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PROGRESSIVE ENLARGEMENTS OF FILTRATIONS WITH PSEUDO-HONEST TIMES[1]

BY LIBO LI AND MAREK RUTKOWSKI[2]

University of Sydney, and University of Sydney and Warsaw University of Technology

We deal with various alternative decompositions of $\mathbb{F}$-martingales with respect to the filtration $\mathbb{G}$, which represents the enlargement of a filtration $\mathbb{F}$ by a progressive flow of observations of a random time that either belongs to the class of pseudo-honest times or satisfies the extended density hypothesis. Several related results from the existing literature are revisited and essentially extended. Results on $\mathbb{G}$-semimartingale decompositions of $\mathbb{F}$-local martingales are crucial for applications in financial mathematics, most notably in the context of credit risk modeling and the study of insider trading where the enlarged filtration plays a vital role. We outline potential applications of our results to problems arising in financial mathematics.

1. Introduction. The goal of this work is to perform a thorough analysis of new classes of random times, which are defined using the properties of conditional distributions with respect to a reference filtration $\mathbb{F}$. Our main motivation comes from the area of financial modeling, where random times are used to describe additional information that arises, for instance, from observations of default events or an insider's knowledge about specific future market events, such as takeovers or mergers, and the associated jumps in assets prices. In the context of credit risk modeling, this paper falls within the so-called *reduced-form approach* (in its classical version also known as the *intensity-based* approach), in which one starts with a desirable form of the $\mathbb{F}$-conditional distribution and first constructs a random time that fits this distribution. The next step is to explore the properties of the market model with an enlarged information flow and to find out whether it is suitable for arbitrage pricing of credit-sensitive instruments. For the purpose of arbitrage pricing and hedging, it is necessary to examine the dynamics of $\mathbb{F}$-adapted processes (most crucially, local martingales) with respect to the progressively enlarged filtration, which is denoted as $\mathbb{G}$ in what follows. For this reason, an essential part of

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our work is devoted to the study of $\mathbb{G}$-semimartingale decompositions of $\mathbb{F}$-local martingales.

We work throughout on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$ satisfying the usual conditions, and we assume that τ is an $\mathbb{R}_+$-valued random time given on this space. For explicit constructions of random times with predetermined probabilistic properties, we refer to Jeanblanc and Song [21] and Li and Rutkowski [35]. Note that an extension of a probability space is typically needed to construct τ, so our notation $(\Omega, \mathcal{F}, \mathbb{P})$ corresponds to a possibly extended probability space. By an *enlargement of* $\mathbb{F}$ *associated with* τ is any filtration $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ satisfying the usual conditions and such that: (i) the inclusion $\mathbb{F} \subset \mathbb{K}$ holds (i.e., $\mathcal{F}_t \subset \mathcal{K}_t$ for all $t \in \mathbb{R}_+$) and (ii) τ is a $\mathbb{K}$-stopping time. Let us recall two particular enlargements, which were extensively studied in the literature; see, for example, Dellacherie and Meyer [9], Jacod [19], Jeanblanc and Le Cam [20], Jeulin [25–27], Jeulin and Yor [28–30] and Yor [44, 45]. Section 8 in the survey by Nikeghbali [38] provides a succinct overview of existing results, but mostly under either the postulate (**C**) that all $\mathbb{F}$-martingales are continuous and/or the postulate (**A**) that the random time τ avoids all $\mathbb{F}$-stopping times.

DEFINITION 1.1. The *initial enlargement* of $\mathbb{F}$ is the filtration $\mathbb{G}^* = (\mathcal{G}^*_t)_{t\geq 0}$ where the σ-field $\mathcal{G}^*_t$ given by the equality $\mathcal{G}^*_t = \bigcap_{s>t}(\sigma(\tau) \vee \mathcal{F}_s)$ for all $t \in \mathbb{R}_+$.

The initial enlargement does not seem to be well suited for a general analysis of properties of a random time with respect to a reference filtration $\mathbb{F}$, since it implies that $\sigma(\tau) \subset \mathcal{G}^*_0$, meaning that all the information about τ is already available at time 0 (although this feature can indeed be justified when dealing with some problems related to the strong form of insider trading). One can thus argue that the following notion of the *progressive enlargement* of $\mathbb{F}$ with a random time τ is more adequate for stating and solving problems in financial mathematics associated with the additional information conveyed by observations a random time τ.

DEFINITION 1.2. The *progressive enlargement* of $\mathbb{F}$ is the minimal enlargement, that is, the smallest filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, satisfying the usual conditions, such that $\mathbb{F} \subset \mathbb{G}$ and τ is a $\mathbb{G}$-stopping time. More explicitly, $\mathcal{G}_t = \bigcap_{s>t} \mathcal{G}^o_s$ where we denote $\mathcal{G}^o_t = \sigma(\tau \wedge t) \vee \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

Recall that for any two filtrations $\mathbb{F} \subset \mathbb{K}$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, we say that the *hypothesis* (H') holds for filtrations $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$ whenever any $(\mathbb{P}, \mathbb{F})$-semimartingale is also a $(\mathbb{P}, \mathbb{K})$-semimartingale; see, for example, Dellacherie and Meyer [9], Jeulin [26], Jeulin and Yor [29] or Yor [44].

The problems of checking whether the hypothesis (H') is satisfied and finding the canonical semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-special semimartingale with respect to a progressive enlargement $\mathbb{G}$ of a filtration $\mathbb{F}$ have attracted a considerable attention and were examined in several papers during the past thirty years. In particular, the following fundamental properties are worth to be recalled:

(i) a $(\mathbb{P}, \mathbb{F})$-semimartingale may fail to be a $(\mathbb{P}, \mathbb{G})$-semimartingale, in general;

(ii) any $(\mathbb{P}, \mathbb{F})$-special semimartingale stopped at τ is a $(\mathbb{P}, \mathbb{G})$-special semimartingale;

(iii) any $(\mathbb{P}, \mathbb{F})$-special semimartingale is a $(\mathbb{P}, \mathbb{G})$-special semimartingale when τ is an *honest time* with respect to a filtration $\mathbb{F}$; that is, the random variable τ is an end of some $\mathbb{F}$-optional set.

Furthermore, by the classic result due to Jacod [19], the hypothesis (H') holds for the initial enlargement of $\mathbb{F}$ provided that τ is an *initial time*. Recall that a random time is called *initial* with respect to a filtration $\mathbb{F}$ if there exists a measure η on $(\bar{\mathbb{R}}_+, \mathcal{B}(\bar{\mathbb{R}}_+))$ such that the $(\mathbb{P}, \mathbb{F})$-conditional distributions of τ are absolutely continuous with respect to η, that is, $F_{du,t} \ll \eta(du)$. It is common to say that the *density hypothesis* is satisfied when τ is an initial time with respect to $\mathbb{F}$.

The hypothesis (H') should be contrasted with the stronger *hypothesis* (H) for $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{P}$, which is also frequently referred to as the *immersion property* between $\mathbb{F}$ and $\mathbb{K}$. This hypothesis, which stipulates that any $(\mathbb{P}, \mathbb{F})$-local martingale is also a $(\mathbb{P}, \mathbb{K})$-local martingale, was first studied in the paper by Brémaud and Yor [4] and subsequently used in numerous papers, especially in the context of credit risk modeling. For obvious reasons, we are not in a position to discuss these papers here, although some results from them will be quoted or at least referred to in the sequel. Let us only mention that in the seminal paper by Jeulin and Yor [28] (see also Jeulin and Yor [29]), the authors derived the $(\mathbb{P}, \mathbb{G})$-semimartingale decomposition of the stopped process $U_{\tau\wedge t}$ for any random time τ and any $(\mathbb{P}, \mathbb{F})$-local martingale U. They also obtained the $(\mathbb{P}, \mathbb{G})$-semimartingale decomposition of an arbitrary $(\mathbb{P}, \mathbb{F})$-local martingale U under an additional assumption that τ is an honest time with respect to the filtration $\mathbb{F}$. Further results on successive progressive enlargement with honest times, which can be found in Jeulin [25], were used recently by Nikeghbali [39] in his study of semimartingale decomposition of pseudo-stopping times constructed from generalizations of William's example [42]. Results on the semimartingale decomposition after τ for nonhonest times were recently extended to the case of initial times by El Karoui et al. [11], Jeanblanc and Le Cam [20] and Kchia et al. [31].

The main hypotheses examined in the present work are the *hypothesis* (*HP*) and the *extended density hypothesis* [the *hypothesis* (*ED*), for short], as specified in Definitions 2.2 and 2.5. For convenience, the corresponding classes of random times are termed *pseudo-honest times* and *pseudo-initial times*, respectively. The hypothesis (*HP*) is clearly less restrictive than the hypothesis (H) and is known to hold, in particular, when a random time is constructed using the *multiplicative approach* (see Li and Rutkowski [35]), as well as for the alternative construction developed in Jeanblanc and Song [21]. It was also shown in [35] that, under mild technical assumption, the hypothesis (*HP*) is equivalent to the *separability* of the $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ; see Definition 2.4. The hypothesis (*ED*) extends the density hypothesis; it is introduced in order to avoid the awkward assumption on strict positivity of $(\mathbb{P}, \mathbb{F})$-conditional distribution of the random time τ. It is worth to point out that most results obtained for initial times can be extended to this new setting.

We will argue that honest times and random times satisfying the hypothesis (*HP*) share some common probabilistic features (that is why we propose the term *pseudo-honest times*), but they also differ in many respects. We would like to stress that although the hypothesis (*HP*) maybe not have deep probabilistic interpretation enjoyed by honest times, but it is more suitable for arbitrage-free pricing, as was shown recently by Imkeller [17], Zwierz [46] and Fontana et al. [12] that arbitrage opportunities exists after τ in a market endowed with a filtration progressively enlarged with an honest time. Therefore, one of the purposes of this paper is to show that under the hypothesis (*HP*) and some mild conditions, one can retain to some extent the distributional property and form of the $\mathbb{G}$-decomposition, while still have an arbitrage-free market with the progressively enlarged filtration.

The paper is organized as follows. In Section 2, we recall some basic properties of $(\mathbb{P}, \mathbb{F})$-conditional distributions of random times and enlarged filtrations. We also provide an alternative characterization of the progressive enlargement $\mathbb{G}$, which is used in Section 3 to compute conditional expectations of $\mathbb{G}$-adapted processes under (*HP*) and (*ED*) hypotheses. Subsequently, in Section 4, we provide sufficient conditions for a $\mathbb{G}$-adapted process to be a $(\mathbb{P}, \mathbb{G})$-martingale. Explicit computations of the $\mathbb{G}$-compensator [i.e., the $(\mathbb{P}, \mathbb{G})$-dual predictable projection] of the indicator process $H_t = \mathbb{1}_{\{\tau \le t\}}$ are provided in Section 5. Main results of this paper are established in Section 6 in which the validity of the hypothesis (H') is studied for the progressive enlargement of the underlying filtration $\mathbb{F}$ through either a pseudo-honest or a pseudo-initial random time. We extend there several related results from the existing literature.

In Theorem 6.2, we obtain a general semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-martingale with respect to the progressively enlarged filtration $\mathbb{G}$ when τ is assumed to be a pseudo-honest time. Particular examples of this decomposition are subsequently examined in Section 6.2, where we work under the assumption that a random time is constructed through the multiplicative approach developed in [35], that is, using either a predictable or an optional multiplicative system associated with a predetermined Azéma submartingale F. In Section 6.3, we deal with the corresponding results for pseudo-initial times. Finally, in Section 7, we discuss the financial interpretation of the hypothesis (*HP*), and we demonstrate that this class is suitable (under some mild conditions) for arbitrage-free pricing. The paper concludes with a brief study of the *information drift* associated with the utility maximization problem, which was previously studied by Ankirchner and Imkeller [1]. The proofs of some auxiliary results are omitted; they can be found in working paper [34].

2. Random times and filtrations.

In this section, we deal with the most pertinent properties of random times and the associated enlargements of a reference filtration $\mathbb{F}$. For more details, we refer to [35] where, in particular, various constructions of a random time with a predetermined Azéma supermartingale are examined. The interested reader may also consult papers by Jeanblanc and Song [21, 22] for closely related results.

2.1. *Properties of conditional distributions.*

Let us first introduce the notation for several characteristics of a finite random time τ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The $(\mathbb{P}, \mathbb{F})$-supermartingale $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ is commonly known as the *Azéma supermartingale* of τ. We will sometimes refer to the $(\mathbb{P}, \mathbb{F})$-submartingale $F = 1 - G$ as the *Azéma submartingale* of τ. The $(\mathbb{P}, \mathbb{F})$*-conditional distribution of* τ is the random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ given by

$$F_{u,t} = \mathbb{P}(\tau \le u|\mathcal{F}_t) \qquad \forall u, t \in \bar{\mathbb{R}}_+. \tag{2.1}$$

The following definition characterizes the class of all conditional distributions of a random time.

DEFINITION 2.1. A random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is said to be a $(\mathbb{P}, \mathbb{F})$*-conditional distribution* if it satisfies:

(i) for every $u \in \bar{\mathbb{R}}_+$ and $t \in \bar{\mathbb{R}}_+$, we have $0 \le F_{u,t} \le 1$, $\mathbb{P}$-a.s.;
(ii) for every $u \in \bar{\mathbb{R}}_+$, the process $(F_{u,t})_{t\in\bar{\mathbb{R}}_+}$ is a $(\mathbb{P}, \mathbb{F})$-martingale;
(iii) for every $t \in \bar{\mathbb{R}}_+$, the raw (i.e., nonadapted) process $(F_{u,t})_{u\in\bar{\mathbb{R}}_+}$ is right-continuous and increasing, with $F_{\infty,t} = 1$.

Note that for every $u \in \bar{\mathbb{R}}_+$, conditions (i)–(ii) in Definition 2.1 imply that $F_{u,\infty} = \lim_{t\to\infty} F_{u,t}$ and $F_{u,t} = \mathbb{E}_{\mathbb{P}}(F_{u,\infty}|\mathcal{F}_t)$ for every $t \in \bar{\mathbb{R}}_+$. Since (iii) yields $F_{u,t} \le F_{s,t}$ for all $u \le s$, the raw process $(F_{u,\infty})_{u\in\bar{\mathbb{R}}_+}$ is increasing and thus it admits a càdlàg version. It is known that for any random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ there exists a random time τ on some suitable extension of the space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that (2.1) holds.

Let us examine some properties of conditional distributions of random times. Throughout this section, by a $(\mathbb{P}, \mathbb{F})$-conditional distribution, we mean any random field $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ satisfying Definition 2.1. We first recall the classic hypothesis (H), which was studied in numerous papers (see, e.g., Brémaud and Yor [4] or Elliott et al. [10]), and its generalization termed the hypothesis (HP) [it is obvious that the hypothesis (H) implies (HP)]. In the rest of the paper, we assume (when possible) all random functions and processes are taken to be their regularized version.

DEFINITION 2.2. A $(\mathbb{P}, \mathbb{F})$-conditional distribution $(F_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ is said to satisfy:

(i) the *hypothesis* (H) whenever for all $0 \le u \le s < t$

$$F_{u,s} = F_{u,t}; \tag{2.2}$$

(ii) the *hypothesis* (*HP*) whenever for all $0 \le u < s < t$

$$F_{u,s} F_{s,t} = F_{s,s} F_{u,t}. \tag{2.3}$$

It was shown in [35] that any honest time satisfies the hypothesis (*HP*), and in fact, an $\mathcal{F}_\infty$-measurable random time τ is an honest time if and only if it satisfies the hypothesis (*HP*). This motivates us to say that a random time is a *pseudo-honest time* with respect to $\mathbb{F}$ whenever the $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ satisfies the hypothesis (*HP*).

REMARK 2.1. Let us observe that if $F_{u,t}$ satisfies the hypothesis (*HP*), then for all $0 \le u \le s \le t$,

$$\frac{F_{u,s}}{F_{s,s}} F_{s,t} = F_{u,t}. \tag{2.4}$$

Note that the inclusion $\{F_{s,s} = 0\} \subset \{F_{u,s} = 0\}$ is valid for all $0 \le u \le s$ and, by convention, $0/0 = 0$. More generally, the inclusion $\{F_{u,s} = 0\} \subset \{F_{u,t} = 0\}$ is known to hold for all $u \le s \le t$; see the proof of Lemma 2.2 in [35].

Let us recall the concept of the *complete separability* of a $(\mathbb{P}, \mathbb{F})$-conditional distribution; see [21, 35].

DEFINITION 2.3. We say that a $(\mathbb{P}, \mathbb{F})$-conditional distribution $(F_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$ is *completely separable* if there exists a positive, $\mathbb{F}$-adapted, increasing process K and a positive $(\mathbb{P}, \mathbb{F})$-martingale L such that $F_{u,t} = K_u L_t$ for every $u, t \in \mathbb{R}_+$ such that $0 \le u \le t$.

It is easily seen that the complete separability of $F_{u,t}$ implies that the hypothesis (*HP*) holds. Indeed, we have that $F_{u,s} F_{s,t} = (K_u L_s)(K_s L_t) = (K_s L_s) \times (K_u L_t) = F_{s,s} F_{u,t}$ for all $0 \le u < s < t$. In this setup, the Doob–Meyer decomposition of F can be easily calculated in terms of processes K and L; for more details, see Section 5.1. However, it appears, that the property of complete separability is too restrictive, since it does not cover all cases of our interest. This motivates the weaker concept of *separability* (note that it is termed the *partial separability* in [21]).

DEFINITION 2.4. We say that a $(\mathbb{P}, \mathbb{F})$-conditional distribution $(F_{u,t})_{u,t \in \bar{\mathbb{R}}_+}$ is *separable at* $v \ge 0$ if there exist a positive $(\mathbb{P}, \mathbb{F})$-martingale $(L^v_t)_{t \in \mathbb{R}_+}$ and a positive, $\mathbb{F}$-adapted, increasing process $(K^v_u)_{u \in [v,\infty)}$ such that the equality $F_{u,t} = K^v_u L^v_t$ holds for every $v \le u \le t$. A $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$ is called *separable* whenever it is separable at all $v > 0$.

