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BROWNIAN EARTHWORM[1]

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We prove that the distance between two reflected Brownian motions, driven by the same white noise, outside a sphere in a 3-dimensional flat torus does not converge to 0, a.s., if the radius of the sphere is sufficiently small, relative to the size of the torus.

1. Introduction. This article is partly motivated by a natural phenomenon. We would like to analyze the effect of a randomly moving earthworm on the soil. The soil is pushed aside by the earthworm. What is the asymptotic distribution of soil particles when time goes to infinity? Is the soil compacted, or are soil particles more or less evenly spread over the region, especially when the earthworm is small compared to the size of the region? The answer seems to depend on the shape of the earthworm; for example, we believe that the soil is compacted if the "earthworm" is cubical. In our toy model, the earthworm is represented by a sphere following a Brownian path. We conjecture that in this model, the soil particles will be more or less evenly spread over the region. Our rigorous results in this paper partly justify these heuristic claims. We will next state the model in rigorous terms and then present a theorem and some conjectures. We will also briefly review related results. The earthworm picture will be mathematically interpreted after Conjecture 1.6.

Let $\mathbb{T}_1$ be the flat d-dimensional torus with side length 2, that is, $\mathbb{T}_1$ is the cube $\{(x_1, \dots, x_d) \in \mathbb{R}^d : |x_k| \le 1 \text{ for } k = 1, \dots, d\}$, with the opposite sides identified in the usual way. Let $\mathcal{B}(x, r)$ denote the open ball with center x and radius r. For $0 < r < 1$, let $D = \mathbb{T}_1 \setminus \overline{\mathcal{B}(0, r)}$. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D = \partial \mathcal{B}(0, r)$. Let B be a standard d-dimensional Brownian motion, $x_0, y_0 \in \overline{D}$, $x_0 \ne y_0$ and consider the following Skorokhod equations:

$$X_t = x_0 + B_t + \int_0^t \mathbf{n}(X_s)\, dL_s^X \qquad \text{for } t \ge 0, \tag{1.1}$$

$$Y_t = y_0 + B_t + \int_0^t \mathbf{n}(Y_s)\, dL_s^Y \qquad \text{for } t \ge 0. \tag{1.2}$$

Here L^X is the local time of X on ∂D. In other words, L^X is a nondecreasing continuous process which does not increase when X is in D, that is,

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$\int_0^\infty \mathbf{1}_D(X_t)\, dL_t^X = 0$, a.s. Equation (1.1) has a unique pathwise solution (X, L^X) such that $X_t \in \overline{D}$ for all $t \geq 0$; see [11]. The reflected Brownian motion X is a strong Markov process. The same remarks apply to (1.2), so (X, Y) is also strong Markov. Note that on any time interval (s, t) such that $X_u \in D$ and $Y_u \in D$ for all $u \in (s, t)$, we have $X_u - Y_u = X_s - Y_s$ for all $u \in (s, t)$.

For $x, y \in \mathbb{T}_1$, we use $\operatorname{dist}(x, y)$ to denote the geodesic distance between x and y in the torus $\mathbb{T}_1$.

THEOREM 1.1. *When the dimension $d = 3$, there is $r_0 > 0$ such that for every $r \leq r_0$ and every $x_0 \neq y_0$, we have $\limsup_{t\to\infty} \operatorname{dist}(X_t, Y_t) > 0$, a.s.*

An analogous problem was considered in [5] for planar domains D. It was proved that if D is a bounded domain with a smooth boundary and at most one hole, then $\lim_{t\to\infty} \operatorname{dist}(X_t, Y_t) = 0$, a.s. It is not known whether there exists a two-dimensional domain D such that we have $\limsup_{t\to\infty} \operatorname{dist}(X_t, Y_t) > 0$ with positive probability.

Note that by the pathwise uniqueness of the solutions to (1.1)–(1.2), 0 is an absorbing state for the distance process $\operatorname{dist}(X_t, Y_t)$; that is, if $\operatorname{dist}(X_{t_0}, Y_{t_0}) = 0$, then $\operatorname{dist}(X_t, Y_t) = 0$ for all $t \geq t_0$. Theorem 1.1 says that $\operatorname{dist}(X_t, Y_t)$ never enters the absorbing state 0 nor converges to 0 as $t \to \infty$. Since D is compact, this suggests that $\operatorname{dist}(X_t, Y_t)$ fluctuates and is a "recurrent" process. We suspect that (X_t, Y_t) has a stationary probability distribution but this does not follow from recurrence alone. Hence, we propose the following

CONJECTURE 1.2. *When the dimension $d = 3$, there is $r_0 > 0$ such that for $r \leq r_0$ the process (X, Y) has a stationary distribution Q which does not charge the diagonal $\{(x, x) : x \in \overline{D}\}$. There is only one stationary distribution for (X, Y) which does not charge the diagonal.*

Since (1.1)–(1.2) have a unique pathwise solution, if $x_0 = y_0$, then $X_t = Y_t$ for all $t \geq 0$, a.s. It follows that (X, Y) has a unique stationary distribution Q' supported on the diagonal, characterized by the fact that the distribution of X under Q' is uniform in D.

Our state space D for reflected Brownian motion is a subset of a torus because three-dimensional Brownian motion is transient so the result analogous to Theorem 1.1 for the complement of a ball in $\mathbb{R}^3$ is not interesting. Moreover, the boundary of D has no other component besides $\partial\mathcal{B}(0, r)$ so the relative position of X and Y is determined solely by the interaction of the processes with $\partial\mathcal{B}(0, r)$.

PROBLEM 1.3. *Is Theorem 1.1 valid when the dimension $d = 2$?*

The reader may find it paradoxical that we can prove Theorem 1.1 in 3 dimensions, but the analogous result in 2 dimensions is stated as an open problem.

The reason is that the proof depends in a crucial way on the sign of a certain "Lyapunov exponent" $\lambda_\rho^* = 1 + \lambda_\rho$ where $\rho := 1/r$ and λ_ρ is defined in Theorem 3.1(ii) relative to the domain D. We prove in Lemma 3.2 that λ_ρ^* is positive for D if $d = 3$ and ρ is large. In the 2-dimensional case, the analogous exponent is equal to 0 [5], Proposition 2.3, and this critical value makes the problem harder. We could have defined the domain D as $\mathbb{T}_1 \setminus A$, with A being not necessarily a ball. It is easy to see that for many sets A, for example, those that are bounded, smooth and close to a polyhedron, λ^* is negative. It was shown in [5] that in 2-dimensional space, negative λ^* implies that $\lim_{t\to\infty} \operatorname{dist}(X_t, Y_t) = 0$, a.s. In such a case, (X, Y) does not have a stationary distribution with some mass outside the diagonal. It is not known whether there is a 2-dimensional domain, bounded or unbounded, with positive λ^*. This is related to another open problem that we have already mentioned—it is not known whether there exists a two-dimensional domain D such that $\limsup_{t\to\infty} \operatorname{dist}(X_t, Y_t) > 0$ with positive probability. Theorem 1.1 shows that this is the case for a subset of a three-dimensional torus. We believe that the theorem also holds in some bounded subsets of $\mathbb{R}^3$, but we will not provide a rigorous proof. We make this claim more precise in the following conjecture.

CONJECTURE 1.4. *Suppose that* $\mathcal{B}(x_j, r) \subset \mathcal{B}(0, 1)$ *for* $j = 1, \dots, k$, *and let* $D_1 = \mathcal{B}(0, 1) \setminus \bigcup_{j=1}^k \overline{\mathcal{B}(x_j, r)} \subset \mathbb{R}^3$. *If k is sufficiently large and* $(\min_{1\le j\le k}(1 - |x_j|) + \min_{1\le i<j\le k} |x_i - x_j|)/r$ *is sufficiently large, then Theorem* 1.1 *holds for* D_1.

Suppose that Conjecture 1.2 is true, that is, for some $r_0 > 0$ and all $r \le r_0$, the process (X, Y) has a stationary distribution Q which does not charge the diagonal. This stationary measure Q depends on r, the radius of the ball deleted from the torus $\mathbb{T}_1$, so we can write Q_r to emphasize this dependence.

CONJECTURE 1.5. *The measures Q_r converge to the uniform probability distribution on* $(\mathbb{T}_1)^2$ *as* $r \to 0$.

Next, we consider the flow X_t^x of reflected Brownian motions, defined for $x \in \overline{D}$ by

$$X_t^x = x + B_t + \int_0^t \mathbf{n}(X_s^x)\, dL_s^x \qquad \text{for } t \ge 0. \tag{1.3}$$

Here L^x is the local time of X^x on ∂D. Equation (1.3) have unique pathwise solutions (X^x, L^x) for all x simultaneously because the construction of the solution to the Skorokhod equation given in [11] is deterministic. Let $|A|$ denote the Lebesgue measure of a set A and $\mathbf{Q}_{r,t}(A) = |\{x \in D : X_t^x \in A\}|$. We note that $\mathbf{Q}_{r,t}$ is a random measure. For the definitions of a random measure and weak convergence of random measures, see, for example, [10]; we will not review these notions here as they are not used in the core of our paper.

CONJECTURE 1.6. *The measures $\mathbf{Q}_{r,t}$ converge to a random measure $\mathbf{Q}_r$ on $\mathbb{T}_1 \setminus \mathcal{B}(0,r)$ when $t \to \infty$, in the sense of weak convergence of random measures. Random measures $\mathbf{Q}_r$ converge weakly to the uniform measure on $\mathbb{T}_1$ when $r \to 0$, in probability.*

In the context of (1.3), the earthworm picture is obtained by interpreting $\mathcal{B}(0,r) - B_t$ as a Brownian earthworm and $X_t^x - B_t$ as the location of a displaced soil particle.

For an extensive review of related results, see [4]. Some of those results will be recalled in Section 2.4. The present article is, philosophically speaking, a mirror image of [5]. That article analyzed domains where $\operatorname{dist}(X_t, Y_t)$ converged to 0, while the present article analyzes domains where the opposite is true. It was proved in [8, 9] that, under mild technical assumptions on the domain, reflected Brownian motions X and Y do not coalesce in a finite time. A series of papers by Pilipenko [13, 14, 16] discuss stochastic flows of reflected processes. The article [15] is posted on Math ArXiv; it is a review and discussion of Pilipenko's previously published results.

We will now outline the idea of the proof of our main result, Theorem 1.1. When the distance between the two solutions to the Skorokhod problem X and Y is small, it changes in two distinct ways. It increases at a rate proportional to the local time spent by the processes on ∂D, due to the fact that ∂D is curved and, therefore, the directions in which X and Y are pushed are slightly different. The distance between the two processes has negative jumps at the ends of excursions of X and Y from ∂D because the difference between the two processes is not (approximately) parallel to ∂D at the ends of excursions; hence the local time push has a different effect on the two trajectories. A discrete version of these ideas is expressed in a formal way in (2.3) below. The origin of these ideas goes back at least to the paper by Airault [1]. The continuous rate of increase of the distance between X and Y is greater than the combined effect of negative jumps over long periods of time, on average, for the domain D—this is the main estimate of this paper, derived in Section 3. The main body of the paper is devoted to detailed arguments showing that all modes of behavior of the two processes not captured by the above description but theoretically possible (such as coupling of the two processes at a finite time) have negligibly small probability.

The rest of the paper is organized as follows. Section 2 is a review of known results needed in this paper, including a review of excursion theory in Section 2.3, some technical estimates from [4, 6] in Section 2.4 and preliminary analysis of the coupling. The paper is based in an essential way on the exact and explicit evaluation of the Lyapunov exponent λ_ρ. The calculation is presented in Section 3. The proof of Theorem 1.1 is given in Section 4; it consists of several lemmas.

2. Preliminaries.

2.1. *General.* For a process Z, a set A and a point a in the state space of Z, let $T_A^Z = \inf\{t \geq 0 : Z_t \in A\}$, $T_a^Z = \inf\{t \geq 0 : Z_t = a\}$ and $\tau_A^Z = \inf\{t \geq 0 : Z_t \notin A\}$. By the Brownian scaling, if $\{X_t; t \geq 0\}$ is the reflecting Brownian motion on $\mathbb{T}_1 \setminus \overline{\mathcal{B}(0,r)}$ driven by Brownian motion B_t, then $\{r^{-1}X_{r^2 t}; t \geq 0\}$ is the reflecting Brownian motion on $(r^{-1}\mathbb{T}_1) \setminus \overline{\mathcal{B}(0,1)}$ driven by Brownian motion $r^{-1}B_{r^2 t}$. For notational convenience, throughout the remaining part of this paper, we fix $\rho = 1/r > 1$ and take $\mathbb{T}_\rho$ to be the flat 3-dimensional torus with side length $2\rho > 2$, that is, $\mathbb{T}_\rho$ is the cube $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_k| \leq \rho, k = 1, 2, 3\}$, with the opposites sides identified in the usual way, and let $D = \mathbb{T}_\rho \setminus \overline{\mathcal{B}(0,1)}$.

2.2. *Linear structure in torus.* In Section 1, we used notation normally reserved for elements of linear spaces, such as vector sum (e.g., $X_s - Y_s$) and norm (e.g., $|X_t - Y_t|$). We will now make this convention precise. Note that the torus $\mathbb{T}_\rho$ can be represented as the quotient $(\mathbb{R}/(2\rho\mathbb{Z}))^3$. For $x \in \mathbb{T}_\rho$, let A_x denote the set of all points in $\mathbb{R}^3$ which correspond to x. For $x, y \in \mathbb{T}_\rho$, we choose $x_1 \in A_x$ and $y_1 \in A_y$ with the minimal distance $|x_1 - y_1|$ among all such pairs. Then we let $x - y = x_1 - y_1$ and $\operatorname{dist}(x, y) = |x - y| = |x_1 - y_1|$.

2.3. *Review of excursion theory.* This section contains a brief review of excursion theory needed in this paper. See, for example, [12] for the foundations of the theory in the abstract setting and [3] for the special case of excursions of Brownian motion. Although Burdzy [3] does not discuss reflected Brownian motion, all results we need from his book readily apply in the present context. We will use two different, but closely related, "exit systems." The first one, presented below, is a simple exit system representing excursions of a single reflected Brownian motion from ∂D. The second exit system encodes the information about both processes X and Y, but it is essentially equivalent to the first exit system. We will introduce and use the second exit system in step 2.3 of the proof of Lemma 4.2. Our review applies to general domains D with smooth boundaries, but we will assume that D is the torus with the unit ball removed, as in Section 2.1.

Let $\mathbb{P}^{x_0}$ denote the distribution of the process X defined by (1.1), and let $\mathbb{E}^{x_0}$ be the corresponding expectation. Let $\mathbb{P}_D^x$ denote the distribution of Brownian motion starting from $x \in D$ and killed upon exiting D.

An "exit system" for excursions of the reflected Brownian motion X from ∂D is a pair (L_t^*, H^x) consisting of a positive continuous additive functional L_t^* of X and a family of "excursion laws" $\{H^x\}_{x \in \partial D}$. Let $\mathbf{\Delta}$ denote the "cemetery" point outside $\overline{D}$, and let $\mathcal{C}$ be the space of all functions $f : [0, \infty) \to \overline{D} \cup \{\mathbf{\Delta}\}$ which are continuous and take values in $\overline{D}$ on some interval $[0, \zeta)$, and are equal to $\mathbf{\Delta}$ on $[\zeta, \infty)$. For $x \in \partial D$, the excursion law H^x is a σ-finite (positive) measure on $\mathcal{C}$, such that the canonical process is strong Markov on (t_0, ∞), for every $t_0 > 0$, with the transition probabilities $\mathbb{P}_D^{\cdot}$. Moreover, H^x gives zero mass to paths which do not start from x. We will be concerned only with the "standard" excursion laws; see Definition 3.2 of [3]. For every $x \in \partial D$ there exists a unique standard excursion law H^x in D, up to a multiplicative constant.

Excursions of X from ∂D will be denoted e or e_s, that is, if $s < u$, $X_s, X_u \in \partial D$, and $X_t \notin \partial D$ for $t \in (s,u)$, then $e_s = \{e_s(t) = X_{t+s}, t \in [0, u-s)\}$ and $\zeta(e_s) = u - s$. By convention, $e_s(t) = \mathbf{\Delta}$ for $t \geq \zeta(e_s)$, so $e_t \equiv \mathbf{\Delta}$ if $\inf\{s > t : X_s \in \partial D\} = t$.

Let $\sigma_t = \inf\{s \geq 0 : L_s^* \geq t\}$ and $\mathcal{E}_u = \{e_s : s < \sigma_u\}$. Let I be the set of left endpoints of all connected components of $(0,\infty) \setminus \{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula of [12]. For every $x \in \overline{D}$, every bounded predictable process V_t and every universally measurable function $f : \mathcal{C} \to [0,\infty)$ that vanishes on excursions e_t identically equal to $\mathbf{\Delta}$, we have

$$\mathbb{E}^x\left[\sum_{t\in I} V_t \cdot f(e_t)\right] = \mathbb{E}^x \int_0^\infty V_{\sigma_s} H^{X(\sigma_s)}(f)\,ds = \mathbb{E}^x \int_0^\infty V_t H^{X_t}(f)\,dL_t^*. \tag{2.1}$$

Here and elsewhere $H^x(f) = \int_{\mathcal{C}} f\,dH^x$. Intuitively speaking, (2.1) says that the right continuous version $\mathcal{E}_{t+}$ of the process of excursions is a Poisson point process on the local time scale with variable intensity $H^{\cdot}(f)$.

The normalization of the exit system is somewhat arbitrary. For example, if (L_t^*, H^x) is an exit system, and $c \in (0,\infty)$ is a constant, then $(cL_t^*, (1/c)H^x)$ is also an exit system. One can even make c dependent on $x \in \partial D$. Theorem 7.2 of [3] shows how to choose a "canonical" exit system; that theorem is stated for the usual planar Brownian motion, but it is easy to check that both the statement and the proof apply to the reflected Brownian motion. According to that result, we can take L_t^* to be the continuous additive functional whose Revuz measure is a constant multiple of the surface area measure dx on ∂D and H^x's to be standard excursion laws normalized so that

$$H^x(A) = \lim_{\delta\downarrow 0} \frac{1}{\delta}\mathbb{P}_D^{x+\delta\mathbf{n}(x)}(A) \tag{2.2}$$

for any event A in a σ-field generated by the process on an interval $[t_0,\infty)$, for any $t_0 > 0$. The Revuz measure of L^X is the measure $dx/(2|D|)$ on ∂D, that is, if the initial distribution of X is the uniform probability measure μ on D, then $\mathbb{E}^\mu \int_0^1 \mathbf{1}_A(X_s)\,dL_s^X = \int_A dx/(2|D|)$ for any Borel set $A \subset \partial D$. It has been shown in [5] that $L_t^* = L_t^X$.

2.4. *Differentiability of stochastic flow of reflected Brownian motions.* It was proved in [2, 4, 15], in somewhat different settings, that the stochastic flow of reflected Brownian motions is differentiable in the initial condition. We will use this result, and we will also need a key estimate from [4] that was partly developed in [6]. First, we will recall some notation from [4]. The notation may seem somewhat awkward in the present context because it was developed for complicated arguments. We leave most of this notation unchanged to help the reader consult the results in [4].

We consider ∂D to be a smooth, properly embedded, orientable hypersurface (i.e., submanifold of codimension 1) in $\mathbb{R}^3$, endowed with a smooth unit normal inward vector field $\mathbf{n}$. We consider ∂D as a Riemannian manifold with the induced metric. We use the notation $\langle \cdot, \cdot \rangle$ for both the Euclidean inner product on $\mathbb{R}^3$ and its restriction to the tangent space $\mathcal{T}_x \partial D$ for any $x \in \partial D$, and $|\cdot|$ for the associated norm. For any $x \in \partial D$, let $\pi_x : \mathbb{R}^3 \to \mathcal{T}_x \partial D$ denote the orthogonal projection onto the tangent space $\mathcal{T}_x \partial D$, so $\pi_x \mathbf{z} = \mathbf{z} - \langle \mathbf{z}, \mathbf{n}(x) \rangle \mathbf{n}(x)$, and let $\mathcal{S}(x) : \mathcal{T}_x \partial D \to \mathcal{T}_x \partial D$ denote the shape operator (also known as the Weingarten map), which is the symmetric linear endomorphism of $\mathcal{T}_x \partial D$ associated with the second fundamental form. It is characterized by $\mathcal{S}(x)\mathbf{v} = -\partial_{\mathbf{v}} \mathbf{n}(x)$ for $\mathbf{v} \in \mathcal{T}_x \partial D$, where $\partial_{\mathbf{v}}$ denotes the ordinary Euclidean directional derivative in the direction of $\mathbf{v}$.

Recall that $\boldsymbol{\Delta}$ is an extra "cemetery point" outside $\overline{D}$, so that we can send processes killed at a finite time to $\boldsymbol{\Delta}$. For $s \geq 0$ such that $X_s \in \partial D$ we let $\zeta(e_s) = \inf\{t > 0 : X_{s+t} \in \partial D\}$. Here e_s is an excursion starting at time s, that is, $e_s = \{e_s(t) = X_{t+s}, t \in [0, \zeta(e_s))\}$. We let $e_s(t) = \boldsymbol{\Delta}$ for $t \geq \zeta(e_s)$, so $e_t \equiv \boldsymbol{\Delta}$ if $\zeta(e_s) = 0$.

Let σ_t^X be the inverse of local time L_t^X, that is, $\sigma_t^X = \inf\{s \geq 0 : L_s^X \geq t\}$, and $\mathcal{E}_b = \{e_s : s < \sigma_b^X\}$. For $b, \varepsilon > 0$, let $\{e_{u_1}, e_{u_2}, \dots, e_{u_m}\}$ be the set of all excursions $e \in \mathcal{E}_b$ with $|e(0) - e(\zeta-)| \geq \varepsilon$. We assume that excursions are labeled so that $u_k < u_{k+1}$ for all k, and we let $\ell_k = L_{u_k}^X$ for $k = 1, \dots, m$. We also let $u_0 = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0 = 0$, $\ell_{m+1} = b$ and $\Delta \ell_k = \ell_{k+1} - \ell_k$. Let $x_k = e_{u_k}(\zeta-)$ be the right endpoint of excursion e_{u_k} for $k = 1, \dots, m$ and $x_0 = X_{u_0}$.

For $\mathbf{v}_0 \in \mathbb{R}^3$, let

$$\mathbf{v}_b = \exp\big(\Delta \ell_m \mathcal{S}(x_m)\big)\pi_{x_m} \cdots \exp\big(\Delta \ell_1 \mathcal{S}(x_1)\big)\pi_{x_1} \exp\big(\Delta \ell_0 \mathcal{S}(x_0)\big)\pi_{x_0} \mathbf{v}_0. \tag{2.3}$$

Note that all concepts based on excursions e_{u_k} depend implicitly on $\varepsilon > 0$, which is often suppressed in the notation. Let $\mathcal{A}_b^\varepsilon$ denote the linear mapping $\mathbf{v}_0 \to \mathbf{v}_b$.

