling unique?*

1.2. *Connection with the literature*

Couplings between random measures which can be interpreted as transportation problems between random measures have been considered by several authors. In [34] and [32] Last and Thorisson resp. Last analyze equivariant transports between random measures in a rather general setting. They establish deep connections to Palm theory and mass stationarity. In the recent article [33], Last, Mörters and Thorisson construct an equivariant transport between two diffuse random measures to build an unbiased shift of Brownian motion. They also derive some moment estimates on the typical transport distance.

Monge couplings between the Lebesgue measure and a point process appear in the context of *fair allocations* which also have a connection to Palm theory, see [25]. They have been investigated and constructed e.g. in [12,22,25,38,42] and references therein.

Matchings of two independent Poisson processes, which correspond to Monge solutions, have been intensely explored in [23,24]. There are still a couple of challenging open questions. Solving the conjecture on optimal couplings between two Poisson processes might help solve some of them.

Tessellations and power diagrams have been studied e.g. in [3,35] and references therein.

To our knowledge, [26] is so far the only article analyzing couplings of two random measures with the additional constraint of optimality. [26] investigates the particular situation of semicouplings between the Lebesgue measure and an equivariant simple point process. Even though that setup is very special some of the ideas (the use of local optimality in the uniqueness proof and the insight to use a symmetrization to prove Theorem 1.2) can be used in this article as well. However, due to the different focus and more complicated structure they have to be combined with new techniques.

1.3. *Outline*

In the next section we give a sketch of the main ideas used to prove the different results. In Section 3 we introduce the setting and objects we work with. Moreover, we show that there always exists a Kantorovich solution to the optimal transport problem. In Section 4 we examine the problem of optimal semicouplings on bounded sets which is necessary to understand before we proceed to the unbounded case. In Section 5 we prove Theorem 1.1 the uniqueness of Monge solutions. Theorem 1.2 and Theorem 1.3 are proved in Section 6. In particular we will see how equivariance naturally appears in the case that G is amenable. In all these sections we always assume that the second marginal has intensity $\beta \le 1$. In Section 7 we treat the case of intensity $\beta \ge 1$. In Section 8 we show that $\mathbb{W}_p$ defines a metric on $\mathcal{P}_p$ implying sufficient conditions for two random measures to have finite $\mathbb{W}_p$ transport distance.

2. Sketch of main proofs

2.1. *Theorem 1.1*

We take the standard approach in optimal transport, namely we show that every optimizer is concentrated on the graph of a function. As the set of all optimizers is convex this implies uniqueness. To this end, we consider the restrictions of optimizers to bounded sets. By equivariance, we can deduce that this restriction has to be the optimal coupling between its marginals which are finite measures. Then, classical optimal transport implies the existence of a transport map proving the theorem.

2.2. *Theorems 1.2 and 1.3*

The main problem to overcome is to control in the limit the contribution to the cost of the transport into a *fixed* set A. Due to the potentially huge normalizing factor in (1.3) it might happen that without noticing a relevant change of transport cost the mass transported into A escapes to infinity in the limit.

The key idea to solve this problem is to consider different approximating sequences at the same time and take a suitable convex combination, cf. (1.4). The big advantage of this approach is that the convex combination symmetrizes the problem so that the transport cost of these semicouplings can be uniformly bounded by $\mathfrak{c}_\infty$. Doing the same for two or more disjoint sets $(g_i B_0)_{i=1}^n$ we need to take care that the mass transported into these sets does also come from disjoint sets, i.e. if we sum up the different semicouplings, in the limit, we still want to end up with a semicoupling of $\lambda^\bullet$ and $\mu^\bullet$. This is exactly what amenability is doing for us.

Having proved Theorem 1.2 and knowing that the limit is unique by Theorem 1.3 we can deduce that each of the sequences of semicouplings used in the symmetrization (1.4) already has to converge to the optimal coupling. Finally, as all these couplings are concentrated on the graph of a function we can infer Theorem 1.3.

3. Set-up

In this section we will explain the general set-up, some basic concepts and derive the first result, the existence of Kantorovich solutions.

3.1. *The setting*

(M, d, m) denotes a geodesic Polish metric measure space (i.e. (M, d) is a complete, separable metric space with Radon measure m), with geodesic metric d and Borel measure m. The Borel sets on M will be denoted by $\mathcal{B}(M)$. Given a map S and a measure ρ we denote the push forward of ρ under S by $S_*\rho$, i.e. $S_*\rho(A) = \rho(S^{-1}(A))$ for any Borel set A. Given any product $X = \prod_{i=1}^n X_i$ of measurable spaces, the projection onto the ith space will be denoted by π_i. Given a set $A \subset M$ its complement will be denoted by $\complement A$ and the indicator function of A by 1_A.

We will assume that there is a group G of isometries acting on M under which the measure m is invariant. For a set $A \subset M$ we write $\tau_g A := gA = \{ga\colon\ a \in A\}$. For a point $x \in M$ its *orbit* under the group action of G is defined as $Gx = \{gx\colon\ g \in G\}$. Its *stabilizer* is defined as $G_x = \{g \in G\colon\ gx = x\}$ the elements of G that fix x.

Definition 3.1 (Group action). *Let G act on M. We say that the action is*

- properly discontinuous *if for any $x \in M$ and any compact $K \subset M$, $gx \in K$ for only finitely many $g \in G$,*
- cocompact *if M/G is compact in the quotient topology,*
- free *if $gx = x$ for one $x \in M$ implies $g = id$, that is the stabilizer for every point is trivial.*

We will assume that the group action is properly discontinuous, cocompact and free. By Theorem 3.5 in [9] this already implies that G is finitely generated and therefore countable.

Definition 3.2 (Fundamental region). *A measurable subset $B_0 \subset M$ is defined to be a fundamental region for G if*

(i) $\bigcup_{g\in G} gB_0 = M$,
(ii) $B_0 \cap gB_0 = \varnothing$ *for all $id \neq g \in G$.*

The family $\{gB_0\colon\ g \in G\}$ is also called tessellation of M.

There are many different choices of fundamental regions. We fix a point p and choose a suitable subset of its Dirichlet region which is the cell containing p in the Voronoi tessellation defined by Gp (e.g. see [30]). We call this set B_0. The specific choice is not important, because by invariance of m each fundamental region has the same volume and therefore defines a tiling of M in pieces of equal volume. Indeed, we have the following lemma.

Lemma 3.3. *Let F_1 and F_2 be two fundamental regions for G. Assume $m(F_1) < \infty$. Then $m(F_1) = m(F_2)$.*

Proof. As $F_1 \cap gF_2$ and $F_1 \cap hF_2$ are disjoint for $g \neq h$ by the defining property of fundamental regions we have

$$m(F_1) = \sum_{g \in G} m(F_1 \cap gF_2) = \sum_{g \in G} m\bigl(g^{-1}F_1 \cap F_2\bigr) = m(F_2).$$

□

For examples of fundamental regions we refer to Section 1.1.

Remark 3.4. *There are many possible different choices for m. Indeed, take any measure $m_0 \in \mathcal{M}(B_0)$ and define $m_g(A) := m_0(g^{-1}A)$. Then the measure $m := \sum_{g \in G} m_g$ is invariant under the action of G.*

By scaling of the measure m we can and will assume that $m(B_0) = 1$. This assumption is just made to simplify some notations.

Definition 3.5. *A countable discrete group G is called amenable if for any finite subset W of G and every $\varepsilon > 0$, there is another finite subset W^* of G such that for any $g \in W$ it holds that*

$$\frac{|gW^* \triangle W^*|}{|W^*|} \leq \varepsilon.$$

Denote by $\triangle$ the symmetric difference and for $A \subset G$ we denote the cardinality of A by $|A|$.

Proposition 3.6. *Let G be a countable discrete group. The following are equivalent*:

(i) *G is amenable.*
(ii) *There exists a sequence of nonempty finite sets F_n such that for all $g \in G$*

$$\frac{|gF_n \triangle F_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty.$$

The sequence $(F_n)_{n \in \mathbb{N}}$ is called Følner sequence (sometimes also summing sequence).

Lemma 3.7. *There exists a Følner sequence $(F_n)_{n \in \mathbb{N}}$ such that its inverse $(F_n^{-1})_{n \in \mathbb{N}}$ is also a Følner sequence (actually a right Følner sequence). Moreover, $(F_n)_{n \in \mathbb{N}}$ can be taken to be exhausting. In particular, for any $h \in G$ and $r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $hF_r \subset F_n$.*

For a proof of these statements and an introduction to amenability we refer to [44], especially Section 4 and Theorem 4.16.

Several times we will use a rather simple but very powerful tool, the *mass transport principle*. It already appeared in the proof of Lemma 3.3. It is a kind of conservation of mass formula for invariant transports.

Lemma 3.8 (Mass transport principle). *Let $f : G \times G \to \mathbb{R}_+$ be a function which is invariant under the diagonal action of G, that is $f(u, v) = f(gu, gv)$ for all $g, u, v \in G$. Then we have*

$$\sum_{v \in G} f(u, v) = \sum_{v \in G} f(v, u).$$

Proof.

$$\sum_{v \in G} f(u, v) = \sum_{g \in G} f(u, gu) = \sum_{g \in G} f\bigl(g^{-1}u, u\bigr) = \sum_{v \in G} f(v, u).$$

□

For a more general version we refer to [7] and [34].

Recall the disintegration theorem for finite measures (e.g. see Theorem 5.1.3 in [2] or III-70 in [14]).

Theorem 3.9 (Disintegration of measures). *Let X, Y be Polish spaces, and let γ be a finite Borel measure on $X \times Y$. Denote by μ and ν the marginals of γ on the first and second factor respectively. Then, there exist two measurable families of probability measures $(\gamma_x)_{x\in X}$ and $(\gamma_y)_{y\in Y}$ such that*

$$\gamma(\mathrm{d}x, \mathrm{d}y) = \gamma_x(\mathrm{d}y)\mu(\mathrm{d}x) = \gamma_y(\mathrm{d}x)\nu(\mathrm{d}y).$$

3.2. *Couplings and semicouplings*

For each Polish space X the set of Radon measures on X – equipped with its Borel σ-field – will be denoted by $\mathcal{M}(X)$. Given any ordered pair of Polish spaces X, Y and measures $\lambda \in \mathcal{M}(X)$, $\mu \in \mathcal{M}(Y)$ we say that a measure $q \in \mathcal{M}(X \times Y)$ is a *semicoupling* of λ and μ, briefly $q \in \Pi_s(\lambda, \mu)$, iff the (first and second, resp.) marginals satisfy

$$(\pi_1)_*q \le \lambda, \qquad (\pi_2)_*q = \mu,$$

that is, iff $q(A \times Y) \le \lambda(A)$ and $q(X \times B) = \mu(B)$ for all Borel sets $A \subset X$, $B \subset Y$. The semicoupling q is called *coupling*, briefly $q \in \Pi(\lambda, \mu)$, iff in addition

$$(\pi_1)_*q = \lambda.$$

In particular, $q(\cdot \times Y) \ll \lambda$. Hence, there is a density $\rho : X \to [0, 1]$ such that $(\pi_1)_*q = \rho \cdot \lambda$. If q happens to be concentrated on the graph of a function T defined on a subset A of full λ-measure, i.e. $q = (id, T)_*\lambda$, we extend T to the whole of X by adding a cemetery point $\eth$ to Y and setting $T(x) = \eth$ for all $x \notin A$. We often think of $Y \cup \{\eth\}$ as the one-point compactification of Y.

See also [18] for the related concept of *partial coupling*.

3.3. *Random measures on M*

We endow $\mathcal{M}(M)$ with the vague topology, i.e. we test against continuous functions with compact support. For more details on vague convergence we refer to [28] or [5].

The action of G on M induces an action of G on $\mathcal{M}(M \times \cdots \times M)$ by push forward with the map τ_g:

$$(\tau_g)_*\lambda(A_1, \ldots, A_k) = \lambda\big(g^{-1}(A_1), \ldots, g^{-1}(A_k)\big) \quad \forall A_1, \ldots, A_k \in \mathcal{B}(M), k \in \mathbb{N}.$$

A random measure on M is a random variable $\lambda^\bullet$ (the notation with the "$\bullet$" is intended to make it easier to distinguish random and nonrandom measures, nonrandom measures will usually be denoted by $\lambda, \mu, \ldots$) modeled on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ taking values in $\mathcal{M}(M)$. It can also be regarded as a kernel from Ω to M. Therefore, we write either $\lambda^\omega(A)$ or $\lambda(\omega, A)$ depending on which property we want to stress. For convenience, we will assume that Ω is a compact metric space and $\mathfrak{A}$ its completed Borel field. These technical assumptions are only made to simplify the presentation.

We call a random measure $\lambda^\bullet$ absolutely continuous iff it is absolutely continuous to m on M for a.e. $\omega \in \Omega$. It is called diffusive iff it has no atoms almost surely. It is called a point process if it takes values in the subset of all locally finite counting measures on M. The point process is simple iff $\mu^\omega(\{x\}) \in \{0, 1\}$ for every $x \in M$ and a.e. $\omega \in \Omega$. The intensity measure of a random measure $\lambda^\bullet$ is the measure on M defined by $A \mapsto \mathbb{E}[\lambda^\bullet(A)]$.

A random measure $\lambda^\bullet : \Omega \to \mathcal{M}(M)$ is called G-invariant or just invariant if the distribution of $\lambda^\bullet$ is invariant under the action of G, that is, iff

$$(\tau_g)_*\lambda^\bullet \overset{(d)}{=} \lambda^\bullet$$

for all $g \in G$. A random measure $q^\bullet : \Omega \to \mathcal{M}(M \times M)$ is called invariant if its distribution is invariant under the diagonal action of G.

Let $\mathfrak{G}$ be the discrete σ-algebra of G. If $(\Omega, \mathfrak{A})$ admits a measurable flow $\theta_g : \Omega \to \Omega$, $g \in G$, that is a $\mathfrak{A} \otimes \mathfrak{G} - \mathfrak{A}$-measurable mapping $(\omega, g) \mapsto \theta_g\omega$ with θ_0 the identity on Ω and

$$\theta_g \circ \theta_h = \theta_{gh}, \quad g, h \in G,$$

then a random measure $\lambda^\bullet : \Omega \to \mathcal{M}(M)$ is called G-equivariant or just equivariant iff

$$\lambda(\theta_g\omega, gA) = \lambda(\omega, A),$$

for all $g \in G, \omega \in \Omega, A \in \mathcal{B}(M)$. We can think of $\lambda(\theta_g\omega, \cdot)$ as $\lambda(\omega, \cdot)$ shifted by g. Indeed, let $\mathfrak{M}$ be the cylindrical σ-algebra generated by the evaluation functionals $A \mapsto \mu(A), A \in \mathcal{B}(M), \mu \in \mathcal{M}$. As in Example 2.1 of [34], consider the measurable space $(\Omega, \mathfrak{A}) = (\mathcal{M}, \mathfrak{M})$ and define for $\mu \in \mathcal{M}, g \in G$ the measure $\theta_g\mu(A) = \mu(g^{-1}A)$. Then, $\{\theta_g, g \in G\}$ is a measurable flow and the identity is an equivariant measure. A random measure $q^\bullet : \Omega \to \mathcal{M}(M \times M)$ is called equivariant iff

$$q^{\theta_g\omega}(gA, gB) = q^\omega(A, B),$$

for all $g \in G, \omega \in \Omega, A, B \in \mathcal{B}(M)$.

Example 3.10. *Let $q^\bullet$ be an equivariant random measure on $M \times M$ given by $q^\omega = (id, T^\omega)_*\lambda^\omega$ for some measurable map $T^\bullet$ and some equivariant random measure $\lambda^\bullet$. The equivariance condition*

$$\int_A 1_B(y)\delta_{T^{\theta_g\omega}(gx)}\big(\mathrm{d}(gy)\big)\lambda^{\theta_g\omega}(\mathrm{d}x) = q^{\theta_g\omega}(gA, gB)$$
$$= q^\omega(A, B) = \int_A 1_B(y)\delta_{T^\omega(x)}(\mathrm{d}y)\lambda^\omega(\mathrm{d}x),$$

translates into an equivariance condition for the transport maps:

$$T^{\theta_g\omega}(gx) = gT^\omega(x).$$

A probability measure $\mathbb{P}$ is called *stationary* iff

$$\mathbb{P} \circ \theta_g = \mathbb{P}$$

for all $g \in G$. Given a measure space $(\Omega, \mathfrak{A})$ with a measurable flow $(\theta_g)_{g\in G}$ and a stationary probability measure $\mathbb{P}$ any equivariant measure is automatically invariant. The advantage of this definition is that the sum of equivariant measures is again equivariant, and therefore also invariant. The sum of two invariant random measures does not have to be invariant (see Remark 3.17).

Definition 3.11 (Intensity). *Let $\mu^\bullet$ be an equivariant random measure and B_0 a fundamental region. We define the intensity β of $\mu^\bullet$ to be $\beta := \mathbb{E}[\mu^\bullet(B_0)]$. We say $\mu^\bullet$ has unit (resp. subunit) intensity if $\beta = 1$ (resp. $\beta \leq 1$).*

Note that by another application of the mass transport principle the intensity does not depend on the specific fundamental region.

Given a random measure, the measure $(\lambda^\bullet\mathbb{P})(\mathrm{d}y, \mathrm{d}\omega) := \lambda^\omega(\mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)$ on $M \times \Omega$ is called *Campbell measure* of the random measure $\lambda^\bullet$.

