SIMPLE RANDOM WALK ON LONG-RANGE PERCOLATION CLUSTERS II: SCALING LIMITS

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We study limit laws for simple random walks on supercritical long-range percolation clusters on $\mathbb{Z}^d$, $d \geq 1$. For the long range percolation model, the probability that two vertices x, y are connected behaves asymptotically as $\|x - y\|_2^{-s}$. When $s \in (d, d + 1)$, we prove that the scaling limit of simple random walk on the infinite component converges to an α-stable Lévy process with $\alpha = s - d$ establishing a conjecture of Berger and Biskup [*Probab. Theory Related Fields* **137** (2007) 83–120]. The convergence holds in both the quenched and annealed senses. In the case where $d = 1$ and $s > 2$ we show that the simple random walk converges to a Brownian motion. The proof combines heat kernel bounds from our companion paper [Crawford and Sly *Probab. Theory Related Fields* **154** (2012) 753–786], ergodic theory estimates and an involved coupling constructed through the exploration of a large number of walks on the cluster.

1. Introduction. The study of stochastic processes in random media has been a focal point of mathematical physics and probability for the past thirty years. One such research problem regards the study of random walk in random environment (RWRE) in its many forms. This subject includes tagged particles in interacting particle systems [17], the study of $\nabla \phi$-fields through the Helffer–Sjöstrand representation [16, 27] and random conductance models.

In this paper we continue the study of simple random walks (SRW) on percolation clusters on the ambient space $\mathbb{Z}^d$. By now, many properties of the nearest neighbor percolation model are understood in the supercritical case. We mention in this context the important work of Kipnis and Varadhan [18], who introduced "the environment viewed from the particle" point of view to derive annealed functional central limit theorems. This work was strengthened in De Masi et al. [12] where it was applied to SRW on nearest neighbor percolation clusters under the annealed law for the walk. Sidoravicious–Sznitman [26] proved an invariance principle for SRW on supercritical percolation clusters for $\mathbb{Z}^d$, $d \geq 4$. Mathieu and Rémy [21] and Barlow [3] proved quenched heat kernel bounds on supercritical percolation

clusters. Early estimates in this direction were obtained by Heicklen and Hoff-
man [15]. Finally Mathieu and Pianitskii [20] and Berger and Biskup [7] extended
[26] to all $d \geq 2$.

We consider a variant of these latter results—scaling limits for SRW on super-
critical long-range percolation clusters on $\mathbb{Z}^d$ (LRP). LRP was first considered by
Schulman in [25] and Zhang et al. [29]. It is a random graph process on $\mathbb{Z}^d$ where,
independently for each pair of vertices $x, y \in \mathbb{Z}^d$, we attach an edge $\langle x, y \rangle$ with
probability $p_{x,y}$. We shall assume an isotropic translation invariant model for the
connection probabilities setting $p_{x,y} = \mathrm{P}(\|x - y\|_2)$ where

$$(1) \qquad\qquad \mathrm{P}(r) \sim Cr^{-s}$$

for some $C \in \mathbb{R}^+$ and large r where $x_n \sim y_n$ denotes $\lim \frac{x_n}{y_n} \to 1$.

Early work on LRP concentrated on characterizing when the process, whose
probability space we denote by $(\Omega, \mathcal{B}, \mu)$, admits an infinite connected cluster in
dimension $d = 1$. There are a number of transitions for the behavior of this event as
a function of s (and in the critical case $s = 2$, the prefactor β, denoted by C here).
The first results were obtained by Schulman [25]: for $s > 2$ there is no infinite
component unless $\mathrm{P}(r) = 1$ for some $r \in \mathbb{N}$. Later Newman and Schulman [23]
proved that if $s < 2$, then if one begins with a percolation measure μ which does
not admit an infinite component a.s., one can adjust (nontrivially) $\mathrm{P}(1)$ to produce
an infinite component. They also demonstrated that this same result holds in the
case $s = 2$ and C sufficiently large. Finally, in a striking paper [2], Aizenman and
Newman address the case $s = 2$, showing that the behavior (in the above sense)
depends on the precise value of the constant C in (1) ($C = 1$ is critical).

More recently, the long-range model gained interest in the context of "small
world phenomena;" see works such as [22, 28] and [8] for discussions. Ben-
jamini and Berger [4] initiated a quantitative study of these models, focusing on
the asymptotics of the diameter on the discrete cycle $\mathbb{Z}/N\mathbb{Z}$. Their motivation re-
garded connections to modeling the topology of the internet; see also [19] for a
different perspective. Further analysis was done in [10].

The study of random walks on LRP clusters was begun in [6], which addresses
recurrence and transience properties of SRW on the infinite component of super-
critical LRP in the general setting where nearest neighbor connections do not exist
with probability 1. In principle, this paper makes use of the transience results estab-
lished therein. Note, however, that our heat kernel bound Theorem 9.1 (Theorem 1
of [11]) may be used alternately to establish this fact. Benjamini, Berger and Yadin
[5] study the spectral gap τ of SRW on $\mathbb{Z}/N\mathbb{Z}$, providing bounds of the form

$$cN^{s-1} \leq \tau \leq CN^{s-1}\log^\delta N;$$

in that case that nearest neighbor connections exist with probability 1.

As was just alluded to, in a companion paper to the present paper [11], we derive
quenched upper bounds for the heat kernel of continuous time SRW on the infinite

component of supercritical LRP clusters on $\mathbb{Z}^d$ when $s \in (d, (d+2) \wedge 2d)$. These estimates are crucial in establishing the quenched limit law of SRW, the main result of this paper. The companion paper also yields a number of results on the geometry of LRP in finite boxes which we make use of here; see Section 9 for details.

The scaling exponent of the connection probabilities determines the limiting behavior of the walk. Smaller values of s produce more long edges, and these edges determine the macroscopic behavior of the walk suggesting a non-Gaussian stable law as the limiting process. To this end, we let $\Gamma_\alpha(t)$ denote d-dimensional isotropic α-stable Lévy motion (formally defined in Section 2). We will assume that the percolation process admits an infinite component μ-a.s. and let Ω_0 denote the set of environments where the origin is in the infinite component with μ_0 the conditional measure on Ω_0. We now state our main result, a quenched limit law for simple random walk on long-range percolation clusters which affirms a conjecture of Berger and Biskup [7] in the case $s \in (d, d+1)$.

THEOREM 1.1. *Let $d \geq 1$ and $s \in (d, d+1)$. Let X_n be the simple random walk on $\omega \in \Omega_0$, and let*

$$X_n(t) = n^{-1/(s-d)} X_{\lfloor nt \rfloor}.$$

Then for μ_0-a.s. every environment ω and $1 \leq q < \infty$ the law of $(X_n(t), 0 \leq t \leq 1)$ on $L^q([0, 1])$ converges weakly to the law of an isotropic α-stable Lévy motion with $\alpha = s - d$.

Some remarks are in order. First, when $\alpha \in (1, 2)$ and $d \geq 2$, much of the machinery we develop applies. However, in that case one has to rely on cancellations between small jumps of the walk to arrive at the limiting stable law. In this paper we do not prove that the latter cancellations occur. With an eye toward future work, we formulate a precise condition (3) regarding these cancellations and in Proposition 3.1 prove the analog of Theorem 1.1 on the basis of (3).

Second, while the natural topology of convergence to a non-Gaussian stable law is the Skorohod topology, we note that convergence in that sense does not hold. There exist times at which the walk crosses a particular long edge of the graph an even number of times on a small time scale. These events do not appear in the limit law, but do preclude convergence in the Skorohod sense. We thus adopt the L^q topology as it does not see these spurious discontinuities; see Section 3 for more details.

Our proof proceeds by revealing the randomness of the environment as the walk explores the cluster of the origin, occasionally encountering macroscopic edges which constitute the main contribution in the limit. To make this approach rigorous we require the precise heat kernel upper bounds and structural picture established in our companion paper [11] together with ergodic theory estimates which

guarantee that new vertices are encountered at a constant rate over time. By combining these estimates with a highly involved coupling, and performing this construction simultaneously for a large number of independent walks, we establish the quenched convergence.

In the case $d = 1$ we establish a sharp transition in the scaling limit at $s = 2$. In the case $s > 2$ there is no infinite component unless $P(r) = 1$ for some $r \in \mathbb{N}$ [25]; hence we make the assumption that nearest neighbor edges are included with probability 1. We prove the quenched law converges weakly to the law of Brownian motion in the space $C([0, 1])$ under the uniform norm.

THEOREM 1.2.   *Let $d = 1$ and $s > 2$ be fixed. Assume that* (1) *and $P(1) = 1$ hold, and let $\omega \in \Omega$. Let X_n be the simple random walk on $\omega \in \Omega_0$, and let*

$$X_n(t) = \frac{1}{\sqrt{n}}\big(X_{\lfloor nt \rfloor} + \big(tn - \lfloor tn \rfloor\big)(X_{\lfloor nt \rfloor + 1} - X_{\lfloor nt \rfloor})\big).$$

Then for μ-a.s. environments ω the law of $(X_n(t), 0 \le t \le 1)$ in $C([0, 1])$ converges weakly to $(K B(t), 0 \le t \le 1)$ where $B(t)$ is standard Brownian motion, and K is a constant depending on the connection probabilities.

The rest of the paper is divided as follows. In Section 2 we introduce the basic notation used for the remainder of the paper and describe the "environment exploration" process which is at the core of the proof. The result relies on a technical coupling construction defined in Section 5. This is the heart of our proof and is described in the proof overview in Section 3. Section 4 lists certain a priori bounds needed to show the coupling works with high probability. Sections 6 and 7 detail how the coupling can be used to derive Theorem 1.1. Sections 8 and 9 are devoted to justifying various technical lemmas of Sections 4, 5, 6 and 7. In particular, to guarantee that our coupling works with high probability, we need to rule out various rare events and establish ergodic theorems for the walks. Finally we prove Theorem 1.2 in Section 10.

2. Notations, basic objects.   In this section, we describe the notation used in our proofs. We denote by $\Omega = \{0, 1\}^{\mathcal{E}}$ the sample space of environments for LRP on $\mathbb{Z}^d$ where $\mathcal{E}$ is the edge set of unordered pairs in $\mathbb{Z}^d$. Let μ denote the product measure on Ω determined by connection probabilities p_{xy} satisfying equation (1). We assume the p_{xy} are *percolating*, that is, μ admits an infinite component. It was proved in [1] that in this case the infinite component is unique, and we denote this component by $\mathcal{C}^{\infty}(\omega)$. We let Ω_0 denote the subset of environments where the origin is in the infinite component and let μ_0 be the induced measure of Ω_0,

$$\mu_0(\cdot) = \mu\big(\cdot | 0 \in \mathcal{C}^{\infty}(\omega)\big).$$

Throughout we assume that $s \in (d, \infty)$ with $d \ge 1$ which ensures finite degrees almost surely: letting $d^{\omega}(x)$ denote the degree of x in ω, when $s \in (d, \infty)$,

$\mathbb{E}_\mu[d^\omega(0)] < \infty$, and it follows that μ-a.s., for all $x \in \mathbb{Z}^d$, $d^\omega(x) < \infty$. Let us use the notation $B_L(v) = \{x \in \mathbb{Z}^d : \|x - v\|_\infty \leq L\}$ and henceforth $\alpha = s - d$. Here, we are using the standard notation $\|\cdot\|_p$ for the ℓ^p norm on $\mathbb{R}^d$.

The simple random walk on ω is the walk which moves to the uniformly chosen neighbor of the current location at each step [have transition kernel $P^\omega(x, y) = \frac{\delta(1 - \omega_{\langle x, y \rangle})}{d^\omega(x)}$]. We let $(X_i)_{i \in \mathbb{N}}$ denote the random walk trajectory generated by $P^\omega(x, y)$ with $X_0 = 0$. For $\ell \in \mathbb{N}$, we let $(X_i^\ell)_{i, \ell \in \mathbb{N}}$ denote independent copies of the walk on the same environment ω. Studying the joint *annealed* law over many ℓ plays a crucial role in our proof of the quenched law.

It will at times be convenient to work under the "degree-biased" measure ν on environments Ω, given by

$$\nu(A) = \mathbb{E}_\mu[\mathbb{1}\{A\}d^\omega(0)]/E_\mu[d^\omega(0)]$$

and ν_0, given by

$$\nu_0(A) = \nu(A|0 \in \mathcal{C}^\infty(\omega)).$$

These measures are important since the process on environments described below is stationary relative to them. Generically we use the notation $\mathbb{P}$ to denote the underlying probability distribution, and $\mathbb{E}$, the corresponding expectation. The actual meaning of this notation should be clear from context. There is one exception to this rule: the joint law of $(\omega, (X^\ell)_\ell)$ depends on the distribution on environments, of which we have the four choices μ, μ_0, ν, ν_0. To emphasize which choice is employed, we use subscripts. Thus we have $\mathbb{P}_{\mu_0}$, $\mathbb{E}_\nu$, etc. By definition, omission of the subscript indicates that we are using the measure μ. In Lemma 9.4 we establish bounds relating these measures.

Let us now describe the limiting processes. For $\alpha \in (0, 2)$ we recall that an isotropic α-stable Lévy motion $\Gamma(t) : t \in \mathbb{R}^+$ is (up to a single parameter) the unique cádlág stochastic process with state space $\mathbb{R}^d$ having stationary independent increments and the self-similarity property $\Gamma_{at} \overset{d}{=} a^{1/\alpha}\Gamma_t$.

These are non-Gaussian processes whose marginal distributions have power law tails with index α. If Y is a $\mathbb{Z}^d$ valued random vector such that $\mathbb{P}(Y = y) \sim \|y\|_2^{-s}$ for $s = \alpha + d$, then Y is in the domain of attraction of an isotropic α-stable law. For convenience we will normalize $\Gamma_\alpha(t)$ so that it is the limit law of associated with random vectors Y with $\mathbb{P}(Y = y) \sim \mathbb{P}(\|y\|_2)$. We refer the reader to [24] for more information.

2.1. *Environment exploration process.*

In Section 5, for each $k \in \mathbb{N}$ we provide a coupling construction between $(\omega, (X_i^\ell)_{\ell \in k^3, i \in [2^k]})$ and an i.i.d. family of variables which represent increments of a family of discrete processes converging to i.i.d. copies α-stable Lévy motions. For this purpose it is important to have an alternate description of the law of $(\omega, (X_i^\ell)_{\ell \in k^3, i \in [2^k]})$ under $\mathbb{P}_\mu$.

The description we use is a version of an environment exploration process in which the family of walks reveals the edges of the long-range percolation cluster as it encounters new vertices. We emphasize that the reader should be aware that the standard definition of the environment exploration process is not the one we use and rather we reveal extra local edges of the process for the purpose of our coupling.