It was proved in Theorem 3.2 in [6] that for every $b > 0$, a.s., the limit $\mathcal{A}_b := \lim_{\varepsilon \to 0} \mathcal{A}_b^\varepsilon$ exists and it is a linear mapping of rank 2. For any $\mathbf{v}_0$, with probability 1, $\mathcal{A}_b^\varepsilon \mathbf{v}_0 \to \mathcal{A}_b \mathbf{v}_0$ as $\varepsilon \to 0$, uniformly in b on compact sets.

Recall the stochastic flow X_t^x of reflected Brownian motions defined in (1.3). By Theorem 3.1 of [4], for every $x \in D$, $b > 0$ and compact set $K \subset \mathbb{R}^3$, we have a.s.,

$$\lim_{\varepsilon \to 0} \sup_{\mathbf{v} \in K} \big| \big(X_{\sigma_b^x}^{x+\varepsilon \mathbf{v}} - X_{\sigma_b^x}^x\big)/\varepsilon - \mathcal{A}_b \mathbf{v} \big| = 0, \tag{2.4}$$

where $\sigma_b^x = \inf\{t \geq 0 : L_t^{X^x} \geq b\}$. Informally speaking, the last formula says that $y \to X_{\sigma_b^x}^y$ is differentiable, that is, the stochastic flow X is differentiable in the space variable. Formula (2.3) represents a discrete approximation to the derivative $\mathcal{A}_b$. According to that formula, the approximation to the derivative is a composition of two types of linear mappings. After the k-th excursion, the projection on the tangent plane to ∂D at the endpoint of the kth excursion is added to the composition. Between excursions, the derivative expands or contracts (in the sense of the exponential function of a linear mapping) at the rate proportional to the curvature of ∂D at the point where the most recent excursion ended.

Consider some $b > 0$, and let $\sigma_* = \inf\{t \geq 0 : L_t^X \vee L_t^Y \geq b\}$. Thus defined σ_* is different from the random variable denoted by the same symbol in [4]. Article [4] is concerned with a stochastic flow, and σ_* denotes in that paper, roughly speaking, the time when at least one of the local times corresponding to reflected Brownian motions in the flow exceeds a certain level. The results and arguments given in [4] can be applied in our paper with our definition of σ_* because we are concerned only with two reflected Brownian motions X and Y.

For $\varepsilon_* > 0$, let

$$\{e_{t_1^*}, e_{t_2^*}, \dots, e_{t_{m^*}^*}\} = \{e_t \in \mathcal{E}_b : |e_t(0) - e_t(\zeta-)| \geq \varepsilon_*, t < \sigma_*\}. \tag{2.5}$$

These excursions are labeled so that $t_k^* < t_{k+1}^*$ for all k. We let $\ell_k^* = L_{t_k^*}^X$ for $k = 1, \dots, m^*$. We also let $t_0^* = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0^* = 0$, $\ell_{m^*+1}^* = L_{\sigma_*}^X$ and $\Delta \ell_k^* = \ell_{k+1}^* - \ell_k^*$. Let $x_k^* = e_{t_k^*}(\zeta-)$ for $k = 1, \dots, m^*$, and $x_0^* = X_{t_0^*}$. Let

$$\mathcal{I}_k = \exp(\Delta \ell_k^* \mathcal{S}(x_k^*)) \pi_{x_k^*}.$$

The arguments in [4] were given only for $b = 1$, but it is easy to see that they apply equally to any fixed value of $b > 0$.

Let $\mathbb{P}^{x_0, y_0}$ denote the distribution of the solution (X, Y) to (1.1)–(1.2), and let $\mathbb{E}^{x_0, y_0}$ denote the corresponding expectation.

Fix an arbitrarily small $c_3 > 0$. By (3.161) and (3.167) of [4], there exist $c_4, c_5, c_6, \varepsilon_0 > 0$, $\beta_1 \in (1, 4/3)$ and $\beta_2 \in (0, 4/3 - \beta_1)$ such that if $X_0 = x$, $Y_0 = y$, $|x - y| = \varepsilon < \varepsilon_0$ and $\varepsilon_* = c_4 \varepsilon$, then

$$|(Y_{\sigma_*} - X_{\sigma_*}) - \mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(Y_0 - X_0)| \leq |\Lambda| + \Xi, \tag{2.6}$$

where $|\Lambda| < c_3 \varepsilon$, $\mathbb{P}^{x,y}$-a.s., and

$$\mathbb{P}^{x,y}(|\Xi| > c_5 \varepsilon^{\beta_1}) \leq c_6 \varepsilon^{\beta_2}. \tag{2.7}$$

The meaning of Λ and Ξ is not important in the present paper. These random variables arise in the decomposition of the difference on the left-hand side of (2.6). The random variable Λ is "large" because it is bounded by a constant multiple of ε to power 1; on the positive side, this bound is deterministic. The random variable

Ξ is "small" because it is (typically) smaller than ε^{β_1} with $\beta_1 > 1$, but this bound does not hold with probability 1.

2.5. *Some path properties of couplings*. If no confusion may arise, x_0 and y_0 will be suppressed in the notation $\mathbb{P}^{x_0,y_0}$, $\mathbb{E}^{x_0,y_0}$ and $\mathbb{P}^{x_0,y_0}$-a.s., and we will use the notation "$\mathbb{P}$," "$\mathbb{E}$" and "a.s."

The next lemma says that if the two processes X and Y are close to each other and almost parallel to ∂D then they will stay almost parallel to ∂D as long as they do not move far away from the current position. The proof is based on an idea that will be used several times in this article; see steps 2.1, 2.2, 2.4 and 2.6 of the proof of Lemma 4.2. The argument is concerned with an interval where only one of the processes can have some local time push. The analysis of the relative positions of the two processes at the beginning and the end of the interval, and the direction of the local time push, leads to a (desired) contradiction. The idea is graphically illustrated in Figure 2 below (step 2.2 of the proof of Lemma 4.2) because that implementation yields the most convincing picture.

LEMMA 2.1. *Suppose that $x_1 \in \partial D$, $c_1 \in (0, 1/100)$, and let $D_1 = D \cap \mathcal{B}(x_1, c_1/4)$. Assume that $x_0, y_0 \in D_1$ and $|\langle x_0 - y_0, \mathbf{n}(x_1)\rangle| \le c_1|x_0 - y_0|$. Let $T_1 = \tau^X_{D_1} \wedge \tau^Y_{D_1}$. Suppose that X and Y solve (1.1)–(1.2) with $X_0 = x_0$ and $Y_0 = y_0$. Then a.s., $|\langle X_t - Y_t, \mathbf{n}(x_1)\rangle| \le c_1|X_t - Y_t|$ for all $t \le T_1$.*

PROOF. Observe that for $x_2 \in \partial D \cap D_1$ and $y_2 \in D_1$ we have $\langle x_2 - y_2, \mathbf{n}(x_1)\rangle \le c_1|x_2 - y_2|/2$. Moreover, for any $x_3 \in \partial D \cap D_1$, the angle between $\mathbf{n}(x_1)$ and $\mathbf{n}(x_3)$ is less than $c_1/2$ radians.

Assume that $|\langle X_t - Y_t, \mathbf{n}(x_1)\rangle| > c_1|X_t - Y_t|$ for some $t \le T_1$. We will show that this assumption leads to a contradiction. Let

$$T_2 = \inf\{t \ge 0 : |\langle X_t - Y_t, \mathbf{n}(x_1)\rangle| > c_1|X_t - Y_t|\}.$$

By assumption and the pathwise uniqueness of solutions to (1.1)–(1.2), $T_2 < T_1$ and $|X_{T_2} - Y_{T_2}| > 0$. We have $|\langle X_{T_2} - Y_{T_2}, \mathbf{n}(x_1)\rangle| = c_1|X_{T_2} - Y_{T_2}|$ so at most one of the points X_{T_2} and Y_{T_2} belongs to the boundary of D. At least one of these points belongs to ∂D because $t \to |\langle X_t - Y_t, \mathbf{n}(x_1)\rangle|/|X_t - Y_t|$ is constant over intervals where neither X nor Y visit ∂D. Suppose without loss of generality that $X_{T_2} \in \partial D$. Then, by the opening remarks, $\langle X_{T_2} - Y_{T_2}, \mathbf{n}(x_1)\rangle \le c_1|X_{T_2} - Y_{T_2}|/2$, and therefore,

$$T_2 = \inf\{t \ge 0 : \langle X_t - Y_t, \mathbf{n}(x_1)\rangle < -c_1|X_t - Y_t|\}. \tag{2.8}$$

In particular, $\langle X_{T_2} - Y_{T_2}, \mathbf{n}(x_1)\rangle = -c_1|X_{T_2} - Y_{T_2}|$. Let

$$T_3 = \inf\{s > T_2 : Y_s \in \partial D\} \wedge T_1.$$

Then $T_2 < T_3$ and $L^Y_{T_3} = L^Y_{T_2}$. Hence, for $t \in [T_2, T_3]$, we have

$$\begin{aligned}\langle X_t - Y_t, \mathbf{n}(x_1)\rangle &= \left\langle X_{T_2} - Y_{T_2} + \int_{T_2}^t \mathbf{n}(X_s)\, dL^X_s, \mathbf{n}(x_1)\right\rangle \\ &\geq -c_1|X_{T_2} - Y_{T_2}| + c_1\big(L^X_t - L^X_{T_2}\big) \\ &\geq -c_1|X_{T_2} - Y_{T_2}| + c_1\left|\int_{T_2}^t \mathbf{n}(X_s)\, dL^X_s\right| \\ &\geq -c_1\left|X_{T_2} - Y_{T_2} + \int_{T_2}^t \mathbf{n}(X_s)\, dL^X_s\right| \\ &= -c_1|X_t - Y_t|,\end{aligned}$$

contradicting the definition of T_2 in view of (2.8). This completes the proof of the lemma. □

LEMMA 2.2. *If $x, y \in \overline{D}$ and $x \neq y$, then $\mathbb{P}^{x,y}(X_t \neq Y_t$, for every $t \geq 0) = 1$.*

PROOF. The proof of the lemma consists of two main steps. The first step uses a result on differentiability of the stochastic flow of reflected Brownian motions. According to this result, under some assumptions, the derivative of the stochastic flow is a nontrivial linear mapping. Hence, different trajectories in the stochastic flow do not collide. This argument applies directly only when the starting points of X and Y are "almost parallel" to ∂D. The general case, presented in step 2 below, is dealt with by reducing it to the first case at an appropriate stopping time.

Assume that for some distinct $x, y \in \overline{D}$, $X_t = Y_t$ for some $t < \infty$, with positive probability. A standard application of the strong Markov property shows that there must exist $r \in (0, 1/200)$, $x_1 \in \partial D$ and $y_1 \in \overline{D}$ such that if we write $D_1 = \overline{D} \cap \mathcal{B}(x_1, r/8)$ and $T_1 = \tau^X_{D_1} \wedge \tau^Y_{D_1}$, then $\mathbb{P}^{x_1,y_1}(\exists t \in [0, T_1]: X_t = Y_t) > 0$. Note that necessarily $y_1 \in D_1$.

Step 1. Suppose that $r \in (0, 1/100)$, $x_1 \in \partial D$, $y_1 \in D_1$ and $x_1 \neq y_1$. In this step, we will consider the case when $|\langle x_1 - y_1, \mathbf{n}(x_1)\rangle| \leq (r/2)|x_1 - y_1|$.

Let $K_\delta = (x_1 + \mathcal{T}_{x_1}\partial D) \cap \partial\mathcal{B}(x_1, \delta)$ and $K^0_\delta = \mathcal{T}_{x_1}\partial D \cap \partial\mathcal{B}(0, \delta)$. Recall the stochastic flow X^x_t of reflected Brownian motions defined in (1.3), and note that $(X_t, Y_t) = (X^{x_1}_t, X^{y_1}_t)$ under $\mathbb{P}^{x_1,y_1}$. Let $\widehat{\sigma}_b = \inf\{t \geq 0: L^{X^{x_1}}_t \geq b\}$. According to Theorem 3.2 of [6] and its proof, for any fixed $b > 0$, $\mathcal{A}_b$ has rank 2. In fact, the proof shows more than that, namely, $\mathbb{P}^{x_1}$-a.s., $\inf_{\mathbf{v}\in K^0_\delta} |\mathcal{A}_b(\mathbf{v})| > 0$. This and (2.4) imply that for any $b > 0$,

$$\lim_{\delta\to 0} \mathbb{P}^{x_1}\Big(\inf_{\mathbf{v}\in K^0_\delta} \big|X^{x_1+\mathbf{v}}_{\widehat{\sigma}_b} - X^{x_1}_{\widehat{\sigma}_b}\big|/|\mathbf{v}| > 0\Big) = 1.$$

Since the stochastic differential equation (1.1) has a unique strong solution, if $X^x_t = X^y_t$ for some t, then $X^x_s = X^y_s$ for all $s \geq t$, a.s. Hence, the last formula can be strengthened as follows:

$$\lim_{\delta\to 0}\mathbb{P}^{x_1}\Big(\inf_{\mathbf{v}\in K^0_\delta}\inf_{0\le t\le\widehat{\sigma}_b}\big|X_t^{x_1+\mathbf{v}}-X_t^{x_1}\big|/|\mathbf{v}|>0\Big)=1.$$

For every $k\ge 1$ find $\delta_k>0$ such that

$$\mathbb{P}^{x_1}\Big(\inf_{\mathbf{v}\in K^0_{\delta_k}}\inf_{0\le t\le\widehat{\sigma}_b}\big|X_t^{x_1+\mathbf{v}}-X_t^{x_1}\big|/|\mathbf{v}|>0\Big)\ge 1-2^{-k}. \tag{2.9}$$

It follows from Lemmas 3.3 and 3.4 of [4] and their proofs that there exist stopping times S_k such that $S_k\to\infty$ as $k\to\infty$, and $|X_t^x-X_t^y|\le k|X_0^x-X_0^y|$ for al $x,y\in\overline{D}$ and $t\in[0,S_k]$, a.s. We can assume without loss of generality that $\delta_k\to 0$ as $k\to\infty$. We make $\delta_k>0$ smaller, if necessary, so that $|X_t^{x_1}-X_t^z|\le r/8$, for all $k\ge 1$, $z\in K_{\delta_k}$ and $t\in[0,S_k]$, a.s. By passing to a subsequence, if necessary, we may assume that

$$\mathbb{P}(S_k>\widehat{\sigma}_b)\ge 1-2^{-k}. \tag{2.10}$$

If we let $T_2=T_1\wedge\widehat{\sigma}_b$,

$$\begin{aligned}F_k^1&=\{\big|X_t^{x_1}-X_t^z\big|\le r/8,\forall z\in K_{\delta_k},t\in[0,T_2]\},\\ F_k^2&=\Big\{\inf_{\mathbf{v}\in K^0_{\delta_k}}\inf_{0\le t\le T_2}\big|X_t^{x_1+\mathbf{v}}-X_t^{x_1}\big|/|\mathbf{v}|>0\Big\},\\ F_k&=F_k^1\cap F_k^2,\end{aligned}$$

then, by (2.9) and (2.10), $\mathbb{P}(F_k)\ge 1-2^{-k+1}$.

We will argue that if F_k holds, then for all $t\in[0,T_2]$ and $z\in K_{\delta_k}$,

$$\big|\langle X_t-Y_t,\mathbf{n}(x_1)\rangle\big|\le(r/2)|X_t-Y_t|, \tag{2.11}$$

$$\big|\langle X_t-X_t^z,\mathbf{n}(x_1)\rangle\big|\le r\big|X_t-X_t^z\big|, \tag{2.12}$$

$$\big|\langle X_t^z-Y_t,\mathbf{n}(x_1)\rangle\big|\le r\big|X_t^z-Y_t\big|. \tag{2.13}$$

We obtain (2.11) from our assumption that $|\langle x_1-y_1,\mathbf{n}(x_1)\rangle|\le(r/2)|x_1-y_1|$ and Lemma 2.1. If F_k^1 holds, then $X_t^z\in\mathcal{B}(x_1,r/4)$ for all $t\in[0,T_2]$ and $z\in K_{\delta_k}$. Hence, (2.12) follows from Lemma 2.1 applied with $c_1=r$. The claim holds for all $z\in K_{\delta_k}$ simultaneously because Lemma 2.1 is deterministic. We can make $\delta_k>0$ smaller, if necessary, so that $|\langle z-y_1,\mathbf{n}(x_1)\rangle|\le r|z-y_1|$ for all k and all $z\in K_{\delta_k}$. Once again, we apply Lemma 2.1 with $c_1=r$ and conclude that (2.13) holds true.

Estimates (2.11)–(2.13) have the following topological consequences. Recall that $\pi_{x_1}\mathbf{z}$ denotes the projection of $\mathbf{z}$ on $\mathcal{T}_{x_1}\partial D$. Assuming that F_k holds and $t\le T_2$, the set $\Gamma_t=\pi_{x_1}\{X_t^x,x\in K_{\delta_k}\}$ is a closed loop that contains $\pi_{x_1}X_t$ inside. When t goes from 0 to T_2, $\pi_{x_1}X_t$, $\pi_{x_1}Y_t$ and Γ_t evolve continuously. If $X_t=Y_t$ for some $t\le T_2$, then we must have $\pi_{x_1}Y_s=\pi_{x_1}X_s^x$ for some $k\ge 1$, $x\in K_{\delta_k}$ and $0\le s\le t$. This and (2.13) imply that $Y_s=X_s^x$. Hence, $X_t=Y_t=X_t^x$. But this means that

F_k^2 does not hold. Since $\mathbb{P}(F_k) \geq 1 - 2^{-k+1}$, we conclude that the probability that there exists $t \in [0, T_2]$ such that $X_t = Y_t$ is less than 2^{-k+1}. Since k and b are arbitrarily large, $\mathbb{P}^{x_1,y_1}(\exists t \in [0, T_1] : X_t = Y_t) = 0$.

Step 2. Suppose that $r \in (0, 1/200)$, $x_1 \in \partial D$, $y_1 \in D_1$ and $x_1 \neq y_1$. In this step, we no longer assume that $|\langle x_1 - y_1, \mathbf{n}(x_1)\rangle| \leq (r/2)|x_1 - y_1|$. Also, note that we assume that $r \in (0, 1/200)$ while in step 1 we assumed that $r \in (0, 1/100)$.

Suppose that $\mathbb{P}^{x_1,y_1}(\exists t \in [0, T_1] : Y_t = X_t) = p_1 > 0$. We will show that this assumption leads to a contradiction. Let

$$A = \{y \in \overline{D} : |x_1 - y| = |x_1 - y_1|, \langle x_1 - y, \mathbf{n}(x_1)\rangle = \langle x_1 - y_1, \mathbf{n}(x_1)\rangle\}.$$

The set A is a circle, possibly with a zero radius. If the radius of A is 0, that is, if A contains only y_1, then $x_1 - y_1$ is parallel to $\mathbf{n}(x_1)$. It is easy to see that for any $t_0 > 0$, with probability 1, there exists time $t \in (0, t_0 \wedge T_1)$ such that $X_t \neq Y_t$, $X_t \in \partial D$, $X_t - Y_t$ is not parallel to $\mathbf{n}(X_t)$, and t is the terminal time of an excursion of X from ∂D. Let U_r be the smallest such t greater than $r > 0$. We can apply the strong Markov property at time U_r, for every rational time $r > 0$, and the result proved below for the case when A does not reduce to a single point to show that X and Y will not meet before T_1.

Hence, we will assume from now on that the set A is a circle with a nonzero radius. Choose n distinct points $y_1, \ldots, y_n$ in A, with $n > 2/p_1$. Let $T_1^{y_j} = \tau_{D_1}^X \wedge \tau_{D_1}^{X^{y_j}}$. By our assumption and symmetry, $\mathbb{P}^{x_1,y_j}(\exists t \in [0, T_1^{y_j}] : X_t = X_t^{y_j}) = p_1$. It follows that for some $j \neq k$,

$$\mathbb{P}(\exists t \in [0, T_1^{y_j}] : X_t = X_t^{y_j}, \text{ and } \exists s \in [0, T_1^{y_k}] : X_s = X_s^{y_k}) > 0.$$

If the event in the last formula holds, then for $u = s \vee t$ we have $X_u^{y_j} = X_u^{y_k}$ and $u \leq \tau_{D_1}^X = \tau_{D_1}^{X^{y_j}} = \tau_{D_1}^{X^{y_k}}$. In other words, we have shown that if $T_1^{y_j,y_k} = \tau_{D_1}^{X^{y_j}} \wedge \tau_{D_1}^{X^{y_k}}$, then $\mathbb{P}^{y_j,y_k}(\exists t \in [0, T_1^{y_j,y_k}] : X_t^{y_j} = X_t^{y_k}) > 0$. We will prove that this leads to a contradiction. If the processes X^{y_j} and X^{y_k} do not hit ∂D before $T_1^{y_j,y_k}$, then of course they do not meet before $T_1^{y_j,y_k}$. If one of them hits ∂D before time $T_1^{y_j,y_k}$, then we can suppose without loss of generality that $T_3 := T_{\partial D}^{X^{y_j}} \leq T_{\partial D}^{X^{y_k}} \wedge T_1^{y_j,y_k}$. Then $|\langle X_{T_3}^{y_j} - X_{T_3}^{y_k}, \mathbf{n}(X_{T_3}^{y_j})\rangle| \leq (r/4)|X_{T_3}^{y_j} - X_{T_3}^{y_k}|$. Since $T_3 \leq T_1^{y_j,y_k}$, $\mathcal{B}(x_1, r/8) \in \mathcal{B}(X_{T_3}^{y_j}, r/4)$. Let $T_4 = \tau^{X^{y_j}}_{\mathcal{B}(X_{T_3}^{y_j}, r/4)} \wedge \tau^{X^{y_k}}_{\mathcal{B}(X_{T_3}^{y_j}, r/4)}$. By step 1, applied with $2r$ in place of r, and the strong Markov property applied at T_3,

$$\begin{aligned}\mathbb{P}^{y_j,y_k}(\exists t \in [0, T_1^{y_j,y_k}] : X_t^{y_j} = X_t^{y_k}) &\leq \mathbb{P}^{y_j,y_k}(\exists t \in [0, T_4] : X_t^{y_j} = X_t^{y_k}) \\ &= \mathbb{P}^{y_j,y_k}(\exists t \in [T_3, T_4] : X_t^{y_j} = X_t^{y_k}) = 0.\end{aligned}$$

This contradicts our earlier assertion and finishes the proof. □

The next lemma is almost the same as a lemma that appeared in [5]. It says that at the time when the local time reaches a fixed level, the difference between the processes X and Y is very likely to be "almost parallel" to ∂D.