The next definition gives a natural extension of the *density hypothesis*, which was introduced by Jacod [19] and subsequently studied by numerous authors; see, for example, [11, 20]. Since random times satisfying the density hypothesis are called *initial times*, we find it natural to say that a random time is a *pseudo-initial time* whenever it satisfies Definition 2.5. From [19], the hypothesis (H') is known to hold for the initial (and thus the progressive) enlargement of a filtration $\mathbb{F}$ with an initial time.

DEFINITION 2.5. A $(\mathbb{P}, \mathbb{F})$-conditional distribution $F_{u,t}$ is said to satisfy the *extended density hypothesis* [or, briefly, the *hypothesis* (ED)] if there exists a random field $(m_{s,t})_{s\geq 0, t\geq s}$ and an $\mathbb{F}$-adapted, increasing process D with $D_{0-} = 0$ and such that, for all $0 \leq u \leq t$,

$$F_{u,t} = \int_{[0,u]} m_{s,t}\, dD_s \tag{2.5}$$

and, for every $s \in \mathbb{R}_+$, the process $(m_{s,t})_{t\geq s}$ is a positive $(\mathbb{P}, \mathbb{F})$-martingale.

Let us note that the complete separability is in fact a subcase of the extended density hypothesis. To see this, it suffices to simply take $D_s = K_s$ and $m_{s,t} = L_t$ for all $s \leq t$. We will now give a nontrivial example of a random time satisfying the extended density hypothesis.

EXAMPLE 2.1. Let X be a positive $\mathcal{F}_\infty$-measurable random variable and Λ a continuous, $\mathbb{F}$-adapted, increasing process. Assume that ξ is a random variable uniformly distributed on $[0, 1]$ and independent of $\mathcal{F}_\infty$. Let us define the random time τ by setting

$$\tau = \inf\{u \geq 0 : 1 - e^{-X\Lambda_u} > \xi\}.$$

Then we obtain, for all $u \leq t$,

$$\begin{aligned}
\mathbb{P}(\tau \leq u|\mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}(1 - e^{-X\Lambda_u}|\mathcal{F}_t) \\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^u Xe^{-X\Lambda_s}\, d\Lambda_s \Big| \mathcal{F}_t\right) \\
&= \int_0^u \mathbb{E}_{\mathbb{P}}(Xe^{-X\Lambda_s}|\mathcal{F}_t)\, d\Lambda_s = \int_0^u m_{s,t}\, dD_s,
\end{aligned}$$

where we define $m_{s,t} := \mathbb{E}_{\mathbb{P}}(Xe^{-X\Lambda_s}|\mathcal{F}_t)$ for all $s \leq t$ and we set $D = \Lambda$.

As one might guess, the results obtained under the extended density hypothesis are similar to those proven under the usual density hypothesis, that is, for the initial times. Nevertheless, it is convenient to introduce it here, since it will allow us to circumvent an awkward nondegeneracy condition of the $(\mathbb{P}, \mathbb{F})$-conditional distribution of a random time, which is needed, for instance, in the proof of Theorem 4.1. Moreover, it is worth noting that the extended density hypothesis is also satisfied when a pseudo-honest time is constructed through the multiplicative construction, as shown in Remark 2.1 in [34]; for a special case, see also Theorem 5.2 in Jeanblanc and Song [21]. Hence the study of pseudo-initial times is intimately linked to our main goal, which is to examine the properties of pseudo-honest times.

2.2. *Enlargements of filtrations.* We will now analyze the basic properties of various enlargements of $\mathbb{F}$ associated with a random time τ. When studying semimartingale decompositions of processes stopped at τ, it is common to use, at least implicitly, the following concept, which was formally introduced by Guo and Zeng [15].

DEFINITION 2.6. An enlargement $\mathbb{K}$ of a filtration $\mathbb{F}$ is said to be *admissible before* τ if the equality $\mathcal{K}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}$ holds for every $t \in \mathbb{R}_+$.

In the case of a general (i.e., not necessarily honest) random time, we find it convenient to introduce the following notion, stemming from a remark in Meyer [37]. Recall that the initial enlargement $\mathbb{G}^*$ was introduced in Definition 1.1.

DEFINITION 2.7. The family $\widehat{\mathbb{G}} = (\widehat{\mathcal{G}}_t)_{t\in\mathbb{R}_+}$ is defined by setting, for all $t \in \mathbb{R}_+$,

$$\widehat{\mathcal{G}}_t = \big\{A \in \mathcal{G} | \exists A_t \in \mathcal{F}_t,\, A_t^* \in \mathcal{G}_t^* \text{ s.t. } A = \big(A_t \cap \{\tau > t\}\big) \cup \big(A_t^* \cap \{\tau \le t\}\big)\big\}.$$

We note that, for all $t \in \mathbb{R}_+$,

$$\widehat{\mathcal{G}}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}, \qquad \widehat{\mathcal{G}}_t \cap \{\tau \le t\} = \mathcal{G}_t^* \cap \{\tau \le t\}. \tag{2.6}$$

It can be checked that the σ-field $\widehat{\mathcal{G}}_t$ is uniquely characterized by the conditions of (2.6). The next elementary result shows that the family $\widehat{\mathbb{G}}$ coincides in fact with the progressive enlargement $\mathbb{G}$, which was introduced in Definition 1.2; for the proof of the lemma, we refer to [34].

LEMMA 2.1. *For any random time τ the progressive enlargement $\mathbb{G}$ coincides with the filtration $\widehat{\mathbb{G}}$.*

It is easy to see that the filtration $\widehat{\mathbb{G}}$ is admissible before τ. When dealing with a semimartingale decomposition of an $\mathbb{F}$-martingale after τ, we will use the following definition.

DEFINITION 2.8. We say that an enlargement $\mathbb{K}$ is *admissible after* τ if the equality $\mathcal{K}_t \cap \{\tau \le t\} = \mathcal{G}_t^* \cap \{\tau \le t\}$ holds for every $t \in \mathbb{R}_+$.

It is clear that the filtration $\widehat{\mathbb{G}}$ (and thus also $\mathbb{G}$) is admissible after τ for any random time. Note also that if an enlargement $\mathbb{K}$ is admissible before and after τ then necessarily $\mathbb{K} = \mathbb{G}$. For the proof of the next elementary lemma, we refer to [34].

LEMMA 2.2. *For any integrable, $\mathcal{G}_\infty$-measurable random variable X and any enlargement $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ admissible after τ we have that, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}X|\mathcal{K}_t) = \lim_{s\downarrow t}\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\leq t\}}X|\sigma(\tau)\vee\mathcal{F}_s\big). \tag{2.7}$$

3. Conditional expectations under progressive enlargements. In the rest of the paper, we work under the assumption that the $(\mathbb{P}, \mathbb{F})$-conditional distribution of a random time τ satisfies either the hypothesis (*HP*) or the hypothesis (*ED*), which were introduced in Definitions 2.2 and 2.5, respectively. In addition, the special case of the complete separability will be examined as well. We will need the following auxiliary result, which ensures that the processes $\mathbb{1}_{\{\tau>t\}}(G_t)^{-1}$ and $\mathbb{1}_{\{\tau\leq t\}}(F_t)^{-1}$ are well-defined; for the proof of Lemma 3.1, see [34].

LEMMA 3.1. *The following inclusions are satisfied, for every $t \in \mathbb{R}_+$:*

(i) $\{\tau > t\} \subset \{G_t > 0\}$, $\mathbb{P}$-*a.s. and*
(ii) $\{\tau \leq t\} \subset \{F_t > 0\}$, $\mathbb{P}$-*a.s.*

REMARK 3.1. Let us set, by convention, $0/0 = 0$. Hence, by Lemma 3.1, the quantities $\mathbb{1}_{\{\tau>t\}}G_t^{-1}$ and $\mathbb{1}_{\{\tau\leq t\}}F_t^{-1}$ are well-defined for all t, $\mathbb{P}$-a.s.

3.1. *Conditional expectations for pseudo-honest times*. For a fixed $T > 0$, we consider the map $U_T : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, and we use the notation $(u, \omega) \mapsto U_{u,T}(\omega)$. We postulate that U_T is a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_T$-measurable map, so that $U_{\tau,T}$ is a $\sigma(\tau) \vee \mathcal{F}_T$-measurable random variable. The following result corresponds to Theorem 3.1 in El Karoui et al. [11], where the case of the density hypothesis was studied.

LEMMA 3.2. *Let $U_{\cdot,T} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_T$-measurable map. Assume that τ is a pseudo-honest time and the random variable $U_{\tau,T}$ is $\mathbb{P}$-integrable. Then*:

(i) *for every $t \in [0, T)$, we have that*

$$\mathbb{E}_{\mathbb{P}}(U_{\tau,T}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\widetilde{U}_{t,T} + \mathbb{1}_{\{\tau\leq t\}}\widehat{U}_{\tau,t,T},$$

where

$$\widetilde{U}_{t,T} = (G_t)^{-1}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}U_{\tau,T}|\mathcal{F}_t) = (G_t)^{-1}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(t,\infty]} U_{v,T}\,dF_{v,T}\bigg|\mathcal{F}_t\bigg) \tag{3.1}$$

and, for all $0 \leq u \leq t < T$,

$$\widehat{U}_{u,t,T} = (F_t)^{-1}\mathbb{E}_{\mathbb{P}}(F_{t,T}U_{u,T}|\mathcal{F}_t); \tag{3.2}$$

(ii) *if, in addition*, $F_{u,t}$ *is completely separable so that* $F_{u,t} = K_u L_t$ *for* $u \le t$, *then* (3.1) *yields*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T \ge \tau > t\}} U_{\tau,T} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\biggl(L_T \int_{(t,T]} U_{v,T} \, dK_v \Big| \mathcal{F}_t \biggr)$$

and (3.2) *becomes*

$$\widehat{U}_{u,t,T} = (L_t)^{-1} \mathbb{E}_{\mathbb{P}}(L_T U_{u,T} | \mathcal{F}_t).$$

PROOF. The derivation of (3.1) is rather standard. Note that the hypothesis (*HP*) is not needed here and we may take $t \in [0, T]$. It suffices to take $U_{u,T} = g(u)\mathbb{1}_A$ for a Borel measurable map $g : \mathbb{R}_+ \to \mathbb{R}$ and an event $A \in \mathcal{F}_T$ such that the random variable $U_{\tau,T} = g(\tau)\mathbb{1}_A$ is $\mathbb{P}$-integrable.

Using part (i) in Lemma 3.1 and the well known formula for the conditional expectation with respect to $\mathcal{G}_t$, we obtain

$$\begin{aligned}
\mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}(U_{\tau,T} | \mathcal{G}_t) &= \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} U_{\tau,T} | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)} \\
&= \mathbb{1}_{\{\tau > t\}} (G_t)^{-1} \mathbb{E}_{\mathbb{P}}\bigl(\mathbb{1}_A \mathbb{E}_{\mathbb{P}}\bigl(\mathbb{1}_{\{\tau > t\}} g(\tau) | \mathcal{F}_T\bigr) | \mathcal{F}_t\bigr) \\
&= \mathbb{1}_{\{\tau > t\}} (G_t)^{-1} \mathbb{E}_{\mathbb{P}}\biggl(\mathbb{1}_A \int_{(t,\infty]} g(v) \, dF_{v,T} \Big| \mathcal{F}_t\biggr) \\
&= \mathbb{1}_{\{\tau > t\}} (G_t)^{-1} \mathbb{E}_{\mathbb{P}}\biggl(\int_{(t,\infty]} g(v) \mathbb{1}_A \, dF_{v,T} \Big| \mathcal{F}_t\biggr) \\
&= \mathbb{1}_{\{\tau > t\}} (G_t)^{-1} \mathbb{E}_{\mathbb{P}}\biggl(\int_{(t,\infty]} U_{v,T} \, dF_{v,T} \Big| \mathcal{F}_t\biggr) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_{t,T},
\end{aligned}$$

where $\widetilde{U}_{t,T}$ is given by (3.1). Let us take any $t \in [0, T)$. To establish (3.2), we need to evaluate $\mathbb{E}_{\mathbb{P}}(U_{\tau,T} | \mathcal{G}_t)$ on the event $\{\tau \le t\}$. An application of Lemma 2.2 yields

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}} U_{\tau,T} | \mathcal{G}_t) = \lim_{s \downarrow t} \mathbb{E}_{\mathbb{P}}\bigl(\mathbb{1}_{\{\tau \le t\}} U_{\tau,T} | \sigma(\tau) \vee \mathcal{F}_s\bigr).$$

We first compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}} U_{\tau,T} | \sigma(\tau) \vee \mathcal{F}_s)$ for $0 \le t < s < T$. Recall that the hypothesis (*HP*) means that the equality $F_{u,s} F_{s,T} = F_{s,s} F_{u,T}$ holds for all $0 \le u < s < T$, which implies that $F_{s,T} \, dF_{u,s} = F_s \, dF_{u,T}$ for any fixed $s < T$ and all $u \in [0, t]$.

Hence, for any bounded, $\sigma(\tau) \vee \mathcal{F}_s$-measurable random variable $H_{\tau,s}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}} H_{\tau,s} U_{\tau,T}) &= \mathbb{E}_{\mathbb{P}}\bigl(\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}} H_{\tau,s} U_{\tau,T} | \mathcal{F}_T)\bigr) \\
&= \mathbb{E}_{\mathbb{P}}\biggl(\int_{[0,t]} H_{u,s} U_{u,T} \, dF_{u,T}\biggr) \\
&= \mathbb{E}_{\mathbb{P}}\biggl(\int_{[0,t]} H_{u,s} (F_s)^{-1} F_{s,T} U_{u,T} \, dF_{u,s}\biggr)
\end{aligned}$$

$$= \mathbb{E}_{\mathbb{P}}\left(\int_{[0,t]} H_{u,s}(F_s)^{-1}\mathbb{E}_{\mathbb{P}}(F_{s,T}U_{u,T}|\mathcal{F}_s)\,dF_{u,s}\right)$$

$$= \mathbb{E}_{\mathbb{P}}\left(\int_{[0,t]} H_{u,s}\widehat{U}_{u,s,T}\,dF_{u,s}\right) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}H_{\tau,s}\widehat{U}_{\tau,s,T})$$

since $\{\tau \le t\} \subset \{\tau \le s\} \subset \{F_s > 0\}$, $\mathbb{P}$-a.s.; see part (ii) in Lemma 3.1.

This in turn yields

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}|\mathcal{G}_t) &= \lim_{s\downarrow t}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}U_{\tau,T}|\sigma(\tau)\vee\mathcal{F}_s)\\
&= \lim_{s\downarrow t}\mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,s,T}\\
&= \lim_{s\downarrow t}\mathbb{1}_{\{\tau\le t\}}(F_s)^{-1}\mathbb{E}_{\mathbb{P}}(F_{s,T}U_{u,T}|\mathcal{F}_s)_{u=\tau}\\
&= \mathbb{1}_{\{\tau\le t\}}(F_t)^{-1}\mathbb{E}_{\mathbb{P}}(F_{t,T}U_{u,T}|\mathcal{F}_t)_{u=\tau} = \mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,t,T},
\end{aligned}$$

where the penultimate equality holds by the right-continuity of the filtration $\mathbb{F}$ and the right-continuity of processes F and $F_{\cdot,T}$. This completes the proof of part (i). For part (ii), we observe that if, in addition, the random field $F_{u,t}$ is completely separable, then the asserted formulae follow from equations (3.1) and (3.2). $\square$

3.2. *Conditional expectations for pseudo-initial times*. Under the extended density hypothesis, we establish the following counterpart of Lemma 3.2.

LEMMA 3.3. *Let $U_{\cdot,T} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_T$-measurable map. If τ is a pseudo-initial time, and the random variable $U_{\tau,T}$ is $\mathbb{P}$-integrable, then for every $t \in [0, T)$,*

$$\mathbb{E}_{\mathbb{P}}(U_{\tau,T}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\widetilde{U}_{t,T} + \mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,t,T},$$

where

$$\widetilde{U}_{t,T} = (G_t)^{-1}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}U_{\tau,T}|\mathcal{F}_t) = (G_t)^{-1}\mathbb{E}_{\mathbb{P}}\left(\int_{(t,\infty]} U_{v,T}\,dF_{v,T}\Big|\mathcal{F}_t\right)$$

and, for every $0 \le u \le t < T$,

$$\widehat{U}_{u,t,T} = (m_{u,t})^{-1}\mathbb{E}_{\mathbb{P}}(m_{u,T}U_{u,T}|\mathcal{F}_t). \tag{3.3}$$

PROOF. It suffices to revise the proof of Lemma 3.2 on the event $\{\tau \le t\}$. Let us first observe that, by Definition 2.5, for every $u \in \mathbb{R}_+$, the process $(m_{u,t})_{t\ge u}$ is a positive $(\mathbb{P}, \mathbb{F})$-martingale, and thus we have that, for all $u \le t \le s \le T$,

$$\{m_{u,t} = 0\} \subset \{m_{u,s} = 0\} \subset \{m_{u,T} = 0\}.$$

Recall also that, by convention, we set $0/0 = 0$, and thus $\widehat{U}_{u,s,T}$ is well defined.