From now on we will always assume that we are given two equivariant random measures $\lambda^\bullet$ and $\mu^\bullet$ modeled on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ admitting a measurable flow $(\theta_g)_{g\in G}$ such that $\mathbb{P}$ is stationary. We will assume that Ω is a compact metric space. Moreover, we will assume that $\lambda^\bullet$ and $\mu^\bullet$ are almost surely not the zero measure. Note that the invariance implies that $\mu^\omega(M) = \lambda^\omega(M) = \infty$ for almost every ω (e.g. see Proposition 12.1.VI in [13]).

3.4. *Semicouplings of $\lambda^\bullet$ and $\mu^\bullet$*

A semicoupling of the random measures $\lambda^\bullet$ and $\mu^\bullet$ is a measurable map $q^\bullet : \Omega \to \mathcal{M}(M \times M)$ s.t. for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$q^\omega \text{ is a semicoupling of } \lambda^\omega \text{ and } \mu^\omega.$$

Its Campbell measure is given by $Q = q^\bullet \mathbb{P} \in \mathcal{M}(M \times M \times \Omega)$. Q is a semicoupling between the Campbell measures $\lambda^\bullet \mathbb{P}$ and $\mu^\bullet \mathbb{P}$ in the sense that

$$Q(M \times \cdot \times \cdot) = \mu^\bullet \mathbb{P} \quad \text{and} \quad Q(\cdot \times M \times \cdot) \leq \lambda^\bullet \mathbb{P}.$$

Q could also be regarded as semicoupling between $\lambda^\bullet \mathbb{P}$ and $\mu^\bullet \mathbb{P}$ on $M \times \Omega \times M \times \Omega$ which is concentrated on the diagonal of $\Omega \times \Omega$. We will always identify these semicouplings with measures on $M \times M \times \Omega$.

Given such a semicoupling $Q \in \mathcal{M}(M \times M \times \Omega)$ we can disintegrate (see Theorem 3.9) Q to get a measurable map $q^\bullet : \Omega \to \mathcal{M}(M \times M)$ which is a semicoupling of $\lambda^\bullet$ and $\mu^\bullet$.

According to this one-to-one correspondence between $q^\bullet$ – semicoupling of $\lambda^\bullet$ and $\mu^\bullet$ – and $Q = q^\bullet \mathbb{P}$ – semicoupling of $\lambda^\bullet \mathbb{P}$ and $\mu^\bullet \mathbb{P}$ – we will freely switch between them. And quite often, we will simply speak of *semicouplings of* $\lambda^\bullet$ *and* $\mu^\bullet$.

We denote the set of all semicouplings between $\lambda^\bullet$ and $\mu^\bullet$ by $\Pi_s(\lambda^\bullet, \mu^\bullet)$. The set of all equivariant semicouplings between $\lambda^\bullet$ and $\mu^\bullet$ will be denoted by $\Pi_{es}(\lambda^\bullet, \mu^\bullet)$.

A *factor* of some random variable X is a random variable Y which is measurable with respect to $\sigma(X)$. This is equivalent to the existence of a deterministic function f with $Y = f(X)$. In other words, a factor is a rule such that given X we can construct Y. A *factor semicoupling* is a semicoupling of $\lambda^\bullet$ and $\mu^\bullet$ which is a factor of $\lambda^\bullet$ and $\mu^\bullet$.

3.5. *The Monge–Kantorovich problem*

Let λ, μ be two *probability measures* on M. Moreover, let a cost function $c : M \times M \to \mathbb{R}$ be given. The Monge–Kantorovich problem is to find a minimizer of

$$\int_{M \times M} c(x, y) q(\mathrm{d}x, \mathrm{d}y)$$

among all couplings q of λ and μ. A minimizing coupling is called *optimal coupling*. If the optimal coupling q is induced by a transportation map, i.e. $q = (id, T)_* \lambda$, we say that q is a solution to the Monge problem. We always assume that the cost function $c(x, y) = \vartheta(d(x, y))$ is such that there is a unique solution to the Monge problem between λ and μ whenever $\lambda \ll m$. There are very general results on the existence and uniqueness of the solution to the Monge problem for which we refer to Chapters 9 and 10 of [51]. To be more concrete we refer to a uniqueness result for compact Riemannian manifolds due to McCann [40] and an uniqueness result by Cavalletti and Huesmann [11] on metric measure spaces satisfying the measure contraction property (for a definition we refer to [43,47]; an example for such a space is any Riemannian manifold with lower Ricci curvature bound but also some sub Riemannian manifolds, e.g. the Heisenberg group [27]).

It can be shown that any optimal coupling is concentrated on a c-cyclically monotone set. A set $A \subset X \times X$ is called (c-)cyclically monotone if for all $n \in \mathbb{N}$ and $(x_i, y_i)_{i=1}^n \in A^n$ it holds that

$$\sum_{i=1}^n c(x_i, y_i) \leq \sum_{i=1}^n c(x_i, y_{i+1}),$$

where $y_1 = y_{n+1}$. If the cost function is reasonably well behaved (continuous is more than sufficient, see [6]), also the reverse direction holds. Any coupling which is concentrated on a c-cyclically monotone plan is optimal. For further details and applications of mass transport theory we refer to [45,50,51].

3.6. *Cost functionals*

Throughout this article, ϑ will be a strictly increasing, continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\vartheta(0) = 0$ and $\lim_{r\to\infty} \vartheta(r) = \infty$. Given a *scale function* ϑ as above we define the *cost function*

$$c(x, y) = \vartheta\big(d(x, y)\big)$$

on $M \times M$, the *cost functional*

$$\mathsf{Cost}(q) = \int_{M\times M} c(x,y)q(\mathrm{d}x,\mathrm{d}y)$$

on $\mathcal{M}(M \times M)$ and the *mean cost functional*

$$\mathfrak{Cost}(Q) = \int_{M\times M\times\Omega} c(x,y)Q(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega)$$

on $\mathcal{M}(M \times M \times \Omega)$.

Remark 3.12. *We need the cost function c to be invariant under the action of G in order to stay in a G-invariant setting. Otherwise a result like Corollary 4.8 will not be true. Therefore, we consider cost functions of this special form.*

In Section 4, we will show existence and uniqueness of optimal semicouplings on bounded sets, cf. Theorem 4.7, relating the minimizers of Cost and $\mathfrak{Cost}$ in a natural way.

For a bounded Borel set $A \subset M$, the *transportation cost on A* is given by the random variable $\mathsf{C}_A : \Omega \to [0,\infty]$ as

$$\mathsf{C}_A(\omega) := \mathsf{Cost}\big(q_A^\omega\big) = \inf\big\{\mathsf{Cost}\big(q^\omega\big):\ q^\omega \text{ semicoupling of } \lambda^\omega \text{ and } 1_A\mu^\omega\big\}.$$

Lemma 3.13.

(i) *If $A_1,\dots,A_n$ are disjoint then $\forall\omega\in\Omega$*

$$\mathsf{C}_{\bigcup_{i=1}^n A_i}(\omega) \geq \sum_{i=1}^n \mathsf{C}_{A_i}(\omega).$$

(ii) *If $A_1 = gA_2$ for some $g \in G$, then C_{A_1} and C_{A_2} are identically distributed.*

Proof. Property (ii) follows directly from the joint invariance of $\lambda^\bullet$ and $\mu^\bullet$. The intuitive argument for (i) is, that minimizing the cost on $\bigcup_i A_i$ is more restrictive than doing it separately on each of the A_i. The more detailed argument is the following. Given any semicoupling q^ω of λ^ω and $1_{\bigcup_i A_i}\mu^\omega$ then for each i the measure $q_i^\omega := 1_{M\times A_i}q^\omega$ is a semicoupling of λ^ω and $1_{A_i}\mu^\omega$. Choosing q^ω as the minimizer of $\mathsf{C}_{\bigcup_{i=1}^n A_i}(\omega)$ yields

$$\mathsf{C}_{\bigcup_i A_i}(\omega) = \mathsf{Cost}\big(q^\omega\big) = \sum_i \mathsf{Cost}\big(q_i^\omega\big) \geq \sum_i \mathsf{C}_{A_i}(\omega).$$

□

3.7. *Optimality*

The standard notion of optimality – minimizers of Cost or $\mathfrak{Cost}$ – is not well adapted to our setting. Due to the almost sure infinite mass of λ^ω and μ^ω the total transportation cost will typically be infinite. Hence, we need to introduce a different notion which we explain in this section.

The collection of admissible sets is defined as

$$\mathrm{Adm}(M) = \left\{B \in \mathcal{B}(M):\ \exists I \subset G, 1 \leq |I| < \infty, E \text{ fundamental region: } B = \bigcup_{g\in I} gE\right\}.$$

Definition 3.14. *For a semicoupling $q^\bullet$ between $\lambda^\bullet$ and $\mu^\bullet$ the* mean transportation cost *of $q^\bullet$ is defined by*

$$\mathfrak{C}\big(q^\bullet\big) := \sup_{B\in\mathrm{Adm}(M)} \frac{1}{m(B)}\mathbb{E}\left[\int_{M\times B} c(x,y)q^\bullet(\mathrm{d}x,\mathrm{d}y)\right].$$

This definition is slightly different from the definition given in [26]. However, specializing to the case $M = \mathbb{R}^d$ and $G = \mathbb{Z}^d$ using Lemma 3.13 we can recover the definition given in [26], see also Corollary 6.5.

Definition 3.15. *A semicoupling $q^\bullet$ between $\lambda^\bullet$ and $\mu^\bullet$ is called*

(i) asymptotically optimal *iff*

$$\mathfrak{C}(q^\bullet) = \inf_{\tilde{q}^\bullet \in \Pi_{es}(\lambda^\bullet, \mu^\bullet)} \mathfrak{C}(\tilde{q}^\bullet) =: \mathfrak{c}_{e,\infty},$$

(ii) optimal *iff $q^\bullet$ is equivariant and asymptotically optimal.*

We will also use several times the quantity

$$\inf_{\tilde{q}^\bullet \in \Pi_s(\lambda^\bullet, \mu^\bullet)} \mathfrak{C}(\tilde{q}^\bullet) =: \mathfrak{c}_\infty.$$

Obviously $\mathfrak{c}_\infty \le \mathfrak{c}_{e,\infty}$.

Note that the set of optimal semicouplings is convex. This will be useful for the proof of uniqueness. Moreover, we would like to stress that the existence of one asymptotically optimal semicoupling implies the existence of many of them. Hence, the additional constraint of equivariance is necessary and will be shown to be natural.

Remark 3.16. *Equivariant semicouplings $q^\bullet$ are invariant. Hence, they are asymptotically optimal iff*

$$\mathfrak{C}(q^\bullet) = \mathbb{E}\left[\int_{M\times B_0} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right] = \mathfrak{c}_{e,\infty}.$$

Because of the invariance, the supremum does not play any role. Moreover, for two different fundamental regions B_0 and $\tilde{B}_0$ define

$$f(g, h) = \mathbb{E}\left[\mathsf{Cost}\left(1_{M\times(gB_0\cap h\tilde{B}_0)} q^\bullet\right)\right].$$

Then, for $k \in G$ and equivariant $q^\bullet$ we have $f(g, h) = f(kg, kh)$. Hence, we can apply the mass transport principle, Lemma 3.8, to get

$$\begin{aligned}\mathbb{E}\left[\int_{M\times B_0} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right] &= \sum_{h\in G} f(id, h) \\ &= \sum_{g\in G} f(g, id) = \mathbb{E}\left[\int_{M\times \tilde{B}_0} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right].\end{aligned}$$

Thus, the specific choice of fundamental region is not important for the cost functional $\mathfrak{C}(\cdot)$ if we restrict to equivariant semicouplings.

Remark 3.17. *The notion of optimality explains why we restrict to stationary probability measures and equivariant random measures. If $\lambda^\bullet$ and $\mu^\bullet$ are just invariant, there does not have to be any invariant semicoupling between them. Indeed, take $\lambda^\bullet$ a Poisson point process of unit intensity in $\mathbb{R}^d$. It can be written as $\mu^\omega = \sum_{\xi\in\Xi(\omega)} \delta_\xi$. Define $\lambda^\omega := \sum_{\xi\in\Xi(\omega)} \delta_{-\xi}$ to be the Poisson process that we get if we reflect the first one at the origin. Then $\lambda^\bullet$ and $\mu^\bullet$ are invariant but not jointly invariant, e.g. consider the set $[0, 1)^d \times [-1, 0)^d$, and not both of them can be equivariant.*

3.8. *Kantorovich solutions*

Given that the mean transportation cost is finite the existence of a Kantorovich solution can be shown by an abstract compactness result. A similar reasoning is used to prove Corollary 11 in [23].

Proposition 3.18. *Let $\lambda^\bullet$ and $\mu^\bullet$ be two equivariant random measures on M with intensities* 1 *and $\beta \leq 1$ respectively. Assume that the optimal mean transportation cost is finite*, $\inf_{q^\bullet \in \Pi_{es}(\lambda^\bullet,\mu^\bullet)} \mathfrak{C}(q^\bullet) = \mathfrak{c}_{e,\infty} < \infty$, *then there exists some equivariant semicoupling $q^\bullet$ between $\lambda^\bullet$ and $\mu^\bullet$ with $\mathfrak{C}(q^\bullet) = \mathfrak{c}_{e,\infty}$.*

It is not important that $\lambda^\bullet$ has intensity one. By symmetry one intensity has to dominate the other. By scaling, we can then always assume that $\lambda^\bullet$ has intensity one and $\mu^\bullet$ has intensity less or equal than one.

Proof of Proposition 3.18. As $\mathfrak{c}_{e,\infty} < \infty$ there is a sequence $q_n^\bullet \in \Pi_{es}(\lambda^\bullet, \mu^\bullet)$ such that $\mathfrak{C}(q_n^\bullet) = c_n \searrow \mathfrak{c}_{e,\infty}$. Moreover, we can assume that the transportation cost is uniformly bounded by $c_n \leq 2\mathfrak{c}_{e,\infty} =: c$ for all n. We claim that there is a subsequence $(q_{n_k}^\bullet)_{k\in\mathbb{N}}$ of $(q_n^\bullet)_{n\in\mathbb{N}}$ converging to some $q^\bullet \in \Pi_{es}(\lambda^\bullet, \mu^\bullet)$ with $\mathfrak{C}(q^\bullet) = \mathfrak{c}_{e,\infty}$. We prove this in four steps:

(i) *The functional $\mathfrak{C}(\cdot)$ is lower semicontinuous*: It is sufficient to prove that the functional $\mathsf{Cost}(\cdot)$ is lower semicontinuous. Let $(\rho_n)_{n\in\mathbb{N}}$ be any sequence of couplings between finite measures converging to some measure ρ in the vague topology. If $\mathsf{Cost}(\rho_n) = \infty$ for all n we are done. Hence, we can assume, that the transportation cost are bounded. Let $(B_0)_r$ denote the r-neighbourhood of B_0. For $k \in \mathbb{N}$ let $\phi_k : M \times M \to [0, 1]$ be nice cut off functions with $\phi_k(x, y) = 1$ on $(B_0)_k \times (B_0)_k$ and $\phi_k(x, y) = 0$ if $x \in \complement((B_0)_{k+1})$ or $y \in \complement((B_0)_{k+1})$. Then, we have using continuity of the cost function $c(x, y)$ and by the definition of vague convergence

$$
\begin{aligned}
\liminf_{n\to\infty} \mathsf{Cost}(\rho_n) &= \liminf_{n\to\infty} \int_{M\times M} c(x, y)\rho_n(\mathrm{d}x, \mathrm{d}y) \\
&= \liminf_{n\to\infty} \sup_{k\in\mathbb{N}} \int_{M\times M} \phi_k(x, y)\, c(x, y)\rho_n(\mathrm{d}x, \mathrm{d}y) \\
&\geq \sup_k \liminf_{n\to\infty} \int_{M\times M} \phi_k(x, y)c(x, y)\rho_n(\mathrm{d}x, \mathrm{d}y) \\
&= \sup_k \int_{M\times M} \phi_k(x, y)c(x, y)\rho(\mathrm{d}x, \mathrm{d}y) = \mathsf{Cost}(\rho).
\end{aligned}
$$

Applying this to $1_{M\times B_0} q_n^\bullet$ shows the lower semicontinuity of $\mathfrak{C}(\cdot)$.

(ii) *The sequence $(q_n^\bullet\mathbb{P})_{n\in\mathbb{N}}$ is vaguely relatively compact in $\mathcal{M}(M \times M \times \Omega)$*: Put $f \in C_c(M \times M \times \Omega)$. According to Theorem A2.3 of [28] we have to show $\sup_{n\in\mathbb{N}} q_n^\bullet\mathbb{P}(f) \leq L_f < \infty$ for some constant L_f. To this end let $A \subset M$ compact be such that $\mathrm{supp}(f) \subset A \times M \times \Omega$ and $A \in \mathrm{Adm}(M)$. We estimate

$$
\begin{aligned}
\int_{M\times M\times\Omega} f(x, y, \omega)q_n^\omega(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) &\leq \|f\|_\infty \lambda^\bullet\mathbb{P}(A \times \Omega) \\
&\leq \|f\|_\infty m(A) =: L_f.
\end{aligned}
$$

Hence, there is some measure $q^\bullet$ and a subsequence $q_{n_k}^\bullet$ with $q_{n_k}^\bullet\mathbb{P} \to q^\bullet\mathbb{P}$ in the vague topology on $\mathcal{M}(M \times M \times \Omega)$. By lower semicontinuity, we have $\mathfrak{C}(q^\bullet) \leq \liminf \mathfrak{C}(q_{n_k}^\bullet) = \mathfrak{c}_{e,\infty}$. Now we have a candidate.

