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CHARACTERIZATION OF POSITIVELY CORRELATED SQUARED GAUSSIAN PROCESSES

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We solve a conjecture raised by Evans in 1991 on the characterization of the positively correlated squared Gaussian vectors. We extend this characterization from squared Gaussian vectors to permanental vectors. As side results, we obtain several equivalent formulations of the property of infinite divisibility for squared Gaussian processes.

1. Introduction. A random vector $(\psi_j)_{1\leq j\leq n}$ of $\mathbb{R}^n$ is said to be "associated" or "positively correlated" if for every couple of increasing functions F, G from $\mathbb{R}^n$ into $\mathbb{R}$ (i.e., F and G are increasing in each variable)

$$\mathbb{E}\big(FG\big((\psi_j)_{1\leq j\leq n}\big)\big) \geq \mathbb{E}\big(F\big((\psi_j)_{1\leq j\leq n}\big)\big)\mathbb{E}\big(G\big((\psi_j)_{1\leq j\leq n}\big)\big). \tag{1.1}$$

In 1982, Pitt [16] has shown that a centered Gaussian vector $\eta = (\eta_i)_{1\leq i\leq n}$ is "positively correlated" iff the entries of its covariance matrix are all nonnegative, which means that the Gaussian vector is positively correlated in the usual sense. To distinguish between the two meanings for positive correlation, we will keep the writing "positively correlated," in inverted commas, to refer to the definition (1.1).

In 1991, Evans [9] conjectured that given a centered Gaussian vector $\eta = (\eta_i)_{1\leq i\leq n}$, the squared centered Gaussian vector $\eta^2 = (\eta_i^2)_{1\leq i\leq n}$ is "positively correlated" iff there exists a function σ from $\{1 \leq i \leq n\}$ into $\{-1, 1\}$ such that $(\sigma(i)\eta_i)_{1\leq i\leq n}$ is positively correlated.

We prove the following:

THEOREM 1.1. *A squared centered Gaussian vector is "positively correlated" if and only if it is infinitely divisible.*

Evans condition for a squared centered Gaussian vector to be "positively correlated" is hence necessary but not sufficient. Indeed, several necessary and sufficient conditions for a squared centered Gaussian vector to be infinitely divisible have been established that allow to see this. The first one was found by Griffiths [12] in 1983, simplified then by Bapat [1]. This condition has been translated in terms of Green function of Markov processes by Eisenbaum and Kaspi [6]. Another version

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of this condition has been established by Vere-Jones [17]. We will use Vere-Jones characterization of infinitely divisible squared Gaussian vectors to establish three other equivalent necessary and sufficient conditions for a squared centered Gaussian process with continuous covariance to be "positively correlated." One extends the definition (1.1) from vectors to processes by saying that a process is "positively correlated" if all its finite-dimensional marginals are "positively correlated."

Eisenbaum and Kaspi's characterization stems from the desire to understand which were the Gaussian processes involved in Dynkin's isomorphism theorem [4]. Here is a brief presentation of the content of this theorem. Consider a symmetric transient Markov process X with state space E and 0-potential density (i.e., Green function) $(g(x, y), (x, y) \in E \times E)$. The function g is positive definite. Denote by $(\eta_x)_{x\in E}$ a centered Gaussian process with covariance g, independent of X. For a, b in E such that $g(a, b) > 0$, denote by $\tilde{\mathbb{P}}_{ab}$ the probability under which X starts at a and dies at its last visit to b. Besides, X admits a local time process. Denote by $(\tilde{L}^{ab}(x), x \in E)$ the process of the total accumulated local times under $\tilde{\mathbb{P}}_{ab}$. Then according to Dynkin's isomorphism theorem, the process $(\tilde{L}^{ab}(x) + \frac{1}{2}\eta_x^2, x \in E)$ has the same law as $(\frac{1}{2}\eta_x^2, x \in E)$ under the measure $\frac{1}{\mathbb{E}[\eta_a\eta_b]}\mathbb{E}[\eta_a\eta_b, \cdot]$.

This identity in law immediately raises two questions: Which are the centered Gaussian processes with a covariance equal to a Green function? Which are the centered Gaussian processes η such that the law of η^2 under $\mathbb{E}[\eta_a\eta_b, \cdot]$ is a positive measure?

An answer to the first question has been given in [6] (completed then in [5]; see (1.3) below) under the following form. Given a centered Gaussian process $(\eta_x)_{x\in E}$ with a continuous positive definite covariance $(G(x, y), (x, y) \in E \times E)$, $(\eta_x^2)_{x\in E}$ *is infinitely divisible if and only if there exist a real nonnegative measurable function* d *on* E *and a function* g *on* E^2 *such that*

$$G(x, y) = d(x)g(x, y)d(y) \tag{1.2}$$

and g *is the Green function of a symmetric transient Markov process.*

The corollary below actually provides three alternative formulations to this answer. One of them is our solution to Evans conjecture for processes. Another one answers also to the second question. To introduce the remaining one, we will use the following definition.

DEFINITION 1.2. A random process $(\phi_t)_{t\in E}$ is said to satisfy *Fortuyin Kasteleyn Ginibre's inequality* (FKG inequality) if for some reference positive measure m, for every integer n, every $t = (t_1, t_2, \dots, t_n)$ in E^n, $(\phi_{t_1}, \phi_{t_2}, \dots, \phi_{t_n})$ has a density with respect to $m(dx_1)\cdots m(dx_n)$ product measure on $\mathbb{R}^n$ denoted by h_t such that for every x, y in $\mathbb{R}^n$,

$$h_t(x)h_t(y) \le h_t(x \wedge y)h_t(x \vee y),$$

where $x \wedge y = (x_1 \wedge y_1, x_2 \wedge y_2, \dots, x_n \wedge y_n)$ and $x \vee y = (x_1 \vee y_1, x_2 \vee y_2, \dots, x_n \vee y_n)$.

