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ERGODICITY OF POISSON PRODUCTS AND APPLICATIONS

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In this paper we study the Poisson process over a σ-finite measure-space equipped with a measure preserving transformation or a group of measure preserving transformations. For a measure-preserving transformation T acting on a σ-finite measure-space X, the *Poisson suspension* of T is the associated probability preserving transformation T_* which acts on realization of the Poisson process over X. We prove ergodicity of the *Poisson-product* $T \times T_*$ under the assumption that T is ergodic and conservative. We then show, assuming ergodicity of $T \times T_*$, that it is impossible to deterministically perform natural equivariant operations: *thinning*, *allocation* or *matching*. In contrast, there are well-known results in the literature demonstrating the existence of *isometry* equivariant thinning, matching and allocation of homogenous Poisson processes on $\mathbb{R}^d$. We also prove ergodicity of the "first return of left-most transformation" associated with a measure preserving transformation on $\mathbb{R}_+$, and discuss ergodicity of the Poisson-product of measure preserving group actions, and related spectral properties.

1. Introduction. It is straightforward that the distribution of a homogenous Poisson point process on $\mathbb{R}^d$ is preserved by isometries. In the literature, various *translation-equivariant* and *isometry-equivariant* operations on Poisson process have been considered:

- *Poisson thinning*: A (deterministic) *Poisson-thinning* is a rule for selecting a subset of the points in the Poisson process which are equal in distribution to a lower intensity homogenous Poisson process. Ball [4] demonstrated a deterministic Poisson-thinning on $\mathbb{R}$ which was *translation equivariant*—that is, if a translation is applied to the original process, the new points selected are translations of the original ones by the same vector. This was extended and refined by Holroyd, Lyons and Soo [11] to show that for any $d \geq 1$, there is an *isometry-equivariant* Poisson-thinning on $\mathbb{R}^d$.
- *Poisson allocation*: Given a realization ω of a Poisson process on $\mathbb{R}^d$, a *Poisson allocation* partitions $\mathbb{R}^d$ up to measure 0 by assigning to each point in ω a *cell* which is a finite-measure subset of $\mathbb{R}^d$. Hoffman, Holroyd and Peres [9] constructed an isometry-equivariant allocation scheme for any stationary point process of finite intensity. The above allocation scheme had the characteristic

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property of being "stable." Subsequent work demonstrated isometry-equivariant Poisson allocations with other nice properties such as connectedness of the allocated cells [15] or good stochastic bounds on the diameter of the cells [5].

- *Poisson matching*: A *Poisson matching* is a deterministic scheme which finds a perfect matching of two identically distributed independent Poisson processes. Different isometry-equivariant Poisson matching schemes have been constructed [10, 12].

Consider a transformation of $\mathbb{R}^d$ which preserves Lebesgue measure. Does there exist a Poisson thinning which is equivariant with respect to the given transformation? What about an equivariant Poisson allocation or matching?

To have a couple of examples in mind, consider the following transformations $T_{\mathrm{RW}}, T_{\mathrm{Boole}} : \mathbb{R} \to \mathbb{R}$ of the real line given by

$$T_{\mathrm{RW}}(x) = \lfloor x \rfloor + (2x \bmod 1) - 1 + 2 \cdot 1_{(0,1/2]}(x \bmod 1) \tag{1}$$

and

$$T_{\mathrm{Boole}}(x) = x - \frac{1}{x} \tag{2}$$

T_{Boole} is known as Boole's transformation. It is a is a classical example of an ergodic transformation preserving Lebesgue measure. See [3] for a proof of ergodicity and discussions of this transformation. You may notice that T_{RW} is isomorphic to the shift map on the space of forward trajectories of the simple random walk on $\mathbb{Z}$.

From our perspective, it is natural (although mathematically equivalent) to consider an abstract standard σ-finite measure space $(X, \mathcal{B}, \mu)$, instead of $\mathbb{R}^d$ with Lebesgue measure. We consider a Poisson point process on this space, which denoted by $(X^*, \mathcal{B}^*, \mu^*)$. Any measure preserving transformation $T : X \to X$ naturally induces a map $T_* : X^* \to X^*$ on the Poisson process. This transformation T_* is the *Poisson suspension* of T [17].

We prove the following theorem:

THEOREM 1.1. *Let $T : X \to X$ be any conservative and ergodic measure preserving transformation of $(X, \mathcal{B}, \mu)$ with $\mu(X) = \infty$. There does not exist a T-equivariant Poisson thinning, allocation or matching.*

We prove Theorem 1.1 by studying ergodic properties of the map $T \times T_*$, which acts on the product space $(X \times X^*, \mathcal{B} \times \mathcal{B}^*, \mu^* \times \mu)$. We refer to this system as the *Poisson-product* associated with T. The space $X \times X^*$ can be considered as a countable set of "indistinguishable" points in X, with a unique "distinguished" point. The Poisson-product $T \times T_*$ acts on this by applying the same map T to each point, including the distinguished point.

Our main result about Poisson-products is the following theorem:

THEOREM 1.2. *Let $(X, \mathcal{B}, \mu, T)$ be a conservative, measure-preserving transformation with $\mu(X) = \infty$. Then the Poisson-product $T \times T_*$ is ergodic if and only if T is ergodic.*

Before concluding the introduction and proceeding with the details, we recall a couple of results regarding nonexistence of certain equivariant operations on Poisson processes. Evans proved in [6] that with respect to any noncompact group of *linear* transformations there is no invariant Poisson-thinning on $\mathbb{R}^d$. Gurel-Gurevich and Peled proved the nonexistence of translation equivariant *Poisson thickening* on the real line [7], which means that there is no measurable function on realizations of the a homogenous Poisson process that sends a Poisson process to a higher intensity homogenous Poisson process.

This paper is organized as follows: In Section 2 we briefly provide some terminology and necessary background. Section 3 contains a short proof of Theorem 1.2 stated above, based on previous work in ergodic theory. In Section 4 we prove any T-equivariant thinning is trivial, assuming $T \times T_*$ is ergodic. In Section 5 we show that under the same assumptions there are no T-equivariant Poisson allocations or Poisson matchings, using an intermediate result about nonexistence of positive equivariant maps into L^1. Section 6 discusses the "leftmost position transformation" and contains a proof of ergodicity, yet another application of Theorem 1.2. Section 7 is a discussion of ergodicity of Poisson products for measure preserving group actions.

2. Preliminaries. In this section we briefly recall some definitions and background from ergodic theory required for the rest of the paper. We also recall some properties of the Poisson point process on a σ-finite measure space.

2.1. *Ergodicity, conservative transformations and induced transformations.* Throughout this paper $(X, \mathcal{B}, \mu)$ is a standard σ-finite measure space. We will mostly be interested in the case where $\mu(X) = \infty$. Also throughout the paper, $T : X \to X$ is a measure preserving transformation, unless explicitly stated otherwise, where T denotes an action of a group by measure preserving transformations of $(X, \mathcal{B}, \mu)$. The collection of measurable sets of positive measure by will be denoted by $\mathcal{B}^+ := \{B \in \mathcal{B} : \mu(B) > 0\}$.

