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DISTANCE BETWEEN TWO SKEW BROWNIAN MOTIONS AS A S.D.E. WITH JUMPS AND LAW OF THE HITTING TIME

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In this paper, we consider two skew Brownian motions, driven by the same Brownian motion, with different starting points and different skewness coefficients. We show that we can describe the evolution of the distance between the two processes with a stochastic differential equation. This S.D.E. possesses a jump component driven by the excursion process of one of the two skew Brownian motions. Using this representation, we show that the local time of two skew Brownian motions at their first hitting time is distributed as a simple function of a Beta random variable. This extends a result by Burdzy and Chen [*Ann. Probab.* **29** (2001) 1693–1715], where the law of coalescence of two skew Brownian motions with the same skewness coefficient is computed.

1. Presentation of the problem. Consider $(B_t)_{t\geq 0}$ a standard Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where the filtration satisfies the usual right continuity and completeness conditions. Recall that the skew Brownian motion $X^{x,\beta}$ is defined as the solution of the stochastic differential equation with singular drift coefficient,

$$X_t^{x,\beta} = x + B_t + \beta L_t^0(X^{x,\beta}), \tag{1.1}$$

where $\beta \in (-1, 1)$ is the skewness parameter, $x \in \mathbb{R}$ and $L_t^0(X^{x,\beta})$ is the symmetric local time at 0,

$$L_t^0(X^{x,\beta}) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[-\varepsilon,\varepsilon]}(X_s^{x,\beta})\, ds.$$

It is known that a strong solution of equation (1.1) exists, and pathwise uniqueness holds as well; see [3, 10]. Remark that in [5] it is shown that $X^{x,\beta}$ can be obtained as the limit of diffusion processes $X^{x,\beta,n}$ with smooth coefficients. Indeed, if one mollifies the singularity due to the local time, the following diffusion processes can be defined:

$$X_t^{x,\beta,n} = x + B_t + \frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right) \int_0^t n\phi(nX_s^{x,\beta,n})\, ds,$$

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where ϕ is any symmetric positive function with support on $[-1/2, 1/2]$ and having unit mass. Then, the almost sure convergence of some sub-sequence X^{x,β,n_k} to $X^{x,\beta}$ is shown in [5], Theorem 1.7.

The skew Brownian motion is an example of a process partially reflected at some frontier. It finds applications in the fields of stochastic modelization and of numerical simulations, especially as it is deeply connected to diffusion processes with noncontinuous coefficients; see [12] and references therein. The structure of the flow of a reflected, or partially reflected, Brownian motion has been the subject of several works; see, for example, [2, 4]. The long time behavior of the distance between reflected Brownian motions with different starting points has been largely studied too; see, for example, [6, 8].

Actually, a quite intriguing fact about the solutions of (1.1) is that they do not satisfy the usual flow property of differential equations, which prevents two solutions with different initial positions from meeting in finite time. Indeed, it is shown in [2] that, almost surely, the two paths $t \mapsto X_t^{x,\beta}$ and $t \mapsto X_t^{0,\beta}$ meet at a finite random time. Moreover, the law of the values of the local times of these processes at this instant of coalescence are computed in [5].

In this paper, we study the time dynamic of the distance between the two processes X^{0,β_1} and X^{x,β_2} where the skewness parameters β_1, β_2 are possibly different. We show that, after some random time change, the distance between the two processes is a Markov process, solution to an explicit stochastic differential equation with jumps; see Theorem 1 below. The dynamic of this stochastic differential equation enables us to compute the law of the hitting time of zero for the distance between the two skew Brownian motions. Consequently, we can draw informations about the hitting time of the two skew Brownian motions.

More precisely, let us denote $T^\star$ the first instant where X^{x,β_2} and X^{0,β_1} meet and define the quantity $U^\star = L^0_{T^\star}(X^{0,\beta_1})$. For $x > 0$, $0 < \beta_1, \beta_2 < 1$, we show, in Theorem 3 below, that the random variable $\frac{x}{\beta_1 U^\star}$ is distributed with a Beta law. This extends the result of Burdzy and Chen [5] where the law of the hitting time was computed under the restriction $\beta_1 = \beta_2$. We study also the situation where $-1 < \beta_2 < 0 < \beta_1 < 1$ and $x > 0$. In this case, we show that the random variable $\frac{\beta_1 U^\star}{x}$ is distributed with a Beta law (Theorem 4).

The organization of the paper is as follows. In Section 2, we precisely state our main results.

Sections 3 and 4 are devoted to the proofs of the results in the case $0 < \beta_1$, $\beta_2 < 1$. In Section 3, we introduce our fundamental tool, which is the process $u \mapsto X^{x,\beta_2}_{\tau_u^0(X^{0,\beta_1})}$, where $\tau_u^0(X^{0,\beta_1})$ is the inverse local time of X^{0,β_1}. This process is a measurement of the distance between X^{0,β_1} and X^{x,β_2}. We prove that this process is solution of some explicit stochastic differential equation with jumps, driven by the Poisson process of the excursions of X^{0,β_1}. In Section 4, we show

how the dynamic of this process enables us to compute the law of the hitting time of the two skew Brownian motions.

In Section 5, we sketch the proofs of our results in the situation $-1 < \beta_2 < 0 < \beta_1 < 1$. For the sake of shortness, we will only put the emphasis on the main differences with the case $0 < \beta_1, \beta_2 < 1$.

Some of the technical results needed in the proofs are postponed to an Appendix.

2. Main results.

Consider the two skew Brownian motions,

$$X_t^{x,\beta_2} = x + B_t + \beta_2 L_t^0(X_t^{x,\beta_2}), \tag{2.1}$$

$$X_t^{0,\beta_1} = B_t + \beta_1 L_t^0(X_t^{0,\beta_1}), \tag{2.2}$$

with $x > 0$. We introduce the cadlag process defined as

$$Z_u^{x,\beta_1,\beta_2} = X^{x,\beta_2}_{\tau_u(X^{0,\beta_1})}, \tag{2.3}$$

where $\tau_u(X^{0,\beta_1})$ is the inverse of the local time, given as

$$\tau_u(X^{0,\beta_1}) = \inf\{t \geq 0 \mid L_t^0(X^{0,\beta_1}) > u\}.$$

Note that, since $X^{0,\beta_1}_{\tau_u(X^{0,\beta_1})} = 0$, we have $Z_u^{x,\beta_1,\beta_2} = X^{x,\beta_2}_{\tau_u(X^{0,\beta_1})} - X^{0,\beta_1}_{\tau_u(X^{0,\beta_1})}$. This explains why we choose below to call Z^{x,β_1,β_2} the "distance process." Our first result shows that the "distance process" is solution to a stochastic differential equation with jumps, driven by the excursion Poisson process of X^{0,β_1}. We need some additional notation before stating it. We introduce $(\mathbf{e}_u)_{u>0}$, the excursion process associated to X^{0,β_1},

$$\mathbf{e}_u(r) = X^{0,\beta_1}_{\tau_{u-}(X^{0,\beta_1})+r} \qquad \text{for } r \leq \tau_u(X^{0,\beta_1}) - \tau_{u-}(X^{0,\beta_1}).$$

The Poisson point process $(\mathbf{e}_u)_{u>0}$ takes values in the space $\mathcal{C}_{0\to 0}$ of excursions with finite lifetime, endowed with the usual uniform topology. We denote $\mathbf{n}_{\beta_1}$ the excursion measure associated with X^{0,β_1}.

Let us define $T^\star = \inf\{t \geq 0 \mid X_t^{0,\beta_1} = X_t^{0,\beta_2}\} \in [0,\infty]$ and $U^\star = L^0_{T^\star}(X_t^{0,\beta_1})$. Since X^{x,β_2} and X^{0,β_1} are driven by the same Brownian motion, it is easy to see that they can only meet when $X^{0,\beta_1} = 0$. As a consequence, we have

$$U^\star = \inf\{u \geq 0 \mid Z_u^{x,\beta_1,\beta_2} = 0\} \in [0,\infty] \quad \text{and} \quad Z^{x,\beta_1,\beta_2} > 0 \text{ on } [0, U^\star).$$

Our first result about Z^{x,β_1,β_2} is the following.

THEOREM 1. *Assume $x > 0$ and $0 < \beta_1, \beta_2 < 1$. Almost surely, we have for all $t < U^\star$,*

$$Z_t^{x,\beta_1,\beta_2} = x - \beta_1 t + \sum_{0<u\leq t} \beta_2 \ell(Z_{u-}^{x,\beta_1,\beta_2}, \mathbf{e}_u), \tag{2.4}$$

where $\ell : (0, \infty) \times \mathcal{C}_{0\to 0} \to [0, \infty)$ *is a measurable map.*

For $h > 0$, *we can describe the law of* $\mathbf{e} \mapsto \ell(h, \mathbf{e})$ *under* $\mathbf{n}_{\beta_1}$ *by*

$$\mathbf{n}_{\beta_1}\big(\ell(h, \mathbf{e}) \geq a/\beta_2\big) = \frac{1-\beta_1}{2h}\left(1 + \frac{a}{h}\right)^{-(1+\beta_2)/(2\beta_2)} \qquad \forall a > 0. \tag{2.5}$$

COROLLARY 1. *Assume* $x > 0$ *and* $0 < \beta_1, \beta_2 < 1$. *We have for all* $t < U^\star$,

$$Z_t^{x,\beta_1,\beta_2} = x - \beta_1 t + \int_{[0,t]\times(0,\infty)} a\mu(du, da), \tag{2.6}$$

where $\mu(du, da)$ *is the random jumps measure of* Z^{x,β_1,β_2} *on* $[0, U^\star) \times (0, \infty)$. *The compensator of the measure* $\mu(du, da)$ *is* $du \times \nu(Z_{u-}^{x,\beta_1,\beta_2}, da)$ *with*

$$\nu(h, da) = \frac{\kappa}{h^2}\left(1 + \frac{a}{h}\right)^{-\gamma} 1_{\{a>0\}}\, da, \tag{2.7}$$

where $\kappa = \frac{(1-\beta_1)(1+\beta_2)}{4\beta_2}$ *and* $\gamma = \frac{1+3\beta_2}{2\beta_2}$.

REMARK 1. Theorem 1 fully details the dynamic of the "distance process" before it (possibly) reaches 0. The "distance process" decreases with a constant negative drift, and has positive jumps. Moreover, the value of a jump at time u is a function of the level $Z_{u-}^{x,\beta_1,\beta_2}$ and of the excursion $\mathbf{e}_u$. The image of the excursion measure under this function, with a fixed level $h > 0$, is given by the explicit expression (2.5).

In [2, 5] it is shown that the processes X^{0,β_1} and X^{x,β_2} meet in finite time under some appropriate conditions for the skewness coefficients.

THEOREM 2. *Assume* $x > 0$ *and* $0 < \beta_1, \beta_2 < 1$ *with* $\beta_1 > \frac{\beta_2}{1+2\beta_2}$. *Then the hitting time* $T^\star = \inf\{t > 0 \mid X_t^{0,\beta_1} = X_t^{x,\beta_2}\}$ *is almost surely finite.*

REMARK 2. Actually, in [2] the case $\beta_1 = \beta_2$ is considered with $x > 0$. In [5] the situation $\beta_1 \neq \beta_2$ is treated in the case $x = 0$ and with the condition $\frac{\beta_2}{1+2\beta_2} < \beta_1 < \beta_2$; see [5], Theorem 1.4(iii). Nevertheless, it is rather clear that the additional condition $\beta_1 < \beta_2$ is mainly related to the choice $x = 0$ and could be removed if $x > 0$. We will give below a new proof of Theorem 2.

In [5] the law of $U^\star = L_{T^\star}^0(X^{0,\beta_1})$ is computed in the particular situation $\beta_1 = \beta_2$. In the following theorem, we compute the law without this restriction.

THEOREM 3. *Assume $x > 0$ and $0 < \beta_1, \beta_2 < 1$ with $\beta_1 > \frac{\beta_2}{1+2\beta_2}$. Denote $U^\star = L^0_{T^\star}(X^{0,\beta_1})$. Then the law of $U^\star$ has the density*

$$
\begin{aligned}
p_{U^\star}(x, du) &= \frac{1}{\mathbf{b}(1-\xi^\star, (1-\beta_1)/(2\beta_1))} \frac{\beta_1}{x} \left(\frac{\beta_1 u}{x}\right)^{\xi^\star - 2} \\
&\quad \times \left(1 - \frac{x}{\beta_1 u}\right)^{(1-3\beta_1)/(2\beta_1)} 1_{[x/\beta_1, \infty)}(u)\, du,
\end{aligned} \tag{2.8}
$$

where $\mathbf{b}(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\xi^\star = \frac{1}{2\beta_1} - \frac{1}{2\beta_2}$.

Hence, $\frac{x}{\beta_1 U^\star}$ is distributed as a Beta random variable $\mathcal{B}(1-\xi^\star, \frac{1-\beta_1}{2\beta_1})$.