COROLLARY 1.3. *Let $(\eta_x)_{x\in E}$ be a centered Gaussian process with a continuous positive definite covariance $(G(x, y), (x, y) \in E \times E)$. The following four properties are equivalent*:

(1) *η^2 is infinitely divisible.*
(2) *η^2 is "positively correlated."*
(3) *η^2 satisfies the FKG inequality.*
(4) *For every (a, b) in E^2, the law of η^2 under $\mathbb{E}[\eta_a\eta_b, \cdot]$ is a positive measure.*

Once the question of the characterization of "positively correlated" squared centered Gaussian processes is solved, one may ask the same question for shifted Gaussian processes. In particular, given a centered Gaussian process $(\eta_x)_{x\in E}$ and a real number r, when is the process $((\eta_x + r)^2)_{x\in E}$ "positively correlated"? Thanks to [2] and [8], we know a sufficient condition for the realization of that property: the infinite divisibility of $((\eta_x + r)^2)_{x\in E}$. But there is not known characterization, in terms of the covariance of η, of that condition for a fixed r. Nevertheless, in [5], we have established the following characterization. Assuming that the set E contains more that two elements (see Remark 4.2), let $(\eta_x)_{x\in E}$ be a centered Gaussian process with a continuous covariance

$$\begin{gathered}\big((\eta_x + r)^2\big)_{x\in E} \textit{ is infinitely divisible for every real } r,\\ \textit{if and only if}\\ \textit{the covariance of } \eta \textit{ is the Green function of a transient Markov process.}\end{gathered} \tag{1.3}$$

This will be used to enunciate another sufficient condition for $((\eta_x + r)^2)_{x\in E}$ to be "positively correlated" for every r.

The paper is organized as follows. In Section 2 we prove Theorem 1.1. We then deduce Corollary 1.3. The proofs involve stochastic comparison of squared centered Gaussian vectors. As a side result, for a given covariance G, we give necessary and sufficient conditions for the stochastic monotonicity of the family of squared Gaussian vectors with the resolvents of G for respective covariance.

In Section 3 we extend our characterization of "positively correlated" squared Gaussian vectors to permanental vectors. This extension is legitimated by the fact that a connection, similar to Dynkin isomorphism theorem, has been established in [6], between permanental processes and local times of not necessarily symmetric Markov processes.

In Section 4 we establish an equivalent formulation of (1.3).

As it will be shown in Sections 2, 3 and 4, many properties of Gaussian processes and, more generally, of permanental processes, are hence conditioned to the fact that their kernel is a Green function or not. So it is interesting to mention a way to generate Green functions. This is done in Section 5.

2. Proof of Theorem 1.1 and Corollary 1.3. The proof of Theorem 1.1 will show some other equivalent properties to infinite divisibility for squared Gaussian vectors. To formulate them, we make use of the following definitions.

DEFINITION 2.1. A random vector $(\phi_i)_{1\le i\le n}$ of $\mathbb{R}^n$ *stochastically dominates* another random vector $(\psi_i)_{1\le i\le n}$ of $\mathbb{R}^n$ if for any increasing function F from $\mathbb{R}^n$ into $\mathbb{R}$,

$$\mathbb{E}[F(\phi_1,\phi_2,\dots,\phi_n)]\ge \mathbb{E}[F(\psi_1,\psi_2,\dots,\psi_n)].$$

DEFINITION 2.2. Let $(\phi_{t_1},\phi_{t_2},\dots,\phi_{t_n})$ and $(\psi_{t_1},\psi_{t_2},\dots,\psi_{t_n})$ be two random vectors of $\mathbb{R}^n$, such that there exists a positive measure m on $\mathbb{R}$, such that their laws both admit respective densities h and f with respect to $m(dx_1)\cdots m(dx_n)$ product measure on $\mathbb{R}^n$. If for every x, y in $\mathbb{R}^n$,

$$f(x)h(y)\le f(x\wedge y)h(x\vee y),$$

then one says that $(\phi_{t_1},\phi_{t_2},\dots,\phi_{t_n})$ is strongly stochastically bigger than (or strongly stochastically dominates) $(\psi_{t_1},\psi_{t_2},\dots,\psi_{t_n})$.

One extends this definition to a couple (ϕ,ψ) of real-valued processes indexed by the same set by saying that ϕ is strongly stochastically bigger than ψ when all the finite-dimensional marginals of ϕ and ψ satisfy the above relation.

Strong stochastic domination implies usual stochastic domination.

DEFINITION 2.3. Let C be a positive semi-definite matrix. For $\alpha>0$, one defines the associated α-resolvent matrix as $C_\alpha=(I+\alpha C)^{-1}C$.

We have the following corollary of Theorem 1.1.

COROLLARY 2.4. *Let $\eta=(\eta_i)_{1\le i\le n}$ be a centered Gaussian vector with covariance G, an $n\times n$-positive definite matrix. Denote by $\eta_\alpha=(\eta_\alpha(i))_{1\le i\le n}$ a centered Gaussian vector with covariance G_α. Then the four following points are equivalent*:

(i) *η^2 is infinitely divisible.*

(ii) *The family of vectors $(\eta_\alpha^2)_{\alpha\ge 0}$ is stochastically decreasing as α increases on $\mathbb{R}^+$.*

(iii) *The family of vectors $(\eta_\alpha^2)_{\alpha\ge 0}$ is strongly stochastically decreasing as α increases on $\mathbb{R}^+$.*

(iv) *For every couple (i,j), $1\le i,j\le n$, for every $n\times n$ diagonal matrix D, $(\mathbb{E}[|\eta_\alpha(i)\eta_\alpha(j)|])_{\alpha\ge 0}$ is decreasing as α increases on $\mathbb{R}^+$ when G is replaced by DGD.*

We adopt the following notation from the paper [13]. For C a $n \times n$-positive definite matrix and any measurable function F on $\mathbb{R}^n$, $\mathbb{E}_C[F(\eta)]$ denotes the expectation with respect to a centered Gaussian vector η with covariance matrix C.

PROOF OF THEOREM 1.1. Thanks to [2] or [8], we know that if the vector η^2 is infinitely divisible, then it is "positively correlated." We prove now the converse.