Recall that T is *ergodic* if any set $A \in \mathcal{B}$ which is T-invariant has either $\mu(A) = 0$ or $\mu(A^c) = 0$. Equivalently, T is ergodic if any measurable function $f : X \to \mathbb{R}$ satisfying $f \circ T = f$ μ-almost everywhere is constant on a set of full measure.

A set $W \in \mathcal{B}$ is called a *wandering set* if $\mu(T^{-n}W \cap W) = 0$ for all $n > 0$. The transformation T is called *conservative* if there are no wandering sets in $\mathcal{B}^+$. The *Poincaré recurrence theorem* asserts that any T which preserves a *finite* measure is conservative.

For a conservative T and $A \in \mathcal{B}^+$, the *first return time function* is defined for $x \in A$ by $\varphi_A(x) = \min\{n \geq 1 : T^n(x) \in A\}$. φ_A is finite μ-a.e; this is a direct consequence of T being conservative.

The *induced transformation* on A is defined by $T_A(x) := T^{\varphi_A(x)}(x)$. If T is conservative and ergodic and $A \in \mathcal{B}^+$, $T_A : A \to A$ is a conservative, ergodic transformation of $(A, \mathcal{B} \cap A, \mu|_A)$.

See [1] for a comprehensive introduction to ergodic theory of infinite measure preserving transformations.

2.2. *Cartesian product transformations*. Suppose T is conservative, and $S : Y \to Y$ is a probability preserving transformation of $(Y, \mathcal{C}, \nu)$, namely $\nu(Y) = 1$. It follows (as in Proposition 1.2.4 in [1]) that the *Cartesian product transformation* $T \times S : X \times Y \to X \times Y$ is a conservative, measure-preserving transforation of the Cartesian product measure-space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \times \nu)$.

2.3. *L^∞-eigenvalues of measure preserving transformations*. A function $f \in L^\infty(X, \mathcal{B}, \mu)$ is an *L^∞-eigenfunction* of T if $f \neq 0$ and $Tf = \lambda f$ for some $\lambda \in \mathbb{C}$. The corresponding λ is called an *L^∞-eigenvalue* of T. We briefly recall some well-known results:

If T is ergodic and f is an L^∞-eigenfunction, it follows that $|f|$ is constant almost-everywhere. The L^∞-eigenvalues of T are

$$e(T) := \{\lambda \in \mathbb{C} : \exists f \in L^\infty(X, \mathcal{B}, \mu) f \neq 0 \text{ and } Tf = \lambda f\}.$$

If T is conservative, then $|\lambda| = 1$ for any eigenvalue λ, for otherwise the set

$$\{x \in X : |f(x)| \in (|\lambda|^k, |\lambda|^{k+1}]\}$$

would be a nontrivial wandering set for some $k \in \mathbb{Z}$ if $|\lambda| > 1$. Thus, for any conservative transformation T, $e(T)$ is a subset if the unit sphere

$$\mathbb{S}^1 = \{x \in \mathbb{C} : |x| = 1\}.$$

$e(T)$ is a group with respect to multiplication, and carries a natural Polish topology, with respect to which the natural embedding in $\mathbb{S}^1$ is continuous.

When T preserves a finite measure, $e(T)$ is at most countable. For a general infinite-measure preserving T, however, $e(T)$ can be uncountable, and quite "large," for instance, the arbitrary Hausdorff dimension $\alpha \in (0, 1)$. Importantly for us, however, there are limitations on how "large" $e(T)$ can be. For instance, $e(T)$ is a *weak Dirichlet* set. This means that

$$\liminf_{n \to \infty} \int |1 - \chi_n(s)| \, dp(s) = 0$$

whenever p is a probability measure on $\mathbb{S}^1$ with $p(e(T)) = 1$, and $\chi_n(s) := \exp(2\pi i n s)$. In particular the set $e(T)$ has measure zero with respect to Haar measure on $\mathbb{S}^1$.

We refer the reader to existing literature for further details [1, 2, 16, 19].

2.4. *The L^2-spectrum.* Let $U_T : L^2(\mu) \to L^2(\mu)$ denote the unitary operator defined by $U_T(f) := f \circ T$.

The *spectral type* of a unitary operator U on a Hilbert space H, denoted σ_U, is a positive measure on $\mathbb{S}^1$ satisfying

(a)

$$\langle U^n f, g\rangle = \int_{\mathbb{S}^1} \chi_n(s) h(f, g)(s)\, d\sigma_U(s),$$

where $h : H \times H \to L^1(\sigma_U)$ is a sesquilinear map;

(b) σ_U is minimal with that property, in the sense that it satisfies $\sigma_U \ll \sigma$ for any measure σ on $\mathbb{S}^1$ satisfying (a).

In (b) above and throughout the paper, we write $\mu_1 \ll \mu_2$ to indicate that the measure μ_1 is absolutely continuous with respect to μ_2. If $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$, we say they are in the same measure class.

The spectral type σ_U is defined only up to measure class. Existence of σ_U is a formulation of *the scalar spectral theorem*.

For a measure-preserving transformation T, The *spectral type* of T σ_T is the spectral type of the associated unitary operator U_T on $L^2(\mu)$. For a probability preserving transformation S, the *restricted spectral type* is the spectral type the unitary operator U_S restricted to L^2-functions with integral zero.

Our brief exposition here follows Section 2.5 of [1].

2.5. *Poisson processes and the Poisson suspension.* For a standard σ-finite measure space $(X, \mathcal{B}, \mu)$, $(X^*, \mathcal{B}^*, \mu^*)$ denotes the associated *Poisson point process*, which we now describe. X^* is the space of countable subsets of X. We will typically denote an element of X^* by ω, ω_1, ω_2 and so on. The σ-algebra $\mathcal{B}^*$ is generated by sets of the form

$$\big[|\omega \cap B| = n\big] := \big\{\omega \in X^* : |\omega \cap B| = n\big\} \tag{3}$$

for $n \geq 0$ and $B \in \mathcal{B}$.

The probability measure μ^* is is uniquely defined by requiring that for any pairwise disjoint $A_1, A_2, \dots, A_n \in \mathcal{B}$, if $\omega \in X^*$ is sampled according to μ^*, then $|\omega \cap A_i|$ are jointly independent random variables individually distributed Poisson with expectation $\mu(A_i)$

$$\mu^*\big(|\omega \cap A| = k\big) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}. \tag{4}$$

The underlaying measure μ^* is called the *intensity* of the Poisson process. We will assume that the measure μ has no atoms, namely $\mu(\{x\}) = 0$ for any $x \in X$. This is a necessary and sufficient condition to avoid multiplicity of points almost surely with respect to μ^*.

A Poisson point process can be defined on very general measure spaces, under milder assumptions than "standard." Details of the construction and general properties of Poisson processes can be found, for instance, in [13, 14].