LEMMA 2.3. *For any $b > 0$ and $\beta_1 \in (0,1)$ there exist $c_0, \beta_2, \varepsilon_1 > 0$ such that if $\varepsilon \le \varepsilon_1$, $x, y \in \overline{D}$ and $|x - y| = \varepsilon$, then*

$$\mathbb{P}^{x,y}\left(\frac{|\langle Y_{\sigma_b^X} - X_{\sigma_b^X}, \mathbf{n}(X_{\sigma_b^X})\rangle|}{|Y_{\sigma_b^X} - X_{\sigma_b^X}|} \ge c_0 \varepsilon^{\beta_1}\right) \le \varepsilon^{\beta_2}. \tag{2.14}$$

PROOF. The proof is similar to the proof of Lemma 4.6 in [5], so we only sketch the main ideas. The paper [5] is concerned with 2-dimensional domains, but it is easy to see that the results from that paper that we use here apply to multidimensional domains.

By Lemma 4.1(ii) of [5], $\mathbb{P}(L^Y_{\sigma_b^X} \ge a) \le c_1 e^{-c_2 a}$. Hence, for any $\beta_3 > 0$ and $\beta_4 > 0$ depending on β_3,

$$\mathbb{P}\big(L^Y_{\sigma_b^X} \ge \beta_3 |\log \varepsilon|\big) \le c_1 \exp\big(-c_2 \beta_3 |\log \varepsilon|\big) = c_1 \varepsilon^{\beta_4}.$$

If the event $A_1 := \{L^Y_{\sigma_b^X} \le \beta_3 |\log \varepsilon|\}$ holds, then by Lemma 3.8 of [5],

$$\sup_{t \in [0, \sigma_b^X]} |X_t - Y_t| \le |X_0 - Y_0| \exp\big(c_4(1 + \beta_3 |\log \varepsilon|)\big) \le c_5 \varepsilon^{1 - c_4 \beta_3} = c_5 \varepsilon^{1 - \beta_5},$$

where β_5 is defined as $c_4 \beta_3$. Choose $\beta_3 > 0$ so small that $\beta_5 < \beta_1$, and we can find β_6 such that $\beta_1 < \beta_6 < 1 - \beta_5$.

Let $T_1 = \inf\{t \ge 0 : X_t \in \partial D\}$ and $\{V_t, 0 \le t \le \sigma_b^X - T_1\} := \{X_{\sigma_b^X - t}, 0 \le t \le \sigma_b^X - T_1\}$. If we condition on the values of X_{T_1} and $X_{\sigma_b^X}$, the process V is a reflected Brownian motion in D starting from $X_{\sigma_b^X}$ and conditioned to approach X_{T_1} at its lifetime. It is easy to see that $\mathbb{P}(|X_{T_1} - X_{\sigma_b^X}| \le \varepsilon^{\beta_1}) \le c_6 \varepsilon^{\beta_1}$.

Suppose that the event $A_2 := \{\mathrm{dist}(X_{T_1}, X_{\sigma_b^X}) \ge \varepsilon^{\beta_1}\}$ holds. Conditionally on this event, the probability that V does not spend at least ε^{β_6} units of local time on the boundary of ∂D before leaving the ball $\mathcal{B}(V_0, \varepsilon^{\beta_1})$ is bounded by $c_7 \varepsilon^{\beta_6 - \beta_1}$. Let A_3 be the event that V spends ε^{β_6} or more units of local time on the boundary of ∂D before leaving the ball $\mathcal{B}(V_0, \varepsilon^{\beta_1})$. Let $T_2 = \sup\{t \le \sigma_b^X : X_t \notin \mathcal{B}(V_0, \varepsilon^{\beta_1})\}$. If A_1 and A_3 hold, then Y must hit ∂D at some time $t \in [T_2, \sigma_b^X]$ because $\varepsilon^{\beta_6} > c_5 \varepsilon^{1-\beta_5}$ for small ε; that is, the amount of push given to X exceeds the maximum distance between the two processes. We also have $X_{\sigma_b^X} \in \partial D$. The maximum angle between normal vectors at points of $\partial D \cap \mathcal{B}(V_0, \varepsilon^{\beta_1})$ is less than $c_8 \varepsilon^{\beta_1}$. A modification of Lemma 2.1 shows that $|\langle X_{\sigma_b^X} - Y_{\sigma_b^X}, \mathbf{n}(X_{\sigma_b^X})\rangle| \le |X_{\sigma_b^X} - Y_{\sigma_b^X}| c_9 \varepsilon^{\beta_1}$. Recall that, by Lemma 2.2, $|X_{\sigma_b^X} - Y_{\sigma_b^X}| > 0$, a.s. We have shown that the complement of the event in (2.14) occurs if $A_1 \cap A_2 \cap A_3$ holds. Since $\mathbb{P}((A_1 \cap A_2 \cap A_3)^c) \le c_1 \varepsilon^{\beta_4} + c_6 \varepsilon^{\beta_1} + c_7 \varepsilon^{\beta_6 - \beta_1}$, the lemma follows. $\square$

3. The sign of the Lyapunov exponent. This section is devoted to the calculation of the "Lyapunov exponent" for the exterior of a three-dimensional ball. In our model, the Lyapunov exponent is represented by $1+\lambda_\rho$ where λ_ρ is defined in Theorem 3.1(ii). This is a three-dimensional analogue of an exponent defined in [5] for two-dimensional domains. The sign of this exponent—positive for the domain D—has the fundamental importance for this article.

Recall that H^x is the excursion law for X in D, and π_x denotes the projection on the plane tangent to ∂D at $x \in \partial D$. For an excursion e and nonzero vector $\mathbf{v} \in \mathbb{R}^3$, we let $f_{\mathbf{v}}(e) = \log|\pi_{e(\zeta-)}(\mathbf{v})| - \log|\mathbf{v}|$. Note that $f_{\mathbf{v}}(e) \le 0$. Let $D_1 = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$, and let $(\widehat{L}_t, \widehat{H}^x)$ be the exit system for reflected Brownian motion $\widehat{X}$ in D_1.

THEOREM 3.1. (i) *For every $x \in \partial D_1$ and $\mathbf{v} \in \mathcal{T}_x \partial D_1$, $|\mathbf{v}| > 0$,*

$$\widehat{H}^x(f_{\mathbf{v}}(e)) = \sqrt{2} - 1 - \log(1+\sqrt{2}).$$

(ii) *Let $\lambda_\rho(x,\mathbf{v}) = H^x(f_{\mathbf{v}}(e))$. We have uniformly in $x \in \partial D$ and $\mathbf{v} \in \mathcal{T}_x \partial D$, $|\mathbf{v}| > 0$,*

$$\lim_{\rho\to\infty} \lambda_\rho(x,\mathbf{v}) = \lim_{\rho\to\infty} H^x(f_{\mathbf{v}}(e)) = \sqrt{2} + \log 2 - 2 - \log(1+\sqrt{2}) \approx -0.774013.$$

The actual value of the Lyapunov exponents comes from a computation presented in the Appendix, which leads to the following lemma.

LEMMA 3.2. *We have*

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\pi} \frac{1}{16\pi} \frac{\sin\alpha}{\sin^3(\alpha/2)} \log(\sin^2\beta + \cos^2\beta\cos^2\alpha)^{1/2}\, d\alpha\, d\beta \\ &\quad = \sqrt{2} - 1 - \log(1+\sqrt{2}) \end{aligned} \tag{3.1}$$

and

$$\int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi} (\sin\alpha) \log(\sin^2\beta + \cos^2\beta\cos^2\alpha)^{1/2}\, d\alpha\, d\beta = \log 2 - 1. \tag{3.2}$$

PROOF OF THEOREM 3.1. (i) We will derive a formula for the expectation of a random variable under the excursion law from the well-known formula for the density of the harmonic measure.

Let $\tau_A^X = \inf\{t \ge 0 : X_t \notin A\}$. Recall that $\mathbb{P}_{D_1}^x$ denotes the distribution of Brownian motion starting from x and killed at the time $\tau_{D_1}^X$. Let μ_r denote the uniform probability distribution on the sphere $\mathcal{B}(0,r)$; we will abbreviate $\mu_1 = \mu$. An explicit formula for the harmonic measure in D_1 is given in [17], Theorem 3.1, page 102. That formula implies that

$$\mathbb{P}_{D_1}^x(X(\tau_{D_1}^X -) \in dy) = a(x)|x-y|^{-3}\mu(dy) \tag{3.3}$$

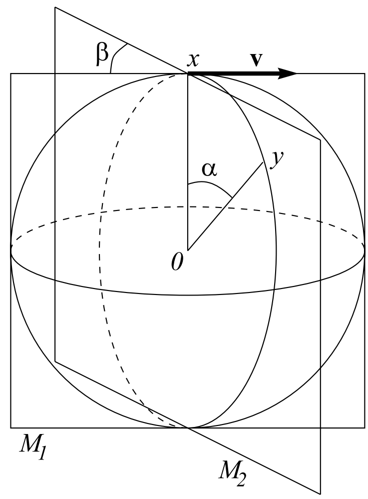

FIG. 1. *The spherical coordinates α and β used in the derivation of length reduction of the vector* **v**.

for $x \in D_1$ and $y \in \partial\mathcal{B}(0,1)$, where $a(x)$ is such that for $x, y \in \partial\mathcal{B}(0,1)$,

$$\lim_{\delta\downarrow 0} \frac{\mathbb{P}_{D_1}^{x+\delta\mathbf{n}(x)}(X(\tau_{D_1}^X -) \in dy)}{2\delta|x+\delta\mathbf{n}(x)-y|^{-3}\mu(dy)} = 1.$$

We use this and (2.2) to see that for $x, y \in \mathcal{B}(0,1)$,

$$\widehat{H}_{D_1}^x\big(e(\zeta-) \in dy\big) = 2|x-y|^{-3}\mu(dy). \tag{3.4}$$

Note that, by symmetry, $\widehat{H}^x(f_{\mathbf{v}}(e))$ does not depend on $x \in \partial D_1$ and $\mathbf{v} \in \mathcal{T}_x\partial D_1$, so we can fix arbitrarily $x \in \partial D_1$ and $\mathbf{v} \in \mathcal{T}_x\partial D_1$ with $|\mathbf{v}| > 0$. We will express $\mu(dy)$ and $f_{\mathbf{v}}(e)$ using spherical coordinates. Let α denote the angle between the radii of $\mathcal{B}(0,1)$ going from 0 to x and y in $\partial\mathcal{B}(0,1)$. Let M_1 be the plane that contains $\mathbf{v}$ and 0, and let M_2 be the plane that contains $0, x$ and y. Let β be the angle between M_1 and M_2; see Figure 1. The uniform probability measure on the sphere $\partial\mathcal{B}(0,1)$ can be represented as

$$\mu(dy) = (2\pi)^{-1}\, d\beta (1/2) \sin\alpha\, d\alpha. \tag{3.5}$$

We have $|x-y| = 2\sin(\alpha/2)$, so (3.4)–(3.5) yield

$$\begin{aligned} \widehat{H}_{D_1}^x\big(e(\zeta-) \in dy\big) &= 2\big(2\sin(\alpha/2)\big)^{-3}(2\pi)^{-1}\, d\beta(1/2)\sin\alpha\, d\alpha \\ &= \frac{1}{16\pi}\frac{\sin\alpha}{\sin^3(\alpha/2)}\, d\alpha\, d\beta. \end{aligned} \tag{3.6}$$

It is elementary (although somewhat tedious) to check that

$$\frac{|\pi_y(\mathbf{v})|}{|\mathbf{v}|} = \big(\sin^2\beta + \cos^2\beta\cos^2\alpha\big)^{1/2}.$$

If $e(\zeta-)=y$, then

$$f_{\mathbf{v}}(e)=\log\bigl|\pi_{e(\zeta-)}(\mathbf{v})\bigr|-\log|\mathbf{v}|=\log\bigl(\sin^2\beta+\cos^2\beta\cos^2\alpha\bigr)^{1/2}. \tag{3.7}$$

We combine this formula with (3.6) to see that

$$\widehat{H}^x_{D_1}\bigl(f_{\mathbf{v}}(e)\bigr)=\int_0^{2\pi}\int_0^{\pi}\frac{1}{16\pi}\frac{\sin\alpha}{\sin^3(\alpha/2)}\log\bigl(\sin^2\beta+\cos^2\beta\cos^2\alpha\bigr)^{1/2}\,d\alpha\,d\beta.$$

Part (i) of the theorem follows from this formula and Lemma 3.2.

(ii) We will divide excursions into two families—these that return to ∂D relatively soon, and those that travel far away from ∂D. The first part of the following argument shows that the excursions which travel far away are likely to hit ∂D at a random point distributed almost uniformly over ∂D. Excursions from ∂D which do not travel far away contribute to the estimate about as much as excursions from ∂D_1.

First, we will show that the harmonic measure in a spherical shell has a density very close to a constant, under some assumptions. Let $S(r,R)=\mathcal{B}(0,R)\setminus\overline{\mathcal{B}(0,r)}$ denote the spherical shell with center 0, inner radius r and outer radius R. Let $h(r,R;x,y)$ be the density of harmonic measure in $S(r,R)$ restricted to $\partial\mathcal{B}(0,r)$; more precisely, let

$$h(r,R;x,y)=\frac{\mathbb{P}^x_{S(r,R)}(X_{\tau^X_{S(r,R)}}\in dy)}{\mu_r(dy)}$$

for $x\in S(r,R)$ and $y\in\partial\mathcal{B}(0,r)$. For fixed r,R and y, the function $x\to h(r,R;x,y)$ is harmonic in $S(r,R)$. By the Harnack principle, there exists $c_1>0$ such that for any positive harmonic function f in $\mathcal{B}(0,1)$, we have $c_1<f(v)/f(z)<1/c_1$ for all $v,z\in\mathcal{B}(0,1/2)$. By scaling, for any $r>0$ and for any positive harmonic function f in $\mathcal{B}(0,r)$, we have $c_1<f(v)/f(z)<1/c_1$ for all $v,z\in\mathcal{B}(0,r/2)$. We can find a finite number N such that there exist $x_k\in\partial\mathcal{B}(0,2r)$, $k=1,\dots,N$, such that $\partial\mathcal{B}(0,2r)\subset\bigcup_{1\le k\le N}\mathcal{B}(x_k,r/2)$. Then the standard chaining argument shows that for $R\ge 3r$ and every positive harmonic function f in $S(r,R)$, we have $c_1^N<f(v)/f(z)<1/c_1^N$ for all $v,z\in\mathcal{B}(0,2r)$. Let $c_2=c_1^N$. Consider a large integer m. As a particular case of the last formula, we obtain that

$$c_2<h\bigl(2^k,2^m;x,y\bigr)/h\bigl(2^k,2^m;v,y\bigr)<1/c_2 \tag{3.8}$$

for $0\le k\le m-2$, $y\in\partial\mathcal{B}(0,2^k)$ and $x,v\in\partial\mathcal{B}(0,2^{k+1})$. By the strong Markov property for Brownian motion applied at the hitting time of $\partial\mathcal{B}(0,2^{k+1})$,

$$h\bigl(2^k,2^m;x,y\bigr)=\int_{\partial\mathcal{B}(0,2^{k+1})}h\bigl(2^k,2^m;v,y\bigr)h\bigl(2^{k+1},2^m;x,v\bigr)\mu_{2^{k+1}}(dv)$$

for $0\le k\le m-3$, $y\in\partial\mathcal{B}(0,2^k)$ and $x\in\partial\mathcal{B}(0,2^{k+2})$. This, (3.8) and Lemma 6.1 of [7] imply, using the same argument as at the end of the proof of Theorem 6.1 in [7], that for any $c_3<1$ arbitrarily close to 1 there exists m_0 such that for $m\ge m_0$,

$$c_3<h\bigl(1,2^m;x,y\bigr)/h\bigl(1,2^m;v,y\bigr)<1/c_3$$

for $y \in \partial\mathcal{B}(0,1)$ and $x, v \in \partial\mathcal{B}(0, 2^{m-1})$. By applying a rotation, we obtain the following variant of the above result. For any $c_3 < 1$ arbitrarily close to 1 there exists m_0 such that for $m \ge m_0$,

$$c_3 < h(1, 2^m; x, y)/h(1, 2^m; x, z) < 1/c_3 \tag{3.9}$$

for $y, z \in \partial\mathcal{B}(0,1)$ and $x \in \partial\mathcal{B}(0, 2^{m-1})$.

Suppose that ρ used in the definition of D satisfies $2^{m+1} \le \rho \le 2^{m+2}$ for some $m \ge m_0$. Let

$$\begin{aligned} T_1 &= 0, \\ U_k &= \inf\{t > T_k : X_t \in \partial S(1, 2^m)\}, \qquad k \ge 1, \\ T_k &= \inf\{t > U_{k-1} : X_t \in \partial\mathcal{B}(0, 2^{m-1})\}, \qquad k \ge 2. \end{aligned}$$

Then for $x \in \partial\mathcal{B}(0, 2^{m-1})$ and $y \in \partial D = \partial\mathcal{B}(0,1)$,

$$\begin{aligned} \mathbb{P}_D^x(X_{T_{\partial D}^X} \in dy) &= \sum_{k=1}^{\infty} \mathbb{P}_D^x(X_{U_k} \in dy; X_{U_j} \in \partial\mathcal{B}(0, 2^m), j < k) \\ &= \sum_{k=1}^{\infty} \mathbb{E}_D^x(\mathbb{P}_{S(1,2^m)}^{X_{T_k}}(X_{U_k} \in dy)\mathbf{1}_{\{X_{U_j} \in \partial\mathcal{B}(0,2^m), j<k\}}) \\ &= \sum_{k=1}^{\infty} \mathbb{E}_D^x(h(1, 2^m; X_{T_k}, y)\mu(dy)\mathbf{1}_{\{X_{U_j} \in \partial\mathcal{B}(0,2^m), j<k\}}). \end{aligned}$$

This and (3.9) imply that

$$c_3 < \mathbb{P}_D^x(X_{T_{\partial D}^X} \in dy)/\mathbb{P}_D^x(X_{T_{\partial D}^X} \in dz) < 1/c_3$$

for $y, z \in \partial\mathcal{B}(0,1)$ and $x \in \partial\mathcal{B}(0, 2^{m-1})$. The last estimate and the strong Markov property of excursion laws applied at the hitting time $T_{\partial\mathcal{B}(0,2^{m-1})}$ of $\partial\mathcal{B}(0, 2^{m-1})$ show that

$$\begin{aligned} c_3 &< H^x(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta) \\ &\quad /H^x(e(\zeta-) \in dz; T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta) < 1/c_3 \end{aligned} \tag{3.10}$$

for $x, y, z \in \partial\mathcal{B}(0,1)$. Informally speaking, for sufficiently large m (and ρ), the density of $H^x(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta)$ is arbitrarily close to a constant on ∂D.

The probability that 3-dimensional Brownian motion starting from $x + \delta\mathbf{n}(x)$, $x \in \partial\mathcal{B}(0,1)$, will never return to $\partial\mathcal{B}(0,1)$ is equal to $1 - (1+\delta)^{-1}$. This and (2.2) imply that for any $c_4 > 0$ there exists m_1 such that for $m \ge m_1$ and $x \in \partial D$,

$$1 - c_4 < H^x(T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta) < 1 + c_4.$$

It follows from this and (3.10) that for any $c_5 > 0$ and sufficiently large ρ, we have for $x, y \in \partial D$,

$$1 - c_5 < H^x\big(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta\big)/\mu(dy) < 1 + c_5. \tag{3.11}$$

We have by continuity of probability that

$$\lim_{m\to\infty} H^x\big(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} > \zeta\big) = \widehat{H}^x\big(e(\zeta-) \in dy\big). \tag{3.12}$$

Note that the above limit is monotone.

We have

$$\begin{aligned} H^x\big(f_{\mathbf{v}}(e)\big) &= \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) H^x\big(e(\zeta-) \in dy\big) \\ &= \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) H^x\big(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} > \zeta\big) \\ &\quad + \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) H^x\big(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta\big). \end{aligned} \tag{3.13}$$

It follows from (3.12), monotone convergence theorem and part (i) of this theorem that

$$\begin{aligned} &\lim_{m\to\infty} \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) H^x\big(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} > \zeta\big) \\ &\quad = \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) \widehat{H}^x\big(e(\zeta-) \in dy\big) = \widehat{H}^x\big(f_{\mathbf{v}}(e)\big) \\ &\quad = \sqrt{2} - 1 - \log(1 + \sqrt{2}). \end{aligned} \tag{3.14}$$

We combine (3.5), (3.7), (3.11) and Lemma 3.2 to obtain

$$\begin{aligned} &\lim_{m\to\infty} \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) H^x\big(e(\zeta-) \in dy; T_{\partial\mathcal{B}(0,2^{m-1})} < \zeta\big) \\ &\quad = \int_{\partial D} \big(\log\big|\pi_y(\mathbf{v})\big| - \log|\mathbf{v}|\big) \mu(dy) \\ &\quad = \int_0^{2\pi} \int_0^{\pi} \frac{1}{4\pi} \sin\alpha \log\big(\sin^2\beta + \cos^2\beta\cos^2\alpha\big)^{1/2} \, d\alpha\, d\beta = \log 2 - 1. \end{aligned}$$

Part (ii) of the theorem follows from this formula, (3.13) and (3.14). □

4. Recurrence of synchronous couplings in 3-dimensional torus. The natural scale for our arguments is the combination of the local time scale and the logarithmic scale. The reason is that when the "real" time reaches a fixed level, the vector between X and Y is not parallel to ∂D in any reasonable sense. On the contrary, when the local time reaches a fixed level, the vector between X and Y is approximately parallel to ∂D, in a sense. The last observation is used repeatedly in our arguments. The following definitions introduce the "local time scale."

Let $\sigma_t^X = \inf\{s \geq 0 : L_s^X \geq t\}$, $\sigma_t^Y = \inf\{s \geq 0 : L_s^Y \geq t\}$ and $\sigma_b' = \sigma_b^X \wedge \sigma_b^Y$. The random variable σ_b' was denoted σ_* in Section 2.4 for consistency with the notation of [4]. The new notation, σ_b', is more appropriate for this paper. An alternative formula is $\sigma_b' = \inf\{t \geq 0 : L_t^X \vee L_t^Y \geq b\}$. Let

$$\sigma'_{(k+1)b} = \inf\{t \geq \sigma'_{kb} : (L_t^X - L^X_{\sigma'_{kb}}) \vee (L_t^Y - L^Y_{\sigma'_{kb}}) \geq b\}$$

for $k \geq 1$. Note that, typically, σ'_{kb} is not equal to $\inf\{t \geq 0 : L_t^X \vee L_t^Y \geq kb\}$. Let

$$\begin{aligned} &R_t = |X_t - Y_t|, \qquad M_t = \log R_t, \qquad t \geq 0, \\ &V_k = M_{\sigma'_{kb}}, \qquad k = 0, 1, \ldots. \end{aligned} \tag{4.1}$$

The following lemma shows that over a long time interval, the distance between X and Y is unlikely to decrease.