Assume that η^2 is "positively correlated." Denote by $G = (G(i,j))_{1\le i,j\le n}$ its covariance matrix. For every decreasing function F, H on $\mathbb{R}^n$ we have

$$\mathbb{E}_G\big(FH(\eta^2)\big) \ge \mathbb{E}_G\big(F(\eta^2)\big)\mathbb{E}_G\big(H(\eta^2)\big)$$

and, in particular, for every $\alpha, \varepsilon > 0$

$$\mathbb{E}_G\big(e^{-((\alpha+\varepsilon)/2)\sum_{i=1}^n \eta_i^2}\big) \ge \mathbb{E}_G\big(e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big)\mathbb{E}_G\big(e^{-(\varepsilon/2)\sum_{i=1}^n \eta_i^2}\big). \tag{2.1}$$

Moreover, for any decreasing function F on $\mathbb{R}_+^n$, we have

$$\begin{aligned} &\mathbb{E}_G\big(F(\eta^2)e^{-((\alpha+\varepsilon)/2)\sum_{i=1}^n \eta_i^2}\big) \\ &\qquad \ge \mathbb{E}_G\big(F(\eta^2)e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big)\mathbb{E}_G\big(e^{-(\varepsilon/2)\sum_{i=1}^n \eta_i^2}\big). \end{aligned} \tag{2.2}$$

We make use now of a remark of Marcus and Rosen (Remark 5.2.4, page 200 in [15]) according to which for all measurable function K on $\mathbb{R}^n$,

$$\mathbb{E}_G\big[K(\eta)e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big] = \mathbb{E}_{G_\alpha}\big[K(\eta)\big]\mathbb{E}_G\big[e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big]. \tag{2.3}$$

In particular, we have

$$\mathbb{E}_G\big[F(\eta^2)e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big] = \mathbb{E}_{G_\alpha}\big[F(\eta^2)\big]\mathbb{E}_G\big[e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big]. \tag{2.4}$$

We mention that, unlike for (2.3), one does not need to assume that G is invertible to obtain (2.4) (for a direct proof see the proof of Proposition 3.2 in Section 3).

Thanks to (2.4), (2.2) can be rewritten as

$$\begin{aligned} &\mathbb{E}_{G_{\alpha+\varepsilon}}\big[F(\eta^2)\big]\mathbb{E}_G\big[e^{-((\alpha+\varepsilon)/2)\sum_{i=1}^n \eta_i^2}\big] \\ &\qquad \ge \mathbb{E}_{G_\alpha}\big[F(\eta^2)\big]\mathbb{E}_G\big[e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big]\mathbb{E}_G\big(e^{-(\varepsilon/2)\sum_{i=1}^n \eta_i^2}\big). \end{aligned}$$

Consequently, for every increasing function F, we obtain, thanks to (2.1),

$$\mathbb{E}_{G_{\alpha+\varepsilon}}\big[F(\eta^2)\big] \le \mathbb{E}_{G_\alpha}\big[F(\eta^2)\big]\frac{\mathbb{E}_G[e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}]\mathbb{E}_G(e^{-(\varepsilon/2)\sum_{i=1}^n \eta_i^2})}{\mathbb{E}_G[e^{-((\alpha+\varepsilon)/2)\sum_{i=1}^n \eta_i^2}]}.$$

Thanks to (2.1), we finally obtain for every increasing, nonnegative function F on $\mathbb{R}_+^n$,

$$\mathbb{E}_{G_{\alpha+\varepsilon}}\big[F(\eta^2)\big] \le \mathbb{E}_{G_\alpha}\big[F(\eta^2)\big]. \tag{2.5}$$

Because of the restriction on the sign of F, the above inequality does not mean stochastic domination but will be sufficient for our purpose. Indeed, for a fixed $\alpha > 0$, note that

$$G_{\alpha+\varepsilon} = (I + \varepsilon G_\alpha)^{-1} G_\alpha.$$

Set $f_\alpha(\varepsilon) = \mathbb{E}_{G_{\alpha+\varepsilon}}[F(\eta^2)]$, and note that f_α is decreasing at 0.

Besides, we set $C_{ij}(G_\alpha) = G_\alpha(i, j)$. We also define a function $\mathcal{F}$ on the set of covariance matrices by setting

$$\mathcal{F}(C) = \mathbb{E}_C\big[F\big(\eta^2\big)\big].$$

In [13], the derivatives of functions of the form $\mathbb{E}_C[H(\eta)]$ with respect to the entries of the matrix are computed. The authors work with a $C^2(\mathbb{R}^n)$-function H which together with its first and second derivatives satisfy a $O(|x|^N)$ growth condition at ∞, for some finite N. For F measurable function on $\mathbb{R}^n_+$ such that the function H defined by $H(x_1, \dots, x_n) = F(x_1^2, \dots, x_n^2)$ satisfies this condition, one easily obtains for $i \neq j$,

$$\frac{\partial \mathcal{F}}{\partial C_{ij}}(C) = 4\mathbb{E}_C\left[\eta_i \eta_j \frac{\partial^2 F}{\partial x_i \, \partial x_j}\big(\eta^2\big)\right] \tag{2.6}$$

and

$$\frac{\partial \mathcal{F}}{\partial C_{ii}}(C) = 2\mathbb{E}_C\left[\eta_i^2 \frac{\partial^2 F}{\partial x_i^2}\big(\eta^2\big)\right] + \mathbb{E}_C\left[\frac{\partial F}{\partial x_i}\big(\eta^2\big)\right]. \tag{2.7}$$

For ε small enough, we have $G_{\alpha+\varepsilon} = \sum_{k=0}^{\infty} (-1)^k \varepsilon^k (G_\alpha)^{k+1}$, hence,

$$C_{ij}(G_{\alpha+\varepsilon}) = \sum_{k=0}^{\infty} (-1)^k \varepsilon^k (G_\alpha)^{k+1}(i, j), \tag{2.8}$$

which is a derivable function of ε at 0. We obtain

$$f'_\alpha(\varepsilon) = \sum_{1 \le i \le j \le n} \frac{\partial \mathcal{F}}{\partial C_{ij}}(G_{\alpha+\varepsilon}) \frac{\partial C_{ij}}{\partial \alpha}(G_{\alpha+\varepsilon}), \tag{2.9}$$

which, thanks to (2.6), (2.7) and (2.8), leads to

$$\begin{aligned} f'_\alpha(0) = {} & -4 \sum_{1 \le i < j \le n} \mathbb{E}_{G_\alpha}\left[\eta_i \eta_j \frac{\partial^2 F}{\partial x_i \, \partial x_j}\big(\eta^2\big)\right] (G_\alpha)^2(i, j) \\ & - \sum_{i=1}^{n} \mathbb{E}_{G_\alpha}\left[2\eta_i^2 \frac{\partial^2 F}{\partial x_i^2}\big(\eta^2\big) + \frac{\partial F}{\partial x_i}\big(\eta^2\big)\right] (G_\alpha)^2(i, i) \end{aligned} \tag{2.10}$$

[we mention that $(G_\alpha)^2(i, j)$ is not $(G_\alpha(i, j))^2$].