REMARK 3. For $\beta_1 = \beta_2$ we retrieve the result of [5]. However, in [5] the cumulative distribution function of $U^\star$ was explicitly derived using a max-stability argument for the law of $U^\star$. By (2.8) we see that for $\beta_1 \neq \beta_2$ the cumulative distribution function cannot be computed explicitly. Actually, arguments similar to [5] do not seem to apply directly here.

The following proposition deals with the finiteness of the hitting time of X^{0,β_1} and X^{x,β_2} when one of the skewness parameters is negative. It can be easily derived from Theorem 2; a proof is given in Section 5.

PROPOSITION 1. *Assume $x > 0$ and $-1 < \beta_2 < 0 < \beta_1 < 1$. Then $T^\star$ is almost surely finite.*

Assume $x > 0$ and $-1 < \beta_1 < 0 < \beta_2 < 1$. Then $T^\star = \infty$ almost surely.

We can compute the law of the hitting time when the skewness parameters have different signs.

THEOREM 4. *Assume $x > 0$ and $-1 < \beta_2 < 0 < \beta_1 < 1$; then the law of $U^\star = L^0_{T^\star}(X^{0,\beta_1})$ has the density*

$$
\begin{aligned}
p_{U^\star}(x, du) &= \frac{1}{\mathbf{b}((\beta_2 - 1)/(2\beta_2), (1-\beta_1)/(2\beta_1))} \frac{\beta_1}{x} \left(\frac{\beta_1 u}{x}\right)^{-(1+\beta_2)/(2\beta_2)} \\
&\quad \times \left(1 - \frac{\beta_1 u}{x}\right)^{(1-3\beta_1)/(2\beta_1)} 1_{[0, x/\beta_1]}(u)\, du.
\end{aligned} \tag{2.9}
$$

Hence, $\frac{\beta_1 U^\star}{x}$ is distributed as a Beta random variable $\mathcal{B}(\frac{\beta_2 - 1}{2\beta_2}, \frac{1-\beta_1}{2\beta_1})$.

It remains to study the case where $\beta_1 < 0$, $\beta_2 < 0$. We have the following result, which will be deduced from the previous ones.

COROLLARY 2. *Assume $x > 0$ with $-1 < \beta_1, \beta_2 < 0$ and $|\beta_2| > \frac{|\beta_1|}{1+2|\beta_1|}$. Then $T^\star$ is finite and $(1 - \frac{\beta_1 L^0_{T^\star}(X^{0,\beta_1})}{x})^{-1}$ is distributed as a product of two independent Beta variables.*

REMARK 4. Remark that the condition $x > 0$ in the previous results is essentially irrelevant. Indeed if $x < 0$, we may set $\widetilde{X}^x = -X^x$, $\widetilde{X}^0 = -X^0$, $\widetilde{\beta}_1 = -\beta_1$ and $\widetilde{\beta}_2 = -\beta_2$. This simple transformation reduces the situation to one of those studied in Theorems 3 and 4 or Corollary 2.

Throughout the paper, the parameter β_1 is associated to the process starting from 0 and β_2 to the one starting from $x > 0$, so we will, from now on, suppress the dependence upon the skewness parameters and write X^0, X^x, Z^x for X^{0,β_1}, X^{x,β_2}, Z^{x,β_1,β_2}. Moreover, we shall only consider the inverse of local time for the process X^0, and hence we shall write τ_u for $\tau_u(X^0)$ when no confusion is possible.

Let us introduce, for $u \geq 0$, the sigma field

$$\mathcal{G}_u = \mathcal{F}_{\tau_u}. \tag{2.10}$$

With these notation, the process $(Z^x_u)_{u\geq 0}$ is $(\mathcal{G}_u)_{u\geq 0}$ adapted. Moreover we can see that its law defines a Markov semi group. Indeed, we can use the a.s. relation $\tau_l(X^0_{\tau_h+\cdot}) = \tau_{h+l}(X^0) - \tau_h(X^0)$ to get

$$Z^x_{h+l} = X^x_{\tau_{h+l}(X^0)} = X^x_{\tau_h(X^0)+\tau_l(X^0_{\tau_h+\cdot})}. \tag{2.11}$$

Then, using the pathwise uniqueness for the skew equations, we see that the law of $(X^x_{\tau_h+\cdot}, X^0_{\tau_h+\cdot})$, conditional on $\mathcal{F}_{\tau_h}$, is the law of solutions to (2.1) and (2.2) starting from $(X^x_{\tau_h}, X^0_{\tau_h}) = (X^x_{\tau_h}, 0)$. This fact with (2.11) shows that the law of $(Z^x_u)_{u\geq 0}$ defines a Markov semi group.

Consequently, we remark that $U^\star = \inf\{u \geq 0 \mid Z^x_u = 0\}$ is the a hitting time of a Markov process. This is the crucial fact that allows us to compute the law of $U^\star$.

In the next section, we will study the dynamics of the Markov process Z^x and, in particular, give the proof of Theorem 1. For simplicity, we have decided to focus the paper mainly on the situation $0 < \beta_1, \beta_2 < 1$. This restriction especially holds true in the Sections 3 and 4 below.

3. Stochastic differential equation with jumps characterization of Z^x (case $0 < \beta_1, \beta_2 < 1$). In this section we assume that we are in the situation $0 < \beta_1, \beta_2 < 1$. We will show that Z^x is solution to some stochastic differential equation governed by the excursion point Poisson process of X^0.

First, we recall some basic facts about the excursion theory.

3.1. *Excursions of a skew Brownian motion.* Consider $X^{0,\beta}$ a skew Brownian motion starting from 0, and introduce the inverse of its local time $\tau_u(X^{0,\beta}) = \inf\{t \geq 0 \mid L_t^0(X^{0,\beta}) > u\}$. Recall that the excursion process $(\mathbf{e}_u)_{u>0}$ associated to $X^{0,\beta}$ is $\mathbf{e}_u(r) = X^{0,\beta}_{\tau_{u-}(X^{0,\beta})+r}$, for $r \leq \tau_u(X^{0,\beta}) - \tau_{u-}(X^{0,\beta})$. The Poisson point process $(\mathbf{e}_u)_{u>0}$ takes values in the space $\mathcal{C}_{0\to 0}$ of excursions. For $\mathbf{e} \in \mathcal{C}_{0\to 0}$ we denote $R(\mathbf{e})$ the lifetime of the excursion and recall that by definition $\mathbf{e}$ does not hit zero on $(0, R(\mathbf{e}))$, and $\mathbf{e}(r) = 0$ for $r \geq R(\mathbf{e})$.

If we denote $\mathbf{n}_\beta$ the excursion measure of the $X^{0,\beta}$, we have the formula, for any Borel subset A of $\mathcal{C}_{0\to 0}$,

$$\mathbf{n}_\beta(A) = \frac{(1+\beta)}{2}\mathbf{n}_{|\mathrm{B.M.}|}(A) + \frac{(1-\beta)}{2}\mathbf{n}_{|\mathrm{B.M.}|}(-A), \tag{3.1}$$

where $\mathbf{n}_{|\mathrm{B.M.}|}$ is the excursion measure for the absolute value of a Brownian motion. Let us recall some useful facts on the excursion measure $\mathbf{n}_{|\mathrm{B.M.}|}$, that are immediate from well-known properties of the excursion measure of a standard Brownian motion.

First, we recall the law of the height of an excursion (e.g., see Chapter 12 in [13]),

$$\mathbf{n}_{|\mathrm{B.M.}|}(\mathbf{e} \text{ reaches } h) = \frac{1}{h} \qquad \text{for } h > 0. \tag{3.2}$$

Second, we recall that in the case of a standard Brownian motion, the law of the excursion after reaching some fixed level h, is the same as the law of a Brownian motion starting from h before it hits 0.

In the sequel, we will use these properties in a context that we now describe.

Let $G : \mathcal{C}([0,\infty), \mathbb{R}) \to \mathbb{R}_+$ be some measurable functional on the canonical Wiener space. For $h \in \mathbb{R}$ denote $T^h(\mathbf{e}) = \inf\{s \mid \mathbf{e}_s = h\}$, and let $w_r^h := w_r^h(\mathbf{e}) := \mathbf{e}_{T^h(\mathbf{e})+r} - h$ for $r \leq R(\mathbf{e}) - T^h(\mathbf{e}) = T^{-h}(w^h(\mathbf{e}))$ be the shifted part of the excursion after T^h. Then by Theorem 3.5, page 491 in [13], for $h > 0$,

$$\begin{aligned} &\frac{\mathbf{n}_{|\mathrm{B.M.}|}[G(\mathbf{e}_{(T^h(\mathbf{e})+\cdot)\wedge R(\mathbf{e})} - h)1_{\{\mathbf{e} \text{ reaches } h\}}]}{\mathbf{n}_{|\mathrm{B.M.}|}[\mathbf{e} \text{ reaches } h]} \\ &\quad = \frac{\mathbf{n}_{|\mathrm{B.M.}|}[G(w^h_{\cdot\wedge T^{-h}(w^h)})1_{\{\mathbf{e} \text{ reaches } h\}}]}{\mathbf{n}_{|\mathrm{B.M.}|}[\mathbf{e} \text{ reaches } h]} \\ &\quad = \int_{\mathcal{C}([0,\infty),\mathbb{R})} G(w_{\cdot\wedge T^{-h}(w)})\, d\mathbb{W}(w), \end{aligned}$$

where $\mathbb{W}$ is the standard Wiener measure.

Using (3.1) we deduce that for $h \neq 0$,

$$\frac{\mathbf{n}_\beta[G(w^h_{\cdot\wedge T^{-h}(w^h)})1_{\{\mathbf{e} \text{ reaches } h\}}]}{\mathbf{n}_\beta[\mathbf{e} \text{ reaches } h]} = \int_{\mathcal{C}([0,\infty),\mathbb{R})} G(w_{\cdot\wedge T^{-h}(w)})\, d\mathbb{W}(w). \tag{3.3}$$

3.2. *Representation of the local time of X^x as a functional of the excursion process of X^0.* The next proposition shows that $L^0_{\tau_u}(X^x)$ is a functional of $(\mathbf{e}_u)_{u>0}$ the excursion process of X^0 [recall that $\tau_u = \tau_u(X^{0,\beta_1})$].

PROPOSITION 2. *Almost surely, one has the representation for all $t < U^\star$,*

$$L^0_{\tau_t}(X^x) = \sum_{0<u\le t} \ell(X^x_{\tau_{u-}}, \mathbf{e}_u), \tag{3.4}$$

where $\ell : (0,\infty)\times \mathcal{C}_{0\to 0} \to [0,\infty)$ is a measurable map.

For $h>0$, we can describe the law of $\mathbf{e}\mapsto \ell(h,\mathbf{e})$ under $\mathbf{n}_{\beta_1}$ by

$$\mathbf{n}_{\beta_1}\big(\ell(h,\mathbf{e})\ge a\big) = \frac{1-\beta_1}{2h}\left(1+\frac{\beta_2 a}{h}\right)^{-(1+\beta_2)/(2\beta_2)} \qquad \forall a>0. \tag{3.5}$$

PROOF. Before turning to a rigorous proof, let us give some insight about the representation (3.4). Given $u>0$ such that $\tau_u - \tau_{u-} > 0$, using (2.2), we have for $r \le R(\mathbf{e}_u)$,

$$\begin{aligned}\mathbf{e}_u(r) = X^0_{\tau_{u-}+r} &= X^0_{\tau_{u-}} + B_{\tau_{u-}+r} - B_{\tau_{u-}} + \beta_1\big[L^0_{\tau_{u-}+r}(X^0) - L^0_{\tau_{u-}}(X^0)\big]\\ &= B_{\tau_{u-}+r} - B_{\tau_{u-}}.\end{aligned}$$

Recalling (2.1), we deduce

$$\begin{aligned}X^x_{\tau_{u-}+r} &= X^x_{\tau_{u-}} + B_{\tau_{u-}+r} - B_{\tau_{u-}} + \beta_2\big[L^0_{\tau_{u-}+r}(X^x) - L^0_{\tau_{u-}}(X^x)\big]\\ &= X^x_{\tau_{u-}} + \mathbf{e}_u(r) + \beta_2\big[L^0_{\tau_{u-}+r}(X^x) - L^0_{\tau_{u-}}(X^x)\big]\\ &= X^x_{\tau_{u-}} + \mathbf{e}_u(r) + \beta_2 L^0_r(X^x_{\tau_{u-}+\cdot}).\end{aligned} \tag{3.6}$$

Relation (3.6) shows that $(X^x_{\tau_{u-}+r})_{r<R(\mathbf{e}_u)}$ satisfies a skew Brownian motion type of equation, but governed by the excursion path $\mathbf{e}_u$, and starting from the value $X^x_{\tau_{u-}}$. By solving this equation, we will show that the process $(X^x_{\tau_{u-}+r})_{r<R(\mathbf{e}_u)}$ can be obtained as a functional of the excursion $\mathbf{e}_u$ and of the initial value $X^x_{\tau_{u-}}$. As a consequence, the local time $L^0_{\tau_u}(X^x) - L^0_{\tau_{u-}}(X^x)$ will be written too as a functional $\ell(X_{\tau^x_{u-}}, \mathbf{e}_u)$. We give a rigorous proof of these facts in the first two steps below. Remark that in general, it is not true that equation $\widehat{X}_r = h + \mathbf{e}(r) + \beta_2 L^0_r(\widehat{X})$ admits a unique solution for all $h\in\mathbb{R}$ and all $\mathbf{e}\in\mathcal{C}_{0\to 0}$. This makes the rigorous construction a bit delicate. We shall perform this construction in Step 1 below.