To make various measurability statements in the following sections more transparent, we assume the following technical condition: There is a fixed sequence $\{\beta_n\}_{n=1}^{\infty}$ of countable partitions of X into $\mathcal{B}$-measurable sets, such that β_{n+1} refines β_n, with the additional property that the mesh of these partitions goes to 0, namely,

$$\lambda(\beta_n) := \sup\{\mu(B) : B \in \beta_n\} \to 0 \qquad \text{as } n \to \infty.$$

We assume that $\mathcal{B} = \bigvee_{n=1}^{\infty} \sigma(\beta_n)$ is the σ-algebra generated by the union of these partitions. For instance, if $(X, \mathcal{B}, \mu)$ is the real line with Lebesgue measure on the Borel sets, we can take β_n to be the partition into half-open intervals with endpoints on the lattice $\frac{1}{2^n}\mathbb{Z}$.

The σ-algebra $\mathcal{B}^*$ can now be defined by

$$\mathcal{B}^* = \bigvee_{n=1}^{\infty} \beta_n^*,$$

where β_n^* is the σ-algebra generated by sets of the form (3) with $B \in \beta_n$ and $n \in \{0, 1, 2, \ldots\}$. Different sequences β_n with the above properties will not change the completion with respect to μ^* of the resulting σ-algebra $\mathcal{B}^*$.

The *Poisson suspension* of a measure preserving map $T : X \to X$, is the natural map obtained by applying T on X^*. As in [17], we denote it by $T_* : X^* \to X^*$. This transformation is formally defined by

$$T_*(\omega) = \{T(x) : x \in \omega\}.$$

T_* is a probability-preserving transformation of $(X^*, \mathcal{B}^*, \mu^*)$.

The following proposition relates the spectral measures of T and T_* [17]:

PROPOSITION 2.1. *If σ is the spectral-type of T, the restricted spectral type of T_* is given by*

$$\sigma_{T_*} = \sum_{n \geq 1} \frac{1}{n!} \sigma^{\otimes n}.$$

It is a classical result that a probability-preserving transformation is ergodic if and only if its restricted spectral type has no atom at $\lambda = 1$, and is *weakly mixing* if and only if its restricted spectral type has no atoms in $\mathbb{S}^1$ (this property is also equivalent to ergodicity of $T \times T$). It follows that T_* is ergodic if and only if T_* is weakly mixing if and only if there are no T-invariant sets of finite measure in $\mathcal{B}^+$ [17].

In the following sections we will use the map $\pi : X \times X^* \to X^*$ given by

$$\pi(x, \omega) = \{x\} \cup \omega. \tag{5}$$

The map π defined by (5) is a measurable map from between the measure spaces $(X \times X^*, \mathcal{B} \otimes \mathcal{B}^*)$ and $(X^*, \mathcal{B}^*)$. This is can be verified directly using the following equalities of sets:

$$\pi^{-1}\big[|\omega \cap A| = 0\big] = (X \setminus A) \times \big[|\omega \cap A| = 0\big]$$

and

$$\pi^{-1}\big[|\omega \cap A| = n\big] = \big((X \setminus A) \times \big[|\omega \cap A| = n\big]\big) \cup \big(A \times \big[|\omega \cap A| \in \{n-1, n\}\big]\big)$$

for $A \in \mathcal{B}$ and $n \in \mathbb{N}$.

In fact, π is a ∞*-factor map* between the measure preserving maps $T \times T_*$ and T_*, in the sense of Chapter 3 of [1]: This means that $\pi \circ T_* = (T \times T_*) \circ \pi$ and for $A \in \mathcal{B}^*$

$$\big(\mu \times \mu^*\big) \circ \pi^{-1}(A) = \begin{cases} 0, & \text{if } \mu^*(A) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

3. Ergodicity of Poisson product for conservative transformations. We now provide a proof of Theorem 1.2. The argument we use is an adaptation of [2]. To prove our result, we invoke the following condition for ergodicity of Cartesian products, due to M. Keane:

THEOREM (The ergodic multiplier theorem). *Let S be a probability preserving transformation and T a conservative, ergodic, nonsingular transformation. $S \times T$ is ergodic if and only if $\sigma_S(e(T)) = 0$, where*:

- *σ_S is the restricted spectral type of S*;
- *$e(T)$ is the group of L_∞-eigenvalues of T.*

A proof of this result is provided, for instance, in Section 2.7 of [1].

By Proposition 2.1, the restricted spectral-type of the Poisson suspension T_* is a linear combination of convolution powers of the spectral type of T.

We make use of the following basic lemma about convolution of measures and equivalence of measure classes. A short proof is provided here for the sake of completeness:

LEMMA 3.1. *Let μ_1 and μ_2 be Borel probability measures on $\mathbb{S}^1$ with the same null-sets. For any Borel probability measure ν on $\mathbb{S}^1$, the measures $\mu_1 * \nu$ and $\mu_2 * \nu$ have the same null-sets.*

PROOF. We will prove that $\mu_1 \ll \mu_2$ implies that $\mu_1 * \nu \ll \mu_2 * \nu$ which suffices by symmetry.

We assume $\mu_1 \ll \mu_2$, and show that for any $\varepsilon > 0$, there exists $\delta > 0$ so that any set $A \in \mathcal{P}(\mathbb{S}^1)$ with $(\mu_1 * \nu)(A) \geq \varepsilon$ has $(\mu_2 * \nu)(A) \geq \delta$.

Fix $\varepsilon > 0$ and choose any $A \in \mathcal{B}(\mathbb{S}^1)$ with $(\mu_1 * \nu)(A) \geq \varepsilon$. It follows that

$$\nu\left(\left\{x \in \mathbb{S}^1 : \mu_1(A \cdot x) \geq \frac{\varepsilon}{2}\right\}\right) \geq \frac{\varepsilon}{2}.$$

Since $\mu_1 \ll \mu_2$, there exists $\delta' > 0$ so that $\mu_1(B) \geq \frac{\varepsilon}{2}$ implies $\mu_2(B) \geq \delta'$. Thus,

$$\nu(\{x \in \mathbb{S}^1 : \mu_2(A \cdot x) \geq \delta'\}) \geq \frac{\varepsilon}{2}.$$

It follows that $(\mu_2 * \nu)(A) \geq \delta' \cdot \frac{\varepsilon}{2}$, which establishes the claim with $\delta = \delta' \cdot \frac{\varepsilon}{2}$. □

From this we deduce the following lemma.

LEMMA 3.2. *Let T be a conservative, measure-preserving transformation. For any $n \geq 1$, the group $e(T)$ acts nonsingularly on $\sigma_T^{\otimes n}$, the nth convolution power of the restricted spectral type of T.*

PROOF. Our claim is that

$$\forall t \in e(T) \qquad \sigma_T^{\otimes n} \sim \delta_t * \sigma_T^{\otimes n}, \tag{6}$$

where δ_t denotes dirac measure at t, and $\sim$ denotes equivalence of measure classes. For $n = 1$, a proof can be found in [2, 8].