(iii) *$q^\bullet$ is equivariant*: Take any continuous compactly supported $f \in C_c(M \times M \times \Omega)$. By definition of vague convergence

$$
\int f(x, y, \omega)q_{n_k}^\omega(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) \to \int f(x, y, \omega)q^\omega(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega).
$$

As all the $q_{n_k}^\bullet$ are equivariant, we have for any $g \in G$

$$
\begin{aligned}
&\int f(x, y, \omega)q_{n_k}^\omega(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) \\
&\quad = \int f\big(g^{-1}x, g^{-1}y, \omega\big)q_{n_k}^{\theta_g\omega}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)
\end{aligned}
$$

$$= \int f\big(g^{-1}x, g^{-1}y, (\theta_g)^{-1}\omega\big) q^{\omega}_{n_k}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)$$
$$\to \int f\big(g^{-1}x, g^{-1}y, (\theta_g)^{-1}\omega\big) q^{\omega}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega),$$

where we used stationarity of $\mathbb{P}$ in the second equality. Putting this together, we have for any $g \in G$

$$\int f(x, y, \omega) q^{\omega}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) = \int f\big(g^{-1}x, g^{-1}y, \omega\big) q^{\theta_g\omega}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega).$$

Hence, $q^{\bullet}$ is equivariant.

(iv) *$q^{\bullet}$ is a semicoupling of $\lambda^{\bullet}$ and $\mu^{\bullet}$*: Fix $h \in C_c(M \times \Omega)$. Put $A \subset M$ compact such that $\mathrm{supp}(h) \subset A \times \Omega$ and $A \in \mathrm{Adm}(M)$. Denote the R-neighbourhood of A by A_R. By the uniform bound on transportation cost we have

$$q^{\bullet}_n\mathbb{P}\big(\complement(A_R), A, \Omega\big) \le m(A)\frac{c}{\vartheta(R)}, \tag{3.1}$$

uniformly in n. Let $f_R : M \to [0, 1]$ be a continuous compactly supported function such that $f_R(x) = 1$ for $x \in A_R$ and $f_R(x) = 0$ for $x \in \complement A_{R+1}$. As $q^{\bullet}_n\mathbb{P}$ is a semicoupling of $\lambda^{\bullet}$ and $\mu^{\bullet}$ we have due to monotone convergence

$$\int_{M\times\Omega} h(y, \omega)\mu^{\omega}(\mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) = \int_{M\times M\times\Omega} h(y, \omega) q^{\omega}_n(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)$$
$$= \lim_{R\to\infty} \int_{M\times M\times\Omega} f_R(x)h(y, \omega) q^{\omega}_n(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega).$$

Because of the uniform bound (3.1) we have

$$\left|\int_{M\times\Omega} h(x, \omega)\mu^{\omega}(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega) - \int_{M\times M\times\Omega} f_R(x)h(y, \omega) q^{\omega}_{n_k}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)\right|$$
$$\le m(A)\frac{c \cdot \|h\|_\infty}{\vartheta(R)}.$$

Taking first the limit of $n_k \to \infty$ and then the limit of $R \to \infty$ we conclude using vague convergence and monotone convergence that

$$0 = \lim_{R\to\infty} \lim_{k\to\infty} \left|\int_{M\times\Omega} h(y, \omega)\mu^{\omega}(\mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) \right.$$
$$\left. - \int_{M\times M\times\Omega} f_R(x)h(y, \omega) q^{\omega}_{n_k}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)\right|$$
$$= \lim_{R\to\infty} \left|\int_{M\times\Omega} h(y, \omega)\mu^{\omega}(\mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) - \int_{M\times M\times\Omega} f_R(x)h(y, \omega) q^{\omega}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)\right|$$
$$= \left|\int_{M\times\Omega} h(y, \omega)\mu^{\omega}(\mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) - \int_{M\times M\times\Omega} h(y, \omega) q^{\omega}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)\right|.$$

This shows that the second marginal equals $\mu^{\bullet}$. For the first marginal we have for any nonnegative $k \in C_c(M \times \Omega)$

$$\int_{M\times\Omega} k(x, \omega) q^{\omega}_{n_k}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) \le \int_{M\times\Omega} k(x, \omega)\lambda^{\omega}(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega).$$

In particular, using the function f_R from above we have,

$$\int_{M\times\Omega} f_R(y)k(x, \omega) q^{\omega}_{n_k}(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) \le \int_{M\times\Omega} k(x, \omega)\lambda^{\omega}(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega).$$

Taking the limit $n_k \to \infty$ yields by vague convergence

$$\int_{M\times\Omega} f_R(y)k(x,\omega)q^\omega(\mathrm{d}x,\mathrm{d}y)\mathbb{P}(\mathrm{d}\omega) \le \int_{M\times\Omega} k(x,\omega)\lambda^\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega).$$

Finally taking the supremum over R shows that $q^\bullet$ is indeed a semicoupling of $\lambda^\bullet$ and $\mu^\bullet$. □

Remark 3.19.

(i) *The optimal semicoupling constructed in the last proposition need not be a factor if the optimal semicoupling is not unique.*
(ii) *The same proof shows the existence of optimal semicouplings between $\lambda^\bullet$ and $\mu^\bullet$ with intensities* 1 *and $\beta \ge 1$ respectively. In this case the "semi" is on the side of $\mu^\bullet$* (*see also Section* 7).

Lemma 3.20. *Let $q^\bullet$ be an invariant semicoupling of two random measures $\lambda^\bullet$ and $\mu^\bullet$ with intensities* 1 *and $\beta \le 1$ respectively. Then, $q^\bullet$ is a coupling iff $\beta = 1$.*

Proof. This is another application of the mass transport principle, Lemma 3.8. Let B_0 be a fundamental region and define $f(g,h) = \mathbb{E}[q^\bullet(gB_0, hB_0)]$. By invariance of $q^\bullet$, we have $f(g,h) = f(kg,kh)$ for any $k \in G$. Hence, we get

$$1 = \mathbb{E}\big[\lambda^\bullet(B_0)\big] \ge \mathbb{E}\big[q^\bullet(B_0, M)\big] = \sum_{g\in G} f(id, g) = \sum_{h\in G} f(h, id) = \mathbb{E}\big[q^\bullet(M, B_0)\big] = \beta.$$

We have equality iff $\beta = 1$. If $\beta = 1$ replacing B_0 by gB_0 we get that $\mathbb{E}[\lambda^\bullet(gB_0)] = \mathbb{E}[q^\bullet(gB_0, M)]$ for all $g \in G$. As B_0 induces a tessellation (cf. Definition 3.2) and $q^\omega(\cdot, M) \le \lambda^\omega(\cdot)$ by the definition of semicouplings this implies that for all bounded sets $A \subset M$ we have $q^\omega(A, M) = \lambda^\omega(A)$ for $\mathbb{P}$-almost all ω. Indeed, if there was a true inequality, there has to be a $g \in G$ such that $q^\omega(gB_0, M) < \lambda^\omega(gB_0)$ on a set of positive $\mathbb{P}$-measure; a contradiction to the assumption $\beta = 1$. □

Remark 3.21. *The remark above applies again. Considering the case of intensity $\beta \ge 1$ gives that $q^\bullet$ is a coupling iff $\beta = 1$.*

4. Optimal semicouplings on bounded sets

Before we can tackle the Monge problem between infinite measures we need to understand optimal semicouplings between finite measures on bounded sets.

The goal of this section is to prove Theorem 4.7, the existence and uniqueness result for optimal semicouplings between $\lambda^\bullet$ and $\mu^\bullet$ restricted to a bounded set. The strategy will be to first prove existence and uniqueness of optimal semicouplings $q = q^\omega$ for deterministic measures $\lambda = \lambda^\omega$ and $\mu = \mu^\omega$. Secondly, we will show that the map $\omega \mapsto q^\omega$ is measurable, which will allow us to deduce Theorem 4.7.

Optimal semicouplings are solutions of a twofold optimization problem: the optimal choice of a density $\rho \le 1$ of the first marginal λ and subsequently the optimal choice of a coupling between $\rho\lambda$ and μ. This twofold optimization problem can also be interpreted as a transport problem with free boundary values.

Throughout this section, we fix the cost function $c(x,y) = \vartheta(d(x,y))$ with ϑ – as before – being a strictly increasing, continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\vartheta(0) = 0$ and $\lim_{r\to\infty}\vartheta(r) = \infty$. As already mentioned, we additionally assume that the optimal transportation problem between two compactly supported probability measures λ and μ such that $\lambda \ll m$ has a unique solution given by a transportation map, i.e. the optimal coupling is given by $q = (id, T)_*\lambda$. For sufficient conditions for this to hold we refer to Section 3.5.

Fix two deterministic measures $\lambda = f \cdot m$ for some compactly supported density f (in particular $\lambda \ll m$) and an arbitrary finite measure μ with $\mathrm{supp}(\mu) \subset A$ for some compact set A such that $\mu(M) \le \lambda(M) < \infty$. We are looking for minimizers of

$$\mathsf{Cost}(q) = \int c(x,y)q(\mathrm{d}x,\mathrm{d}y)$$

under all semicouplings q of λ and μ. The key step is a nice observation by Figalli, namely Proposition 2.4 in [18]. The version we state here is adapted to our setting.

Proposition 4.1 (Figalli). *Let q be a* Cost *minimizing semicoupling between λ and μ with $\lambda \ll m$. Write $f_q \cdot m = (\pi_1)_* q$. Consider the Monge–Kantorovich problem*:

$$\textit{minimize } C(\gamma) = \int_{M\times M} c(x, y)\gamma(\mathrm{d}x, \mathrm{d}y)$$

among all γ which have λ and $\mu + (f - f_q)\cdot m$ as first and second marginals, respectively. Then, the unique minimizer is given by

$$q + (id \times id)_*(f - f_q)\cdot m.$$

This allows us to show that all minimizers of Cost are concentrated on the same graph which also gives us uniqueness. Recall the cemetery point introduced in Section 3.2.

Proposition 4.2. *There is a unique* Cost *minimizing semicoupling between λ and μ. It is given by $q = (id, T)_*(\rho\cdot\lambda)$ for some measurable map $T : M \to M \cup \{\eth\}$ and density ρ.*

Proof. (i) Similar to the proof of Proposition 3.18 it can be seen that the functional $\mathsf{Cost}(\cdot)$ is lower semicontinuous on $\mathcal{M}(M \times M)$ wrt weak convergence of measures.

(ii) Let $\mathcal{O}$ denote the set of all semicouplings of λ and μ and $\mathcal{O}_1$ denote the set of all semicouplings q satisfying $\mathsf{Cost}(q) \le 2\inf_{q\in\mathcal{O}} \mathsf{Cost}(q) =: 2c$. Then $\mathcal{O}_1$ is relatively compact wrt weak topology (cf. proof of Proposition 3.18).

(iii) The set $\mathcal{O}$ is closed wrt weak topology. Indeed, if $q_n \to q$ then $(\pi_1)_* q_n \to (\pi_1)_* q$ and $(\pi_2)_* q_n \to (\pi_2)_* q$. Hence $\mathcal{O}_1$ is compact and Cost attains its minimum on $\mathcal{O}$. Let q denote one such minimizer. Note that q is an optimal coupling between its marginals. Its first marginal is absolutely continuous to m. By assumption (the results cited in Section 3.5), there is a measurable map $T : M \to M \cup \{\eth\}$ and densities $\tilde{f}_q, f_q$ such that $q = (id, T)_*(\tilde{f}_q \cdot \lambda) = (id, T)_*(f_q \cdot m)$.

(iv) Given a minimizer of Cost, say q. By Proposition 4.1, $\tilde{q} := q + (id, id)_*(f - f_q)\cdot m$ solves

$$\min C(\gamma) = \int c(x, y)\gamma(\mathrm{d}x, \mathrm{d}y)$$

under all γ which have λ and $\mu + (f - f_q)m$ as first respectively second marginals, where $f_q \cdot m = (\pi_1)_* q$ as above. Again by assumption, there is a measurable map S such that $\tilde{q} = (id, S)_*\lambda$. That is, $\tilde{q}$ and in particular q are concentrated on the graph of S. By definition $\tilde{q} = q + (id, id)_*(f - f_q)\cdot m$ and, therefore, we must have $S(x) = x$ on $\{f > f_q\}$.

(v) This finally allows us to deduce uniqueness. By the previous step, we know that any convex combination of optimal semicouplings is concentrated on a graph. This implies that all optimal semicouplings are concentrated on the *same* graph. Moreover, Proposition 4.1 implies that if we do not transport all the λ mass in one point we leave it where it is. Hence, all optimal semicouplings choose the same density ρ of λ and therefore coincide. Let us make this precise.

Assume there are two optimal semicouplings q_1 and q_2. Then $q_3 := \frac{1}{2}(q_1 + q_2)$ is optimal as well. By the previous step for any $i \in \{1, 2, 3\}$, we get maps S_i such that q_i is concentrated on the graph of S_i. Moreover, we have $S_3(x) = x$ on the set $\{f > f_{q_3}\} = \{f > f_{q_1}\} \cup \{f > f_{q_2}\}$, where again $f_{q_i} \cdot m = (\pi_1)_* q_i$. As q_3 is concentrated on the graph of S_3 and q_3 is a convex combination of q_1 and q_2, both q_1 and q_2 must also be concentrated on the graph of S_3 (the support of q_3 is the union of the supports of q_1 and q_2). Hence, we have $S_3 = S_i$ on $\{f_{q_i} > 0\}$ for $i = 1, 2$. This gives, that $S_3 = S_1 = S_2$ on $\{f_{q_1} > 0\} \cap \{f_{q_2} > 0\}$.

We still need to show that $\{f_{q_1} > 0\} = \{f_{q_2} > 0\}$. Put $A_1 := \{f_{q_1} > f_{q_2}\}$ and $A_2 := \{f_{q_2} > f_{q_1}\}$ and assume $m(A_1) > 0$. As $A_1 \subset \{f > f_{q_2}\}$ we know that $S_3(x) = x$ on A_1 and similarly $S_3(x) = x$ on A_2. Now consider

$$A := S_3^{-1}(A_1) = \big(A \cap \{f_{q_1} = f_{q_2}\}\big) \cup (A \cap A_1) \cup (A \cap A_2).$$

As $S_3(A_2) \subset A_2$ and $A_1 \cap A_2 = \varnothing$ we have $A \cap A_2 = \varnothing$. Therefore, we can conclude

$$\begin{aligned}
\mu(A_1) &= (S_3)_* f_{q_1} m(A_1) = f_{q_1} m(A_1) + f_{q_1} m\big(A \cap \{f_{q_1} = f_{q_2}\}\big) \\
&> f_{q_2} m(A_1) + f_{q_2} m\big(A \cap \{f_{q_1} = f_{q_2}\}\big) \\
&= (S_3)_* f_{q_2} m(A_1) = \mu(A_1),
\end{aligned}$$

which is a contradiction, proving $q_1 = q_2$. □

Remark 4.3. *Let $q = (id, T)_*(\rho\lambda)$ be the optimal semicoupling of λ and μ. If μ happens to be discrete, we have $\rho(x) \in \{0, 1\}$ m-almost everywhere.*

We showed the existence and uniqueness of optimal semicouplings between deterministic measures. The next step in the proof of Theorem 4.7 is to show the measurability of the mapping $\omega \mapsto \Phi(\lambda^\omega, 1_A\mu^\omega) = q_A^\omega$ the unique optimal semicoupling between λ^ω and $1_A\mu^\omega$. The mapping $\omega \mapsto (\lambda^\omega, 1_A\mu^\omega)$ is measurable by definition of random measures. Hence, we have to show that $(\lambda^\omega, 1_A\mu^\omega) \mapsto \Phi(\lambda^\omega, 1_A\mu^\omega)$ is measurable. We will show a bit more, namely that this mapping is actually continuous. We start with a simple but important observation about optimal semicouplings.

Denote the one-point compactification of M by $M \cup \{\eth\}$ and let $\tilde\vartheta(r)$ be such that it is equal to $\vartheta(r)$ on a very large interval, say $[0, K]$, and then tends continuously to zero such that $\tilde c(x, \eth) = \tilde\vartheta(d(x, \eth)) = \lim_{r\to\infty} \tilde\vartheta(r) = 0$ for any $x \in M$. By a slight abuse of notation, we also write $\eth : M \to \{\eth\}$ for the map $x \mapsto \eth$.

Lemma 4.4. *Let two measures λ and μ on M be given such that $\infty > \lambda(M) = N \geq \mu(M) = \alpha$ and assume there is a ball $B(x, K/2)$ such that* $\text{supp}(\lambda), \text{supp}(\mu) \subset B(x, K/2)$. *Then, q is an optimal semicoupling between λ and μ wrt to the cost function $c(\cdot,\cdot)$ iff $\tilde q = q + (id, \eth)_*(1 - f_q)\cdot\lambda$ is an optimal* coupling *between λ and $\tilde\mu = \mu + (N - \alpha)\delta_\eth$ wrt the cost function $\tilde c(\cdot,\cdot)$, where $(\pi_1)_* q = f_q\lambda$.*

Proof. Let q be any semicoupling between λ and μ. Then $\tilde q = q + (id, \eth)_*(1 - f_q)\cdot\lambda$ defines a coupling between λ and $\tilde\mu$. Moreover, the transportation cost of the semicoupling and the one of the coupling are exactly the same, that is $\mathsf{Cost}(q) = \mathsf{Cost}(\tilde q)$. Hence, q is optimal iff $\tilde q$ is optimal. □

This allows to deduce the continuity of Φ from the classical theory of optimal transportation.