We choose now to take $F(x) = \sqrt{x_i x_j}$, with $i \neq j$. We first check that (2.6) and (2.7) still hold. Indeed, the formulas computed in [13] are still available for $H(x) = |x_i x_j|$. For this choice (2.10) gives

$$\mathbb{E}_{G_\alpha}\big[\mathrm{sgn}(\eta_i \eta_j)\big](G_\alpha)^2(i, j) \geq 0. \tag{2.11}$$

Note that for every $(\lambda_k)_{1\leq k\leq n}$ in $\mathbb{R}^n$, the vector $(\lambda_k^2 \eta_k^2)_{1\leq k\leq n}$ is also "positively correlated." Consequently, setting $\lambda = \mathrm{Diag}((\lambda_k)_{1\leq k\leq n})$, one can replace G_α by $\lambda G_\alpha \lambda$ in (2.11) to obtain

$$\mathrm{sgn}(\lambda_i \lambda_j)\mathbb{E}_{G_\alpha}\big(\mathrm{sgn}(\eta_i \eta_j)\big)\lambda_i \lambda_j \sum_{k=1}^{n} G_\alpha(i, k)\lambda_k^2 G_\alpha(k, j) \geq 0,$$

which is equivalent to

$$\sum_{k=1}^{n} \lambda_k^2 \mathbb{E}_{G_\alpha}\big(\mathrm{sgn}(\eta_i \eta_j)\big) G_\alpha(i, k) G_\alpha(k, j) \geq 0.$$

Since this is true for every λ, we have

$$\mathbb{E}_{G_\alpha}\big(\mathrm{sgn}(\eta_i \eta_j)\big) G_\alpha(i, k) G_\alpha(k, j) \geq 0 \qquad \text{for every } i, j, k \text{ with } i \neq j. \tag{2.12}$$

We choose to take $k = i$ and obtain

$$G_\alpha(i, j)\mathbb{E}_{G_\alpha}\big[\mathrm{sgn}(\eta_i \eta_j)\big] \geq 0, \tag{2.13}$$

which together with (2.12) leads to

$$G_\alpha(j, i) G_\alpha(i, k) G_\alpha(k, j) \geq 0 \qquad \text{for every } i, j, k \text{ with } i \neq j. \tag{2.14}$$

We show now that this condition implies that there exists σ_α from $\{1, 2, \dots, n\}$ into $\{-1, 1\}$ such that for every i, j,

$$\sigma_\alpha(i) G_\alpha(i, j) \sigma_\alpha(j) \geq 0. \tag{2.15}$$

We do it by recurrence on the size of the matrix G_α. Assume that our claim is true at rank n and suppose that G_α is a $(n + 1) \times (n + 1)$-covariance matrix. We just need to define $\sigma_\alpha(n + 1)$. For every j, k in $\{1, 2, \dots, n\}$, we have $\sigma_\alpha(j)\sigma_\alpha(k) G_\alpha(j, k) \geq 0$. Since $G_\alpha(n+1, j) G_\alpha(j, k) G_\alpha(k, n+1) \geq 0$, we obtain $\sigma_\alpha(j)\sigma_\alpha(k) G_\alpha(n + 1, j) G_\alpha(n + 1, k) \geq 0$. Consequently, $\sigma_\alpha(j) G_\alpha(n + 1, j)$ has a constant sign independent of j, $1 \leq j \leq n$, that we denote by $\sigma_\alpha(n + 1)$. This implies immediately that $\sigma_\alpha(j) G_\alpha(n + 1, j)\sigma_\alpha(n + 1) \geq 0$.

We then easily check that our claim holds for $n = 3$.

For a real positive number β and a $m \times m$-matrix $M = (M_{i,j})_{1\leq i,j\leq m}$, the quantity $\mathrm{per}_\beta(M)$ is defined as follows: $\mathrm{per}_\beta(M) = \sum_{\tau \in \mathcal{S}_m} \beta^{\nu(\tau)} \prod_{i=1}^{m} M_{i,\tau(i)}$ where $\mathcal{S}_m$ is the set of the permutations on $\{1, 2, \dots, m\}$, and $\nu(\tau)$ is the signature of τ.

For every integer m, every $k_1, k_2, \dots, k_m$ in $\{1, 2, \dots, n\}$ and every $\beta > 0$, we hence have

$$\begin{aligned}\mathrm{per}_\beta\big((G_\alpha(k_i, k_j))_{1\le i,j\le m}\big) &= \sum_{\tau\in\mathcal{S}_m} \beta^{\nu(\tau)} \prod_{i=1}^n G_\alpha(k_i, k_{\tau(i)})\\ &= \sum_{\tau\in\mathcal{S}_m} \beta^{\nu(\tau)} \prod_{i=1}^n \sigma_\alpha(k_i)\sigma_\alpha(k_{\tau(i)})G_\alpha(k_i, k_{\tau(i)}) \ge 0,\end{aligned}$$

which is a sufficient condition for η^2 to be infinitely divisible thanks to the Vere-Jones criteria [17] (this criteria is recalled at the beginning of Section 3). □

PROOF OF COROLLARY 1.3. One can easily notice that (1) is equivalent to (3). Indeed, according to Bapat [1], a centered Gaussian vector $(\eta_i)_{1\le i\le n}$ with nonsingular covariance matrix G is such that $(\eta_i^2)_{1\le i\le n}$ is infinitely divisible iff there exists a signature matrix σ [a diagonal matrix such that $\sigma(i,i) = -1$ or 1] such that $\sigma G^{-1}\sigma$ is a M-matrix (i.e., its off-diagonal entries are nonpositive). Thanks to [14], we know that this is also a necessary and sufficient condition for $(\eta_i^2)_{1\le i\le n}$ to satisfy the Fortuyin–Kasteleyn–Ginibre's inequality. Note that there is no need of the continuity of the covariance to then conclude on the equivalence between (1) and (3).

Thanks to Theorem 1.1, we hence immediately have the equivalence of (1), (2) and (3). Note that we did not have to use the well-known fact that (3) implies (2) [11].

Under the assumption of continuity of G, we know thanks to [6] that (1) is realized iff for every x, y in E, $G(x, y) = d(x)g(x, y)d(y)$, with d a nonnegative measurable function on E and g the Green function of some transient Markov process. Denote by $(\tilde\eta_x, x \in E)$ a centered Gaussian process with covariance g. Thanks to Dynkin's isomorphism theorem, we know that for every a, b in E, the law of $(\tilde\eta_x^2)_{x\in E}$ under $\mathbb{E}[\tilde\eta_a\tilde\eta_b, \cdot]$ is a positive measure. Since $(\eta_x)_{x\in E} \overset{(\mathrm{law})}{=} (d(x)\tilde\eta_x)_{x\in E}$, we see that (4) is realized.