For each k, we let

$$\rho = \rho_k = \begin{cases} k^{-200/(1-\alpha)} 2^{k/\alpha}, & \alpha \in (0, 1), \\ \lambda 2^{k/\alpha}, & \alpha \in [1, 2), \end{cases}$$

with λ a small constant to be taken to 0. The quantity ρ represents the minimum for the macroscopic length scale. When $\alpha \in (0, 1)$, the contribution to the total variation of X_i from jumps of size less than ρ is negligible under the rescaling by $2^{k/\alpha}$ as $k \to \infty$; see Lemma 9.11. When $\alpha \in [1, 2)$ this is no longer true and as a consequence, we do not give a full proof of the analog to Theorem 1.1. However, we prepare the ground for future work by giving a proof of result under the assumption that the small edges contribution is negligible; see Proposition 3.1. Fix $\delta \in (0, 1)$ (further restrictions will be placed on δ below). For a vertex $x \in \mathbb{Z}^d$ let V_x denote the set $\{y \in \mathbb{Z}^d : \|x - y\|_\infty \le 2^{\delta k}\}$. For $0 \le i \le 2^k$ and $1 \le \ell \le k^3$ define the σ-algebras $\mathcal{F}_{i,\ell}$ inductively as follows:

- Let $\mathcal{F}_{0,1}$ be the σ-algebra generated by $\{\omega_{0,x} : x \in \mathbb{Z}\}$ and $\{\omega_{x,y} : x, y \in V_0\}$.
- For $1 \le i \le 2^k$,

$$\mathcal{F}_{i,\ell} = \mathcal{F}_{i-1,\ell} \vee \sigma\{\{X_i^t\} \cup \{\omega_{X_i^\ell, y} : y \in \mathbb{Z}^d\} \cup \{\omega_{x,y} : x, y \in V_{X_i^\ell}\}$$

$$\cup \{\omega_{x,y} : z \in \mathbb{Z}^d, x, y \in V_z, \omega_{X_i^\ell, z} = 1, \|X_i^\ell - z\|_\infty > \rho\}$$

$$\cup \{\omega_{y,z} : y, z \in \mathbb{Z}^d, \omega_{X_i^\ell, z} = 1, \|X_i^\ell - z\|_\infty > \rho\}\}.$$

- $\mathcal{F}_{0,\ell} = \mathcal{F}_{2^k, \ell-1}$ for $\ell \ge 2$.

This σ-algebra encodes the edges revealed by the process by the first $\ell - 1$ walks and after the ℓth walk reaches the ith step. It includes short edges in the surrounding neighborhood of the walk which are used by the process to determine the coupling. Denote

$$\mathcal{F}_{i,\ell}^- = \mathcal{F}_{i-1,\ell} \vee \sigma\{\{\omega_{X_i^\ell, y} : |y| \le \rho\} \cup \{\omega_{X_i^\ell, y} : y \in \mathbb{Z}^d\} \cup \{\omega_{x,y} : x, y \in V_{X_i^\ell}\}\},$$

that is, ignoring new edges of length greater than ρ. Let $\mathcal{W}_{i,\ell}$ denote the set of vertices visited by the first $\ell - 1$ walks up to time 2^k plus the vertices visited by walk ℓ up to time i. Let

$$\mathcal{W}_{i,\ell}^+ = \mathcal{W}_{i,\ell} \cup \{x \in \mathbb{Z}^d : y \in \mathcal{W}_{i,\ell}, \omega_{x,y} = 1, \|x - y\|_\infty > \rho\}.$$

3. Outline of the proof. Before giving technical details we would like to discuss the important ideas and difficulties of our approach. The main theorem can be separated into two issues: identification of the limit law for X_i under $\mathbb{P}_{\mu_0}$ and proof that this limit law coincides with the limit law for X_i under the quenched measure P^ω for almost every $\omega \in \Omega_0$.

We use a coupling constructed under the measure $\mathbb{P}_\mu$ so that edges in ω are independent. As a consequence we need to relate results in $\mathbb{P}_\mu$ to those $\mathbb{P}_{\mu_0}$. Note that this particular difficulty disappears if we make the a priori assumption that $\mathbb{P}(1) = 1$. This simplifying assumption will be used in this discussion, although the general case is given in the actual proof.

The power law scaling of the connections probabilities gives the probability of a long edge according to

$$\mathbb{P}_\mu\big(\exists y \in \mathbb{Z}^d : \|y\|_\infty > R, \omega_{\langle 0, y \rangle = 1}\big) \sim R^{-\alpha}.$$

By passing to the degree-biased measure $\mathbb{P}_\nu$, it follows that

$$\mathbb{E}_\nu\left[\sum_{i=1}^T \mathbb{1}\big\{\exists y \in \mathbb{Z}^d : \|y\|_\infty > R, \omega_{\langle X_i, y \rangle = 1}\big\}\right] \lesssim T R^{-\alpha},$$

and the asymptotic holds for $\mathbb{P}_\mu$ as well. Further, the a priori knowledge that the process is transient, ergodic theory and reversibility imply that this gives the correct order of magnitude. In other words, the largest edges encountered by X_i in time T are $O(T^{1/\alpha})$. This calculation allows us to determine the right length scales: we expect to see a nontrivial limiting process under the scaling

$$X_n(t) = n^{-1/\alpha} X_{\lfloor nt \rfloor}.$$

Our proof will examine time scales of length $n = 2^k$ for $k \in \mathbb{Z}^d$ which is sufficient due to the self-similarity of the limit law.

We make a key use of the assumption that $s \in (d, d+1)$ as follows. For any n, let $\rho = \rho(n) = n^{(1-\varepsilon)/\alpha}$. We separate the increments of the walk, $(X_i - X_{i-1})_{i \leq n}$, according to ρ. In Lemma 9.11, we show that

$$(2) \qquad n^{-1/\alpha} \sum_{i=1}^n \|X_i - X_{i-1}\|_\infty \mathbb{1}\big\{\|X_i - X_{i-1}\|_\infty \leq \rho\big\} \to 0$$

in probability under $\mathbb{P}_\mu$ (giving rates of convergence in the proof). Thus, if we characterize the behavior of X_i as it encounters edges greater than ρ, then the annealed limit law will follow.

When $\alpha \geq 1$ equation (2) does not hold. In effect, we ask too much by taking the absolute values of the increments as there is much cancelation between the small edges. The issue is an exact analog to those which arise in the construction of α-stable Lévy motions. We present the argument in such a way that the only ingredient missing for a full proof when $\alpha \in [1, 2), d \geq 2$ will be a replacement

of (2). Our small jump assumption says that for any $\kappa > 0$ there exists $\lambda^*(\kappa)$ such that if $0 < \lambda \leq \lambda^*(\kappa)$ and $\rho_k = \lambda 2^{k/\alpha}$, then

$$
\mathbb{P}\left[\left\{\mathbb{P}\left[\max_{1 \leq n \leq 2^k} \frac{1}{2^{k/\alpha}} \sum_{i=1}^{n} (X_i - X_{i-1})\mathbb{1}\{\|X_i - X_{i-1}\|_\infty \leq \rho_k\} \Big| \omega \right] > \kappa\right\}\right]
$$

(3)

$$
\leq \frac{1}{k^3},
$$

when $k > k'(\kappa, \lambda)$. Assuming this we have the following analog to Theorem 1.1.

PROPOSITION 3.1.   *Let $d \geq 2$ and $s \in [d+1, d+2)$. Let X_n be the simple random walk on $\omega \in \Omega_0$, and let*

$$
X_n(t) = n^{-1/(s-d)} X_{\lfloor nt \rfloor}.
$$

Then for μ_0-a.s. every environment ω and $1 \leq q < \infty$, if equation (3) holds, then the law of $(X_n(t), 0 \leq t \leq 1)$ on $L^q([0, 1])$ converges weakly to the law of an isotropic α-stable Lévy motion with $\alpha = s - d$.

Establishing (3) will be the subject of future work.

Now let us consider the path of a typical walk. Suppose that at step $0 \leq i < n$ the walk reaches a vertex v not previously encountered. One of the key observations of this paper is that almost all vertices of distance ρ or greater have not previously been visited by the path, so we can effectively treat the long edges coming out of v as being chosen according to the connection probabilities $p_{v,y}$ independent of the past. We may even treat the local neighborhood of radius n^δ of a distant endpoint y as being independent of the past as well, provided $\delta > 0$ is sufficiently small.

This indicates that we may treat the behavior of excursions of X_i near edges of length at least ρ as being asymptotically independent. Estimates which quantify this claim are stated in Section 5 and are proved in Section 9. Moreover, because of transience, on the macroscopic scale the only excursions which play a role in the limit come from long edges that are crossed in odd number of times.

The observations of the previous paragraph motivate the following analysis of the walk in the local neighborhood of a long edge. Let $v = X_i$ and y denote the other endpoint of the long edge connected to v (there is only one long edge with high probability). Consider the restrictions $\mathcal{C}_v, \mathcal{C}_y$ of the percolation cluster to the balls $B_{n^\delta}(v), B_{n^\delta}(y)$, respectively. Using the heat kernel estimates from [11], the number of times the walk crosses the edge (v, y) is approximately (as $n \to \infty$) the same as the number of times it crosses the edge before leaving $\mathcal{C}_v \cup \mathcal{C}_y$. This latter quantity can be determined by the degrees of v and y and certain local return times which measure the chance the walk inside $\mathcal{C}_v$ (resp., $\mathcal{C}_y$) returns to v (resp., y) before time n^γ for a suitably chosen $\gamma > 0$. In order to make use of this information our proof analyses the number of times the walks encounter a new vertex of given degree and local return probability.

In Section 5 these considerations lead us to construct a coupling between X_i and a second process $\hat{X}_i$ having only jumps of length at least ρ which tracks the displacement of the excursions of X_i which cross long edges an odd number of times. This construction is at the heart of the proof and is explained in detail in that section (in fact, the coupling occurs for k^3 i.i.d. copies of X_i). The important point in the coupling is that the increments of $\hat{X}_i$ are close to discrete versions of an i.i.d. family of α stable Lévy motions. However, the number of increments which contribute to the position $\hat{X}_j$ at time j depend on the underlying walk X_i in a highly nontrivial way: through the number of new vertices the walks encounters of a given degree and local return probability.

To take care of this time dependence, we introduce a *third* process $\mathfrak{X}_i$. The process $\mathfrak{X}_i$ uses the same increments as $\hat{X}_i$ but is time deterministic. We show that $\mathfrak{X}_i$ and $\hat{X}_i$ are close by showing an ergodic theorem for the number of new vertices of each degree and return probability type. Finally, the increments of $\mathfrak{X}_i$ are independent and in the domain of attraction of (a multiple of) the stable variable $\Gamma_\alpha(1)$, and so this third process converges weakly to Γ_α.

To pass from an annealed limit law to a quenched limit law we will use the law of large numbers. The idea is to first extend the indicated coupling to k^3 walks $(X_i^\ell)_{i \in [2^k], \ell \in [k^3]}$ and apply the Chernoff bound to the *quenched* law of $\tilde{X}_k(t) = 2^{-k/(s-d)} X_{\lfloor 2^k t \rfloor}$. We will not enter into further details here, except to emphasize that this approach works precisely because the k^3 walks intersect relatively few times if $s \in (d, d+1)$. If $s \in [d+1, d+2)$, this needs to be qualified by saying that intersections are unlikely near encounters with long edges. In particular the observation that the distribution of long edges from a new vertex is essentially independent of the past still applies to the exploration process for k^3 walks. Large deviations estimates now imply that the law of the walk given ω converges to the α-stable law.

Finally, let us comment on the topology in which the limit law holds. As was previously mentioned, there will be long edges which the walk X_i crosses in even number of times in on a short $O(1)$ time scale. As a result convergence in the Skorohod topology does not hold. Instead we prove convergence in the topology of $L^q([0, 1])$ in which these spikes do not affect the limit law. An alternative way of dealing with this issue would be to define the limit as

$$X^*(t) = n^{-1/\alpha} X_{m(n) \lfloor nt/m(n) \rfloor}$$

for some sequence of integers $m(n) \to \infty$ as $n \to \infty$. This has the effect of only sampling the process every m steps, and since m grows with n the spikes will mostly be between step mi and step $m(i + 1)$ for some i. It is not difficult to modify our proof to show that $X^*(t)$ converges in the Skorohod topology, but we omit this for space considerations.

4. A priori estimates. In order to establish our main coupling we need to bound the probability of several types of unlikely events and also to prove certain ergodic theorems for the number of new vertices encountered by the random walks. The proofs are postponed to Section 9.

We define $F^*(\rho, k)$ to be the probability that the first two walks both encounter the same long edge,

$$F^*(\rho, k) = \{\exists x, v \in \mathbb{Z}^d, i, j \in [2^k], \omega_{x,v} = 1, \|v\|_\infty \geq \rho, X_i^1 \text{ and } X_j^2 \in \{x, v\}\}.$$

The following lemma establishes that this is unlikely which allows us to treat new long edges as essentially being independent.

PROPOSITION 4.1 (Pairs of walks rarely intersect at long edges). *There exists $\varepsilon > 0$ (independent of ρ) and a constant $c = c(\varepsilon)$ such that*

$$\mathbb{P}_\mu\big(F^*(\rho, k)\big) < c\big[\rho^{-\varepsilon} + 2^{(1-\varepsilon)k}\rho^{1-o(1)}\big].$$

For a proof of this fact, see Section 9.

We now define a number of events involving a single path that we wish to exclude. For $\gamma, \delta > 0$, let

$$A(\rho) = \{\exists v \in \mathbb{Z}^d, \omega_{0,v} = 1, \|v\|_\infty > \rho\}$$

be the event there is a long edge at the origin, let

$$B(\rho, \gamma, k) = \{\forall v \in \mathbb{Z}^d : \omega_{0,v} = 1, \|v\|_\infty > \rho, v \notin \{X_1, \ldots, X_{2\gamma k+1}\}\}$$

be the event that there is a long edge at the origin and the walk does not visit the other end until time $2^{\gamma k+1}$. We let C denote the event that the walk leaves the ball of radius $2^{\delta k}$ before time $2^{\gamma k}$

$$C(\delta, \gamma, k) = \Big\{\max_{0 \leq t \leq 2^{\gamma k}} \|X_t\|_\infty > 2^{\delta k}\Big\}.$$

We define

$$D(\rho, k) = \{\exists v \in \mathbb{Z}^d, \omega_{0,v} = 1, |v| > \rho,$$
$$\exists J \in [2^k] \text{ s.t. } X_J = v, (0, v) \notin \{(X_i, X_{i+1})\}_{i \leq J}\}$$

to be the event that there is a long edge at the origin, and the walk reaches the other end of the edge without traversing it. Next define

$$E(\rho, \delta, k) = \{\exists v, x : \|v\|_\infty \geq \rho, \min(\|x - v\|_\infty, \|x\|_\infty) \geq 2^{\delta k},$$
$$\omega_{0,v} = 1 \text{ and either } \omega_{x,v} = 1 \text{ or } \omega_{0,x} = 1\}$$

to be the event that there is a long edge at the origin and one of the endpoints is connected to another edge of length at least $2^{\delta k}$. Let

$$F(\rho, \gamma, k) = \{\exists i, 2^{\gamma k+1} \leq i \leq 2^k,$$
$$\exists v \in \mathbb{Z}^d \text{ s.t. } \omega_{0,v} = 1 \text{ and } \|v\|_\infty \geq \rho, X_i \in \{0, v\}\}$$

denote the event that there is a long edge at the origin and the walk returns to either end of the edge at any time after $2^{\gamma k+1}$ steps. Finally let

$$G(\rho, \gamma, \delta, k) = A(\rho) \cap B(\rho, \gamma, k) \cap C(\delta, \gamma, k)$$

denote the event that there is a long edge at the original and the walk leave the leaves a ball of radius $2^{\delta k}$ in time $2^{\gamma k+1}$ without taking the long edge.