Lemma 4.5. *Given a sequence of measures $(\lambda_n)_{n\in\mathbb{N}}$ converging vaguely to some λ, all absolutely continuous to m with $\lambda_n(M) = \lambda(M) = \infty$. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of finite measures converging weakly to some finite measure μ, all concentrated on some bounded set $A \subset M$. Let q_n be the optimal semicoupling between λ_n and μ_n and q be the optimal semicoupling between λ and μ. Then, q_n converges weakly to q. In particular, the map $(\lambda, \mu) \mapsto \Phi(\lambda, \mu) = q$ is continuous.*

Proof. (i) As the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to μ and μ is finite, we can assume that $\sup_n \mu_n(M), \mu(M) \leq \alpha < \infty$. Moreover, there is a bounded set B with $\lambda(\partial B) = 0$ such that $q_n(\cdot, A), q(\cdot, A)$ are concentrated on B. Indeed, as λ has infinite mass there is a $r \geq 5\,\text{diam}(A)$ such that the r-neighborhood A_r of A satisfies $\lambda(A_r) \geq 5\alpha$. Then clearly $q(\cdot, A)$ is concentrated on A_{2r}. Assume the contrary, then q transports a positive amount of mass at least distance $2r$. However, inside of A_r there is at least an amount of 4α of λ-measure which is not used by q, a contradiction to optimality. By vague convergence of λ_n to λ we can assume that (at least for large n) also $q_n(\cdot, A)$ is concentrated on A_{2r}. By enlarging A_{2r} if necessary, we can assume that its boundary has λ-measure zero.

(ii) Note that $\lambda(\partial B) = 0$ implies that $\lambda_n(B) =: N_n \to \lambda(B) = N$ as $n \to \infty$. Hence, by rescaling λ_n and μ_n by $\frac{N}{N_n}$ we can assume that $N_n = N$. Put $\tilde\lambda_n := 1_B\lambda_n$ and $\tilde\lambda := \lambda$.

(iii) Now we are in a setting where we can apply Lemma 4.4. Set $K = 2r$ and define $\tilde\vartheta, \tilde\mu_n, \tilde\mu$ as above. Then $\tilde q_n$ and $\tilde q$ are optimal couplings between $\tilde\lambda_n$ and $\tilde\mu_n$ and $\tilde\lambda$ and $\tilde\mu$ respectively wrt to the cost function $\tilde c(\cdot,\cdot)$. The cost function $\tilde c$ is continuous and M and $M \cup \{\eth\}$ are Polish spaces. Hence, we can apply the stability result of the classical optimal transportation theory (e.g. Theorem 5.20 in [51]) to conclude that $\tilde q_n \to \tilde q$ weakly and therefore $q_n \to q$ weakly. □

Take a pair of equivariant random measure $(\lambda^\bullet, \mu^\bullet)$ with $\lambda^\omega \ll m$ as usual. For a given $\omega \in \Omega$ we want to apply the results of the previous lemma to a fixed realization $(\lambda^\omega, \mu^\omega)$. Then, for any bounded Borel set $A \subset M$, there is a unique optimal semicoupling q_A^ω between λ^ω and $1_A\mu^\omega$, that is, a unique minimizer of the cost function Cost among all semicouplings of λ^ω and $1_A\mu^\omega$.

Lemma 4.6. *For each bounded Borel set $A \subset M$ the map $\omega \mapsto q_A^\omega$ is measurable.*

Proof. We saw that the map $\Phi : (\lambda^\omega, 1_A\mu^\omega) = q_A^\omega$ is continuous. By definition of random measures the map $\omega \mapsto (\lambda^\omega, 1_A\mu^\omega)$ is measurable. Hence, the map

$$\omega \mapsto \Phi\big(\lambda^\omega, 1_A\mu^\omega\big) = q_A^\omega$$

is measurable. □

The uniqueness and measurability of q_A^ω allows us to finally deduce

Theorem 4.7.

(i) *For each bounded Borel set $A \subset M$ there exists a unique semicoupling Q_A of $\lambda^\bullet\mathbb{P}$ and $(1_A\mu^\bullet)\mathbb{P}$ which minimizes the mean cost functional $\mathfrak{Cost}(\cdot)$.*
(ii) *The measure Q_A can be disintegrated as $Q_A(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\omega) := q_A^\omega(\mathrm{d}x, \mathrm{d}y)\mathbb{P}(\mathrm{d}\omega)$ where for $\mathbb{P}$-a.e. ω the measure q_A^ω is the unique minimizer of the cost functional $\mathsf{Cost}(\cdot)$ among the semicouplings of λ^ω and $1_A\mu^\omega$.*
(iii) $\mathfrak{Cost}(Q_A) = \int_\Omega \mathsf{Cost}(q_A^\omega)\mathbb{P}(\mathrm{d}\omega)$.

The proof is a direct consequence of the results and arguments in this section. We omit the details.

The first part of the theorem, the existence and uniqueness of an optimal semicoupling, is very much in the spirit of an analogous result by Figalli [18] on existence and (if enough mass is transported) uniqueness of an optimal partial coupling. However, in our case the second marginal is arbitrary whereas in [18] it is absolutely continuous.

Corollary 4.8. *The optimal semicouplings $Q_A = q_A^\bullet\mathbb{P}$ are equivariant in the sense that*

$$Q_{gA}(gC, gD, \theta_g\omega) = Q_A(C, D, \omega),$$

for any $g \in G$ and $C, D \in \mathcal{B}(M)$.

Proof. This is a consequence of the equivariance of $\lambda^\bullet$ and $\mu^\bullet$ and the fact that q_A^ω is a deterministic function of λ^ω and $1_A\mu^\omega$. □

5. Monge solutions

The aim of this section is to prove Theorem 1.1, the uniqueness of optimal semicouplings. Moreover, the representation of optimal semicouplings, that we get as a byproduct of the uniqueness statement, allows to draw several conclusions about the geometry of the cells of the induced allocations.

Throughout this section we fix a pair of equivariant random measures $\lambda^\bullet$ and $\mu^\bullet$ of unit resp. subunit intensity on some metric measure space (M, d, m) together with a cost function $c(x, y) = \vartheta(d(x, y))$ such that the Monge problem has a unique solution (e.g. see Section 3.5). We assume that the optimal mean transportation cost (see Definition 3.15) $\mathfrak{c}_{e,\infty}$ are finite and that $\lambda^\bullet$ is absolutely continuous. Then, by Proposition 3.18, there is at least one optimal semicoupling. Recall the definition of cyclical monotonicity from Section 3.5.

Proposition 5.1. *Given a semicoupling q^ω of λ^ω and μ^ω for fixed $\omega \in \Omega$, then the following properties are equivalent.*

(i) *For each bounded Borel set $A \subset M$, the measure $1_{M\times A}q^\omega$ is the unique optimal coupling of the measures $q^\omega(\cdot, A)$ and $1_A\mu^\omega$ (cf. Figure 1).*

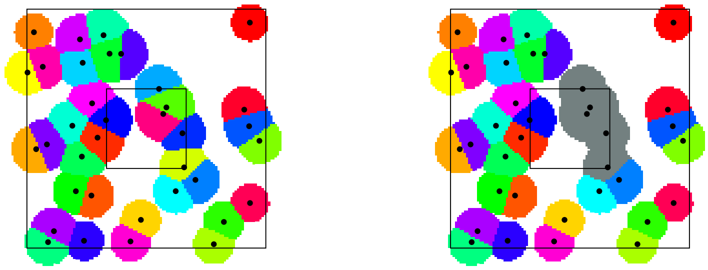

Fig. 1. The left picture is a semicoupling of Lebesgue and 36 points with cost function $c(x, y) = |x - y|^4$. In the right picture, the five points within the small cube can choose new partners from the mass that was transported to them in the left picture (corresponding to the measure λ_A^ω). If the semicoupling on the left hand side is locally optimal, then the points in the small cube on the right hand side will choose from the gray region exactly the partners they have in the left picture.

(ii) *The support of q^ω is c-cyclically monotone.*
(iii) *There exists a nonnegative density ρ^ω and a c-cyclically monotone map $T^\omega : \{\rho^\omega > 0\} \to M$ such that on $\{\rho^\omega > 0\} \times M$*

$$q^\omega = \big(id, T^\omega\big)_* \big(\rho^\omega \lambda^\omega\big). \tag{5.1}$$

By definition, a map T is c-cyclically monotone iff the closure of its graph $\{(x, T(x)) \colon x \in \{\rho^\omega > 0\}\}$ is a c-cyclically monotone set.

The proof is a direct consequence of the characterization of optimal couplings by cyclically monotone sets together with the assumption that the Monge problem has a unique solution. For a proof (in a slightly different setting) we refer to Section 3 of [26].

Remark 5.2. *Put $A^\omega = \{\rho^\omega > 0\}$. As before (see Section 3.2), any* transport map *$T^\omega : A^\omega \to M$ as above will be extended to a map $T^\omega : M \to M \cup \{\eth\}$. Then (5.1) reads*

$$q^\omega = \big(id, T^\omega\big)_* \rho^\omega \lambda^\omega \quad \text{on } M \times M. \tag{5.2}$$

Moreover, we put $c(x, T^\omega(x)) = c(x, \eth) := 0$ for $x \in M \setminus A^\omega$. If we know a priori that $\rho^\omega(x) \in \{0, 1\}$ almost surely (5.2) simplifies to

$$q^\omega = \big(id, T^\omega\big)_* \lambda^\omega \quad \text{on } M \times M. \tag{5.3}$$

Definition 5.3. *A semicoupling $Q = q^\bullet \mathbb{P}$ of $\lambda^\bullet$ and $\mu^\bullet$ is called* locally optimal *iff some (hence every) property of the previous proposition is satisfied for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Remark 5.4.

(i) *As in [26] asymptotic optimality is not sufficient for uniqueness and local optimality does not imply asymptotic optimality. A simple example for the last statement is the coupling $q(z, z') = 1_{\mathbb{Z}}(z)\delta_{z+5}(z')$ of $\mu = \sum_{z \in \mathbb{Z}} \delta_z$ with itself. For the cost function $c(x, y) = |x - y|^2$ this is locally optimal but certainly not asymptotically optimal.*
(ii) *The name* local optimality *might be misleading in the context of semicouplings. Consider a Poisson process $\mu^\bullet$ of intensity $1/2$ and let $q^\bullet$ be an optimal coupling between $1/2m$ and $\mu^\bullet$. Then, it is locally optimal (see Theorem 5.5) according to this definition. However, as we leave half of the m-measure laying around we can everywhere locally produce a coupling with less cost. In short, the optimality does not refer to the choice of density. It only refers to the (local) optimality of the transport of the chosen density.*
(iii) *Note that local optimality – in contrast to asymptotic optimality and equivariance – is not preserved under convex combinations. It is an open question if local optimality and asymptotic optimality together imply optimality.*

However, we have

Theorem 5.5. *Every optimal semicoupling between* $\lambda^\bullet$ *and* $\mu^\bullet$ *is locally optimal.*

The proof of this theorem is similar to the respective result in [26]. Assume the contrary. Then there is some bounded set A on which we can improve the semicoupling locally. By equivariance, this translates to a global improvement. Hence, the semicoupling could not have been asymptotically optimal, a contradiction. We omit the details.

Theorem 5.6. *Assume that* $\mu^\bullet$ *has intensity one, then there is a unique optimal coupling between* $\lambda^\bullet$ *and* $\mu^\bullet$.

Proof. Assume we are given two optimal couplings $q_1^\bullet$ and $q_2^\bullet$. Then also $q^\bullet := \frac{1}{2}q_1^\bullet + \frac{1}{2}q_2^\bullet$ is an optimal coupling because asymptotic optimality and equivariance are stable under convex combination. Hence, by the previous theorem all three couplings – $q_1^\bullet$, $q_2^\bullet$ and $q^\bullet$ – are locally optimal. Thus, for a.e. ω by the results of Proposition 5.1 there exist maps $T_1^\omega, T_2^\omega, T^\omega$ such that

$$\begin{aligned}\delta_{T^\omega(x)}(\mathrm{d}y)\lambda^\omega(\mathrm{d}x) &= q^\omega(\mathrm{d}x,\mathrm{d}y)\\ &= \left(\frac{1}{2}\delta_{T_1^\omega(x)}(\mathrm{d}y) + \frac{1}{2}\delta_{T_2^\omega(x)}(\mathrm{d}y)\right)\lambda^\omega(\mathrm{d}x).\end{aligned}$$

This, however, implies $T_1^\omega(x) = T_2^\omega(x)$ for a.e. $x \in M$. Thus $q_1^\omega = q_2^\omega$. (By Lemma 3.20 we know that every invariant semicoupling between $\lambda^\bullet$ and $\mu^\bullet$ has to be a coupling.) By Proposition 3.18 there exists an optimal coupling; hence it is unique. □

Before we can prove the uniqueness of optimal semicouplings we have to translate Proposition 4.1 to this setting.

Proposition 5.7. *Assume* $\mu^\bullet$ *has intensity* $\beta \le 1$ *and let* $q^\bullet$ *be an optimal semicoupling between* $\lambda^\bullet$ *and* $\mu^\bullet$. *Let* $(\pi_1)_* q^\bullet = \rho \cdot \lambda^\bullet$ *for some density* $\rho : \Omega \times M \to [0,1]$. *Then,*

$$\tilde{q}^\bullet = q^\bullet + (id \times id)_*\big((1-\rho)\cdot\lambda^\bullet\big)$$

is the unique optimal coupling between $\lambda^\bullet$ *and* $\hat{\mu}^\bullet := \mu^\bullet + (1-\rho)\cdot\lambda^\bullet$.

Proof. Because $q^\bullet$ is equivariant by assumption also $\rho\lambda^\bullet(\cdot) = q^\bullet(\cdot, M)$ is equivariant. But then $\hat{\mu}^\bullet = \mu^\bullet + (1-\rho)\cdot\lambda^\bullet$ is equivariant. The intensity (cf. Definition 3.11) of $\hat{\mu}$ equals, by the mass transport principle Lemma 3.8,

$$\hat{\mu}(B_0) = \beta + 1 - \mathbb{E}\big[q^\bullet(B_0, M)\big] = \beta + 1 - \mathbb{E}\big[q^\bullet(M, B_0)\big] = 1.$$

Moreover, by assumption we have $\mathfrak{C}(\tilde{q}^\bullet) = \mathfrak{C}(q^\bullet) < \infty$ which implies

$$\inf_{\kappa^\bullet \in \Pi_e(\lambda^\bullet, \hat{\mu}^\bullet)} \mathfrak{C}\big(\kappa^\bullet\big) < \infty.$$

Hence, we can apply the previous theorem and get a unique optimal coupling $\kappa^\bullet$ between $\lambda^\bullet$ and $\hat{\mu}^\bullet$ given by $\kappa^\bullet = (id, S)_*\lambda^\bullet$. Moreover,

$$\mathfrak{C}\big(\kappa^\bullet\big) \le \mathfrak{C}\big(\tilde{q}^\bullet\big) = \mathfrak{C}\big(q^\bullet\big).$$

Because $S_*\lambda^\bullet = \hat{\mu}^\bullet$ there is a density f such that $S_*(f\cdot\lambda^\bullet) = (1-\rho)\cdot\lambda^\bullet$. Indeed, for any $g \in G$ we can disintegrate

$$1_{M\times gB_0}\kappa^\omega(\mathrm{d}x,\mathrm{d}y) = \kappa_y^{\omega,g}(\mathrm{d}x)\big(\mu^\omega(\mathrm{d}y) + \big(1-\rho^\omega(y)\big)\lambda^\omega(\mathrm{d}y)\big).$$

The measure $\sum_{g\in G}\kappa_y^{\omega,g}(\mathrm{d}x)((1-\rho^\omega(y))\lambda^\omega(\mathrm{d}y))$ does the job. In particular this implies that

$$\tilde{\kappa}^\bullet = (id \times S)_*\big((1-f)\cdot\lambda^\bullet\big)$$

is a semicoupling between $\lambda^\bullet$ and $\mu^\bullet$. The mean transportation cost of $\tilde{\kappa}^\bullet$ are bounded above by the mean transportation cost of $\kappa^\bullet$ as we just transport less mass. Hence, we have

$$\mathfrak{C}(\tilde{\kappa}^\bullet) \leq \mathfrak{C}(\kappa^\bullet) \leq \mathfrak{C}(\tilde{q}^\bullet) = \mathfrak{C}(q^\bullet).$$

As $q^\bullet$ was assumed to be optimal, hence asymptotically optimal, we must have equality everywhere. By uniqueness of optimal couplings this implies that $\tilde{q}^\bullet = \kappa^\bullet$ almost surely. □

Lemma 5.8. *Assume $\mu^\bullet$ has intensity $\beta \leq 1$ and let $q^\bullet = (id, T)_*(\rho \cdot \lambda^\bullet)$ be an optimal semicoupling between $\lambda^\bullet$ and $\mu^\bullet$. Then, for $\mathbb{P}$-almost all ω on the set $\{0 < \rho^\omega < 1\}$ we have $T^\omega(x) = x$.*

Proof. Just as in the previous proposition consider $\tilde{q}^\bullet = (id, S)_*\lambda^\bullet$ the optimal coupling between $\lambda^\bullet$ and $\hat{\mu}^\bullet$. $\tilde{q}^\bullet$ is concentrated on the graph of S and therefore also $q^\bullet$ has to be concentrated on the graph of S. In particular, this shows that $S = T$ almost everywhere almost surely (we can safely extend T by S on $\{\rho = 0\}$). But on $\{\rho < 1\}$ we have $S(x) = x$. Hence, we also have $T(x) = x$ on $\{0 < \rho < 1\}$. □

This finally enables us to prove uniqueness of Monge solutions.