Therefore, for all $t \le s \le T$ and any bounded, $\sigma(\tau) \vee \mathcal{F}_s$-measurable random variable $H_{\tau,s}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}} H_{\tau,s} U_{\tau,T}) &= \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}} H_{\tau,s} U_{\tau,T} | \mathcal{F}_T)\big) \\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,\infty]} \mathbb{1}_{\{u\le t\}} H_{u,s} U_{u,T}\, dF_{u,T}\bigg) \\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]} H_{u,s} m_{u,T} U_{u,T}\, dD_u\bigg) \\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]} H_{u,s} (m_{u,s})^{-1} \mathbb{E}_{\mathbb{P}}(m_{u,T} U_{u,T} | \mathcal{F}_s) m_{u,s}\, dD_u\bigg) \\
&= \mathbb{E}_{\mathbb{P}}\bigg(\int_{[0,t]} H_{u,s} \widehat{U}_{u,s,T}\, dF_{u,s}\bigg) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}} H_{\tau,s} \widehat{U}_{\tau,s,T}).
\end{aligned}$$

By passing to the limit and using similar arguments as in the proof of Lemma 3.2, we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}} U_{\tau,T} | \mathcal{G}_t) &= \lim_{s\downarrow t} \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\le t\}} U_{\tau,T} | \sigma(\tau) \vee \mathcal{F}_s\big) = \lim_{s\downarrow t} \mathbb{1}_{\{\tau\le t\}} \widehat{U}_{\tau,s,T} \\
&= \lim_{s\downarrow t} \mathbb{1}_{\{\tau\le t\}} (m_{u,s})^{-1} \mathbb{E}_{\mathbb{P}}(m_{u,T} U_{u,T} | \mathcal{F}_s)_{u=\tau} \\
&= \mathbb{1}_{\{\tau\le t\}} (m_{u,t})^{-1} \mathbb{E}_{\mathbb{P}}(m_{u,T} U_{u,T} | \mathcal{F}_t)_{u=\tau} = \mathbb{1}_{\{\tau\le t\}} \widehat{U}_{\tau,t,T},
\end{aligned}$$

which establishes (3.3) and thus completes the proof of the lemma. □

4. Properties of $\mathbb{G}$-local martingales. Let us consider a map $\widehat{U}: \mathbb{R}_+^2 \times \Omega \to \mathbb{R}$ where we use the notation $(u,t,\omega) \mapsto \widehat{U}_{u,t}(\omega)$. We say that $\widehat{U}$ is an $\mathbb{F}$-*optional map* when it is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{O}(\mathbb{F})$-measurable, where $\mathcal{O}(\mathbb{F})$ is the $\mathbb{F}$-optional σ-field in $\mathbb{R}_+ \times \Omega$. In that case, the map $\widehat{U}_{\cdot,t}$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_t$-measurable and the process $(\widehat{U}_{t,t})_{t\ge 0}$ is $\mathbb{F}$-optional, in the usual sense. We will sometimes need an additional assumption that the process $(\widehat{U}_{t,t})_{t\ge 0}$ is $\mathbb{F}$-predictable.

4.1. $\mathbb{G}$-*local martingales for pseudo-honest times*. Consider an arbitrary random time τ such that the process $G = 1 - F$ is the Azéma supermartingale of τ. We denote by $G = M - A$ the Doob–Meyer decomposition of G. Then the dual $(\mathbb{P}, \mathbb{F})$-predictable projection of the indicator process $H_t = \mathbb{1}_{\{\tau\le t\}}$ satisfies $H^p = A$.

The following result, which corresponds to Propositions 5.1 and 5.6 in El Karoui et al. [11], is an important step toward establishing a $(\mathbb{P}, \mathbb{G})$-semimartingale decomposition of a $(\mathbb{P}, \mathbb{F})$-local martingale.

LEMMA 4.1. *The process* $(C_{s,u} = (F_u)^{-1} F_{s,u})_{u \geq s}$ *is positive, decreasing and* $\mathbb{F}$*-adapted.*

PROOF. Indeed, from (2.3) we obtain the equality

$$C_{s,u} = (F_u)^{-1} F_{s,u} = (F_{u,t})^{-1} F_{s,t},$$

which holds for all $0 \leq s < u < t$, where $(F_{u,t})_{u \geq 0}$ is an increasing. □

THEOREM 4.1. *Assume that* τ *is a pseudo-honest time and* $0 < F_{u,t} \leq 1$ *for every* $0 < u \leq t$. *Let* U^* *be a* $\mathbb{G}$*-adapted and* $\mathbb{P}$*-integrable process given by the following expression*:

$$U_t^* = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t + \mathbb{1}_{\{\tau \leq t\}} \widehat{U}_{\tau,t}, \tag{4.1}$$

where $\widetilde{U}$ *is an* $\mathbb{F}$*-adapted*, $\mathbb{P}$*-integrable process, and* $\widehat{U}$ *is an* $\mathbb{F}$*-optional map such that for every* $t \in \mathbb{R}_+$ *the random variable* $\widehat{U}_{\tau,t}$ *is* $\mathbb{P}$*-integrable and the process* $(\widehat{U}_t := \widehat{U}_{t,t})_{t \geq 0}$ *is* $\mathbb{F}$*-predictable. Assume, in addition, that the following conditions are satisfied*:

(i) *the process* $(W_t)_{t \geq 0}$ *is a* $(\mathbb{P}, \mathbb{F})$*-local martingale where*

$$W_t = \widetilde{U}_t G_t + \int_{(0,t]} \widehat{U}_v \, dF_v; \tag{4.2}$$

(ii) *for any fixed* $u, s \geq 0$, *the process* $(F_{s,t} \widehat{U}^0_{u,t})_{t \geq u \vee s}$ *is a* $(\mathbb{P}, \mathbb{F})$*-local martingale where we denote* $\widehat{U}^0_{u,t} = \widehat{U}_{u,t} - \widehat{U}_{u,u}$ *for every* $0 \leq u \leq t$.

Then the process $(U_t^*)_{t \geq 0}$ *is a* $(\mathbb{P}, \mathbb{G})$*-local martingale.*

PROOF. Since the proof proceeds along the similar lines as the proofs of Propositions 5.1 and 5.6 in El Karoui et al. [11], we will focus on computations, and for the details regarding suitable localization and measurability arguments we refer to [11]. We start by noting that the following decomposition is valid:

$$\begin{aligned} U_t^* &= U_t^* \mathbb{1}_{\{\tau > t\}} + U_\tau^* \mathbb{1}_{\{\tau \leq t\}} + \bigl(U_t^* - U_\tau^*\bigr) \mathbb{1}_{\{\tau \leq t\}} \\ &= \widetilde{U}_t \mathbb{1}_{\{\tau > t\}} + \widehat{U}_{\tau,\tau} \mathbb{1}_{\{\tau \leq t\}} + (\widehat{U}_{\tau,t} - \widehat{U}_{\tau,\tau}) \mathbb{1}_{\{\tau \leq t\}}. \end{aligned}$$

It is thus enough to examine the following two subcases, corresponding to conditions (i) and (ii), respectively:

(a) the case of a process U^* stopped at τ;

(b) the case of a process U^* such that $U^*_{\tau \wedge t} = 0$ for all $t \geq 0$.

Case (a). We first assume that a $\mathbb{G}$-adapted process U^* is stopped at τ, specifically,

$$U_t^* = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t + \mathbb{1}_{\{\tau \leq t\}} \widehat{U}_\tau, \tag{4.3}$$

where $(\widetilde{U}_t)_{t\geq 0}$ is an $\mathbb{F}$-adapted process, and $(\widehat{U}_t := \widehat{U}_{t,t})_{t\geq 0}$ is an $\mathbb{F}$-predictable process. We start by observing that, for every $0 \leq s < t$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}\widehat{U}_\tau|\mathcal{G}_s) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{s<\tau\leq t\}}\widehat{U}_\tau|\mathcal{G}_s) + \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau|\mathcal{G}_s) \\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\widehat{U}_v\,dH_v\Big|\mathcal{F}_s\biggr) + \mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau \\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\widehat{U}_v\,dA_v\Big|\mathcal{F}_s\biggr) + \mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau \\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\widehat{U}_v\,dF_v\Big|\mathcal{F}_s\biggr) + \mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau.
\end{aligned}$$

Hence, for every $0 \leq s < t$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(U^*_t|\mathcal{G}_s) &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}(\widetilde{U}_t G_t|\mathcal{F}_s) \\
&\quad + \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\widehat{U}_v\,dF_v\Big|\mathcal{F}_s\biggr) + \mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau \\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}(W_t - W_s|\mathcal{F}_s) \\
&\quad + \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}(\widetilde{U}_s G_s|\mathcal{F}_s) + \mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau \\
&= \mathbb{1}_{\{\tau>s\}}\widetilde{U}_s + \mathbb{1}_{\{\tau\leq s\}}\widehat{U}_\tau = U^*_s,
\end{aligned}$$

where we used the assumption that the process W given by (4.2) is a $(\mathbb{P},\mathbb{F})$-martingale. We conclude that the process U^* given by (4.3) is a $(\mathbb{P},\mathbb{G})$-martingale.

Case (b). Let us denote $\widehat{U}^0_{v,t} = \widehat{U}_{v,t} - \widehat{U}_{v,v}$ for $0 \leq v \leq t$. Consider a $\mathbb{G}$-adapted process U^* given by $U^*_t = \mathbb{1}_{\{\tau\leq t\}}\widehat{U}^0_{\tau,t}$ where $\widehat{U}^0_{t,t} = 0$. We need to show that the equality $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}\widehat{U}^0_{\tau,t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau\leq s\}}\widehat{U}^0_{\tau,s}$ holds for every $0 \leq s < t$. From part (i) in Lemma 3.2, we obtain, for every $0 \leq s < t$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\leq t\}}\widehat{U}^0_{\tau,t}|\mathcal{G}_s) &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\widehat{U}^0_{v,t}\,dF_{v,t}\Big|\mathcal{F}_s\biggr) \\
&\quad + \mathbb{1}_{\{\tau\leq s\}}(F_s)^{-1}\mathbb{E}_{\mathbb{P}}(F_{s,t}\widehat{U}^0_{u,t}|\mathcal{F}_s)_{|u=\tau} \\
&= I_1 + I_2.
\end{aligned}$$

Let us examine I_1. We first assume that $s > 0$. Recall that we assume that the hypothesis (*HP*) holds and $0 < F_{v,t} \leq 1$ for every $0 < v \leq t$.

Hence, for $0 < s \leq v \leq t$, we can write $dF_{v,t} = F_{s,t}\,d(F_{v,v}F^{-1}_{s,v}) = F_{s,t}\,dD_{s,v}$ where, by Lemma 4.1, the process $(D_{s,v} = F_{v,v}F^{-1}_{s,v})_{v\geq s}$ is increasing and $\mathbb{F}$-adapted. Consequently,

$$\begin{aligned}
I_1 &= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\mathbb{E}_{\mathbb{P}}(F_{s,t}\widehat{U}^0_{v,t}|\mathcal{F}_v)\,dD_{s,v}\Big|\mathcal{F}_s\biggr) \\
&= \mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}F_{s,v}\widehat{U}^0_{v,v}\,dD_{s,v}\Big|\mathcal{F}_s\biggr) = 0,
\end{aligned}$$

where we first used condition (ii) and subsequently the equality $\widehat{U}^0_{v,v} = 0$. It remains to examine the case $s = 0$. We denote $\widehat{U}^{0+}_{\tau,t} = \max(\widehat{U}^0_{\tau,t}, 0)$ and $\widehat{U}^{0-}_{\tau,t} = \max(-\widehat{U}^0_{\tau,t}, 0)$. Then, for all $t > 0$,

$$\begin{aligned}
I_1 &= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(0,t]} \widehat{U}^0_{v,t}\, dF_{v,t}\Big|\mathcal{F}_0\biggr) \\
&= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\mathbb{E}_{\mathbb{P}}\Bigl(\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\bigl(\mathbb{1}_{\{s\le\tau\le t\}}\widehat{U}^0_{\tau,t}|F_t\bigr)|\mathcal{F}_0\Bigr) \\
&= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\mathbb{E}_{\mathbb{P}}\Bigl(\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\bigl(\mathbb{1}_{\{s\le\tau\le t\}}\widehat{U}^{0+}_{\tau,t}|F_t\bigr)|\mathcal{F}_0\Bigr) \\
&\quad - \mathbb{E}_{\mathbb{P}}\Bigl(\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\bigl(\mathbb{1}_{\{s\le\tau\le t\}}\widehat{U}^{0-}_{\tau,t}|F_t\bigr)|\mathcal{F}_0\Bigr).
\end{aligned}$$

Using the monotone convergence theorem for conditional expectations, condition (ii) and the equality $\widehat{U}^0_{v,v} = 0$, we obtain

$$\begin{aligned}
I_1 &= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]} \widehat{U}^0_{v,t}\, dF_{v,t}\Big|\mathcal{F}_0\biggr) \\
&= \mathbb{1}_{\{\tau>0\}}(G_0)^{-1}\lim_{s\downarrow 0}\mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]} \mathbb{E}_{\mathbb{P}}\bigl(F_{s,t}\widehat{U}^0_{v,t}|\mathcal{F}_v\bigr)\, dD_{s,v}\Big|\mathcal{F}_0\biggr) = 0.
\end{aligned}$$

For I_2, using again condition (ii), we obtain, for $0 \le u \le s < t$,

$$\begin{aligned}
I_2 &= \mathbb{1}_{\{\tau\le s\}}(F_s)^{-1}\mathbb{E}_{\mathbb{P}}\bigl(F_{s,t}\widehat{U}^0_{u,t}|\mathcal{F}_s\bigr)_{|u=\tau} \\
&= \mathbb{1}_{\{\tau\le s\}}(F_s)^{-1}F_{s,s}\bigl(\widehat{U}^0_{u,s}\bigr)_{|u=\tau} = \mathbb{1}_{\{\tau\le s\}}\widehat{U}^0_{\tau,s}.
\end{aligned}$$

We conclude that the process $(\mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t})_{t\ge 0}$ is a $(\mathbb{P}, \mathbb{G})$-martingale, and thus the proof of the proposition is complete. $\square$

The following corollary to Theorem 4.1 deals with the special case when the process U given by (4.1) is continuous at τ. It is easy to check that under the assumptions of Corollary 4.1 the process $(\widehat{U}_t := \widehat{U}_{t,t})_{t\ge 0}$ is $\mathbb{F}$-predictable.

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1 *we postulate*, *in addition*, *that the equality* $\widetilde{U}_{t-} = \widehat{U}_{t,t}$ *holds for every* $t \in \mathbb{R}_+$. *Then the process* U^* *is continuous at* τ, *and condition* (i) *in Theorem* 4.1 *can be replaced by the following condition*:

(i′) *the process* $(W_t)_{t\ge 0}$ *is a* $(\mathbb{P}, \mathbb{F})$-*local martingale where*

$$W_t = \widetilde{U}_t G_t + \int_{(0,t]} \widetilde{U}_{u-}\, dF_u. \tag{4.4}$$

To establish another corollary to Theorem 4.1, we assume that $F_{u,t}$ is completely separable.

COROLLARY 4.2. *Under the assumptions of Theorem 4.1 we postulate, in addition, that the* $(\mathbb{P},\mathbb{F})$*-conditional distribution of* τ *satisfies* $F_{u,t}=K_uL_t$ *for all* $0\le u\le t$, *where* K *is a positive*, $\mathbb{F}$*-adapted, increasing process and* L *is a positive* $(\mathbb{P},\mathbb{F})$*-martingale. Then condition* (ii) *in Theorem 4.1 can be replaced by the following condition*:

(ii′) *For every* $u\ge 0$, *the process* $(W_{u,t}=L_t\widehat{U}^0_{u,t})_{t\ge u}$ *is a* $(\mathbb{P},\mathbb{F})$*-martingale.*

4.2. $\mathbb{G}$*-local martingales for pseudo-initial times.* It was necessary to assume in Theorem 4.1 that the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is nondegenerate, since the random measure $D_{s,u}:=F_{u,u}(F_{s,u})^{-1}$ is not always well defined when the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is degenerate. In order to circumvent this technical assumption, one can postulate instead that $F_{u,t}$ satisfies the hypothesis (*ED*). In the next result, we work under the setup of Theorem 4.1, but we no longer assume that $0<F_{u,t}\le 1$ for every $0<u\le t$.