We want to show that these events do not occur at any time up to time 2^k. For any event $E \subset \Omega^{\mathbb{Z}}$, let $\mathbf{T}^{-i} \cdot E = \{\underline{\omega} : T^i \cdot \underline{\omega} \in E\}$. For any of the events $O \in \{A, \dots, G\}$ defined above, let $O_i := \mathbf{T}^{-i} O$ and let

$$\mathscr{G}(\rho, \gamma, \delta, k) = \bigcup_{i=0}^{2^k} G_i(\rho, \gamma, \delta, k),$$

$$\mathscr{D}(\rho, k) = \bigcup_{i=0}^{2^k} D_i(\rho, k),$$

$$\mathscr{E}(\rho, \delta, k) = \bigcup_{i=0}^{2^k} E_i(\rho, \delta, k),$$

$$\mathscr{F}(\rho, \gamma, k) = \bigcup_{i=0}^{2^k} F_i(\rho, \gamma, k).$$

Note that all these events are increasing in γ. The following proposition shows we can exclude these events with high probability.

PROPOSITION 4.2.   *Let $d \geq 1$ and $s \in (d, (d+2) \wedge 2d)$ be fixed, and let $\rho > 2^{k/(2(s-d))}$. For any $\delta \in (0, 1)$, there exists $\gamma, \varepsilon > 0$ such that*

$$\mathbb{P}_\mu\big(\mathscr{G}(\rho, \gamma, \delta, k) \cup \mathscr{D}(\rho, k) \cup \mathscr{E}(\rho, \delta, k) \cup \mathscr{F}(\rho, \gamma, k)\big)$$
$$< C\big[\rho^{-\varepsilon} + 2^{k(1-\varepsilon)}\rho^{-\alpha(1-\eta)}\big],$$

where $\eta \in (0, 1)$ is arbitrary, and $C > 0$ depends on δ, η, d, s.

We note that this estimate is of most interest when $\omega \in \Omega_0$, but holds under the measure μ as well; whenever the origin is in a finite component it will be a small component with high probability by Theorem 2 of [11].

4.1. *Ergodic estimates.*   In this subsection we state the ergodic estimates for the number of new vertices encountered by the paths. The proofs are given in Section 8. Let N_i be the indicator of the event that X_i is the first visit to that vertex, that is,

$$N_i = \mathbb{1}\{X_i \notin \{X_0, \dots, X_{i-1}\}\} = \mathbb{1}\{\omega_i \notin \{\omega_0, \dots, \omega_{i-1}\}\}.$$

For a vertex v we let $p_v = p_v(\omega)$ denote the quenched probability that a random walk started from v will ever return there, that is,

$$P_v^\omega(X_t = v \text{ for some } t \in \mathbb{N}).$$

When $v = 0$, we will sometimes use the shorthand $p(\omega) = p_0(\omega)$. The notion of the "type" of a vertex and the frequency with which new vertices of each type are encountered plays a crucial role in our proof. The type is determined by its degree and local return probability. For $0 = q_0 < q_1 < \cdots < q_J < 1$ a finite increasing sequence in $[0, 1]$ and $m \in \mathbb{N}$ let us denote by $C_{q_{j-1},q_j,m}$ the quantity

$$(4) \qquad C_{q_{j-1},q_j,m} = \mathbb{P}_{v_0}\big(X_i \neq 0 \text{ for } i > 0, p_0(\omega) \in (q_{j-1}, q_j), \mathrm{d}^\omega(0) = m\big)$$

and let

$$(5) \qquad\qquad\qquad C^\star = \mathbb{P}_{v_0}(X_i \neq 0 \text{ for } i > 0)$$

denote the annealed escape probability. By reversibility, this is also the rate at which a walk sees new vertices when started in the infinite component. Our main ergodic estimate shows that they indeed give the long-run frequency of new vertices of a particular type. For any subinterval $[a, b] = \mathcal{A} \subseteq [0, 1]$, $\mathcal{M} \subseteq \mathbb{N}$, denote the number of vertices of a given type by

$$N_t^{\mathcal{A},\mathcal{M}} := \sum_{i=1}^t N_i \mathbb{1}\big\{p_{X_i} \in \mathcal{A}, \mathrm{d}^\omega(X_i) \in \mathcal{M}\big\}.$$

For the ℓth walk of the ensemble $(X_i^\ell)_{i \in [2^k], \ell \in [k^3]}$, we denote these by N_t^ℓ and $N_t^{\ell,\mathcal{A},\mathcal{M}}$. Let $L(\mathcal{A}, \mathcal{M}) \in \Omega^{\mathbb{Z}}$ denote the event

$$L(\mathcal{A}, \mathcal{M}) := \{\underline{\omega} : \omega_0 \notin \{\omega_i : i \leq -1\}, p(\omega_0) \in \mathcal{A}, \mathrm{d}(\omega_0) \in \mathcal{M}\}.$$

The following events require that the number of new vertices up to time t for all $1 \leq t \leq 2^k$ is close to what we expect. Let $\mathcal{H}_{k,\chi}^\ell$ denote the event that

$$\sup_{\mathcal{A},\mathcal{M}} \max_{1 \leq t \leq 2^k} \big|N_t^{\ell,\mathcal{A},\mathcal{M}} - t\mathbb{P}_{v_0}(L(\mathcal{A},\mathcal{M}))\big| \leq \chi 2^k$$

and let $\mathcal{H}_{k,\chi}$ denote the event

$$\left\{\frac{1}{k^3} \sum_{\ell=1}^{k^3} \mathbb{1}\{\mathcal{H}_{k,\chi}^\ell\} > 1 - \chi\right\}.$$

Roughly this event says that the ergodic theorem bound holds by time 2^k for most of the independent copies of the random walk. The following lemma shows that $\mathcal{H}_{k,\chi}$ holds for all but finitely many values of k almost surely, and indeed this is even the case for a suitable χ_k converging to 0.

LEMMA 4.3. *For any $\chi > 0$,*

$$(6) \qquad \lim_{k' \to \infty} \mathbb{P}_{\mu_0} \left(\bigcap_{k \geq k'} \mathcal{H}_{k, \chi} \right) = 1$$

and hence there exists $\chi_k > 0$ with $\chi_k \downarrow 0$ such that

$$(7) \qquad \lim_{k' \to \infty} \mathbb{P}_{\mu_0} \left(\bigcap_{k \geq k'} \mathcal{H}_{k, \chi_k} \right) = 1.$$

5. Main coupling.   Recall $(X_i^\ell)_{\ell, i \in \mathbb{N}}$ denote independent copies of the random walk started from the origin with respect to the same environment ω. In this section we define and analyze the main tool of our proof, a coupling of $(X_i^\ell)_{\ell=1}^{k^3}$ to a new sequence of walks $(\hat{X}_i^\ell)_{\ell=1}^{k^3}$ on $\mathbb{Z}^d$ which will be simpler to handle. The coupling involves revealing the environment as the walks progress, as well as a complicated bookkeeping of the long edges each of the walks in the process encounters and the number of times these edges are traversed. In particular, when the process arrives at a vertex which has an edge of length greater than ρ, we keep track of the edge's size as well as the degree and local escape probabilities of each end of the edge.

Recall that $p_v = p_v(\omega)$ denotes the return probability the vertex v in the environment ω, and let $(\mathcal{P}, \mathcal{D})$ denote the distribution of the random vector $(p_0(\omega), d^\omega(0))$ under $\mathbb{P}_\mu$.

For the remainder of our proof, it is important that $1 + \delta d < d/(s - d)$, so that each walk history, thickened by a neighborhood of radius $2^{\delta k}$, takes up volume at most $2^{(1+\delta d)k}$. Because of the power law tail distribution, endpoints of new long edges can occur essentially anywhere in a region of volume $2^{kd/\alpha} \gg 2^{(1+\delta d)k}$ and are very unlikely to be close to the history of the walk. For definiteness, let us fix, for the remainder of our proof,

$$(8) \qquad \delta = 1/2 \left(\frac{1}{\alpha} - \frac{1}{d} \right) \wedge 1/2.$$

Note that for $s \in (d, (d+2) \wedge 2d)$, $\delta > 0$. We may then choose γ sufficiently small so that the results of Section 4 hold for the pair (δ, γ) and any choice of edge size cutoff ρ.

For a vertex v and each $k \in \mathbb{N}$ denote $\tilde{p}_v = \tilde{p}_v(k)$ the probability that a walk started from v and conditioned to stay in the set $\{u : \|v - u\|_\infty < 2^{k\delta}\}$ returns to v before time $2^{k\gamma}$ and set to 1 if v has no neighbors within distance $2^{k\delta}$ which we call the local return probability.

As mentioned in Section 3, when $\alpha \in (0, 1)$ we set $\rho = \rho_k = 2^{k/\alpha}/k^{200/(1-\alpha)}$. When $\alpha \in [1, 2)$ and $d \geq 2$, we set $\rho_k = \lambda 2^{k/\alpha}$ where λ is a small positive constant which we will eventually send to 0. We let $\tilde{d}^\omega(v) := \#\{u : \|v - u\|_\infty \leq 2^{\delta k}\}$, and let $(\mathcal{P}(k), \mathcal{D}(k))$ denote the joint distribution of $(\tilde{p}_v, \tilde{d}^\omega(v))$ of the origin under $\mathbb{P}_\mu$. We observe that $(\mathcal{P}(k), \mathcal{D}(k))$ converges in distribution to $(\mathcal{P}, \mathcal{D})$ as k tends to ∞.

We will prove a more quantitative statement in Lemma 9.12. The purpose of localizing the degrees and escape probabilities is that when a walk arrives a new vertex v we can first reveal these local quantities before revealing if v has any long connections.

For technical reasons, we discretize the the joint distribution $(\mathcal{P}, \mathcal{D})$ as follows. Recall the definitions of $C^\star$ and $C_{q_{j-1}, q_j, j}$ from equations (5) and (4). For each positive integer J, we choose a sequence $0 = q_0 < q_1 < \cdots < q_J < 1$ so that the distribution $\mathcal{P}$ does not have any atoms on the q_i and so that for some sequence ψ_J converging to 0 we have that $\frac{1}{1-q_i} - \frac{1}{1-q_{i+1}} < \psi_J$ and

$$(9) \qquad \bar{C} = C^\star - \sum_{j=1}^{J} \sum_{m=1}^{J} C_{q_{j-1}, q_j, j}$$

for all J.

5.1. *Coupling variables.* In our construction of the coupling of the walks, the random variables we use will depend on the local neighborhood the walk is in. To define the coupling, we introduce several sequences of random variables. For each $i \geq 1$, $(j, m) \in [J]^2 \cup \{(0, 0)\}$, $\ell \in [k^3]$ and $x \in \mathbb{Z}^d$ define $w_i^{\ell, j, m}(x)$ to be independent Bernoulli random variables with probability

$$\mathbb{P}(w_i^{\ell, j, m}(x) = 1) = \mathbb{P}(\|x\|_2).$$

Also let $w_i(x)$ (with no superscripts) be independent Bernoulli random variables with the same probabilities. These random variables will be coupled with the newly revealed edges found by the exploration process.

We will denote the geometric distribution by $\mathbb{P}(\mathrm{Geom}(p) = r) = (1 - p)^r p$ for $r \geq 0$. Let $U_0, U_1, \ldots$ be an i.i.d. sequence of uniform $[0, 1]$ random variables, and define a geometric process as

$$(10) \qquad R(t) = \min\{i \geq 0 : U_i < t\}.$$

Then $R(t)$ is a decreasing integer valued stochastic process on $[0, 1)$ with marginals given by $\mathrm{Geom}(t)$. For each $i, \ell \geq 0$ and $(j, m) \in [J]^2 \cup \{(0, 0)\}$, let $R_i^{\ell, j, m}(t)$ and $\tilde{R}_i^{\ell, j, m}(t)$ be independent copies of $R(t)$. These processes will be used to decide how many times the walk crosses a long edge before escaping and never returning to the neighborhood.

Finally for each $i \geq 0$, $(j, m) \in [J]^2 \cup \{(0, 0)\}$ and $\ell \in [k^3]$, let $(\mathfrak{r}_i^{\ell, j, m}, \mathfrak{d}_i^{\ell, j, m})$ be independent variables, distributed as $(\mathcal{P}(k), \mathcal{D}(k))$. When a new long edge is encountered by a walk of the process, the local neighborhood of the other side of the long edge will generally be unrevealed and so independent of the walk so far and will be coupled with the $(\mathfrak{r}_i^{\ell, j, m}, \mathfrak{d}_i^{\ell, j, m})$.

5.2. *Coupling construction.* Using the random variables defined above, we now show how we couple the environment ω and the sequence of k^3 walks with the variables defined in the previous subsection. In the following section we will then use these to couple the walks to a family of processes $(\hat{X}_i^\ell)$. We will reveal the edges of the graph either as the sequence of walks encounters them, or if they are in some local neighborhood of the vertices the walks do encounter.

The key point we analyze is the behavior of each walk after it encounters a new long edge of size greater than ρ_k. Thus we define the coupling construction by two sets of rules: one which will be used for most of the walk and a second special phase which begins when a new long edge is encountered and which then runs for $2^{k\gamma+1}$ steps.