Step 1: *Construction of solutions to the skew equation driven by a single excursion.*

Consider $\mathcal{C}([0,\infty),\mathbb{R})$ the canonical space endowed with $\mathbb{W}$ the measure of the standard Brownian motion (starting at zero). Since the skew Brownian motion equation admits a unique strong solution, we know that there exists a solution $(\mathcal{X}_r)_{r\ge 0}$ to the equation

$$\mathcal{X}_r(\omega) = \omega_r + \beta_2 L^0_r\big(\mathcal{X}(\omega)\big) \tag{3.7}$$

as long as $\omega \in \widehat{\Omega}$ where $\widehat{\Omega}$ is some subset of $\mathcal{C}([0,\infty),\mathbb{R})$ with $\mathbb{W}(\widehat{\Omega}) = 1$. Moreover, we may assume that

$$\forall \omega \in \widehat{\Omega}, \forall r > 0 \qquad \mathcal{X}_r(\omega) = \mathcal{X}_r(\omega_{\cdot\wedge r}). \tag{3.8}$$

For $h > 0$, define $T^h(\omega) = \inf\{u > 0 \mid w_u = h\}$, and one can easily see that, for any $h > 0$, the process $\mathcal{X}_{\cdot\wedge T^h(\omega)}(\omega)$ is some functional of $\omega_{\cdot\wedge T^h(\omega)}$. (It can be seen, e.g., and up to restricting $\widehat{\Omega}$, using that (3.7) is a limit of S.D.E. with smooth coefficients; for this last point, see [5] Theorem 1.7.) With slight abuse of notation, we write $\mathcal{X}_{\cdot\wedge T^h(\omega)}(\omega) = \mathcal{X}_{\cdot\wedge T^h(\omega)}(\omega_{\cdot\wedge T^h(\omega)})$.

Define $\widehat{\Omega}^h = \{(\omega_{r\wedge T^h(\omega)})_{r\geq 0} \mid \omega \in \widehat{\Omega}\}$, by construction,

$$\mathbb{W}\big(\{\omega \in \mathcal{C}\big([0,\infty),\mathbb{R}\big) \mid \omega_{\cdot\wedge T^h(\omega)} \in \widehat{\Omega}^h\}\big) = \mathbb{W}(\widehat{\Omega}) = 1. \tag{3.9}$$

Now, we construct a solution of the skew equation driven by the "generic" excursion $\mathbf{e}$ and starting from an arbitrary value $h > 0$ as follows.

For $h > 0$ and $\mathbf{e} \in \mathcal{C}_{0\to 0}$:

- If $\mathbf{e}$ does not reach $-h$, we simply set

$$\widehat{X}_r(h,\mathbf{e}) = h + \mathbf{e}(r) \qquad \text{for } r \in \big[0, R(\mathbf{e})\big]. \tag{3.10}$$

- If $\mathbf{e}$ reaches $-h$ and $w^{-h} \in \widehat{\Omega}^h$, we denote

$$T^{-h}(\mathbf{e}) = \inf\big\{r \mid \mathbf{e}(r) = -h\big\} \quad \text{and} \quad \omega^{-h}_{\cdot} = \mathbf{e}\big(T^{-h}(\mathbf{e}) + \cdot\big) + h.$$

  Note that $\omega^{-h}_0 = 0$, that $\omega^{-h}_{\cdot} = \omega^{-h}_{\cdot\wedge T^h(\omega^{-h})}$, and that $R(\mathbf{e}) - T^{-h}(\mathbf{e}) = T^h(\omega^{-h})$. In this case, we set

$$\widehat{X}_r(h,\mathbf{e}) = \begin{cases} h + \mathbf{e}(r), & \text{for } r \leq T^{-h}(\mathbf{e}), \\ \mathcal{X}_{r-T^{-h}(\mathbf{e})}\big(\omega^{-h}_{\cdot\wedge T^h(\omega^{-h})}\big), & \text{for } r \in \big(T^{-h}(\mathbf{e}), R(\mathbf{e})\big]. \end{cases} \tag{3.11}$$

- If $\mathbf{e}$ reaches $-h$ and $w^{-h} \notin \widehat{\Omega}^h$, we arbitrarily set

$$\widehat{X}_r(h,\mathbf{e}) = h \qquad \text{for all } r \in \big[0, R(\mathbf{e})\big]. \tag{3.12}$$

Remark that (3.11) simply means that after the excursion reaches $-h$, we use the solution of the skew equation defined on the canonical space, treating the part of the excursion after $T^{-h}(\mathbf{e})$ as the realization of the Brownian motion when such construction is feasible.

Remark that, in cases where $\widehat{X}$ is defined by (3.10) and (3.11), one may write with a slight abuse of notation

$$\widehat{X}_r(h,\mathbf{e}) = \widehat{X}_r\big(h,\mathbf{e}(\cdot\wedge r)\big) \qquad \forall r > 0, \tag{3.13}$$

where we used (3.8).

Since for $\omega \in \widehat{\Omega}$, $\mathcal{X}(\omega)$ satisfies (3.7), and the local time of $\widehat{X}(\mathbf{e},h)$ does not increase before $T^{-h}(\mathbf{e})$, we have $\forall \mathbf{e} \in \mathcal{C}_{0\to 0}$, such that $\mathbf{e}$ reaches $-h$ with $\omega^{-h} \in \widehat{\Omega}^h$:

$$\widehat{X}_r(h,\mathbf{e}) = h + \mathbf{e}(r) + \beta_2 L^0_r\big(\widehat{X}(h,\mathbf{e})\big) \qquad \forall r \leq R(\mathbf{e}). \tag{3.14}$$

We now show that for $h > 0$ fixed, the $\mathbf{n}_{\beta_1}$ measure of the set of excursions $\mathbf{e}$ where (3.14) is possibly not satisfied is zero. Indeed, using the fundamental property (3.3) of the excursion measure,

$$\mathbf{n}_{\beta_1}\big(\mathbf{e}\big(T^{-h}(\mathbf{e}) + \cdot\big) + h \notin \widehat{\Omega}^h \mid \mathbf{e} \text{ reaches } -h\big) \tag{3.15}$$

$$= \mathbf{n}_{\beta_1}\big(\omega^{-h}_{\cdot\wedge T^h(\omega^{-h})} \notin \widehat{\Omega}^h \mid \mathbf{e} \text{ reaches } -h\big)$$

$$= \mathbb{W}\big(\{\omega \mid \omega_{\cdot\wedge T^h(\omega)} \notin \widehat{\Omega}^h\}\big) = 0, \tag{3.16}$$

where we have used (3.9).

We finally define the local time of $\widehat{X}(h, \mathbf{e})$ during the lifetime of the excursion in the following way:

- If $\mathbf{e}$ does not reach $-h$, we simply set

$$\ell(h, \mathbf{e}) = 0. \tag{3.17}$$

- If $\mathbf{e}$ reaches $-h$ and $w^{-h} \in \widehat{\Omega}^h$,

$$\ell(h, \mathbf{e}) = \begin{cases} 0, & \text{for } R(\mathbf{e}) \le T^{-h}(\mathbf{e}), \\ L^0_{R(\mathbf{e}) - T^{-h}(\omega^{-h})}\big(\mathcal{X}\big(\omega^{-h}_{\cdot\wedge T^h(\omega^{-h})}\big)\big), & \\ \qquad \text{for } r \in \big(T^{-h}(\mathbf{e}), R(\mathbf{e})\big]. & \end{cases} \tag{3.18}$$

- If $\mathbf{e}$ reaches $-h$ and $w^{-h} \notin \widehat{\Omega}^h$, we arbitrarily set

$$\ell(h, \mathbf{e}) = 0. \tag{3.19}$$

Step 2: *Proof of relation* (3.4).

For $s < T^\star$, we have $X^x_s > X^0_s$, and we deduce that the local time of X^x does not increase on the set $\{s < T^\star \mid X^0_s = 0\} = [0, T^\star) \setminus \bigcup_{u<U^\star}]\tau_{u-}, \tau_u[$. Hence, we have for $t < U^\star$ the relation

$$L^0_{\tau_t}(X^x) = \sum_{0<u\le t} \big[L^0_{\tau_u}(X^x) - L^0_{\tau_{u-}}(X^x)\big].$$

Now it is clear that (3.4) will be proved if we show that almost surely,

$$L^0_{\tau_u}(X^x) - L^0_{\tau_{u-}}(X^x) = \ell\big(X^x_{\tau_{u-}}, \mathbf{e}_u\big) \qquad \text{for all } u \text{ with } \tau_u - \tau_{u-} > 0. \tag{3.20}$$

In Appendix A.1, we show that it is possible to construct $\tilde{X}^x$ satisfying (3.20), and such that $\tilde{X}^x$ and X^x are indistiguishable up to time $T^\star$. This is enough to prove (3.4).

Step 3: *Law of* $\mathbf{e} \mapsto \ell(h, \mathbf{e})$ *under the excursion measure* ($h > 0$ *fixed*).

Let $a > 0$, using the fundamental property (3.3) of the excursion measure and the definitions (3.7), (3.18), we have

$$\frac{\mathbf{n}_{\beta_1}(\ell(h, \mathbf{e}) > a; \mathbf{e} \text{ reaches } -h)}{\mathbf{n}_{\beta_1}(\mathbf{e} \text{ reaches } -h)} = \mathbb{W}\big(L^0_{T^h(\omega)}(\mathcal{X}(\omega)) > a\big), \tag{3.21}$$

where $\mathcal{X}$ solves $\mathcal{X}_r = \omega_r + \beta_2 L_r^0(\mathcal{X})$, and $(\omega_r)_{r\geq 0}$ is a standard Brownian motion under $\mathbb{W}$. We can compute

$$\begin{aligned}
&\mathbb{W}\bigl(L^0_{T^h(\omega)}(\mathcal{X}(\omega)) > a\bigr)\\
&\quad = \mathbb{W}\bigl(\text{no excursion of } \mathcal{X}_\cdot \text{ crossed over } h + \beta_2 L^0_\cdot(\mathcal{X})\\
&\qquad\qquad \text{before } L^0_\cdot(\mathcal{X}) \text{ reaches } a\bigr)\\
&\quad = \exp\biggl(-\int_0^a \mathbf{n}_{\beta_2}\bigl[\mathbf{e} \text{ reaches level } (h+\beta_2 u)\bigr]\,du\biggr),
\end{aligned}$$

where in the last line we have used that the measure of excursion of the process $\mathcal{X}$ is $\mathbf{n}_{\beta_2}$, together with standard computations on Poisson processes. Recalling (3.1)–(3.2), we have $\mathbf{n}_{\beta_2}[\mathbf{e} \text{ reaches level } (h+\beta_2 u)] = \frac{1+\beta_2}{2(h+\beta_2 u)}$ and we easily get that

$$\mathbb{W}\bigl(L^0_{T^h(\omega)}(\mathcal{X}(\omega)) > a\bigr) = \biggl(1+\frac{\beta_2 a}{h}\biggr)^{-(1+\beta_2)/(2\beta_2)}. \tag{3.22}$$

Finally, we remark that

$$\begin{aligned}
&\mathbf{n}_{\beta_1}\bigl(\ell(h,\mathbf{e}) > a\bigr)\\
&\quad = \mathbf{n}_{\beta_1}\bigl(\ell(h,\mathbf{e}) > a; \mathbf{e} \text{ reaches } -h\bigr)\\
&\quad = \mathbf{n}_{\beta_1}(\mathbf{e} \text{ reaches } -h)\biggl(1+\frac{\beta_2 a}{h}\biggr)^{-(1+\beta_2)/(2\beta_2)} \qquad \text{using (3.21)–(3.22)},\\
&\quad = \frac{1-\beta_1}{2h}\biggl(1+\frac{\beta_2 a}{h}\biggr)^{-(1+\beta_2)/(2\beta_2)} \qquad \text{by (3.1)–(3.2)}.
\end{aligned}$$

The proof of the Proposition 2 is complete. □

3.3. *Representation of the "distance process" as a jump Markov process* (*proof of Theorem* 1 *and Corollary* 1).