PROPOSITION 4.1. *Suppose that* τ *is a pseudo-initial time, and* (2.5) *holds with a positive random field* $(m_{s,t})_{t\ge s}$ *and an* $\mathbb{F}$*-adapted increasing process* D. *Then condition* (ii) *in Theorem 4.1 can be replaced by the following condition*:

(ii*) *for every* $u\ge 0$, *the process* $(m_{u,t}\widehat{U}^0_{u,t})_{t\ge u}$ *is a* $(\mathbb{P},\mathbb{F})$*-local martingale.*

PROOF. We only need to adjust the proof of Theorem 4.1 in case (b). Let $U^*_t=\mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t}$ where $\widehat{U}^0_{t,t}=0$. Using Lemma 3.3, we obtain, for every $0\le s<t$ (recall that $\{m_{u,s}=0\}\subset\{m_{u,t}=0\}$)

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau\le t\}}\widehat{U}^0_{\tau,t}|\mathcal{G}_s\big)&=\mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]}\widehat{U}^0_{u,t}\,dF_{u,t}\bigg|\mathcal{F}_s\bigg)\\&\qquad+\mathbb{1}_{\{\tau\le s\}}(m_{u,s})^{-1}\mathbb{E}_{\mathbb{P}}\big(m_{u,t}\widehat{U}^0_{u,t}|\mathcal{F}_s\big)_{|u=\tau}\\&=I_1+I_2.\end{aligned}$$

The integral I_1 satisfies

$$\begin{aligned}I_1&=\mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]}\widehat{U}^0_{u,t}\,dF_{u,t}\bigg|\mathcal{F}_s\bigg)\\&=\mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]}m_{u,t}\widehat{U}^0_{u,t}\,dD_u\bigg|\mathcal{F}_s\bigg)\\&=\mathbb{1}_{\{\tau>s\}}(G_s)^{-1}\mathbb{E}_{\mathbb{P}}\bigg(\int_{(s,t]}\mathbb{E}_{\mathbb{P}}\big(m_{u,t}\widehat{U}^0_{u,t}|\mathcal{F}_u\big)\,dD_u\bigg|\mathcal{F}_s\bigg)=0,\end{aligned}$$

where to obtain the last equality we first use assumption (ii*) and next the equality $\widehat{U}^0_{u,u}=0$. The integral I_2 simplifies to

$$\begin{aligned}I_2&=\mathbb{1}_{\{\tau\le s\}}(m_{u,s})^{-1}\mathbb{E}_{\mathbb{P}}\big(m_{u,t}\widehat{U}^0_{u,t}|\mathcal{F}_s\big)_{|u=\tau}\\&=\mathbb{1}_{\{\tau\le s\}}(m_{u,s})^{-1}m_{u,s}\big(\widehat{U}^0_{u,s}\big)_{|u=\tau}=\mathbb{1}_{\{\tau\le s\}}\widehat{U}_{\tau,s}.\end{aligned}$$

We conclude that the process U^* is a $(\mathbb{P},\mathbb{G})$-martingale, as was required to show. $\square$

5. Compensators of the indicator process. Our next goal is to compute the $(\mathbb{P}, \mathbb{G})$-dual predictable projection of the indicator process $H_t = \mathbb{1}_{\{\tau \le t\}}$ where, as usual, we denote by $\mathbb{G}$ the progressive enlargement of $\mathbb{F}$ with a random time τ. Recall that the Doob–Meyer decomposition of G is denoted by $G = M - A$. Then the $(\mathbb{P}, \mathbb{F})$-dual predictable projection [i.e., the $(\mathbb{P}, \mathbb{F})$-*compensator*] of H, denoted as H^p, coincides with the $\mathbb{F}$-predictable, increasing process A. To find the $(\mathbb{P}, \mathbb{G})$-dual predictable projection [i.e., the $(\mathbb{P}, \mathbb{G})$-*compensator*] of H, it is enough to apply the following classic result, due to Jeulin and Yor [28] (see also Guo and Zeng [15]), and to compute explicitly the $(\mathbb{P}, \mathbb{F})$-compensator of H.

THEOREM 5.1. *Let τ be a random time with the Azéma supermartingale G. Then the $(\mathbb{P}, \mathbb{G})$-compensator of H equals*

$$H_t^{p,\mathbb{G}} = \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, dH_u^p, \tag{5.1}$$

meaning that the process $H - H^{p,\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-martingale.

5.1. *Compensator of H under complete separability.* Let us first examine the case where the $(\mathbb{P}, \mathbb{F})$-conditional distribution of τ under $\mathbb{P}$ is completely separable, that is, $F_{u,t} = K_u L_t$. We assume, in addition, that the increasing process K is $\mathbb{F}$-predictable. It is worth noting that both assumptions are satisfied in the construction of τ based on a predictable multiplicative system, provided that $G_t < 1$ for all $t > 0$; see [35]. By applying the integration by parts formula to F and using the assumption that K is an $\mathbb{F}$-predictable process, we obtain

$$F_t = K_t L_t = K_0 L_0 + \int_{(0,t]} K_u\, dL_u + \int_{(0,t]} L_{u-}\, dK_u.$$

Hence, by the uniqueness of the Doob–Meyer decomposition, we conclude that $dA_u = L_{u-}\, dK_u$. Consequently, using the Jeulin–Yor formula (5.1), we obtain

$$H_t^{p,\mathbb{G}} = \int_{(0,t\wedge\tau]} \frac{L_{u-}}{1 - L_{u-}K_{u-}}\, dK_u.$$

5.2. *Compensator of H for a pseudo-initial time.* Assume now that τ is a pseudo-initial time, that is, the hypothesis (*ED*) holds; see Definition 2.5. Let the process m be given as $(m_t := m_{t,t})_{t\ge 0}$, and let ${}^p m$ denote the $(\mathbb{P}, \mathbb{F})$-predictable projection of m. For brevity, we will use the notation $X \overset{\text{mart}}{=} Y$ whenever $X - Y$ is a $(\mathbb{P}, \mathbb{F})$-local martingale. In the following, it is assumed that the process $\int m_u\, dD_u$ is of integrable variation. Let us set $m_t^D := \int_{(0,t]} (m_{u,t} - m_u)\, dD_u$ for all $t \ge 0$.

LEMMA 5.1. *The process m_t^D is a $(\mathbb{P}, \mathbb{F})$-martingale.*

PROOF. By splitting the integral and taking the conditional expectation, we obtain, for any $s \le t$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(m_t^D|\mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}(m_{u,t}-m_u)\,dD_u\bigg|\mathcal{F}_s\biggr)\\
&\qquad + \mathbb{E}_{\mathbb{P}}\biggl(\int_{(0,s]}(m_{u,t}-m_u)\,dD_u\bigg|\mathcal{F}_s\biggr)\\
&= \mathbb{E}_{\mathbb{P}}\biggl(\int_{(s,t]}\mathbb{E}_{\mathbb{P}}(m_{u,t}-m_u|\mathcal{F}_u)\,dD_u\bigg|\mathcal{F}_s\biggr)\\
&\qquad + \mathbb{E}_{\mathbb{P}}\biggl(F_{s,t}-F_{0,t}-\int_{(0,s]}m_u\,dD_u\bigg|\mathcal{F}_s\biggr)\\
&= F_{s,s}-F_{0,s}-\int_{(0,s]}m_u\,dD_u = \int_{(0,s]}(m_{u,s}-m_u)\,dD_u = m_s^D,
\end{aligned}$$

since, for any fixed s, the process $F_{s,t} = \int_{[0,s]} m_{u,t}\,dD_u$ is assumed to be a $(\mathbb{P},\mathbb{F})$-martingale; see Definitions 2.1 and 2.5. $\square$

PROPOSITION 5.1. *Assume that τ is a pseudo-initial time, and the process m is a special semimartingale with the canonical decomposition $m = N + P$ where N is the local martingale part. If the predictable covariation of N and $D - D^p$ exists, then the $(\mathbb{P},\mathbb{F})$-compensator of H is given by the formula*

$$H_t^p = \langle N, D - D^p\rangle_t + \int_{(0,t]} {}^p m_u\,dD_u^p, \tag{5.2}$$

where ${}^p m$ is the $(\mathbb{P},\mathbb{F})$-predictable projection of m.

PROOF. It suffices to compute the Doob–Meyer decomposition of F. Using (2.5) and the canonical decomposition of m, we obtain

$$\begin{aligned}
F_t &= \int_{[0,t]} m_{u,t}\,dD_u\\
&= m_{0,t}\Delta D_0 + \int_{(0,t]}(m_{u,t}-m_{u,u})\,dD_u + \int_{(0,t]}N_u\,dD_u + \int_{(0,t]}P_u\,dD_u\\
&= m_{0,t}\Delta D_0 + m_t^D + [N,D]_t + \int_{(0,t]}(N_{u-}+P_u)\,dD_u,
\end{aligned}$$

where in the second equality we used the fact that $[N,D]_t = \int_{(0,t]}\Delta N_u\,dD_u$ (see Proposition 9.3.7.1 in [24]). It is clear that the process $m_{0,t}\Delta D_0$ is a $(\mathbb{P},\mathbb{F})$-martingale, whereas m_t^D is a $(\mathbb{P},\mathbb{F})$-martingale in view of Lemma 5.1. Using the $(\mathbb{P},\mathbb{F})$-dual predictable projection of D, we can write

$$\begin{aligned}
\int_{(0,t]}(N_{u-}+P_u)\,dD_u &= \int_{(0,t]}(N_{u-}+P_u)\,d\bigl(D_u-D_u^p\bigr) + \int_{(0,t]}(N_{u-}+P_u)\,dD_u^p\\
&\overset{\text{mart}}{=} \int_{(0,t]}(N_{u-}+P_u)\,dD_u^p.
\end{aligned}$$

Finally, the $\mathbb{F}$-predictable covariation of N and $D - D^p$ is assumed to exist and

$$[N, D] = [N, D - D^p] + [N, D^p] \overset{\text{mart}}{=} \langle N, D - D^p \rangle,$$

where the second equality holds since:

(a) the processes N and $D - D^p$ are $(\mathbb{P}, \mathbb{F})$-local martingales, so that the process $[N, D - D^p] - \langle N, D - D^p \rangle$ is a $(\mathbb{P}, \mathbb{F})$-local martingale;

(b) by Yœurp's lemma (see the proof of Proposition 9.3.7.1 in [24]), the process $[N, D^p]$ is a $(\mathbb{P}, \mathbb{F})$-local martingale since N is $(\mathbb{P}, \mathbb{F})$-local martingale, and D^p is a predictable process of finite variation.

We have thus shown that

$$F_t \overset{\text{mart}}{=} \langle N, D - D^p \rangle_t + \int_{(0,t]} (N_{u-} + P_u)\, dD_u^p,$$

where the right-hand side is the $\mathbb{F}$-predictable process of finite variation. Now, let $(T_n)_{n \in \mathbb{N}}$ be a localizing sequence such that M and all the $(\mathbb{P}, \mathbb{F})$-local martingales defined above are uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingales once stopped at T_n. Then we can conclude that, for every $n \in \mathbb{N}$,

$$\begin{aligned} A_t^{T_n} &= \langle N, D - D^p \rangle_t^{T_n} + \int_{(0,t]} (N_{u-} + P_u) \mathbb{1}_{[\![0, T_n]\!]}(u)\, dD_u^p \\ &= \langle N, D - D^p \rangle_t^{T_n} + \int_{(0,t]} {}^p m_u \mathbb{1}_{[\![0, T_n]\!]}(u)\, dD_u^p, \end{aligned}$$

where the last equality follows from the equalities ${}^p(m^{T_n}) = {}^p(N^{T_n}) + {}^p(P^{T_n}) = N_-^{T_n} + P^{T_n}$ (see, in particular, Theorem 4.5 in [38]) and (note that the process $\mathbb{1}_{[\![0, T_n]\!]}$ is $\mathbb{F}$-predictable)

$${}^p(m^{T_n}) \mathbb{1}_{[\![0, T_n]\!]} = {}^p(m^{T_n} \mathbb{1}_{[\![0, T_n]\!]}) = {}^p(m \mathbb{1}_{[\![0, T_n]\!]}) = {}^p m \mathbb{1}_{[\![0, T_n]\!]}.$$

To complete the proof of equality (5.2), it suffices to let $n \to \infty$. $\square$

REMARK 5.1. As an example, consider the case where the process D is predictable. Then we obtain from (5.2)

$$A_t = H_t^p = \int_{(0,t]} {}^p m_u\, dD_u,$$

and it is enough to require that ${}^p m$ exists.

6. Hypothesis (H') and semimartingale decompositions. The aim of this section is to analyze the validity of the classic hypothesis (H') for progressive enlargements associated with pseudo-honest and pseudo-initial times. We establish here the main results of this work, Theorems 6.2 and 6.6, and we study the case of the multiplicative construction of a random time associated with a predetermined Azéma submartingale.

Let us first recall the general definition, in which $\mathbb{K}$ stands for any enlargement of $\mathbb{F}$, that is, any filtration such that $\mathbb{F} \subset \mathbb{K}$.

DEFINITION 6.1. We say that the *hypothesis* (H') is satisfied by $\mathbb{F}$ and its enlargement $\mathbb{K}$ if any $(\mathbb{P}, \mathbb{F})$-semimartingale is also a $(\mathbb{P}, \mathbb{K})$-semimartingale.

For exhaustive studies of the hypothesis (H') the interested reader is referred to Jeulin [26], who examined a general case as well as honest times, and Jacod [19], who worked under the density hypothesis and covered the initial times. The latter study was recently extended by Kchia and Protter [32], who dealt with the progressive enlargement with a general stochastic process, and not only the indicator process of a random time.

As is well known, to establish the hypothesis (H') between $\mathbb{F}$ and any enlargement $\mathbb{K}$, it suffices to show that any bounded $(\mathbb{P}, \mathbb{F})$-martingale is a $(\mathbb{P}, \mathbb{K})$-semimartingale; see Yor [44]. This crucial observation follows, for instance, from the Jacod–Mémin decomposition of a $(\mathbb{P}, \mathbb{F})$-semimartingale; $X = X_0 + K + B + N$ where K represents large jumps, B is predictable of finite variation and N is a local martingale with jumps bounded by 1; see, for example, page 3 in Jeulin [26]. One can then show that, under the hypothesis (H'), any bounded $(\mathbb{P}, \mathbb{F})$-martingale is in fact a special $(\mathbb{P}, \mathbb{K})$-semimartingale and this in turn implies that, more generally, any special $(\mathbb{P}, \mathbb{F})$-semimartingale remains a special $(\mathbb{P}, \mathbb{K})$-semimartingale. Therefore, assuming that the hypothesis (H') holds for $\mathbb{F}$ and $\mathbb{K}$, the natural goal is thus to find the canonical semimartingale decomposition with respect to the enlarged filtration $\mathbb{K}$ of a bounded $(\mathbb{P}, \mathbb{F})$-martingale. In addition, if the hypothesis (H') for $\mathbb{F}$ and $\mathbb{K}$ fails to hold, then the goal is to describe the class of $(\mathbb{P}, \mathbb{F})$-semimartingales that remain also $(\mathbb{P}, \mathbb{K})$-semimartingales. For an exhaustive study of these problems, the reader may consult Jeulin [26].

Let us now focus on the case of the progressive enlargement of a filtration $\mathbb{F}$ with a random time. It is well known, in particular, that for any random time τ with values in $\bar{\mathbb{R}}_+$ and any $(\mathbb{P}, \mathbb{F})$-local martingale U, the stopped process U^τ is a $(\mathbb{P}, \mathbb{G})$-semimartingale; see Yor [44] and Jeulin and Yor [28]. Before proceeding to computations of semimartingale decompositions, we first state a theorem, which recalls and summarizes results established by Jeulin and Yor; for parts (ii) and (iv), see Theorem 1 and Lemma 4 in Jeulin and Yor [28], respectively; for part (iii), see Proposition 4.16 in Jeulin [26].

THEOREM 6.1. *Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with an arbitrary random time τ*:

(i) *if U is a $(\mathbb{P}, \mathbb{F})$-local martingale, then the stopped process U^τ is a $(\mathbb{P}, \mathbb{G})$-special semimartingale*;

(ii) *the process*

$$U_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\big(\langle U, M\rangle_u + \breve{U}^p_u\big) \tag{6.1}$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale, where $\breve{U}^p$ stands for the dual $\mathbb{F}$-predictable projection of the process $\breve{U}_t = \Delta U_\tau \mathbb{1}_{\{\tau\le t\}}$;

(iii) *the process*

$$U_{t\wedge\tau} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\langle U, \bar{M}\rangle_u \tag{6.2}$$

is a $(\mathbb{P},\mathbb{G})$*-local martingale, where* $\bar{M}$ *is the unique BMO martingale such that* $\mathbb{E}_{\mathbb{P}}(N_\tau) = \mathbb{E}_{\mathbb{P}}(N_\infty \bar{M}_\infty)$ *for every bounded* $(\mathbb{P},\mathbb{F})$*-martingale* N;

(iv) *let* U *be a* $(\mathbb{P},\mathbb{F})$*-local martingale. Denote by* $\breve{U}^p$ *the dual* $(\mathbb{P},\mathbb{F})$*-predictable projection of the process*

$$\breve{U}_t = \int_{(0,t]} \Delta U_u\, dH_u = \Delta U_\tau \mathbb{1}_{\{\tau\le t\}}.$$

Then the process

$$\int_{(0,t]} \Delta U_u\, dH_u - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\breve{U}^p_u = \Delta U_\tau \mathbb{1}_{\{\tau\le t\}} - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\breve{U}^p_u \tag{6.3}$$

is a $(\mathbb{P},\mathbb{G})$*-local martingale.*

REMARK 6.1. It is known that $G = \bar{M} - \bar{A}$ where $\bar{A} = H^o$ is the dual $\mathbb{F}$-optional projection of H. Under the assumption (**C**) (i.e., when all $\mathbb{F}$-martingales are continuous) and/or the assumption (**A**) (i.e., when the random time τ avoids all $\mathbb{F}$-stopping times), we have that $\bar{M} = M$ and thus also $H^p = H^o$ or, equivalently, $A = \bar{A}$.

6.1. *Hypothesis* (H') *for the progressive enlargement.* Our goal is to show that if a random time τ given on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ satisfies the hypothesis (*HP*), and its $(\mathbb{P},\mathbb{F})$-conditional distribution is positive, then the $\mathbb{G}$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-semimartingale can be computed explicitly. From Theorem 6.1, we know that for any $(\mathbb{P},\mathbb{F})$-local martingale U, the stopped process U^τ is a $(\mathbb{P},\mathbb{G})$-special semimartingale with explicitly known canonical decomposition. Therefore, it will be enough to focus on the behavior of a process U after τ. It should be stressed that we do not claim that the hypothesis (*HP*) implies that the hypothesis (H') between $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$ holds, in general, since certain additional assumptions will be imposed when deriving alternative versions of $\mathbb{G}$-semimartingale decompositions of a $(\mathbb{P},\mathbb{F})$-local martingale.

The main result of this section, Theorem 6.2, furnishes an explicit $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-local martingale U when τ is a pseudo-honest time, that is, satisfies the hypothesis (*HP*). This result can be seen as a counterpart of Proposition 5.9 in El Karoui et al. [11] who dealt with the case of an initial time (see also Jeanblanc and Le Cam [20] who examined both initial and honest times). In the special case of the multiplicative construction (see Corollary 6.3 and Remark 6.5), it is also related to Theorem 7.1 in Jeanblanc and Song [21] and Section 7 in Jeanblanc and Song [23].