As part of our coupling process we will define several auxiliary indicator or "flag" variables to track certain events through the coupling. Roughly, they are described as follows:

- New long jump: $\mathcal{A}_{i,\ell}$ represents that in step i of walk ℓ a new long edge has been encountered.
- Phase of the coupling: $\mathcal{A}_{i,\ell}^*$ will indicate the phase that the walk is in, with a value 1 indicating that we are in the special phase and have recently encountered a long edge.
- Error variables: $\mathcal{B}_{i,\ell}$ represents that one of several types of rare events occurred which we loosely describe as an "error." We restrict our attention to paths constructed when no errors occur.
- New vertex variables: $\mathcal{N}_i^{\ell,j,m}$ for $(j,m) \in [J]^2 \cup \{(0,0)\}$ will denote whether a new vertex has been found and what its local return probability $\tilde{p}_v$ and local degree $\tilde{d}^\omega(v)$ are.

More precisely, we set $\mathcal{A}_{i-1,\ell}^*$ to be the indicator of the event that for some $i - 2^{\gamma k+1} - 1 \leq i' < i - 1$ that $\mathcal{A}_{i',\ell} = 1$ indicating a recent long jump. This will indicate which phase we are to use. Let v denote X_{i-1}^ℓ.

Main phase: In this case $\mathcal{A}_{i-1,\ell}^* = 0$. We now describe how a new step from $v = X_{i-1}^\ell$ to X_i^ℓ is chosen.

Case 1 (*already visited vertices*): Suppose that $v \in \mathcal{W}_{i-2,\ell}^+$.

By definition, in this case one of the first ℓ walks has already visited v or has visited a neighbor of v at distance more than ρ. Either way, the entire local neighborhood of v has already been revealed and the walk chooses X_i^ℓ uniformly amongst the neighbors of v.

- Set $\mathcal{A}_{i-1,\ell} = 0$.
- Set $\mathcal{B}_{i-1,\ell} = 1$ if there exists $y \in \mathbb{Z}^d$ with $\omega_{v,y} = 1$ and $|v - y| > \rho$. We will call this a *type one* error. Otherwise set $\mathcal{B}_{i-1,\ell} = 0$.
- Set $\mathcal{N}_{i-1}^{\ell,j,m} = 0$ for all $(j,m) \in [J]^2 \cup \{(0,0)\}$.

CLAIM 5.1.   *There exists $\varepsilon > 0$ such for each ℓ we have that $\mathbb{P}(\exists 0 \leq i \leq 2^k - 1 : \mathcal{B}_{i-1,\ell} = 1) = o(2^{-k\varepsilon})$.*

PROOF.   This event occurs if at some time $i \in [2^k - 1]$, a long edge e of length greater than ρ is encountered in the main phase which has already been encountered by walk ℓ or one of the previous walks. If it were previously encountered by this walk (and not by a previous one), then it must have started a special phase (see below), and so must have been encountered some time before time $i - 2^{\gamma k}$. The probability of this event is bounded by Proposition 4.2. The bound is completed using Proposition 4.1 and a union bound to account for possible intersections of different walks at long edges.   □

Case 2 (*new vertices*): Suppose next that $v \notin \mathcal{W}^+_{i-2,\ell}$, and so we are at a new vertex. First reveal all edges not already revealed in the local neighborhood of v of radius δ and all edges connected to v of length at most ρ. That is, we reveal

$$\{\omega_{v,y} : \|y\|_\infty \leq \rho\} \quad \text{and} \quad \{\omega_{x,y} : x, y \in \mathbb{Z}^d, \|x - v\|_\infty \leq 2^{\delta k}, \|y - v\|_\infty \leq 2^{\delta k}\}$$

so that we have revealed the σ-algebra $\mathcal{F}^-_{i-1,\ell}$. Both the local escape probability $\tilde{p}_v$ and local degree $\tilde{d}^\omega(v)$ are $\mathcal{F}^-_{i-1,\ell}$ measurable.

We now classify the neighborhood according it its local escape probability and degree. If $\tilde{p}_v \leq q_J$ and $\tilde{d}^\omega(v) \leq J$, then set $j = \min\{j' : q_{j'} > p_v\}$ and $m = \tilde{d}^\omega(v)$; otherwise set $j = 0, m = 0$. Set $\mathcal{N}^{\ell,j,m}_{i-1} = 1$ and the other $\mathcal{N}^{\ell,\cdot,\cdot}_{i-1}$ to 0. Let

$$\iota = \phi^{\ell,j,m}_i = \sum_{i'=0}^{i-1} \mathcal{N}^{\ell,j,m}_{i'}$$

count the number of distinct *new* vertices of type (j, m) which have been encountered so far by the ℓth walk. We note the use of two notations $\iota, \phi^{\ell,j,m}_i$; our use of ι implicitly depends on the fixed triple (ℓ, j, m).

Using this local classification we now reveal the possible long edges from v by coupling them to the w random variables. For each $x \notin \mathcal{W}^+_{i-2,\ell}$ with $\|x - v\|_\infty > \rho$ we have that $\omega_{v,x}$ is so far unrevealed. We couple the environment and the random variables $w^{\ell,j,m}_\iota$ so that for each such x, $\omega_{v,x} = w^{\ell,j,m}_\iota(x - v)$. Our procedure now depends on whether or not any of the $w^{\ell,j,m}_\iota(x - v)$ are nonzero:

(1) *No long edges.* $\sum_{x:\|x-v\|_\infty > \rho} w^{\ell,j,m}_\iota(x - v) = 0$; then we do not encounter a new long edge. Set $\mathcal{A}_{i-1,\ell} = 0$, and $\mathcal{B}_{i-1,\ell} = 0$ and choose the next step of the walk X^ℓ_i uniformly from the neighbors of v.

(2) *Multiple long edges or long edges returning to previous neighborhood.* $\sum_{x:\|x-v\|_\infty > \rho} w^{\ell,j,m}_\iota(x - v) \geq 2$ or $\sum_{x:\|x-v\|_\infty > \rho} w^{\ell,j,m}_\iota(x - v) = 1$ and there exist x, y such that $\|x - y\|_\infty \leq 2^{\delta k+1}$, $w^{\ell,j,m}_\iota(x - v) = 1$ and $y \in \mathcal{W}^+_{i-2,\ell}$. Both of these are unlikely and we will call them *type two* errors. Set $\mathcal{A}_{i-1,\ell} = 0$ and $\mathcal{B}_{i-1,\ell} = 2$ and choose X^ℓ_i uniformly from the neighbors of v.

(3) *Single good long edge.* In the remaining case we have some x with $\|x - v\|_\infty > \rho$ and $\omega_{v,x} = 1$. Moreover, the set of edge indicator variables $\{\omega_{y,z} : y, z \in V_x\}$ are so far unrevealed and are therefore independent of the construction up to this point. The choice of edges here determines the local degree and escape probabilities $(\tilde{p}_x, \tilde{\mathrm{d}}^\omega(x))$ which are distributed according to $(\mathcal{P}(k), \mathcal{D}(k))$. We reveal the edges

$$\{\omega_{y,z} : y, z \in V_x\}$$

and couple them so that $(\tilde{p}_x, \tilde{\mathrm{d}}^\omega(x)) = (\mathfrak{r}_t^{\ell,j,m}, \mathfrak{d}_t^{\ell,j,m})$. We also reveal the remaining edges in $\{\omega_{x,y} : y \in \mathbb{Z}^d\}$. If $\mathrm{d}^\omega(v) \neq \tilde{\mathrm{d}}^\omega(v) + 1$ or $\mathrm{d}^\omega(x) \neq \tilde{\mathrm{d}}^\omega(x) + 1$ set $\mathcal{B}_{i-1,\ell} = 3$, otherwise set $\mathcal{A}_{i-1,\ell} = 1$ and $\mathcal{B}_{i-1,\ell} = 0$.

CLAIM 5.2.   *With δ as in* (8) *there exists $\varepsilon > 0$ such that*

$$P\big(\exists 0 \leq i \leq 2^k - 1, \exists \ell \in [k^3] : \mathcal{B}_{i-1,\ell} = 2\big) = o(2^{-k\varepsilon}).$$

PROOF.   A type 2 error means that either the walk encounters a new vertex with two long edges or that nearby there is a long edge connecting to somewhere already encountered by this or one of the previous walks. First note that the decay of the probabilities of the $w_i(z)$ gives that $\mathbb{P}(\sum_{z:\|z\|_\infty > \rho} w_i(z) \geq 2) \leq 2^{-2k(1-\varepsilon)}$. By a union bound, this implies that we never have $\sum_{x:\|x-v\|_\infty > \rho} w_i^{\ell,j,m}(x - v) \geq 2$ except with probability $o(2^{-k\varepsilon})$. A similar analysis shows that none of the k^3 walks ever encounters a vertex with two long edges except with probability $o(2^{-k\varepsilon})$.

By definition, for any i, ℓ we have a bound on the total number of edges visited as $|\mathcal{W}_{i,\ell}| \leq k^3 2^k$. Now if none of the walks ever reaches a vertex with two long edges, then for any i and ℓ we have that $|\mathcal{W}_{i,\ell}^+| \leq 2k^3 2^k$.

$$\mathbb{P}\Bigg(\sum_{\substack{x:\|x-v\|_\infty > \rho \\ \min_{y \in \mathcal{W}_{i,\ell}^+} \|x-y\|_\infty \leq 2^{k\delta+1}}} w_i^{\ell,j,m}(x - v) \geq 1 \Bigg) \leq \mathbb{P}(\rho) 2^{1+d(k\delta+2)} k^3 2^k$$

$$= o\big(2^{-k(1+\varepsilon)}\big)$$

for some fixed $\varepsilon > 0$. By our choice of δ, this holds because the walks only explore a vanishing proportion of the local area on the length scale ρ. A union bound completes the proof.   □

CLAIM 5.3.   *With*

$$\delta = 1/2\left(\frac{1}{s-d} - \frac{1}{d}\right) \wedge 1/2,$$

there exists $\varepsilon > 0$ such that

$$\mathbb{P}\big(\exists 0 \leq i \leq 2^k - 1, \exists \ell \in [k^3] : \mathcal{B}_{i-1,\ell} = 3\big) = o(2^{-k\varepsilon}).$$

In fact, we may take $\varepsilon = \delta(s - d) - o(1)$.

PROOF.   For the event $\{\mathcal{B}_{i-1,\ell} = 3\}$ to take place, the walk must encounter a vertex v at time i with an edge (v,u) of length at least ρ such that either v or u is attached to another edge of length at least $2^{\delta k}$ which implies that the event $\mathcal{E}(\rho,\delta,k)$ takes place, and hence the bound follows by Proposition 4.2.   □

Special phase: *Coupling procedure after encountering a new long edge.* We now describe the more complicated coupling after a long edge is encountered (when $\mathcal{A}_{i-1,\ell} = 1$). At such an event, the walk is at the vertex $v = X_{i-1}^\ell$, which is connected to a vertex x such that $\|v - x\|_\infty > \rho$. Our coupling ensures that

$$\big(\tilde{p}_x, \tilde{d}^\omega(x)\big) = \big(\mathfrak{r}_\iota^{\ell,j,m}, \mathfrak{d}_\iota^{\ell,j,m}\big),$$

where $\iota = \sum_{i'=1}^{i-1} \mathcal{N}_{i'}^{\ell,j,m}$. For the rest of this subsection, if $\tilde{p}_v \le q_J$ and $\tilde{d}^\omega(v) \le J$, then we denote $j = \min\{j : q_j > p_v\}$ and $m = \tilde{d}^\omega(v)$ and otherwise $j = 0, m = 0$.

If the walk is in the infinite component, transience implies it will cross the edge (v,x) a finite number times, but then escape and never return to the local neighborhood. In the scaling limit the crucial information will be the parity of the number of times the walk crosses the edge. In our coupling this will be determined by the geometric processes $R_t^{\ell,j,m}(t)$ and $\tilde{R}_\iota^{\ell,j,m}(t)$ [where ι represents the number of vertices of type (j,m) the ℓth walk has encountered by time $i-1$].

Let V^* denote the graph with vertices $V_v \cup V_x$ and edges

$$\{(v,x)\} \cup \{(y,z) : y, z \in V_v, \omega_{y,z} = 1\} \cup \{(y,z) : y, z \in V_x, \omega_{y,z} = 1\}$$

and let Y_t denote a random walk on V^* started at v. Recall our choice $\delta = 1/2(\frac{1}{s-d} - \frac{1}{d}) \wedge 1/2$. For γ depending on δ as in Section 4, let

$$\tau^* = \inf\{t > 2^{\gamma k} : \forall t - 2^{\gamma k} \le t' \le t, Y_t \notin \{v, x\}\};$$

that is, τ^* is the first time that the walk Y_t has not been at either v or x in the last $2^{\gamma k}$ steps. What we want to know is, at time τ^*, which side of V^* will Y_t be on (i.e., is Y_{τ^*} in V_v or V_x)?

Since (v,x) is the only edge between V_v and V_x a walk starting from v can do one of three types of excursions:

(1) Move to x with probability $\frac{1}{1+\tilde{d}^\omega(v)}$.

(2) Move to another vertex in V_v; then perform a walk in V_v and return to v in with in the next $2^{\gamma k}$ step with probability $\frac{\tilde{p}_v \tilde{d}^\omega(v)}{1+\tilde{d}^\omega(v)}$.

(3) Move to another vertex in V_v; then perform a walk in V_v and not return to v in the next $2^{\gamma k}$ steps with probability $\frac{(1-\tilde{p}_v)\tilde{d}^\omega(v)}{1+\tilde{d}^\omega(v)}$.

Let R_v be the number of excursions of type (1) made by the walk from v before it makes an excursion of type (3) from v.

Analogous statements hold for walks started from x. Let R_x be the number of excursions of type (1) from x made by the walk from the first time it visits x before

it makes an excursion of type (3) from x. The following claim is immediate from the definitions.

LEMMA 5.4. *The random variables R_v and R_x are independent and distributed, respectively, as*

$$\text{Geom}\left(\frac{(1-\tilde{p}_v)\tilde{d}^\omega(v)}{1+(1-\tilde{p}_v)\tilde{d}^\omega(v)}\right) \quad and \quad \text{Geom}\left(\frac{(1-\tilde{p}_x)\tilde{d}^\omega(x)}{1+(1-\tilde{p}_x)\tilde{d}^\omega(x)}\right).$$

Moreover, if $R_v > R_x$, then $Y_{\tau^*} \in V_x$ while if $R_v \leq R_x$ then $Y_{\tau^*} \in V_v$.