Theorem 5.9. *There exists a unique optimal semicoupling of $\lambda^\bullet$ and $\mu^\bullet$.*

The proof goes along the same lines as the one for Proposition 4.2. We omit the details.

5.1. *Geometry of tessellations induced by fair allocations*

The fact that any optimal semicoupling is locally optimal allows us to say something about the geometries of the cells of fair allocations to a point process $\mu^\bullet$. The following result was already shown for probability measures in Section 4 of [49] and also in [4].

Corollary 5.10. *In the case $\vartheta(r) = r^2$, given an optimal coupling $q^\bullet$ of Lebesgue measure $\mathcal{L}$ and a point process $\mu^\bullet$ of unit intensity in $M = \mathbb{R}^d$ (for a Poisson point process this implies $d \geq 3$ as otherwise the mean transportation cost will be infinite, see Theorem 1.3 in [26]) then for a.e. $\omega \in \Omega$ there exists a convex function $\varphi^\omega : \mathbb{R}^d \to \mathbb{R}$ (unique up to additive constants) such that*

$$q^\omega = (id, \nabla\varphi^\omega)_*\mathcal{L}.$$

In particular, a 'fair allocation rule' is given by the monotone map $T^\omega = \nabla\varphi^\omega$.

Moreover, for a.e. ω and any center $\xi \in \Xi(\omega) := \mathrm{supp}(\mu^\omega)$, the associated cell

$$S^\omega(\xi) = (T^\omega)^{-1}(\{\xi\})$$

is a convex polytope of volume $\mu^\omega(\xi) \in \mathbb{N}$. If the point process is simple then all these cells have volume 1. In particular, T induces a Laguerre-tessellation.

Proof. See Corollary 3.10 of [26]. □

A set $A \subset M$ is called starlike with respect to a point x iff for all $p \in A$ all the points on the minimizing geodesic between x and p lie in A.

Corollary 5.11. *Let (M, d, m) be a Riemannian manifold. In the case $\vartheta(r) = r$, given an optimal coupling $q^\bullet$ of m and a point process $\mu^\bullet$ of unit intensity on M with $\dim(M) \geq 2$, there exists an allocation rule T such that the optimal coupling is given by*

$$q^\omega = (id, T^\omega)_*m.$$

Moreover, for a.e. ω and any center $\xi \in \Xi(\omega) := \operatorname{supp}(\mu^\omega)$, the associated cell

$$S^\omega(\xi) = \left(T^\omega\right)^{-1}\left(\{\xi\}\right)$$

is starlike with respect to ξ.

Remark 5.12. *The existence and uniqueness of an optimal semicoupling between m and a sum of finitely many Dirac-measures for $\vartheta(r) = r$ follows from Lemma* 6.1 *in* [26]. *It is stated for $\mathbb{R}^d$ but the proof is valid for Riemannian manifolds as well.*

Proof of Corollary 5.11. Let $(K_n)_{n\in\mathbb{N}}$ be an increasing exhausting sequence of geodesically convex sets. By Proposition 5.1 we know that $T^\omega = \lim_{n\to\infty} T_n^\omega$, where T_n^ω is an optimal transportation map from some set A_n^ω to K_n. From the classical theory (see [10,19]) we know that,

$$T_n^\omega(x) = \xi_0 \quad \Leftrightarrow \quad -d(x,\xi_0) + b_{\xi_0} > -d(x,\xi) + b_\xi \quad \forall \xi \in \Xi(\omega) \cap K_n, \xi \neq \xi_0,$$

for some (unknown) constants b_{ξ_i}. Hence, the cell can be written as the intersection of the sets $H_j^0 := \{x\colon -d(x,\xi_0) + b_{\xi_0} > -d(x,\xi_j) + b_{\xi_j}\}$. Therefore, it is sufficient to show that for any $z \in H_j^0$ the whole geodesic from z to ξ_0 lies inside H_j^0. For convenience we write $\Phi_0(x) = -d(x,\xi_0) + b_{\xi_0}$ and $\Phi_j(x) = -d(x,\xi_j) + b_{\xi_j}$.

Assume $\xi_0 \in \partial H_j^0$ and w.l.o.g. $b_{\xi_0} = 0$. Then, we have

$$\Phi_0(\xi_0) = 0 = \Phi_j(\xi_0) \quad \Rightarrow \quad b_{\xi_j} = d(\xi_j,\xi_0).$$

The set $N = \{z \in M\colon d(\xi_j,z) = d(\xi_j,\xi_0) + d(\xi_0,z)\}$ is a m-null set. For all $z \notin N$ we have

$$\Phi_j(z) = -d(\xi_j,z) + b_{\xi_j} > -d(\xi_j,\xi_0) + b_{\xi_j} - d(\xi_0,z) = \Phi_0(z).$$

This implies that $m(T_n^{-1}(\xi_i)) = 0$ contradicting the assumption of T being an allocation. Thus, $\xi_0 \notin \partial H_j^0$ and in particular $T(\xi_0) \in \Xi = \operatorname{supp}(\mu)$.

Assume $T(\xi_0) \neq \xi_0$. Then, there is a $\xi_j \neq \xi_0$ such that $T(\xi_0) = \xi_j$, i.e. $\Phi_j(\xi_0) = -d(\xi_0,\xi_j) + b_{\xi_j} > b_{\xi_0} = \Phi_0(\xi_0)$. Then, we have for any $p \in M$, $p \neq \xi_0$

$$-d(p,\xi_j) + b_{\xi_j} \geq -d(p,\xi_0) - d(\xi_0,\xi_j) + b_{\xi_j} > -d(p,\xi_0) + b_{\xi_0}.$$

This implies, that $m(T^{-1}(\xi_0)) = 0$ contradicting the assumption of T being an allocation. Thus, $T(\xi_0) = T_n(\xi_0) = \xi_0$.

Take any $w \in T_n^{-1}(\xi_0)$ (hence, $\Phi_0(w) > \Phi_j(w)$ for all $j \neq 0$) and $p \in M$ such that $d(\xi_0,w) = d(\xi_0,p) + d(p,w)$, i.e. p lies on the minimizing geodesic from ξ_0 to w. Then, we have for any $j \neq 0$ by using the triangle inequality once more

$$\begin{aligned}
-d(p,\xi_0) + b_{\xi_0} &= -d(\xi_0,w) + d(p,w) + b_{\xi_0} \\
&\geq -d(\xi_0,w) + b_{\xi_0} + d(w,\xi_j) - d(p,\xi_j) \\
&> -d(p,\xi_j) + b_{\xi_j},
\end{aligned}$$

which means that $\Phi_0(p) > \Phi_j(p)$ for all $j \neq 0$. Hence, $p \in H_j^0$ proving the claim. □

Remark 5.13.

(i) *Questions on the geometry of the cells of fair allocations are strongly connected to the very difficult problem of the regularity of optimal transportation maps (see* [31,36,37]*). The link is of course the cyclical monotonicity. The geometry of the cells of the "optimal fair allocation" is dictated by the cyclical monotonicity and the optimal choice of cyclically monotone map to get an asymptotic optimal coupling.*

Consider the classical transport problem between two probability measures one being absolutely continuous to m with full support on a convex set and the other one being a convex combination of N Dirac masses. Assume *that the cell being transported to one of the N points is not connected. Then, it is not difficult to imagine that it is possible to smear out the Dirac masses slightly to get two absolutely continuous probability measures (even with very nice densities) but a discontinuous transportation map.*

(ii) *Considering L^p cost on $\mathbb{R}^d$ with $p \notin \{1,2\}$, the cell structure is much more irregular than in the two cases considered above. The cells do not even have to be connected. Indeed, just as in the proof of the two corollaries above it holds also for general p that $T^\omega(x) = \xi_0$ iff $\Phi_0(x) > \Phi_i(x)$ for all $i \neq 0$ where $\Phi_i(x) = -|x-\xi_i|^p + b_i$ for some constants b_i (see also Example* 1.6 *in* [19]). *By considering the sets $\Phi_i \equiv \Phi_0$ it is not difficult to cook up examples of probability measures such that the cells do not have to be connected.*

In the case that $p \in (0,1)$ similar to the case that $p = 1$ we always have that the center of each cell lies in the cell, that is $T(\xi_i) = \xi_i$ for all $\xi_i \in \operatorname{supp}(\mu^\bullet)$ because the cost function defines a metric (see [19]).

(iii) *As was shown by Loeper in Section* 8.1 *of* [36] *the cells induced by the optimal transportation problem in the hyperbolic space between an absolutely continuous measure and a discrete measure with respect to the cost function $c(x,y) = d^2(x,y)$ do not have to be connected. In the same article he shows that for the same problem on the sphere the cells have to be connected. In* [52] *von Nessi studies more general cost functions on the sphere, including the L^p cost function $c(x,y) = d^p(x,y)$. He shows that in general for $p \neq 2$ the cells do not have to connected. This suggests that on a general metric measure space the cell structure will probably be rather irregular.*

6. Limits of optimal semicouplings on bounded sets and equivariance

We fix again a pair of equivariant random measures $\lambda^\bullet$ and $\mu^\bullet$ of unit resp. subunit intensity on some Polish geodesic metric measure space (M,d,m) together with a cost function $c(x,y) = \vartheta(d(x,y))$ such that the Monge problem has a unique solution. We assume that the optimal mean transportation cost (see Definition 3.15) $\mathfrak{c}_{e,\infty}$ is finite and that $\lambda^\bullet$ is absolutely continuous. Also recall $\mathfrak{c}_\infty$ from Definition 3.15 which is bounded by $\mathfrak{c}_{e,\infty}$ and therefore also finite.

In this section, we show how equivariance naturally appears in the limit of optimal semicouplings on bounded sets. In particular, we will prove Theorem 1.2 and Theorem 1.3. To this end, we *additionally* assume that G is amenable and fix an exhausting Følner sequence $(F_n)_{n\in\mathbb{N}}$ with $id = F_0$ (see Definition 3.5 and Lemma 3.7). B_0 denotes a fundamental region and B_n the range of the action of F_n on B_0, i.e. $B_n = \bigcup_{g\in F_n} gB_0 = F_n B_0$. Then, the amenability assumption implies by Proposition 3.6

$$\frac{|CF_r \triangle F_r|}{|F_r|} \to 0 \quad \text{as } r \to \infty, \tag{6.1}$$

for any finite set $C \subset G$. This of course implies for any $g \in G$

$$\frac{m(B_r \triangle gB_r)}{m(B_r)} \to 0 \quad \text{as } r \to \infty.$$

6.1. *Symmetrization and annealed limits*

We want to construct the optimal semicoupling by approximation of optimal semicouplings on bounded sets. The difficulty in this approximation lies in the estimation of the contribution of the fundamental regions gB_0 to the transportation cost of Q_{gB_r}, i.e. what is the contribution of the transport into gB_0 to $\mathfrak{Cost}(Q_{gB_r})$? How can the cost be bounded in order to be able to conclude that the limiting measure still transports the right amount of mass into gB_0? The solution is to mix several optimal semicouplings and thereby get a symmetry which will be very useful (see proof of Lemma 6.1(i)). One can also think of the mixing as an expectation of the random choice of increasing sequences of sets hB_r exhausting M. Moreover, the amenability will allow us to ensure that we do not add up too much mass in the mixing procedure.

For each $g \in G$ and $r \in \mathbb{N}$, recall that Q_{gB_r} denotes the minimizer of $\mathfrak{Cost}$ among the semicouplings of $\lambda^\bullet$ and $1_{gB_r}\mu^\bullet$ as constructed in Theorem 4.7. It inherits the equivariance from $\lambda^\bullet$ and $\mu^\bullet$, namely $Q_{gA}(g\cdot, g\cdot, \theta_g\omega) = Q_A(\cdot,\cdot,\omega)$ (see Corollary 4.8). In particular, the stationarity of $\mathbb{P}$ implies $(\tau_h)_* Q_{gBr} \stackrel{d}{=} Q_{hgB_r}$. Put (cf. Figure 2)

$$Q_g^r(dx, dy, d\omega) := \frac{1}{|F_r|} \sum_{h\in G} 1_{gB_0}(y) Q_{hB_r}(dx, dy, d\omega).$$

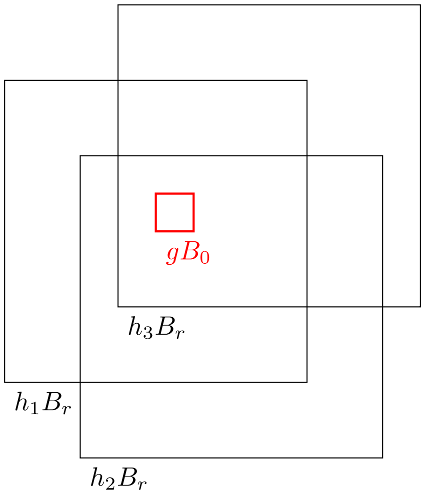

Fig. 2. Schematic picture of the mixing procedure.

Note that the sum on the right hand side is effectively a finite sum because it only contributes nonzero terms if $gB_0 \subset hB_r$, i.e. iff $h \in gF_r^{-1}$.

The measure Q_g^r defines a semicoupling between $\lambda^\bullet$ and $1_{gB_0}\mu^\bullet$. It is in general not optimal but does not depend on any external randomness. Moreover, Q_g^r shares the equivariance properties of the measures Q_{hB_r}.

Lemma 6.1.

(i) *For each $r \in \mathbb{N}$ and $g \in G$*

$$\int_{M\times gB_0\times\Omega} c(x,y)Q_g^r(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega) \le \mathfrak{c}_\infty.$$

(ii) *The family $(Q_g^r)_{r\in\mathbb{N}}$ of probability measures on $M \times M \times \Omega$ is relatively compact in the weak topology.*

(iii) *There exist probability measures Q_g^∞ and a subsequence $(r_l)_{l\in\mathbb{N}}$ such that for all $g \in G$:*

$$Q_g^{r_l} \longrightarrow Q_g^\infty \quad \textit{weakly as } l \to \infty.$$

Proof. (i) Let us fix $g \in G$. Note that $g \in hF_r \Leftrightarrow h^{-1}g \in F_r \Leftrightarrow gh^{-1}g \in gF_r$. Hence, using the equivariance of Q_{hB_r}, Corollary 4.8, and the invariance of c under isometries, in particular under the action of G, we obtain

$$\begin{aligned}
&\int_{M\times gB_0\times\Omega} c(x,y)Q_g^r(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega)\\
&\quad= \frac{1}{|F_r|}\sum_{h:gh^{-1}g\in gF_r}\int_{M\times gB_0\times\Omega} c(x,y)Q_{hB_r}(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega)\\
&\quad= \frac{1}{|F_r|}\sum_{h:gh^{-1}g\in gF_r}\int_{M\times gh^{-1}gB_0\times\Omega} c(x,y)Q_{gh^{-1}hB_r}(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega)\\
&\quad= \frac{1}{|F_r|}\sum_{k\in gF_r}\int_{M\times kB_0\times\Omega} c(x,y)Q_{gB_r}(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega)\\
&\quad= \frac{1}{|F_r|}\mathsf{C}_{gB_r} =: \mathfrak{c}_r \le \mathfrak{c}_\infty,
\end{aligned}$$

by definition of $\mathfrak{c}_\infty$ (see Definition 3.15 and Lemma 3.13).

(ii) In order to prove tightness of $(Q_g^r)_{r\in\mathbb{N}}$, let $(gB_0)_l$ denote the closed l-neighborhood of gB_0 in M. Then,

$$\begin{aligned} Q_g^r\big(\complement(gB_0)_l, gB_0, \Omega\big) &\le \frac{1}{\vartheta(l)}\int_{M\times gB_0\times\Omega} c(x,y)\,Q_g^r(\mathrm{d}x,\mathrm{d}y,\mathrm{d}\omega) \\ &\le \frac{1}{\vartheta(l)}\mathfrak{c}_\infty. \end{aligned}$$

Since $\vartheta(l)\to\infty$ as $l\to\infty$ this proves tightness of the family $(Q_g^r)_{r\in\mathbb{N}}$ on $M\times M\times\Omega$. (Recall that Ω was assumed to be compact from the very beginning.)

(iii) Tightness yields the existence of Q_g^∞ and of a converging subsequence for each $g\in G$. A standard argument ('diagonal sequence') then gives convergence for all $g\in G$ along a common subsequence (G is countable as it is finitely generated). □

Note that the measures Q_g^∞ inherit as weak limits the property $Q_{hg}^\infty(h\cdot, h\cdot, \theta_h\cdot) = Q_g^\infty(\cdot,\cdot,\cdot)$ from the measures Q_g^r (see also the proof of the equivariance property in Proposition 3.18). The next lemma allows us to control the difference in the first marginals of Q_g^∞ and Q_h^∞ for $g\neq h$. This is the first point where we use amenability. Let S be a (finite) generating set for G. We denote the Cayley graph with respect to S by $\Delta(G,S)$ and the associated graph/word metric by $d_\Delta(\cdot,\cdot)$.