Equation (6) follows for $n > 1$ by induction using Lemma 3.1, with $t \in e(T)$, σ_T and $\delta_t * \sigma_T$ substituting for μ_1 and μ_2, respectively, and $\sigma_T^{\otimes(n-1)}$ substituting for ν. □

Completing the proof of Theorem 1.2.

By the ergodic multiplier theorem above, proving ergodicity of the Poisson-product amounts to proving $\sigma_{T_*}(e(T)) = 0$. Since $\sigma_{T_*} = \sum_{n\geq 1} \frac{1}{n!}\sigma_T^{\otimes n}$, it is sufficient to prove that for all $n \geq 1$,

$$\sigma_T^{\otimes n}(e(T)) = 0. \tag{7}$$

A proof that $\sigma_T(e(T)) = 0$ is provided in [8]; see also [2]. This is the case $n = 1$ of equation (7). We also refer to the discussion in Chapter 9 of [16].

For convenience of the reader and in preparation for the discussion in Section 7, we briefly recall the arguments leading to this result: Suppose the contrary, $\sigma_T(e(T)) > 0$. Since $e(T)$ acts nonsingularly on σ_T, it follow that $\sigma_T\,|_{e(T)}$ is a quasi-invariant measure on $e(T)$. Thus, $e(T)$ can be furnished with a locally-compact second-countable topology, respecting the Borel structure inherited from $\mathbb{S}^1$. Haar measure on $e(T)$ must be is equivalent to $\sigma_T|_{e(T)}$. With respect to this topology, we have that $e(T)$ is a locally compact group, continuously embedded in $\mathbb{S}^1$, where the topological embedding is also a group embedding. In this situation, it follows as in [2] that $e(T)$ is either discrete or $e(T) = \mathbb{S}^1$. The possibility that $e(T)$ is discrete is ruled out since this would imply σ_T has atoms, which means T has $L^2(\mu)$ eigenfunctions. This is impossible since T is an ergodic transformation preserving an infinite measure. The alternative is that $e(T) = \mathbb{S}^1$. This is impossible since $e(T)$ is weak Dirichlet, thus must be a null set with respect to Haar measure on $\mathbb{S}^1$ [19].

To prove the equality in (7) for $n > 1$, note that the convolution power of an atom-free measure is itself atom-free and that by Lemma 3.2 above $e(T)$ also acts nonsingularly on $\sigma_T^{\otimes n}$. The result now follows using the same arguments outlined above for the case $n = 1$.

This completes the proof of Theorem 1.2.

4. Nonexistence of equivariant thinning. Here is a formalization of the notion of a (deterministic) *thinning*. This is a $\mathcal{B}^*$-measurable map $\Psi : X^* \to X^*$, satisfying

$$\mu^*\big(\big[\big|\Psi(\omega) \cap B\big| \le |\omega \cap B|\big]\big) = 1 \qquad \forall B \in \mathcal{B}.$$

This essentially means that Ψ is a measurable map on the space X^* of countable sets of X, for which almost-surely $\Psi(\omega) \subset \omega$.

A *Poisson thinning* satisfies the extra condition that $\mu^* \circ \Psi^{-1} = (\theta\mu)^*$ for some $\theta \in (0, 1)$. By $(\theta\mu)^*$ we mean the measure on $(X^*, \mathcal{B}^*)$ which corresponds to a Poisson process with intensity given by $\theta \cdot \mu$. In other words, the law of the countable set $\Psi(\omega)$ is that of a lower-intensity Poisson process.

Given a measure preserving transformation $T : X \to X$, a thinning Ψ is called *T-equivariant* if $\Psi \circ T_* = T_* \circ \Psi$. A thinning Ψ is *trivial* if

$$\mu^*\big(\big[\Psi(\omega) = \varnothing\big]\big) = 1 \quad \text{or} \quad \mu^*\big(\big[\Psi(\omega) = \omega\big]\big) = 1.$$

PROPOSITION 4.1. *Let T be a group-action by measure preserving transformations. If $T \times T_*$ is ergodic, there does not exist a nontrivial T-equivariant thinning.*

PROOF. Suppose by contradiction that Ψ is a nontrivial T-equivariant thinning. Consider the set

$$A = \big\{(x, \omega) \in X \times X^* : x \in \Psi\big(\omega \cup \{x\}\big)\big\}. \tag{8}$$

Measurability of the set A is verified by the following:

$$A = \bigcap_{n=1}^{\infty} \bigcup_{B \in \beta_n} (B \times X^*) \cap \big((\Psi \circ \pi)^{-1}\big[|\omega \cap B| > 0\big]\big) \bmod \mu \times \mu^*,$$

where $\{\beta_n\}_{n=1}^{\infty}$ is a "decreasing net" of countable partitions, as in Section 2.

Since Ψ is T-equivariant, the set A is a $T \times T_*$ invariant set. By ergodicity of $T \times T_*$, either $(\mu \times \mu^*)(A) = 0$ or $(\mu \times \mu^*)(A^c) = 0$.

Intuitively, A is the subset of $X \times X^*$ where applying the thinning Ψ on the union of the "indistinguishable points" with the "distinguished point" does not delete the distinguished point. We will complete the proof by showing that this implies that the thinning Ψ is trivial.

For $j \in \mathbb{N}$, define $\pi_{(j)} : \overbrace{X \times \cdots \times X}^{j} \times X^* \to X^*$ by

$$\pi(x_1, \dots, x_j, \omega) = \bigcup_{k=1}^{j} \{x_k\} \cup \omega.$$

$\pi_{(j)}$ is $\mathcal{B}^{\otimes j} \otimes \mathcal{B}^*$-measurable. This follows from measurability of the map π given by (5), which coincides with $\pi_{(1)}$.

For any $B \in \mathcal{B}$ with $0 < \mu(B) < \infty$, and $j \in \mathbb{N}$, we consider the following probability measures:

(i)

$$\mu^*_{B,j}(\cdot) := \mu^*\big(\cdot \mid [(\omega \cap B) = j]\big).$$

This is a probability measure on $(X^*, \mathcal{B}^*)$ corresponding to a Poisson process with intensity μ, conditioned to have exactly j points in the set B,

(ii)

$$\hat{\mu}_{B,j}(\cdot) := \frac{(\mu \times \mu^*) \mid_{B \times [(\omega \cap B) = j]}}{\mu(B) \cdot \mu^*([\omega \cap B] = j)}(\cdot).$$

$\hat{\mu}_{B,j}$ is a probability measure on $X \times X^*$ given by the product of a random point in B, distributed according to $\mu \mid_B$ and an independent Poisson process with intensity μ, conditioned to have exactly j points inside the set B,

(iii)

$$\tilde{\mu}_{B,j}(\cdot) := \frac{\overbrace{\mu \mid_B \times \cdots \times \mu \mid_B}^{j} \times (\mu \mid_{B^c})^*}{\mu(B)^j}(\cdot).$$

This is the probability on $(X^j \times X^*, \mathcal{B}^{\otimes j} \otimes \mathcal{B}^*)$ which corresponds to j independent random points identically distributed according to $\mu \mid_B$ and an independent Poisson process of intensity $\mu \mid_{B^c}$.