PROOF OF THEOREM 1. Using (2.1) we have $Z_t^x = X_{\tau_t}^x = x + B_{\tau_t} + \beta_2 L^0_{\tau_t}(X^x)$. Now from (2.2), $0 = X^0_{\tau_t} = B_{\tau_t} + \beta_1 t$, and we deduce

$$Z_t^x = x - \beta_1 t + \beta_2 L^0_{\tau_t}(X^x).$$

Hence, relation (3.4) yields to (2.4). Equation (2.5) appears as an immediate consequence of (3.5). □

PROOF OF COROLLARY 1. Representation (2.6) is just the usual way to rewrite the stochastic differential equation with jumps: we transform (2.4) into

(2.6) by defining $\mu(du, da)$ as the sum of Dirac masses $\sum_{u<U^\star, Z^x_{u-}\neq Z^x_u} \delta_{(u,\Delta Z^x_u)}$, and (2.7) appears as a direct consequence of (2.5) and $\Delta Z^x_u = \beta_2 \ell(Z^x_{u-}, \mathbf{e}_u)$. □

REMARK 5. From Corollary 1 we deduce that the rate for the jumps of Z^x is given by

$$\mathbb{P}\big(Z^x \text{ jumps on } [t, t+dt] \mid Z^x_{t-} = h\big) = \frac{1-\beta_1}{2h}\, dt. \tag{3.23}$$

Conditionally on $Z^x_{t-} = h$, the law of the jumps is given by

$$\frac{1+\beta_2}{2\beta_2 h}\left(1 + \frac{a}{h}\right)^{-\gamma} 1_{(0,\infty)}(a)\, da. \tag{3.24}$$

Remark that the jumps intensity of the process Z^x is proportional to $1/Z^x$. If a jump occurs at time t, then the size of the jump is proportional to Z^x_{t-}. Informally, $\Delta Z^x_t \overset{\text{law}}{=} Z^x_{t-} J$ where J has the density (3.24) with $h = 1$.

REMARK 6. As a consequence of Remark 5, the number of jumps on $[0, t]$ is finite for $t < U^\star$ since the jump activity is bounded when $Z^x > \varepsilon$.

From Corollary 1 , we can deduce that $(Z^x_t)_{t<U^\star}$ is a local submartingale (resp., supermartingale) if $\beta_2 > \beta_1$ (resp., $\beta_2 < \beta_1$). Indeed, with simple computations $\int_{\mathcal{C}_{0\to 0}} \beta_2 \ell(h, \mathbf{e})\, d\mathbf{n}_{\beta_1}(\mathbf{e}) = \int_0^\infty a\nu(h, da) = \beta_2 \frac{1-\beta_1}{1-\beta_2}$ is independent of h. Hence, we can write

$$\begin{aligned} Z^x_t &= x - \beta_1 t + \beta_2 \frac{1-\beta_1}{1-\beta_2} t \\ &\quad + \sum_{u\le t} \beta_2 \ell\big(Z^x_{u-}, \mathbf{e}_u\big) - \int_0^t \int_{\mathcal{C}_{0\to 0}} \beta_2 \ell\big(Z^x_{u-}, \mathbf{e}\big)\, d\mathbf{n}_{\beta_1}(\mathbf{e})\, du \\ &:= x + \frac{\beta_2 - \beta_1}{1-\beta_2} t + M_t, \end{aligned}$$

where $(M_t)_t$ is a compensated jump process, and hence a local martingale. Remark that for $\beta_1 = \beta_2$ the process Z^x is a local martingale.

4. Hitting time of the two skew Brownian motions (case $0 < \beta_1, \beta_2 < 1$, $\beta_1 > \frac{\beta_2}{1+2\beta_2}$). In this section we prove the results of Section 2 corresponding to the situation $0 < \beta_1, \beta_2 < 1$, $\beta_1 > \frac{\beta_2}{1+2\beta_2}$. We start by giving a new proof of Theorem 2 [2, 5] relying on the dynamic of the process Z^x.

4.1. *Finiteness of the hitting time* (*proof of Theorem* 2). Let us show that if $\beta_1 > \frac{\beta_2}{1+2\beta_2}$, then $U^\star < \infty$ almost surely. We apply Ito's formula to the semimartingale $\ln(Z_t^x)$ for $t < U^\star$,

$$\begin{aligned}\ln(Z_t^x) &= \ln(x) + \int_0^t \frac{dZ_u^x}{Z_{u-}^x} + \sum_{u\le t}\left\{\ln(Z_{u-}^x + \Delta Z_u^x) - \ln(Z_{u-}^x) - \frac{\Delta Z_u^x}{Z_{u-}^x}\right\}\\ &= \ln(x) - \int_0^t \frac{\beta_1 du}{Z_u^x} + \sum_{u\le t}\ln\left(1 + \frac{\Delta Z_u^x}{Z_{u-}^x}\right) \qquad \text{by (2.4).}\end{aligned} \tag{4.1}$$

Consider the jump process

$$\mathcal{J}_t = \sum_{u\le t}\ln\left(1+\frac{\Delta Z_u^x}{Z_{u-}^x}\right) = \sum_{u\le t}\ln\left(1+\frac{\beta_2\ell(Z_{u-}^x,\mathbf{e}_u)}{Z_{u-}^x}\right).$$

Its compensator can be easily computed using (2.6) and (2.7). Indeed, we have

$$\int_{\mathcal{C}_{0\to 0}}\ln\left(1+\frac{\beta_2\ell(h,\mathbf{e})}{h}\right)d\mathbf{n}_{\beta_1}(\mathbf{e}) = \int_0^\infty \ln\left(1+\frac{a}{h}\right)\nu(h,da) = \frac{(1-\beta_1)\beta_2}{(1+\beta_2)h},$$

and hence $\widetilde{\mathcal{J}}_t = \mathcal{J}_t - \frac{(1-\beta_1)\beta_2}{(1+\beta_2)}\int_0^t \frac{du}{Z_u^x}$ is a compensated jump process. Using (4.1), we can write

$$\ln(Z_t^x) = \ln(x) + \theta\int_0^t \frac{du}{Z_u^x} + \widetilde{\mathcal{J}}_t, \tag{4.2}$$

with $\theta = \frac{(1-\beta_1)\beta_2}{(1+\beta_2)} - \beta_1 = \frac{\beta_2-\beta_1(1+2\beta_2)}{1+\beta_2} < 0$.

The process $\widetilde{\mathcal{J}}$ is a quadratic pure jumps local martingale, and its bracket is clearly given by

$$[\widetilde{\mathcal{J}},\widetilde{\mathcal{J}}]_t = \sum_{u\le t}\ln\left(1+\frac{\beta_2\ell(Z_{u-}^x,\mathbf{e}_u)}{h}\right)^2.$$

With the help of (2.7), we compute

$$\int_{\mathcal{C}_{0\to 0}}\ln\left(1+\frac{\beta_2\ell(h,\mathbf{e})}{h}\right)^2 d\mathbf{n}_{\beta_1}(\mathbf{e}) = \int_0^\infty \ln\left(1+\frac{a}{h}\right)^2\nu(h,da) = \frac{c}{h}$$

for some constant $c > 0$. We deduce $\langle\widetilde{\mathcal{J}},\widetilde{\mathcal{J}}\rangle_t = c\int_0^t \frac{du}{Z_u^x}$. Using (4.2), we get

$$\frac{\ln(Z_t^x)}{\langle\widetilde{\mathcal{J}},\widetilde{\mathcal{J}}\rangle_t} = \frac{\ln(x)}{\langle\widetilde{\mathcal{J}},\widetilde{\mathcal{J}}\rangle_t} + \frac{\theta}{c} + \frac{\widetilde{\mathcal{J}}_t}{\langle\widetilde{\mathcal{J}},\widetilde{\mathcal{J}}\rangle_t}. \tag{4.3}$$

Suppose now we are on the event $\{U^\star = \infty\}$. Then, either $\{\int_0^\infty \frac{ds}{Z_s^x} < \infty\}$ or $\{\int_0^\infty \frac{ds}{Z_s^x} = \infty\}$.

On the set $\{\int_0^\infty \frac{ds}{Z_s^x} = \infty\}$, we have $\langle \widetilde{\mathcal{J}}, \widetilde{\mathcal{J}} \rangle_\infty = \infty$, and using Kronecker's lemma (see Lemma 3 in the Appendix A.2),

$$\frac{\ln(Z_t^x)}{\langle \widetilde{\mathcal{J}}, \widetilde{\mathcal{J}} \rangle_t} \xrightarrow{t\to\infty} \frac{\theta}{c} < 0.$$

Thus, $\mathbb{P}$ a.s. on the set $\langle \widetilde{\mathcal{J}}, \widetilde{\mathcal{J}} \rangle_\infty = \infty$, there exist t_0 and $\eta > 0$ such that $\ln(Z_t^x) \le -\eta \int_0^t \frac{du}{Z_u^x}$ for all $t \ge t_0$. In view of Lemma 4 in Appendix A.3, this is not possible.

On the set $\{\int_0^\infty \frac{du}{Z_u^x} < \infty\} = \{\langle \widetilde{\mathcal{J}}, \widetilde{\mathcal{J}} \rangle_\infty < \infty\}$, using Kronecker's lemma and (4.3), we see that $\ln(Z_t^x)$ converges as $t \to \infty$. This is clearly in contradiction with the finiteness of the integral $\int_0^\infty \frac{du}{Z_u^x}$.

Hence, by contradiction, we have proved that $U^\star < \infty$ a.s., and thus Theorem 2 is shown.

4.2. *Computation of the law of the hitting time* (*proof of Theorem* 3). The main tool in order to compute the law of $U^\star$ is Z^x. Let us denote A the generator of the process Z^x given by

$$Af(h) = -\beta_1 f'(h) + \int_0^\infty [f(h+a) - f(h)]\nu(h, da), \tag{4.4}$$

$$= -\beta_1 f'(h) + \int_0^\infty [f(h+a) - f(h)]\frac{\kappa}{h^2}\left(1 + \frac{a}{h}\right)^{-\gamma} da \tag{4.5}$$

for $h > 0$ and f an element of $\mathcal{C}^1(0,\infty)$ bounded on $[0,\infty)$. Using representation (2.6)–(2.7), it is clear that $f(Z_t^x) - \int_0^t Af(Z_u^x)\,du$ with $t < U^\star$ is a compensated jump process and thus a local martingale.

Before turning to the heart of the proof, we need to prove several lemmas in the next section.

4.2.1. *Dynkin's formula*. Our first lemma shows a "Dynkin's formula" that relates the generator of the process with $U^\star$.

For $\lambda > 0$ we denote $u_\lambda(x) = \mathbb{E}_x[e^{-\lambda U^\star}]$ where the subscript x emphasizes the dependence upon the starting point of the process Z^x.

LEMMA 1 (Dynkin's formula). (1) *The function $x \mapsto u_\lambda(x)$ is $\mathcal{C}^\infty(0,\infty)$ and satisfies* $\lim_{x\to 0} u_\lambda(x) = 1$, *and* $|u_\lambda(x)| \le e^{-\lambda x/\beta_1}$. *Moreover, the derivatives of u_λ decay exponentially near ∞ and satisfy $x^k u_\lambda^{(k)}(x) = O(1)$ near* 0 (*for any $k \ge 0$*).

(2) *The function u_λ is solution to the integro-differential equation*

$$Au_\lambda(x) = \lambda u_\lambda(x) \qquad \textit{for all } x > 0. \tag{4.6}$$

PROOF. (1) First we show that $x \mapsto u_\lambda(x)$ is a smooth function. Denote $(U_n^x)_n$ the successive jumps of $(Z_u^x)_{u<U^{\star,x}}$, where again we stress the dependence upon x as we write $U^{\star,x} = U^\star = \sup_n U_n^x$.

Since Z^x evolves with the constant negative drift $-\beta_1\,dt$ and jumps with the infinitesimal probability (3.23), we can easily compute the law of U_1^x.

$$\mathbb{P}(U_1^x \ge t) = \left(1 - \frac{\beta_1 t}{x}\right)^{(1-\beta_1)/(2\beta_1)}. \tag{4.7}$$

As a result the law of U_1^x is equal to the law of xU_1^1. Moreover, the law of the jump of Z_t^x is proportional to Z_{t-}^x (see Remark 5), and this implies that the law of $Z^x_{U_1^x}$ and $xZ^1_{U_1^1}$ are equal. Consequently, we deduce that the processes $(Z_t^x)_{t\le U_1^x}$ and $(xZ^1_{t/x})_{t\le xU_1^1}$ have the same law. Then, by induction, it can be seen that the two processes (Z_t^x) and $(xZ^1_{t/x})$ have the same law up to their respective nth jump time. Letting n tend to infinity, we deduce

$$(Z_t^x)_{t<U^{\star,x}} \overset{\text{law}}{=} (xZ^1_{t/x})_{t<xU^{\star,1}} \quad\text{and}\quad U^{\star,x} = xU^{\star,1}. \tag{4.8}$$

Now, by definition $u_\lambda(x) = \mathbb{E}[e^{-\lambda U^{\star,x}}] = \mathbb{E}[e^{-\lambda x U^{\star,1}}]$. In turn, $x \mapsto u_\lambda(x)$ is clearly a $\mathcal{C}^\infty(0,\infty)$ function. Moreover, using $U^{\star,1} < \infty$ almost surely, and Lebesgue's theorem we get $u_\lambda(x) \xrightarrow{x\to 0} \mathbb{E}[e^0] = 1$.