LEMMA 4.1. *For any $c_0 > 0$, $\beta_1 \in (0,1)$ and $p < 1$ there exist $c_1, b, \varepsilon_1 > 0$ such that if $\varepsilon \leq \varepsilon_1$, $x_0 \in \partial D$, $y_0 \in \overline{D}$, $|x_0 - y_0| = \varepsilon$, $X_0 = x_0$, $Y_0 = y_0$ and*

$$\frac{|\langle y_0 - x_0, \mathbf{n}(x_0)\rangle|}{|y_0 - x_0|} \leq c_0 \varepsilon^{\beta_1}, \tag{4.2}$$

then

$$\mathbb{P}^{x_0, y_0}(V_1 - V_0 \geq c_1) \geq p.$$

PROOF. It suffices to prove the lemma for $c_0 = 1$. To see this, choose any $\beta_1^* \in (0, \beta_1)$ and note that $c_0 \varepsilon^{\beta_1} \leq \varepsilon^{\beta_1^*}$ for some $\varepsilon_* > 0$ and all $\varepsilon \in (0, \varepsilon_*)$. Hence, if the lemma is proved for β_1^* in place of β_1, with 1 in place of c_0 and for $\varepsilon < \varepsilon_1$, then it also holds for β_1, c_0 and $\varepsilon < \varepsilon_1 \wedge \varepsilon_*$.

Step 1. In this step, the distance between X and Y is approximated by a sum of increments related to excursions. The rate of increase (or decrease) of the distance is expressed using excursion theory-based calculations from Section 3.

Recall the results from [4] reviewed in Section 2.4. Suppose that $\varepsilon_* > 0$, $x_0 \in \partial D$, $\mathbf{v} \in \mathcal{T}_{x_0} \partial D$, $|\mathbf{v}| = 1$, $X_0 = x_0$ and let e_u be the first excursion of X from ∂D with $|e_u(0) - e_u(\zeta-)| \geq \varepsilon_*$. Let $x_1 = e_u(\zeta-)$ and $\alpha = 3/4$. We will estimate $\mathbb{P}^{x_0}(|x_0 - e_u(0)| \geq \varepsilon_*^\alpha)$ and $\mathbb{E}^{x_0}[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0 - e_u(0)| \leq \varepsilon_*^\alpha\}}]$.

Let $U_0 = \varnothing$, $U_1 = \mathcal{B}(x_0, \varepsilon_*) \cap \partial D$, $U_k = (\mathcal{B}(x_0, k\varepsilon_*) \setminus \mathcal{B}(x_0, (k-1)\varepsilon_*)) \cap \partial D$ for $k \geq 2$ and $T_0 = 0$. Set $j_0 = 1$ and for $k \geq 0$, set

$$\begin{aligned} T_{k+1} &= \inf\{t \geq T_k : X_t \in \partial D \setminus (U_{j_k - 1} \cup U_{j_k} \cup U_{j_k + 1})\}, \\ j_{k+1} &= \min\{i \geq 0 : X_{T_{k+1}} \in U_i\}. \end{aligned}$$

Recall that u denotes the starting time of the first excursion of X from ∂D with $|e_u(0) - e_u(\zeta-)| \geq \varepsilon_*$. Let p_1 be the probability that $|x_0 - e_u(0)| < \varepsilon_*$ and note that $p_1 > 0$. The strong Markov property applied at T_k shows that $\mathbb{P}^{x_0}(u \leq T_{k+1} \mid u \geq$

$T_k) \geq p_1$. It follows that $\mathbb{P}^{x_0}(u \geq T_k) \leq (1-p_1)^k$. For the event $\{|x_0 - e_u(0)| \geq \varepsilon_*^\alpha\}$ to occur, we have to have $u \geq T_k$ with $k \geq \varepsilon_*^\alpha/(2\varepsilon_*)$. It follows that, setting $c_1 = -(1/2)\log(1-p_1) > 0$,

$$\mathbb{P}^{x_0}\big(|x_0 - e_u(0)| \geq \varepsilon_*^\alpha\big) \leq (1-p_1)^{\varepsilon_*^\alpha/(2\varepsilon_*)} = \exp\big(-c_1\varepsilon_*^{\alpha-1}\big). \tag{4.3}$$

Let $\beta = 5/8$ and note that if $|x_1 - x_0| \leq \varepsilon_*^\beta$, then $|\log|\pi_{x_1}\mathbf{v}|| \leq c_2\varepsilon_*^{2\beta}$. Hence,

$$\begin{aligned}
&\mathbb{E}^{x_0}\big[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0-e_u(0)|\leq\varepsilon_*^\alpha\}}\big]\\
&\quad= \mathbb{E}^{x_0}\big[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0-e_u(0)|\leq\varepsilon_*^\alpha\}}\mathbf{1}_{\{|x_1-x_0|\leq\varepsilon_*^\beta\}}\big]\\
&\qquad+ \mathbb{E}^{x_0}\big[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0-e_u(0)|\leq\varepsilon_*^\alpha\}}\mathbf{1}_{\{|x_1-x_0|\geq\varepsilon_*^\beta\}}\big]\\
&\quad\leq c_2\varepsilon_*^{2\beta} + \mathbb{E}^{x_0}\big[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0-e_u(0)|\leq\varepsilon_*^\alpha\}}\mathbf{1}_{\{|x_1-x_0|\geq\varepsilon_*^\beta\}}\big].
\end{aligned} \tag{4.4}$$

It follows from (3.4) and (3.11) that for large ρ, small ε_*, $|x_1 - x_0| \geq \varepsilon_*^\beta$ and $|x_0 - x| \leq \varepsilon_*^\alpha$,

$$\frac{H^x(e(\zeta-)\in dx_1)}{H^{x_0}(e(\zeta-)\in dx_1)} \leq \frac{|x-x_1|^{-3}}{|x_0-x_1|^{-3}} \leq \frac{(\varepsilon_*^\beta - \varepsilon_*^\alpha)^{-3}}{\varepsilon_*^{-3\beta}} \leq 1 + 6\varepsilon_*^{\alpha-\beta}. \tag{4.5}$$

Let $c_* = \sqrt{2} + \log 2 - 2 - \log(1+\sqrt{2}) \approx -0.77$ be the constant in the statement of Theorem 3.1(ii). Theorem 3.1(ii), the exit system formula (2.1) and (4.5) imply that for any $c_4 \in (-c_*, 1)$ and $c_3 \in (0, c_4 + c_*)$, all large ρ and small $\varepsilon_* > 0$,

$$\begin{aligned}
&\mathbb{E}^{x_0}\big[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0-e_u(0)|\leq\varepsilon_*^\alpha\}}\mathbf{1}_{\{|x_1-x_0|\geq\varepsilon_*^\beta\}}\big]\\
&\quad= \mathbb{E}^{x_0}\frac{H^{X_u}(|\log|\pi_{e(\zeta-)}\mathbf{v}||\mathbf{1}_{\{|x_0-X_u|\leq\varepsilon_*^\alpha\}}\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*^\beta\}})}{H^{X_u}(\mathbf{1}_{\{|e(\zeta-)-X_u|\geq\varepsilon_*\}})}\\
&\quad\leq \frac{(1+6\varepsilon_*^{\alpha-\beta})H^{x_0}(|\log|\pi_{e(\zeta-)}\mathbf{v}||\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*^\beta\}})}{H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*\}})}\\
&\quad\leq \frac{(1+6\varepsilon_*^{\alpha-\beta})(c_3 + |\sqrt{2}+\log 2 - 2 - \log(1+\sqrt{2})|)}{H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*\}})}\\
&\quad\leq c_4/H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*\}}).
\end{aligned}$$

We combine the last estimate and (4.4) to obtain

$$\mathbb{E}^{x_0}\big[|\log|\pi_{x_1}\mathbf{v}||\mathbf{1}_{\{|x_0-e_u(0)|\leq\varepsilon_*^\alpha\}}\big] \leq c_2\varepsilon_*^{2\beta} + c_4/H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*\}}). \tag{4.6}$$

Recall the notation from the paragraph containing (2.5). Consider an arbitrary $\mathbf{v}_0 \in \mathbb{R}^3$. Since ∂D is a sphere with the unit radius, $\mathcal{S}(x)$ is the identity operator so

$\mathcal{I}_k = \exp(\Delta\ell_k^*)\pi_{x_k^*}$ and, therefore,

$$\begin{aligned}\mathcal{I}_{m^*} \circ \cdots \circ \mathcal{I}_0(\mathbf{v}_0) &= \exp\Bigl(\sum_{0\le k\le m^*} \Delta\ell_k^*\Bigr)\pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\\ &= \exp(\ell_{m^*+1}^*)\pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\\ &= \exp(L_{\sigma_b'}^X)\pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0).\end{aligned} \tag{4.7}$$

We will estimate the above quantity, starting with the composition of projection operators. We have

$$\begin{aligned}&\log\bigl|\pi_{x_{m^*}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|\\ &\quad = \sum_{1\le k\le m^*} \bigl(\log\bigl|\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr| - \log\bigl|\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|\bigr)\\ &\qquad + \log\bigl|\pi_{x_0^*}(\mathbf{v}_0)\bigr|.\end{aligned} \tag{4.8}$$

By the strong Markov property applied at the excursion endpoint $s_{k-1} := t_{k-1}^* + \zeta(e_{t_{k-1}^*})$, the conditional distribution of

$$\log\bigl|\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr| - \log\bigl|\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|$$

given $\mathcal{F}_{s_{k-1}}$ is the same as that of $|\log|\pi_{x_1}\mathbf{v}||$, introduced at the beginning of the proof. Let

$$F_k = \bigl\{\bigl|x_{k-1}^* - e_{t_k^*}(0)\bigr| \le \varepsilon_*^\alpha\bigr\}.$$

We see that the events F_k, $k \ge 1$, are independent and so are the random variables

$$\bigl|\log\bigl|\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr| - \log\bigl|\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|\bigr|\mathbf{1}_{F_k}. \tag{4.9}$$

It follows from (4.6) that

$$\begin{aligned}&\mathbb{E}^{x_0}\bigl[\bigl|\log\bigl|\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr| - \log\bigl|\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|\bigr|\mathbf{1}_{F_k} \mid \mathcal{F}_{s_{k-1}}\bigr]\\ &\quad \le c_2\varepsilon_*^{2\beta} + c_4/H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\ge\varepsilon_*\}}).\end{aligned}$$

Thus the process

$$\begin{aligned}N_n &= n\bigl(c_2\varepsilon_*^{2\beta} + c_4/H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\ge\varepsilon_*\}})\bigr)\\ &\quad - \sum_{1\le k\le n} \bigl|\log\bigl|\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr| - \log\bigl|\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|\bigr|\mathbf{1}_{F_k}\end{aligned}$$

is a submartingale. By the optional stopping theorem, $\mathbb{E}^{x_0}N_{m^*} \ge 0$, so

$$\begin{aligned}&\mathbb{E}^{x_0}\Bigl[\sum_{1\le k\le m^*} \bigl|\log\bigl|\pi_{x_k^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr| - \log\bigl|\pi_{x_{k-1}^*} \circ \cdots \circ \pi_{x_0^*}(\mathbf{v}_0)\bigr|\bigr|\mathbf{1}_{F_k}\Bigr]\\ &\quad \le \mathbb{E}^{x_0}m^*\bigl(c_2\varepsilon_*^{2\beta} + c_4/H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\ge\varepsilon_*\}})\bigr).\end{aligned} \tag{4.10}$$

Formula (3.4) implies that $H^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*\}})\leq c_5/\varepsilon_*$. It follows from the definition of m^* and the exit system formula (2.1) that m^* has the Poisson distribution with the expected value $bH^{x_0}(\mathbf{1}_{\{|e(\zeta-)-x_0|\geq\varepsilon_*\}})$. These observations and (4.10) yield for some $c_6>0$, any $c_7\in(c_4,1)$ and small ε_*,

$$\begin{aligned}
&\mathbb{E}^{x_0}\Bigl[\sum_{1\leq k\leq m^*}\bigl|\log|\pi_{x_k^*}\circ\cdots\circ\pi_{x_0^*}(\mathbf{v}_0)|-\log|\pi_{x_{k-1}^*}\circ\cdots\circ\pi_{x_0^*}(\mathbf{v}_0)|\bigr|\prod_{1\leq j\leq m^*}\mathbf{1}_{F_j}\Bigr]\\
&\quad\leq\mathbb{E}^{x_0}\Bigl[\sum_{1\leq k\leq m^*}\bigl|\log|\pi_{x_k^*}\circ\cdots\circ\pi_{x_0^*}(\mathbf{v}_0)|-\log|\pi_{x_{k-1}^*}\circ\cdots\circ\pi_{x_0^*}(\mathbf{v}_0)|\bigr|\mathbf{1}_{F_k}\Bigr]\\
&\quad\leq\bigl(c_6\varepsilon_*^{2\beta-1}+c_4\bigr)b\leq c_7b.
\end{aligned}\tag{4.11}$$

In addition, since we are dealing with a sum of i.i.d. random variables given in (4.9), and the sum has a Poisson number m^* of terms with large mean, it is easy to see that for any $c_8\in(c_7,1)$ and $p_2>0$ there exist b_1 and ε_0 such that for $b\geq b_1$ and $\varepsilon_*\leq\varepsilon_0$,

$$\begin{aligned}
&\mathbb{P}^{x_0}\Bigl(\sum_{1\leq k\leq m^*}\bigl|\log|\pi_{x_k^*}\circ\cdots\circ\pi_{x_0^*}(\mathbf{v}_0)|-\log|\pi_{x_{k-1}^*}\circ\cdots\circ\pi_{x_0^*}(\mathbf{v}_0)|\bigr|\mathbf{1}_{F_k}\\
&\qquad\qquad\geq c_8b\Bigr)\leq p_2.
\end{aligned}\tag{4.12}$$

A similar argument based on the strong Markov property applied at times s_k and the optional stopping theorem for submartingales, combined with (4.3), gives

$$\mathbb{P}^{x_0}\Bigl(\bigcup_{1\leq k\leq m^*}F_k^c\Bigr)\leq\mathbb{E}^{x_0}m^*\exp\bigl(-c_1\varepsilon_*^{\alpha-1}\bigr)\leq c_9b\exp\bigl(-c_1\varepsilon_*^{\alpha-1}\bigr)\varepsilon_*^{-1}.\tag{4.13}$$

Step 2. We will use a result from a different paper to show that the discrete approximation of the distance between X and Y employed in the previous step is sufficiently accurate for our purposes.

Recall the notation from Section 2.4. We copy below (2.6)–(2.7) because these estimates are crucial to the present argument. Fix an arbitrarily small $c_{10}>0$. There exist $c_{11},c_{12},c_{13},\varepsilon_0>0$, $\beta_1\in(1,4/3)$ and $\beta_2\in(0,4/3-\beta_1)$ such that if $X_0=x$, $Y_0=y$, $|x-y|=\varepsilon<\varepsilon_0$ and $\varepsilon_*=c_{11}\varepsilon$, then

$$\bigl|(Y_{\sigma_b'}-X_{\sigma_b'})-\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)\bigr|\leq|\Lambda|+\Xi,\tag{4.14}$$

where $|\Lambda|<c_{10}\varepsilon$, $\mathbb{P}^{x,y}$-a.s., and

$$\mathbb{P}^{x,y}\bigl(|\Xi|>c_{12}\varepsilon^{\beta_1}\bigr)\leq c_{13}\varepsilon^{\beta_2}.\tag{4.15}$$

We have

$$
\begin{aligned}
&\log\bigl|\pi_{x^*_{m^*}}\circ\cdots\circ\pi_{x^*_0}(Y_0-X_0)\bigr| \\
&\quad=\sum_{1\le k\le m^*}\bigl(\log\bigl|\pi_{x^*_k}\circ\cdots\circ\pi_{x^*_0}(Y_0-X_0)\bigr| \\
&\qquad\qquad -\log\bigl|\pi_{x^*_{k-1}}\circ\cdots\circ\pi_{x^*_0}(Y_0-X_0)\bigr|\bigr) \\
&\qquad +\log\bigl|\pi_{x^*_0}(Y_0-X_0)\bigr|.
\end{aligned}
\tag{4.16}
$$

Note that $x^*_0=x_0$. It follows from (4.2) that

$$
\log\varepsilon-\log\bigl|\pi_{x^*_0}(Y_0-X_0)\bigr|=\log\varepsilon-\log\bigl|\pi_{x_0}(y_0-x_0)\bigr|\le c_{14}\varepsilon^{2\beta_1}.
\tag{4.17}
$$

We combine this with (4.12), (4.13) and (4.16) to see that for any $c_{15}\in(c_7,1)$ and $p_2>0$, there exists b_2 such that for any $b\ge b_2$, there exists $\varepsilon_1>0$ such that for $\varepsilon\le\varepsilon_1$,

$$
\mathbb{P}\bigl(\bigl|\log\bigl|\pi_{x^*_{m^*}}\circ\cdots\circ\pi_{x^*_0}(Y_0-X_0)\bigr|-\log|Y_0-X_0|\bigr|\ge c_{15}b\bigr)\le p_2.
\tag{4.18}
$$

A special case of (4.7) is

$$
\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)=\exp\bigl(L^X_{\sigma'_b}\bigr)\pi_{x^*_{m^*}}\circ\cdots\circ\pi_{x^*_0}(Y_0-X_0).
$$

This implies that

$$
\log\bigl|\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)\bigr|=L^X_{\sigma'_b}+\log\bigl|\pi_{x^*_{m^*}}\circ\cdots\circ\pi_{x^*_0}(Y_0-X_0)\bigr|.
$$

Recall that $|X_0-Y_0|=|x_0-y_0|=\varepsilon$. On the event $\{\sigma'_b=\sigma^X_b\}$ we have $L^X_{\sigma'_b}=b$ so, in view of (4.17) and (4.18), for any $c_{16}\in(c_{15},1)$, $c_{17}=1-c_{16}>0$ and $p_3>0$, there exists b_3 such that for any $b\ge b_3$, there exists $\varepsilon_2>0$ such that for $\varepsilon\le\varepsilon_2$,

$$
\begin{aligned}
&\mathbb{P}^{x_0,y_0}\bigl(\log\bigl|\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)\bigr|-b-\log\varepsilon\le -c_{16}b\text{ and }\sigma'_b=\sigma^X_b\bigr) \\
&\quad=\mathbb{P}^{x_0,y_0}\bigl(\log\bigl|\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)\bigr|\le(1-c_{16})b+\log\varepsilon \\
&\qquad\qquad\text{and }\sigma'_b=\sigma^X_b\bigr) \\
&\quad=\mathbb{P}^{x_0,y_0}\bigl(\bigl|\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)\bigr|\exp(c_{17}b)\text{ and }\sigma'_b=\sigma^X_b\bigr) \\
&\quad\le p_3.
\end{aligned}
\tag{4.19}
$$

Recall from (4.14) that we can assume that $|\Lambda|\le c_{10}\varepsilon$, a.s. It follows from (4.15) that for small ε, $\mathbb{P}(|\Xi|\ge c_{10}\varepsilon)<p_3$. These remarks and (4.19) imply that

$$
\begin{aligned}
&\mathbb{P}^{x,y}\bigl(\bigl|\mathcal{I}_{m^*}\circ\cdots\circ\mathcal{I}_0(Y_0-X_0)\bigr|-|\Lambda|-|\Xi|\le\varepsilon\bigl(\exp(c_{17}b)-2c_{10}\bigr)\text{ and }\sigma'_b=\sigma^X_b\bigr) \\
&\quad\le 2p_3.
\end{aligned}
$$

We combine this estimate with (4.14) to see that

$$\begin{aligned}(4.20)\quad &\mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b^X} - X_{\sigma_b^X})\big| \le \varepsilon\big(\exp(c_{17}b) - 2c_{10}\big) \text{ and } \sigma_b' = \sigma_b^X\big)\\ &\quad = \mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le \varepsilon\big(\exp(c_{17}b) - 2c_{10}\big) \text{ and } \sigma_b' = \sigma_b^X\big) \le 2p_3.\end{aligned}$$

We choose large b_4 so that for $b \ge b_4$, $c_{18} = c_{18}(b) := \exp(c_{17}b) - 2c_{10} > 1$. We can now write (4.20) as

$$\begin{aligned}(4.21)\quad &\mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le c_{18}\varepsilon \text{ and } \sigma_b' = \sigma_b^X\big)\\ &\quad = \mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b^X} - X_{\sigma_b^X})\big| \le c_{18}\varepsilon \text{ and } \sigma_b' = \sigma_b^X\big) \le 2p_3.\end{aligned}$$

Recall that $x_0 \in \partial D$, $y_0 \in \overline{D}$, and let $T' = \inf\{t \ge 0 : |X_t| = |Y_t|\}$. Note that the distributions of $\{(X_t, Y_t), t \ge T'\}$ and $\{(Y_t, X_t), t \ge T'\}$ are symmetric. Moreover, $Y_t \notin \partial D$ for $t < T'$ and, therefore, $L^Y_{T'} = 0$. It follows from this and (4.21) that

$$\begin{aligned}&\mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le c_{18}\varepsilon, T' \le \sigma_b' \text{ and } \sigma_b' = \sigma_b^Y\big)\\ &\quad = \mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le c_{18}\varepsilon, T' \le \sigma_b' \text{ and } \sigma_b' = \sigma_b^X\big) \le 2p_3\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le c_{18}\varepsilon\big)\\ (4.22)\quad &\quad \le \mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le c_{18}\varepsilon, \text{ and } \sigma_b' = \sigma_b^X\big)\\ &\qquad + \mathbb{P}^{x_0,y_0}\big(\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| \le c_{18}\varepsilon, T' \le \sigma_b' \text{ and } \sigma_b' = \sigma_b^Y\big) \le 4p_3.\end{aligned}$$

Let $c_{19} = \log c_{18} > 0$. Then

$$\mathbb{P}^{x_0,y_0}(V_1 - V_0 \le c_{19}) = \mathbb{P}^{x_0,y_0}\big(\log\big|(Y_{\sigma_b'} - X_{\sigma_b'})\big| - \log\varepsilon \le \log c_{18}\big) \le 4p_3.$$

Since $p_3 > 0$ is arbitrarily small and $c_{19} > 0$, the lemma is proved. □

The following lemma estimates the distribution of the increment of the logarithm of the distance between X and Y. The assertion of the lemma has two parts. One part says that the distribution is close to the distribution of an integrable random variable. The other part shows the that error of approximation is small in an appropriate sense. Recall notation from (4.1).