From the properties of the Poisson process, it directly follows that the probability measures defined above are related as follows:

$$\hat{\mu}_{B,j} \circ \pi^{-1} = \tilde{\mu}_{B,j+1} \circ \pi_{(j)}^{-1} = \mu^*_{B,j+1} \tag{9}$$

and

$$\hat{\mu}_{B,j} = \tilde{\mu}_{B,j+1} \circ \pi_{[2,j]}^{-1}, \tag{10}$$

where $\pi_{[2,j]} : \overbrace{X \times \cdots \times X}^{j} \times X^* \to X \times X^*$ is given by

$$\pi_{[2,j]}(x_1, \ldots, x_j, \omega) = \left(x_1, \bigcup_{k=2}^{j} \{x_k\} \cup \omega \right).$$

In particular, it follows that $\pi_{(j)}$ is a nonsingular map for all $j \geq 1$, in the sense that the inverse image of a μ^*-null set is always $\overbrace{\mu \times \cdots \mu}^{j} \times \mu^*$-null.

Assuming Ψ is not a trivial thinning implies that there exist $B \in \mathcal{B}$ with $0 < \mu(B) < \infty$ so that

$$\mu^*\bigl(0 < \bigl|\Psi(\omega) \cap B\bigr| < |\omega \cap B|\bigr) > 0.$$

It follows that for some $j > 1$,

$$\mu^*_{B,j}\left(0 < \frac{|\Psi(\omega) \cap B|}{|\omega \cap B|} < 1\right) > 0. \tag{11}$$

Now by (9) and (10), using symmetry of $\tilde{\mu}_{B,j}$ with respect to the variables $(x_1, \ldots, x_j)$, it follows that the probability $\hat{\mu}_{B,j}(x \in \Psi(\pi(x, \omega)))$ is equal to the expectation of $\frac{|\Psi(\omega) \cap B|}{|\omega \cap B|}$ under $\mu^*_{B,j}$. By (11) this expectation must be strictly positive and smaller than one. This contradicts triviality of the set A: Either $(\mu \times \mu^*)(A) = 0$ in which case $\hat{\mu}_{B,j}(x \in \Psi(\pi(x, \omega))) = 0$ or $(\mu \times \mu^*)(A^c) = 0$ in which case $\hat{\mu}_{B,j}(x \in \Psi(\pi(x, \omega))) = 1$. □

5. Nonexistence of equivariant allocation and matching. The aim of this section is to establish the nonexistence of T-equivariant Poisson allocation and Poisson matching, under an ergodicity assumption of a certain extension of T. Combined with Theorem 1.2, this will establish the last part of Theorem 1.1.

We begin with an intermediate result about measure-preserving systems. Consider a measurable function $\Phi : X \to L^1(\mu)$, sending $x \in X$ to $\Phi_x \in L^1(\mu)$, which is T-equivariant in the sense that $\Phi_{Tx} \circ T = \Phi_x$. Such a function Φ can be interpreted as a T-equivariant "mass allocation" scheme. For instance, on $X = \mathbb{R}^d$ with Lebesgue measure, $\Phi_x(y) = 1_{B_1(x)}(y)$ and $\Phi_x(y) = \exp(-\|x - y\|)$ both define isometry-equivariant "mass allocations." The later can be considered a "fractional allocation," in the sense that it obtains values in the interval $(0, 1)$. Nonexistence of T-equivariant Poisson allocation and Poisson matching will be a consequence of the following:

PROPOSITION 5.1. *Let T be a measure-preserving group action on $(X, \mathcal{B}, \mu)$. If $T \times T_*$ is ergodic, and $\mu(X) = \infty$, any T-equivariant measurable function $\Phi : X \to L^1(\mu)$ must be equal to 0 μ-a.e.*

PROOF. Suppose $\Phi : X \to L^1(\mu)$ satisfies $\Phi_{Tx} \circ T = \Phi_x$. Note that ergodicity of T implies that $\|\Phi_x\|_{L^1(\mu)}$ is constant μ-a.e, as this is a T-invariant function. Consider the function $F : X \times X^* \to \mathbb{R}$ given by

$$F(x,\omega) = \sum_{y \in \omega} |\Phi_x(y)|.$$

We verify that F indeed coincides with a $\mathcal{B} \otimes \mathcal{B}^*$-measurable function on a set of full $\mu \times \mu^*$-measure.

Indeed,

$$\Phi_x = \sum_{B \in \beta_1} \sum_{y \in \omega \cap B} |\Phi_x(y)|,$$

by Martingale convergence,

$$\sum_{y \in \omega \cap B} |\Phi_x(y)| = \lim_{n \to \infty} E_{\mu^*}\Big(\sum_{y \in \omega \cap B} |\Phi_x(y)| \mid \beta_n^*\Big)$$

for $\mu \times \mu^*$-almost-every (x,ω). For $B \in \beta_1$ and $n \geq 1$ we have

$$E_{\mu^*}\Big(\sum_{y \in \omega \cap B} |\Phi_x(y)| \mid \beta_n^*\Big) = \sum_{D \in \beta_n \cap B} E_{\mu^*}\Big(\sum_{y \in (\omega \cap D)} |\Phi_x(y)|\Big),$$

and the right-hand side is clearly $\mathcal{B} \times \beta_n^*$-measurable.

Let

$$\tilde{F}(x) := \int |F(x,\omega)| \, d\mu^*(\omega) = \int \sum_{y \in \omega} |\Phi_x(y)| \, d\mu^*(\omega),$$

and it follows from the definition of μ^* that $\tilde{F} = \|\Phi_x\|_{L^1(\mu)}$. Thus, by ergodicity of T, $\tilde{F}$ is equal to a nonzero (finite) constant μ-almost everywhere. In particular, F is finite $\mu \times \mu^*$-almost everywhere.

Observe that F is $T \times T_*$-invariant, so by ergodicity of $T \times T_*$ must be constant $\mu \times \mu^*$-a.e. On the other hand, for any $\varepsilon > 0$ and $M > 0$, we have $F(x,\omega) > M$ whenever $(x,\omega) \in X \times X^*$ satisfy $|\omega \cap \{y \in X : |\Phi_x(y)| > \varepsilon\}| > \frac{M}{\varepsilon}$. From the definition of the Poisson process, it thus follows that

$$(\mu \times \mu^*)([F \geq M]) \geq \mu(\{x \in X : \|\Phi_x\|_{L^1(\mu)} \geq \varepsilon\}) \cdot \frac{\varepsilon^{M/\varepsilon}}{M!} \exp\Big(-\frac{M}{\varepsilon}\Big).$$

Because the right-hand side is strictly positive for any $M > 0$, whenever $\varepsilon > 0$ is sufficiently small, it follows that F is not essentially bounded, which contradicts F being almost-everywhere constant. □

Together with Theorem 1.2, Proposition 5.1, immediately gives the following corollary, which does not seem to involve Poisson processes at all:

COROLLARY 5.2. *Let $T : X \to X$ be a conservative and ergodic measure preserving transformation of $(X, \mathcal{B}, \mu)$ with $\mu(X) = \infty$. Any measurable function $\Phi : X \to L^1(\mu)$ satisfying $\Phi_{Tx} \circ T = \Phi_x$ must be equal to* 0 *μ-a.e.*

We now turn to define and establish a nonexistence result for equivariant Poisson allocations:

By a *Poisson allocation rule* we mean a $\mathcal{B}^* \otimes \mathcal{B}$-measurable map $\Upsilon : X \times X^* \to L^1(\mu)$ satisfying the following properties:

(A1) *nonnegativity:* $\Upsilon_{(x,\omega)}(y) \geq 0$;
(A2) *partition of unity:* $\sum_{x\in\omega}(y)\Upsilon_{(x,\omega)} = 1$ μ^*-a.e.;
(A3) $\Upsilon_{(x,\omega)} \equiv 0$ if $x \notin \omega$.