To see that (4) implies (1), note first that for every x, y in E, $G(x, y) \ge 0$. Denote by $\mathbb{G}$ the matrix $(G(x_i, x_j))_{1\le i,j\le n}$ for $x_1, x_2, \dots, x_n$ in E and denote by $\mathbb{G}_\alpha$ the α-resolvent matrix associated to $\mathbb{G}$. We note that, thanks to (2.3), for every $\alpha > 0$, we have for every a and b in $\{x_1, \dots, x_n\}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{G}}\big[\eta_a\eta_b e^{-(\alpha/2)\sum_{i=1}^n \eta_{x_i}^2}\big] &= \mathbb{E}_{\mathbb{G}_\alpha}[\eta_a\eta_b]\mathbb{E}_{\mathbb{G}}\big[e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big]\\ &= \mathbb{G}_\alpha(a, b)\mathbb{E}_{\mathbb{G}}\big[e^{-(\alpha/2)\sum_{i=1}^n \eta_i^2}\big].\end{aligned}$$

Hence, for every $\alpha > 0$, and every a and b, $\mathbb{G}_\alpha(a, b) \ge 0$, which, according to Vere-Jones (Proposition 4.5 in [17], recalled at the beginning of Section 3), is a sufficient condition for $(\eta_{x_i}^2, 1 \le i \le n)$ to be infinitely divisible. Since this is true for every $x_1, \dots, x_n$, we conclude that η^2 is infinitely divisible. □

PROOF OF COROLLARY 2.4. We start by noting that the density f_α of η_α^2 with respect to the Lebesgue measure is connected to the density f_0 of η^2. Indeed, thanks to (2.3), we have for a.e. $x = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}_+^n$,

$$f_\alpha(x) = \frac{e^{-(\alpha/2)\sum_{i=1}^n x_i}}{\mathbb{E}[\exp\{-(\alpha/2)\sum_{i=1}^n \eta_i^2\}]} f_0(x). \tag{2.16}$$

Assume now that (i) is satisfied. Thanks to Corollary 1.3, this implies that η^2 satisfies the FKG inequality. Thanks to (2.16), one obtains for $\alpha < \beta$ and every x, y in $\mathbb{R}_+^n$,

$$f_\alpha(x) f_\beta(y) \le f_\alpha(x \vee y) f_\beta(x \wedge y),$$

which leads to (iii).

Now (iii) implies (ii) and (ii) implies (iv). The proof of Theorem 1.1 shows that (iv) implies (i). □

3. The nonsymmetric case. A real-valued positive vector $(\psi_i, 1 \le i \le n)$ is a permanental vector if its Laplace transform satisfies for every $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathbb{R}_+^n$,

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_i\right\}\right] = |I + \alpha G|^{-1/\beta}, \tag{3.1}$$

where I is the $n \times n$-identity matrix, α is the diagonal matrix $\mathrm{Diag}((\alpha_i)_{1 \le i \le n})$, $G = (G(i,j))_{1 \le i,j \le n}$ and β is a fixed positive number.

Such a vector $(\psi_i, 1 \le i \le n)$ is a permanental vector with kernel $(G(i,j), 1 \le i, j \le n)$ and index β.

Permanental vectors represent a natural extension of squared centered Gaussian vectors. Indeed, for $\beta = 2$ and G covariance matrix, (3.1) is the Laplace transform of a squared centered Gaussian vector.

Thanks to Vere-Jones (Proposition 4.5 in [17]), we know that there exists a nonnegative random vector with Laplace transform given by (3.1) if and only if:

(I) All the real eigenvalues of G are nonnegative.
(II) For every $\alpha > 0$, set $G_\alpha = (I + \alpha G)^{-1} G$, then G_α is β-positive.

A $n \times n$-matrix $M = (M(i,j))_{1 \le i,j \le n}$ is said to be β-positive if for every integer m, every $k_1, k_2, \dots, k_m$ in $\{1, 2, \dots, n\}$

$$\mathrm{per}_\beta\big((M(k_i, k_j))_{1 \le i,j \le m}\big) \ge 0,$$

where for any $m \times m$-matrix $A = (A(i,j))_{1 \le i,j \le m}$, the quantity $\mathrm{per}_\beta(A)$ is defined as follows: $\mathrm{per}_\beta(A) = \sum_{\tau \in \mathcal{S}_m} \beta^{\nu(\tau)} \prod_{i=1}^m A_{i,\tau(i)}$, with $\mathcal{S}_m$ the set of the permutations on $\{1, 2, \dots, m\}$, and $\nu(\tau)$ the signature of τ.

Obviously, a permanental vector with kernel G is infinitely divisible if and only if it satisfies the Vere-Jones conditions for every $\beta > 0$.

Note that the kernel of a permanental vector is not uniquely determined. We have proved in [6] that a permanental vector is infinitely divisible iff it admits as kernel the Green function of some transient Markov process.

THEOREM 3.1. *Let ψ be a permanental vector with index 2 and kernel G. The two following properties are equivalent*:

(1) *ψ is infinitely divisible.*
(2) *ψ is "positively correlated."*

To prove Theorem 3.1, we need the following preliminary proposition, that will be established at the end of this section.

PROPOSITION 3.2. *For $\beta > 0$, let M be a $n \times n$ matrix such that there exists a random nonnegative vector $\psi = (\psi(1), \psi(2), \dots, \psi(n))$ with Laplace transform*

$$\mathbb{E}\big(e^{-(1/2)\sum_{i=1}^n x_i\psi(i)}\big) = |I + xM|^{-1/\beta}$$

for every $(x_1, x_2, \dots, x_n)$ in R_+^n. Set for every $\alpha \geq 0$, $M_\alpha = M(I + \alpha M)^{-1}$.

There exists a nonnegative random vector $\psi_\alpha = (\psi_\alpha(1), \psi_\alpha(2), \dots, \psi_\alpha(n))$ with Laplace transform

$$\mathbb{E}\big(e^{-(1/2)\sum_{i=1}^n x_i\psi_\alpha(i)}\big) = |I + xM_\alpha|^{-1/\beta}.$$

The law of ψ_α is absolutely continuous with respect to the law of ψ. Moreover, for every bounded measurable functional F on $\mathbb{R}_+^n$, we have

$$\mathbb{E}\big[F(\psi_\alpha)\big] = \mathbb{E}\bigg[\frac{\exp\{-(\alpha/2)\sum_{i=1}^n \psi(i)\}}{\mathbb{E}[\exp\{-(\alpha/2)\sum_{i=1}^n \psi(i)\}]} F(\psi)\bigg].$$

PROOF OF THEOREM 3.1. Let G be a $n \times n$-matrix such that there exists a permanental vector with index 2 and kernel G. For any measurable function F on $\mathbb{R}_+^n$, $\mathbb{E}_G[F(\psi)]$ denotes the expectation with respect to a permanental vector ψ with covariance matrix G and index 2.