Next, by (2.4) and the positivity of the jumps of Z^x, one must have $U^{\star,x} \ge x/\beta_1$ a.s. and thus $u_\lambda(x) = \mathbb{E}[e^{-\lambda U^{\star,x}}] \le e^{-\lambda x/\beta_1}$. In the same way, from $u_\lambda^{(k)}(x) = \mathbb{E}[(-\lambda U^{\star,1})^k e^{-\lambda x U^{\star,1}}]$ we easily deduce the exponential decay of $u_\lambda^{(k)}$ near ∞. Finally, the boundedness $x^k u_\lambda^{(k)}(x)$ is clear too from the latter representation of $u_\lambda^{(k)}(x)$.

(2) We now prove equation (4.6). Recall notation (2.10) and consider the martingale $(M_t)_{t\ge 0}$ defined as $M_t = \mathbb{E}[e^{-\lambda U^{\star,x}} \mid \mathcal{G}_{t\wedge U^{\star,x}}]$ for $t \ge 0$. We can write

$$\begin{aligned} M_t 1_{\{t<U^{\star,x}\}} &= \mathbb{E}[e^{-\lambda U^{\star,x}} \mid \mathcal{G}_{t\wedge U^{\star,x}}]1_{\{t<U^{\star,x}\}} \\ &= \mathbb{E}[e^{-\lambda U^{\star,x}} 1_{\{t<U^{\star,x}\}} \mid \mathcal{G}_{t\wedge U^{\star,x}}]1_{\{t<U^{\star,x}\}} \\ &= \mathbb{E}[e^{-\lambda U^{\star,x}} 1_{\{t<U^{\star,x}\}} \mid \mathcal{G}_t]1_{\{t<U^{\star,x}\}}. \end{aligned} \tag{4.9}$$

For $t > 0$, denote $U^\star(Z^x_{t+\cdot}) = \inf\{s \ge 0 \mid Z^x_{t+s} \le 0\}$, and remark that on the set $U^{\star,x} > t$ we have $U^\star(Z^x_{t+\cdot}) = U^{\star,x} - t$ and thus $e^{-\lambda U^{\star,x}}1_{\{t<U^{\star,x}\}} = e^{-\lambda[U^\star(Z^x_{t+\cdot})+t]}1_{\{t<U^{\star,x}\}}$. As a result, using (4.9) with the Markov property at time t, we deduce

$$M_t 1_{\{t<U^{\star,x}\}} = u_\lambda(Z_t^x)e^{-\lambda t}1_{\{t<U^{\star,x}\}}. \tag{4.10}$$

We now consider $M_t 1_{\{t\ge U^{\star,x}\}}$. We can write $M_t 1_{\{t\ge U^{\star,x}\}} = \mathbb{E}[e^{-\lambda U^{\star,x}} \mid \mathcal{G}_{t\wedge U^{\star,x}}]1_{\{t\ge U^{\star,x}\}} = \mathbb{E}[e^{-\lambda U^{\star,x}} \mid \mathcal{G}_{U^{\star,x}}]1_{\{t\ge U^{\star,x}\}} = e^{-\lambda U^{\star,x}}1_{\{t\ge U^{\star,x}\}}$. Using $u_\lambda(Z^x_{U^{\star,x}}) = 1$ and (4.10), we deduce

$$M_t = u_\lambda(Z^x_{t\wedge U^{\star,x}})e^{-\lambda(t\wedge U^{\star,x})}. \tag{4.11}$$

Relation (4.11) shows that $(u_\lambda(Z^x_{t\wedge U^{\star,x}})e^{-\lambda(t\wedge U^{\star,x})})_{t\geq 0}$ is a martingale. We apply Ito's formula to the process $t\mapsto u_\lambda(Z^x_t)e^{-\lambda t}$ and find for $t<U^{\star,x}$,

$$u_\lambda(Z^x_t)e^{-\lambda t}=u_\lambda(x)+\int_0^t e^{-\lambda s}\bigl[Au_\lambda(Z^x_s)-\lambda u_\lambda(Z^x_s)\bigr]\,ds+N_t, \tag{4.12}$$

where $N_t=\sum_{0\leq s\leq t}[u_\lambda(Z^x_{s-}+\Delta_s Z^x)-u_\lambda(Z^x_{s-})]-\int_0^t\int_0^\infty[u_\lambda(Z^x_{s-}+a)-u_\lambda(Z^x_{s-})]\nu(Z^x_s,da)\,ds$ is a local martingale. By uniqueness of the Doob–Meyer decomposition, the predictive finite variation part is zero in representation (4.12), and we deduce

$$\int_0^t e^{-\lambda s}\bigl[Au_\lambda(Z^x_s)-\lambda u_\lambda(Z^x_s)\bigr]\,ds=0 \qquad \forall t<U^{\star,x} \text{ almost surely.}$$

Differentiating the latter and using the almost sure continuity of $s\mapsto Z^x_s$ at zero, we finally obtain $Au_\lambda(x)=\lambda u_\lambda(x)$ for all $x>0$. □

REMARK 7. In the proof of Lemma 1 we have shown that the laws of the processes $t\mapsto Z^x_t$ and $t\mapsto xZ^1_{t/x}$ coincide until they reach zero. This is not surprising, since one can show using (2.7) that the compensator of the point processes $t\mapsto Z^x_t+\beta_1 t$ and $t\mapsto x(Z^1_{t/x}+\beta_1 t/x)$ are the same. For point processes with finite intensities, it is known that the compensator characterizes the law of the process; see Theorem 1.26 in Chapter III of [11].

We now show that the integro-differential equation (4.6) can be transformed into an ordinary differential equation. Related techniques were used in [7] for computing the ruin time of Levy processes. In [7], a crucial fact is that the generator of a Levy process acts as a multiplier in the Fourier domain. Such simplifications in the Fourier domain do not occur for the generator of the process Z^x; however, the multiplicative invariance of the process (see Remark 7) suggests the use of the Mellin transform.

LEMMA 2. *The function $x\mapsto u_\lambda(x)$ is the solution to*

$$\beta_1 x u''_\lambda(x)+u'_\lambda(x)\bigl(\lambda x+\beta_1\xi^\star\bigr)-\lambda(\gamma-2)u_\lambda(x)=0 \qquad \text{for all } x\in(0,\infty),$$

where $\xi^\star=\frac{1}{2\beta_1}-\frac{1}{2\beta_2}$ and the constant γ was defined in Corollary 1.

PROOF. The main idea is that the generator A acts as a kind of multiplier for the Mellin transform. Let us recall that for $f:[0,\infty)\to\mathbb{R}$, one defines the Mellin transform of f as

$$\mathcal{M}[f](\xi)=\int_0^\infty x^{\xi-1}f(x)\,dx$$

for all $\xi\in\mathbb{C}$ such that the latter integral is well defined. It is clear that if f is bounded and with exponential decay near ∞, then $\xi\mapsto\mathcal{M}[f](\xi)$ is well defined and holomorphic on the half plane $\{\xi\in\mathbb{C}\mid \mathrm{Re}(\xi)>0\}$.

For such functions f, we recall the four following properties which are easily derived from the definition of the Mellin transform:

$$\mathcal{M}[x \mapsto f(x(1+y))](\xi) = (1+y)^{-\xi}\mathcal{M}[f](\xi) \qquad \text{for } \operatorname{Re}(\xi) > 0; \tag{4.13}$$

$$\mathcal{M}[x \mapsto f(x)/x](\xi) = \mathcal{M}[f](\xi - 1) \qquad \text{for } \operatorname{Re}(\xi) > 1; \tag{4.14}$$

$$\mathcal{M}[f'](\xi) = (1-\xi)\mathcal{M}[f](\xi - 1) \qquad \text{if } f \in \mathcal{C}^1(0,\infty) \text{ and } \operatorname{Re}(\xi) > 1; \tag{4.15}$$

$$\mathcal{M}[x \mapsto xf'(x)](\xi) = -\xi\mathcal{M}[f](\xi) \qquad \text{if } f \in \mathcal{C}^1(0,\infty) \text{ and } \operatorname{Re}(\xi) > 0. \tag{4.16}$$

Now, using the expression of the generator (4.5) with a simple change of variable, we have

$$Af(x) = -\beta_1 f'(x) + \int_0^\infty [f(x(1+y)) - f(x)]\frac{\kappa}{x}(1+y)^{-\gamma}\,dy.$$

Using Fubini's theorem and properties (4.13)–(4.15) we deduce that for $\operatorname{Re}(\xi) > 1$,

$$\begin{aligned}
\mathcal{M}[Af](\xi) &= \beta_1(\xi - 1)\mathcal{M}[f](\xi - 1) \\
&\qquad + \left(\kappa\int_0^\infty [(1+y)^{-\xi+1} - 1](1+y)^{-\gamma}\,dy\right)\mathcal{M}[f](\xi - 1) \\
&= \left[\beta_1(\xi - 1) + \frac{\kappa}{\xi + \gamma - 2} - \frac{\kappa}{\gamma - 1}\right]\mathcal{M}[f](\xi - 1) \\
&= Q(\xi)\mathcal{M}[f](\xi - 1)
\end{aligned}$$

with $Q(\xi)$ being the rational function $Q(\xi) = \frac{\beta_1(\xi-1)(\xi-\xi^\star)}{(\xi+\gamma-2)}$ and $\xi^\star = \frac{1}{2\beta_1} - \frac{1}{2\beta_2}$.

Now we turn back to the solution of the equation $Au_\lambda = \lambda u_\lambda$ and apply the Mellin transform on both sides of this equality. We deduce

$$Q(\xi)\mathcal{M}[u_\lambda](\xi - 1) = \lambda\mathcal{M}[u_\lambda](\xi) \qquad \forall \xi \text{ with } \operatorname{Re}(\xi) > 1.$$

From the definition of Q, we obtain

$$\beta_1(\xi - 1)(\xi - \xi^\star)\mathcal{M}[u_\lambda](\xi - 1) = \lambda(\xi + \gamma - 2)\mathcal{M}[u_\lambda](\xi),$$

and using (4.15)–(4.16) this equation can be transformed into

$$\begin{aligned}
&-\beta_1\xi\mathcal{M}[u'_\lambda(x)](\xi) + \beta_1\xi^\star\mathcal{M}[u'_\lambda](\xi) \\
&\qquad = \lambda(\gamma - 2)\mathcal{M}[u_\lambda](\xi) - \lambda\mathcal{M}[x \mapsto xu'_\lambda(x)](\xi), \qquad \operatorname{Re}(\xi) > 1.
\end{aligned}$$

We apply again (4.16) with the choice $f = u'_\lambda$. Remark that, even if $f = u'_\lambda$ is not bounded near 0, it is easy to see that property (4.16) is still valid, using $u'_\lambda(x) \overset{x\to 0}{=} O(1/x)$ and $\operatorname{Re}(\xi) > 1$. We deduce the following relation for all ξ with $\operatorname{Re}(\xi) > 1$:

$$\begin{aligned}
&\beta_1\mathcal{M}[x \mapsto xu''_\lambda(x)](\xi) + \beta_1\xi^\star\mathcal{M}[u'_\lambda](\xi) \\
&\qquad = \lambda(\gamma - 2)\mathcal{M}[u_\lambda](\xi) - \lambda\mathcal{M}[x \mapsto xu'_\lambda(x)](\xi).
\end{aligned} \tag{4.17}$$

Since equality (4.17) holds true for any ξ in the half plane $\mathrm{Re}(\xi) > 1$, we may invert the Mellin transform, and we deduce the lemma. $\square$

4.2.2. *Proof of Theorem* 3. By Lemma 2, the function $u_\lambda(x) = \mathbb{E}_x[e^{-\lambda U^\star}]$ is the solution to the equation $\beta_1 x u_\lambda''(x) + u_\lambda'(x)(\lambda x + \beta_1 \xi^\star) - \lambda(\gamma - 2)u_\lambda(x) = 0$, for all $x \in (0, \infty)$. Then, if we define

$$w_\lambda(x) = \frac{\lambda}{\beta_1} u_\lambda\left(-\frac{x\beta_1}{\lambda}\right) \qquad \text{for } x < 0, \tag{4.18}$$

it is simple to check that w_λ is solution to Kummer's equation

$$x w_\lambda''(x) + w_\lambda'(x)[\xi^\star - x] + (\gamma - 2) w_\lambda(x) = 0 \qquad \text{for all } x \in (-\infty, 0). \tag{4.19}$$

Moreover, from Lemma 1, this solution w_λ satisfies the boundary condition $\lim_{x\to 0} w_\lambda(x) = \frac{\lambda}{\beta_1}$ and $|w_\lambda(x)| \le \frac{\lambda}{\beta_1} e^{-|x|}$ for $x < 0$.