If $x \in \omega$, we think of $\Upsilon_{(x,\omega)}$ as the "the cell allocated to x." Properties (A1) and (A2) above guarantee that Υ essentially takes values in the interval $[0, 1]$. The three above properties together express the statement that $\Upsilon_{(\cdot,\omega)}$ corresponds to a partition of X up to a null set between the points in ω, which assigns each $x \in \omega$ finite mass. For a "proper" allocation, we would require that $\Phi_{(x,\omega)}$ only takes values in $\{0, 1\}$, but this extra requirement is not necessary in order to prove our result.

For it is often useful to consider a wider class of Poisson allocation rules, where $\Upsilon_{(x,\omega)}$ is undefined for a null set of (x, ω)'s, and Υ is only measurable with respect to the $\mu \times \mu^*$-completion of the σ-algebra $\mathcal{B}^* \otimes \mathcal{B}$. However, conditions (A2) and (A3) above apply to $\mu \times \mu^*$-null sets, so we need to be careful and restate them as follows:

(A1) *nonnegativity:* $\Upsilon_{(x,\omega)}(y) \geq 0$;
(A2′) *partition of unity:* $\int_X \Upsilon_{(x,\omega)}\, d\mu(x) = 1$ μ^*-a.e.;
(A3′) $\int_A \Upsilon_{(x,\omega)}\, d\mu(x) \equiv 0$ μ^*-a.e on $\{\omega \in X^* : \omega \cap A = \varnothing\}$ whenever $A \in \mathcal{B}$.

A poisson allocation Υ is T-equivariant if $\Upsilon_{(Tx,T_*\omega)} \circ T = \Upsilon_{(x,\omega)}$.

PROPOSITION 5.3. *Let T be a group-action by measure preserving transformations, and denote $S := T \times T_*$. If $S \times S_*$ is ergodic, there does not exist a T-equivariant Poisson-allocation.*

PROOF. Given a Poisson allocation $\Upsilon : X \times X^* \to L^1(\mu)$, we will define a $T \times T_*$-equivariant function $\Phi : X \times X^* \to L^1(\mu \times \mu^*)$, which by ergodicity of $S = T \times T_*$ will contradict Proposition 5.1. This is given by

$$\Phi_{(x,\omega)}(y, \omega_2) = \Upsilon_{(x,\omega\cup\{x\})}(y).$$

It follows directly that

$$\|\Phi_{(x,\omega)}\|_{L^1(\mu\times\mu^*)} = \|\Upsilon_{(x,\omega\cup\{x\})}\|_{L^1(\mu)},$$

which is positive and finite $\mu \times \mu^*$-a.e.

Measurability of Φ follows from the measurability assumptions on Υ and from measurability of the map $(x,\omega) \to \{x\} \cup \omega$. $\square$

We now consider the existence of equivariant Poisson matching schemes:

Given a pair of independent Poisson processes realizations a (deterministic) *Poisson matching* assigns a perfect matching (or bijection) between the points of the two realizations, almost surely. To formalize this we define a Poisson matching as a measurable-function $\Psi : X^* \times X^* \to (X \times X)^*$, satisfying the following:

(M1)

$$\mu^*\bigl(\bigl\{\omega_2 \in X^* : \bigl|\Psi(\omega_1,\omega_2) \cap (B_1 \times B_2)\bigr| \le \min\bigl\{|\omega_1 \cap B_1|, |\omega_2 \cap B_2|\bigr\}\bigr\}\bigr) = 1$$

for μ^*-a.e ω_1 and all $B_1, B_2 \in \mathcal{B}$;

(M2)

$$\mu^*\bigl(\bigl\{\omega_2 \in X^* : \bigl|\Psi(\omega_1,\omega_2) \cap (B_1 \times X)\bigr| = |\omega_1 \cap B_1|\bigr\}\bigr) = 1$$

for μ^*-a.e ω_1 and all $B_1 \in \mathcal{B}$;

(M3)

$$\mu^*\bigl(\bigl\{\omega_1 \in X^* : \bigl|\Psi(\omega_1,\omega_2) \cap (X \times B_2)\bigr| = |\omega_2 \cap B_2|\bigr\}\bigr) = 1$$

for μ^*-a.e ω_2 and all $B_2 \in \mathcal{B}$.

PROPOSITION 5.4. *Under the assumptions of Proposition 5.3, there does not exist a nontrivial T-equivariant Poisson matching.*

PROOF. Suppose Ψ is a T-equivariant Poisson matching. We will define a "fractional" T-equivariant Poisson allocation $\Upsilon : X \times X^* \to L^1(\mu)$, contradicting Proposition 5.3.

The (implicit) definition of Υ is given by

$$\int_A \Upsilon_{(x,\omega_1)}(y)\, d\mu(y) = \mu^*\bigl(\bigl\{\omega_2 : \bigl|\Psi(\omega_1,\omega_2) \cap \bigl(\{x\} \times A\bigr)\bigr| > 0\bigr\}\bigr) \tag{12}$$

for all $A \in \mathcal{B}$, $\omega_1 \in X^*$ and $x \in X$.

In other words, if $x \in \omega_1$, $\Upsilon_{(x,\omega_1)}$ is the density with respect to Lebesgue measure of the conditional distribution of the *partner* of x under the matching Ψ, given ω_1. This defines Υ up to a null set.

It follows from the properties of Ψ that Υ satisfies the conditions (A1), (A2′) and (A3′) above.

Thus, Υ is indeed a Poisson allocation. Because Ψ is a T-equivariant matching, it follows directly that Υ is a T-equivariant allocation. $\square$

To complete the proof of the last part of Theorem 1.1, we note that if T is a conservative and ergodic measure-preserving transformation, $S = T \times T_*$ is also conservative and ergodic by Theorem 1.2, and so $S \times S_*$ is also ergodic, again by Theorem 1.2.

6. The leftmost position transformation. In this section $X = \mathbb{R}_+$ is the set of positive real numbers, $\mathcal{B}$ is the Borel σ-algebra on X and μ is Lebesgue measure on the positive real numbers. $T : X \to X$ is an arbitrary conservative, ergodic, Lebesgue-measure-preserving map of the positive real numbers.