THEOREM 6.2. *Assume that τ is a pseudo-honest time such that, for every $s \geq 0$, the bounded $(\mathbb{P}, \mathbb{F})$-martingale $(F_{s,u})_{u \geq s}$ is strictly positive. If U is a $(\mathbb{P}, \mathbb{F})$-local martingale, then the process U^* is a $(\mathbb{P}, \mathbb{G})$-local martingale where*

$$\begin{aligned} U_t^* = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\big(\langle U, M\rangle_u + \breve{U}_u^p\big) \\ - \int_{(t\wedge s,t]} (F_{s,u-})^{-1}\, d\langle U, F_{s,\cdot}\rangle_u \Big|_{s=\tau}. \end{aligned} \tag{6.4}$$

Hence U is a $(\mathbb{P}, \mathbb{G})$-special semimartingale, and equality (6.4) yields its canonical decomposition.

Note that $(t \wedge \tau, t] = \varnothing$ on the event $\{\tau \geq t\}$, since manifestly $(t \wedge s, t] = \varnothing$ for all $s \geq t$. Before proceeding to the proof of Theorem 6.2, we make some remarks and prove a preliminary lemma.

REMARK 6.2. Recall that any local martingale is locally in the space $\mathcal{H}^1$. If N is a BMO martingale, then by the Fefferman inequality (see Revuz and Yor [41]), there exists a constant c such that for any local martingale U

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^\infty \big|d[U, N]_t\big|\right) \leq c\|U\|_{\mathcal{H}^1}\|N\|_{\mathrm{BMO}}.$$

Consequently, the process $[U, N]$ is locally of integrable variation, and its compensator $\langle U, N\rangle$ is well defined.

REMARK 6.3. The Azéma supermartingale G is generated by the $\mathbb{F}$-predictable, increasing process A, in the sense that, for every $t \geq 0$,

$$G_t = \mathbb{E}_{\mathbb{P}}(A_\infty|\mathcal{F}_t) - A_t = M_t - A_t.$$

This implies that the process $M_t = \mathbb{E}_{\mathbb{P}}(A_\infty|\mathcal{F}_t)$ is a BMO martingale since $G \leq 1$; see Proposition 10.13 in [16]. It is also well known that any bounded martingale is a BMO martingale; see Proposition 10.11 in [16].

As a first step toward the proof of Theorem 6.2, we establish the existence of the integrals appearing in right-hand side of (6.4).

LEMMA 6.1. *Under the assumptions of Theorem 6.2, the integrals in the right-hand side of equality (6.5) are well defined.*

PROOF. The process $(G_t)^{-1}\mathbb{1}_{\{\tau>t\}}$ is known to have, with probability 1, finite left-hand limits for all $t \in \mathbb{R}_+$ (see Yor [44]), and thus it possess a finite left-hand limit at τ. Hence the first integral in (6.4) is a well-defined $\mathbb{G}$-adapted process of finite variation. Next, let us check that for all $u \leq t$ the integral

$Z_{u,t} = \int_{(u,t]} (F_{u,v-})^{-1} \, d\langle U, F_{u,\cdot}\rangle_v$ is well-defined as well. To this end, we proceed as follows. Under the standing assumption that $(F_{u,t})_{t\geq u}$ is a strictly positive process, the stochastic logarithm

$$\mathcal{L}(F_{u,\cdot}) = \int_{(u,t]} (F_{u,v-})^{-1} \, dF_{u,v}$$

is well-defined. For the existence of the predictable bracket $\langle U, \mathcal{L}(F_{u,\cdot})\rangle$, it is sufficient to check that the càdlàg process $\mathcal{L}(F_{u,\cdot})$ is a locally bounded martingale, and for this purpose it is enough to show that the jump process $\Delta\mathcal{L}(F_{u,\cdot})_t = (F_{u,t-})^{-1}\Delta F_{u,t}$ is locally bounded for $t \geq u$. The latter property is clear since the left-continuous process $(F_{u,t-})^{-1}$ is locally bounded [we use here the property that $(F_{u,t})_{t\geq u}$ is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale] and the jumps of $F_{u,\cdot}$ are obviously bounded by 1. We conclude that the integral $Z_{u,t}$ is well defined since

$$Z_{u,t} = \int_{(u,t]} (F_{u,v-})^{-1} \, d\langle U, F_{u,\cdot}\rangle_v = \int_{(u,t]} d\big\langle U, \mathcal{L}(F_{u,\cdot})\big\rangle_v.$$

Moreover, the process $(Z_{u,t})_{t\geq u}$ is of locally integrable variation, since

$$\mathbb{E}_{\mathbb{P}}\left(\int_u^\infty \big| d\big\langle U, \mathcal{L}(F_{u,\cdot})\big\rangle_v \big| \right) \leq c\|U\|_{\mathcal{H}^1} \|\mathcal{L}(F_{u,\cdot})\|_{\mathrm{BMO}},$$

where the local martingale U is locally in $\mathcal{H}^1$, and the locally bounded martingale $\mathcal{L}(F_{u,\cdot})$ is locally in the space BMO. □

We are now ready to prove Theorem 6.2. Note that parts (ii) and (iii) of Theorem 6.1 are not employed in the proof of Theorem 6.2, although we use part (iv) in Theorem 6.1 to compensate the jump of U at τ.

PROOF OF THEOREM 6.2. Despite the fact that the present setup is more general than the one studied in the paper by El Karoui et al. [11], some steps in our proof are analogous to those employed in the proof of Proposition 5.9 in [11]. We start by noting that we may and do assume, without loss of generality, that U is a uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale.

In view of part (i) in Theorem 6.1, it suffices to show that if U is a $(\mathbb{P}, \mathbb{F})$-local martingale, then the process U^* is a $(\mathbb{P}, \mathbb{G})$-local martingale where

$$\begin{aligned} U_t^* = U_t &- \int_{(0,t\wedge\tau]} (G_{u-})^{-1} \, d\langle U, M\rangle_u - \Delta U_\tau \mathbb{1}_{\{\tau\leq t\}} \\ &- \int_{(t\wedge s,t]} (F_{s,u-})^{-1} \, d\langle U, F_{s,\cdot}\rangle_u \bigg|_{s=\tau}. \end{aligned} \tag{6.5}$$

We note that U^* satisfies the equality $U^* = U - B^*$ where the process B^* is defined by

$$B_t^* = \widetilde{B}_t \mathbb{1}_{\{\tau>t\}} + \widehat{B}_{\tau,t} \mathbb{1}_{\{\tau\leq t\}},$$

where in turn the $\mathbb{F}$-predictable process $\widetilde{B}$ equals

$$\widetilde{B}_t = \int_{(0,t]} (G_{u-})^{-1}\, d\langle U, M\rangle_u, \tag{6.6}$$

and the map $\widehat{B}$ is given by the following expression, for all $0 \le u \le t$,

$$\widehat{B}_{u,t} = \widetilde{B}_u + \Delta U_u + \int_{(u,t]} (F_{u,v-})^{-1}\, d\langle U, F_{u,\cdot}\rangle_v,$$

where $\Delta U_u = U_u - U_{u-}$. We need to show that the process

$$U_t^* = \mathbb{1}_{\{\tau>t\}}\widetilde{U}_t + \mathbb{1}_{\{\tau\le t\}}\widehat{U}_{\tau,t} = (U_t - \widetilde{B}_t)\mathbb{1}_{\{\tau>t\}} + (U_t - \widehat{B}_{\tau,t})\mathbb{1}_{\{\tau\le t\}}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale. We observe that $\widehat{U}_{u,t} = U_t - \widehat{B}_{u,t}$ is an $\mathbb{F}$-optional map, and the process $\widehat{U}_{t,t} = U_{t-} - \widetilde{B}_t$ is $\mathbb{F}$-predictable. Hence the assumptions of Theorem 4.1 are satisfied.

We thus see that in order to demonstrate that U^* is a $(\mathbb{P},\mathbb{G})$-local martingale, it suffices to show that: (i) the process

$$(U_t - \widetilde{B}_t)G_t - \int_{(0,t]} (U_{u-} - \widetilde{B}_u)\, dG_u \tag{6.7}$$

is a $(\mathbb{P},\mathbb{F})$-local martingale, and (ii) for any fixed $u,s \ge 0$, the process $(F_{s,t}\widehat{U}^0_{u,t})_{t\ge s\vee u}$ is a $(\mathbb{P},\mathbb{F})$-local martingale. For brevity, we will use the notation $X \overset{\text{mart}}{=} Y$ whenever the process $X - Y$ is a $(\mathbb{P},\mathbb{F})$-local martingale.

To establish the property (i), we observe that

$$\begin{aligned} d\big((U_t - \widetilde{B}_t)G_t\big) &= U_{t-}\, dG_t + G_{t-}\, dU_t + d[U,G]_t - \widetilde{B}_t\, dG_t - G_{t-}\, d\widetilde{B}_t \\ &\overset{\text{mart}}{=} U_{t-}\, dG_t + d[U,G]_t - \widetilde{B}_t\, dG_t - G_{t-}\, d\widetilde{B}_t. \end{aligned}$$

Consequently,

$$\begin{aligned} d\big((U_t - \widetilde{B}_t)G_t\big) - (U_{t-} - \widetilde{B}_{t-})\, dG_t &\overset{\text{mart}}{=} d[U,G]_t - d\langle U, M\rangle_u \\ &= d[U,B]_t + d[U,M]_t - d\langle U,M\rangle_u. \end{aligned}$$

Hence the process given by formula (6.7) is a $(\mathbb{P},\mathbb{F})$-local martingale since, by Yœurp's lemma, the process $[U,B]_t$ is a $(\mathbb{P},\mathbb{F})$-local martingale, and the process $[U,M] - \langle U,M\rangle$ is a $(\mathbb{P},\mathbb{F})$-local martingale as well. We conclude that the property (i) is valid in the present setup.

For (ii), we fix $u \ge 0$, and we observe that, for every $t \ge u$,

$$\widehat{U}^0_{u,t} = U_t - \widehat{B}_{u,t} - \widehat{U}_{u,u} = U_t - Z_{u,t} - V_u, \tag{6.8}$$

where we denote $V_u = \widetilde{B}_u + \Delta U_u + \widehat{U}_{u,u}$ and $(Z_{u,t})_{t\ge u}$ is the $\mathbb{F}$-predictable process given by

$$Z_{u,t} = \int_{(u,t]} (F_{u,v-})^{-1}\, d\langle U, F_{u,\cdot}\rangle_v. \tag{6.9}$$

Let us now fix $u, s \geq 0$. By applying the integration by parts formula, we obtain, for $t \geq s$,

$$
\begin{aligned}
d\big(F_{s,t}\widehat{U}^0_{u,t}\big) &= U_{t-}\,dF_{s,t} + F_{s,t-}\,dU_t + d[U, F_{s,\cdot}]_t \\
&\quad - Z_{u,t}\,dF_{s,t} - F_{s,t-}\,dZ_{u,t} - V_u\,dF_{s,t},
\end{aligned}
$$

where we used the fact that the process $(Z_{u,t})_{t\geq u}$ is $\mathbb{F}$-predictable. Recall that the processes U and $(F_{s,t})_{t\geq s}$ are $(\mathbb{P},\mathbb{F})$-martingales. Therefore, to show that the process $(F_{s,t}\widehat{U}^0_{u,t})_{t\geq u\vee s}$ is a $(\mathbb{P},\mathbb{F})$-local martingale, it is enough to check that, for all $u \leq s \leq t$ and $s \leq u \leq t$,

$$
d[U, F_{s,\cdot}]_t - F_{s,t-}\,dZ_{u,t} \overset{\text{mart}}{=} d\langle U, F_{s,\cdot}\rangle_t - F_{s,t-}\,dZ_{u,t} \overset{\text{mart}}{=} 0. \tag{6.10}
$$

Since we assumed that τ is a pseudo-honest time, using Remark 2.1, we obtain, for all $u \leq s \leq t$,

$$
\begin{aligned}
F_{s,t-}\,dZ_{u,t} &= \frac{F_{s,t-}}{F_{u,t-}}\,d\langle U, F_{u,\cdot}\rangle_t = \frac{F_{s,t-}}{F_{u,t-}}\,d\left\langle U, \frac{F_{u,s}}{F_{s,s}}F_{s,\cdot}\right\rangle_t \\
&= \frac{F_{s,t-}F_{u,s}}{F_{u,t-}F_{s,s}}\,d\langle U, F_{s,\cdot}\rangle_t = d\langle U, F_{s,\cdot}\rangle_t.
\end{aligned}
$$

Similarly, for all $s \leq u \leq t$, we get

$$
\begin{aligned}
F_{s,t-}\,dZ_{u,t} &= \frac{F_{s,t-}}{F_{u,t-}}\,d\langle U, F_{u,\cdot}\rangle_t = \frac{F_{s,u}F_{u,t-}}{F_{u,u}F_{u,t-}}\,d\langle U, F_{u,\cdot}\rangle_t \\
&= \frac{F_{s,u}}{F_{u,u}}\,d\langle U, F_{u,\cdot}\rangle_t = d\left\langle U, \frac{F_{s,u}}{F_{u,u}}F_{u,\cdot}\right\rangle_t = d\langle U, F_{s,\cdot}\rangle_t.
\end{aligned}
$$

This shows that $d\langle U, F_{s,\cdot}\rangle_t - F_{s,t-}\,dZ_{u,t} = 0$, and thus the second equality (6.10) is trivially satisfied. This means, of course, that the property (ii) is satisfied. Using Theorem 4.1, we thus conclude that the process U^* given by formula (6.5) is a $(\mathbb{P},\mathbb{G})$-local martingale. This in turn implies that the process U^* given by formula (6.4) is a $(\mathbb{P},\mathbb{G})$-local martingale, as was required to demonstrate. □

The next result is borrowed from the paper by Kchia et al. [31] who examined the case of any two enlargements that coincide after a random time τ. However, for simplicity of presentation, their result (see Theorem 3 in [31]) is stated here for the special case of the progressive enlargement $\mathbb{G}$ and the initial enlargement $\mathbb{G}^*$, which are known to coincide after τ. For brevity, we write hereafter

$$
\widetilde{C} = \langle U, M\rangle + \breve{U}^p, \tag{6.11}
$$

where by $\breve{U}^p$ we denote the dual $\mathbb{F}$-predictable projection of the process $\breve{U}_t = \Delta U_\tau \mathbb{1}_{\{\tau\leq t\}}$.

THEOREM 6.3. *Let U be a $(\mathbb{P},\mathbb{F})$-local martingale. Suppose that B is a $\mathbb{G}^*$-predictable process of finite variation such that $U-B$ is a $(\mathbb{P},\mathbb{G}^*)$-local martingale. Then the process*

$$U_t - \int_{(0,t\wedge\tau]} \frac{1}{G_{u-}}\, d\widetilde{C}_u - \int_{(t\wedge\tau,t]} dB_u \tag{6.12}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale. Hence U is a $(\mathbb{P},\mathbb{G})$-special semimartingale.

6.2. *Special cases of pseudo-honest times.* From Theorem 6.2, we know that the hypothesis (H') holds for a pseudo-honest time with a strictly positive $(\mathbb{P},\mathbb{F})$-conditional distribution. Moreover, this result furnishes also a general expression for the canonical decomposition with respect to $\mathbb{G}$ for an arbitrary $(\mathbb{P},\mathbb{F})$-local martingale. Of course, any $\mathbb{F}$-predictable process of finite variation is also a $\mathbb{G}$-predictable process of finite variation, and thus it suffices to focus on the canonical decomposition with respect to $\mathbb{G}$ of a $(\mathbb{P},\mathbb{F})$-local martingale, rather than a $(\mathbb{P},\mathbb{F})$-special semimartingale.

Our next goal is to examine some useful consequences of Theorem 6.2 under alternative additional assumptions imposed on a pseudo-honest time under consideration. In particular, we will compare various semimartingale decompositions for pseudo-honest times with their classic counterparts for honest times, which were established by Barlow [2], Jeulin and Yor [28], and Jeulin and Yor [29].

For the reader's convenience, we first recall the most pertinent results regarding the case of an honest time. It was shown by Barlow (see Theorem 3.10 in [2]) and Yor (see Theorem 4 in [44]) that the hypothesis (H') holds for $\mathbb{F}$ and its progressive enlargement with an honest time. The following result summarizes the well-known properties of the progressive enlargement with an honest time; see Theorem A in Barlow [2], Theorem 2 in Jeulin and Yor [28] and Theorem 15 in Jeulin and Yor [29].

THEOREM 6.4. *Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with an honest time τ. If U is a $(\mathbb{P},\mathbb{F})$-local martingale, then U is a $(\mathbb{P},\mathbb{G})$-special semimartingale and the following statements hold:*

(i) *the process*

$$U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\widetilde{C}_u, \tag{6.13}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale where $\widetilde{C}$ is given by (6.11);

(ii) *the process*

$$U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\langle U,\bar{M}\rangle_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\langle U,\bar{M}\rangle_u \tag{6.14}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale, where $\bar{M}$ is the $(\mathbb{P},\mathbb{F})$-martingale of class BMO introduced in part (iii) *of Theorem 6.1.*

6.2.1. *Completely separable case.* As a first special case of a pseudo-honest time, we consider the situation where the $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time τ is completely separable; see Definition 2.3. We obtain the following immediate corollary to Theorem 6.2. Corollary 6.1 will later be exemplified through the predictable multiplicative construction of a random time; see Corollary 6.3.

COROLLARY 6.1. *Under the assumptions of Theorem 6.2, we postulate, in addition, that the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is completely separable, that is, $F_{u,t} = K_u L_t$ for every $0 \le u \le t$ where L is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process U^* is a $(\mathbb{P},\mathbb{G})$-local martingale where*

$$U_t^* = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u - \int_{(t\wedge\tau,t]} (L_{u-})^{-1}\, d\langle U, L\rangle_u. \tag{6.15}$$

PROOF. We note that under the assumptions of Corollary 6.1, formula (6.5) reduces to (6.15). □

6.2.2. *Predictable multiplicative construction.* We will now show that if τ is a pseudo-honest time with a nondegenerate $(\mathbb{P},\mathbb{F})$-conditional distribution, then under certain technical assumptions, the $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of a $(\mathbb{P},\mathbb{F})$-local martingale is analogous to the one derived by other authors for an honest time and reported in Theorem 6.4. It is worth stressing that the present setup is manifestly different from the one covered by Theorem 6.4, and in fact, our results do not cover the case of an honest time.