LEMMA 4.2. *For any $\beta_1 \in (0, 1/2)$ there exist $\beta_2, b, c_1, \varepsilon_1 > 0$ and a cumulative distribution function $G : \mathbb{R} \to [0, 1]$ satisfying $\int_{-\infty}^{\infty} |a|\, dG(a) < \infty$ and such that if $\varepsilon \le \varepsilon_1$, $x_0 \in \partial D$, $y_0 \in \overline{D}$, $|x_0 - y_0| = \varepsilon$, $X_0 = x_0$, $Y_0 = y_0$ and*

$$(4.23)\quad \frac{|\langle y_0 - x_0, \mathbf{n}(x_0)\rangle|}{|y_0 - x_0|} \le \varepsilon^{\beta_1},$$

then there exists an event F such that

$$(4.24)\quad \mathbb{P}^{x_0,y_0}\big(F^c\big) \le c_1\varepsilon^{\beta_2},$$

$$(4.25)\quad \mathbb{P}^{x_0,y_0}\big(|V_1 - V_0|\mathbf{1}_F \le a\big) \le G(a), \qquad a \in \mathbb{R}.$$

PROOF. *Step* 1. This step is devoted to a review of upper bounds on the rate of growth of the distance between X and Y. It also contains a list of definitions (notation) used throughout the rest of the proof.

Fix b as in Lemma 4.1, some β_3 and β_4 such that $\beta_1 < \beta_3 < \beta_4 < 1/2$, and consider the condition

$$\frac{|\langle y_0 - x_0, \mathbf{n}(x_0)\rangle|}{|y_0 - x_0|} \le c_2 \varepsilon^{\beta_4}, \tag{4.26}$$

where $c_2 = 200 \cdot 2^{\beta_4}$. Note that $c_2 \varepsilon^{\beta_4} < \varepsilon^{\beta_1}$ for small $\varepsilon > 0$. It follows from (4.22) that for some $p_1 \in (0, 1)$, $\varepsilon_1 > 0$ and $c_3 = c_3(b) > 0$, if $|x_0 - y_0| = \varepsilon \le \varepsilon_1$ and either (4.23) or (4.26) holds, then

$$\mathbb{P}^{x_0, y_0}\big(|Y_{\sigma_b'} - X_{\sigma_b'}| \le c_3 \varepsilon\big) \le p_1. \tag{4.27}$$

Lemma 3.4 of [4] and its proof show that there exists $c_4 > 0$ such that for all $x, y \in \overline{D}$ and $t \ge 0$, we have $\mathbb{P}^{x,y}$-a.s.,

$$|X_t - Y_t| \le \exp\big(c_4\big(L_t^X + L_t^Y\big)\big)|x - y|. \tag{4.28}$$

By the Markov property, for any fixed $t, s \ge 0$, a.s.,

$$|X_{t+s} - Y_{t+s}| \le \exp\big(c_4\big(L_{t+s}^X - L_t^X + L_{t+s}^Y - L_t^Y\big)\big)|X_t - Y_t|. \tag{4.29}$$

Since the last formula holds for all rational $t, s \ge 0$ simultaneously, a.s., and X and Y are continuous, the inequality actually holds for all random times $t, s \ge 0$ (not necessarily stopping times). We obtain from (4.28),

$$\inf_{0 \le t \le \sigma_b'} |X_t - Y_t| \ge \exp(-2c_4 b)|X_{\sigma_b'} - Y_{\sigma_b'}|. \tag{4.30}$$

Let $c_5 = \exp(2c_4 b)$ and $c_6 = c_3 c_5^{-1}$. It follows from (4.27) and (4.30) that

$$\mathbb{P}^{x_0, y_0}\Big(\inf_{0 \le t \le \sigma_b'} |Y_t - X_t| \le c_6 \varepsilon\Big) \le p_1, \tag{4.31}$$

and for any random time $T \in [0, \sigma_b']$,

$$\sup_{T \le t \le \sigma_b'} |Y_t - X_t| \le c_5 |Y_T - X_T|, \qquad \mathbb{P}^{x_0, y_0}\text{-a.s.} \tag{4.32}$$

In particular,

$$\sup_{0 \le t \le \sigma_b'} |Y_t - X_t| \le c_5 \varepsilon, \qquad \mathbb{P}^{x_0, y_0}\text{-a.s.} \tag{4.33}$$

We set $c_7 = (-1 - 2c_4 b) \wedge \log c_6$ and $c_8 = e^{c_7}$. Hence,

$$c_8 c_5 \le \exp(-1 - 2c_4 b + 2c_4 b) < 1/2. \tag{4.34}$$

Obviously, (4.31) implies that, assuming that either (4.23) or (4.26) holds,

$$\mathbb{P}^{x_0, y_0}\Big(\inf_{0 \le t \le \sigma_b'} |Y_t - X_t| \le c_8 \varepsilon\Big) \le p_1. \tag{4.35}$$

Let $U_0 = 0$ and

$$S_1 = \inf\{t \geq 0 : M_t - M_0 \leq c_7\} = \inf\{t \geq 0 : |X_t - Y_t| \leq c_8 |X_0 - Y_0|\}.$$

Here and later, $\inf \varnothing = \infty$. Note that at least one of the processes X and Y must belong to ∂D at time S_1. We proceed by induction. First assume that $X_{S_k} \in \partial D$. Let $z_k \in \partial D$ be the point such that $\mathbf{n}(z_k) = \frac{Y_{S_k} - X_{S_k}}{|Y_{S_k} - X_{S_k}|}$, and for some $c_9 > 0$ (to be specified later) and $k \geq 1$, let

$$\begin{aligned}
U_k &= \inf\{t \geq S_k : Y_t \in \partial D\}, \\
S_k &= \inf\{t \geq U_{k-1} : M_t - M_{U_{k-1}} \leq c_7\} \\
&= \inf\{t \geq U_{k-1} : |X_t - Y_t| \leq c_8 |X_{U_{k-1}} - Y_{U_{k-1}}|\}, \\
F_k &= \{S_k < \sigma_b'\}, \\
\mathcal{G}_k &= \sigma(B_t, t \leq U_k), \\
J_k &= n \in \mathbb{Z} \text{ such that } 2^{-n} \leq |X_{S_k} - z_k| < 2^{-n+1}, \\
d_k &= |X_{U_{k-1}} - Y_{U_{k-1}}| \qquad (\text{hence, } d_0 = |X_0 - Y_0| = \varepsilon), \\
I_k &= \{2^{-J_k} \geq d_k^{\beta_3}\}, \\
S_k^* &= \inf\{t \geq S_k : |X_t - X_{S_k}| \geq d_k^{\beta_4}\}, \\
C_k &= \{U_k \leq S_k^*\}, \\
G_k &= \{|X_{U_k} - Y_{U_k}| \geq c_9 2^{-J_k} d_k\}, \\
K_k &= \left\{ \frac{|\langle X_{U_k} - Y_{U_k}, \mathbf{n}(Y_{U_k})\rangle|}{|X_{U_k} - Y_{U_k}|} \leq c_2 |X_{U_k} - Y_{U_k}|^{\beta_4} \right\}, \\
A_k &= F_k \cap I_k \cap C_k \cap G_k \cap K_k, \\
A_k^+ &= \bigcap_{j \leq k} A_j.
\end{aligned}$$

If $X_{S_k} \notin \partial D$, then we must have $Y_{S_k} \in \partial D$, and we apply all the above definitions with the roles of X and Y interchanged. In the rest of the proof, we will discuss only the case when $X_{S_k} \in \partial D$. Our arguments hold in the other case by symmetry.

Step 2. In this step we will prove that, for some $c_{10}, c_{11} < \infty$, $k \geq 1$ and m such that $2^{-m} \geq d_k^{\beta_3}$, on A_{k-1}^+,

$$\mathbb{P}(A_k^c \cap F_k \mid \mathcal{G}_{k-1}) \leq c_{10} d_k^{\beta_3}, \tag{4.36}$$

$$\mathbb{P}(\{J_k \geq m\} \cap F_k \mid \mathcal{G}_{k-1}) \leq c_{11} 2^{-m}. \tag{4.37}$$

Informally speaking, we will show that some events are unlikely. Given that they do not happen, we will find good estimates for the distance between X and Y. This step is subdivided into further substeps because we have to analyze several families of "unusual" events and show that they all have small probabilities. The first substep will show that "long" excursions are unlikely.

Step 2.1. Let $U_k' = S_k^* \wedge U_k$. Note that $U_k' = U_k$ on C_k and $U_k' = S_k^*$ on C_k^c. If F_k holds, then $d_k \le c_5\varepsilon$, by (4.33). By the definition of S_k, $|X_{S_k} - Y_{S_k}| = c_8 d_k$, if $S_k < \infty$. We have assumed that $X_{S_k} \in \partial D$ so $\operatorname{dist}(Y_{S_k}, \partial D) \le c_8 d_k$. We apply Lemma 3.2 of [5] to the process Y at the stopping time S_k to see that for some $c_{12} > 0$,

$$\mathbb{P}\big(|Y_{S_k} - Y_{U_k'}| \ge d_k^{\beta_4}/3\big) \le c_{12} d_k^{1-\beta_4}. \tag{4.38}$$

We will show that if $\varepsilon_1 > 0$ is sufficiently small and $\varepsilon < \varepsilon_1$, then $L^X_{U_k'} - L^X_{S_k} \le 3(1+c_5)d_k^{\beta_4}$. Suppose that the last inequality does not hold, and let $T_k^* = \inf\{t \ge S_k : L_t^X - L_{S_k}^X = 3(1+c_5)d_k^{\beta_4}\}$. Then by assumption we have $T_k^* \le U_k' \le S_k^*$. Assuming that F_k holds and using (4.33),

$$|X_{S_k} - Y_{S_k}| = c_8 d_k \le c_5 \varepsilon. \tag{4.39}$$

Hence, if ε is sufficiently small, then $c_8 d_k \le d_k^{\beta_4}/3$ and, therefore,

$$|X_{S_k} - Y_{S_k}| \le d_k^{\beta_4}/3. \tag{4.40}$$

It follows from the definitions of U_k and U_k' that $L^Y_{U_k'} - L^Y_{S_k} = 0$. If ε is small then d_k is small and $3(1+c_5)d_k^{\beta_4} < b \wedge 1/100$. So the definition of T_k^*, (4.40) and (4.28) imply that

$$|X_{T_k^*} - Y_{T_k^*}| \le c_5 d_k^{\beta_4}. \tag{4.41}$$

For all $t \in [S_k, T_k^*] \subset [S_k, S_k^*]$ such that $X_t \in \partial D$, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(X_{S_k})$ is less than $2d_k^{\beta_4}$. It follows that the angle between $\int_{S_k}^{T_k^*} \mathbf{n}(X_t)\, dL_t^X$ and $\mathbf{n}(X_{S_k})$ is also smaller than $2d_k^{\beta_4} < 1/50$. Moreover, the length of $\int_{S_k}^{T_k^*} \mathbf{n}(X_t)\, dL_t^X$ is greater than $2(1+c_5)d_k^{\beta_4}$. Recall that $Y_t \notin \partial D$ for $t \in [S_k, T_k^*]$. Thus $\int_{S_k}^{T_k^*} \mathbf{n}(Y_t)\, dL_t^Y = 0$ and, therefore,

$$X_{T_k^*} - Y_{T_k^*} = X_{S_k} - Y_{S_k} + \int_{S_k}^{T_k^*} \mathbf{n}(X_t)\, dL_t^X.$$

This relation, the fact that $X_{S_k} \in \partial D$, (4.40), (4.41) and our observations about the direction and length of $\int_{S_k}^{T_k^*} \mathbf{n}(X_t)\, dL_t^X$ imply that $Y_{T_k^*}$ must be at least $(1+c_5)d_k^{\beta_4}$ units inside the ball $\mathcal{B}(0,1)$. This is impossible so the claim that $L^X_{U_k'} - L^X_{S_k} \le 3(1+c_5)d_k^{\beta_4}$ is proved.

Recall that, assuming that F_k holds, $c_8 d_k \le c_5 \varepsilon$. Hence, if $\varepsilon_1 > 0$ is sufficiently small and $\varepsilon < \varepsilon_1$, then $3(1+c_5)d_k^{\beta_4} < b$. Since $L^X_{U_k'} - L^X_{S_k} \le 3(1+c_5)d_k^{\beta_4}$, $L^Y_{U_k'} -$

$L^Y_{S_k} = 0$ and $|X_{S_k} - Y_{S_k}| = c_8 d_k$, we have by (4.29), for all $t \in [S_k, U'_k]$,

$$|X_t - Y_t| \le c_5 c_8 d_k < d_k^{\beta_4}/3. \tag{4.42}$$

In particular, $|X_{U'_k} - Y_{U'_k}| < d_k^{\beta_4}/3$. This, the definitions of S^*_k and U'_k and (4.40) imply that, assuming that C_k does not hold, $|Y_{S_k} - Y_{S^*_k}| = |Y_{S_k} - Y_{U'_k}| \ge d_k^{\beta_4}/3$. This and (4.38) imply that, on A^+_{k-1},

$$\mathbb{P}(C^c_k \cap F_k \mid \mathcal{G}_{k-1}) \le c_{13} d_k^{1-\beta_4}. \tag{4.43}$$

We record, for future reference, the following variants of (4.42). If $C_k \cap F_k$ holds, then $U'_k = U_k$ and for any random time $R \in [S_k, U_k]$ and all $t \in [R, U_k]$,

$$|X_t - Y_t| \le c_5 c_8 |X_R - Y_R|. \tag{4.44}$$

It follows from (4.29) that if $C_k \cap F_k$ holds, then for all $t \in [U_{k-1}, U_k]$,

$$|X_t - Y_t| \le c_5 |X_{U_{k-1}} - Y_{U_{k-1}}| = c_5 d_k. \tag{4.45}$$

Since $|X_{S_k} - Y_{S_k}| = c_8 d_k$, if F_k holds, then we have by (4.29) and (4.34), for all $t \in [S_k, \sigma'_b]$,

$$|X_t - Y_t| \le c_5 c_8 d_k < d_k/2. \tag{4.46}$$

Step 2.2. The intuitive meaning of the technical estimate in this step is that if the vector between X and Y is close to the normal to ∂D, then Y must have traveled a long distance since it last visited ∂D.

Assume that F_k holds. Let $U^*_k = \sup\{t < U_k : Y_t \in \partial D\}$ and $\widetilde{U}_k = U_{k-1} \vee U^*_k$. It is easy to see that, a.s., $U^*_k \le \widetilde{U}_k < S_k < U_k$, for $k \ge 2$. (We will limit our discussion to the case $k \ge 2$; the case $k = 1$ requires minor modifications so we omit the proof.) Random times U^*_k and U_k are the endpoints of an excursion of Y from ∂D. Suppose that $J_k \ge m$ and $2^{-m} \ge d_k^{\beta_3}$. By (4.32), $d_k \le c_5 \varepsilon$ so, assuming that $\varepsilon_1 > 0$ is small and $\varepsilon \le \varepsilon_1$, we have $c_8 d_k \le d_k^{\beta_3} \le 2^{-m}$. We have $|X_{S_k} - z_k| \le 2^{-m+1}$ and, using (4.39),

$$|Y_{S_k} - z_k| \le |X_{S_k} - z_k| + |X_{S_k} - Y_{S_k}| \le 2^{-m+1} + c_8 d_k \le 2^{-m+2}. \tag{4.47}$$

Suppose that $\sup_{\widetilde{U}_k \le t \le S_k, X_t \in \partial D} |Y_t - z_k| \le c_{14} := 1/400$. We will show that this assumption leads to a contradiction. The assumption and (4.33) imply that, for small ε, $\sup_{\widetilde{U}_k \le t \le S_k, X_t \in \partial D} |X_t - z_k| \le 2c_{14}$. This in turn implies that for all $t \in [\widetilde{U}_k, S_k]$ such that $X_t \in \partial D$, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(z_k)$ is less than $4c_{14}$. It follows that the angle between $\int_{\widetilde{U}_k}^{S_k} \mathbf{n}(X_t)\, dL^X_t$ and $\mathbf{n}(z_k)$ is also smaller than $4c_{14}$. Note that $Y_t \notin \partial D$ for $t \in [\widetilde{U}_k, S_k]$ by the definition of $\widetilde{U}_k$. Thus $\int_{\widetilde{U}_k}^{S_k} \mathbf{n}(Y_t)\, dL^Y_t = 0$ and, therefore,

$$X_{S_k} - Y_{S_k} = X_{\widetilde{U}_k} - Y_{\widetilde{U}_k} + \int_{\widetilde{U}_k}^{S_k} \mathbf{n}(X_t)\, dL^X_t. \tag{4.48}$$

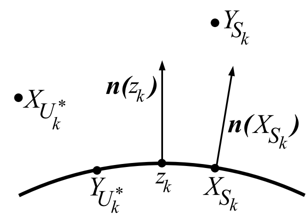

FIG. 2. *In analysis of possible locations of X and Y, we can ignore Brownian oscillations because they are common to both X and Y and hence do not affect their relative position. Consider the case $\widetilde{U}_k = U_k^*$. On the interval $[U_k^*, S_k]$, only X gets "local time push" on ∂D because Y does not visit ∂D between these times. The direction of the push is always close to $\mathbf{n}(z_k)$. The picture represents an impossible configuration—it is impossible for $X_{U_k^*}$ to be "above" $Y_{U_k^*}$ and for X_{S_k} to be "below" Y_{S_k} if X is pushed in the "upward" direction between times U_k^* and S_k.*

Recall that $X_{S_k} - Y_{S_k}$ is a positive multiple of $-\mathbf{n}(z_k)$ and $X_{S_k} \in \partial D$. Assume that K_{k-1} holds. If $\widetilde{U}_k = U_k^*$, then $Y_{\widetilde{U}_k} \in \partial D$. Next consider the case $\widetilde{U}_k > U_k^*$. In this case, $X_{U_{k-1}} \in \partial D$ and, assuming that $\varepsilon > 0$ is small, the vector $Y_{U_{k-1}} - X_{U_{k-1}}$ is almost orthogonal to $\mathbf{n}(X_{U_{k-1}})$. More precisely, K_{k-1} implies that $\operatorname{dist}(Y_{U_{k-1}}, \partial D) \le 2c_2 d_k^{1+\beta_4}$. These observations and the fact that the angle between $\int_{\widetilde{U}_k}^{S_k} \mathbf{n}(X_t)\, dL_t^X$ and $\mathbf{n}(z_k)$ is smaller than $4c_{14}$ show that (4.48) cannot be true; see Figure 2. This contradiction implies that $\sup_{\widetilde{U}_k \le t \le S_k, X_t \in \partial D} |Y_t - z_k| \ge c_{14}$. We combine this with (4.47) to see that $\sup_{\widetilde{U}_k \le t \le S_k, X_t \in \partial D} |Y_t - Y_{S_k}| \ge c_{15} := c_{14}/2$, for some m_1 and all $m \ge m_1$. Suppose that s_1 is such that $\widetilde{U}_k \le s_1 \le S_k$, $X_{s_1} \in \partial D$ and $|Y_{s_1} - Y_{S_k}| \ge c_{15}$. Then either $|Y_{\widetilde{U}_k} - Y_{S_k}| \ge c_{15}/2$ or $|Y_{\widetilde{U}_k} - Y_{s_1}| \ge c_{15}/2$. Since $X_{S_k} \in \partial D$, it follows that there exists s_2 such that $\widetilde{U}_k \le s_2 \le S_k$, $X_{s_2} \in \partial D$ and $|Y_{\widetilde{U}_k} - Y_{s_2}| \ge c_{15}/2$. We record this for future reference. There exists m_1 such that if $m \ge m_1$, $F_k \cap K_{k-1}$ holds, $J_k \ge m$ and $2^{-m} \ge d_k^{\beta_3}$, then

$$\sup_{\widetilde{U}_k \le t \le S_k, X_t \in \partial D} |Y_t - Y_{\widetilde{U}_k}| \ge c_{15}/2. \tag{4.49}$$

Step 2.3. We will show that if Y comes close to ∂D, then it is not likely to hit ∂D far from this point.

Assume that $C_k \cap F_k$ holds. Recall notation related to excursions from Section 2.3. We will apply excursion theory to excursions of the Markov process (Y, X) from $\partial D \times \overline{D}$. From the intuitive point of view, the exit system representing these excursions is equivalent to the exit system for excursions of Y from ∂D. We use the "richer" version of excursion theory so that we can discuss the relationship of excursions of Y and the process X. We will use the same notation H^z for excursion laws of the process (Y, X) as for excursion laws of the process Y since the two families of excursion laws can be clearly identified with each other. All estimates of H^z-measures of events given in this proof hold uniformly in $z \in \partial D$, so we will write $H^{\cdot}$ for such uniform bounds.

Consider an arbitrary $c_{16} \in (0, c_{15}/2)$, and for the moment, consider d_k a fixed number. In the following definitions, e will represent excursions of the second component of (X, Y) from ∂D. Let

$$T = \inf\{t \geq 0 : \mathrm{dist}(e(t), \partial D) \leq c_5 d_k, |e(0) - e(t)| \geq c_{16}, X_t \in \partial D\},$$

$$\widetilde{A} = \Big\{T < \zeta, \sup_{T<t<\zeta} |e(\zeta-) - e(t)| \geq d_k^{\beta_4}/4\Big\}.$$

An application of Lemma 3.2 of [5] and (2.2) give

$$H^{\cdot}(T < \zeta) \leq H^{\cdot}\Big(\sup_{0<t<\zeta} |e(t) - e(0)| \geq c_{16}\Big) \leq c_{17}. \tag{4.50}$$

Another application of Lemma 3.2 of [5] and the strong Markov property applied at the stopping time T yield

$$H^{\cdot}(\widetilde{A} \mid T < \zeta) = H^{\cdot}\Big(\sup_{T<t<\zeta} |e(\zeta-) - e(t)| \geq d_k^{\beta_4}/4 \mid T < \zeta\Big) \leq c_{18} d_k^{1-\beta_4}.$$

We combine this and (4.50) to see that

$$H^{\cdot}(\widetilde{A}) \leq c_{19} d_k^{1-\beta_4}. \tag{4.51}$$

Now we go back to the original definition of d_k—we treat it again as a random variable. Note that $t \to |X_t - Y_t|$ is a predictable process, so $\sum_k d_k \mathbf{1}_{t\in(U_{k-1},U_k]}$ is a predictable process. This, (4.51) and the exit system formula (2.1) imply that the probability that there exists an excursion of Y belonging to the set $\widetilde{A}$ and starting in the time interval $[U_{k-1}, U_k \wedge \sigma_b']$ is less than $bc_{19} d_k^{1-\beta_4}$.