In order to have a concrete example for such transformation T in hand, the reader can consider the unsigned version of Boole's transformation, given by $T(x) = |x - \frac{1}{x}|$. We define the following function:

$$t_1 : X^* \to X \qquad \text{by } t_1(\omega) = \inf \omega. \tag{13}$$

The map t_1 is well defined on a set of full μ^*-measure, namely whenever $\omega \neq \varnothing$. Note that $t_1(\omega)$ is the leftmost point of ω whenever ω is a discrete countable subset of $\mathbb{R}_+$. The map t_1 is $\mathcal{B}^*$-measurable since

$$t_1^{-1}(a, b) = \{\omega \in X^* : \omega \cap (0, a] = \varnothing \text{ and } \omega \cap (a, b) \neq \varnothing\}.$$

From this, it also follows directly that

$$\mu^* \circ t_1^{-1}(a, b) = e^{-\mu(0,a)}\big(1 - e^{-\mu(a,b)}\big) = e^{-a} - e^{-b}.$$

In particular it follows that $\mu^* \circ t^{-1} \ll \mu$.

Define the *leftmost return time* $\kappa : X^* \to \mathbb{N} \cup \{+\infty\}$ by

$$\kappa(\omega) = \inf\{k \geq 1 : t_1\big(T_*^k(\omega)\big) = T^k\big(t_1(\omega)\big)\}. \tag{14}$$

μ^*-almost surely, $\kappa(\omega)$ is the smallest positive number of iterations of T_* which must be applied to ω in order for the leftmost point to return to the leftmost location. A priori, κ_T is could be infinite. Nevertheless, we will soon show that when T is conservative and measure preserving, κ is finite μ^*-almost surely. Finally, the *leftmost position transformation* associated with T, $T_*^\kappa : \omega \to \omega$, is defined by

$$T_*^\kappa(\omega) := T_*^{\kappa(\omega)}(\omega).$$

This is the map of X_* obtained by reapplying T_* till once again there are no points to the left of the point which was originally leftmost.

The reminder of this section relates the leftmost transformation associated with T with the Poisson-product $T \times T_*$.

Let

$$X_0 = \{(x, \omega) \in X \times X^* : \omega \cap (0, x] = \varnothing\}. \tag{15}$$

The set X_0 is simply the subset of $X \times X^*$ in which the "distinguished point" is strictly to the left of any "undistinguished point." The formula below verifies measurability of X_0:

$$X_0 = \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}} \left(\left(q - \frac{1}{n}, q + \frac{1}{n}\right) \times \left\{\omega \in X^* : \omega \cap \left(0, q + \frac{2}{n}\right) = \varnothing\right\}\right) \bmod \mu \times \mu^*.$$

PROPOSITION 6.1. *Let $T:\mathbb{R}_+ \to \mathbb{R}_+$ be conservative and Lebesgue-measure-preserving. Then the leftmost position transformation associated with T is well defined and is isomorphic to the induced map of the Poisson product on the set X_0 defined by equation* (15),

$$\big(X^*, \mathcal{B}^*, \mu_*, T_*^{\kappa}\big) \cong \big(X_0, \mathcal{B}_0, \mu_0, (T \times T_*)_{X_0}\big),$$

where $\mu_0 = (\mu \times \mu_*)\,|_{X_0}$ is the restriction of the measure product $\mu \times \mu_*$ to the set X_0, and $\mathcal{B}_0 = (\mathcal{B} \otimes \mathcal{B}^*) \cap X_0$ is the restriction of the σ-algebra on the product space to subset of X_0.

In particular, $\mu_0(X_0) = 1$, so $(X_0, \mathcal{B}_0, \mu_0)$ is a probability space.

PROOF. Consider the map $\pi_0 : X_0 \to X^*$ which is the restriction to X_0 of the map $\pi(x, \omega) = \{x\} \cup \omega$ described in Section 2.5 above.

For a nonempty, discrete $\omega \in X^*$ we have

$$\pi_0^{-1}(\omega) = \big(t_1(\omega), \omega \setminus t_1(\omega)\big).$$

Thus π_0 is invertible on a set of full μ^*-measure in X^*.

As T is conservative and T_* is a probability preserving transformation, the Poisson product $T \times T_*$ is also conservative. We will show below that $\mu \times \mu^*(X_0) > 0$. Therefore, the return time φ_{X_0} is finite almost everywhere on X_0.

Since $\kappa \circ \pi_0 = \pi_0 \circ \varphi_{X_0}$, it follows that κ is finite μ^*-a.e.

We also have

$$\pi_0\big(T^n x, T_*^n \omega\big) = T_*^n\big(\pi_0(x, \omega)\big)$$

whenever (x, ω) and $(T^n x, T_*^n \omega)$ are in X_0. Thus,

$$\pi_0 \circ (T \times T_*)_{X_0} = T_*^{\kappa} \circ \pi_0.$$

It remains to check that $\pi_0^{-1}\mu^* = \mu_0$. It is sufficient to verify that $\mu^*(A) = \mu_0(\pi_0^{-1}(A))$ for sets $A \in \mathcal{B}^*$ of the form

$$A = \bigcap_{k=1}^{N} \big[|\omega \cap A_k| = n_k\big],$$

where $A_i = (a_{i-1}, a_i]$, $0 = a_0 < a_1 < a_2 < \cdots < a_N$ and $n_k \geq 0$ for $k = 1, \ldots N$.

Given the definition of μ^*, this amounts to an exercise in elementary calculus. By definition of μ^*,

$$\mu^*(A) = \prod_{k=1}^{N} \frac{\mu(A_k)^{n_k}}{n_k!} \exp\big(-\mu(A_k)\big),$$

which simplifies to

$$\mu^*(A) = \exp(-a_N) \prod_{k=1}^{N} \frac{(a_k - a_{k-1})^{n_k}}{n_k!}. \tag{16}$$

Assuming the n_k's are not all zero, let k the smallest index for which $n_k > 0$. We have

$$\pi_0^{-1}(A) = \bigcap_{j \neq k} (X \times [|\omega \cap A_j| = n_j]) \\ \cap \bigcup_{x \in A_k} \{x\} \times \big(\big[|\omega \cap [a_{k-1}, x)| = 0\big] \cap \big[|\omega \cap [x, a_k)| = n_k - 1\big]\big).$$

Thus

$$\mu_0(\Phi^{-1}(A)) = T_0 \int_{A_k} \exp(-(x - a_{k-1})) \exp(-(a_k - x)) \frac{(a_k - x)^{n_k - 1}}{(n_k - 1)!} \, dx,$$

where

$$T_0 = \prod_{j \neq k} \frac{(a_j - a_{j-1})^{n_j}}{n_j!} \exp(a_j - a_{j-1}).$$

Integrating this rational function of a single variable, we see that the last expression is equal to the expression on right-hand side of (16).

In particular, it follows that $\mu_0(X_0) = 1$.