COROLLARY 6.2. *Let the assumptions of Theorem 6.2 be satisfied. Assume, in addition, that, for every $s \ge 0$, the decreasing process $(C_{s,u} = (F_u)^{-1} F_{s,u})_{u\ge s}$ is $\mathbb{F}$-predictable. If U is a $(\mathbb{P},\mathbb{F})$-local martingale, then the process U^* given by*

$$U_t^* = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u + \int_{(t\wedge\tau,t]} ({}^pF_u)^{-1}\, d\langle U, M\rangle_u \tag{6.16}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale.

PROOF. To show that the second integral in the right-hand side of (6.16) is a well-defined process of locally integrable variation, we observe that $0 < F_{u-} \le {}^pF_u$ (since F is a submartingale) and thus

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\biggl(\int_0^\infty \mathbb{1}_{\{\tau<u\}} ({}^pF_u)^{-1}\, d\bigl|\langle U, M\rangle\bigr|_u\biggr) \\
&\quad = \mathbb{E}_{\mathbb{P}}\biggl(\int_0^\infty F_{u-} ({}^pF_u)^{-1}\, d\bigl|\langle U, M\rangle\bigr|_u\biggr) \\
&\quad \le \mathbb{E}_{\mathbb{P}}\biggl(\int_0^\infty \bigl|d\langle U, M\rangle_t\bigr|\biggr) \le c\|U\|_{\mathcal{H}^1}\|M\|_{\mathrm{BMO}},
\end{aligned}$$

where we used Fefferman's inequality in the last inequality. To obtain (6.16) from (6.4), we start by noting that $C_{s,u}$ is an $\mathbb{F}$-predictable multiplicative system associated with a positive submartingale F; see Meyer [36] or Li and Rutkowski [35]. By assumption, the process F_t is strictly positive for $t > 0$ and thus, by Theorem 4.1 and Corollary 4.1 in [35], the unique $\mathbb{F}$-predictable multiplicative system $C_{s,u}$ associated with F satisfies the following stochastic differential equation:

$$dC_{s,u} = -C_{s,u-}({}^pF_u)^{-1}\, dA_u = -C_{s,u}(F_{u-})^{-1}\, dA_u, \tag{6.17}$$

where the second equality follows from the equality $C_{s,u}{}^pF_u = C_{s,u-}F_{u-}$; see formula (17) in [35]. Since the decreasing process $(C_{s,u})_{u\geq s}$ is assumed to be $\mathbb{F}$-predictable, the integration by parts formula yields, for any fixed $s \geq 0$,

$$dF_{s,u} = C_{s,u}\, dF_u + F_{u-}\, dC_{s,u} = -C_{s,u}\, dM_u,$$

where the second equality follows from (6.17) and the Doob–Meyer decomposition $dF_t = dA_t - dM_t$. We only need to focus on the last term in formula (6.4). We obtain, for all $s \leq t$,

$$\begin{aligned}\int_{(s,t]} (F_{s,u-})^{-1}\, d\langle U, F_{s,\cdot}\rangle_u &= -\int_{(s,t]} (F_{u-}C_{s,u-})^{-1} C_{s,u}\, d\langle U, M\rangle_u \\ &= -\int_{(s,t]} \big({}^pF_u\big)^{-1}\, d\langle U, M\rangle_u\end{aligned}$$

as was required to show. $\square$

REMARK 6.4. Let us consider the situation of Corollary 6.3, and let us assume, in addition, that the avoidance property (**A**) holds, that is, τ avoids all $\mathbb{F}$-stopping times. Then ${}^pF = F_-$ and the $(\mathbb{P}, \mathbb{F})$-local martingale U is continuous at τ. Hence (6.16) becomes

$$U_t^* = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\langle U, M\rangle_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\langle U, M\rangle_u \tag{6.18}$$

since, under assumption (**A**), the martingales M and $\bar{M}$ are known to coincide as well. Note that under assumption (**A**) alternative semimartingale decompositions (6.13) and (6.14) obtained for an honest time reduce to (6.16) as well.

We will now describe a particular instance where the assumptions of Corollary 6.3 are satisfied. For the reader's convenience, we first briefly summarize the main steps in an explicit construction of a random time associated with an arbitrary Azéma submartingale F, as developed in [35]. We now assume that we are given a predetermined Azéma submartingale F, that is, an arbitrary submartingale $F = (F_t)_{t\in\bar{\mathbb{R}}_+}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, satisfying the inequalities $0 \leq F_t \leq 1$ for every $t \in \mathbb{R}_+$ and with $F_\infty = 1$. The predictable multiplicative construction of a random time τ associated with F runs as follows:

- We start by establishing the existence of an $\mathbb{F}$-predictable multiplicative system $\widehat{C}_{u,t}$ associated with a positive submartingale F (see Meyer [36] and Theorem 4.1 in [35]).
- Subsequently, using a (possibly nonunique) $\mathbb{F}$-predictable multiplicative system $\widehat{C}_{u,t}$, we define the unique $(\mathbb{P},\mathbb{F})$-conditional distribution $\widehat{F}_{u,t}$ by setting (see Theorem 4.2 and Lemma 5.1 in [35])

$$\widehat{F}_{u,t} = \begin{cases} \mathbb{E}_{\mathbb{P}}(F_u|\mathcal{F}_t), & t \in [0,u), \\ \widehat{C}_{u,t} F_t, & t \in [u,\infty]. \end{cases}$$

- Finally, we construct a random time τ on the extended probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{\mathbb{P}})$ such that $\widehat{\mathbb{P}}(\tau \le t|\mathcal{F}_t) = F_t$ for all $t \in \mathbb{R}_+$; see Theorem 5.1 in [35].

Recall that $\widehat{\mathbb{P}}$ is chosen in such a way that the probability measures $\widehat{\mathbb{P}}$ and $\mathbb{P}$ coincide on the filtration $\mathbb{F}$. It is also worth noting that if $G = M - A$ is the Doob–Meyer decomposition of the Azéma supermartingale $G = 1 - F$ then, using the uniqueness of the Doob–Meyer decomposition, we deduce that $M_t = \mathbb{E}_{\widehat{\mathbb{P}}}(H^p_\infty|\mathcal{F}_t)$ and $A = H^p$. It is clear that Corollary 6.2 can now be applied to the random time τ constructed as above. Specifically, we are in a position to establish the following result, in which we use the fact that, under stronger assumptions on F, the unique $\mathbb{F}$-predictable multiplicative system $\widehat{C}_{u,t}$ associated with F is known explicitly; see [35].

COROLLARY 6.3. *Assume that $F_t > 0$ and $F_{t-} > 0$ for every $t > 0$ and a pseudo-honest time τ is constructed using the unique $\mathbb{F}$-predictable multiplicative system associated with F. If U is a $(\widehat{\mathbb{P}},\mathbb{F})$-local martingale, then the process U^* given by* (6.16) *is a $(\widehat{\mathbb{P}},\mathbb{G})$-local martingale.*

PROOF. The statement follows immediately from Corollary 6.2. Alternatively, it can also be deduced from Corollary 6.1. To see this, we start by noting that Proposition 5.1 in [35] implies that $F_{u,t}$ is completely separable with $L = F\mathcal{E}$ where $\mathcal{E}$ is the Doléans exponential (see formula (30) in [35])

$$\mathcal{E}_t = \mathcal{E}_t\biggl(-\int_{(0,\cdot]} ({}^pF_s)^{-1}\, dA_s\biggr),$$

so that $d\mathcal{E}_t = -\mathcal{E}_{t-}({}^pF_t)^{-1}\, dA_t$. It is also known that $\mathcal{E}_t\, {}^pF_t = \mathcal{E}_{t-}F_{t-} = L_{t-}$; this is a consequence of formula (17) in [35]. Since $\mathcal{E}$ is an $\mathbb{F}$-predictable process of finite variation, by applying the integration by parts formula, we obtain

$$dL_t = \mathcal{E}_t\, dF_t + F_{t-}\, d\mathcal{E}_t = ({}^pF_t)^{-1} L_{t-}\, dF_t - F_{t-}\mathcal{E}_{t-}({}^pF_t)^{-1}\, dA_t.$$

Consequently, we also have that

$$\begin{aligned}(L_{t-})^{-1}\, dL_t &= -({}^pF_t)^{-1}\, dM_t + ({}^pF_t)^{-1}\, dA_t + (L_{t-})^{-1} F_{t-}\mathcal{E}_{t-}({}^pF_t)^{-1}\, dA_t \\ &= -({}^pF_t)^{-1}\, dM_t,\end{aligned}$$

and this in turn implies

$$(L_{t-})^{-1}\, d\langle U, L\rangle_t = -\big({}^p F_t\big)^{-1}\, d\langle U, M\rangle_t.$$

To complete the proof, it suffices to apply Corollary 6.1. □

REMARK 6.5. Corollary 6.3 corresponds to Theorem 7.1 in Jeanblanc and Song [21] who work under the assumption that $G_t = N_t e^{-\Lambda_t}$ where N is a positive local martingale and Λ is a continuous increasing process. Consequently, the martingale part M in the Doob–Meyer decomposition of G satisfies $dM_t = e^{-\Lambda_t}\, dN_t$. Moreover, they postulate that $G_t < 1$ and $G_{t-} < 1$ for every $t > 0$. It is shown in [21] that one may construct a random time τ on the product space $\Omega \times \bar{\mathbb{R}}_+$ and with respect to a suitably defined probability measure $\mathbb{Q}$ such that the equality $\mathbb{Q} = \mathbb{P}$ is satisfied on $\mathbb{F}$ and $\mathbb{Q}(\tau > t | \mathcal{F}_t) = G_t$ for all $t \in \bar{\mathbb{R}}_+$. Jeanblanc and Song also show (see Theorem 7.1 in [21]) that if a process U is a $(\mathbb{P}, \mathbb{F})$-local martingale, then the process U^* given by (6.18) is a $(\mathbb{P}, \mathbb{G})$-local martingale. More precisely, under their assumptions, formula (6.18) becomes

$$U^*_t = U_t - \int_{(0, t\wedge\tau]} \frac{e^{-\Lambda_u}}{G_{u-}}\, d\langle U, N\rangle_u + \int_{(t\wedge\tau, t]} \frac{e^{-\Lambda_u}}{F_{u-}}\, d\langle U, N\rangle_u.$$

For more general results in this vein, the interested reader is also referred to Theorems 4.2 and 4.4 in Jeanblanc and Song [22].

It is worth noting that, in the setup considered in [21], the canonical solution τ satisfies the "local density hypothesis in canonical form" or, in other words, the hypothesis (*ED*) is satisfied; see Theorem 5.1 in [21]. This implies that for any $\mathbb{F}$-stopping time T (for the definition of $\mathcal{E}_t(u)$, see Corollary 3.1 in [21])

$$\mathbb{Q}(\tau = T | \mathcal{F}_\infty) = \int_{[\![T]\!]} N_u \mathcal{E}_\infty(u) e^{-\Lambda_u}\, d\Lambda_u = N_T \mathcal{E}_\infty(T) e^{-\Lambda_T}\, \Delta\Lambda_T = 0,$$

where we also used the assumption made in [21] that the process Λ is continuous. We conclude that the avoidance property (**A**) holds. As a consequence, the dual $\mathbb{F}$-predictable and $\mathbb{F}$-optional projections of H coincide and thus also $M = \bar{M}$. In our general setting, the assumption that Λ is continuous is equivalent to the assumption that the $\mathbb{F}$-predictable process A which generates G is continuous. Under this assumption, Theorem 7.1 in [21] can be deduced from Corollary 6.3.

6.2.3. *Optional multiplicative construction*. In the next special case, we assume that a pseudo-honest time τ is constructed using an $\mathbb{F}$-optional multiplicative system associated with a predetermined Azéma submartingale F. We thus start by supposing that we are given an Azéma submartingale F. In order to construct an $\mathbb{F}$-optional multiplicative system associated with F, we will proceed along the same lines as in Section 4.3 in [35].

Specifically, we begin by assuming that F is given as $F_t = \mathbb{P}(\widehat{\tau} \le t | \mathcal{F}_t)$ for some random time $\widehat{\tau}$. Next, an $\mathbb{F}$-optional multiplicative system associated with F is defined. We may and do assume, without loss of generality, that an auxiliary random time $\widehat{\tau}$ was constructed using an $\mathbb{F}$-predictable multiplicative system $(\widehat{C}_{u,t})_{u,t\ge 0}$ associated with F. Hence this additional requirement is not restrictive.

More formally, to construct an $\mathbb{F}$-optional multiplicative system associated with F, we set $\widehat{H} = \mathbb{1}_{[\![\widehat{\tau},\infty[\![}$ and we define $\widehat{A} = \widehat{H}^o$ and $\widehat{M}_t = \mathbb{E}_{\mathbb{P}}(\widehat{H}^o_\infty | \mathcal{F}_t)$. As in Remark 6.1, we note that the equality $G = \widehat{M} - \widehat{A}$ yields an $\mathbb{F}$-optional decomposition of G. We define the random field $(C_{u,t})_{u,t\in\bar{\mathbb{R}}_+}$ by setting $C_{u,t} = 1$ for all $u \ge t$ and, for all $t \ge u$,

$$dC_{u,t} = -C_{u,t-}(F_t)^{-1}\, d\widehat{A}_t. \tag{6.19}$$

Then, from Corollary 4.2 in [35], the random field $(C_{s,u})_{s,u\in\bar{\mathbb{R}}_+}$ is an $\mathbb{F}$-optional multiplicative system associated with F. The $(\mathbb{P},\mathbb{F})$-conditional distribution of a random time is now defined by

$$F_{u,t} = \begin{cases} \mathbb{E}_{\mathbb{P}}(F_u | \mathcal{F}_t), & t \in [0,u), \\ C_{u,t} F_t, & t \in [u,\infty]. \end{cases}$$

Finally, the random time τ can be constructed using once again Theorem 5.1 in [35]. It is then not difficult to check that τ is a pseudo-honest time. It is important to emphasize that it is not asserted here that the equality $\widehat{A} = H^o$ holds where, as usual, we write $H = \mathbb{1}_{[\![\tau,\infty[\![}$. Therefore, the $\mathbb{F}$-optional decomposition of $G = \bar{M} - \bar{A}$, which is obtained for a random time τ as outlined in Remark 6.1, does not coincide with the $\mathbb{F}$-optional decomposition $G = \widehat{M} - \widehat{A}$, which is associated with an auxiliary random time $\widehat{\tau}$.

COROLLARY 6.4. *Let the assumptions of Theorem 6.2 be satisfied by a pseudo-honest time τ constructed using the $\mathbb{F}$-optional multiplicative system given by* (6.19). *If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process*

$$U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\langle U,\bar{M}\rangle_u + \int_{(t\wedge\tau,t]} (F_{u-})^{-1}\, d\langle U,\widehat{M}\rangle_u \tag{6.20}$$

is a $(\mathbb{P},\mathbb{G})$-local martingale.

PROOF. The first integral in (6.20) is dealt with as in part (iii) of Theorem 6.1. For the second integral in (6.20), we start by noting that it is a well-defined process of locally integrable variation, since

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\biggl(\int_0^\infty \mathbb{1}_{\{\tau<t\}}(F_{t-})^{-1}\, d\bigl|\langle U,\widehat{M}\rangle\bigr|_t\biggr)\\
&\quad = \mathbb{E}_{\mathbb{P}}\biggl(\int_0^\infty F_{t-}(F_{t-})^{-1}\, d\bigl|\langle U,\widehat{M}\rangle\bigr|_t\biggr)\\
&\quad \le \mathbb{E}_{\mathbb{P}}\biggl(\int_0^\infty \bigl|d\langle U,\widehat{M}\rangle_t\bigr|\biggr) \le c\|U\|_{\mathcal{H}^1}\|\widehat{M}\|_{\mathrm{BMO}},
\end{aligned}$$

where the local martingale U is locally in $\mathcal{H}^1$ and $\widehat{M}$ is the BMO martingale. Since the process $(C_{s,u})_{u\ge s}$ is decreasing, the integration by parts formula yields, for any fixed $s \ge 0$,

$$dF_{s,u} = C_{s,u-}\,dF_u + F_u\,dC_{s,u} = -C_{s,u-}\,d\widehat{M}_u,$$

where the second equality follows from (6.19) and the decomposition $G = \widehat{M} - \widehat{A}$. Hence, for all $s \le t$,

$$\begin{aligned}\int_{(s,t]} (F_{s,u-})^{-1}\,d\langle U, F_{s,\cdot}\rangle_u &= -\int_{(s,t]} (F_{u-}C_{s,u-})^{-1} C_{s,u-}\,d\langle U, \widehat{M}\rangle_u \\ &= -\int_{(s,t]} (F_{u-})^{-1}\,d\langle U, \widehat{M}\rangle_u.\end{aligned}$$

To conclude the proof, it suffices to combine Theorem 6.2 with part (iii) in Theorem 6.1. $\square$

6.3. *Hypothesis* (H') *for pseudo-initial times*. Let us finally examine the case of a pseudo-initial time. We first quote from Li [33] a useful result showing that the hypothesis (H') is met, that is, any $(\mathbb{P},\mathbb{F})$-semimartingale is also a $(\mathbb{P},\mathbb{G})$-semimartingale. Note that Theorem 6.5 is a direct extension of the classic result due to Jacod [19], who dealt with the case of an initial time (i.e., a random time satisfying the density hypothesis).