We know from [1] that Kummer's equation (4.19) admits two independent solutions,

$$y_1(x) = M(2 - \gamma, \xi^\star, x), \tag{4.20}$$

$$\begin{aligned} y_2(x) &= e^x U(\xi^\star + \gamma - 2, \xi^\star, -x) \\ &= \frac{1}{\Gamma(\xi^\star + \gamma - 2)} \int_1^\infty e^{xt} t^{1-\gamma} (t-1)^{\xi^\star + \gamma - 3}\, dt, \end{aligned} \tag{4.21}$$

where M and U are the confluent hypergeometric functions; for the definition of these functions see formulas 13.1.2 and 13.1.3 in Chapter 13 of [1], and see formula 13.2.6 of [1] for the integral representation of U. The asymptotic behavior of the fundamental solutions can be found, using equation 13.1.5 in [1],

$$y_1(x) \sim_{x\to-\infty} \frac{\Gamma(\xi^\star)}{\Gamma(\xi^\star + \gamma - 2)} (-x)^{\gamma - 2},$$

and using equation 13.5.2 in [1],

$$y_2(x) \sim_{x\to-\infty} e^x (-x)^{2-\gamma-\xi^\star}.$$

From the exponential decay of w_λ, we deduce that w_λ is proportional to y_2. Hence, using (4.21) we get

$$w_\lambda(x) = c y_2(x) = \frac{c}{\Gamma(\xi^\star + \gamma - 2)} \int_1^\infty e^{xt} (t-1)^{\xi^\star + \gamma - 3} t^{1-\gamma}\, dt$$

for some $c \in \mathbb{R}$.

The condition $\beta_1 > \frac{\beta_2}{1 + 2\beta_2}$, equivalent to $\xi^\star < 1$, is sufficient for the finiteness of $U(\xi^\star + \gamma - 2, \xi^\star, 0)$ and $U(\xi^\star + \gamma - 2, \xi^\star, 0) = \frac{\Gamma(1-\xi^\star)}{\Gamma(\gamma - 1)}$ (see formulas 13.5.10–13.5.12 in [1]). We deduce that $c = \frac{\Gamma(\gamma-1)}{\Gamma(1-\xi^\star)} \frac{\lambda}{\beta_1}$ and

$$w_\lambda(x) = \frac{\Gamma(\gamma - 1)}{\Gamma(1 - \xi^\star)\Gamma(\xi^\star + \gamma - 2)} \frac{\lambda}{\beta_1} \int_1^\infty e^{xt} (t-1)^{\xi^\star + \gamma - 3} t^{1-\gamma}\, dt. \tag{4.22}$$

From (4.18) and (4.22), we deduce

$$
\begin{aligned}
&\mathbb{E}_x\big[e^{-\lambda U^\star}\big] \\
&\quad = u_\lambda(x) \\
&\quad = \frac{\Gamma(\gamma-1)}{\Gamma(1-\xi^\star)\Gamma(\xi^\star+\gamma-2)}\frac{\lambda}{\beta_1}\int_1^\infty e^{-\lambda x t/\beta_1}(t-1)^{\xi^\star+\gamma-3}t^{1-\gamma}\,dt \\
&\quad = \frac{\Gamma(\gamma-1)}{\Gamma(1-\xi^\star)\Gamma(\xi^\star+\gamma-2)}\int_{x/\beta_1}^\infty e^{-\lambda u}\left(\frac{\beta_1 u}{x}-1\right)^{\xi^\star+\gamma-3}\left(\frac{\beta_1 u}{x}\right)^{1-\gamma}du,
\end{aligned}
$$

where the last line comes from a simple change of variable.

Identification of the Laplace's transform shows that the law of $U^\star$ admits the density $\frac{p_{U^\star}(x,du)}{du} = \frac{\Gamma(\gamma-1)}{\Gamma(1-\xi^\star)\Gamma(\xi^\star+\gamma-2)}(\frac{\beta_1 u}{x}-1)^{\xi^\star+\gamma-3}(\frac{\beta_1 u}{x})^{1-\gamma}1_{\{u\geq x/\beta_1\}}$. Expression (2.8) is obtained with simple algebra and recalling $\gamma = \frac{1+3\beta_2}{2\beta_2}$.

5. Case of negative skewness coefficients. In this section, we give sketches of the proofs of the results of Section 2 corresponding to the situations where one of the skewness parameters may be negative.

5.1. *Proof of Proposition* 1. We will use the following comparison result (see Theorem 3.1 in [5]): if $X^{x,\beta}$ and $X^{x,\beta'}$ are two solutions of the skew Brownian motion equation (1.1) with $-1<\beta<\beta'<1$, then $\mathbb{P}(X_t^{x,\beta}\leq X_t^{x,\beta'}, \forall t\geq 0)=1$.

In the case $\beta_1>0$, choose $\beta=\beta_2$ and β' such that $0<\beta'$ and $\frac{\beta'}{1+2\beta'}<\beta_1$ (this is always possible since $\beta_1>0$). The comparison $X^{x,\beta_2}\leq X^{x,\beta'}$ ensures that the hitting time of X^{x,β_2} and X^{0,β_1} is smaller than the hitting time of $X^{x,\beta'}$ and X^{0,β_1}. The result of Proposition 1, in the case $\beta_2<0<\beta_1$, follows then from Theorem 2.

In the case $\beta_1<0<\beta_2$, it is clear that $X^{x,\beta_2}-X^{0,\beta_1}$ remains greater than x and thus $T^\star=\infty$ almost surely.

5.2. *Sketches of the proof of Theorem* 4. The proof of Theorem 4 follows the same route as the proof of Theorem 3: we first determine the dynamic of Z^x and then characterize the solution of the associated Dynkin equation with help of the Mellin transform. Finally, we identify the Laplace transform of the law of $U^\star$ with an explicit solution of Kummer's equation. Let us give some more details.

In the case $-1<\beta_2<0<\beta_1<1$, by a proof similar to the one of Proposition 2, we can show that

$$
L^0_{\tau_t}(X^x) = \sum_{0<u\leq t}\ell\big(X^x_{\tau_{u-}}, \mathbf{e}_u\big),
$$

where the law of the functional ℓ under the excursion measure is

$$
\mathbf{n}_{\beta_1}\big(\ell(h,\mathbf{e})\geq a\big) = \frac{1-\beta_1}{2h}\left(1+\frac{\beta_2 a}{h}\right)^{-(1+\beta_2)/(2\beta_2)}1_{[0,h/|\beta_2|]}(a) \qquad \forall a>0.
$$

We deduce that the dynamic of the process $Z_t^x = X_{\tau_t}^x$ is as follows:

$$\begin{aligned} Z_t^x &= x - \beta_1 t + \sum_{0<u\le t} \beta_2 \ell(Z_{u-}^x, \mathbf{e}_u) \\ &= x - \beta_1 t - \int_{[0,t]\times(0,\infty)} a\mu(du, da), \end{aligned}$$

where the compensator of the random measure $\mu(du, da)$ is $du \times \nu(Z_{u-}^x, da)$ with

$$\nu(h, da) = \frac{|\kappa|}{h^2}\left(1 - \frac{a}{h}\right)^{-\gamma} 1_{[0,h]}(a)\, da, \qquad \kappa = \frac{(1-\beta_1)(1+\beta_2)}{4\beta_2}$$

and

$$\gamma = \frac{1+3\beta_2}{2\beta_2}.$$

Remark that both the drift and jumps of Z^x are negative, yielding to an almost sure finite hitting time of the level 0. Especially, it is clear that the support of $U^\star$ is included in $[0, \frac{x}{\beta_1}]$. The generator of the process Z^x now writes

$$Af(h) = -\beta_1 f'(h) + \int_0^h [f(h-a) - f(h)] \frac{|\kappa|}{h^2}\left(1 - \frac{a}{h}\right)^{-\gamma} da \qquad \text{for } h > 0.$$

Consider $u_\lambda(x) = \mathbb{E}[e^{-\lambda U^{\star,x}}]$ where we use the notation of the proof of Lemma 1. Exactly as in Lemma 1, we can prove that u_λ satisfies Dynkin's formula $Au_\lambda = \lambda u_\lambda$. Moreover, as in Lemma 1, we can check that $U^{\star,x}$ and $xU^{\star,1}$ have the same law and hence $u_\lambda(x) = \mathbb{E}[e^{-\lambda x U^{\star,1}}]$. As a result, the function $x \mapsto u_\lambda(x)$ is the Laplace transform of the law of a random variable with the compact support $[0, \frac{\lambda}{\beta_1}]$.

Then, the integro-differential equation $Au_\lambda = \lambda u_\lambda$ can be transformed to an ordinary differential equation by applying Mellin's transform as in the proof of Lemma 2. One finds exactly the same equation as in the case of positive skewness coefficients,

$$\beta_1 x u_\lambda''(x) + u_\lambda'(x)(\lambda x + \beta_1 \xi^\star) - \lambda(\gamma - 2) u_\lambda(x) = 0 \qquad \text{for all } x \in (0,\infty), \tag{5.1}$$

with $\xi^\star = \frac{1}{2\beta_1} - \frac{1}{2\beta_2}$ and $\gamma = \frac{1+3\beta_2}{2\beta_2}$.

Let us stress that some additional technical difficulties arise for the application of Mellin's transform when $\beta_2 < 0$. Indeed, contrary to the case of Section 4, we cannot show the exponential decay of u_λ (due to the different shape of the support of the law of $U^\star$). In order to define and manipulate Mellin's transform of u_λ on some sufficiently large strip, some preliminary bounds on the decay of u_λ have to be established. Lemma 5 of the Appendix ensures that all the necessary computations are allowed.

As in Section 4.2.2, if one sets $w_\lambda(x) = \frac{\lambda}{\beta_1} u_\lambda(-\frac{x\beta_1}{\lambda})$ for $x < 0$, then w_λ is solution to Kummer's equation (4.19). A fundamental system of solution to Kummer's equation is given by y_1 and y_2; see (4.20)–(4.21). For $\beta_2 < 0$, the solution y_1 admits an integral representation (see Formula 13.2.1 in [1]),

$$y_1(x) = \frac{\Gamma(\xi^\star)}{\Gamma(\xi^\star + \gamma - 2)\Gamma(2-\gamma)} \int_0^1 e^{xt} t^{1-\gamma} (1-t)^{\xi^\star + \gamma - 3}\, dt. \tag{5.2}$$

Comparing (4.21) and (5.2) with the fact that u_λ is the Laplace transform of some function with compact support, we deduce that w_λ is proportional to y_1. From the condition $w_\lambda(0) = \frac{\lambda}{\beta_1} u_\lambda(0) = \frac{\lambda}{\beta_1}$, we get

$$w_\lambda(x) = \frac{\lambda}{\beta_1} \frac{\Gamma(\xi^\star)}{\Gamma(\xi^\star + \gamma - 2)\Gamma(2-\gamma)} \int_0^1 e^{xt} t^{1-\gamma} (1-t)^{\xi^\star + \gamma - 3}\, dt. \tag{5.3}$$

Using $\mathbb{E}_x[e^{-\lambda U^\star}] = u_\lambda(x) = \frac{\beta_1}{\lambda} w_\lambda(-\frac{\lambda x}{\beta_1})$ with a few computations, one can deduce (2.9).

5.3. *Proof of Corollary* 2 ($x > 0, \beta_1 < 0, \beta_2 < 0$). Set $\widetilde{X}^{-x} = -X^x$, $\widetilde{X}^0 = -X^0$ and $\widetilde{B} = -B$, then

$$\begin{aligned} \widetilde{X}_t^{-x} &= -x + \widetilde{B}_t + |\beta_2| L_t^0(\widetilde{X}^{-x}), \\ \widetilde{X}_t^0 &= \widetilde{B}_t + |\beta_1| L_t^0(\widetilde{X}^0), \end{aligned}$$

so that we are now dealing with positive skewness coefficients, but a negative starting value $-x$. Denote $T^0 = \inf\{t > 0 \mid \widetilde{X}_t^{-x} = 0\}$. We define $\widehat{X}_t^0 = \widetilde{X}_{T_0+t}^{-x}$, $\widehat{X}_t = \widetilde{X}^0_{T^0+t}$ and $\widehat{B}_t = \widetilde{B}_{T^0+t} - \widetilde{B}_{T^0}$. These processes are solutions to

$$\begin{aligned} \widehat{X}_t &= \widehat{X}_0 + \widehat{B}_t + |\beta_1| L_t^0(\widehat{X}), \\ \widehat{X}_t^0 &= \widehat{B}_t + |\beta_2| L_t^0(\widehat{X}^0), \end{aligned}$$

where $\widehat{X}_0$ is independent of $(\widehat{B}_t)_{t\geq 0}$. Note that for these new processes the role of the skewness parameter has been exchanged, and the starting point of $\widehat{X}$ is a positive random variable. Let us introduce $\widehat{T}^\star = \inf\{t \geq 0 : \widehat{X}_t = \widehat{X}_t^0\} = T^\star - T^0$.