We already know, thanks to [2] or [8], that (1) implies (2). We show that (2) implies (1). Assume that ψ is "positively correlated." Thanks to Proposition 3.2, for every measurable function F on $\mathbb{R}_+^n$,

$$\mathbb{E}_G\big[F(\psi)e^{-(\alpha/2)\sum_{i=1}^n \psi_i}\big] = \mathbb{E}_{G_\alpha}\big[F(\psi)\big]\mathbb{E}_G\big[e^{-(\alpha/2)\sum_{i=1}^n \psi_i}\big]. \tag{3.2}$$

Similarly as in the proof of Theorem 1.1, one hence obtains that for every nonnegative increasing function F on $\mathbb{R}_+^n$,

$$\mathbb{E}_{G_\alpha}\big[F(\psi)\big] \leq \mathbb{E}_G\big[F(\psi)\big]. \tag{3.3}$$

Now we use the fact noticed in [17] that for every $i \neq j$, $G_{ij}G_{ji} \geq 0$. Remark that for every permanental vector (ψ_i, ψ_j) with index 2 and kernel the 2×2-matrix C, we have for every function F,

$$\mathbb{E}_C\big[F(\psi_i, \psi_j)\big] = \mathbb{E}_{\overline{C}}\big[F\big(\eta_i^2, \eta_j^2\big)\big] \tag{3.4}$$

with the covariance matrix $\overline{C}$ defined by $\overline{C}_{ii} = C_{ii}$, $\overline{C}_{jj} = C_{jj}$ and $\overline{C}_{ij} = \sqrt{C_{ij}C_{ji}}$.

Indeed, to prove (3.4), one just compares the respective Laplace transform of the two random couples and checks that for every 2×2-diagonal matrix x with nonnegative entries,

$$|I + xC| = |I + x\overline{C}|.$$

Choosing $F(x) = \sqrt{x_i x_j}$ on $\mathbb{R}_+^n$, we obtain, thanks to (3.3),

$$\mathbb{E}_{G_\alpha}[\sqrt{\psi_i \psi_j}] \leq \mathbb{E}_G[\sqrt{\psi_i \psi_j}],$$

which together with (3.4) leads to

$$\mathbb{E}_{\overline{G}_\alpha}\Big[\sqrt{\eta_i^2 \eta_j^2}\Big] \leq \mathbb{E}_{\overline{G}}\Big[\sqrt{\eta_i^2 \eta_j^2}\Big],$$

where $\overline{G}_\alpha$ is the 2×2-matrix defined by $\overline{G}_\alpha(i,i) = G_\alpha(i,i)$, $\overline{G}_\alpha(j,j) = G_\alpha(j,j)$ and $\overline{G}_\alpha(i,j) = \sqrt{G_\alpha(i,j)G_\alpha(j,i)}$.

Setting $f(\alpha) = \mathbb{E}_{\overline{G}_\alpha}[\sqrt{\eta_i^2 \eta_j^2}]$, we know that f is decreasing at 0. Using the same arguments as in the proof of Theorem 1.1, for α small enough, we have $f'(\alpha) = -4\mathbb{E}_{\overline{G}_\alpha}[\operatorname{sgn}(\eta_i \eta_j)] \frac{\partial \overline{G}_\alpha(i,j)}{\partial \alpha}$, with $\frac{\partial \overline{G}_\alpha(i,j)}{\partial \alpha} = \frac{1}{2}(G_\alpha(i,j)G_\alpha(j,i))^{-1/2}\{G_\alpha(i,j) \times G'_\alpha(j,i) + G_\alpha(j,i)G'_\alpha(i,j)\}$.

Hence, we obtain

$$f'(0) = -\frac{1}{\overline{G}(i,j)}\mathbb{E}_{\overline{G}}[\operatorname{sgn}(\eta_i \eta_j)]\{G^2(i,j)G(j,i) + G^2(j,i)G(i,j)\}.$$

Consequently, we must have

$$\mathbb{E}_{\overline{G}}[\operatorname{sgn}(\eta_i \eta_j)]\{G^2(i,j)G(j,i) + G^2(j,i)G(i,j)\} \geq 0.$$

Note that since the couple (η_i^2, η_j^2) is always infinitely divisible, we have, using (2.13), $\mathbb{E}_{\overline{G}}[\operatorname{sgn}(\eta_i \eta_j)] \geq 0$. Hence, we have

$$G^2(i,j)G(j,i) + G^2(j,i)G(i,j) \geq 0. \tag{3.5}$$

Remark that for every $(\lambda_1, \lambda_2, \dots, \lambda_n)$ in $\mathbb{R}_+^n$, the permanental vector $(\lambda_1\psi_1, \lambda_2\psi_2, \dots, \lambda_n\psi_n)$ is also "positively correlated." Since

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \alpha_i \lambda_i \psi_i\right\}\right] = |I + \alpha\lambda G|^{-1/2},$$

$(\lambda_1\psi_1, \lambda_2\psi_2, \ldots, \lambda_n\psi_n)$ admits λG for the kernel. In particular, λG satisfies (3.5), which gives

$$\sum_{k=1}^{n} \lambda_k \{G(i,j)G(j,k)G(k,i) + G(j,i)G(i,k)G(k,j)\} \geq 0$$

and, consequently, we obtain for every i, j, k with $i \neq j$,

$$\{G(j,i)G(j,k)G(k,i) + G(j,i)G(i,k)G(k,j)\} \geq 0.$$

Since $G(i,j)G(j,i) \geq 0$, $G(j,k)G(k,j) \geq 0$ and $G(i,k)G(k,i) \geq 0$, the two terms $G(i,j)G(j,k)G(k,i)$ and $G(j,i)G(i,k)G(k,j)$ have the same sign. Their sum can be nonnegative only if they are both nonnegative. We have obtained for every i, j, k

$$G(j,i)G(j,k)G(k,i) \geq 0.$$

By substituting $(\alpha + \varepsilon)$ to α in (3.2), one obtains similarly for every $\alpha > 0$

$$G_\alpha(j,i)G_\alpha(j,k)G_\alpha(k,i) \geq 0.$$

We can then develop the same argument as in the proof of Theorem 1.1 from (2.14) until the conclusion that ψ has to be infinitely divisible. □

REMARK 3.3. Note that the proof of Theorem 3.1 shows that a permanental vector $(\psi_i)_{1 \leq i \leq n}$ is infinitely divisible if and only if for every i, j, $1 \leq i, j \leq n$, for every $n \times n$ nonnegative diagonal matrix λ, $\mathbb{E}_{(\lambda G)_\alpha}[\sqrt{\psi_i \psi_j}]$ is a decreasing function of α on $\mathbb{R}^+$.