THEOREM 6.5. *If τ is a pseudo-initial time then the hypothesis (H') is satisfied by $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$.*

PROOF. The arguments used in the demonstration of the theorem combine the idea of the original proof of Jacod's theorem under the standard density hypothesis (see [19]) with the time change. For details, the interested reader is referred to the proof of Theorem 5.6.5 in Li [33]. $\square$

REMARK 6.6. It is not obvious that the hypothesis (H') is satisfied between $\mathbb{F}$ and the initial enlargement of $\mathbb{F}$ with τ, since $(m_{s,t})_{t\ge s}$ is only assumed to be a $(\mathbb{P},\mathbb{F})$-martingale for $t \ge s$.

We will now derive the $\mathbb{G}$-semimartingale decomposition for a progressive enlargement with a pseudo-initial time. Note that the $\mathbb{G}$-semimartingale decomposition established in the literature under the density hypothesis by Jeanblanc and Le Cam [20] (see also Kchia et al. [31] who employed their Theorem 6.3) can be obtained as a special case of equality (6.21) by postulating that the increasing process D in Definition 2.5 is nonrandom. Our proof of decomposition (6.21) is based on Theorem 6.2.

THEOREM 6.6. *Let τ be a pseudo-initial time. If U is a $(\mathbb{P},\mathbb{F})$-local martingale then the process U^* is a $(\mathbb{P},\mathbb{G})$-local martingale where*

$$U_t^* = U_t - \int_{(0,t\wedge\tau]} (G_{u-})^{-1}\, d\widetilde{C}_u + \int_{(t\wedge s,t]} (m_{s,u-})^{-1}\, d\langle U, m_{s,\cdot}\rangle_u \bigg|_{s=\tau}. \tag{6.21}$$

PROOF. We maintain the notation introduced in the proof of Theorem 6.2. In view of Proposition 4.1 and the arguments used in the proof of Theorem 6.2, it suffices to show that, for any fixed $u \ge 0$, the process $(m_{u,t}\widehat{U}^0_{u,t})_{t\ge u}$ is a $(\mathbb{P},\mathbb{F})$-local martingale, where we set $\widehat{U}^0_{u,t} = U_t - Z_{u,t} - V_u$ [see (6.8)] where in turn $V_u = \widetilde{B}_u + \Delta U_u + \widehat{U}_{u,u}$, with the process $\widetilde{B}$ defined by (6.6) and $Z_{u,t}$ given by the following expression [see (6.9)]

$$Z_{u,t} = \int_{(u,t]} (m_{u,v-})^{-1}\, d\langle U, m_{u,\cdot}\rangle_v.$$

By applying the integration by parts formula, we obtain

$$\begin{aligned} d\big(m_{u,t}\widehat{U}^0_{u,t}\big) &= U_{t-}\, dm_{s,t} + m_{s,t-}\, dU_t + d\langle U, m_{s,\cdot}\rangle_t \\ &\quad - Z_{u,t-}\, dm_{s,t} - m_{u,t}\, dZ_{u,t} - V_u\, dm_{u,t} \\ &= U_{t-}\, dm_{s,t} + m_{s,t-}\, dU_t - Z_{u,t-}\, dm_{u,t} - V_u\, dm_{u,t}, \end{aligned}$$

which is clearly a $(\mathbb{P},\mathbb{F})$-local martingale for $t \ge u$. □

REMARK 6.7. Let us assume, in addition, that there exists a positive $(\mathbb{P},\mathbb{F})$-martingale L and a positive, $\mathbb{F}$-adapted process a such that the equality $m_{s,t} = L_t a_s$ holds for all $s \le t$. Then

$$F_{u,t} = \int_{[0,u]} m_{s,t}\, dD_s = L_t \int_{[0,u]} a_s\, dD_s = K_u L_t,$$

where $K_u = \int_{[0,u]} a_s\, dD_s$, and thus the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is completely separable. Consequently, formula (6.21) can also be deduced from Corollary 6.1.

COROLLARY 6.5. *Let τ be a pseudo-initial time. Assume that there exists a positive $(\mathbb{P},\mathbb{F})$-martingale L and a positive, $\mathbb{F}$-adapted process a such that $m_{s,t} = L_t a_s$ for all $s \le t$. Then the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ is completely separable. Furthermore, if U is a $(\mathbb{P},\mathbb{F})$-local martingale, then the process U^* given by (6.21) is a $(\mathbb{P},\mathbb{G})$-local martingale.*

PROOF. It suffices to apply Corollary 6.1 and observe that, for any fixed s, the equality $m_{s,u} = a_s L_u$ holds. Hence formulas (6.15) and (6.21) are equivalent. □

7. Applications to financial mathematics. In the final section, we briefly outline applications of some of our general results established in the preceding sections to specific problems arising in financial mathematics.

7.1. *Credit risk modeling*. We will first comment on the plausible financial interpretation of the hypothesis (*HP*) in credit risk modeling. It is worth noting in this regard that it was fairly common in the financial literature to work under the stronger hypothesis (H), which enforces a very special dependence structure on default time and other processes arising in the model, for instance, the price processes of nondefaultable securities; for a discussion of the hypothesis (H), see, for instance, Elliott et al. [10] and Blanchet-Scalliet and Jeanblanc [3]. In particular, one can show that under hypothesis (H) the future dynamics of prices of nondefaultable securities are insensitive with respect to the occurrence of the default event. Let us stress that this very restrictive and practically dubious feature is relaxed when working under the hypothesis (*HP*). To illustrate this claim, the let us consider an arbitrary $\mathbb{F}$-adapted process S and let us set, for all $s \leq t$,

$$J_{s,t} := \mathbb{E}_{\mathbb{P}}(S_t - S_s | \mathcal{F}_s).$$

Note that $J_{s,t}$ can be interpreted as the expected return on the asset S with $\mathbb{F}$-adapted price process given the present information at time s conveyed by prices of nondefaultable securities. Furthermore, we also define, for all $u \leq s \leq t$,

$$\begin{aligned} J^u_{s,t} &:= \mathbb{E}_{\mathbb{P}}\big(S_t - S_s | \{\tau \leq u\} \cap \mathcal{F}_s\big) \\ &= J_{s,t} + (F_{u,s})^{-1} \mathbb{E}_{\mathbb{P}}\big((S_t - S_s)(F_{u,t} - F_{u,s}) | \mathcal{F}_s\big), \end{aligned}$$

which is the expected return on the asset S given the current information $\mathcal{F}_s$ and the event $\{\tau \leq u\}$. Then we obtain, for all $u \leq s \leq t$,

$$\begin{aligned} J^u_{s,t} &= J_{s,t} + (C_{u,s} F_s)^{-1} \mathbb{E}_{\mathbb{P}}\big((S_t - S_s)(C_{u,s} C_{s,t} F_t - C_{u,s} F_s) | \mathcal{F}_s\big) \\ &= J_{s,t} + (F_s)^{-1} \mathbb{E}_{\mathbb{P}}\big((S_t - S_s)(C_{s,t} F_t - F_s) | \mathcal{F}_s\big), \end{aligned}$$

which makes it clear that $J^u_{s,t} \neq J_{s,t}$, in general, but the equality $J^s_{s,t} = J^u_{s,t}$ holds for all $u \leq s \leq t$. The financial interpretation is a follows: the dynamics of nondefaultable securities may change after default; however, knowing that the default has already occurred, the expected return on the asset S is insensitive with respect to exact timing of the default event. It is still up for discussion whether this feature is sufficiently flexible for credit risk modeling. We would also like to point out that Remark 7.1 below (or Lemma 4.6 of Coculescu et al. [7]) emphasize the fact that for the purpose of arbitrage pricing of credit-risky securities, the $\mathcal{F}_\infty$-measurability of a default time might not be a desirable property, unless τ is an $\mathbb{F}$-stopping time (as in structural models of credit risk).

Regarding honest times, it is easy to see that if τ is an honest time, then the equality $J^u_{s,t} = J^s_{s,t}$ holds for all $u \leq s \leq t$. It was shown by Imkeller [17] and Zwierz [46] (see also the recent work by Fontana et al. [12]) that there is no arbitrage strictly before τ; however, arbitrage opportunities at τ and immediately after τ exist in a market model with the information given by the progressive enlargement of $\mathbb{F}$ with τ. Therefore, if one is only interested in pricing/trading of financial instruments strictly before default or modeling insider trading, then a model with an honest time could be suitable. This should be contrasted, however, with a growing interest in dynamics of price processes and pricing after default, especially in the context of the counterparty credit risk where the closeout amount at default needs to be evaluated in order to assess the so-called *credit value adjustment*; for an example of the CVA computations, see Brigo et al. [5].

We stress that a random time satisfying the hypothesis (*HP*) allows the modeler to retain the property that $J^s_{s,t} = J^u_{s,t}$, while still giving the possibility (under some mild conditions) to ensure the arbitrage-free property in the enlarged filtration. In particular, it was shown by Gapeev et al. [13], who worked in the Brownian filtration setup, that it is possible to construct a random time τ satisfying the hypothesis (*HP*) and such that there exists an equivalent probability measure under which the hypothesis (H) holds. This is indeed a crucial property, since, as was shown by Coculescu et al. [7], the hypothesis (H) is a sufficient condition for a market model with enlarged filtration to be arbitrage-free (obviously, this goal cannot be achieved when working with an honest time). In the foregoing subsection, we will re-examine this issue within the present more general framework.

7.2. *Arbitrage-free property of a market model.* In the recent paper by Coculescu et al. [7], the existence of an equivalent probability measure under which the immersion property holds was shown to be a sufficient condition for a market model with enlarged filtration to be arbitrage-free, assuming that the underlying market model based on the filtration $\mathbb{F}$ enjoys this property. In Proposition 7.3, we will show that if τ satisfies the hypothesis (*HP*) (or, more precisely, when the complete separability of the conditional distribution $F_{u,t}$ holds) then, under mild technical assumptions, the result from [7] can be applied to the progressive enlargement $\mathbb{G}$ associated with τ. In particular, if a random time τ is used to model the moment of occurrence of a default event, this result establishes the arbitrage-free property of a credit risk model in which the usual hypothesis (H) may fail to hold.

7.2.1. *Immersion property under an equivalent probability measure.* Before studying the case of a progressive enlargement, we first summarize briefly some results from Coculescu et al. [7] and make pertinent comments. Suppose that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with arbitrary filtrations $\mathbb{F}$ and $\mathbb{K}$ such that $\mathbb{F} \subset \mathbb{K}$. Let $\mathcal{I}(\mathbb{P})$ stand for the class of all probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{F}$ and $\mathbb{K}$ satisfy the immersion property under $\mathbb{Q}$. Recall that the immersion property for $\mathbb{F}$ and $\mathbb{K}$ under $\mathbb{Q}$ stipulates that any $(\mathbb{Q}, \mathbb{F})$-local martingale is a $(\mathbb{Q}, \mathbb{K})$-local martingale. The following lemma was established in [7].

LEMMA 7.1. *Assume that the class $\mathcal{I}(\mathbb{P})$ is nonempty. Then for every $\widetilde{\mathbb{Q}} \in \mathcal{I}(\mathbb{P})$ there exists a probability measure $\mathbb{Q}$ equivalent to $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F})$ and such that the following conditions are met*:

(i) *the Radon–Nikodým density process* $\eta_t := \frac{d\mathbb{Q}}{d\mathbb{Q}}|_{\mathcal{G}_t}$ *is* $\mathbb{F}$*-adapted*;
(ii) *the probability measures* $\mathbb{Q}$ *and* $\mathbb{P}$ *coincide on* $\mathbb{F}$;
(iii) $\mathbb{Q}$ *belongs to* $\mathcal{I}(\mathbb{P})$;
(iv) *every* $(\mathbb{P}, \mathbb{F})$*-local martingale is a* $(\mathbb{Q}, \mathbb{K})$*-local martingale*.

Note that in Lemma 7.1 we may replace the probability measure $\mathbb{P}$ by any probability measure equivalent to $\mathbb{P}$. This means that the assumption that the class $\mathcal{I}(\mathbb{P})$ is nonempty is fairly strong; it implies that for any probability measure $\mathbb{P}'$ there exists a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{P}'$ on $(\Omega, \mathcal{F})$, coinciding with $\mathbb{P}'$ on $\mathbb{F}$, and such that the immersion property between $\mathbb{F}$ and $\mathbb{K}$ holds under $\mathbb{Q}'$. It is thus natural to ask under which (nontrivial) circumstances the class $\mathcal{I}(\mathbb{P})$ is nonempty. A partial answer to this question for the progressive enlargement will be provided in Proposition 7.3.

REMARK 7.1. It is known from Lemma 4.6 in [7], the property that the class $\mathcal{I}(\mathbb{P})$ is nonempty is not satisfied by $\mathbb{F}$ and the progressive enlargement $\mathbb{G}$ when τ is an $\mathcal{F}_\infty$-measurable random time (e.g., an honest time with respect to $\mathbb{F}$), unless it is an $\mathbb{F}$-stopping time (so that $\mathbb{F} = \mathbb{G}$ and the immersion property is trivially satisfied).

The next result, also borrowed from [7], provides a complete characterization of nonemptiness of the class $\mathcal{I}(\mathbb{P})$.

PROPOSITION 7.1. *The following conditions are equivalent*:

(i) *the class* $\mathcal{I}(\mathbb{P})$ *is nonempty*;
(ii) *there exists a probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *such that every* $(\mathbb{P}, \mathbb{F})$*-local martingale is a* $(\mathbb{Q}, \mathbb{K})$*-local martingale*.

7.2.2. *Martingale measures via the hypothesis* (H). Suppose now we are given a $(\mathbb{P}, \mathbb{F})$-semimartingale X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{M}(\mathbb{P}, \mathbb{F})$ stand for the class of all $\mathbb{F}$-*local martingale measures* for X, meaning that a probability measure $\mathbb{Q}$ belongs to $\mathcal{M}(\mathbb{P}, \mathbb{F})$ whenever (i) $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and (ii) X is a $(\mathbb{Q}, \mathbb{F})$-local martingale. Let $\mathbb{K}$ be any enlargement of the filtration $\mathbb{F}$. We denote by $\mathcal{M}(\mathbb{P}, \mathbb{K})$ the class of $\mathbb{K}$-*local martingale measures* for X. In [7], the authors assumed that the class $\mathcal{M}(\mathbb{P}, \mathbb{F})$ is nonempty and they searched for sufficient conditions ensuring that the class $\mathcal{M}(\mathbb{P}, \mathbb{K})$ is nonempty as well.

One possibility is to postulate that the immersion property holds under some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and to infer that it is also valid under some $\mathbb{F}$-local martingale measure. Obviously, any $\mathbb{F}$-local martingale measure under which the immersion property holds is also a $\mathbb{K}$-local martingale measure.

A particular example of an $\mathbb{F}$-local martingale measure under which the immersion property holds can be produced using Lemma 7.1, leading to the following result, also due to Coculescu et al. [7]; see Corollary 4.6 therein.

PROPOSITION 7.2. (i) *The classes* $\mathcal{M}(\mathbb{P},\mathbb{F})$ *and* $\mathcal{I}(\mathbb{P})$ *are nonempty if and only if the set* $\mathcal{M}(\mathbb{P},\mathbb{F})\cap\mathcal{I}(\mathbb{P})$ *is nonempty.*

(ii) *If the classes* $\mathcal{M}(\mathbb{P},\mathbb{F})$ *and* $\mathcal{I}(\mathbb{P})$ *are nonempty, then the class* $\mathcal{M}(\mathbb{P},\mathbb{K})$ *is nonempty.*

REMARK 7.2. An inspection of the proof of Proposition 7.2 (i.e., Corollary 4.6 in [7]) shows that the process X plays no essential role [except, of course, for the assumption that the class $\mathcal{M}(\mathbb{P},\mathbb{F})$ is nonempty]. Note also that the probability measure $\mathbb{Q}$ constructed in the above-mentioned proof has the property that every $(\mathbb{P}',\mathbb{F})$-local martingale is also an $(\mathbb{Q},\mathbb{F})$-local martingale (since $\mathbb{P}'$ and $\mathbb{Q}$ coincide on $\mathbb{F}$) and thus a $(\mathbb{Q},\mathbb{K})$-local martingale [since $\mathbb{Q}\in\mathcal{I}(\mathbb{P}')$].

REMARK 7.3. The class $\mathcal{M}(\mathbb{P},\mathbb{F})$ can be replaced in part (i) of Proposition 7.2 by any subset $\mathcal{P}$ of probability measures equivalent to $\mathbb{P}$ and such that the following implication holds: if $\mathbb{P}'\in\mathcal{P}$ and $\mathbb{P}''=\mathbb{P}'$ on $\mathbb{F}$ then $\mathbb{P}''\in\mathcal{P}$.

To summarize the conclusions from Coculescu et al. [7], the following conditions are equivalent:

(C.1) $\mathcal{M}(\mathbb{P},\mathbb{F})$ and $\mathcal{I}(\mathbb{P})$ are nonempty;

(C.2) $\mathcal{M}(\mathbb{P},\mathbb{F})\cap\mathcal{I}(\mathbb{P})$ is nonempty;

(C.3) $\mathcal{M}(\mathbb{P},\mathbb{F})$ is nonempty and there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F})$ such that every $(\mathbb{P},\mathbb{F})$-local martingale is a $(\mathbb{Q},\mathbb{K})$-local martingale.

In view of Proposition 7.2, any of the conditions in (C.1)–(C.3) implies that the class $\mathcal{M}(\mathbb{P},\mathbb{K})$ is nonempty. It is thus natural to refer to any of the conditions in (C.1)–(C.3) as the *no-arbitrage condition* for the market model with an enlarged filtration $\mathbb{K}$.