PROOF.   The type of excursion depends only on steps from V_v and V_x, respectively, and thus R_v and R_x are independent. By definition the time τ^* occurs in the first type 3 excursion, and hence R_v and R_x determines which side the walk is on at time τ^*. Note that the asymmetry between $R_v > R_x$ and $R_v \leq R_x$ comes from the fact that the walk starts at v.   □

So we may couple the random walk Y_t to the process (and in particular X^ℓ) so that

$$R_v = R_i^{\ell,j,m}\left(\frac{(1-\tilde{p}_v)\tilde{d}^\omega(v)}{1+(1-\tilde{p}_v)\tilde{d}^\omega(v)}\right), \qquad R_x = \tilde{R}_i^{\ell,j,m}\left(\frac{(1-\tilde{p}_x)\tilde{d}^\omega(x)}{1+(1-\tilde{p}_x)\tilde{d}^\omega(x)}\right).$$

Now construct the random walk step by step from X_{i-1+t}^ℓ to X_{i+t}^ℓ for $0 \leq t \leq 2^{\gamma k+1} - 1$ as follows:

(1) If X_{i-1+t} is a vertex not already visited, then reveal any unrevealed edges in the set

$$\{\omega_{X_i^\ell,y} : y \in \mathbb{Z}^d\} \cup \{\omega_{x,y} : x, y \in V_{X_i^\ell}\}.$$

(2) If there exists $y \in \mathbb{Z}^d$ with $\omega_{X_{i-1+t}^\ell,y} = 1$ and $\|X_{i-1+t}^\ell - y\|_\infty > \rho$ and $\{X_{i-1+t}^\ell, y\} \neq \{v, x\}$, then set $\mathcal{B}_{i-1+t,\ell} = 4$.

(3) Choose X_{i+t}^ℓ uniformly amongst the neighbors of X_{i-1+t}^ℓ. If $X_{i-1+t}^\ell = Y_t$ and the edge $(X_{i-1+t}^\ell, X_{i+t}^\ell)$ is in the graph V^*, then couple so that $X_{i+t}^\ell = Y_{t+1}$.

Let

$$\tau = \begin{cases} 2^{\gamma k+1}, & \forall 1 \leq t \leq 2^{\gamma k+1} - 1, X_{i+t}^\ell = Y_t, \\ \min\{1 \leq t < 2^{\gamma k+1} : X_{i+t}^\ell \neq Y_t\}, & \text{otherwise,} \end{cases}$$

and set $\mathcal{B}_{i-1+\tau,\ell} = 5$ if $\tau \neq 2^{\gamma k+1}$. Note that conditional on X_{i-1+t}^ℓ staying within V^* up to time $i + 2^{\gamma k+1}$ and it making no steps between V_v and V_x except along the edge (v, x), then $\tau = 2^{\gamma k+1}$.

Let $\mathcal{K}$ denote the event that $\tau = 2^{\gamma k+1}$, that $\tau^* < 2^{\gamma k+1}$ and that $Y_t \notin \{v, x\}$ for all $\tau^* \leq t \leq 2^{\gamma k+1}$. If $\mathcal{K}$ does not hold, set $\mathcal{B}_{i-1+2^{\gamma k+1}}^\ell = 6$. Finally, inside the

special phase we set all $\mathcal{N}^{\ell,j,m}_{i-1+t} = 0$, $\mathcal{A}_{i-1+t,\ell} = 0$ and set $\mathcal{B}_{i-1+t,\ell} = 0$ unless otherwise stated. By definition $\mathcal{A}^*_{i-1+t,\ell} = 1$ inside the special phase.

The following lemma is immediate from the construction.

LEMMA 5.5. *On the event $\mathcal{K}$ we have that $X^\ell_{i-1+2^{\gamma k+1}} \in V_x$ if $R_v > R_x$ and* $X^\ell_{i-1+2^{\gamma k+1}} \in V_v$ *if $R_v \leq R_x$.*

We now bound the probability of errors of type 4, 5 or 6 (i.e., $\mathcal{B}^\ell_i \in \{4, 5, 6\}$).

CLAIM 5.6. *With*

$$\delta = 1/2 \left(\frac{1}{s-d} - \frac{1}{d} \right) \wedge 1/2,$$

there exists $\gamma, \varepsilon > 0$ so that we have

$$P\big(\exists 0 \leq i \leq 2^k - 1, \exists \ell \in [k^3] : \mathcal{B}_{i-1,\ell} = \{4, 5, 6\} \big) = o\big(2^{-k\varepsilon} \big).$$

PROOF.   For an error of type 4 to occur the walk must encounter 2 long jumps of length at least ρ within $2^{\gamma k+1}$ steps. For an error of type 5 to occur it must encounter a long edge and then leave the $2^{\delta k}$ neighborhood of that edge in time less than $2^{\gamma k+1}$. Finally an error of type 6 to occur implies that the walk returns to a long edge after an excursion of at least $2^{\gamma k}$. The probability of each of these events is bounded by Proposition 4.2 establishing the claim.   □

This completes the coupling. We denote by $\mathcal{G}$ the event that no errors occurred in the coupling (i.e., $\sum_{\ell=1}^{k^3} \sum_{i=0}^{2^k} \mathcal{B}^\ell_i = 0$). Combining Claims 5.1, 5.2, 5.3 and 5.6 we have the following result.

LEMMA 5.7. *There exists $\varepsilon > 0$ such that $P(\mathcal{G}) \geq 1 - O(2^{-\varepsilon k})$.*

We now use this coupling as the basis for establishing the scaling limit.

6. Limiting processes.   In this section we construct a series of approximations of X^ℓ_i culminating in a stable process. The first approximation is a process $\hat{X}^\ell_i$ which contains only the macroscopic vectors which correspond to long edges crossed an odd number of times. This approximation still has a complex dependence on the environment and previous walks. Thus, the second approximation step replaces $\hat{X}^\ell_i$ with $\mathfrak{X}^\ell_i$, a process with independent increments in the domain of attraction of a stable law. Finally we approximate $\mathfrak{X}^\ell_i$ with $\Gamma^\ell_\alpha(t)$ which are independent α-stable Lévy motions. Throughout this section we use the notation $|x| = \|x\|_\infty$ for $x \in \mathbb{R}^d$.

6.1. *Approximating* $(X_i^\ell)_{i,\ell}$ *by* $(\hat{X}_i^\ell)_{i,\ell}$. For the first approximation step we construct a process $(\hat{X}_i^\ell)_{i,\ell}$ and compare this family to the underlying walks directly. We emphasize that the bounds obtained in this coupling *do not* require that 0 is in the infinite component. Indeed, when 0 is not in the infinite component then its component will, with high probability, have diameter less than ρ, and so both processes $(X_i^\ell)_{i,\ell}$ and $(\hat{X}_i^\ell)_{i,\ell}$ will be close to 0 after spatial space by $2^{-k/(s-d)}$.

For $i \in [2^k]$, $(j, m) \in [J]^2 \cup \{(0, 0)\}$ and $\ell \in [k^3]$, define

$$\sigma_i^{\ell,j,m} = \mathbb{1}\left\{ R_i^{\ell,j,m}\left(\frac{(1 - \tilde{p}_{X_{i-1}^\ell})m}{1 + (1 - \tilde{p}_{X_{i-1}^\ell})m}\right) > \tilde{R}_i^{\ell,j,m}\left(\frac{(1 - \mathfrak{r}_i^{\ell,j,m})\mathfrak{d}_i^{\ell,j,m}}{1 + (1 - \mathfrak{r}_i^{\ell,j,m})\mathfrak{d}_i^{\ell,j,m}}\right) \right\},$$

which determines on which side of V^* the walk ended, and in particular, whether it contributes to the scaling limit. We want to sandwich these indicators between random variables which depend only on j and not $\tilde{p}_{X_{i-1}}$. To this end we define

$$\sigma_i^{+,\ell,j,m} = \mathbb{1}\left\{ R_i^{\ell,j,m}\left(\frac{(1 - q_j)m}{1 + (1 - q_j)m}\right) > \tilde{R}_i^{\ell,j,m}\left(\frac{(1 - \mathfrak{r}_i^{\ell,j,m})\mathfrak{d}_i^{\ell,j,m}}{1 + (1 - \mathfrak{r}_i^{\ell,j,m})\mathfrak{d}_i^{\ell,j,m}}\right) \right\}$$

and

$$\sigma_i^{-,\ell,j,m} = \mathbb{1}\left\{ R_i^{\ell,j,m}\left(\frac{(1 - q_{j-1})m}{1 + (1 - q_{j-1})m}\right) > \tilde{R}_i^{\ell,j,m}\left(\frac{(1 - \mathfrak{r}_i^{\ell,j,m})\mathfrak{d}_i^{\ell,j,m}}{1 + (1 - \mathfrak{r}_i^{\ell,j,m})\mathfrak{d}_i^{\ell,j,m}}\right) \right\}$$

for $i \in [2^k]$, $(j, m) \in [J]^2$ and $\ell \in [k^3]$. Recall that if $\tilde{p}_v \le q_J$ and $\tilde{\mathrm{d}}^\omega(v) \le J$, then we denote $j = \min\{j' : q'_j > p_v\}$ and $m = \tilde{\mathrm{d}}^\omega(v)$ and otherwise $j = 0, m = 0$. Then for this choice of (j, m), $q_{j-1} \le \tilde{p}_{X_{i-1}^\ell} \le q_j$, and it follows that $\sigma_i^{-,\ell,j,m} \le \sigma_i^{\ell,j,m} \le \sigma_i^{+,\ell,j,m}$ as $R(t)$ is decreasing.

LEMMA 6.1.   *For each* $(j, m) \in [J]^2$ *the following limit exists:*

$$\varsigma_{j,m,J} := \lim_{k \to \infty} \mathbb{P}\left(R_1^{1,j,m}\left(\frac{(1 - q_j)m}{1 + (1 - q_j)m}\right) > \tilde{R}_1^{1,j,m}\left(\frac{(1 - \mathfrak{r}_1^{1,j,m})\mathfrak{d}_1^{1,j,m}}{1 + (1 - \mathfrak{r}_1^{1,j,m})\mathfrak{d}_1^{1,j,m}}\right) \right).$$

PROOF.   Note that $(\mathfrak{r}_1^{1,j,m}, \mathfrak{d}_1^{1,j,m})$ depends on k. It is taken from the distribution $(\mathcal{P}(k), \mathcal{D}(k))$. By Lemma 9.12 it follows that $(\mathcal{P}(k), \mathcal{D}(k))$ converges to $(\mathcal{P}, \mathcal{D})$ which completes the result.   □

Define

$$Z_i^{+,\ell,j,m} = \sigma_i^{+,\ell,j,m} \sum_{|x| > \rho} x w_i^{\ell,j,m}(x)$$

and similarly define $Z_i^{-,\ell,j,m}$ and $Z_i^{\ell,j,m}$ replacing $\sigma_i^{+,\ell,j,m}$ with $\sigma_i^{-,\ell,j,m}$ and $\sigma_i^{\ell,j,m}$, respectively. Recall that for each $i \in [2^k]$, $(j,m) \in [J]^2 \cup \{(0,0)\}$ and $\ell \in [k^3]$, we defined

$$\phi_i^{\ell,j,m} = \sum_{i'=0}^{i-1} \mathcal{N}_{i'}^{\ell,j,m}$$

and now define

$$\hat{X}_i^\ell = \sum_{(j,m)\in[J]^2\cup\{(0,0)\}} \sum_{i'=1}^{\phi_i^{\ell,j,m}} Z_{i'}^{\ell,j,m}.$$

We will show that under the event $\mathcal{G}$ (which was defined at Lemma 5.7) we can jointly couple the paths X^ℓ and $\hat{X}^\ell$ with high probability in the L^q norm. The coupling will be stronger at times in the main phase than in the special phase: in the latter phase, the paths may differ more as the walk X^ℓ may traverse a long edge multiple times while in $\hat{X}^\ell$ the corresponding jump only occurs once. Denote the set of times in the main phase as $\mathcal{I}^\ell = \{1 \le i \le 2^k : \mathcal{A}_{i-1,\ell}^* = 0\}$, and let $\mathcal{X}^\ell = \{1 \le i \le 2^k : \mathcal{A}_{i-1,\ell} = 1\}$ denote the set of times the coupling enters the special phase. Finally define

$$Z_{\max}^\ell = \sum_{i=1}^{2^k} \sum_{(j,m)\in[J]^2\cup\{(0,0)\}} \sum_{\substack{x\in\mathbb{Z}^d \\ |x|\ge\rho_k}} |x| w_i^{\ell,j,m}(x),$$

which we will use as an overall bound on the total jumps. In the following lemma we control the coupling in the main and special phases separately. In particular, it will turn out that the coupling is $o(2^{k/(s-d)})$ in the main phase, but only $O(2^{k/(s-d)})$ in the special phase.

LEMMA 6.2. *When $\alpha \in (0,1)$ or when $\alpha \in [1,2)$ and condition (3) holds, there exists $\delta_0 > 0$ depending on d,s so that for all $\delta \in (0,\delta_0)$, there exists $\gamma, \varepsilon > 0$ and for any $\kappa > 0$ [and in the case of $\alpha \ge 1$ $\exists \lambda^*(s,d,\kappa) > 0$ such that for all $0 < \lambda < \lambda^*$] and for large enough k,*

$$(11) \qquad \mathbb{P}\big(|[2^k] \setminus \mathcal{I}^\ell| \ge 2^{2\gamma k}|\mathcal{G}\big) < 2^{-\varepsilon k}$$

that

$$(12) \qquad \mathbb{P}\Big(\#\Big\{\ell \in [k^3] : \max_{i\in\mathcal{I}^\ell} 2^{-k/(s-d)}|\hat{X}_i^\ell - X_i^\ell| > \kappa\Big\} > \kappa k^3 |\mathcal{G}\Big) \le 2k^{-3}$$

and

$$\mathbb{P}\Big(\#\Big\{\ell \in [k^3] : \max_{i\in[2^k]\setminus\mathcal{I}^\ell} 2^{-k/(s-d)}|\hat{X}_i^\ell - X_i^\ell| > \kappa + \frac{Z_{\max}}{2^{-k/(s-d)}}\Big\} > \kappa k^3 |\mathcal{G}\Big)$$

$$(13) \qquad \le 2k^{-3}.$$

PROOF. By our construction, on the event $\mathcal{G}$ that there are no errors in our coupling, we have $|[2^k] \setminus \mathcal{I}^\ell| \le 2^{\gamma k+1}|\mathcal{X}^\ell|$. Now a vertex is in $\mathcal{X}^\ell$ if and only if it is a new vertex where the walk encounters a long jump, and hence $|\mathcal{X}^\ell|$ is stochastically dominated by the binomial random variable $B(2^k, p)$ where

$$p = \mathbb{P}\left(\sum_{i=1}^{2^k} \sum_{(j,m)\in[J]^2\cup\{(0,0)\}} \sum_{|y|>\rho} w_1^{\ell,j,m}(y) = 1\right) \le \sum_{|y|>\rho} \mathbb{P}(|y|) = 2^{-k(1+o(1))}$$

and so by the Azuma–Hoeffding inequality (with room to spare)

$$(14) \qquad\qquad \mathbb{P}\big(|\mathcal{X}^\ell| \ge 2^{\gamma k-1}|\mathcal{G}\big) < 2^{-\varepsilon k};$$

hence equation (11) holds.