PROOF OF PROPOSITION 3.2. We note that

$$I + xM_\alpha = I + xM(I + \alpha M)^{-1} = \big(I + (x + \alpha)M\big)(I + \alpha M)^{-1},$$

where $x + \alpha$ means $(x_1 + \alpha, x_2 + \alpha, \ldots, x_n + \alpha)$. Taking the determinant of each part of this equation and then the power $(-1/\beta)$ gives

$$|I + xM_\alpha|^{-1/\beta} = \mathbb{E}\big(Xe^{-(1/2)\sum_{i=1}^{n} x_i \psi(i)}\big),$$

where X is the positive random variable with expectation 1 defined by

$$X = \exp\left\{-\frac{\alpha}{2}\sum_{i=1}^{n}\psi(i)\right\} \Big/ \mathbb{E}\left[\exp\left\{-\frac{\alpha}{2}\sum_{i=1}^{n}\psi(i)\right\}\right].$$

Hence, ψ_α exists and has the law of ψ under $\mathbb{E}(X, \cdot)$. □

4. The shifted case. Given a centered Gaussian process $(\eta_x)_{x\in E}$ and a real number r, we write $(\eta+r)^2$ for $((\eta_x+r)^2)_{x\in E}$. Thanks to [2] and [8], we know that

$$\begin{gathered} \textit{If } (\eta+r)^2 \textit{ is infinitely divisible for every real } r, \\ \textit{then} \\ (\eta+r)^2 \textit{ is "positively correlated" for every real } r. \end{gathered} \tag{4.1}$$

The following theorem gives another sufficient condition for $(\eta+r)^2$ to be "positively correlated." It can also be seen as an alternative characterization of Gaussian processes with a covariance equal to the Green function of a Markov process. We assume that E contains more than two elements.

THEOREM 4.1. *Let $(\eta_x)_{x\in E}$ be a centered Gaussian process with a continuous positive definite covariance. The following properties are equivalent*:

(1) *The covariance of η is the Green function of a transient Markov process.*

(2) *The family of processes $((\eta+r)^2, r\geq 0)$ is strongly stochastically increasing as r increases on $\mathbb{R}^+$.*

The definition of a strong stochastic comparison is given at the beginning of Section 2 (Definition 2.2).

PROOF OF THEOREM 4.1. (1) $\Longrightarrow$ (2): Assuming (1), we know that for every positive integer n and every $(x_i)_{1\leq i\leq n}$ in E^n, the covariance matrix G of the vector $(\eta_{x_i})_{1\leq i\leq n}$ is the inverse of a diagonally dominant M-matrix (see [6]), that is, setting $G^{-1}=M$, all the entries of G are nonnegative, all the off-diagonal entries of M are nonpositive, and for every k, $\sum_{i=1}^n M_{ki}\geq 0$. The fact that G^{-1} is an M-matrix implies that for every $\beta=(\beta_i)_{1\leq i\leq n}$ and $\alpha=(\alpha_i)_{1\leq i\leq n}$ in $\mathbb{R}_+^n$, such that $\alpha_i\geq\beta_i$, we have, using a result of Fang and Hu (Theorem 2.3 in [10]),

$$\big((\eta_{x_i}+(G\alpha)_i)^2\big)_{1\leq i\leq n} \text{ strongly stochastically dominates } \big((\eta_{x_i}+(G\beta)_i)^2\big)_{1\leq i\leq n}.$$

Since G^{-1} is diagonally dominant, we know that the vector $G^{-1}\mathbb{1}$, where $\mathbb{1}$ is the vector $(1,1,\dots,1)^t$ of $\mathbb{R}_+^n$, belongs to $\mathbb{R}_+^n$. Hence, we can choose to take $\alpha=rM\mathbb{1}$ and $\beta=r'M\mathbb{1}$, with $r\geq r'$, to obtain

$$\big((\eta_{x_i}+r)^2\big)_{1\leq i\leq n} \text{ strongly stochastically dominates } \big((\eta_{x_i}+r')^2\big)_{1\leq i\leq n}.$$

By definition, this means that the sequence of processes $((\eta+r)^2, r>0)$ increases with r with respect to the strong stochastic order.

(2) $\Longrightarrow$ (1): Conversely, for $r>0$ fixed and n positive integer, denote by $(f_r(x), x\in\mathbb{R}_+^n)$ the density of the vector $((\eta_{x_i}+r)^2)_{1\leq i\leq n}$. By assumption for every (r,r') such that $r>r'$, we have for every x,y in $\mathbb{R}_+^n$,

$$f_r(x)f_{r'}(y)\leq f_r(x\vee y)f_{r'}(x\wedge y).$$

By integrating the above inequality with respect to $\frac{1}{\sqrt{2\pi}}e^{-r^2/2}\,dr$, one obtains

$$h(x)f_{r'}(y) \le h(x \vee y)f_{r'}(x \wedge y),$$

where $(h(x), x \in \mathbb{R}_+^n)$ is the density of the vector $((\eta_{x_i} + N)^2)_{1\le i\le n}$, with N standard Gaussian variable independent of η.

One integrates then this last inequality with respect to $\mathbb{P}(N \in dr')$ to obtain

$$h(x)h(y) \le h(x \vee y)h(x \wedge y),$$

which means that the vector $((\eta_{x_i} + N)^2, 1 \le i \le n)$ satisfies the FKG inequality. Thanks to Theorem 1.1, this vector is hence infinitely divisible. Since this is true for every n and every $(x_i)_{1\le i\le n}$, the process $((\eta_x + N)^2, x \in E)$ is infinitely divisible. We use now the assumption on the continuity of the covariance of η to claim that, thanks to [5], this can be so only if the covariance of η is the Green function of a Markov process. □

REMARK 4.2. The case of Gaussian couples has to be studied as a particular case. Indeed, in [5], we have shown that, given a centered Gaussian couple (η_x, η_y), the couple $((\eta_x + r)^2, (\eta_y + r)^2)$ is infinitely divisible for every r, if and only if

$$\mathbb{E}(\eta_x\eta_y) \ge 0 \quad \text{and} \quad \mathbb{E}(\eta_x\eta_y) \le \mathbb{E}(\eta_x^2)\mathbb{E}(\eta_y^2).$$

But one can use the two-dimensional case to show that the converse of (4.1) is false. Indeed, consider a centered Gaussian couple (η_x, η_y) with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ such that $|\rho| < 1$. Then according to Corollary 3.1 of Fang and Hu [10], for every r $((\eta_x + r)^2, (\eta_y + r)^2)$ satisfies the FKG inequality. In particular, $((\eta_x + r)^2, (\eta_y + r)^2)$ is "positively correlated" for every r. But choosing $\rho < 0$, we see that $((\eta_x + r)^2, (\eta_y + r)^2)$ cannot be infinitely divisible for every r.