Lemma 6.2.

(i) *For all $l>0$ there exist numbers $\varepsilon_r(l)$ with $\varepsilon_r(l)\to 0$ as $r\to\infty$ s.t. for all $g,g'\in G$ and all $r\in\mathbb{N}$*

$$\frac{1}{|F_r|}\sum_{h\in g'F_r^{-1}} Q_{hB_r}(A) \le \frac{1}{|F_r|}\sum_{h\in gF_r^{-1}} Q_{hB_r}(A) + \varepsilon_r\big(d_\Delta(g,g')\big)\cdot \sup_{h\in g'F_r^{-1}} Q_{hB_r}(A)$$

for any Borel set $A\subset M\times M\times\Omega$.

(ii) *For all $g_1,\dots,g_n\in G$, all $r\in\mathbb{N}$ and all Borel sets $A\subset M$, $D\subset\Omega$*

$$\sum_{i=1}^n Q_{g_i}^r(A,M,D) \le \left(1+\sum_{i=1}^n \varepsilon_r\big(d_\Delta(g_1,g_i)\big)\right)\cdot\lambda(D,A),$$

where $\lambda(D,A) := \int_D\int_A \lambda^\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$.

Proof. (i) Fix $g,g'\in G$ with $d_\Delta(g,g')=l$. By Lemma 3.7 also $(F_n^{-1})_{n\in\mathbb{N}}$ is a Følner sequence. Moreover,

$$\{h\in G:\ g,g'\in hF_r\} = \{gF_r^{-1}\}\cap\{g'F_r^{-1}\}.$$

Put

$$\varepsilon_r(l) = \max_{g:d_\Delta(id,g)=l}\frac{|gF_r^{-1}\,\triangle\,F_r^{-1}|}{|F_r|}.$$

By the amenability assumption on G we have $\varepsilon_r(l)\to 0$ as $r\to\infty$ and therefore

$$\frac{|\{h\in G:\ g,g'\in hF_r\}|}{|F_r|} \ge 1-\varepsilon_r(l)\to 1 \quad \text{as } r\to\infty.$$

This directly implies that for each Borel set $A\subset M\times M\times\Omega$

$$\frac{1}{|F_r|}\sum_{h\in g'F_r^{-1}} Q_{hB_r}(A) \le \frac{1}{|F_r|}\sum_{h\in gF_r^{-1}} Q_{hB_r}(A) + \varepsilon_r\big(d_\Delta(g,g')\big)\cdot \sup_{h\in g'F_r^{-1}} Q_{hB_r}(A).$$

(ii) According to the previous part (i), for each Borel sets $A \subset M$, $D \subset \Omega$

$$\begin{aligned}\sum_{i=1}^{n} Q^r_{g_i}(A, M, D) &= \sum_{i=1}^{n} \frac{1}{|F_r|} \sum_{h \in g_i F_r^{-1}} Q_{hB_r}(A, g_i B_0, D) \\ &\le \sum_{i=1}^{n} \Bigg(\frac{1}{|F_r|} \sum_{h \in g_1 F_r^{-1}} Q_{hB_r}(A, g_i B_0, D) + \varepsilon_r\big(d_\Delta(g_1, g_i)\big) \cdot \sup_{h \in g_i F_r^{-1}} Q_{hB_r}(A, g_i B_0, D) \Bigg) \\ &\le \left(1 + \sum_{i=1}^{n} \varepsilon_r\big(d_\Delta(g_1, g_i)\big)\right) \lambda(D, A).\end{aligned}$$

□

Having these results at our hands we can argue as in [26] (Theorem 4.3 and Corollary 4.4) to get the following results. We omit further details.

Theorem 6.3. *The measure $Q^\infty := \sum_{g \in G} Q^\infty_g$ is an optimal semicoupling of $\lambda^\bullet$ and $\mu^\bullet$.*

Corollary 6.4 (Theorem 1.2).

(i) *For $r \to \infty$, the sequence of measures $Q^r := \sum_{g \in G} Q^r_g$, $r \in \mathbb{N}$, converges vaguely to the unique optimal semicoupling Q^∞.*

(ii) *For each $g \in G$ and $r \in \mathbb{N}$ put*

$$\tilde{Q}^r_g(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\omega) := \frac{1}{|F_r|} \sum_{h \in g F_r^{-1}} Q_{hB_r}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\omega).$$

The sequence $(\tilde{Q}^r_g)_{r \in \mathbb{N}}$ converges vaguely to the unique optimal semicoupling Q^∞.

In particular, Q^∞ is equivariant. This shows that the requirement that optimal semicouplings need to be equivariant is very natural and not random at all.

Corollary 6.5. *Denote the set of all semicouplings of $\lambda^\bullet$ and $\mu^\bullet$ by Π_s. Then the following holds*

$$\begin{aligned}&\inf_{q^\bullet \in \Pi_s} \liminf_{r \to \infty} \frac{1}{m(B_r)} \mathbb{E}\left[\int_{M \times B_r} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right] \\ &\quad = \liminf_{r \to \infty} \inf_{q^\bullet \in \Pi_s} \frac{1}{m(B_r)} \mathbb{E}\left[\int_{M \times B_r} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right].\end{aligned} \tag{6.2}$$

In particular, we have

$$\mathfrak{c}_\infty = \inf_{q^\bullet \in \Pi_s} \mathfrak{C}\big(q^\bullet\big) = \inf_{q^\bullet \in \Pi_{es}} \mathfrak{C}\big(q^\bullet\big) = \mathfrak{c}_{e,\infty}.$$

Proof. For any semicoupling $q^\bullet$ we have due to the supremum in the definition of $\mathfrak{C}(\cdot)$ that

$$\liminf_{r \to \infty} \frac{1}{m(B_r)} \mathbb{E}\left[\int_{M \times B_r} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right] \le \mathfrak{C}\big(q^\bullet\big).$$

Hence, the left hand side in (6.2) is bounded from above by $\inf_{q^\bullet \in \Pi_s} \mathfrak{C}(q^\bullet)$. However, we just constructed an equivariant semicoupling, the unique optimal semicoupling Q^∞ which attains equality, i.e. with $Q^\infty = q^\bullet \mathbb{P}$

$$\liminf_{r \to \infty} \frac{1}{m(B_r)} \mathbb{E}\left[\int_{M \times B_r} c(x, y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right] = \mathfrak{C}\big(Q^\infty\big).$$

Hence, the left hand side in (6.2) equals $\inf_{q^\bullet \in \Pi_s} \mathfrak{C}(q^\bullet)$.

The right hand side equals $\liminf_{r\to\infty} \mathfrak{c}_r$ which is bounded by $\mathfrak{c}_\infty = \inf_{q^\bullet\in\Pi_s} \mathfrak{C}(q^\bullet)$ by Lemma 3.13. By our construction, the asymptotic transportation cost of Q^∞ is bounded by the right hand side, i.e.

$$\mathfrak{C}\big(Q^\infty\big) \le \liminf_{r\to\infty} \mathfrak{c}_r$$

by Lemma 6.1. Hence, also the right hand side of (6.2) equals $\inf_{q^\bullet\in\Pi_s} \mathfrak{C}(q^\bullet)$. Thus, we have equality. □

Remark 6.6.

(i) *Because of the uniqueness of the optimal semicoupling the limit of the sequence Q^r does not depend on the choice of fundamental region. The approximating sequence $(Q^r)_{r\in\mathbb{N}}$ does of course depend on B_0 and also the choice of Følner sequence.*

(ii) *In the construction of the semicoupling Q^∞ we only used finite transportation cost, invariance of Q_A in the sense that $(\tau_h)_* Q_A \stackrel{d}{=} Q_{hA}$ and the amenability assumption on G. The only specific property of $\lambda^\bullet$ and $\mu^\bullet$ that we used is the uniqueness of the semicoupling on bounded sets which makes is easy to choose a good optimal semicoupling Q_{gB_r}. Hence, we can use the same algorithm to construct an optimal semicoupling between two arbitrary random measures. In particular this shows, that $\mathfrak{c}_\infty = \mathfrak{c}_{e,\infty}$ (see also Proposition 3.18).*

Indeed, given two arbitrary equivariant measures $\nu^\bullet$ and $\mu^\bullet$ of unit respectively subunit intensity. For any $r\in\mathbb{N}$ let $Q_{B_r} = q^\bullet_{B_r}\mathbb{P}$ be an optimal semicoupling between $\nu^\bullet$ and $1_{B_r}\mu^\bullet$. In particular, we made some measurable choice of optimal semicoupling for each ω (they do not have to be unique), e.g. like in Corollary 5.22 *of* [51]. Define Q_{gB_r} *via* $q^{\theta_g\omega}_{gB_r}(d(gx), d(gy)) := q^\omega_{B_r}(\mathrm{d}x, \mathrm{d}y)$. *Due to equivariance, this is again a measurable choice of optimal semicouplings. Stationarity of $\mathbb{P}$ implies $(\tau_h)_* Q_{B_r} \stackrel{d}{=} Q_{hB_r}$. Hence, by the same construction there is some optimal semicoupling Q^∞ of $\nu^\bullet$ and $\mu^\bullet$ with cost bounded by $\mathfrak{c}_\infty$.*

6.2. *Quenched limits*

According to Section 5, the unique optimal semicoupling between $\lambda^\bullet$ and $\mu^\bullet$ can be represented on $M\times M\times\Omega$ as

$$Q^\infty(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\omega) = \delta_{T(x,\omega)}(\mathrm{d}y)\rho^\omega(x)\lambda^\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$$

by means of a measurable map

$$T : M\times\Omega \to M\cup\{\eth\},$$

and a density ρ^ω defined uniquely almost everywhere (also recall Section 3.2 for the notion of cemetery point $\eth$). Similarly, for each $g\in G$ and $r\in\mathbb{N}$, by Theorem 4.7, there exists a measurable map

$$T_{g,r} : M\times\Omega \to M\cup\{\eth\}$$

and a density $\rho^\omega_{g,r}$ such that the measure

$$Q_{gB_r}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}\omega) = \delta_{T_{g,r}(x,\omega)}(\mathrm{d}y)\rho^\omega_{g,r}\lambda^\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$$

on $M\times M\times\Omega$ is the unique optimal semicoupling of $\lambda^\bullet$ and $1_{gB_r}\mu^\bullet$.

Theorem 6.7 (Theorem 1.3). *For every $g\in G$*

$$T_{g,r}(x,\omega) \to T(x,\omega) \quad \textit{as } r\to\infty \textit{ locally in } \lambda^\bullet\otimes\mathbb{P}\textit{-measure}.$$

The claim relies on the following two lemmas. For the first one we use amenability of G once more. The second one is a slight modification (and extension) of a result in [1].

Lemma 6.8.

(i) *Fix $\omega\in\Omega$ and take two disjoint bounded Borel sets $A, B\subset M$. Let $q^\omega_A = (id, T^\omega_A)_*(\rho^\omega_A\lambda^\omega)$ be the optimal semicoupling between λ^ω and $1_A\mu^\omega$. Similarly, let q^ω_B and $q^\omega_{A\cup B}$ be the unique optimal semicouplings between*

λ^ω and $1_B\mu^\omega$ respectively $1_{A\cup B}\mu^\omega$ with transport maps T_B^ω and $T_{A\cup B}^\omega$ and densities ρ_B^ω and $\rho_{A\cup B}^\omega$. Then, it holds that

$$\rho_{A\cup B}^\omega(x) \geq \max\{\rho_A^\omega(x), \rho_B^\omega(x)\} \quad \lambda^\omega \text{ a.s.}$$

(ii) *For any $g \in G$ and $r \in \mathbb{N}$ we have $\rho_{g,r}^\omega(x) \leq \rho^\omega(x)(\lambda^\bullet \otimes \mathbb{P})$ a.s.*
(iii) *For any $g \in G$ we have $\lim_{r\to\infty} \rho_{g,r}^\omega(x) \nearrow \rho^\omega(x)\mathbb{P}$ a.s. locally in λ^ω.*

Proof. (i) Firstly, note that if $\{\rho_A^\omega > 0\} \cap \{\rho_B^\omega > 0\} = \varnothing$ we have $\rho_{A\cup B}^\omega = \rho_A^\omega + \rho_B^\omega$. Because of the symmetry in A and B it is sufficient to prove that $\rho_{A\cup B}^\omega \geq \rho_B^\omega$. The proof is rather technical and involves an iterative choice of possibly different densities.

For simplicity of notation we will suppress ω and write $f = \rho_B$ and $h = \rho_{A\cup B}$ and $T = T_B$, $S = T_{A\cup B}$. We will show the claim by contradiction. Assume there is a set D of positive λ measure such that $f(x) > h(x)$ on D. Put $f_+ := (f - h)_+$ and $\mu_1 := T_*(f_+\lambda)$. Let $h_1 \leq h$ be such that $S_*(h_1\lambda) = \mu_1$, that is h_1 is a subdensity of h such that $T_*(f_+\lambda) = S_*(h_1\lambda)$ (for finding this density we can use disintegration as in the proof of Proposition 5.7).

If $1_{\{h_1>0\}}h > f$ on some set D_1 of positive λ measure, we are done. Indeed, as f is the unique Cost minimizing choice for the semicoupling between λ and $1_B\mu$ the transport $S_*(1_{D_1}h_1\lambda) =: \tilde{\mu}_1$ must be more expensive than the respective transport $T_*(1_{\tilde{D}_1} f_+\lambda) = \tilde{\mu}_1$ for some suitable set $\tilde{D}_1 \subset \{f > h\}$. Hence, $q_{A\cup B}$ cannot be minimizing and therefore not optimal, a contradiction.

If $1_{\{h_1>0\}}h \leq f$ we can assume wlog that $T_*(h_1\lambda) = \mu_2$ and μ_1 are singular to each other. Indeed, if they are not singular we can choose a different h_1 because $1_B\mu$ has to get its mass from somewhere. To be more precise, if $\tilde{h} \leq h_1$ is such that $T_*(\tilde{h}\lambda) \leq \mu_1$ we have $T_*((f_+ + \tilde{h})\lambda) > \mu_1$. Therefore, there must be some density h' such that $h' + h_1 \leq h$ and $S_*((h' + h_1)\lambda) = T_*((f_+ + \tilde{h})\lambda)$. Because, $f_+ > 0$ on some set of positive measure and $T_*(f\lambda) \leq S_*(h\lambda)$, there must be such an h_1 as claimed.

Take a density $h_2 \leq h$ such that $S_*(h_2\lambda) = \mu_2$. If $1_{\{h_2>0\}}h > f$ on some set D_2 of positive λ measure, we are done. Indeed, the optimality of q_B implies that the choice of f_+ and h_1 is cheaper than the choice of h_1 and h_2 for the transport into $\mu_1 + \mu_2$ (or maybe subdensities of these).

If $1_{\{h_2>0\}}h \leq f$ and $\{h_2 > 0\} \cap \{f_+ > 0\}$ has positive λ measure, we get a contradiction of optimality of $q_{A\cup B}$ by cyclical monotonicity. Otherwise, we can again assume that $T_*(h_2\lambda) =: \mu_3$ and μ_2 are singular to each other. Hence, we can take a density $h_3 \leq h$ such that $S_*(h_3\lambda) = \mu_3$.

Proceeding in this manner, because $f_+\lambda(M) = h_i\lambda(M) > 0$ for all i and by the finiteness of $q_B(M, M)$ one of the following two alternatives must happen

- there is j such that $1_{\{h_j>0\}}h > f$ on some set of positive λ measure,
- there are $j \neq i$ such that $\{h_j > 0\} \cap \{h_i > 0\}$ on some set of positive λ measure with $f_+ = h_0$.

Both cases lead to a contradiction by using the optimality of q_B, either by producing a cheaper semicoupling (in the first case) or by arguing via cyclical monotonicity (in the second case).

(ii) Fix ω, g and r. Denote the density of the first marginal of $\tilde{Q}_g^l$ (cf. Corollary 6.4) by $\zeta_{g,l}^\omega$. It is a convex combination of $\rho_{h,l}^\omega$ with $h \in gF_l^{-1}$. For $h \in G$ with $gF_r \subset hF_{r+n}$ we have by part (i) that $\rho_{g,r}^\omega \leq \rho_{h,r+n}^\omega$. Therefore, the contribution of $\rho_{g,r}^\omega(x)$ to $\zeta_{g,r+n}^\omega(x)$ is at least $1/|F_{r+n}|$ times the cardinality of $\{h\colon gF_r \subset hF_{r+n}\}$. Then it holds that

$$\big|\{h\colon gF_r \subset hF_{r+n}\}\big| = \big|\{h\colon F_r \subset hF_{r+n}\}\big| = \left|\bigcap_{f\in F_r}\{fF_{r+n}^{-1}\}\right|.$$

Because

$$\left|\left(\bigcap_{f\in F_r}\{fF_{r+n}^{-1}\}\right) \triangle F_{r+n}^{-1}\right| \leq \sum_{f\in F_r}\big|\{fF_{r+n}^{-1}\} \triangle F_{r+n}^{-1}\big|,$$

by amenability, we can deduce that

$$\big|\{h\colon gF_r \subset hF_{r+n}\}\big|/|F_{r+n}| \to 1 \quad \text{as } n \to \infty.$$

Fix $\varepsilon > 0$. If $\rho^\omega_{g,r} > \varepsilon + \rho^\omega$ on some positive $(\lambda^\bullet \otimes \mathbb{P})$-set, we have that $\zeta^\omega_{g,r+n}(x) > \rho^\omega(x) + \varepsilon/2$ on some positive $(\lambda^\bullet \otimes \mathbb{P})$-set for all n such that $\frac{|\{h: gF_r \subset hF_{r+n}\}|}{|F_{n+r}|} \geq 1 - \varepsilon/2$, because $\rho^\omega_{g,r} \leq 1$ and thus $\rho^\omega \leq 1 - \varepsilon$. Denote this set by A, so $A \subset M \times \Omega$. Then, we have $\tilde{Q}^{r+n}_g(A \times M) > Q^\infty(A \times M) + \varepsilon/2$ for all n big enough. However, this is a contradiction to the vague convergence of $\tilde{Q}^r_g$ to Q^∞ which was shown in Corollary 6.4.