Assume that $J_k = m$ for some $m \geq m_1$ such that $2^{-m} \geq d_k^{\beta_3}$. Recall that we have assumed that $C_k \cap F_k$ holds. Let

$$S_k^1 = \inf\{t \geq \widetilde{U}_k : X_t \in \partial D, |Y_t - Y_{\widetilde{U}_k}| \geq c_{16}\},$$

$$S_k^+ = \inf\{t \geq S_k^1 : |X_t - X_{S_k^1}| \geq d_k^{\beta_4}/2\},$$

$$C_k^1 = \{U_k \leq S_k^+\}.$$

Note that $S_k^1 \leq S_k \leq U_k$ because of (4.49) (recall that $c_{16} < c_{15}/2$). This implies that $S_k^+ \leq S_k^*$.

If $(C_k^1)^c$ holds, then $|X_{S_k^+} - Y_{S_k^+}| \leq c_5 d_k$, by (4.42), (4.45) and the fact that $S_k^+ \leq S_k^*$. Under the same assumptions, we also have $|X_{S_k^1} - Y_{S_k^1}| \leq c_5 d_k$ because $S_k^1 \leq S_k^+$. Suppose that $\varepsilon_1 > 0$ is so small that for $\varepsilon < \varepsilon_1$ we have $c_5 d_k < d_k^{\beta_4}/8$. Then $|Y_{S_k^1} - Y_{S_k^+}| \geq d_k^{\beta_4}/4$, by the definition of S_k^+ and the triangle inequality.

Suppose that $\widetilde{U}_k = U_k^*$. Since $|Y_{S_k^1} - Y_{S_k^+}| \geq d_k^{\beta_4}/4$, the excursion of Y starting at U_k^* belongs to the set $\widetilde{A}$. We have proved that the probability that there exists an excursion of Y belonging to the set $\widetilde{A}$ and starting in the time interval $[U_{k-1}, U_k \wedge \sigma_b']$ is less than $bc_{19}d_k^{1-\beta_4}$. Thus, on A_{k-1}^+,

$$\mathbb{P}(\{\widetilde{U}_k = U_k^*\} \cap \{J_k = m\} \cap (C_k^1)^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}) \le c_{19}bd_k^{1-\beta_4}.$$

Now suppose that $\widetilde{U}_k = U_{k-1}$. If we replace $\widetilde{U}_k$ with U_{k-1} in the definition of S_k^1, then this random time becomes a stopping time, and we can apply the strong Markov property at such modified S_k^1. By Lemma 3.2 of [5] and the strong Markov property applied at the modified S_k^1, the probability of $\{\sup_{s,t\in[S_k^1,U_k]} |Y_s - Y_t| \ge d_k^{\beta_4}/4\}$ is bounded by $c_{20}d_k^{1-\beta_4}$. Since $(C_k^1)^c \cap C_k \cap F_k$ implies $\{|Y_{S_k^1} - Y_{S_k^+}| \ge d_k^{\beta_4}/4\}$, we obtain on A_{k-1}^+,

$$\mathbb{P}(\{\widetilde{U}_k = U_{k-1}\} \cap \{J_k = m\} \cap (C_k^1)^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}) \le c_{20}d_k^{1-\beta_4}.$$

Combining this with the previous case yields

$$\mathbb{P}(\{J_k = m\} \cap (C_k^1)^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}) \le c_{21}d_k^{1-\beta_4}.$$

If we apply this estimate with m defined by $2^{-m} < d_k^{\beta_3} \le 2^{-m+1}$, then we obtain, on A_{k-1}^+,

$$\mathbb{P}(I_k^c \cap (C_k^1)^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}) \le c_{21}d_k^{1-\beta_4}. \tag{4.52}$$

There is no b on the right-hand side of the last estimate because b was fixed, so it can be absorbed into the constant c_{21}. There will be some other places in the proof where we absorb b into the constant.

Step 2.4. We will show that the process Y is unlikely to hit the boundary close to the point where the normal vector is parallel to the original vector from X to Y, assuming that X is in ∂D at the initial time.

It is elementary to check that if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are nonzero vectors, then the angle $\angle(-\mathbf{v}, \mathbf{v} - \mathbf{w})$ is greater than the angle $\angle(\mathbf{v}, \mathbf{w})$.

Suppose that the event $\{J_k = m\} \cap C_k \cap C_k^1 \cap F_k$ occurred for some m such that $2^{-m} \ge d_k^{\beta_3}$. Note that $S_k^1 \le S_k$ because of (4.49). Let α_k be the angle between $X_{S_k^1} - Y_{S_k^1}$ and $X_{S_k} - Y_{S_k}$.

Suppose that $\alpha_k > 16 \cdot 2^{-J_k}$. We will show that this assumption leads to a contradiction. For all $t \in [S_k^1, S_k]$ such that $X_t \in \partial D$, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(X_{S_k})$ is smaller than $d_k^{\beta_4}$ because C_k^1 holds so $S_k \le U_k \le S_k^+$, and therefore, the definition of S_k^+ implies that $\sup_{t\in[S_k^1,S_k]} |X_t - X_{S_k^1}| \le d_k^{\beta_4}/2$. It follows that the angle between $\int_{S_k^1}^{S_k} \mathbf{n}(X_t)\, dL_t^X$ and $\mathbf{n}(X_{S_k})$ is also smaller than $d_k^{\beta_4}$. Since $J_k = m$, the angle between $\mathbf{n}(z_k)$ and $\mathbf{n}(X_{S_k})$ is smaller than or equal to 2^{-m+2}. This is equivalent to saying that the angle between $Y_{S_k} - X_{S_k}$ and $\mathbf{n}(X_{S_k})$ is smaller than or equal to 2^{-m+2}. It follows that the angle between $\int_{S_k^1}^{S_k} \mathbf{n}(X_t)\, dL_t^X$ and $Y_{S_k} - X_{S_k}$

is smaller than $2^{-m+2} + d_k^{\beta_4} \le 2^{-m+2} + d_k^{\beta_3} \le 2^{-m+2} + 2^{-m} < 2^{-m+3}$. Note that $Y_t \notin \partial D$ for $t \in [S_k^1, S_k]$. Thus $\int_{S_k^1}^{S_k} \mathbf{n}(Y_t)\,dL_t^Y = 0$ and therefore,

$$X_{S_k} - Y_{S_k} = X_{S_k^1} - Y_{S_k^1} + \int_{S_k^1}^{S_k} \mathbf{n}(X_t)\,dL_t^X. \tag{4.53}$$

We will identify some elements of the above formula with vectors $\mathbf{v}$ and $\mathbf{w}$ in the opening remark in this step, namely, $\mathbf{v} = X_{S_k} - Y_{S_k}$ and $\mathbf{w} = X_{S_k^1} - Y_{S_k^1}$. Then $\int_{S_k^1}^{S_k} \mathbf{n}(X_t)\,dL_t^X = \mathbf{v} - \mathbf{w}$ and $\alpha_k = \angle(\mathbf{v}, \mathbf{w}) = \angle(X_{S_k^1} - Y_{S_k^1}, X_{S_k} - Y_{S_k}) > 16 \cdot 2^{-J_k} = 2^{-m+4}$. This and the fact that $\angle(-\mathbf{v}, \mathbf{v} - \mathbf{w}) = \angle(Y_{S_k} - X_{S_k}, \int_{S_k^1}^{S_k} \mathbf{n}(X_t)\,dL_t^X) < 2^{-m+3}$ yield a contradiction. Hence we must have $\alpha_k \le 2^{-J_k+4}$ if $\{J_k = m\} \cap C_k \cap C_k^1 \cap F_k$ holds.

If C_k^1 occurred, then $\sup_{S_k^1 \le t \le U_k} |X_t - X_{S_k^1}| \le d_k^{\beta_4}/2$. Since $S_k \in [S_k^1, U_k]$, it follows that $|X_{U_k} - X_{S_k}| \le d_k^{\beta_4}$. Recall that $|X_{U_k} - Y_{U_k}| \le c_5 d_k$ by (4.32). This implies that, for small $\varepsilon_1 > 0$,

$$|Y_{U_k} - X_{S_k}| \le |X_{U_k} - X_{S_k}| + |X_{U_k} - Y_{U_k}| \le d_k^{\beta_4} + c_5 d_k \le 2 d_k^{\beta_4}. \tag{4.54}$$

Let $z_k^1 \in \partial D$ be defined by $\mathbf{n}(z_k^1) = (Y_{S_k^1} - X_{S_k^1})/|Y_{S_k^1} - X_{S_k^1}|$. Since $\alpha_k \le 2^{-J_k+4}$, we have $|z_k - z_k^1| \le c_{22} 2^{-J_k} = c_{22} 2^{-m}$. We have assumed that $2^{-m} \ge d_k^{\beta_3}$, so (4.54) implies that

$$\begin{aligned} |Y_{U_k} - z_k^1| &\le |Y_{U_k} - X_{S_k}| + |X_{S_k} - z_k| + |z_k - z_k^1| \\ &\le 2 d_k^{\beta_4} + 2^{-J_k+1} + c_{22} 2^{-m} = 2 d_k^{\beta_4} + 2^{-m+1} + c_{22} 2^{-m} \le c_{23} 2^{-m}. \end{aligned}$$

We have shown that the event $\{J_k = m\} \cap C_k \cap C_k^1 \cap F_k$ implies

$$\{|Y_{U_k} - z_k^1| \le c_{23} 2^{-m}\}. \tag{4.55}$$

Suppose that $\widetilde{U}_k = U_k^*$. In this case, we will estimate the probability of the event in (4.55) using excursion theory. Recall the remarks and conventions from the beginning of step 2.3. Let $T^1 = \inf\{t \ge 0 : |e(t) - e(0)| \ge c_{16}\}$, $z^2 \in \partial D$ be the point such that $\mathbf{n}(z^2) = (e(T^1) - X_{T^1})/|e(T^1) - X_{T^1}|$ and

$$\widehat{A} = \{e : T^1 < \zeta, |e(\zeta-) - z^2| \le c_{23} 2^{-m}\}.$$

The number of excursions starting before σ_b' and such that $T^1 < \zeta$ is Poisson with the mean bounded by $c_{24} b$, by (2.1) and the right-hand side of (4.50). We can assume that $c_{16} > 0$ is arbitrarily small. If c_{16} is sufficiently small, then it is easy to see that the angle between $e(T^1) - X_{T^1}$ and $\mathbf{n}(e(0))$ must be bounded below by a strictly positive constant, and therefore the distance between z^2 and $e(T^1)$ must be bounded below by $c_{25} > 0$. By the strong Markov property applied at T^1, given the values of $e(T^1)$ and z^2 and assuming that $|e(T^1) - z^2| \ge c_{25}$, the probability that

$e(\zeta-) \in \mathcal{B}(z^2, c_{23}2^{-m})$ is smaller than $c_{26}2^{-m}$, by (3.3). Hence the expected number of excursions in $\widehat{A}$ starting before σ_b' is bounded by $c_{26}b2^{-m}$. This implies that the probability that such an excursion will occur is less than or equal to $c_{26}b2^{-m}$. We have shown that $\{J_k = m\} \cap C_k \cap C_k^1 \cap F_k$ implies $\{|Y_{U_k} - z_k^1| \le c_{23}2^{-m}\}$, so if $\{\widetilde{U}_k = U_k^*\} \cap \{J_k = m\} \cap C_k \cap C_k^1 \cap F_k$ occurs, then the excursion of Y starting at U_k^* belongs to $\widehat{A}$. We conclude that, on A_{k-1}^+,

$$\mathbb{P}(\{\widetilde{U}_k = U_k^*\} \cap \{J_k = m\} \cap C_k \cap C_k^1 \cap F_k \mid \mathcal{G}_{k-1}) \le c_{26}b2^{-m}. \tag{4.56}$$

Next suppose that $\{\widetilde{U}_k = U_{k-1}\} \cap K_{k-1}$ holds. It is easy to see that if $c_{16} > 0$ is sufficiently small, then we can find $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$, the angle between $Y_{S_k^1} - X_{S_k^1}$ and $\mathbf{n}(X_{S_k^1})$ is bounded below by a strictly positive constant. Then the distance between z_k^1 and $Y_{S_k^1}$ is bounded below by $c_{26} > 0$. By the strong Markov property applied at S_k^1, given the values of $Y_{S_k^1}$ and z_k^1 and assuming that $|Y_{S_k^1} - z_k^1| \ge c_{26}$, the probability that $Y_{U_k} \in \mathcal{B}(z_k^1, c_{23}2^{-m})$ is smaller than $c_{27}2^{-m}$, by (3.3). We have shown that $\{J_k = m\} \cap C_k \cap C_k^1 \cap F_k$ implies $\{|Y_{U_k} - z_k^1| \le c_{23}2^{-m}\}$ so, on A_{k-1}^+,

$$\mathbb{P}(\{\widetilde{U}_k = U_{k-1}\} \cap \{J_k = m\} \cap C_k \cap C_k^1 \cap F_k \mid \mathcal{G}_{k-1}) \le c_{27}2^{-m}.$$

We combine this with (4.56) to see that

$$\mathbb{P}(\{J_k = m\} \cap C_k \cap C_k^1 \cap F_k \mid \mathcal{G}_{k-1}) \le c_{28}2^{-m}.$$

Summing over $m \ge m'$, we obtain, on A_{k-1}^+,

$$\mathbb{P}(\{J_k \ge m'\} \cap C_k \cap C_k^1 \cap F_k \mid \mathcal{G}_{k-1}) \le c_{29}2^{-m'}. \tag{4.57}$$

We combine (4.43), (4.52) and (4.57) to see that if $2^{-m} \ge d_k^{\beta_3}$, then on A_{k-1}^+,

$$\begin{aligned}\mathbb{P}(\{J_k \ge m\} \cap F_k \mid \mathcal{G}_{k-1}) &\le \mathbb{P}(C_k^c \cap F_k \mid \mathcal{G}_{k-1}) + \mathbb{P}(I_k^c \cap (C_k^1)^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}) \\ &\quad + \mathbb{P}(\{J_k \ge m\} \cap C_k \cap C_k^1 \cap F_k \mid \mathcal{G}_{k-1}) \\ &\le c_{13}d_k^{1-\beta_4} + c_{21}d_k^{1-\beta_4} + c_{29}2^{-m} \le c_{30}2^{-m};\end{aligned}$$

that is, (4.37) holds.

The following follows from (4.57), with m' defined by $2^{-m'-1} \le d_k^{\beta_3} < 2^{-m'}$. We have on A_{k-1}^+,

$$\mathbb{P}(I_k^c \cap C_k \cap C_k^1 \cap F_k \mid \mathcal{G}_{k-1}) \le c_{29}d_k^{\beta_3}. \tag{4.58}$$

Step 2.5. We will show that the vector from X to Y is very likely to be almost parallel to ∂D at the time U_k.

Assume that $C_k \cap F_k$ holds. Let

$$\widehat{S}_k^j = \inf\{t \geq S_k : |X_t - Y_t| \leq 2^{-j}\},$$
$$\widehat{U}_k^j = \inf\{t \geq \widehat{S}_k^j : |X_t - X_{\widehat{S}_k^j}| \geq 2^{-j\beta_4}\},$$
$$\widehat{C}_k^j = \{U_k \leq \widehat{U}_k^j\}.$$

The following argument is very similar to that in step 2.1. By the definition of $\widehat{S}_k^j$, for large j,

$$|X_{\widehat{S}_k^j} - Y_{\widehat{S}_k^j}| = 2^{-j}. \tag{4.59}$$

Suppose that $\{\widehat{S}_k^j \leq U_k\} \cap (\widehat{C}_k^j)^c$ holds. Then $X_{\widehat{S}_k^j} \in \partial D$ and $\operatorname{dist}(Y_{\widehat{S}_k^j}, \partial D) \leq 2^{-j}$. By Lemma 3.2 of [5],

$$\mathbb{P}\Big(\sup_{\widehat{S}_k^j \leq t \leq U_k} |Y_{\widehat{S}_k^j} - Y_t| \geq 2^{-j\beta_4}/3\Big) \leq c_{31} 2^{-j(1-\beta_4)}. \tag{4.60}$$

It follows from (4.44) that for all $t \in [\widehat{S}_k^j, U_k]$,

$$|X_t - Y_t| \leq c_5 |X_{\widehat{S}_k^j} - Y_{\widehat{S}_k^j}|. \tag{4.61}$$

In particular, for large j, $|X_{\widehat{U}_k^j} - Y_{\widehat{U}_k^j}| \leq c_5 2^{-j} < 2^{-j\beta_4}/3$. This, (4.59) and the definitions of $\widehat{S}_k^j$ and $\widehat{C}_k^j$ imply that, assuming that $\widehat{C}_k^j$ does not hold, $|Y_{\widehat{S}_k^j} - Y_{\widehat{U}_k^j}| \geq 2^{-j\beta_4}/3$. This and (4.60) imply that, on A_{k-1}^+,

$$\mathbb{P}\big((\widehat{C}_k^j)^c \cap \{\widehat{S}_k^j \leq U_k\} \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) \leq c_{32} 2^{-j(1-\beta_4)}. \tag{4.62}$$

Assume that $\widehat{C}_k^j \cap \{\widehat{S}_k^j \leq U_k\}$ holds. Since $X_{\widehat{S}_k^j} \in \partial D$ and $Y_{U_k} \in \partial D$, there is $t \in [\widehat{S}_k^j, U_k]$ such that $\operatorname{dist}(X_t, \partial D) = \operatorname{dist}(Y_t, \partial D)$. Let $\widetilde{S}_k^j$ be the smallest $t \geq \widehat{S}_k^j$ with this property. Let $\widetilde{z} \in \partial D$ be the point closest to $X_{\widetilde{S}_k^j}$ among all points equidistant from $X_{\widetilde{S}_k^j}$ and $Y_{\widetilde{S}_k^j}$. By the definition of $\widehat{C}_k^j$, for all $t \in [\widehat{S}_k^j, U_k]$, we have $|X_t - X_{\widehat{S}_k^j}| \leq 2^{-j\beta_4}$. By (4.61), for all $t \in [\widehat{S}_k^j, U_k]$, we have $|X_t - Y_t| \leq c_5 2^{-j}$. This implies that, for large j, $|\widetilde{z} - X_{\widetilde{S}_k^j}| = |\widetilde{z} - Y_{\widetilde{S}_k^j}| \leq 10 \cdot 2^{-j\beta_4}$. We also have for $t \in [\widetilde{S}_k^j, U_k]$, $|\widetilde{z} - X_t| \leq 20 \cdot 2^{-j\beta_4}$ and $|\widetilde{z} - Y_t| \leq 20 \cdot 2^{-j\beta_4}$. Hence we can apply Lemma 2.1 with $c_1/4 = 20 \cdot 2^{-j\beta_4}$ at the stopping time $\widetilde{S}_k^j$ to see that

$$\big|\langle X_{U_k} - Y_{U_k}, \mathbf{n}(\widetilde{z})\rangle\big| \leq 80 \cdot 2^{-j\beta_4} |X_{U_k} - Y_{U_k}|. \tag{4.63}$$

Since $|\widetilde{z} - Y_{U_k}| \leq 20 \cdot 2^{-j\beta_4}$, the angle between $\mathbf{n}(\widetilde{z})$ and $\mathbf{n}(Y_{U_k})$ is less than $40 \cdot 2^{-j\beta_4}$ for large j. This and (4.63) imply that, for large j,

$$\big|\langle X_{U_k} - Y_{U_k}, \mathbf{n}(Y_{U_k})\rangle\big| \leq 200 \cdot 2^{-j\beta_4} |X_{U_k} - Y_{U_k}|. \tag{4.64}$$

Let j_0 be the largest j such that $\widehat{S}_k^j \le U_k$. Then $|X_{U_k} - Y_{U_k}| \ge 2^{-j_0-1}$, and if the event in (4.64) holds with $j = j_0$, then the following event holds:

$$K_k = \left\{ \frac{|\langle X_{U_k} - Y_{U_k}, \mathbf{n}(Y_{U_k})\rangle|}{|X_{U_k} - Y_{U_k}|} \le c_2 |X_{U_k} - Y_{U_k}|^{\beta_4} \right\},$$

with $c_2 = 200 \cdot 2^{\beta_4}$. It follows from the definitions of S_k and $\widehat{S}_k^j$ that $2^{-j_0} \le 2c_8 d_k$. Thus

$$K_k^c \cap C_k \cap F_k \subset \bigcup_{j:2^{-j} \le 2c_8 d_k} (\widehat{C}_k^j)^c \cap \{\widehat{S}_k^j \le U_k\} \cap C_k \cap F_k.$$

This and (4.62) imply that, on A_{k-1}^+,

$$\begin{aligned} &\mathbb{P}(K_k^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}) \\ &\quad \le \mathbb{P}\Bigl(\bigcup_{j:2^{-j} \le 2c_8 d_k} (\widehat{C}_k^j)^c \cap \{\widehat{S}_k^j \le U_k\} \cap C_k \cap F_k \mid \mathcal{G}_{k-1} \Bigr) \\ &\quad \le \sum_{j:2^{-j} \le 2c_8 d_k} c_{32} 2^{-j(1-\beta_4)} \le c_{33} d_k^{1-\beta_4}. \end{aligned} \tag{4.65}$$

Step 2.6. We will find a lower bound for the distance from X to Y at the time U_k.

Suppose that $I_k \cap C_k \cap F_k$ holds. Recall that $\beta_4 > \beta_3$. Since I_k holds, we have $2^{-J_k} \ge d_k^{\beta_3}$. Assume for now that $d_k^{\beta_3} \le \eta$, where $\eta > 0$ is so small that $d_k^{\beta_4} < (1/100) \wedge d_k^{\beta_3}/(4\pi) \le (1/\pi) 2^{-J_k-1}$. Since C_k is assumed to hold, we have $|X_t - X_{S_k}| \le d_k^{\beta_4}$ for all $t \in [S_k, U_k]$ such that $X_t \in \partial D$, and therefore, for such t, the angle between $\mathbf{n}(X_t)$ and $\mathbf{n}(X_{S_k})$ is smaller than $\pi d_k^{\beta_4}$. It follows that the angle between $\int_{S_k}^{U_k} \mathbf{n}(X_t)\, dL_t^X$ and $\mathbf{n}(X_{S_k})$ is also smaller than $\pi d_k^{\beta_4}$. The angle between $\mathbf{n}(X_{S_k})$ and $\mathbf{n}(z_k)$ is greater than 2^{-J_k}. This implies that the angle between $\int_{S_k}^{U_k} \mathbf{n}(X_t)\, dL_t^X$ and $Y_{S_k} - X_{S_k}$, which is the same as the angle between $\int_{S_k}^{U_k} \mathbf{n}(X_t)\, dL_t^X$ and $\mathbf{n}(z_k)$, is greater than $2^{-J_k} - \pi d_k^{\beta_4} > 2^{-J_k} - 2^{-J_k-1} = 2^{-J_k-1}$. Note that $Y_t \notin \partial D$ for $t \in [S_k, U_k]$ by the definition of U_k. Thus $\int_{S_k}^{U_k} \mathbf{n}(Y_t)\, dL_t^Y = 0$ and, therefore,

$$X_{U_k} - Y_{U_k} = X_{S_k} - Y_{S_k} + \int_{S_k}^{U_k} \mathbf{n}(X_t)\, dL_t^X. \tag{4.66}$$

If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are nonzero vectors and the angle $\angle(\mathbf{v}, \mathbf{w})$ is greater than α then the length of $\mathbf{v} - \mathbf{w}$ is at least $|\mathbf{w}| \sin\alpha$. In view of (4.66), we can apply this observation to $\mathbf{w} = Y_{S_k} - X_{S_k}$ and $\mathbf{v} = \int_{S_k}^{U_k} \mathbf{n}(X_t)\, dL_t^X$, and conclude that $|X_{U_k} - Y_{U_k}| \ge c_{34} 2^{-J_k} |X_{S_k} - Y_{S_k}| = c_{34} 2^{-J_k} c_8 d_k$. We now specify the value of the constant in the definition of G_k to be $c_9 = c_{34} c_8$. With this definition of G_k, we see that we have shown that G_k holds. Hence, assuming that $d_k^{\beta_4} < (1/100) \wedge d_k^{\beta_3}/(4\pi)$, we have on A_{k-1}^+,

$$\mathbb{P}\big(G_k^c \cap I_k \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) = 0. \tag{4.67}$$

Since $\inf\{a \geq 0 : a^{\beta_4} \geq a^{\beta_3}/(4\pi)\} > 0$ and the probability of any event is bounded by 1, we have for some constant $c_{35} < \infty$, on the event $\{d_k^{\beta_4} \geq (1/100) \wedge d_k^{\beta_3}/(4\pi)\} \cap A_{k-1}^+$,

$$\mathbb{P}\big(G_k^c \cap I_k \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) \leq c_{35} d_k^{\beta_3}. \tag{4.68}$$

In view of (4.67), we see that (4.68) holds on A_{k-1}^+.