5. A stability property for Green functions.

THEOREM 5.1. *Let $(g(x, y), (x, y) \in E \times E)$ be the Green function of a transient Markov process. Assume g is continuous, then for every $\beta \ge 1$, $(g^\beta(x, y), (x, y) \in E \times E)$ is also the Green function of a transient Markov process.*

In the case E is finite, the above fact has already been established by Dellacherie et al. [3]. To establish the general case, we first show the following characterization of Green functions, which is an extension of a result on symmetric Green functions (see Theorems 1.2 and 1.3 in [5]).

THEOREM 5.2. *Let G be a continuous function on $E \times E$. The three following points are equivalent*:

(i) *G is the Green function of some Markov process.*

(ii) *For every positive real c, $G+c$ is the kernel of an infinitely divisible permanental process.*

(iii) *$G+1$ is the kernel of an infinitely divisible permanental process.*

PROOF. We follow the proof of Theorem 1.2 and Theorem 1.3 in [5]. We insist only on the arguments that are specific to the nonsymmetric case.

(i) $\Rightarrow$ (ii): Making use of the arguments developed in [5], there exists a recurrent Markov process X such that G represents the 0 potential densities of X killed at its first hitting time of a, a point outside E. We set then $G(a,a)=0=G(a,x)=G(x,a)$ for every x in E. We use then an isomorphism theorem for recurrent Markov processes (Corollary 3.5 in [7]) to claim that for every $c>0$, there exists a permanental process $(\psi_x, x\in E\cup\{a\})$ with kernel $G+c$ and index 2, satisfying for every $r>0$

$$\left(\left(\tfrac{1}{2}\psi_x, x\in E\cup\{a\}\right)\middle|\psi_a=r\right)\overset{\text{(law)}}{=}\left(\tfrac{1}{2}\phi_x+L^x_{\tau_r}, x\in E\cup\{a\}\right),\tag{5.1}$$

where $(\phi_x, x\in E\cup\{a\})$ is a permanental process with kernel G and index 2 independent of X, and $(L^x_{\tau_r}, x\in E\cup\{a\})$ is the local time process of X starting at a, at time $\tau_r=\inf\{s\geq 0: L^a_s>r\}$.

Since G is a Green function, the process ϕ is infinitely divisible (see [7]). Besides, one easily checks that L_{τ_r} is infinitely divisible. Actually, $(L_{\tau_r})_{r>0}$ is a Lévy process and for every $\alpha=(\alpha_i)_{1\leq i\leq n}$ in $\mathbb{R}^n_+$ and $(x_i)_{1\leq i\leq n}$ in $(E\cup\{a\})^n$, we have

$$\mathbb{E}\left(\exp\left\{-\sum_{i=1}^n \alpha_i L^{x_i}_{\tau_r}\right\}\right)=e^{-rF(G,\alpha)},\tag{5.2}$$

where $F(G,\alpha)$ is a nonnegative constant.

Hence, for every $r>0$, $(\psi|\psi_a=r)$ is also infinitely divisible. But (ii) requires the infinite divisibility of ψ. We hence integrate (5.1) with respect to the law of ψ_a to obtain, thanks to (5.2),

$$\mathbb{E}\left(\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i\psi_{x_i}\right\}\right)=\mathbb{E}\left(\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i\phi_{x_i}\right\}\right)\mathbb{E}\left(e^{-F(G,\alpha)\psi_a}\right).$$

Now, ψ_a has the law of a squared Gaussian variable and is hence infinitely divisible. Consequently, for every positive δ, there exists a nonnegative variable Y_δ that we can choose independent of X, such that

$$\left(\mathbb{E}\left(e^{-F(G,\alpha)\psi_a}\right)\right)^\delta=\mathbb{E}\left(e^{-F(G,\alpha)Y_\delta}\right).$$

We hence obtain

$$\mathbb{E}\left(\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i\psi_{x_i}\right\}\right)^\delta=\mathbb{E}\left(\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i\phi_{x_i}\right\}\right)^\delta\mathbb{E}\left(\exp\left\{-\sum_{i=1}^n \alpha_i L^{x_i}_{\tau_{Y_\delta}}\right\}\right),$$

which shows the infinite divisibility of ψ.

To prove that (iii) implies (i), we can directly use the argument given in [5], since symmetry is not required there. And, finally, (ii) obviously implies (iii). $\square$

The following equivalence will help us to show Theorem 5.1.

THEOREM 5.3. *Let G be a continuous function on $E \times E$. Then G is the Green function of a Markov process if and only if for every finite subset F of E the restriction of G to $F \times F$ is the Green function of a Markov process.*

PROOF. One has to establish it only in the nonsymmetric case (in the symmetric case it is a consequence of [6] and [5]). The direct way is known. Conversely, assume that for every finite set F, $G_{|F\times F}$ is a Green function, then thanks to Theorem 5.2, $(G+1)_{|F\times F}$ is the kernel of an infinitely divisible permanental processes with index 2. Hence, there exists a permanental process $(\psi_x, x \in E)$ with kernel $(G(x,y)+1, (x,y) \in E \times E)$ and index 2. Thanks to Theorem 5.2, all the finite-dimensional marginals of ψ are infinitely divisible. Consequently, ψ is infinitely divisible. This implies, thanks to Theorem 5.2, that $(G(x,y)+1, (x,y), (x,y) \in E \times E)$ is the Green function of a Markov process. $\square$

PROOF OF THEOREM 5.1. Thanks to [3], we know that for every finite subset F of E, $(g^\beta(x,y), (x,y) \in F \times F)$ is the Green function of a Markov process. This implies, thanks to Theorem 5.3, that $(g^\beta(x,y), (x,y) \in E \times E)$ is the Green function of a Markov process. $\square$

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