Using the Markov property at the random time T^0, and applying Theorem 3, we get that $\mathbf{B}_1 = (\frac{|\beta_2| L^0_{\widehat{T}^\star}(\widehat{X}^0)}{\widehat{X}_0})^{-1}$ is independent of $\widehat{X}_0$ and distributed as a Beta $\mathcal{B}(1 - (\frac{1}{2|\beta_2|} - \frac{1}{2|\beta_1|}), \frac{1-|\beta_2|}{2|\beta_2|})$ variable.

The random variable $\mathbf{B}_1$ can be related to the local time of the initial process,

$$\begin{aligned} L^0_{T^\star}(X^0) &= L^0_{T^\star}(\widetilde{X}^0) \\ &= \frac{|\beta_2|}{|\beta_1|} L^0_{T^\star}(\widetilde{X}^{-x}) - \frac{x}{|\beta_1|} \\ &= \frac{|\beta_2|}{|\beta_1|} L^0_{\widehat{T}^\star}(\widetilde{X}^{-x}_{T^0+\cdot}) - \frac{x}{|\beta_1|} = \frac{|\beta_2|}{|\beta_1|} L^0_{\widehat{T}^\star}(\widehat{X}^0) - \frac{x}{|\beta_1|} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\widehat{X}_0 \mathbf{B}_1^{-1}}{|\beta_1|} - \frac{x}{|\beta_1|} \\
&= \frac{\widetilde{X}^0_{T^0} \mathbf{B}_1^{-1}}{|\beta_1|} - \frac{x}{|\beta_1|} \\
&= \frac{[x + |\beta_1| L^0_{T^0}(\widetilde{X}^0)] \mathbf{B}_1^{-1}}{|\beta_1|} - \frac{x}{|\beta_1|}.
\end{aligned}
$$

But the law of $L^0_{T^0}(\widetilde{X}^0)$ may be derived by computations similar to those of step 3 in the proof of Proposition 2; see (3.22).

Finally, one finds that $\mathbb{P}(L^0_{T^0}(\widetilde{X}^0) > a) = (1 + \frac{|\beta_1| a}{x})^{-(1+|\beta_2|)/(2|\beta_2|)}$. This means that $\mathbf{B}_2 = [1 + \frac{|\beta_1| L^0_{T^0}(\widetilde{X}^0)}{x}]^{-1}$ is distributed as a $\mathcal{B}(\frac{1+|\beta_2|}{2|\beta_2|}, 1)$ variable. Since $L^0_{T^\star}(X^0) = \frac{x}{|\beta_1|}[\mathbf{B}_2^{-1}\mathbf{B}_1^{-1} - 1]$, the corollary is proved.

APPENDIX

A.1. Proof of relation (3.4). Since for $t < U^\star$, we have the relation

$$
L^0_{\tau_t}(X^x) = \sum_{0<u\le t} [L^0_{\tau_u}(X^x) - L^0_{\tau_{u-}}(X^x)],
$$

it is clear that (3.4) will be proved if we show that almost surely,

$$
L^0_{\tau_u}(X^x) - L^0_{\tau_{u-}}(X^x) = \ell(X^x_{\tau_{u-}}, \mathbf{e}_u) \qquad \text{for all } u \text{ with } \tau_u - \tau_{u-} > 0. \tag{A.1}
$$

Actually, we will construct $\widetilde{X}^x$, indistinguishable of X^x up to $T^\star$, which satisfies the relation (A.1).

Let

$$
U_1 = \inf\{u > 0 \mid \mathbf{e}_u \text{ reaches } -x + \beta_1 u\}.
$$

Then, U_1 is a $(\mathcal{G}_u)_{u\ge 0}$ stopping time, and it is immediate that $U_1 < x/\beta_1$ almost surely. We construct $\widetilde{X}^x$ on $[0, T_1]$ where $T_1 = \tau_{U_1}$ in the following way:

- For $t < \tau_{U_1-}$, we set

$$
\widetilde{X}^x_t = \begin{cases} x + \mathbf{e}_s(t - \tau_{s-}) - \beta_1 s = \widehat{X}_{t-\tau_{s-}}(x - \beta_1 s, \mathbf{e}_s), \\ \qquad \text{if } t \in (\tau_{s-}, \tau_s) \text{ with } s < U_1, \\ x - \beta_1 L^0_t(X^0), \\ \qquad \text{if } t \in [0, \tau_{U_1-}) \setminus \bigcup_{s<U_1} (\tau_{s-}, \tau_s) = \{v < \tau_{U_1-} \mid X^0_v = 0\}. \end{cases} \tag{A.2}
$$

- For $t \in [\tau_{U_1-}, \tau_{U_1}]$, we let

$$
\widetilde{X}^x_t = \widehat{X}_{t-\tau_{U_1-}}(\widetilde{X}^x_{\tau_{U_1-}}, \mathbf{e}_{U_1}) = \widehat{X}_{t-\tau_{U_1-}}(x - \beta_1 U_1, \mathbf{e}_{U_1}) \tag{A.3}
$$

where $\widehat{X}$ was defined in (3.10)–(3.12).

Note that with this definition, $\widetilde{X}_t^x = x + X_t^0 - \beta_1 L_t^0(X^0)$ if $t < \tau_{U_1-}$.

For $t \in [\tau_{U_1-}, \tau_{U_1}]$, we have that

$$L_t^0(\widetilde{X}^x) - L_{\tau_{U_1-}}^0(\widetilde{X}^x) = L_{t-\tau_{U_1-}}^0\big(\widehat{X}(x - \beta_1 U_1, \mathbf{e}_{U_1})\big) \quad \text{for } t \in [\tau_{U_1-}, \tau_{U_1}]. \tag{A.4}$$

Let us check that $\widetilde{X}^x$ satisfies the skew equation. By definition of U_1 we have $\widetilde{X}_t^x > 0$ on $[0, \tau_{U_1-})$. Moreover, $\mathbf{e}_s(t - \tau_s) = X_t^0$ when $t \in (\tau_{s-}, \tau_s)$, together with (2.2) ensures that $\widetilde{X}_t^x = x + B_t$ for $t \in [0, \tau_{U_1-})$. As a result, $\widetilde{X}^x$ satisfies the skew equation on $[0, \tau_{U_1-})$.

We now focus on the interval $[\tau_{U_1-}, \tau_{U_1})$. First, using the so-called "master formula" (Proposition 1.10 page 475 in [13]), we get

$$\begin{aligned} &\mathbb{E}\Big[\sum_{0<s<x/\beta_1} 1_{\{\mathbf{e}\text{ reaches } -x+\beta_1 s\}} 1_{\{\mathbf{e}(T^{-x+\beta_1 s}(\mathbf{e})+\cdot)+x-\beta_1 s\notin\widehat{\Omega}^{x-\beta_1 s}\}}\Big] \\ &\quad = \int_0^{x/\beta_1} \mathbf{n}_{\beta_1}\big(\mathbf{e}\text{ reaches } -x + \beta_1 s; \\ &\qquad\qquad \mathbf{e}\big(T^{-x+\beta_1 s}(\mathbf{e}) + \cdot\big) + x - \beta_1 s \notin \widehat{\Omega}^{x-\beta_1 s}\big)\, ds \\ &\quad = 0, \end{aligned} \tag{A.5}$$

where the latter integral is zero from (3.15)–(3.16). Hence, we deduce that with probability one, for all $u < \frac{x}{\beta_1}$ the relation (3.14) holds true with $\mathbf{e}$ replaced by $\mathbf{e}_u$ and h replaced by $x - \beta_1 u$. As a consequence, it holds true, almost surely, for the excursion occurring at the random time U_1. This leads to the following relation for $t \in [\tau_{U_1-}, \tau_{U_1}]$:

$$\begin{aligned} \widetilde{X}_t^x &= \widetilde{X}_{\tau_{U_1-}}^x + \mathbf{e}_{U_1}(t - \tau_{U_1-}) + \beta_2 L_{t-\tau_{U_1-}}^0\big(\widehat{X}(x - \beta_1 U_1, \mathbf{e}_{U_1})\big) \\ &= \widetilde{X}_{\tau_{U_1-}}^x + \mathbf{e}_{U_1}(t - \tau_{U_1-}) + \beta_2\big[L_t^0(\widetilde{X}^x) - L_{\tau_{U_1-}}^0(\widetilde{X}^x)\big] \qquad \text{by (A.4)} \\ &= \widetilde{X}_{\tau_{U_1-}}^x + X_t^0 + \beta_2\big[L_t^0(\widetilde{X}^x) - L_{\tau_{U_1-}}^0(\widetilde{X}^x)\big] \\ &\qquad \text{by definition of the excursion process} \\ &= \widetilde{X}_{\tau_{U_1-}}^x + B_t + \beta_1 L_t^0(X^0) + \beta_2\big[L_t^0(\widetilde{X}^x) - L_{\tau_{U_1-}}^0(\widetilde{X}^x)\big] \qquad \text{by (2.2)} \\ &= x + B_t + \beta_2 L_t^0(\widetilde{X}^x), \end{aligned}$$

where we have used $\widetilde{X}_{\tau_{U_1-}}^x = x - \beta_1 U_1$, $L_t^0(X^0) = U_1$ and $L_{\tau_{U_1-}}^0(\widetilde{X}^x) = 0$ in the last line. This completes the proof that $\widetilde{X}^x$ almost surely satisfies the skew equation on $[0, T_1]$.

Let us briefly check that $\widetilde{X}^x$ is adapted. Using (3.13) and (A.5), the construction (A.2) and (A.3) ensures that almost surely, for any $t \in [0, T_1]$,

$$\widetilde{X}_t^x = \widehat{X}_{t-\tau_{s-}}\big(x - \beta_1 s; \mathbf{e}_s\big(\cdot \wedge (t - \tau_{s-})\big)\big), \text{ if } t \in (\tau_{s-}, \tau_s) \text{ with } s \in (0, U_1].$$

And the continuity of $\widetilde{X}^x$ implies that the previous equality is in fact satisfied on every closed interval $[\tau_{s-}, \tau_s]$ with $s \in (0, U_1]$. Since $T_1 = \tau_{U_1}$ is a $(\mathcal{F}_t)_{t\geq 0}$ stopping time, this proves that the process $(\widetilde{X}_t^x 1_{t\leq T_1})$ is $(\mathcal{F}_t)_{t\geq 0}$ adapted.

From the pathwise uniqueness for solutions of the skew equation, we deduce that we must have $\widetilde{X}^x = X^x$ on $[0, T_1]$, almost surely.

Then, by the definition of the functional ℓ in the first step of the proof, and by the construction of $\widetilde{X}$, we see that condition (A.1) holds true for $u \leq U^1$. We deduce $L^0_{\tau_t}(X^x) = \sum_{0<u\leq t} \ell(X^x_{\tau_{u-}}, \mathbf{e}_u)$ for $t \leq U_1$. Remark that, since $t \leq U_1$, the only possible nonzero term in this sum is $\ell(X^x_{\tau_{U_1-}}, \mathbf{e}_{U_1})$.

The process $\widetilde{X}^x$ is then constructed recursively. For $i \geq 1$, we let $U_{i+1} = \inf\{u > U_i \mid \mathbf{e}_u \text{ reaches } -\widetilde{X}^x_{T_i} + \beta_1(u - U_i)\}$, we set $T_{i+1} = \tau_{U_{i+1}}$. We define $\widetilde{X}^x$ on $[T_i, T_{i+1}]$ as follows:

$$
\begin{aligned}
&\text{if } t \in [T_i, \tau_{U_{i+1}-}) \text{ with } t \in (\tau_{s-}, \tau_s) \text{ for some } s,\\
&\qquad \widetilde{X}^x_t = \widetilde{X}^x_{T_i} + \mathbf{e}_s(t - \tau_{s-}) - \beta_1(s - U_i);\\
&\text{if } t \in [T_i, \tau_{U_{i+1}-}) \text{ with } X^0_t = 0,\\
&\qquad \widetilde{X}^x_t = \widetilde{X}^x_{T_i} - \beta_1\big(L^0_t(X^0) - U_i\big);\\
&\text{if } t \in [\tau_{U_{i+1}-}, T_{i+1}],\\
&\qquad \widetilde{X}^x_t = \widehat{X}^x_{t-\tau_{U_{i+1}-}}(\widetilde{X}_{\tau_{U_{i+1}-}}, \mathbf{e}_{U_{i+1}})\\
&\qquad\quad\ = \widehat{X}^x_{t-\tau_{U_{i+1}-}}\big(\widetilde{X}_{T_i} - \beta_1(U_{i+1} - U_i), \mathbf{e}_{U_{i+1}}\big).
\end{aligned}
$$

With arguments similar to the one used on $[0, T_1]$, we can prove that $\widetilde{X}^x$ satisfies the skew equation on $[T_i, T_{i+1}]$ and is adapted. Consequently, if X^x and $\widetilde{X}^x$ coincide at the instant T_i, they must coincide almost surely on $[T_i, T_{i+1}]$.