(iii) The last part allows to interpret $\rho^\omega_{g,r}$ as a density of $(\rho^\omega\lambda^\omega)$ instead of as a density of λ^ω. We will adopt this point of view and show that $\rho^\omega_{g,r}$ converges to 1 $\mathbb{P}$ a.s. locally in λ^ω.

Assume that $\rho^\omega_{g,r}(x) \leq \gamma < 1$ for all $r \in \mathbb{N}$. Moreover, assume that there is $k \in G$ and $s \in \mathbb{N}$ such that $\rho^\omega_{k,s}(x) > \gamma$. By Lemma 3.7 there is a $t \in \mathbb{N}$ such that $gF_t \supset kF_s$. The first part of the lemma then implies that $\rho^\omega_{g,t}(x) \geq \rho^\omega_{k,s}(x) > \gamma$ which contradicts the assumption of $\rho^\omega_{g,r}(x) \leq \gamma$. Hence, if we have $\rho^\omega_{g,r}(x) \leq \gamma < 1$ for all $r \in \mathbb{N}$ on a set of positive $(\lambda^\bullet \otimes \mathbb{P})$ measure we must have $\rho^\omega_{k,s}(x) \leq \gamma$ for all $k \in G$ and $s \in \mathbb{N}$ on this set. Denote this set again by A, $A \subset M \times \Omega$. As $\zeta^\omega_{g,r}$ is a convex combination of the densities $\rho^\omega_{h,r}$ it must also be bounded away from 1 by γ on the set A. However, this is again a contradiction to the vague convergence of $\tilde{Q}^r_g$ to Q^∞ shown in Corollary 6.4. □

Lemma 6.9. *Let X, Y be locally compact separable spaces, θ a Radon measure on X and ρ a metric on Y compatible with the topology.*

(i) *For all $n \in \mathbb{N}$ let $T_n, T: X \to Y$ be Borel measurable maps. Put $Q_n(\mathrm{d}x, \mathrm{d}y) := \delta_{T_n(x)}(\mathrm{d}y)\theta(\mathrm{d}x)$ and $Q(\mathrm{d}x, \mathrm{d}y) := \delta_{T(x)}(\mathrm{d}y)\theta(\mathrm{d}x)$. Then,*

$$T_n \to T \quad \text{locally in measure on } X \quad \Longleftrightarrow \quad Q_n \to Q \quad \text{vaguely in } \mathcal{M}(X \times Y).$$

(ii) *More generally, let T and Q be as before whereas*

$$Q_n(\mathrm{d}x, \mathrm{d}y) := \int_{X'} \delta_{T_n(x,x')}(\mathrm{d}y)\theta'(\mathrm{d}x')\theta(\mathrm{d}x)$$

for some probability space $(X', \mathfrak{A}', \theta')$ and suitable measurable maps $T_n: X \times X' \to Y$. Then

$$Q_n \to Q \quad \text{vaguely in } \mathcal{M}(X \times Y) \quad \Longrightarrow \quad T_n(x, x') \to T(x) \quad \text{locally in measure on } X \times X'.$$

For a proof we refer to Section 4 of [26].

Proof Theorem 6.7. Firstly, we will show that the claim holds for ‘sufficiently many’ $g \in G$. We want to apply the previous lemma. Recall from Corollary 6.4 that

$$\tilde{Q}^r_g \to Q^\infty \quad \text{vaguely on } M \times M \times \Omega,$$

where

$$Q^\infty = \delta_T \cdot \rho\lambda^\bullet\mathbb{P} \quad \text{and} \quad \tilde{Q}^r_g = \frac{1}{|F_r|} \sum_{h \in gF_r^{-1}} \delta_{T_{h,r}} \cdot \rho_{h,r}\lambda^\bullet\mathbb{P},$$

with transport maps $T, T_{h,r}: M \times \Omega \to M \cup \{\eth\}$ and densities $\rho, \rho_{h,r}: M \times \Omega \to \mathbb{R}_+$. Lemma 6.8 allows us to interpret $\rho_{h,r}$ as density of the measure $\rho\lambda^\bullet$. Fix $k \in G$ and let θ'_r be the uniform measure on kF_r. Take $\theta = \rho\lambda^\bullet \otimes \mathbb{P}$, $X = M \times \Omega$ and $Y = M \cup \{\eth\}$. For any compact set $K \subset M \times \Omega$ and $\varepsilon > 0$, a slight variant of the second part of Lemma 6.9 implies

$$\lim_{r\to\infty} (\theta \otimes \theta'_r)\big(\big\{(x, \omega, h) \in K \times G\colon \rho^\omega_{h,r}(x) \cdot d\big(T_{h,r}(x, \omega), T(x, \omega)\big) \geq \varepsilon\big\}\big) = 0. \tag{6.3}$$

Let $H \subset G$ be those h for which

$$\lim_{r\to\infty} \theta\big(\big\{(x, \omega) \in K\colon \rho^\omega_{h,r}(x) d\big(T_{h,r}(x, \omega), T(x, \omega)\big) \geq \varepsilon\big\}\big) > 0.$$

Because we know that (6.3) holds, we must have $\lim_{r\to\infty}\theta'_r(H)=0$. Hence, there are countably many $g\in G$ such that

$$\lim_{r\to\infty}\theta\big(\big\{(x,\omega)\in K\colon\ d\big(T_{g,r}(x,\omega),T(x,\omega)\big)\geq\varepsilon\big\}\big)=0,$$

where we used that $\rho^{\omega}_{g,r}\nearrow 1$ $\mathbb{P}$ a.s. locally in λ^{ω}, according to Lemma 6.8. This shows that the theorem holds for those g.

Pick one such $g\in G$. Then the first part of the previous lemma implies

$$Q_{gB_r}\to Q^{\infty}\quad\text{vaguely on } M\times M\times\Omega.$$

This in turn implies that for any $h\in G$ we have $(\tau_h)_*Q_{gB_r}\to(\tau_h)_*Q^{\infty}\overset{(d)}{=}Q^{\infty}$ by invariance of Q^{∞}. Moreover, by Corollary 4.8 we have $(\tau_h)_*Q_{gB_r}\overset{(d)}{=}Q_{hgB_r}$. This means, that for any $h\in G$ we have

$$Q_{hgB_r}\to Q^{\infty}\quad\text{vaguely on } M\times M\times\Omega.$$

Applying once more the first part of the previous lemma proves the theorem. □

Corollary 6.10. *There is a measurable map $\Psi:\mathcal{M}(M)\times\mathcal{M}(M)\to\mathcal{M}(M\times M)$ s.t. $q^{\omega}:=\Psi(\lambda^{\omega},\mu^{\omega})$ denotes the unique optimal semicoupling between λ^{ω} and μ^{ω}. In particular the optimal semicoupling is a factor.*

Proof. We showed that the optimal semicoupling Q^{∞} can be constructed as the unique limit point of a sequence of deterministic functions of $\lambda^{\bullet}$ and $\mu^{\bullet}$. Hence, the map $\omega\mapsto q^{\omega}$ is measurable with respect to the sigma algebra generated by $\lambda^{\bullet}$ and $\mu^{\bullet}$. Thus, there is a measurable map Ψ such that $q^{\bullet}=\Psi(\lambda^{\bullet},\mu^{\bullet})$. □

6.2.1. *Semicouplings of $\lambda^{\bullet}$ and a point process*

If $\mu^{\bullet}$ is known to be a point process the above convergence result can be significantly improved. Just as in Theorem 4.8 and Corollary 4.9 of [26] we get

Theorem 6.11. *For any $g\in G$ and every bounded Borel set $A\subset M$*

$$\lim_{r\to\infty}\big(\lambda^{\bullet}\otimes\mathbb{P}\big)\big(\big\{(x,\omega)\in A\times\Omega\colon\ T_{g,r}(x,\omega)\neq T(x,\omega)\big\}\big)=0.$$

Corollary 6.12. *There exists a subsequence $(r_l)_l$ such that*

$$T_{g,r_l}(x,\omega)\to T(x,\omega)\quad\textit{as } l\to\infty$$

for almost every $x\in M$, $\omega\in\Omega$ and every $g\in G$. Indeed, the sequence $(T_{g,r_l})_l$ is finally stationary. That is, there exists a random variable $l_g:M\times\Omega\to\mathbb{N}$ such that almost surely

$$T_{g,r_l}(x,\omega)=T(x,\omega)\quad\textit{for all } l\geq l_g(x,\omega).$$

7. The other semicouplings

In the previous sections we considered semicouplings between two equivariant random measures $\lambda^{\bullet}$ and $\mu^{\bullet}$ with intensities 1 and $\beta\leq 1$ respectively. In this section we remark on the case that $\mu^{\bullet}$ has intensity $1<\beta\leq\infty$. Then, $q^{\bullet}$ is a semicoupling between $\lambda^{\bullet}$ and $\mu^{\bullet}$ iff for all $\omega\in\Omega$

$$(\pi_1)_*q^{\omega}=\lambda^{\omega}\quad\text{and}\quad(\pi_2)_*q^{\omega}\leq\mu^{\omega}.$$

This will complete the picture of semicouplings with one marginal being absolutely continuous. In the case $\beta<\infty$ we will only prove the key technical lemma, existence and uniqueness of optimal semicouplings on bounded sets. From that result one can deduce following the reasoning of the previous sections the respective results on existence and uniqueness for optimal semicouplings. We will not give the proofs because they are completely the same or become easier as we do not have to worry about densities.

Lemma 7.1. *Let $\rho \in L^1(M,m)$ be a nonnegative density. Let μ be an arbitrary (not necessarily Radon) measure on M with $\mu(M) \ge (\rho \cdot m)(M)$. Then, there is a unique semicoupling q between $(\rho \cdot m)$ and μ minimizing* $\mathsf{Cost}(\cdot)$. *Moreover, $q = (id, T)_*(\rho \cdot m)$ for some measurable cyclically monotone map T.*

Proof. The existence of one Cost minimizing semicoupling q goes along the same lines as for example in Proposition 4.2. Let q_1 be one such minimizer. As q_1 is minimizing it has to be an optimal coupling between its marginals. Therefore, it is induced by a map, that is $q_1 = (id, T_1)_*(\rho \cdot m)$. Let $q_2 = (id, T_2)_*(\rho \cdot m)$ be another minimizer. Then, $q_3 = \frac{1}{2}(q_1 + q_2)$ is minimizing as well. Hence, $q_3 = (id, T_3)_*(\rho \cdot m)$. However, just as in the proof of Theorem 5.9 this implies $T_1 = T_2$ (ρm) almost everywhere and therefore $q_1 = q_2$. □

In the case $\beta = \infty$, similar results can be shown if we take $\mu = \infty \cdot \tilde{\mu}$ for a simple point process $\tilde{\mu}$ with finite intensity. For instance, the optimal semicoupling between m and μ can be shown to be a Monge solution which corresponds to the Voronoi tessellation with respect to $\tilde{\mu}$.

8. Sufficient condition for $\mathfrak{c}_{e,\infty} < \infty$ and stability

In this section, we show a sufficient condition ensuring the finiteness of the optimal mean transportation cost between two equivariant random measures $\lambda^\bullet$ and $\mu^\bullet$. The idea is to reduce this question to the simpler transportation problem between m and $\lambda^\bullet$ (resp. $\mu^\bullet$) by the introduction of a suitable metric.

For two equivariant random measure $\lambda^\bullet, \mu^\bullet$ with intensity one and $c(x,y) = d^p(x,y)$ with $p \in [1,\infty)$ write

$$\mathbb{W}_p^p(\lambda^\bullet, \mu^\bullet) = \inf_{q^\bullet \in \Pi_s(\lambda^\bullet, \mu^\bullet)} \mathfrak{C}(q^\bullet) = \inf_{q^\bullet \in \Pi_s(\lambda^\bullet, \mu^\bullet)} \mathbb{E}\left[\int_{M \times B_0} d^p(x,y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right].$$

We want to establish a triangle inequality for $\mathbb{W}_p$ and therefore restrict to L^p cost functions. We could also extend this to more general cost functions by using Orlicz type norms as developed in [48]. However, to keep notations simple we stick to this case.

In this section, we will assume that all pairs of random measures considered will be equivariant and modeled on the same probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. As usual $\mathbb{P}$ is assumed to be stationary. Moreover, we will always assume without explicitly mentioning it that the mean transportation cost is finite.

Recall the disintegration Theorem 3.9. This will allow us to use the gluing lemma.

Proposition 8.1. *Let $\mu^\bullet, \lambda^\bullet, \xi^\bullet$ be three equivariant random measures of unit intensity.*

(i) $\mathbb{W}_p(\lambda^\bullet, \mu^\bullet) = 0 \Leftrightarrow \lambda^\omega = \mu^\omega$ $\mathbb{P}$*-a.s.*
(ii) $\mathbb{W}_p(\lambda^\bullet, \mu^\bullet) = \mathbb{W}_p(\mu^\bullet, \lambda^\bullet)$.
(iii) $\mathbb{W}_p(\lambda^\bullet, \mu^\bullet) \le \mathbb{W}_p(\lambda^\bullet, \xi^\bullet) + \mathbb{W}_p(\xi^\bullet, \mu^\bullet)$.

Proof. (i) $\mathbb{W}_p(\lambda^\bullet, \mu^\bullet) = 0$ iff there is a coupling of $\lambda^\bullet$ and $\mu^\bullet$ which is entirely concentrated on the diagonal almost surely, that is iff $\lambda^\omega = \mu^\omega$ $\mathbb{P}$-almost surely.

(ii) Let $q^\bullet$ be an optimal coupling between $\lambda^\bullet$ and $\mu^\bullet$. For $g, h \in G$ put

$$f(g,h) = \mathbb{E}\left[\int_{gB_0 \times hB_0} d^p(x,y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right].$$

By equivariance and stationarity, we have $f(g,h) = f(kg, kh)$ for all $k \in G$. Hence, we can apply the mass transport principle, Lemma 3.8.

$$\begin{aligned}\sum_{h \in G} f(g,h) &= \mathbb{E}\left[\int_{gB_0 \times M} d^p(x,y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right] \\ &= \sum_{g \in G} f(g,h) = \mathbb{E}\left[\int_{M \times hB_0} d^p(x,y) q^\bullet(\mathrm{d}x, \mathrm{d}y)\right].\end{aligned}$$

This proves the symmetry.

(iii) The random measures are random variables on some Polish space. Therefore, we can use the gluing lemma (cf. [16] or [51], Chapter 1) to construct an equivariant random measure $q^\bullet$ on $M \times M \times M$ such that

$$(\pi_{1,2})_* q^\bullet \in \Pi_{\mathrm{opt}}\big(\lambda^\bullet, \mu^\bullet\big) \quad \text{and} \quad (\pi_{2,3})_* q^\bullet \in \Pi_{\mathrm{opt}}\big(\mu^\bullet, \xi^\bullet\big),$$

where $\Pi_{\mathrm{opt}}(\lambda^\bullet, \mu^\bullet)$ denotes the set of all optimal couplings between $\lambda^\bullet$ and $\mu^\bullet$. $q^\bullet$ is equivariant as the optimal couplings are equivariant and $q^\bullet$ is glued together along the common marginal of these two couplings.

To be more precise let $q_1^\bullet \in \Pi_{\mathrm{opt}}(\lambda^\bullet, \mu^\bullet)$ and $q_2^\bullet \in \Pi_{\mathrm{opt}}(\mu^\bullet, \xi^\bullet)$. Then, consider $1_{M\times gB_0\times\Omega} q_1^\bullet$ and $1_{gB_0\times M\times\Omega} q_2^\bullet$ to produce with the usual gluing lemma a measure $q_g^\bullet$ on $M \times M \times M \times \Omega$ with the desired marginals on $M \times gB_0 \times M \times \Omega$. As all these sets are disjoint we can add up the different $q_g^\bullet$ yielding $q^\bullet = \sum_{g\in G} q_g^\bullet$ a measure with the desired properties.