7.2.3. *Martingale measures for the progressive enlargement.* We now consider the case where $\mathbb{K}=\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$. Suppose that the hypothesis (H) is not satisfied by the $(\mathbb{P},\mathbb{F})$-conditional distribution $F_{u,t}$ of a random time τ. It is then natural to ask whether there exists a probability measure $\bar{\mathbb{P}}$, which is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F})$ and such that the $(\bar{\mathbb{P}},\mathbb{F})$-conditional distribution $\bar{F}_{u,t}$ of τ satisfies the hypothesis (H). Equivalently, we ask whether there exists a probability measure $\bar{\mathbb{P}}$ equivalent to $\mathbb{P}$ and such that, for all $0\le u\le t$,

$$\bar{F}_{u,u}:=\bar{\mathbb{P}}(\tau\le u|\mathcal{F}_u)=\bar{\mathbb{P}}(\tau\le u|\mathcal{F}_t)=:\bar{F}_{u,t}. \tag{7.1}$$

Under the density hypothesis, the answer to this question is known to be positive; see El Karoui et al. [11] and Grorud and Pontier [14]. By contrast, when τ is assumed to be an honest time then this property never holds, unless τ is an $\mathbb{F}$-stopping time; see Remark 7.1.

In this subsection, we work under the standing assumption that the $(\mathbb{P},\mathbb{F})$-conditional distribution of τ is separable and $F_0 = 0$, so that τ is a pseudo-honest time. Note, however, that the case of an honest time is not covered by the foregoing results, since we also assume from now on that $F_{u,t} > 0$ for all $0 < u \leq t$. Recall also that the complete separability property implies that τ is a pseudo-honest time.

LEMMA 7.2. *Assume that* $(\mathbb{P},\mathbb{F})$*-conditional distribution of* τ *is completely separable, so that* $F_{u,t} = K_u L_t$ *for* $0 \leq u \leq t$. *Let the process* $(Z_t^{\mathbb{G}})_{t\geq 0}$ *be given by*

$$Z_t^{\mathbb{G}} = \widetilde{Z}_t \mathbb{1}_{\{\tau > t\}} + \widehat{Z}_{\tau,t}\mathbb{1}_{\{\tau \leq t\}}, \tag{7.2}$$

where $\widehat{Z}_{u,t} = \frac{F_{u,u}}{F_{u,t}}$ *and*

$$\widetilde{Z}_t = (G_t)^{-1}\biggl(1 - \int_{(0,t]} \widehat{Z}_{u,t}\, dF_{u,t}\biggr) = (G_t)^{-1}\bigl(1 - \mathbb{E}_{\mathbb{P}}(\widehat{Z}_{\tau,t}\mathbb{1}_{\{\tau\leq t\}} | \mathcal{F}_t)\bigr).$$

Then the process $Z^{\mathbb{G}}$ *is a* $(\mathbb{P},\mathbb{G})$*-local martingale.*

PROOF. The proof hinges on an application of Corollary 4.2 and Theorem 4.1. First, $\widehat{Z}_{u,u} = 1$ and thus it is trivially an $\mathbb{F}$-predictable process. Next, we need to check condition (i) in Corollary 4.2, which now reads: the process $(W_t)_{t\geq 0}$ is a $(\mathbb{P},\mathbb{F})$-local martingale where

$$W_t = \widetilde{Z}_t G_t + \int_{(0,t]} \widehat{Z}_{u,u}\, dF_u = \widetilde{Z}_t G_t + F_t. \tag{7.3}$$

We observe that

$$\begin{aligned}\widetilde{Z}_t G_t + F_t &= 1 + F_t - \int_{(0,t]} \widehat{Z}_{u,t}\, dF_{u,t} \\ &= 1 + L_t K_t - \int_{(0,t]} L_u\, dK_u = 1 + L_0 K_0 + \int_{(0,t]} K_{u-}\, dL_u,\end{aligned}$$

so that the process W is indeed a $(\mathbb{P},\mathbb{F})$-local martingale. Finally, we need to check condition (ii) in Corollary 4.2, which takes here the following form: for every $u > 0$, the process $(W_{u,t} = L_t(\widehat{Z}_{u,t} - \widehat{Z}_{u,u}))_{t\geq u}$ is a $(\mathbb{P},\mathbb{F})$-local martingale. To this end, we note that

$$W_{u,t} = L_t \frac{F_{u,u}}{F_{u,t}} - L_t \widehat{Z}_{u,u} = L_t \frac{K_u L_u}{K_u L_t} - L_t \widehat{Z}_{u,u} = L_u - L_t \widehat{Z}_{u,u}, \tag{7.4}$$

which is, obviously, a $(\mathbb{P},\mathbb{F})$-local martingale for $t \geq u$. In view of Theorem 4.1, we conclude that $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-local martingale. $\square$

We will also need the following simple lemma.

LEMMA 7.3. *Assume that τ is a pseudo-honest time. Then for any $\mathbb{F}$-adapted, $\mathbb{P}$-integrable process X we have that, for every $s \le t$,*

$$F_{s,t}\mathbb{E}_{\mathbb{P}}(X_\tau \mathbb{1}_{\{\tau \le s\}}|\mathcal{F}_s) = F_{s,s}\mathbb{E}_{\mathbb{P}}(X_\tau \mathbb{1}_{\{\tau \le s\}}|\mathcal{F}_t). \tag{7.5}$$

PROOF. Note that for $X = 1$ equality (7.5) is trivially satisfied. In general, it suffices to consider an elementary $\mathbb{F}$-adapted process of the form $X_t = \mathbb{1}_A \mathbb{1}_{[u,\infty)}(t)$ for a fixed, but arbitrary, $u \ge 0$ and any event $A \in \mathcal{F}_u$. Obviously, both sides of (7.5) vanish when $u > s$. For any $u \le s$, we obtain

$$\begin{aligned}
F_{s,t}&\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A \mathbb{1}_{[u,\infty)}(\tau)\mathbb{1}_{\{\tau \le s\}}|\mathcal{F}_s\big)\\
&= \mathbb{1}_A F_{s,t}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le s\}} - \mathbb{1}_{\{\tau < u\}}|\mathcal{F}_s)\\
&= \mathbb{1}_A F_{s,t}(F_{s,s} - F_{u-,s}) = \mathbb{1}_A(F_{s,t}F_{s,s} - F_{u-,t}F_{s,s})\\
&= \mathbb{1}_A F_{s,s}(F_{s,t} - F_{u-,t}) = \mathbb{1}_A F_{s,s}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le s\}} - \mathbb{1}_{\{\tau < u\}}|\mathcal{F}_t)\\
&= F_{s,s}\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A \mathbb{1}_{[u,\infty)}(\tau)\mathbb{1}_{\{\tau \le s\}}|\mathcal{F}_t\big),
\end{aligned}$$

where we used condition (2.3) in the third equality. □

PROPOSITION 7.3. *Assume that*:

(i) *the* $(\mathbb{P}, \mathbb{F})$*-conditional distribution of a random time τ is completely separable and $F_0 = 0$*;

(ii) *the process $Z^{\mathbb{G}}$ given by formula* (7.2) *is a positive* $(\mathbb{P}, \mathbb{G})$*-martingale such that $\mathbb{E}_{\mathbb{P}}(Z^{\mathbb{G}}_t|\mathcal{F}_t) = 1$ for every $t \in \mathbb{R}_+$.*

Then the hypothesis (H) holds under $\bar{\mathbb{P}}$ defined by $\bar{\mathbb{P}} = Z^{\mathbb{G}}\mathbb{P}$ and $\bar{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$.

PROOF. It suffices to show that (7.1) holds under $\bar{\mathbb{P}}$, where the probability measure $\bar{\mathbb{P}}$ is defined on $\mathcal{G}_t$ by $d\bar{\mathbb{P}}|_{\mathcal{G}_t} = Z^{\mathbb{G}}_t\, d\mathbb{P}|_{\mathcal{G}_t}$. We observe that, for all $0 \le u \le t$,

$$\begin{aligned}
\bar{F}_{u,u} &= \mathbb{P}\big(Z^{\mathbb{G}}_u \mathbb{1}_{\{\tau \le u\}}|\mathcal{F}_u\big) = \mathbb{P}(\widehat{Z}_{\tau,u}\mathbb{1}_{\{\tau \le u\}}|\mathcal{F}_u)\\
&= (X_u)^{-1}\mathbb{P}(X_\tau \mathbb{1}_{\{\tau \le u\}}|\mathcal{F}_u) = (X_t)^{-1}\mathbb{P}(X_\tau \mathbb{1}_{\{\tau \le u\}}|\mathcal{F}_t)\\
&= \mathbb{P}(\widehat{Z}_{\tau,t}\mathbb{1}_{\{\tau \le u\}}|\mathcal{F}_t) = \mathbb{P}\big(Z^{\mathbb{G}}_t \mathbb{1}_{\{\tau \le u\}}|\mathcal{F}_t\big) = \bar{F}_{u,t},
\end{aligned}$$

where the fourth equality is an immediate consequence of Lemma 7.3. □

An important conclusion from Proposition 7.3 is that if τ is a strictly positive random time such that the complete separability of the $(\mathbb{P}, \mathbb{F})$-conditional distribution holds then, under technical conditions of Proposition 7.3, we have that $\mathcal{I}(\mathbb{P}) \neq \varnothing$. Therefore, part (ii) in Proposition 7.2 can be applied in these circumstances to the progressive enlargement $\mathbb{G}$, leading to the conclusion that the model with the enlarged filtration inherits the arbitrage-free property of the original market model.

7.3. *Maximization of the expected utility.* We conclude this work by outlining an application of the semimartingale decomposition result for a pseudo-initial time in the context of the maximization of the expected utility. For simplicity of presentation, we assume in this subsection that all $(\mathbb{P}, \mathbb{F})$-local martingales are continuous, that is, the assumption (**C**) is valid. We aim to apply the semimartingale decompositions developed in Section 6 to utility maximization and information theory associated with continuous-time market models, as studied, in particular, by Ankirchner and Imkeller [1]. We will need the following definition borrowed from Ankirchner and Imkeller [1] (see Definition 1.1 in [1]). In what follows, the process S can be interpreted as the discounted price of a risky asset. We refer the reader to [1] for the motivation of this definition in the context of the maximization of the expected logarithmic utility from the portfolio's wealth when trading in S.

DEFINITION 7.1. A filtration $\mathbb{F}$ is said to be a *finite utility filtration* for a process S whenever S is a $(\mathbb{P}, \mathbb{F})$-semimartingale with the semimartingale decomposition of the form

$$S_t = U_t + \int_{(0,t]} \phi_u \, d\langle U, U\rangle_u \tag{7.6}$$

for some $(\mathbb{P}, \mathbb{F})$-local martingale U and an $\mathbb{F}$-predictable process ϕ.

REMARK 7.4. Delbaen and Schachermayer [8] showed that for a locally bounded semimartingale S, the existence of an equivalent local martingale measure for S [or, equivalently, the property that S satisfies the no free lunch with vanishing risk (NFLVR) condition] implies that the decomposition of the process S must be of the form given in equation (7.6). It is also interesting to mention that (7.6) is also closely related to the *structure condition* introduced by Schweizer [43] and studied, among others, by Choulli and Stricker [6].

From now on, we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F}$ satisfying the usual conditions and such that $\mathbb{F}$ is a finite utility filtration for a given process S so that decomposition (7.6) holds for some $(\mathbb{P}, \mathbb{F})$-martingale U. The following definition comes from Imkeller [18]; see also Definition 1.2 in [1].

DEFINITION 7.2. Let $\mathbb{K}$ be any enlargement of the filtration $\mathbb{F}$. The $\mathbb{K}$-predictable process ψ such that the process

$$U_t - \int_{(0,t]} \psi_u \, d\langle U, U\rangle_u$$

is a $(\mathbb{P}, \mathbb{K})$-local martingale is called the *information drift* of $\mathbb{K}$ with respect to $\mathbb{F}$.

REMARK 7.5. It was shown in [1] (see Proposition 1.2 therein) that the difference of the maximal expected logarithmic utilities when trading in S is based on two different filtrations $\mathbb{F} \subset \mathbb{K}$ depends only on the information drift of $\mathbb{K}$ with respect to $\mathbb{F}$.

Under the standing assumption (**C**) all $(\mathbb{P}, \mathbb{F})$-local martingales encountered in what follows are locally bounded, and thus locally square-integrable, so that we are in a position to apply the Kunita–Watanabe decomposition theorem; see, for instance, Protter [40]. We will show that the progressive enlargement of $\mathbb{F}$ with a pseudo-initial time τ is once again a finite utility filtration for S and we will compute the information drift of the progressive enlargement $\mathbb{G}$ with respect to $\mathbb{F}$.

PROPOSITION 7.4. *If $\mathbb{F}$ is a finite utility filtration for S, and τ is a pseudo-initial time, then the progressive enlargement $\mathbb{G}$ is a finite utility filtration for S. Furthermore, the information drift of $\mathbb{G}$ with respect to $\mathbb{F}$ is given by the following expression*:

$$\psi_u = \mathbb{1}_{\{\tau \geq u\}}(G_u)^{-1}\eta_u + \mathbb{1}_{\{\tau < u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}, \tag{7.7}$$

where η is given by (7.11) *and the $\mathbb{F}$-predictable processes $(\xi_{s,u})_{s\leq u}$ are defined by the Kunita–Watanabe decompositions*

$$m_{s,t} = \int_{(s,t]} \xi_{s,u}\, dU_u + L_t^s, \tag{7.8}$$

where L^s is a family of $(\mathbb{P}, \mathbb{F})$-local martingales strongly orthogonal to U for every s.

PROOF. By the standing assumption, the filtration $\mathbb{F}$ is a finite utility filtration for the process S. Therefore, the process S admits the $(\mathbb{P}, \mathbb{F})$-semimartingale decomposition (7.6), where U is a $(\mathbb{P}, \mathbb{F})$-local martingale and ϕ is an $\mathbb{F}$-predictable process. To establish the first assertion, it suffices to show that the process S is a $(\mathbb{P}, \mathbb{G})$-semimartingale with the following decomposition:

$$S_t = U_t^* + \int_{(0,t]} \phi_u^*\, d\langle U^*, U^*\rangle_u, \tag{7.9}$$

where U^* is some $(\mathbb{P}, \mathbb{G})$-martingale, and ϕ^* is some $\mathbb{G}$-predictable process. Using Corollary 6.5 and the assumption that all $(\mathbb{P}, \mathbb{F})$-local martingales are continuous (so that $\widetilde{C} = \langle U, M\rangle$), we deduce that the process U^*, which is given by the expression

$$\begin{aligned} U_t^* = U_t &- \int_{(0,t]} \mathbb{1}_{\{\tau \geq u\}}(G_u)^{-1}\, d\langle U, M\rangle_u \\ &- \int_{(t\wedge s,t]} (m_{s,u-})^{-1}\, d\langle U, m_{s,\cdot}\rangle_u \Big|_{s=\tau}, \end{aligned} \tag{7.10}$$

is a $(\mathbb{P}, \mathbb{G})$-local martingale. Recall that we denote by M the $(\mathbb{P}, \mathbb{F})$-local martingale appearing in the Doob–Meyer decomposition of the Azéma supermartingale G. An application of the Kunita–Watanabe decomposition theorem to M and U, after suitable localization if required, yields

$$M_t = \int_{(0,t]} \eta_u\, dU_u + \widehat{L}_t, \tag{7.11}$$

where η is some $\mathbb{F}$-predictable process and a square-integrable $(\mathbb{P},\mathbb{F})$-martingale $\widehat{L}$ is strongly orthogonal to U. It thus follows immediately from (7.11) that

$$d\langle U,M\rangle_u = \eta_u\, d\langle U,U\rangle_u.$$

In the next step, we focus on the $(\mathbb{P},\mathbb{F})$-martingale $(m_{s,t})_{t\ge s}$, for any fixed $s\in\mathbb{R}_+$. Using once again the Kunita–Watanabe decomposition theorem, we deduce that (7.8) holds for all $s\le t$ where $(\xi_{s,u})_{u\ge s}$ is a family of $\mathbb{F}$-predictable processes parametrized by s and L^s is a family of $(\mathbb{P},\mathbb{F})$-local martingales, such that L^s is strongly orthogonal to U for every s. Consequently, by combining (7.8), (7.10) and (7.11), we arrive at the following equalities:

$$\begin{aligned} U_t^* &= U_t - \int_{(0,t]} \mathbb{1}_{\{\tau\ge u\}}(G_u)^{-1}\eta_u\, d\langle U,U\rangle_u + \int_{(0,t]} \mathbb{1}_{\{\tau<u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}\, d\langle U,U\rangle_u \\ &= U_t - \int_{(0,t]} \big(\mathbb{1}_{\{\tau\ge u\}}(G_u)^{-1}\eta_u + \mathbb{1}_{\{\tau<u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}\big)\, d\langle U,U\rangle_u. \end{aligned}$$

It is clear that the equality $\langle U,U\rangle = \langle U^*,U^*\rangle$ holds. Therefore, the canonical $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of U reads

$$U_t = U_t^* + \int_{(0,t]} \psi_u\, d\langle U^*,U^*\rangle_u, \tag{7.12}$$

where the $\mathbb{G}$-predictable process ψ is given by the following expression:

$$\psi_u = \mathbb{1}_{\{\tau\ge u\}}(G_u)^{-1}\eta_u + \mathbb{1}_{\{\tau<u\}}(m_{\tau,u})^{-1}\xi_{\tau,u}.$$

It is now easy to see that the $(\mathbb{P},\mathbb{G})$-semimartingale decomposition of the price process S has indeed the desired form (7.9) with $\phi^* = \phi + \psi$. To obtain equality (7.7), it is enough to use Definition 7.2. The asserted formula follows directly from representation (7.12) and the fact that $\langle U^*,U^*\rangle = \langle U,U\rangle$. □

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SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SYDNEY
SYDNEY, NSW 2006
AUSTRALIA
E-MAIL: liboli99@gmail.com

SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SYDNEY
SYDNEY, NSW 2006
AUSTRALIA
AND
FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
00-661 WARSZAWA
POLAND
E-MAIL: marek.rutkowski@sydney.edu.au