It follows from (4.43), (4.52), (4.58), (4.65) and (4.68) that on A_{k-1}^+,

$$\begin{aligned}
\mathbb{P}\big(A_k^c \cap F_k \mid \mathcal{G}_{k-1}\big) &= \mathbb{P}\big((I_k^c \cup C_k^c \cup G_k^c \cup K_k^c) \cap F_k \mid \mathcal{G}_{k-1}\big) \\
&\leq \mathbb{P}\big(C_k^c \cap F_k \mid \mathcal{G}_{k-1}\big) + \mathbb{P}\big(I_k^c \cap (C_k^1)^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) \\
&\quad + \mathbb{P}\big(I_k^c \cap C_k^1 \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) + \mathbb{P}\big(K_k^c \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) \\
&\quad + \mathbb{P}\big(G_k^c \cap I_k \cap C_k \cap F_k \mid \mathcal{G}_{k-1}\big) \\
&\leq c_{13} d_k^{1-\beta_4} + c_{21} d_k^{1-\beta_4} + c_{29} d_k^{\beta_3} + c_{33} d_k^{1-\beta_4} + c_{35} d_k^{\beta_3} \\
&\leq c_{36} d_k^{\beta_3}.
\end{aligned}$$

This completes the proof of (4.36).

Step 3. The last step of the proof combines the estimates obtained above. Although this part of the proof looks complicated, its beginning consists mostly of elementary combinatorial arguments. The second part is a more or less straightforward translation of the earlier estimates into the language of distributions and stochastic domination.

If F_k holds, then (4.46) shows that $\sup_{t \in [S_k, \sigma_b']} |Y_t - X_t| < d_k/2$. It follows that if $F_k \cap F_{k+1}$ holds, then $U_k \in [S_k, \sigma_b']$ and, therefore,

$$d_{k+1} = |X_{U_k} - Y_{U_k}| \leq \sup_{t \in [S_k, \sigma_b']} |Y_t - X_t| < d_k/2.$$

Hence if the event $\bigcap_{j \leq k-1} F_j$ occurred, then $d_k \leq d_0 2^{-k+1} = \varepsilon 2^{-k+1}$. This, the fact that $A_{k-1}^+ \subset \bigcap_{j \leq k-1} F_j$ and (4.36) imply that

$$\mathbb{P}\big(A_k^c \cap F_k \cap A_{k-1}^+\big) \leq c_{10} d_k^{\beta_3} \leq c_{10} \varepsilon^{\beta_3} 2^{-(k-1)\beta_3}. \tag{4.69}$$

Let

$$F = F_1^c \cup \bigcup_{k=1}^{\infty} \big(F_k \cap F_{k+1}^c \cap A_k^+\big).$$

If $\bigcap_{k=1}^{\infty} F_k$ holds, then

$$\liminf_{k\to\infty} |X_{U_k} - Y_{U_k}| = \liminf_{k\to\infty} d_{k-1} \le \lim_{k\to\infty} \varepsilon 2^{-k-2} = 0, \tag{4.70}$$

so $\inf_{0\le t\le \sigma_b'} |X_t - Y_t| \le \liminf_{k\to\infty} |X_{U_k} - Y_{U_k}| = 0$. The last event has probability 0, according to Lemma 2.2, so $\mathbb{P}(\bigcap_{k=1}^{\infty} F_k) = 0$. Since $F_{k+1} \subset F_k$, there exists at most one N_1 such that $F_{N_1}^c \cup F_{N_1+1}$ fails (in other words, $F_{N_1} \cap F_{N_1+1}^c$ holds). We will write $\bigcap_{k=1}^{\infty} F_k = \{N_1 = \infty\}$ so $\mathbb{P}(N_1 = \infty) = 0$. There exists at most one N_2 such that A_j holds for all $j < N_2$, and A_{N_2} does not hold. Using these definitions of N_1 and N_2, and (4.69), we obtain

$$\begin{aligned}
\mathbb{P}(F^c) &= \mathbb{P}\left(F_1 \cap \bigcap_{k=1}^{\infty}\left(F_k^c \cup F_{k+1} \cup \bigcup_{j\le k} A_j^c\right)\right) \\
&\le \mathbb{P}(F_1 \cap \{N_1 = \infty\}) + \mathbb{P}\left(\bigcap_{k=1}^{\infty}\left(F_k^c \cup F_{k+1} \cup \bigcup_{j\le k} A_j^c\right) \cap \{N_1 < \infty\}\right) \\
&= 0 + \mathbb{P}\left(\bigcap_{k=1}^{\infty}\left(F_k^c \cup F_{k+1} \cup \bigcup_{j\le k} A_j^c\right) \cap \{N_1 < \infty\}\right) \\
&\le \mathbb{P}\left(\bigcup_{n=1}^{\infty}\left(\left(F_n^c \cup F_{n+1} \cup \bigcup_{j\le n} A_j^c\right) \cap \{N_1 = n\}\right)\right) \\
&= \mathbb{P}\left(\bigcup_{n=1}^{\infty}\left(\left(\bigcup_{j\le n} A_j^c\right) \cap \{N_1 = n\}\right)\right) \\
&= \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{n}\left(\left(\bigcup_{j\le m} A_j^c\right) \cap \{N_1 = n, N_2 = m\}\right)\right) \\
&= \mathbb{P}\left(\bigcup_{m=1}^{\infty}\left(\left(\bigcup_{j\le m} A_j^c\right) \cap \{N_1 \ge m, N_2 = m\}\right)\right) \\
&\le \mathbb{P}\left(\bigcup_{m=1}^{\infty} (A_{m-1}^+ \cap A_m^c \cap F_m)\right) \\
&\le \sum_{m=1}^{\infty} \mathbb{P}(A_{m-1}^+ \cap A_m^c \cap F_m) \\
&\le \sum_{m=1}^{\infty} c_{10}\varepsilon^{\beta_3} 2^{-(m-1)\beta_3} \le c_{37}\varepsilon^{\beta_3}.
\end{aligned}$$

This proves (4.24).

Since $F_{k+1} \subset F_k$ and $(F_{k+1} \cap A^+_{k+1}) \subset (F_k \cap A^+_k)$, we have

$$
\begin{aligned}
F &= F_1^c \cup \bigcup_{n=1}^{\infty} (F_n \cap F_{n+1}^c \cap A_n^+) \\
&= F_1^c \cup \bigcup_{n=1}^{k-2} (F_n \cap F_{n+1}^c \cap A_n^+) \cup (F_{k-1} \cap F_k^c \cap A_{k-1}^+) \\
&\quad \cup \bigcup_{n=k}^{\infty} (F_n \cap F_{n+1}^c \cap A_n^+) \\
&\subset F_1^c \cup \bigcup_{n=1}^{k-2} F_{n+1}^c \cup (F_{k-1} \cap F_k^c \cap A_{k-1}^+) \cup \bigcup_{n=k}^{\infty} (F_n \cap A_n^+) \\
&\subset F_{k-1}^c \cup (F_{k-1} \cap F_k^c \cap A_{k-1}^+) \cup (F_k \cap A_k^+).
\end{aligned} \tag{4.71}
$$

Let $T_k = U_k \wedge \sigma_b'$. We make the following three claims:

$$
\begin{aligned}
&\bigl|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\bigr| \\
&\quad \begin{cases} = 0, & \text{if } F_{k-1}^c \text{ holds;} \\ \le c_{38} := c_7 \vee \log c_5, & \text{if } F_{k-1} \cap F_k^c \cap A_{k-1}^+ \text{ holds;} \\ \le c_{39} m, & \text{if } \{J_k = m\} \cap F_k \cap A_k^+ \text{ holds.} \end{cases}
\end{aligned} \tag{4.72}
$$

The first claim follows from the definitions of T_{k-1}, S_{k-1} and F_{k-1}. The second claim follows from the definition of S_k and (4.32) applied with $T = U_{k-1}$. The last claim follows from the fact that $G_k \subset A_k$.

If A_k holds, then K_k holds. Then condition (4.26) is satisfied with $x_0 = X_{U_k}$ and $y_0 = Y_{U_k}$. By the strong Markov property applied at the stopping time U_k, we obtain a formula analogous to (4.35) which implies that

$$\mathbb{P}\bigl(F_k \cap A_k^+ \mid \mathcal{G}_{k-1}\bigr) \le \mathbb{P}(F_k \mid \mathcal{G}_{k-1}) \le p_1$$

on A_{k-1}^+. By the repeated application of the strong Markov property at $U_1, U_2, \dots,$ we obtain

$$\mathbb{P}\bigl(F_k \cap A_k^+\bigr) \le p_1^k. \tag{4.73}$$

This and the second claim in (4.72) imply that

$$\mathbb{P}\bigl(\bigl|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\bigr| \mathbf{1}_{F_{k-1} \cap F_k^c \cap A_{k-1}^+} > c_{38}\bigr) = 0, \tag{4.74}$$

$$
\begin{aligned}
&\mathbb{P}\bigl(\bigl|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\bigr| \mathbf{1}_{F_{k-1} \cap F_k^c \cap A_{k-1}^+} \in (0, c_{38}]\bigr) \\
&\quad \le p_1^{k-1},
\end{aligned} \tag{4.75}
$$

$$
\begin{aligned}
&\mathbb{P}\bigl(\bigl|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\bigr| \mathbf{1}_{F_{k-1} \cap F_k^c \cap A_{k-1}^+} = 0\bigr) \\
&\quad \ge 1 - p_1^{k-1}.
\end{aligned} \tag{4.76}
$$

Since D is bounded, there exists $m_0 > -\infty$ such that $J_k \geq m_0$, a.s. It follows from (4.37) that on A_{k-1}^+,

$$\mathbb{P}(\{J_k \geq m\} \cap F_k \cap A_k \mid \mathcal{G}_{k-1}) \leq c_{40} 2^{-m},$$

so we obtain for $m \geq m_0$, using (4.73) and the third claim in (4.72),

$$\mathbb{P}\big(\big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{\{J_k=m\}\cap F_k \cap A_k^+} > c_{39} m\big) = 0, \tag{4.77}$$

$$\begin{aligned}&\mathbb{P}\big(\big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{\{J_k=m\}\cap F_k \cap A_k^+} \in (0, c_{39} m]\big)\\ &\quad \leq p_1^{k-1} c_{40} 2^{-m},\end{aligned} \tag{4.78}$$

$$\begin{aligned}&\mathbb{P}\big(\big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{\{J_k=m\}\cap F_k \cap A_k^+} = 0\big)\\ &\quad \geq 1 - p_1^{k-1} c_{40} 2^{-m}.\end{aligned} \tag{4.79}$$

Recall from the paragraph following (4.70) that only a finite number of events F_k, $k \geq 1$, hold, a.s. Hence, for some random $k_0 < \infty$ and all $k \geq k_0$, we have $U_k = \sigma_b'$. It follows that $T_n = T_{k_0} = \sigma_b'$ for all $n \geq k_0$, and therefore,

$$|V_1 - V_0| = \left|\sum_{k=1}^{\infty} \log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\right|.$$

This, (4.71) and the first claim in (4.72) imply that

$$\begin{aligned}
|V_1 - V_0|\mathbf{1}_F &= \left|\sum_{k=1}^{\infty} \log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\right|\mathbf{1}_F\\
&\leq \sum_{k=1}^{\infty} \big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{F_{k-1}^c}\\
&\quad + \sum_{k=1}^{\infty} \big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{F_{k-1}\cap F_k^c \cap A_{k-1}^+}\\
&\quad + \sum_{k=1}^{\infty} \big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{F_k \cap A_k^+}\\
&= \sum_{k=1}^{\infty} \big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{F_{k-1}\cap F_k^c \cap A_{k-1}^+}\\
&\quad + \sum_{k=1}^{\infty} \sum_{m \geq m_0} \big|\log|X_{T_k} - Y_{T_k}| - \log|X_{T_{k-1}} - Y_{T_{k-1}}|\big|\mathbf{1}_{\{J_k=m\}\cap F_k \cap A_k^+}.
\end{aligned} \tag{4.80}$$

Let k_0 be such that $p_1^{k-1} + p_1^{k-1} \sum_{m \geq m_0} c_{40} 2^{-m} \leq 1$ for $k \geq k_0$, and let m_1 be such that $\sum_{m \geq m_1} c_{40} 2^{-m} \leq 1$. Let $q' \geq 0$ be such that $q' + \sum_{m \geq m_1} c_{40} 2^{-m} = 1$, and let $q_k \geq 0$ be such that $q_k + p_1^{k-1} + p_1^{k-1} \sum_{m \geq m_0} c_{40} 2^{-m} = 1$, for $k \geq k_0$. Let Z_k, $k \geq 1$, be independent random variables with the following distributions; for $1 \leq k \leq k_0 - 1$,

$$Z_k = \begin{cases} c_{38} + c_{39} m_1, & \text{with probability } q'; \\ c_{39} m, & \text{with probability } c_{40} 2^{-m} \text{ for } m \geq m_1, \end{cases}$$

and for $k \geq k_0$,

$$Z_k = \begin{cases} 0, & \text{with probability } q_k; \\ c_{38}, & \text{with probability } p_1^{k-1}; \\ c_{39} m, & \text{with probability } c_{40} p_1^{k-1} 2^{-m} \text{ for } m \geq m_0. \end{cases}$$

By (4.74)–(4.76), (4.77)–(4.79) and (4.80), the random variable

$$|V_1 - V_0| \mathbf{1}_F = \left| \sum_{k=1}^{\infty} \log |X_{T_k} - Y_{T_k}| - \log |X_{T_{k-1}} - Y_{T_{k-1}}| \right| \mathbf{1}_F$$

is stochastically dominated by $Z_* := \sum_{k \geq 1} Z_k$. We have

$$\begin{aligned} \mathbb{E} Z_* = & \sum_{1 \leq k \leq k_0 - 1} \left((c_{38} + c_{39} m_1) q' + \sum_{m \geq m_1} c_{40} 2^{-m} c_{39} m \right) \\ & + \sum_{k \geq k_0} \left(q_k \cdot 0 + p_1^{k-1} c_{38} + p_1^{k-1} \sum_{m \geq m_0} c_{39} 2^{-m} c_{39} m \right) < \infty. \end{aligned}$$

If we take $G(a)$ to be the cumulative distribution function of Z_*, then the last estimate shows that (4.25) is satisfied. □

The next result is an elementary lemma involving distributions and expectations. Recall the notation from (4.1).

LEMMA 4.3. *For any* $c_0 > 0$, $\beta_1 \in (0, 1/2)$ *there exist* $\beta_2, c_1, c_2, b, \varepsilon_1 > 0$ *such that if* $\varepsilon \leq \varepsilon_1$, $x_0 \in \partial D$, $y_0 \in \overline{D}$, $|x_0 - y_0| = \varepsilon$, $X_0 = x_0$, $Y_0 = y_0$ *and*

$$\frac{|\langle y_0 - x_0, \mathbf{n}(x_0) \rangle|}{|y_0 - x_0|} \leq c_0 \varepsilon^{\beta_1}, \tag{4.81}$$

then there exists an event F *such that*

$$\mathbb{P}^{x_0, y_0}(F^c) \leq c_1 \varepsilon^{\beta_2}, \tag{4.82}$$

$$\mathbb{E}^{x_0, y_0}[(V_1 - V_0) \mathbf{1}_F] \geq c_2. \tag{4.83}$$

PROOF. It suffices to prove the lemma for $c_0 = 1$, by the same argument as the one at the beginning of the proof of Lemma 4.1.

First we prove a general claim. Suppose that a cumulative distribution function $G : \mathbb{R} \to [0, 1]$ satisfies $\int_{-\infty}^{\infty} |a| \, dG(a) < \infty$. Then for every $c_3 > 0$ there exists

$p_1 > 0$ such that if W is a random variable which satisfies $\mathbb{P}(|W| \le a) \le G(a)$ for $a \in \mathbb{R}$ and $\mathbb{P}(W \le c_3) \le p_1$, then $\mathbb{E}W \ge c_3/2$. To see this, let $a_1 > -\infty$ be such that $\int_{(-\infty,a_1]} |a|\, dG(a) < c_3/8$. We choose $p_1 > 0$ so small that $|a_1| p_1 < c_3/8$ and $c_3(1 - p_1) > 3c_3/4$. Then

$$\begin{aligned}\mathbb{E}W &\ge \int_{(-\infty,a_1]} a\, dG(a) - |a_1| p_1 + c_3(1 - p_1)\\ &\ge -c_3/8 - c_3/8 + 3c_3/4 = c_3/2.\end{aligned} \tag{4.84}$$

We will apply this observation to $W = (V_1 - V_0)\mathbf{1}_F$. By Lemma 4.2, there exists an event F such that $\mathbb{P}(F^c) \le \varepsilon^{\beta_2}$ and $\mathbb{P}(|V_1 - V_0|\mathbf{1}_F \le a) \le G(a)$ for $a \in \mathbb{R}$ for some G with $\int_{-\infty}^{\infty} |a|\, dG(a) < \infty$. We can choose small $\varepsilon_1, c_3 > 0$ and apply Lemma 4.1 to obtain

$$\mathbb{P}\big((V_1 - V_0)\mathbf{1}_F \le c_3\big) \le \mathbb{P}(V_1 - V_0 \le c_3) + \mathbb{P}\big(F^c\big) \le p_1/2 + \varepsilon^{\beta_2} \le p_1.$$

We now apply (4.84) to $W = (V_1 - V_0)\mathbf{1}_F$ to see that $\mathbb{E}[(V_1 - V_0)\mathbf{1}_F] \ge c_3/2$. We take $c_2 = c_3/2$ to finish the proof of the lemma. □

PROOF OF THEOREM 1.1. *Step* 1. In this step, we will define, using induction, a pair of stochastic processes similar to X and Y on a sequence of random intervals. At the end of each interval, we check whether the processes have a typical (and desirable) behavior. If so, we let them continue according to the original stochastic differential equations. Otherwise, we insert a jump which brings the processes to a convenient position. We will later argue that the probability of inserting even a single jump is very small. We note that this part of the proof could have been presented in a different way. Instead of inserting jumps, we could have killed the processes at the time when we insert the first jump. This would have made the first step of the argument more natural, but it would make the remaining part of the proof more awkward to present.

Recall that $\sigma'_b = \sigma_b^X \wedge \sigma_b^Y$ and

$$\sigma'_{(k+1)b} = \inf\{t \ge \sigma'_{kb} : \big(L_t^X - L_{\sigma'_{kb}}^X\big) \wedge \big(L_t^Y - L_{\sigma'_{kb}}^Y\big) \ge b\}$$

for $k \ge 1$. Fix $c_0, \varepsilon_1, b, \beta_1 > 0$ and $p < 1$ such that Lemmas 2.3, 4.1 and 4.3 hold with this choice of parameters. Below, the constant c_0 will be denoted c_2.

We will define processes X_t^* and Y_t^* for $t \ge 0$ in an inductive way. Let $X_t^* = X_t$ and $Y_t^* = Y_t$ for $t \in [0, \sigma'_b)$. By Lemma 2.2, $Y_{\sigma'_b} \ne X_{\sigma'_b}$, $\mathbb{P}^{x,y}$-a.s., for any $x, y \in \overline{D}$ such that $x \ne y$. Fix an arbitrary $p_1 > 0$ and choose $c_1 > 0$ such that

$$\mathbb{P}^{x,y}\big(|Y_{\sigma'_b} - X_{\sigma'_b}| \le c_1\big) < p_1. \tag{4.85}$$

Let $F_1 = \{|Y_{\sigma'_b} - X_{\sigma'_b}| \ge c_1\}$. Recall from Section 2.4 that π_x denotes the projection on the plane tangent to ∂D at $x \in \partial D$. Suppose that $\sigma'_b = \sigma_b^X$, recall $c_2 = c_0$ defined above and let

$$A_1 = \left\{ \frac{|\langle Y_{\sigma'_b} - X_{\sigma'_b}, \mathbf{n}(X_{\sigma'_b})\rangle|}{|Y_{\sigma'_b} - X_{\sigma'_b}|} \le c_2 |Y_0 - X_0|^{\beta_1} \right\},$$

$$\begin{cases} Y^*_{\sigma'_b} = Y_{\sigma'_b}, & \text{if } A_1 \cap F_1 \text{ holds}, \\ Y^*_{\sigma'_b} = X_{\sigma'_b} + \pi_{X_{\sigma'_b}}(Y_{\sigma'_b -} - X_{\sigma'_b}) \dfrac{|X_{\sigma'_b} - Y_{\sigma'_b}| \vee c_1}{|\pi_{X_{\sigma'_b}}(Y_{\sigma'_b} - X_{\sigma'_b})|}, \\ \qquad \text{otherwise}. \end{cases} \tag{4.86}$$

Let $\{Y^*_t, t \in [\sigma'_b, \sigma'_{2b})\}$ be the solution to (1.2) with the initial condition given by (4.86) and driven by Brownian motion $\{B_t, t \in [\sigma'_b, \sigma'_{2b})\}$. Let $X^*_t = X_t$ for $t \in [\sigma'_b, \sigma'_{2b})$. Note that, no matter which part of the definition (4.86) is applied, we have $|Y^*_{\sigma'_b} - X^*_{\sigma'_b}| \geq |Y_{\sigma'_b} - X_{\sigma'_b}|$ and

$$\frac{|\langle Y^*_{\sigma'_b} - X^*_{\sigma'_b}, \mathbf{n}(X^*_{\sigma'_b})\rangle|}{|Y^*_{\sigma'_b} - X^*_{\sigma'_b}|} \leq c_2 |Y_0 - X_0|^{\beta_1}. \tag{4.87}$$

We have

$$\mathbb{E}^{x,y} \log\bigl|Y^*_{\sigma'_b} - X^*_{\sigma'_b}\bigr| \geq \log c_1 > -\infty. \tag{4.88}$$

If $\sigma'_b = \sigma^Y_b$, then we exchange the roles of X and Y in the above definitions.