Using a recursion argument, we construct a process $\widetilde{X}^x$ on $[0, \sup_i T_i)$, which is a.s. equal to X^x. Moreover, by construction, relation (A.1) is valid for $u < \sup_i U_i$. To get (3.4) we need to check that, almost surely, $\sup_i U_i = U^\star$ or equivalently that, $\sup_i T_i = T^\star = \inf\{t > 0 \mid X^x_t = X^0_t\}$.

This is immediate if $\sup_{i\geq 1} T_i = \infty$. Assume, by contradiction that the set $\{\sup_{i\geq 1} T_i < \infty; \sup_{i\geq 1} T_i < T^\star\}$ does not have probability zero. On this set, we have $\widetilde{X}^x_t - X^0_t = X^x_t - X^0_t \geq \varepsilon > 0$ for some random ε and t belonging to some random left-neighborhood of $\sup_{i\geq 1} T_i$. But, it can be seen, from the definition of the jump times U_i, that there is only a finite number of jumps when $\widetilde{X}^x_t - X^0_t$ remains above the level ε; see also Remark 6. This is in contradiction with the existence of the accumulation point $\sup_i T_i$, and as a result we deduce that $\sup_i T_i = T^\star$.

A.2. A Kronecker lemma for continuous time local martingales. We were unable to find a reference for the Kronecker Lemma in the context of continuous time local martingale defined on some random interval $[0, U]$; however, see [9] for close results. Hence we give below a short proof of the result.

LEMMA 3. *Let $(\widetilde{\mathcal{J}}_t)_{0\le t<U}$ be a locally square integrable $(\mathcal{G}_t)$-martingale with localizing sequence $\tau_n = \inf\{u \mid \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u \ge n\}$ and $U = \sup_n \tau_n$ (especially $U = \infty$ if $\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_\infty < \infty$). Assume additionally that $(\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_t)_{t\in[0,U)}$ is a continuous process. Then*:

- *On the set $\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_U < \infty$, we have the convergence of $\widetilde{\mathcal{J}}_t$ as $t \to U$.*
- *On the set $\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_U = \infty$, we have $\frac{\widetilde{\mathcal{J}}_t}{\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_t} \xrightarrow{t\to U} 0$.*

PROOF. First, define the event $\Omega_n = \{\omega \mid \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_\infty < n\}$. On this event, $U = \tau_n = \infty$ and $(\widetilde{\mathcal{J}}_t)_{t\ge0} = (\widetilde{\mathcal{J}}_{t\wedge\tau_n})_{t\ge0}$ is a bounded $\mathbf{L}^2$ martingale and thus converges as $t \to \infty$. Since the convergence holds on the set Ω_n for all n, it holds on the set $\{\omega \mid \langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_\infty < \infty\}$.

We now focus on the set $\{\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_U = \infty\}$. Define $\widetilde{N}^n_u = \int_0^{u\wedge\tau_n} \frac{d\widetilde{\mathcal{J}}_u}{1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u}$ which is a $\mathbf{L}^2$-bounded martingale for each n with

$$\langle\widetilde{N}^n, \widetilde{N}^n\rangle_u = \int_0^{u\wedge\tau_n} \frac{d\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u}{(1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u)^2} = 1 - \frac{1}{1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_{u\wedge\tau_n}} \le 1.$$

From this, we easily see that the sequence $(\int_0^{\tau_n} \frac{d\widetilde{\mathcal{J}}_u}{1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u})_{n\ge1}$ is a $\mathcal{G}_{\tau_n}$-martingale sequence which converges to some $\mathbf{L}^2$ variable $\int_0^U \frac{d\widetilde{\mathcal{J}}_u}{1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u}$. As a consequence $\widetilde{N}_u = \int_0^{u\wedge U} \frac{d\widetilde{\mathcal{J}}_u}{1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u}$ is a true $\mathbf{L}^2$-martingale.

We now write for $u_0 < u < U$,

$$\begin{aligned}
\widetilde{\mathcal{J}}_u &= \int_0^u (1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u)\, d\widetilde{N}_u \\
&= \int_0^{u_0} (1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u)\, d\widetilde{N}_u + \int_{u_0}^u (1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u)\, d\widetilde{N}_u \\
&= \int_0^{u_0} (1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u)\, d\widetilde{N}_u - \int_{u_0}^u (\widetilde{N}_v - \widetilde{N}_{u_0})\, d\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u \\
&\quad + \bigl(1+\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u\bigr)(\widetilde{N}_u - \widetilde{N}_{u_0}),
\end{aligned}$$

where we used Ito's formula.

Then, a convenient choice of u_0 determined by the almost sure convergence of $\widetilde{N}_u$ as $u \to U$, with $\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_U = \infty$, easily implies that $\frac{\widetilde{\mathcal{J}}_u}{\langle\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}\rangle_u} \xrightarrow{u\to U} 0$. $\square$

A.3. Technical lemmas.

LEMMA 4. *No locally bounded measurable function $h:[0,\infty) \to \mathbb{R}^{*+}$ exist such that there exists $c > 0$ and $t_0 \ge 0$, satisfying $\forall t \ge t_0$,*

$$\ln\bigl(h(t)\bigr) \le -c \int_{t_0}^t \frac{du}{h(u)}.$$

PROOF. Assume that such a function h exists, and set $g(t) := -\ln(h(t))$. The inequality becomes $g(t) \geq c \int_{t_0}^t e^{g(u)}\,du$. Denote $y(t) = \int_{t_0}^t e^{g(u)}\,du$ which is an increasing function. One has the inequality between Stieljes measures on $[t_0, \infty)$, $d(e^{-cy(t)}) \leq -c\,dt$, that integrates to $e^{-cy(t)} - e^{-cy(t_0)} \leq -c(t - t_0)$. This yields to a contradiction as $t \to \infty$. $\square$

LEMMA 5. *Assume* $-1 < \beta_2 < 0 < \beta_1 < 1$, *and set* ρ *any real number with* $0 < \rho < (1-\gamma) \wedge 1$ (*recall that* $\gamma = \frac{1+3\beta_2}{2\beta_2} < 1$ *is defined in Corollary* 1).

Then, the following functions are bounded on $[0, \infty)$: $x \mapsto u_\lambda(x)$, $x \mapsto x|u_\lambda'(x)|$, $x \mapsto x^2|u_\lambda''(x)|$, $x \mapsto x^{1+\rho}u_\lambda(x)$, $x \mapsto x^{2+\rho}|u_\lambda'(x)|$ *and* $x^{3-\varepsilon}|u_\lambda''(x)|$ *for* $0 < \varepsilon < 1$.

PROOF. We use the notation of the proof of Lemma 1. We have $u_\lambda(x) = \mathbb{E}[e^{-\lambda U^{\star,x}}] = \mathbb{E}[e^{-\lambda x U^{\star,1}}]$, and thus $u_\lambda^{(k)}(x) = \mathbb{E}[(-\lambda U^{\star,1})^k e^{-\lambda x U^{\star,1}}]$ for $k \geq 0$. This clearly implies that $x^k|u_\lambda^{(k)}(x)|$ is a bounded function. For $k = 0, 1, 2$, we get that the first three functions in the statement of the lemma are bounded.

Remark now that $U^{\star,1}$ is almost surely greater than U_1^1, the first jump time of the process $u \mapsto Z_u^x$. The law of U_1^1 is given by (4.7), and one can easily check that $E[(U_1^1)^{-1+\varepsilon}] < \infty$ for $\varepsilon > 0$. We deduce that $E[(U^{\star,1})^{-1+\varepsilon}] < \infty$ for $\varepsilon > 0$. As a consequence,

$$
\begin{aligned}
|u_\lambda^{(k)}(x)| &= |\mathbb{E}[(-\lambda U^{\star,1})^k e^{-\lambda x U^{\star,1}}]| \\
&\leq \mathbb{E}\biggl[\frac{1}{(U^{\star,1})^{1-\varepsilon}} \frac{(\lambda x U^{\star,1})^{k+1-\varepsilon}}{\lambda^{1-\varepsilon} x^{k+1-\varepsilon}} e^{-\lambda x U^{\star,1}}\biggr] \qquad \text{(A.6)} \\
&\leq c x^{-(k+1-\varepsilon)} E\bigl[(U^{\star,1})^{-1+\varepsilon}\bigr] \leq c x^{-(k+1-\varepsilon)}
\end{aligned}
$$

for some constant c independent of x. Using $k = 2$, this shows that the quantity $x^{3-\varepsilon}|u_\lambda''(x)|$ is bounded.

It remains to prove the boundedness of the functions $x \mapsto x^{1+\rho}u_\lambda(x)$ and $x \mapsto x^{2+\rho}|u_\lambda'(x)|$. Clearly, only a control for large values of x is needed. However, this control requires some additional work.

We start by proving that $u_\lambda(x) \leq c x^{-1-\rho}$ for $x > 1$. Using Dynkin's equation $\lambda u_\lambda(x) = A u_\lambda(x)$, we have

$$
\begin{aligned}
\lambda u_\lambda(x) &= -\beta_1 u_\lambda'(x) + \int_0^x \frac{|\kappa|}{x^2}\Bigl(1 - \frac{a}{x}\Bigr)^{-\gamma} \bigl[u_\lambda(x-a) - u_\lambda(x)\bigr]\,da \qquad \text{(A.7)} \\
&= -\beta_1 u_\lambda'(x) - \frac{|\kappa| u_\lambda(x)}{x} \int_0^1 a^{-\gamma}\,da + \int_0^1 \frac{|\kappa|}{x} a^{-\gamma} u_\lambda(xa)\,da,
\end{aligned}
$$

where we performed a change of variables in the last line. From (A.6) with $k = 0$ and $k = 1$, we see that the first two terms in the right-hand side of (A.7) are bounded by $cx^{-2+\varepsilon} = O(x^{-1-\rho})$ if ε is small enough. It remains to control the last term in (A.7). We split the integral $\int_0^1 \frac{|\kappa|}{x} a^{-\gamma} u_\lambda(xa)\, da$ into

$$\frac{|\kappa|}{x}\int_0^{1/x} a^{-\gamma} u_\lambda(xa)\, da + \frac{|\kappa|}{x}\int_{1/x}^1 a^{-\gamma} u_\lambda(xa)\, da.$$

Using the control $u_\lambda(xa) \le 1$ on the first integral and $u_\lambda(xa) \le c(xa)^{-1+\varepsilon}$ on the second one, we get $\int_0^1 \frac{|\kappa|}{x} a^{-\gamma} u_\lambda(xa)\, da \le cx^{\gamma-2} + cx^{-2+\varepsilon} \le cx^{-1-\rho}$. Collecting all terms, we have shown $u_\lambda(x) \le cx^{-1-\rho}$.

To finish the proof of the lemma, we need to establish $|u'_\lambda(x)| \le cx^{-2-\rho}$. If one differentiates relation (A.7) and uses (A.6), for $k = 1$ and $k = 2$, and the control already obtained for $\int_0^1 a^{-\gamma} u_\lambda(xa)\, da$, it can be shown

$$\lambda u'_\lambda(x) = \frac{|\kappa|}{x}\int_0^1 a^{-\gamma+1} u'_\lambda(xa)\, da + O\bigl(x^{-2-\rho}\bigr).$$

Again the integral above can be split into

$$\int_0^{1/x} a^{-\gamma+1} u'_\lambda(xa)\, da + \int_{1/x}^1 a^{-\gamma+1} u'_\lambda(xa)\, da.$$

Using the control $|u'_\lambda(xa)| \le c(xa)^{-1}$ on the first part, and $|u'_\lambda(xa)| \le c(xa)^{-2+\varepsilon}$ on the second part, we get

$$\frac{1}{x}\int_0^1 a^{-\gamma+1} u'_\lambda(xa)\, da \le cx^{-3+\gamma} + cx^{-3+\varepsilon} \le cx^{-(2+\rho)}.$$

Collecting all terms, we have shown $|u'_\lambda(x)| \le cx^{-2-\rho}$. $\square$

REMARK 8. The conclusions of Lemma 5 imply that:

$$\begin{aligned}
&\mathcal{M}[u_\lambda](\xi) \text{ is defined for } \mathrm{Re}(\xi) \in (0, 1+\rho);\\
&\mathcal{M}\bigl[u'_\lambda\bigr](\xi) \text{ is defined for } \mathrm{Re}(\xi) \in (1, 2+\rho);\\
&\mathcal{M}\bigl[x \mapsto x u'_\lambda(x)\bigr](\xi) \text{ is defined for } \mathrm{Re}(\xi) \in (0, 1+\rho);\\
&\mathcal{M}\bigl[x \mapsto x u''_\lambda(x)\bigr](\xi) \text{ is defined for } \mathrm{Re}(\xi) \in (1, 2).
\end{aligned}$$

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