Observe that if $\hat{X}_i^\ell \ne \hat{X}_{i-1}^\ell$, then $i \in \mathcal{X}^\ell$. So suppose that $i \in \mathcal{X}^\ell$ and let v denote X_{i-1}^ℓ. On the event $\mathcal{G}$ we have that v is connected to a unique vertex x such that $|v - x| > \rho$. If $\tilde{p}_v \le q_J$ and $\tilde{d}^\omega(v) \le J$, then, as before, we denote $j = \min\{j' : q_{j'} > p_v\}$ and $m = \tilde{d}^\omega(v)$ and otherwise $j = 0, m = 0$. Recall the notation $\iota = \sum_{i'=1}^{i-1} \mathcal{N}_{i'}^{\ell,j,m}$. Again, under $\mathcal{G}$ the walk stays inside $V^* = V_v \cup V_x$ until at least time $i - 1 + 2^{\gamma k+1}$. At time $i - 1 + 2^{\gamma k+1}$ it is in V_v if $\sigma_\iota^{\ell,j,m} = 0$ and in V_x if $\sigma_\iota^{\ell,j,m} = 1$. Hence on the event $\mathcal{G}$,

$$\big|(X_{i-1+2^{\gamma k+1}}^\ell - X_{i-1}^\ell) - (\hat{X}_{i-1+2^{\gamma k+1}}^\ell - \hat{X}_{i-1}^\ell)\big|$$
$$= \big|X_{i-1+2^{\gamma k+1}}^\ell - X_{i-1}^\ell - (x - v)\sigma_\iota^{\ell,j,m}\big| \le 2^{\delta k}$$

and

$$\max_{0\le h\le 2^{\gamma k+1}-1} \left|\sum_{i'=0}^h (X_{i+i'}^\ell - X_{i+i'-1}^\ell)\mathbb{1}\{\|X_{i+i'}^\ell - X_{i+i'-1}^\ell\|_\infty \le \rho_k\}\right| \le 2^{\delta k}$$

by the definition of V_v and V_x.

Further, on the event $\mathcal{G}$ we have that the displacement from X_{i-1}^ℓ to X_i^ℓ is smaller than ρ when $i \in \mathcal{I}^\ell$. It follows that

$$\max_{i\in\mathcal{I}^\ell}|\hat{X}_i^\ell - X_i^\ell| \le \sum_{i\in\mathcal{X}^\ell} \big|(X_{i-1+2^{\gamma k+1}}^\ell - X_{i-1}^\ell) - (\hat{X}_{i-1+2^{\gamma k+1}}^\ell - \hat{X}_{i-1}^\ell)\big|$$

$$(15) \qquad\qquad + \max_{1\le h\le 2^k}\left|\sum_{\substack{1\le i\le h \\ i\in\mathcal{I}^\ell}} (X_i - X_{i-1})\right|$$

$$\le 2|\mathcal{X}|2^{\delta k} + \max_{1\le h\le 2^k}\left|\sum_{i=1}^h (X_i^\ell - X_{i-1}^\ell)\mathbb{1}\{|X_i - X_{i-1}| \le \rho\}\right|.$$

By Lemma 9.11 when $\alpha \in (0, 1)$ and by condition (3) and large deviations bounds when $\alpha \in [1, 2)$ we have that for sufficiently large k and sufficiently small λ,

$$
\mathbb{P}\left(\#\left\{\ell \in [k^3]: \max_{1 \le h \le 2^k} \frac{1}{2^{k/\alpha}}\left|\sum_{i=1}^{h}(X_i^\ell - X_{i-1}^\ell)\mathbb{1}\{|X_i - X_{i-1}| \le \rho|\right|\right\} > \frac{\kappa k^3}{2}\right)
$$
(16)
$$
\le 2k^{-3}.
$$

Combining (14), (15) and (16) establishes (12).

Suppose $i' \in [2^k] \setminus \mathcal{I}^\ell$ with $i < i' \le i + 2^{\gamma k+1}$. Then given $\mathcal{G}$ and using the notation above, we have that

$$
\hat{X}_{i'}^\ell - \hat{X}_{i-1}^\ell = (x - v)\sigma_i^{\ell, j, m}.
$$

Since $X_{i'}^\ell \in V^*$,

$$
\min\{|X_{i'}^\ell - X_{i-1}^\ell|, |X_{i'}^\ell - X_{i-1}^\ell - (x - v)|\} \le 2^{\delta k}
$$

and $|x - v| \le Z_{\max}^\ell$, so combining this with (12),

$$
\mathbb{P}\left(\#\left\{\ell \in [k^3]: \max_{i \in [2^k] \setminus \mathcal{I}^\ell} \frac{1}{2^{k/\alpha}}|\hat{X}_i^\ell - X_i^\ell| > \kappa + \frac{Z_{\max}}{2^{k/(s-d)}}\right\} > \kappa k^3 \Big| \mathcal{G}\right) \le 2k^{-3},
$$

which completes the proof. □

We now rescale the walks to processes in $D[0, 1]$: let us define, for $0 \le t \le 1$,
$$
X^\ell(t) := 2^{-k/(s-d)} X_{\lfloor t2^k \rfloor}^\ell, \qquad \hat{X}^\ell(t) := 2^{-k/(s-d)} \hat{X}_{\lfloor t2^k \rfloor}^\ell.
$$

COROLLARY 6.3. *For any $1 \le q < \infty$ and $\kappa > 0$ we have that [and in the case of $\alpha \ge 1$ $\exists \lambda^*(s, d, \kappa) > 0$ such that for all $0 < \lambda < \lambda^*$]*

(17) $\mathbb{P}(\#\{\ell \in [k^3]: \|\hat{X}^\ell(t) - X^\ell(t)\|_{L^q([0,1])} \ge \kappa\} > \kappa k^3) \le 6k^{-3}.$

PROOF.   Since

$$
\int_0^1 |\hat{X}^\ell(t) - X^\ell(t)|^q \, dt \le \frac{|[2^k] \setminus \mathcal{I}^\ell|}{2^k} \max_{i \in [2^k] \setminus \mathcal{I}^\ell} |\hat{X}_i^\ell - X_i^\ell|^q + \max_{i \in \mathcal{I}^\ell} |\hat{X}_i^\ell - X_i^\ell|^q.
$$

Lemma 6.2 implies that we have

$$
\mathbb{P}\left(\int_0^1 |\hat{X}^\ell(t) - X^\ell(t)|^q \, dt > 2^{-(1-2\gamma)k}(\kappa + 2^{-k/(s-d)}Z_{\max})^q + \kappa \Big| \mathcal{G}\right) \le \frac{5}{k^3}.
$$

Now by Lemma 6.4 (to be proved next), we have that

$$
\mathbb{P}(2^{-(1-2\gamma)k}(\kappa + 2^{-k/(s-d)}Z_{\max})^q > \kappa) = k^{-6}
$$

with much room to spare. Combining the previous two equations with Lemma 5.7, and taking a union bound over $\ell \in [k^3]$ establishes equation (17). □

The following lemma provides a bound over the terms of the process $w_i(x)$, and therefore of $Z_{\max}$, completing the previous corollary.

LEMMA 6.4.   *There exists a constant $c > 0$ such that for large n,*

$$\mathbb{P}\left(n^{-1/\alpha} \sum_{i=1}^{n} \sum_{\substack{x \in \mathbb{Z}^d, \\ |x| > M}} |x| w_i(x) > y\right)$$

(18)

$$\leq \begin{cases} cy^{-\alpha}, & 0 < \alpha < 1, \\ cy^{-1} \log(e \vee yM^{-1}n^{1/\alpha}), & \alpha = 1, \\ cy^{-1}(1 \vee Mn^{-1/\alpha})^{1-\alpha}, & 1 < \alpha < 2, \end{cases}$$

and hence the variable $Z_{\max}^{\ell}$ satisfies

$$\mathbb{P}(2^{-k/\alpha}|Z_{\max}^{\ell}| > y) \leq \begin{cases} c'y^{-\alpha}, & 0 < \alpha < 1, \\ c'y^{-1/2}\lambda^{-1}, & 1 \leq \alpha < 2, \end{cases}$$

where $c = c(s, d)$ and $c' = c'(s, d, J)$ do not depend on k, n or y.

PROOF.   By the power law decay of $\mathbb{P}(|x|)$ we have that for each i,

$$\mathbb{P}\left(\sum_{|x| > yn^{1/\alpha}} |x| w_i(x) \neq 0\right) \leq \sum_{|x| > yn^{1/\alpha}} \mathbb{P}(|x|) = O(y^{-\alpha}n^{-1})$$

and hence

(19)
$$\mathbb{P}\left(\sum_{i=1}^{n} \sum_{|x| > yn^{1/\alpha}} |x| w_i^{\ell, j, m}(x) \neq 0\right) \leq c_1 y^{-\alpha}.$$

Again using the power law decay of $\mathbb{P}(|x|)$,

$$\mathbb{E}\left(n^{-1/\alpha} \sum_{i=1}^{n} \sum_{M \leq |x| \leq yn^{1/\alpha}} |x| w_i(x)\right)$$

(20)
$$\leq n^{1-1/\alpha} \sum_{M \leq |x| \leq yn^{1/\alpha}} |x| \mathbb{P}(|x|)$$

$$\leq \begin{cases} c_2 y^{1-\alpha}, & 0 < \alpha < 1, \\ c_2 \log(e \vee yn^{1/\alpha}M^{-1}), & \alpha = 1, \\ c_2(Mn^{-1/\alpha})^{1-\alpha}, & 1 < \alpha < 2, \end{cases}$$

and so by Markov's inequality,

$$\mathbb{P}\left(n^{-1/\alpha} \sum_{i=1}^{n} \sum_{|x| \leq yn^{1/\alpha}} |x| w_i(x) \geq y\right)$$

(21)

$$\leq \begin{cases} c_2 y^{-\alpha}, & 0 < \alpha < 1, \\ c_2 y^{-1} \log\left(e \vee \dfrac{yn^{1/\alpha}}{M}\right), & \alpha = 1, \\ c_2 y^{-1}(Mn^{-1/\alpha})^{1-\alpha}, & 1 < \alpha < 2. \end{cases}$$

Combining equations (19) and (21) establishes equation (18). This establishes the required bound on $Z_{\max}^\ell$ since it is equal in distribution to $\sum_{i=1}^{(J^2+1)2^k} \sum_{|x|>\rho_k} |x| \times w_i(x)$. $\square$

6.2. *Approximating* $(\hat{X}_i^\ell)_{i,\ell}$ *by* $(\mathfrak{X}_i^\ell)_{i,\ell}$. Recall the definitions of $C^\star, \bar{C}$ and $C_{q_{j-1},q_j,m}$ from Section 5. The next step is to show that the collection $(\hat{X}_i^\ell)_{i,\ell}$ is well approximated by a family $(\mathfrak{X}_i^\ell)_{i,\ell}$ which has independent increments. This involves two steps. First we replace $Z_{i'}^{\ell,j,m}$ with $Z_{i'}^{+,\ell,j,m}$, the latter having independent increments. The second step is to use ergodic theory to replace $\phi_i^{\ell,j,m}$ by $iC_{q_{j-1},q_j,m}$. Because this second step is a consequence of ergodic considerations, and hence this particular approximation essentially requires us to assume that 0 is in the infinite component, or at least a very large one (which is essentially the same thing).

For any $\chi \in (0,1)$, the event $\mathcal{H}_{k,\chi}^\ell$ defined in Section 4 implies that

$$\max_{1\le t\le 2^k} \sup_{0\le q_1<q_2\le 1, m\ge 1} |N_t^{\ell,[q_1,q_2],\{m\}} - tC_{q_1,q_2,m}| \le \chi 2^k$$

and

$$\max_{1\le t\le 2^k} \left| N_t^\ell - \sum_{(j,m)\in[J^2]} N_t^{\ell,[q_{j-1},q_j],\{m\}} - t\bar{C}_{q_1,q_2,m} \right| \le \chi 2^k.$$

Recall the definition of $\chi_k \to 0$ from Lemma 4.3. The next lemma bounds the difference in our time change.

LEMMA 6.5. *For all J there exists $\varepsilon_k = \varepsilon_{k,J} > 0$ and $k_\star(J)$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and such that for all $(j,m) \in [J]^2$ and for $k > k_\star(J)$,*

$$\mathbb{P}\bigg(\#\bigg\{\ell \in [k^3]: \max_{1\le i\le 2^k} \int_0^1 |\mathbb{1}\{t \ge 2^{-k} iC_{q_{j-1},q_j,m}\}$$
$$- \mathbb{1}\{t \ge 2^{-k}\phi_i^{\ell,j,m}\}| \, dt > \varepsilon_k\bigg\} > \varepsilon_k k^3, \mathcal{H}_{k,\chi_k}\bigg)$$
$$< k^{-3}$$

and

$$\mathbb{P}(\#\{\ell \in [k^3]: |\bar{C} - 2^{-k}\phi_{2^k}^{0,0,\ell}| > \varepsilon_k\} > \varepsilon_k k^3, \mathcal{H}_{k,\chi_k}) < k^{-3}.$$