For $g, h \in G$ put

$$e(g,h) = \mathbb{E}\bigg[\int_{M\times gB_0\times hB_0} d^p(x,z) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg].$$

By equivariance of $q^\bullet$, we have $e(kg, kh) = e(g,h)$ for all $k \in G$. By the mass transport principle, Lemma 3.8, this implies

$$\mathbb{E}\bigg[\int_{M\times B_0\times M} d^p(x,z) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg] = \mathbb{E}\bigg[\int_{M\times M\times B_0} d^p(x,z) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg].$$

Then we can conclude, using the Minkowski inequality

$$\begin{aligned}
\mathbb{W}_p\big(\lambda^\bullet, \xi^\bullet\big) &\le \mathbb{E}\bigg[\int_{M\times M\times B_0} d^p(x,z) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg]^{1/p} \\
&= \mathbb{E}\bigg[\int_{M\times B_0\times M} d^p(x,z) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg]^{1/p} \\
&\le \mathbb{E}\bigg[\int_{M\times B_0\times M} d^p(x,y) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg]^{1/p} \\
&\quad + \mathbb{E}\bigg[\int_{M\times B_0\times M} d^p(y,z) q^\bullet(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\bigg]^{1/p} \\
&= \mathbb{W}_p\big(\lambda^\bullet, \mu^\bullet\big) + \mathbb{W}_p\big(\mu^\bullet, \xi^\bullet\big).
\end{aligned}$$

In the last step we used the symmetry shown in part (ii). □

Remark 8.2. *Note that the first two properties also hold for general cost functions and general semicouplings. The assumption of equal intensity is not needed for these statements.*

Denote the set of all equivariant random measures $\mu^\bullet$ with unit intensity such that $\mathbb{W}_p(m, \mu^\bullet) < \infty$ by $\mathcal{P}_p$. To check that $\mathbb{W}_p(\lambda^\bullet, \mu^\bullet) < \infty$ it is sufficient to show that both measures lie in $\mathcal{P}_p$. Techniques implying $\lambda^\bullet \in \mathcal{P}_p$, based on moment estimates of the random variables $\lambda^\bullet(B_n)$, were developed in [26] for the case of the Poisson process. But they can be used in a much wider setting.

Finally, it might be easier to derive estimates for $\mathbb{W}_p(\mu_n^\bullet, m)$ for some approximating sequence $(\mu_n^\bullet)_{n\in\mathbb{N}}$ of $\mu^\bullet$. To this end, we need to understand the topology induced by $\mathbb{W}_p$.

Proposition 8.3. *Let $(\mu_n^\bullet)_{n\in\mathbb{N}}, \mu^\bullet \in \mathcal{P}_p$ be random measures of intensity one. Let $q_n^\bullet$ denote the optimal coupling between m and $\mu_n^\bullet$ and $q^\bullet$ the optimal coupling between m and $\mu^\bullet$. Consider the following statements.*

(i) $\mathbb{W}_p(\mu_n^\bullet, \mu^\bullet) \to 0$ *as* $n \to \infty$.

(ii) $\mu_n^\bullet \mathbb{P} \to \mu^\bullet \mathbb{P}$ *vaguely and* $\mathbb{W}_p(\mu_n^\bullet, m) \to \mathbb{W}_p(\mu^\bullet, m)$ *as* $n \to \infty$.
(iii) $q_n^\bullet \mathbb{P} \to q^\bullet \mathbb{P}$ *vaguely and* $\mathbb{W}_p(\mu_n^\bullet, m) \to \mathbb{W}_p(\mu^\bullet, m)$ *as* $n \to \infty$.
(iv) $q_n^\bullet \mathbb{P} \to q^\bullet \mathbb{P}$ *vaguely and*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \mathbb{E}\left[\int_{(\complement(B_0)_R)\times B_0} d^p(x,y) q_n^\bullet(\mathrm{d}x,\mathrm{d}y)\right] = 0,$$

where $(B_0)_R$ *denotes the* R*-neighbourhood of* B_0.

Then (i) *implies* (ii). (iii) *and* (iv) *are equivalent and either of them implies* (i).

Proof. (i) $\Rightarrow$ (ii): For any $f \in C_c(M \times \Omega)$ we have to show that $\lim_{n\to\infty} \mathbb{E}[\mu_n(f) - \mu(f)] = 0$. To this end, fix $f \in C_c(M \times \Omega)$ such that $\mathrm{supp}(f) \subset K \times \Omega$ for some compact set K. f is uniformly continuous. Let $\eta > 0$ be arbitrary and set $\varepsilon = \eta/(2m(K))$. Then, there is δ such that $d(x,y) \le \delta$ implies $d(f(x,\omega), f(y,\omega)) \le \varepsilon$. Put $A = \{(x,y)\colon d(x,y) \ge \delta\} \cap M \times K$ and denote by $\kappa_n^\bullet$ an optimal coupling between $\mu_n^\bullet$ and $\mu^\bullet$. By assumption, there is $N \in \mathbb{N}$ such that for all $n > N$ we have $\mathbb{W}_p^p(\mu_n^\bullet, \mu^\bullet) \le \frac{\eta\delta^p}{4\|f\|_\infty m(k)}$. Then, we can estimate for $n > N$

$$\begin{aligned}
\bigl|\mathbb{E}\bigl[\mu_n^\omega(f) - \mu^\omega(f)\bigr]\bigr| &\le \left|\mathbb{E}\left[\int_{M\times M} \bigl(f(x,\omega) - f(y,\omega)\bigr)\kappa_n^\omega(\mathrm{d}x,\mathrm{d}y)\right]\right| \\
&\le \varepsilon \cdot m(K) + \left|\mathbb{E}\left[\int_A \bigl(f(x,\omega) - f(y,\omega)\bigr)\kappa_n^\omega(\mathrm{d}x,\mathrm{d}y)\right]\right| \\
&\le \frac{\eta}{2} + 2\|f\|_\infty \mathbb{E}\bigl[\kappa_n^\bullet(A)\bigr] \\
&\le \frac{\eta}{2} + 2\|f\|_\infty \frac{1}{\delta^p} \mathbb{W}_p^p\bigl(\mu_n^\bullet, \mu^\bullet\bigr) \cdot m(K) \\
&\le \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\end{aligned}$$

The second assertion in (ii) is a direct consequence of the triangle inequality:

$$\mathbb{W}_p\bigl(\mu_n^\bullet, m\bigr) \le \mathbb{W}_p\bigl(\mu_n^\bullet, \mu^\bullet\bigr) + \mathbb{W}_p\bigl(\mu^\bullet, m\bigr)$$

and

$$\mathbb{W}_p\bigl(\mu^\bullet, m\bigr) \le \mathbb{W}_p\bigl(\mu_n^\bullet, \mu^\bullet\bigr) + \mathbb{W}_p\bigl(\mu_n^\bullet, m\bigr).$$

Taking limits yields the claim.

(iii) $\Leftrightarrow$ (iv): By the existence and uniqueness result we know that $q_n^\omega(\mathrm{d}x,\mathrm{d}y) = \delta_{T_n^\omega(x)}(\mathrm{d}y)m(\mathrm{d}x)$ and $q^\omega(\mathrm{d}x,\mathrm{d}y) = \delta_{T^\omega(x)}(\mathrm{d}y)m(\mathrm{d}x)$. In particular, we have that $\mu_n^\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega) = (T_n^\omega)_* m(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$. By Lemma 6.9 we know that the vague convergence of $q_n^\bullet \mathbb{P} \to q^\bullet \mathbb{P}$ implies that $T_n \to T$ locally in $m \otimes \mathbb{P}$ measure. This in turn implies the convergence of $f \circ (id, T_n) \to f \circ (id, T)$ in $m \otimes \mathbb{P}$ measure for any continuous and compactly supported function $f : M \times M \to \mathbb{R}$. Then, it follows that

$$\mathbb{E}\int f\bigl(x, T_n(x)\bigr)m(\mathrm{d}x) \to \mathbb{E}\int f\bigl(x, T(x)\bigr)m(\mathrm{d}x).$$

Let $c_k(x,y)$ be a continuous compactly supported function such that for any $(x,y) \in (B_0)_{k-1} \times B_0$ we have $d^p(x,y) = c_k(x,y)$, for any $x \in \complement(B_0)_k$ we have $c_k(x,y) = 0$ and $c_k(x,y) \le d^p(x,y)$ for all $(x,y) \in M \times M$. Then, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}\left[\int_{\complement((B_0)_R)\times B_0} d^p(x,y) q_n^\bullet(\mathrm{d}x,\mathrm{d}y)\right] \\
&\quad\le \limsup_{n\to\infty}\left(\mathbb{E}\left[\int_{M\times B_0} d^p(x,y) q_n^\bullet(\mathrm{d}x,\mathrm{d}y)\right] - \mathbb{E}\left[\int_{M\times B_0} c_R(x,y) q_n^\bullet(\mathrm{d}x,\mathrm{d}y)\right]\right)
\end{aligned}$$

$$= \mathbb{E}\left[\int_{M\times B_0} d^p(x,y)q^\bullet(\mathrm{d}x,\mathrm{d}y)\right] - \mathbb{E}\left[\int_{M\times B_0} c_R(x,y)q^\bullet(\mathrm{d}x,\mathrm{d}y)\right]$$
$$\le \mathbb{E}\left[\int_{\complement((B_0)_{R-1}\times B_0} d^p(x,y)q^\bullet(\mathrm{d}x,\mathrm{d}y)\right].$$

Taking the limit of $R\to\infty$ proves the implication (iii) $\Rightarrow$ (iv). The other direction is similar.

(iv) $\Rightarrow$ (i): We will show that $\mathbb{W}_p(\mu_n^\bullet,\mu^\bullet)\to 0$ by constructing a not optimal coupling between $\mu_n^\bullet$ and $\mu^\bullet$ whose transportation cost converges to zero. Let T_n, T be the transportation maps from the previous steps. Put $Q_n(\mathrm{d}x,\mathrm{d}y) := (T_n,T)_* m$. This is an equivariant coupling of $\mu_n^\bullet$ and $\mu^\bullet$ because the maps T_n, T are equivariant in the sense that (see also Example 3.10)

$$T^{\theta_g\omega}(x) = gT^\omega\bigl(g^{-1}x\bigr).$$

The transportation cost are given by

$$\mathfrak{C}(Q_n) = \mathbb{E}\left[\int_{B_0\times M} d^p(x,y)Q_n(\mathrm{d}x,\mathrm{d}y)\right] = \mathbb{E}\left[\int_{B_0} d^p\bigl(T_n(x),T(x)\bigr)m(\mathrm{d}x)\right].$$

We divide the integral into four parts. Put $A^R = \{x\colon d(T(x),x)\ge R\}$ and similarly $A_n^R = \{x\colon d(T_n(x),x)\ge R\}$. The four parts will be the integrals over $B_0\cap \complement^a A_n^R \cap \complement^b A^R$ with $a,b\in\{0,1\}$ and $\complement^0 A = A$. We estimate the different integrals separately.

$$\mathbb{E}\left[\int_{B_0\cap\complement A_n^R\cap\complement A^R} d^p\bigl(T_n(x),T(x)\bigr)m(\mathrm{d}x)\right]\to 0,$$

by a similar argument as in the previous step due to the convergence of $T_n\to T$ locally in $m\otimes\mathbb{P}$ measure and the boundedness of the integrand.

$$\mathbb{E}\left[\int_{B_0\cap A_n^R\cap A^R} d^p\bigl(T_n(x),T(x)\bigr)m(\mathrm{d}x)\right]$$
$$\le 2^p\mathbb{E}\left[\int_{B_0\cap A^R} d^p\bigl(x,T(x)\bigr)m(\mathrm{d}x)\right] + 2^p\mathbb{E}\left[\int_{B_0\cap A_n^R} d^p\bigl(x,T_n(x)\bigr)m(\mathrm{d}x)\right].$$

If $d(x,y)\le R$, $d(x,z)\ge R$ and $d(y,z)\le d(x,z)+R+a$ for some constant a ($=\mathrm{diam}(B_0)$), there is a constant C_1, e.g. $C_1 = 2+\mathrm{diam}(B_0)$, such that $d(y,z)\le C_1 d(x,z)$ (because $d(x,z)+R+a\le(2+a)d(x,z)$). This allows to estimate with $(x=x, T(x)=z, T_n(x)=y)$

$$\mathbb{E}\left[\int_{B_0\cap\complement A_n^R\cap A^R} d^p\bigl(T_n(x),T(x)\bigr)m(\mathrm{d}x)\right]\le C_1^p\mathbb{E}\left[\int_{B_0\cap A^R} d^p\bigl(x,T(x)\bigr)m(\mathrm{d}x)\right].$$

Similarly

$$\mathbb{E}\left[\int_{B_0\cap A_n^R\cap\complement A^R} d^p\bigl(T_n(x),T(x)\bigr)m(\mathrm{d}x)\right]\le C_1^p\mathbb{E}\left[\int_{B_0\cap A_n^R} d^p\bigl(x,T_n(x)\bigr)m(\mathrm{d}x)\right].$$

This finally gives

$$\limsup_{n\to\infty}\mathbb{E}\left[\int_{B_0\times M} d^p(x,y)Q_n(\mathrm{d}x,\mathrm{d}y)\right]$$
$$\le \lim_{R\to\infty}\limsup_{n\to\infty}\biggl(2^p\mathbb{E}\left[\int_{B_0\cap A^R} d^p\bigl(x,T(x)\bigr)m(\mathrm{d}x)\right]$$

$$
\begin{aligned}
&+2^p\mathbb{E}\biggl[\int_{B_0\cap A_n^R} d^p\bigl(x,T_n(x)\bigr)m(\mathrm{d}x)\biggr]+C_1^p\mathbb{E}\biggl[\int_{B_0\cap A^R} d^p\bigl(x,T(x)\bigr)m(\mathrm{d}x)\biggr]\\
&+C_1^p\mathbb{E}\biggl[\int_{B_0\cap A_n^R} d^p\bigl(x,T_n(x)\bigr)m(\mathrm{d}x)\biggr]\biggr)\\
=\,&0,
\end{aligned}
$$

by assumption. □

Remark 8.4. *For an equivalence of all statements we would need that* (ii) *implies* (iii). *In the classical theory this is precisely the stability result* (*Theorem* 5.20 *in* [51]). *This result is proven by using the characterization of optimal transports by cyclically monotone supports. However, as mentioned in the discussion on local optimality* (*see Remark* 5.4) *a cyclically monotone support is not sufficient for optimality in this case.*

We do not have real stability in general but we get at least close to it.

Proposition 8.5. *Let $(\lambda_n^\bullet)_{n\in\mathbb{N}}$ and $(\mu_n^\bullet)_{n\in\mathbb{N}}$ be two sequences of equivariant random measures. Let $q_n^\bullet$ be the unique optimal coupling between $\lambda_n^\bullet$ and $\mu_n^\bullet$. Assume that $\lambda_n^\bullet\mathbb{P}\to\lambda^\bullet\mathbb{P}$ vaguely, $\mu_n^\bullet\mathbb{P}\to\mu^\bullet\mathbb{P}$ vaguely and $\sup_n\mathfrak{C}(q_n^\bullet)\le c<\infty$. Then, there is an equivariant coupling $q^\bullet$ of $\lambda^\bullet$ and $\mu^\bullet$ and a subsequence $(q_{n_k}^\bullet)_{k\in\mathbb{N}}$ such that $q_{n_k}^\bullet\mathbb{P}\to q^\bullet\mathbb{P}$ vaguely, the support of $q^\bullet$ is cyclically monotone and*

$$\mathfrak{C}\bigl(q^\bullet\bigr)\le\liminf_{n\to\infty}\mathfrak{C}\bigl(q_n^\bullet\bigr).$$

In particular, if

$$\lim_{n\to\infty}\mathfrak{C}\bigl(q_n^\bullet\bigr)=\inf_{\tilde{q}^\bullet\in\Pi_{es}(\lambda^\bullet,\mu^\bullet)}\mathfrak{C}\bigl(\tilde{q}^\bullet\bigr)$$

then $q^\bullet$ is the/an optimal coupling between $\lambda^\bullet$ and $\mu^\bullet$ and $q_n^\bullet\mathbb{P}\to q^\bullet\mathbb{P}$ vaguely.

The proof is basically the same as for Proposition 3.18. Hence, we omit the details.

Remark 8.6. *The last proposition also holds if we consider semicouplings instead of couplings* (*see Proposition* 3.18).

Example 8.7 (Wiener mosaic). *Let $\mu_0^\bullet$ be a Poisson point process of intensity one on $\mathbb{R}^3$. Let each atom of μ_0 evolve according to independent Brownian motions for some time t. The resulting discrete random measure is again a Poisson point process, denoted by $\mu_t^\bullet$* (*e.g. see page* 404 *of* [15]). *Consider the transport problem between the Lebesgue measure $\mathcal{L}$ and $\mu_t^\bullet$ with cost function $c(x,y)=|x-y|^2$. Let $q_t^\bullet$ be the unique optimal coupling between $\mathcal{L}$ and $\mu_t^\bullet$. Then, $\mathfrak{C}(q_t^\bullet)=\mathbb{W}_2(\mathcal{L},\mu_t^\bullet)=\mathbb{W}_2(\mathcal{L},\mu_s^\bullet)$ for any $s\in\mathbb{R}$ as $\mu_s^\bullet$ and $\mu_t^\bullet$ are both Poisson point processes of intensity one. Moreover, we clearly have $\mu_s^\bullet\mathbb{P}\to\mu_t^\bullet\mathbb{P}$ vaguely as $s\to t$ and therefore $q_s^\bullet\mathbb{P}\to q_t^\bullet\mathbb{P}$ vaguely. By Lemma* 6.9, *this implies the convergence of the transport maps $T_s\to T_t$ locally in $\mathcal{L}\otimes\mathbb{P}$ measure. In particular, we get a continuously moving mosaic.*

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