It remains to check the case that $n_k = 0$ for all $k = 1, \ldots, N$: In this case then $A = [\omega \cap (0, a_N] = 0]$ and

$$\pi_0^{-1}(A) = \{(x, \omega) \in X_0 : x > a_n\}.$$

Thus

$$\mu_0(\pi_0^{-1}(A)) = \int_{[a_N, \infty)} e^{-\mu[x, \infty)} \, d\mu(x) = \exp(-a_N),$$

which is equal to $\mu^*(A)$. $\square$

COROLLARY 6.2. *Let $T : \mathbb{R}_+ \to \mathbb{R}_+$ be a conservative and ergodic Lebesgue-measure-preserving transformation. Then the leftmost position transformation $T_*^\kappa : (\mathbb{R}_+)^* \to (\mathbb{R}_+)$ is an ergodic probability preserving transformation.*

PROOF. Let T be as above. By Proposition 6.1, T_*^κ is isomorphic to the map obtained by inducing the Poisson product $T \times T_*$ onto the set X_0. It is well known that inducing a conservative and ergodic transformation on a set of positive measure results in an ergodic transformation. By Theorem 1.2, $T \times T_*$ is indeed ergodic. $\square$

It would be interesting to establish other ergodic properties of T^κ. For example, what conditions on T are required for T_*^κ to be weakly mixing?

7. Poisson-products and measure-preserving group actions. The purpose of this section is to discuss counterparts of our pervious results on ergodicity of Poisson products, and various equivariant operations in the context of a group of measure preserving transformations. Some motivating examples for this are groups of $\mathbb{R}^n$-isometries, which naturally act on $\mathbb{R}^n$ preserving Lebesgue measure.

Briefly recall the basic setup: We fix a topological group $\mathbb{G}$ and a σ-finite measure space $(X, \mathcal{B}, \mu)$. A measure-preserving $\mathbb{G}$-action T on the σ-finite measure space $(X, \mathcal{B}, \mu)$ is a representation $g \mapsto T_g \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ of $\mathbb{G}$ into the measure preserving automorphisms of $(X, \mathcal{B}, \mu)$.

A $\mathbb{G}$-action T is *ergodic* if for some $A \in \mathcal{B}$, $\mu(T_g A \setminus A) = 0$ for all $g \in \mathbb{G}$ then either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Any measure preserving $\mathbb{G}$-action T induces an action T_* on the Poisson process by probability preserving transformations [18]. The Poisson-product $\mathbb{G}$-action $T \times T_*$ is thus defined the same way as in the case of a single transformation.

The proofs of Propositions 4.1, 5.1, 5.3 and 5.4 above are still valid in this generality.

Let us recall the definition of a conservative $\mathbb{G}$-action: Say $W \in \mathcal{B}$ is a *wandering set* with resect to the action T of a locally-compact group $\mathbb{G}$ if $\mu(T(g, W) \cap W) = 0$ for all g in the complement of some compact $K \subset \mathbb{G}$. Call a $\mathbb{G}$-action *conservative* if there are no nontrivial wandering sets.

If in the statement of Theorem 1.2 we let T be a conservative ergodic $\mathbb{G}$-action for a group other than $\mathbb{Z}$, ergodicity of $T \times T_*$ may fail. This can happen even for conservative and ergodic $\mathbb{Z}^2$-actions, as we demonstrate in the example below:

Let $a, b \in \mathbb{R} \setminus \{0\}$ with $\frac{a}{b} \notin \mathbb{Q}$.Define a $\mathbb{Z}^2$-action T on $\mathbb{R}$ by

$$T_{(m,n)}(x) = x + am + bn \qquad \text{for } (m, n) \in \mathbb{Z}^2.$$

It is a simple exercise to show that the $\mathbb{Z}^2$-action above is both conservative and ergodic. Nevertheless, it is easy to see that $T \times T_*$ is not ergodic, for instance, by noting that

$$\{(x, \omega) \in \mathbb{R} \times \mathbb{R}^* : (x + 1, x - 1) \cap \omega = \varnothing\}$$

is a nontrivial $T \times T_*$-invariant set. Since this action T consists of translations, as noted in the Introduction, there do exist T-equivariant Poisson allocations, Poisson matchings and Poisson thinning.

Although the example above demonstrates Theorem 1.2 does not generalize, for abelian group actions most components of the proof given in Section 3 remain intact. Our next goal is to explain this, and point out where the proof of Theorem 1.2 breaks down for the example above:

Let $\mathbb{G}$ be a locally compact *abelian* group, and let $\widehat{\mathbb{G}}$ denote its dual. Generalizing the discussion in Section 2, the L^∞-*spectra* of a $\mathbb{G}$-action T, denoted $\mathrm{Sp}(T)$, is the set of homomorphisms $\chi : \mathbb{G} \to \mathbb{C}^*$ such that $f(T_g x) = \chi(g) f(x)$ for some nonzero $f \in L^\infty(X, \mu)$. In case $\mathbb{G} = Z$, the spectra is simply the group

L^∞-eigenvalues. As in the case $\mathbb{G} = \mathbb{Z}$ discussed earlier, the L^∞-spectra is a weak-Dirichlet set in $\widehat{\mathbb{G}}$ [19].

The L^2-*spectral type* of T is an equivalence class of Borel measures σ_T on $\widehat{\mathbb{G}}$ for any nonzero $f \in L^2(\mu)$ $\sigma_f \ll \sigma_T$, where the measure σ_f is given by

$$\hat{\sigma}_f(g) = \int f\big(T_g(x)\big)\overline{f(x)}\,d\mu(x).$$

The spectral type of σ_T is the minimal equivalence class of measures on $\widehat{\mathbb{G}}$ with respect to which all the σ_f's are absolutely continuous.

With these definitions, Keane's ergodic multiplier theorem above generalizes as follows: The product of an ergodic measure preserving $\mathbb{G}$-action T and a probability preserving $\mathbb{G}$-action S is ergodic if and only if $\mathrm{Sp}(T)$ is null with respect to the restricted spectral type of σ_T. The discussion in the end of Section 3 following [2, 19] still shows that in this case $\mathrm{Sp}(T)$ must be a locally compact group continuously which embeds continuously in $\widehat{\mathbb{G}}$. However, when $\mathbb{G} \neq \mathbb{Z}$, this does not imply that $\mathrm{Sp}(T)$ is either discrete or equal to $\widehat{\mathbb{G}}$.

Getting back to the example of the $\mathbb{Z}^2$-action T above, we note that for any $\tau \in \mathbb{R}$, the function $f_\tau \in L^\infty(\mathbb{R})$ defined by

$$f_\tau(x) = \exp(i\tau x),$$

is an L^∞ eigenfunction of T, since it satisfies

$$f_\tau\big(T_{(m,n)}(x)\big) = \exp\big(i\tau(x + am + bn)\big) = \chi_{(ta,tb)}(m,n)\exp(i\tau x),$$

where $\chi_{(a,b)}(m,n) = \exp(iam + bn)$. The map $t \to \chi_{(ta,tb)}$ is a continuous group embedding of $\mathbb{R}$ in $\mathrm{Sp}(T) \subsetneq \widehat{\mathbb{Z}^2}$.

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