PROOF. Fix $(j,m) \in [J]^2$, and recall the definitions of $N_i^{\ell,[q_{j-1},q_j],\{m\}}$ and $\phi_i^{\ell,j,m}$. The first is the number of new vertices v with escape probability satisfying $q_{j-1} \le p_v < q_j$ and degree m encountered by path ℓ up to time i. The second is the number of new vertices v not encountered in a previous path with local escape probability $q_{j-1} \le \tilde{p}_v < q_j$ and local degree $\tilde{d}^\omega(v) = m$ encountered at

time $t \in [0, i] \cap \mathcal{I}^\ell$. Then for large enough k,

$$\mathbb{P}\left(\max_{1 \le i \le 2^k} |\phi_i^{\ell, j, m} - iC_{q_{j-1}, q_j, m}| > \frac{1}{2}\varepsilon_k + 4\chi_k 2^k, \mathcal{H}_{k, \chi_k}^\ell\right)$$

$$\le \mathbb{P}\left(\max_{1 \le i \le 2^k} |N_i^{\ell, [q_{j-1}, q_j], \{m\}} - \phi_i^{\ell, j, m}| > \frac{1}{2}\varepsilon_k + 3\chi_k 2^k, \mathcal{H}_{k, \chi_k}^\ell\right),$$

which follows from the definition of $\mathcal{H}_{k, \chi_k}^\ell$. Further, the right-hand side is bounded by

$$\le \mathbb{P}(|[2^k] \setminus \mathcal{I}^\ell| \ge 2^{2\gamma k}, \mathcal{H}_{k, \chi_k}^\ell)$$

$$+ \mathbb{P}\left(\#\left\{i \le 2^k : |p_{X_i^\ell} - \tilde{p}_{X_i^\ell}| > \frac{1}{k}\right\}\right.$$

$$+ \#\left\{i \le 2^k : \tilde{d}^\omega(X_i^\ell) \ne d^\omega(X_i^\ell)\right\} > \frac{1}{k}2^k, \mathcal{H}_{k, \chi_k}^\ell\right)$$

$$+ \mathbb{P}\left(\left||\{X_i^\ell : 1 \le i \le 2^k\} \cap \bigcup_{\ell'=1}^{\ell-1} \{X_i^{\ell'} : 1 \le i \le 2^k\}\right| > \frac{1}{k}2^k, \mathcal{H}_{k, \chi_k}^\ell\right)$$

$$+ \mathbb{P}\left(N_i^{\ell, [q_{j-1}-1/k, q_{j-1}+1/k], \{m\}} > \frac{1}{4}\varepsilon_k, \mathcal{H}_{k, \chi_k}^\ell\right)$$

$$+ \mathbb{P}\left(N_i^{\ell, [q_j - 1/k, q_j + 1/k], \{m\}} > \frac{1}{4}\varepsilon_k, \mathcal{H}_{k, \chi_k}^\ell\right)$$

$$\le k^{-5}.$$

The first quantity in the sum is bounded by Lemma 6.2, the second is bounded by Lemma 9.12 and the third is bounded by Lemma 9.10. The final two terms are 0 as long as ε_k tends to 0 slowly by the definition of $\mathcal{H}_{k, \chi_k}^\ell$ since $\mathcal{P}$ does not have an atom at q_{j-1} or q_j. Noting that $N_i^\ell - \sum_{(j,m) \in [J^2]} N_i^{\ell, q_{j-1}, q_j, m}$ and $\phi_i^{\ell, 0, 0}$ make up the remainder of the new vertices, we similarly have

$$\mathbb{P}\left(\max_{1 \le i \le 2^k} |\phi_i^{\ell, 0, 0} - i\bar{C}| > 8\chi_k 2^k, \mathcal{H}_{k, \chi_k}^\ell\right) \le k^{-5}.$$

Taking ε_k converging to 0 sufficiently slowly and taking a union bound completes the result by the definition of $\mathcal{H}_{k, \chi_k}$. □

We now define a new process

$$\mathfrak{X}_i^\ell := \sum_{(j,m) \in [J]^2} \sum_{i'=1}^{iC_{q_{j-1}, q_j, m}} Z_{i'}^{+, \ell, j, m}$$

$$= \sum_{(j,m) \in [J]^2} \sum_{i'=1}^{iC_{q_{j-1}, q_j, m}} \sigma_{i'}^{+, \ell, j, m} \sum_{|x| > \rho} x w_{i'}^{\ell, j, m}(x)$$

and $\mathfrak{X}^{\ell}(t) := 2^{-k/(s-d)}\mathfrak{X}^{\ell}_{\lfloor t2^k \rfloor}$. Note that the $\mathfrak{X}^{\ell}_i$ are independent for different ℓ, have independent increments depending only on the coupling variables only. We show that, using the previous lemma, we may couple $\hat{X}^{\ell}$ and $\mathfrak{X}^{\ell}$ together in such a way that they are close in $L^q[0, 1]$.

LEMMA 6.6. *For each $\varepsilon > 0$ and when $\alpha \geq 1$ for each $\lambda > 0$, there exists $J^*(s, d, \varepsilon, \lambda)$ and $k^*(s, d, \varepsilon, \lambda, J)$ so that if $J > J^*$ and $k > k^*$, then*

$$\mathbb{P}(\#\{\ell \in [k^3] : \|\hat{X}(t) - \mathfrak{X}(t)\|_{L^q[0,1]} > \varepsilon\} > \varepsilon k^3, \mathcal{H}_{k,\chi_k}) = O(k^{-3}).$$

PROOF.   We have that

$$\|\hat{X}^{\ell}(t) - \mathfrak{X}^{\ell}(t)\|_{L^q[0,1]} \leq \mathcal{U}_1^{\ell} + \mathcal{U}_2^{\ell},$$

where

$$\mathcal{U}_1^{\ell} := 2^{-k/(s-d)}\left\|\sum_{(j,m)\in[J]^2}\sum_{i'=1}^{\phi^{\ell,j,m}_{\lfloor t2^k \rfloor}} Z^{\ell,j,m}_{i'} - \sum_{(j,m)\in[J]^2}\sum_{i'=1}^{\lfloor t2^k \rfloor C_{q_{j-1},q_j,m}} Z^{\ell,j,m}_{i'}\right\|_{L^q[0,1]}$$

and

$$\mathcal{U}_2^{\ell} := 2^{-k/(s-d)}\left\|\sum_{i'=1}^{\phi^{0,0,\ell}_{\lfloor t2^k \rfloor}} Z^{\ell,0,0}_{i'}\right\|_{L^q[0,1]}$$

$$+ 2^{-k/(s-d)}\left\|\sum_{(j,m)\in[J]^2}\sum_{i'=1}^{\lfloor t2^k \rfloor C_{q_{j-1},q_j,m}} Z^{\ell,j,m}_{i'} - Z^{+,\ell,j,m}_{i'}\right\|_{L^q[0,1]}.$$

Now $\mathcal{U}_1^{\ell}$ is bounded above by

$$\frac{1}{2^{k/(s-d)}}\sum_{(j,m)\in[J]^2}\sum_{i=1}^{2^k}\left[\int_0^1 |\mathbb{1}\{t \geq 2^{-k}iC_{q_{j-1},q_j,m}\} - \mathbb{1}\{t \geq 2^{-k}\phi^{\ell,j,m}_i\}|\,dt\right]^{1/q}$$

$$\times \sum_{|x|\geq \rho_k} |x|\mathbb{1}_{\{w^{\ell,j,m}_i(x)=1\}}$$

$$\leq 2^{-k/(s-d)}$$

$$\times Z^{\ell}_{\max}\max_{1\leq i\leq 2^k}\left[\int_0^1 |\mathbb{1}\{t \geq 2^{-k}iC_{q_{j-1},q_j,m}\} - \mathbb{1}\{t \geq 2^{-k}\phi^{\ell,j,m}_i\}|\,dt\right]^{1/q}$$

and so by Lemmas 6.4 and 6.5 and Chernoff bounds for large k,

$$(22) \qquad \mathbb{P}\left(\#\left\{\ell \in [k^3] : \mathcal{U}_1^{\ell} > \frac{\varepsilon}{2}\right\} > \frac{\varepsilon}{2}k^3, \mathcal{H}_{k,\chi_k}\right) < 2k^{-3}.$$

Now define

$$\mathcal{U}_3^\ell := 2^{-k/(s-d)} \sum_{i=1}^{2\bar{C}2^k} \sum_{|x|>\rho} |x| w_i^{\ell,0,0}(x)$$

$$(23) \qquad + 2^{-k/(s-d)} \sum_{(j,m)\in[J]^2} \sum_{i=1}^{2^k C_{q_{j-1},q_j,m}} \left(\sigma_i^{+,\ell,j,m} - \sigma_i^{-,\ell,j,m}\right) \sum_{|x|>\rho} |x| w_i^{\ell,j,m}(x)$$

$$\overset{d}{=} 2^{-k/(s-d)} \sum_{i=1}^{\mathcal{M}_k^\ell} \sum_{|x|>\rho} |x| w_i(x),$$

where $\overset{d}{=}$ denotes equality in distribution, and we define

$$\mathcal{M}_k^\ell = 2\psi_J 2^k + \sum_{(j,m)\in[J]^2} \sum_{i=1}^{2^k C_{q_{j-1},q_j,m}} \left(\sigma_i^{+,\ell,j,m} - \sigma_i^{-,\ell,j,m}\right),$$

where we used the fact that the $\sigma_i^{+,\ell,j,m}$ and $\sigma_i^{-,\ell,j,m}$ are independent of the $w_i^{\ell,j,m}$. Provided that we have that $\phi_{2^k}^{\ell,0,0} \le 2\bar{C}2^k \le 2\psi_J 2^k$, we have that $\mathcal{U}_2^\ell \le \mathcal{U}_3^\ell$. Now by the definition of $\sigma_i^{\pm,\ell,j,m}$,

$$\mathbb{P}(\sigma_i^{+,\ell,j,m} \ne \sigma_i^{-,\ell,j,m}) \le \mathbb{E} R_i^{\ell,j,m} \left(\frac{(1-q_j)m}{1+(1-q_j)m}\right)$$

$$- R_i^{\ell,j,m} \left(\frac{(1-q_{j-1})m}{1+(1-q_{j-1})m}\right)$$

$$= \frac{1}{(1-q_j)m} - \frac{1}{(1-q_{j-1})m} \le \psi_J$$

since the $R_i^{\ell,j,m}$ are geometric. Hence by standard Chernoff estimates,

$$\mathbb{P}\left(\sum_{(j,m)\in[J]^2} \sum_{i=1}^{2^k C_{q_{j-1},q_j,m}} \sigma_i^{+,\ell,j,m} - \sigma_i^{-,\ell,j,m} \ge 2\psi_J 2^k\right) = O(k^{-3})$$

since $\sum_{(j,m)\in[J]^2} C_{q_{j-1},q_j,m} \le C^\star < 1$. By Lemma 6.5 and the previous equation, we have

$$\mathbb{P}\left(\#\{\ell \in [k^3] : \mathcal{M}_k^\ell > 4\psi_J 2^k\} > \frac{\varepsilon k^3}{4}, \mathcal{H}_{k,\chi_k}^\ell\right) < O(k^{-3}).$$

Now by Lemma 6.4, for fixed λ in the case $\alpha \geq 1$, we have that if J is sufficiently large, then ψ_J can be made arbitrarily small, and so for large enough k,

$$\mathbb{P}\left(2^{-k/(s-d)} \sum_{i=1}^{4\psi_J 2^k} \sum_{|x|>\rho_k} |x| w_i(x) > \frac{\varepsilon}{2}\right) < \varepsilon/8,$$

when $J \geq J^*$. Again by large deviations and by equation (23) we have that

$$(24) \qquad \mathbb{P}\left(\#\left\{\ell \in [k^3] : \mathcal{U}_2^\ell > \frac{\varepsilon}{2}\right\} > \frac{\varepsilon k^3}{2}, \mathcal{H}_{k,\chi_k}\right) < O(k^{-3}).$$

Combining equations (22) and (24) completes the result. $\square$

6.3. *Approximating* $(\mathfrak{X}_i^\ell)_{i,\ell}$ *by α-stable laws* $\Gamma_\alpha^\ell(t)$. Finally we approximate $\mathfrak{X}_i^\ell$ by a stable law. So that the distribution of the increments does not depend on k, we set

$$\underline{\mathfrak{X}}_i^\ell = \sum_{(j,m)\in[J]^2} \sum_{i'=1}^{iC_{q_{j-1},q_j,m}} \sigma_{i'}^{+,\ell,j,m} \sum_{x\in\mathbb{R}^d} x w_{i'}^{\ell,j,m}(x)$$

$$= \mathfrak{X}_i^\ell + \sum_{(j,m)\in[J]^2} \sum_{i'=1}^{iC_{q_{j-1},q_j,m}} \sigma_{i'}^{+,\ell,j,m} \sum_{|x|\leq\rho} x w_{i'}^{\ell,j,m}(x)$$

and $\underline{\mathfrak{X}}^\ell(t) := 2^{-k/(s-d)} \underline{\mathfrak{X}}_{\lfloor t2^k \rfloor}^\ell$.

LEMMA 6.7.   *The difference of* $\mathfrak{X}^\ell(t)$ *and* $\underline{\mathfrak{X}}^\ell(t)$ *satisfies*

$$\mathbb{E}\big[\|\mathfrak{X}^\ell(t) - \underline{\mathfrak{X}}^\ell(t)\|_{L^\infty[0,1]}\big] \leq \begin{cases} o(1), & 0 < \alpha < 1, \\ c\lambda^{1-\alpha/2}, & 1 \leq \alpha < 2, \end{cases}$$

for $c = c(s,d)$ *and* k *large enough.*

PROOF.   By symmetry $\mathfrak{X}_i^\ell - \underline{\mathfrak{X}}_i^\ell$ is a martingale, and so by Doobs's maximal inequality,

$$\mathbb{E}\big[\|\mathfrak{X}^\ell(t) - \underline{\mathfrak{X}}^\ell(t)\|_{L^\infty[0,1]}\big] \leq \mathbb{E}\big[\|\mathfrak{X}^\ell(t) - \underline{\mathfrak{X}}^\ell(t)\|_{L^\infty[0,1]}^2\big]^{1/2}$$

$$\leq 2d\mathbb{E}\big[|\mathfrak{X}^\ell(1) - \underline{\mathfrak{X}}^\ell(1)|^2\big]^{1/2}$$

$$\leq \begin{cases} o(1), & 0 < \alpha < 1, \\ c\lambda^{1-\alpha/2}, & 1 \leq \alpha < 2, \end{cases}$$

as required since $\sum_{(j,m)\in[J]^2} C_{q_{j-1},q_j,m} \leq 1$. $\square$

Since $\underline{\mathfrak{X}}_i^\ell$ is a process with i.i.d. increments, in this subsection we can determine the scaling limit of $\underline{\mathfrak{X}}^\ell(t)$ under the $L^q([0,1))$ topology as well as in the Skorohod

topology in $D[0, 1]$. Note that for this step, we can proceed without reference to the Long Range Percolation model, and as a consequence, the meaning of $\mathbb{P}$ is completely unambiguous (i.e., independent of the measure on environments). Let

$$K_J = \sum_{(j,m)\in[J]^2} \varsigma_{j,m,J} C_{q_{j-1},q_j,m},$$

and ς is defined in Lemma 6.1.