The following formulas are a part of the inductive definition, to be continued below. Let

$$\begin{aligned} &\sigma^*_0 = 0, \\ &\sigma^*_{kb} = \inf\bigl\{t \geq \sigma^*_{(k-1)b} : \bigl(L^{X^*}_t - L^{X^*}_{\sigma^*_{(k-1)b}}\bigr) \wedge \bigl(L^{Y^*}_t - L^{Y^*}_{\sigma^*_{(k-1)b}}\bigr) \geq b\bigr\}, \qquad k \geq 1, \\ &R^*_t = \bigl|X^*_t - Y^*_t\bigr|, \qquad M^*_t = \log R^*_t, \qquad t \geq 0, \\ &V^*_k = M^*_{\sigma^*_{kb}}, \qquad k = 0, 1, \ldots \end{aligned}$$

In view of (4.87), we can apply Lemma 4.3 to the process $\{(X^*_t, Y^*_t), t \in [\sigma^*_b, \sigma^*_{2b})\}$ to conclude that there exist $c_3 > 0$ and an event $F_2 \in \sigma(B_t, t \in [\sigma^*_b, \infty))$ such that, on the event $\{R^*_{\sigma^*_b} \leq \varepsilon_1\}$,

$$\begin{aligned} \mathbb{P}\bigl(F^c_2 \mid X^*_{\sigma^*_b}, Y^*_{\sigma^*_b}\bigr) &\leq \bigl(R^*_{\sigma^*_b}\bigr)^{\beta_2}, \\ \mathbb{E}\bigl[\bigl(V^*_2 - V^*_1\bigr)\mathbf{1}_{F_2} \mid X^*_{\sigma^*_b}, Y^*_{\sigma^*_b}\bigr] &\geq c_3. \end{aligned}$$

We proceed with the inductive definition. Suppose that F_k, X^*_t and Y^*_t are already defined for some $k \geq 2$ and $t \in [0, \sigma^*_{kb})$. Suppose that $\sigma^*_{kb} = \inf\{t \geq \sigma^*_{(k-1)b} : L^{X^*}_t - L^{X^*}_{\sigma^*_{(k-1)b}} \geq b\}$, and let

$$A_k = \left\{ \frac{|\langle Y^*_{\sigma^*_{kb}-} - X^*_{\sigma^*_{kb}-}, \mathbf{n}(X^*_{\sigma^*_{kb}-})\rangle|}{|Y^*_{\sigma^*_{kb}-} - X^*_{\sigma^*_{kb}-}|} \leq c_2 |Y_{\sigma^*_{(k-1)b}} - X_{\sigma^*_{(k-1)b}}|^{\beta_1} \right\},$$

$$\begin{cases} Y^*_{\sigma^*_{kb}} = Y^*_{\sigma^*_{kb}-}, & \text{on } A_k \cap F_k, \\ Y^*_{\sigma^*_{kb}} = X^*_{\sigma^*_{kb}-} + \pi_{X^*_{\sigma^*_{kb}-}}(Y^*_{\sigma^*_{kb}-} - X^*_{\sigma^*_{kb}-}) & \\ \qquad \times \dfrac{|X^*_{\sigma^*_{kb}-} - Y^*_{\sigma^*_{kb}-}| \vee |X^*_{\sigma^*_{(k-1)b}} - Y^*_{\sigma^*_{(k-1)b}}|}{|\pi_{X^*_{\sigma^*_{kb}-}}(Y^*_{\sigma^*_{kb}-} - X^*_{\sigma^*_{kb}-})|}, & \text{otherwise.} \end{cases} \tag{4.89}$$

Let $\{(X^*_t, Y^*_t), t \in [\sigma^*_{kb}, \sigma^*_{(k+1)b})\}$ be the solution to (1.1)–(1.2) with the initial conditions given by $X^*_{\sigma^*_{kb}} = X^*_{\sigma^*_{kb}-}$ and (4.89), and driven by Brownian motion $\{B_t, t \in [\sigma^*_{kb}, \sigma^*_{(k+1)b})\}$. No matter which part of the definition (4.89) is applied, we have

$$|Y^*_{\sigma^*_{kb}} - X^*_{\sigma^*_{kb}}| \ge |Y^*_{\sigma^*_{kb}-} - X^*_{\sigma^*_{kb}-}| \tag{4.90}$$

and

$$\frac{|\langle Y^*_{\sigma^*_{kb}} - X^*_{\sigma^*_{kb}}, \mathbf{n}(X^*_{\sigma^*_{kb}})\rangle|}{|Y^*_{\sigma^*_{kb}} - X^*_{\sigma^*_{kb}}|} \le c_2 |Y_{\sigma^*_{(k-1)b}} - X^*_{\sigma^*_{(k-1)b}}|^{\beta_1}. \tag{4.91}$$

If $\sigma^*_{kb} = \inf\{t \ge \sigma^*_{(k-1)b} : L^{Y^*}_t - L^{Y^*}_{\sigma^*_{(k-1)b}} \ge b\}$, then we exchange the roles of X and Y in the above definitions.

In view of (4.91), we can apply Lemma 4.3 to the process $\{(X^*_t, Y^*_t), t \in [\sigma^*_{kb}, \sigma^*_{(k+1)b})\}$ to conclude that there exists an event $F_{k+1} \in \sigma(B_t, t \in [\sigma^*_{kb}, \infty))$ such that, on the event $\{R^*_{\sigma^*_{kb}} \le \varepsilon_1\}$,

$$\begin{aligned} &\mathbb{P}(F^c_{k+1} \mid X^*_{\sigma^*_{kb}}, Y^*_{\sigma^*_{kb}}) \le (R^*_{\sigma^*_{kb}})^{\beta_2}, \\ &\mathbb{E}[(V^*_{k+1} - V^*_k)\mathbf{1}_{F_{k+1}} \mid X^*_{\sigma^*_{kb}}, Y^*_{\sigma^*_{kb}}] \ge c_3. \end{aligned} \tag{4.92}$$

Step 2. We will show that the probability of the undesirable events F^c_k and A^c_k is very small.

Definition (4.89) implies that on F^c_{k+1}, we have $V^*_{k+1} \ge V^*_k$. This and the strong Markov property imply that on the event $\{R^*_{\sigma^*_{kb}} \le \varepsilon_1\}$,

$$\begin{aligned} &\mathbb{E}[V^*_{k+1} - V^*_k \mid \sigma((X^*_t, Y^*_t), t \le \sigma^*_{kb})] \\ &\quad = \mathbb{E}[V^*_{k+1} - V^*_k \mid X^*_{\sigma^*_{kb}}, Y^*_{\sigma^*_{kb}}] \\ &\quad = \mathbb{E}[(V^*_{k+1} - V^*_k)\mathbf{1}_{F_{k+1}} \mid X^*_{\sigma^*_{kb}}, Y^*_{\sigma^*_{kb}}] \\ &\qquad + \mathbb{E}[(V^*_{k+1} - V^*_k)\mathbf{1}_{F^c_{k+1}} \mid X^*_{\sigma^*_{kb}}, Y^*_{\sigma^*_{kb}}] \\ &\quad \ge c_3 + 0 = c_3 > 0. \end{aligned} \tag{4.93}$$

Let $K_1 = \inf\{k \ge 1 : \sup_{t \in [\sigma^*_{kb}, \sigma^*_{(k+1)b}]} R^*_t \ge \varepsilon_1\}$ and $\widetilde{V}_k = V^*_{k \wedge K_1}$. It follows from the definition of V^*_k's that all these random variables are bounded above by a finite constant because D has a finite diameter. The estimate (4.88) implies that $\mathbb{E}V_1^* > -\infty$. It follows from this and (4.93) that $\mathbb{E}|\widetilde{V}_k| < \infty$ for all k and $\{\widetilde{V}_k, k \geq 1\}$ is a submartingale. Thus, $\widetilde{V}_k$ cannot converge to $-\infty$ with positive probability.

For any fixed j, we will estimate the number of k such that $\widetilde{V}_k \in [j, j+1]$.

Let $c_4 = \sup_{x,y\in\overline{D}} \log|x-y|$ and note that $c_4 < \infty$. We will argue that for any $c_5 \in (-\infty, c_4)$, one can choose $\varepsilon_1 > 0$ so small that if $|x-y| \leq \varepsilon_1$, then $\sup_k \widetilde{V}_k \leq c_5$, $\mathbb{P}^{x,y}$-a.s. Let $S = \inf\{t \geq 0 : R_t^* \geq \varepsilon_1\}$ and note that $S \in [\sigma^*_{K_1 b}, \sigma^*_{(K_1+1)b}]$. By (4.29) and the remark following it, for some $c_6 < \infty$,

$$\begin{aligned}\sup_{t\in[0,\sigma^*_{(K_1+1)b}]} R_t^* &\leq \varepsilon_1 \exp\bigl(c_6\bigl(L^{X^*}_{\sigma^*_{(K_1+1)b}} - L^{X^*}_S + L^{Y^*}_{\sigma^*_{(K_1+1)b}} - L^{Y^*}_S\bigr)\bigr)\\ &\leq \varepsilon_1 \exp\bigl(c_6(b+b)\bigr).\end{aligned}$$

It follows that, for small ε_1, a.s.,

$$\sup_k \widetilde{V}_k \leq \log \varepsilon_1 + 2c_6 b \leq c_5. \tag{4.94}$$

Consider any $c_5 \in (-\infty, c_4)$, assume that $\log\varepsilon_1 + 2c_6 b \leq c_5$ and fix an integer $j \leq c_5$. Let $U_1 = 0$ and

$$\begin{aligned}\widehat{U}_k &= \inf\bigl\{n \geq U_k : \widetilde{V}_n \notin [j-1, j+2]\bigr\}, \qquad k \geq 1,\\ U_k &= \inf\bigl\{n \geq \widehat{U}_k : \widetilde{V}_n \in [j, j+1]\bigr\}, \qquad k \geq 2,\\ K_2^j &= \sup\{k : U_k < \infty\},\end{aligned}$$

with the convention that $\inf\varnothing = \infty$. The random variable K_2^j is bounded above by the sum of the number of upcrossings of the interval $[j-1, j]$ and the number of downcrossings of the interval $[j+1, j+2]$ by the process $\widetilde{V}_k$. By the upcrossing inequality, in view of (4.94),

$$\mathbb{E}K_2^j \leq \mathbb{E}\bigl(\widetilde{V}_\infty - (j-1)\bigr)^+ + \mathbb{E}\bigl(\widetilde{V}_\infty - (j+1)\bigr)^+ + 1 \leq 2(c_5 - j + 2). \tag{4.95}$$

Suppose that $\widetilde{V}_{U_k} \in [j, j+1]$ for some k. Let k_0 be the smallest integer greater than $3/c_7$, where c_7 has the same value as c_1 in Lemma 4.1. Let p_2 have the same value as p in Lemma 4.1. We will apply Lemma 4.1 to estimate $\widetilde{V}_{n+1} - \widetilde{V}_n$; this can be done because of (4.90) and (4.91). By Lemma 4.1 and the strong Markov property applied at the stopping times σ^*_{nb}, $n = U_k, U_k+1, \ldots,$ we see that for $c_7, p_2 > 0$ as chosen above and $p_3 := p_2^{k_0+1}$,

$$\mathbb{P}\bigl(\widetilde{V}_{n+1} - \widetilde{V}_n \geq c_7, n = U_k, U_k+1, \ldots, U_k + k_0 \mid X^*_{\sigma^*_{U_k b}}, Y^*_{\sigma^*_{U_k b}}\bigr) \geq p_2^{k_0+1} = p_3.$$

If the event in the last formula occurs, then the process $\widetilde{V}$ will leave the interval $[j-1, j+2]$ in at most k_0+1 steps, so $\widehat{U}_k - U_k \leq k_0 + 1$ in this case. If the process $\{\widetilde{V}_m, m \geq k\}$ does not leave $[j-1, j+2]$ in k_0+1 steps, then we apply the same argument again, this time using stopping times $U_k + k_0 + 1, \ldots, U_k + 2k_0 + 1$.

By induction, the probability that the process $\{\widetilde{V}_m, m \geq k\}$ does not leave $[j-1, j+2]$ in $r(k_0+1)$ steps is at most $(1-p_3)^r$. It follows that $(\widehat{U}_k - U_k)/(k_0+1)$ is majorized by a geometric random variable with mean $1/p_3$ and, therefore, $\mathbb{E}[\widehat{U}_k - U_k \mid X^*_{\sigma^*_{U_k b}}, Y^*_{\sigma^*_{U_k b}}] \leq (k_0+1)/p_3$. Let K_3^j be the number of k such that $\widetilde{V}_k \in [j, j+1]$. We combine the last estimate with (4.95) to see that

$$\mathbb{E}K_3^j \leq 2(c_5 - j + 2)(k_0+1)/p_3. \tag{4.96}$$

This, (4.85), (4.92) and (4.94) yield

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{k\geq 1} F_k^c\Big) &\leq \mathbb{E}\Big[\sum_{k\geq 1} \mathbf{1}_{F_k^c}\Big] = \mathbb{E}\mathbf{1}_{F_1^c} + \sum_{k\geq 2} \mathbb{E}\mathbf{1}_{F_k^c} \\
&\leq p_1 + \sum_{j\leq c_5} \mathbb{E}\Big[\sum_{k:\,\widetilde{V}_{k-1}\in[j,j+1]} \mathbb{E}\big(\mathbf{1}_{F_k^c} \mid \widetilde{V}_{k-1} \in [j, j+1]\big)\Big] \\
&\leq p_1 + \sum_{j\leq c_5} \mathbb{E}\Big[\sum_{k:\,\widetilde{V}_{k-1}\in[j,j+1]} e^{(j+1)\beta_2}\Big] \\
&\leq p_1 + \sum_{j\leq c_5} e^{(j+1)\beta_2} 2(c_5 - j + 2)(k_0+1)/p_3.
\end{aligned}$$

By (4.96) and Lemma 2.3, for some $\beta_3 > 0$,

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{k\geq 1} A_k^c\Big) &\leq \mathbb{E}\Big[\sum_{k\geq 1} \mathbf{1}_{A_k^c}\Big] = \sum_{k\geq 1} \mathbb{E}\mathbf{1}_{A_k^c} \\
&= \sum_{j\leq c_5} \mathbb{E}\Big[\sum_{k:\,\widetilde{V}_{k-1}\in[j,j+1]} \mathbb{E}\big(\mathbf{1}_{A_k^c} \mid \widetilde{V}_{k-1} \in [j, j+1]\big)\Big] \\
&\leq \sum_{j\leq c_5} \mathbb{E}\Big[\sum_{\widetilde{V}_{k-1}\in[j,j+1]} e^{(j+1)\beta_3}\Big] \\
&\leq \sum_{j\leq c_5} e^{(j+1)\beta_3} 2(c_5 - j + 2)(k_0+1)/p_3.
\end{aligned}$$

We combine the last two estimates to obtain

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{k\geq 1} A_k^c \cup F_k^c\Big) \\
&\quad \leq p_1 + \sum_{j\leq c_5} \big(e^{(j+1)\beta_2} + e^{(j+1)\beta_3}\big) 2(c_5 - j + 2)(k_0+1)/p_3.
\end{aligned} \tag{4.97}$$

Consider an arbitrarily small $p_4 > 0$. The probability p_1 in (4.85) may be chosen to be smaller than $p_4/2$. We make the sum in (4.97) smaller than $p_4/2$ by taking

$c_5 > -\infty$ sufficiently small. Then, assuming that $\log\varepsilon_1 + 2c_6 b \le c_5$,

$$\mathbb{P}\biggl(\bigcup_{k\ge 1} A_k^c \cup F_k^c\biggr) \le p_4. \tag{4.98}$$

Step 3. This step contains soft arguments translating estimates that show that the distance between X and Y has a tendency to grow into a statement about the almost sure behavior of the distance process.

Recall that $R_t = |X_t - Y_t|$ and let $T_a^R = \inf\{t \ge 0 : R_t = a\}$. Recall that $\widetilde{V}_k$ does not converge to $-\infty$ at a finite or infinite time, a.s. If all events $A_k \cap F_k$, $k \ge 1$, hold, then $X_t^* = X_t$ and $Y_t^* = Y_t$ for all $t \ge 0$. This and (4.98) imply that for any $p_4 > 0$, there exists $\varepsilon_1 > 0$ such that for any $x, y \in \overline{D}$, $x \ne y$, we have $\mathbb{P}^{x,y}(T_{\varepsilon_1}^R < T_0^R) \ge 1 - p_4$.

The process R_t is continuous for all $t \ge 0$, a.s. because the processes X_t and Y_t are continuous.

Suppose that for some $x \ne y$, $p_5 := \mathbb{P}^{x,y}(T_0^R < \infty) > 0$. We will show that this assumption leads to a contradiction. For $j \ge 1$, let $S_j = \inf\{t \ge 0 : R_t \le 2^{-j}\}$ and

$$G_j = \bigl\{\inf\{t \ge S_j : R_t = \varepsilon_1\} < \inf\{t \ge S_j : R_t = 0\}\bigr\}.$$

Fix any j_0 such that $0 < 2^{-j_0} < R_0 \wedge \varepsilon_1$. If $T_0^R < \infty$, then $S_j < \infty$ for all $j \ge j_0$. It follows from the strong Markov property applied at S_j that $\mathbb{P}^{x,y}(\{S_j < \infty\} \cap G_j) \ge p_5(1 - p_4)$ for $j \ge j_0$. Since $\{S_{j+1} < \infty\} \cap G_{j+1} \subset \{S_j < \infty\} \cap G_j$, we have $\mathbb{P}^{x,y}(\bigcap_{j\ge j_0}(\{S_j < \infty\} \cap G_j)) \ge p_5(1 - p_4) > 0$. If the event $\bigcap_{j\ge j_0}(\{S_j < \infty\} \cap G_j)$ holds, then R has a discontinuity at T_0^R. Since R is continuous a.s., we have a contradiction which proves that for any $x \ne y$, $\mathbb{P}^{x,y}(T_0^R < \infty) = 0$.

Now suppose that $p_6 := \mathbb{P}(\lim_{t\to\infty} R_t = 0) > 0$. If $\lim_{t\to\infty} R_t = 0$, then $S_j < \infty$ for all $j \ge j_0$. We can argue as above to show that

$$\mathbb{P}^{x,y}\biggl(\Bigl\{\lim_{t\to\infty} R_t = 0\Bigr\} \cap \bigcap_{j\ge j_0} (\{S_j < \infty\} \cap G_j)\biggr) \ge p_6(1 - p_4) > 0.$$

If the events $\{T_0^R < \infty\}^c$ and $\bigcap_{j\ge j_0}(\{S_j < \infty\} \cap G_j)$ hold, then $\limsup_{t\to\infty} R_t > 0$. Hence, $\mathbb{P}(\lim_{t\to\infty} R_t = 0 \text{ and } \limsup_{t\to\infty} R_t > 0) > 0$. We have a contradiction which proves that for any $x \ne y$, $\mathbb{P}^{x,y}(\lim_{t\to\infty} R_t = 0) = 0$. □

APPENDIX

PROOF OF LEMMA 3.2. We have

$$\int_0^{2\pi} \log\bigl((\sin^2\beta + \cos^2\beta\cos^2\alpha)^{1/2}\bigr)\,d\beta$$
$$= \int_0^{2\pi} \log\bigl((\cos^2\beta + \sin^2\beta\cos^2\alpha)^{1/2}\bigr)\,d\beta$$

$$= \int_0^\pi \int_1^{\cos^2\alpha} \frac{\sin^2\beta}{\cos^2\beta + u\sin^2\beta}\, du\, d\beta$$

$$(A.1) \qquad = \int_1^{\cos^2\alpha} \int_{-\infty}^{\infty} \frac{1}{1-u}\left(\frac{1}{x^2+u} - \frac{1}{x^2+1}\right) dx\, du (x = \cot\beta)$$

$$= \int_1^{\cos^2\alpha} \frac{1}{\sqrt{u}(1-u)}\left[\arctan\left(\frac{x}{\sqrt{u}}\right) - \sqrt{u}\arctan(x)\right]_{-\infty}^{\infty} du$$

$$= \pi \int_1^{\cos^2\alpha} \frac{1}{\sqrt{u}(1+\sqrt{u})}\, du$$

$$= 2\pi \log\left(\frac{1}{2} + \frac{1}{2}|\cos\alpha|\right),$$

which implies

$$\int_0^{2\pi}\int_0^\pi \frac{1}{4\pi}\sin\alpha \log\left((\sin^2\beta + \cos^2\beta\cos^2\alpha)^{1/2}\right) d\alpha\, d\beta$$

$$= \int_0^{\pi/2} \sin\alpha \log\left((1+\cos\alpha)/2\right) d\alpha$$

$$= \int_{1/2}^1 2\log y\, dy \left(y = (1+\cos\alpha)/2\right)$$

$$= \log 2 - 1.$$

This proves (3.2).

We use (A.1) again to see that

$$\int_0^{2\pi}\int_0^\pi \frac{1}{16\pi}\frac{\sin\alpha}{\sin^3(\alpha/2)} \log\left((\sin^2\beta + \cos^2\beta\cos^2\alpha)^{1/2}\right) d\alpha\, d\beta$$

$$= \frac{1}{4}\int_0^\pi \frac{\cos(\alpha/2)}{\sin(\alpha/2)^2} \log\left(\frac{1}{2} + \frac{1}{2}|\cos\alpha|\right) d\alpha$$

$$= \int_0^{\pi/4} \frac{\cos u}{\sin^2 u}\log(\cos u)\, du + \int_{\pi/4}^{\pi/2} \frac{\cos u}{\sin^2 u}\log(\sin u)\, du (u = \alpha/2)$$

$$= -\left[\frac{\log(\cos u)}{\sin u} + \log\left(\frac{1+\sin u}{\cos u}\right)\right]_0^{\pi/4} - \left[\frac{\log(\sin u)}{\sin u} + \frac{1}{\sin u}\right]_{\pi/4}^{\pi/2}$$

$$= \sqrt{2} - 1 - \log(1+\sqrt{2}).$$

This proves (3.1). □

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