LEMMA 6.8.   *For $1 \leq q < \infty$, the weak limit of $\underline{\mathfrak{X}}^\ell(t)$ in the Skorohod or L^q topology is $K_J^{1/\alpha}\Gamma_\alpha(t)$.*

PROOF.   The random variable $\sum_x x w_i^{\ell,j,m}(x)$ is in the domain of attraction of $\Gamma_\alpha(1)$, and hence we have that $\sigma_i^{+,\ell,j,m} \sum_{x\in\mathbb{Z}^d} x w_i^{\ell,j,m}(x)$ is in the domain of attraction of $(\mathbb{E}\sigma_i^{+,\ell,j,m})^{1/\alpha}\Gamma_\alpha(1)$. As $\varsigma_{j,m,J} = \lim_k \mathbb{E}\sigma_i^{+,\ell,j,m}$, this implies that

$$2^{-k/\alpha} \sum_{i'=1}^{2^k t C_{q_{j-1},q_j,m}} \sigma_{i'}^{+,\ell,j,m} \sum_x x w_{i'}^{\ell,j,m}(x)$$

converges weakly in both the Skorohod topology on $D[0, 1]$ and the $L^q([0, 1])$ topology to $(\varsigma_{j,m,J} C_{q_{j-1},q_j,m})^{1/\alpha}\Gamma_\alpha(t)$. Since for different (j, m) the sums are independent, it follows that

$$2^{-k/\alpha} \sum_{(j,m)\in[J]^2} \sum_{i'=1}^{2^{-k} t C_{q_{j-1},q_j,m}} \sigma_{i'}^{+,\ell,j,m} \sum_{|x|\leq\rho} x w_{i'}^{\ell,j,m}(x)$$

converges weakly to $K_J^{1/\alpha}\Gamma_\alpha(t)$. $\square$

Ultimately for our coupling, we need to take J going to infinity and as such in the following lemma, we show that K_J converges as $J \to \infty$.

LEMMA 6.9.   *The following limit exists*:

(25) $$K := \lim_{J\to\infty} K_J = \lim_{J\to\infty} \sum_{(j,m)\in[J]^2} \varsigma_{j,m,J} C_{q_{j-1},q_j,m}$$

with $0 < K < 1$.

PROOF.   For $m \geq 1$ and $q \in [0, 1]$, let $\eta^m(q) = C^{0,q,m}$ and

$$\xi(q, m) = \mathbb{P}\left(R\left(\frac{(1-q)m}{1 + (1-q)m}\right) > \tilde{R}\left(\frac{(1-\mathfrak{r})\mathfrak{d}}{1 + (1-\mathfrak{r})\mathfrak{d}}\right)\right),$$

where R and $\tilde{R}$ are independent Geometric processes defined in equation (10) and where $(\mathfrak{r}, \mathfrak{d})$ is distributed according to $(\mathcal{P}, \mathcal{D})$. Since since $R(t)$ is decreasing in t, it follows that ξ is increasing in q. Further define

$$K = \sum_{m=1}^{\infty} \int_0^1 \xi(q, m) \, d\eta^m(q),$$

which we interpret as a Riemannn–Stieltjes integral. We remark that regardless of its size this is well defined since each summand is well defined and positive. Now if $q_{j-1} \le q \le q_j$, then since $R(t)$ is a geometric process,

$$0 \le \varsigma_{j,m,J} - \xi(q, m)$$

$$\le \mathbb{P}\left(R\left(\frac{(1-q_j)m}{1+(1-q_j)m} \right) \neq R\left(\frac{(1-q)m}{1+(1-q)m} \right) \right)$$

$$\le \mathbb{E} R\left(\frac{(1-q_j)m}{1+(1-q_j)m} \right) - R\left(\frac{(1-q)m}{1+(1-q)m} \right)$$

$$= \frac{1}{(1-q_j)m} - \frac{1}{(1-q)m} \le \psi_J$$

with ψ_J the error tolerance defined in (9). Hence we have that

$$|K_J - K| \le \sum_{m=1}^{\infty} \int_{q_J}^1 \xi(q, m) \, d\eta^m(q) + \sum_{m=J+1}^{\infty} \int_0^{q_J} \xi(q, m) \, d\eta^m(q)$$

$$+ \sum_{m=1}^{J} \sum_{j=1}^{J} \int_{q_{j-1}}^{q_j} \varsigma_{j,m,J} - \xi(q, m) \, d\eta^m(q)$$

$$\le 2\psi_J,$$

since $\sum_{m=1}^{\infty} \int_{q_J}^1 1 \, d\eta^m(q) + \sum_{m=J+1}^{\infty} \int_0^{q_J} 1 \, d\eta^m(q) = \bar{C} \le \psi_J$. The fact that ψ_J converges to 0 establishes that K_J converges to K. Now since the walk is transient under $\mathbb{P}_{\nu_0}$, all return probabilities are strictly less than 1. Each walk arrives at new vertices at a constant rate by the ergodic theorems below which establishes that $K > 0$. But by definition $K \le C^\star$, the total rate at which new vertices are encountered, which is strictly less than 1. $\square$

6.4. *The complete coupling along dyadic subsequences.*   We now combine all the results of the section to prove the full coupling between the walks $X^\ell(t)$ and isotropic α-stable Lévy motion $\Gamma^\ell(t)$.

THEOREM 6.10.   *For each $\varepsilon > 0$ and $1 \le q < \infty$ for k sufficiently large, there is a coupling of the random walks $(X^\ell(t))_{\ell=1}^{[k^3]}$ with independent α-stable Lévy*

motions $(\Gamma^\ell(t))_{\ell=1}^{[k^3]}$, *satisfying*

$$\mathbb{P}(\#\{\ell \in [k^3] : \|X^\ell(t) - K^{1/(s-d)}\Gamma^\ell(t)\|_{L^q([0,1])} > \varepsilon, \mathcal{H}^\ell_{k,\chi k}\} > \varepsilon k^3) \leq \frac{1}{k^2}.$$

PROOF.   If we take $\lambda > 0$ small enough and take J large enough by Corollary 6.3 and Lemma 6.6, we have that

$$(26) \quad \mathbb{P}\left(\#\left\{\ell \in [k^3] : \|X^\ell(t) - \mathfrak{X}^\ell(t)\|_{L^q[0,1]} > \frac{\varepsilon}{3}, \mathcal{H}^\ell_{k,\chi k}\right\} > \frac{\varepsilon k^3}{3}\right) = O(k^{-3}).$$

By Lemma 6.8 we have that $\underline{\mathfrak{X}}^\ell(t)$ converges weakly to $K_J^{1/(s-d)}\Gamma^\ell(t)$ in $L^q[0,1]$. Hence by Skorohod's almost sure representation theorem (see, e.g., [9]) we can couple Γ^1 and $\mathfrak{X}^1$ so that when k is sufficiently large $\mathbb{P}(\|\underline{\mathfrak{X}}^1(t) - K_J^{1/(s-d)}\Gamma^1(t)\|_{L^q[0,1]} > \varepsilon/4) < \varepsilon/5$. For small enough $\lambda > 0$, by Lemma 6.7 we can then couple $\mathfrak{X}^1(t)$ and $K_J^{1/(s-d)}\Gamma^1(t)$ so that

$$\mathbb{P}(\|\mathfrak{X}^1(t) - K_J^{1/(s-d)}\Gamma^1(t)\|_{L^q[0,1]} > \varepsilon/3) < \varepsilon/4.$$

As the $\mathfrak{X}^\ell(t)$ and $\Gamma^\ell(t)$ are separately independent and identically distributed, we can extend this coupling so that by Chernoff bounds,

$$(27) \quad \mathbb{P}\left(\#\{\ell \in [k^3] : \|\mathfrak{X}^\ell(t) - K_J^{1/(s-d)}\Gamma^\ell(t)\|_{L^q[0,1]} > \varepsilon/3\} > \frac{\varepsilon k^3}{3}\right) = O(k^{-3}).$$

Finally since K_J converges to K, we have that for large enough J,

$$\mathbb{P}(\|K_J^{1/(s-d)}\Gamma^1(t) - K^{1/(s-d)}\Gamma^1(t)\|_{L^q[0,1]} > \varepsilon/3) < \varepsilon/4$$

and hence again using Chernoff bounds,

$$\mathbb{P}\left(\#\left\{\ell \in [k^3] : \|K_J^{1/(s-d)}\Gamma^\ell(t) - K^{1/(s-d)}\Gamma^\ell(t)\|_{L^q[0,1]} > \frac{\varepsilon}{3}\right\} > \frac{\varepsilon k^3}{3}\right)$$

$$(28) \qquad = O\left(\frac{1}{k^3}\right).$$

Combining equations (26), (27) and (28) completes the proof.   □

7. Proof of Theorem 1.1.   We begin with a standard topological lemma for separable Banach spaces whose proof we include for completeness.

LEMMA 7.1.   *Let $\mathcal{M}$ be a complete separable metric space with distance $d(\cdot, \cdot)$, and let μ be a Borel probability measure on $\mathcal{M}$ measurable with respect to the σ-algebra generated by the open subsets of $\mathcal{M}$. Then for any $\varepsilon_1 > 0$, there exists an $\varepsilon_2 > 0$ and a finite collection of disjoint measurable subsets $S_1, \ldots, S_M$ such that:*

- *the union contains most of the mass of the distribution* $\mu(\bigcup_{i=1}^{M} S_i) > 1 - \varepsilon_1$;
- *the S_i are not too large*, $\sup_{x,y \in S_i} d(x, y) < \varepsilon_1$;
- *the S_i are well separated*, $\inf_{i \neq j, x \in S_i, y \in S_j} d(x, y) > \varepsilon_2$.

PROOF.   By separability there exists a countable dense subset $(x_i)_{i \in \mathbb{N}}$. For any sequence $(r_i)_{i \in \mathbb{N}}$, $r_i \in (\varepsilon_1/2, \varepsilon_1)$, the family of open balls $B_{r_i}(x_i)$ covers $\mathcal{M}$. Further, it is possible to choose these r_i so that $\mu(\partial B_{r_i}(x_i)) = 0$ for all i [$\partial B_{r_i}(x_i)$ denotes the boundary of the ball]. For such a choice, consider the usual disjoint decomposition $S_1' = B_{r_1}(x_1)$, $S_i' = B_{r_i}(x_i) \setminus (\bigcup_{j=1}^{i-1} S_j')$. By construction, $\mu(\text{int } S_i') = \mu(S_i')$ (where int A denotes the interior of A), and the sequence $(\text{int } S_i')_i$ gives a disjoint family of open sets whose union has full measure. Thus we may find $M > 0$ so that

$$\mu\left(\bigcup_{i \leq M} \text{int } S_i'\right) \geq 1 - \varepsilon_1/2.$$

By continuity of μ, there is $\varepsilon_2 > 0$ so that if

$$S_i = \{x \in \text{int } S_i' : d(x, \partial S_i') \geq \varepsilon_2\},$$

then

$$\mu\left(\bigcup_{i \leq M} S_i\right) \geq 1 - \varepsilon_1. \qquad \square$$

Applying the previous lemma, for each integer m, choose a finite collection of disjoint measurable subsets $S_1^m, \ldots, S_{M(m)}^m$ of $L^q([0, 1])$ such that $\mathbb{P}(K^{1/(s-d)} \times \Gamma^1(t) \in \bigcup_{i=1}^{M} S_i) > 1 - \frac{1}{m}$ and $\sup_{x,y \in S_i} \|x - y\|_{L^q[0,1]} < \frac{1}{m}$. We set $\upsilon(m) = \inf_{i \neq j, x \in S_i, y \in S_j} \|x - y\|_{L^q[0,1]}$ and let S_i^{m+} denote the enlarged set

$$S_i^{m+} := \left\{ x \in L^q([0, 1]) : \inf_{y \in S_i^m} \|x - y\|_{L^q[0,1]} \leq \min\left\{ \frac{1}{m}, \frac{1}{3}\upsilon(m) \right\} \right\}.$$

By construction, these enlargements are still disjoint. The following lemma shows that for $\mathbb{P}_{\mu_0}$-a.e. environment, the random walk distribution places enough weight on these enlarged sets.

LEMMA 7.2.   *For each $m > 0$ and $1 \leq i \leq M(m)$, and for μ_0 almost every environment $\omega \in \Omega_0$,*

$$\liminf_n \mathbb{P}\left(n^{-1/\alpha} X_{\lfloor nt \rfloor}^1 \in S_i^{m+} | \omega\right) - \mathbb{P}\left(K^{1/\alpha} \Gamma^1(t) \in S_i^m\right) \geq 0.$$

PROOF.   First observe that for any $r > 0$, by the self-similarity property $(\frac{K}{r})^{1/\alpha} \Gamma^\ell(rt)$ is equal in distribution to $K^{1/\alpha} Z^\ell(t)$, so

(29) $$\mathbb{P}\left(\left(\frac{K}{r}\right)^{1/\alpha} \Gamma^\ell(rt) \in S_i^m\right) = \mathbb{P}\left(K^{1/\alpha} \Gamma^1(t) \in S_i^m\right).$$

Now fix $\varepsilon > 0$ and k a positive integer and set $\theta = \min\{\frac{1}{m}, \frac{1}{3}\upsilon(m)\}$. By Theorem 6.10 we may choose there exists a coupling of $(X^\ell(t))_{\ell=1}^{[k^3]}$ and $(\Gamma(t))_{\ell=1}^{[k^3]}$ satisfying

$$\mathbb{P}\left(\#\{\ell \in [k^3] : \|X^\ell(t) - K^{1/\alpha}\Gamma^\ell(t)\|_{L^q([0,1])} > 2^{1/\alpha+1/q}\theta, \mathcal{H}^\ell_{k,\chi_k}\} > \frac{\varepsilon k^3}{4}\right) \leq k^{-2}.$$

Now for $2^{k-1} < n \leq 2^k$, we have that

$$\int_0^1 \left|\frac{X^\ell_{\lfloor nt\rfloor}}{n^{1/\alpha}} - \left(\frac{K2^k}{n}\right)^{1/\alpha}\Gamma^\ell\left(\frac{nt}{2^k}\right)\right|^q dt$$

$$= \left(\frac{n}{2^k}\right)^{-1}\int_0^{n/2^k}\left|\frac{X^\ell_{\lfloor 2^k t\rfloor}}{n^{1/\alpha}} - \left(\frac{K2^k}{n}\right)^{1/\alpha}\Gamma^\ell(t)\right|^q dt$$

$$\leq \left(\frac{n}{2^k}\right)^{-1-q/\alpha}\int_0^1 |X^\ell(t) - K^{1/\alpha}\Gamma^\ell(t)|^q dt.$$

Hence since $\frac{n}{2^k} > \frac{1}{2}$,

$$\|2^{-k/\alpha}X^\ell_{\lfloor 2^k t\rfloor} - K^{1/\alpha}\Gamma^\ell(t)\|_{L^q[0,1]} < 2^{1/\alpha+1/q}\theta$$

implies that

$$\left\|n^{-1/\alpha}X^\ell_{\lfloor nt\rfloor} - \left(\frac{K2^k}{n}\right)^{1/\alpha}\Gamma^\ell\left(\frac{n}{2^k}t\right)\right\|_{L^q[0,1]} < \theta.$$

Thus

$$\mathbb{P}\left(\#\left\{\ell \in [k^3] : \max_{1/2<n/2^k\leq 1}\left\|\frac{X^\ell_{\lfloor nt\rfloor}}{n^{1/\alpha}} - \left(\frac{K2^k}{n}\right)^{1/\alpha}\Gamma^\ell\left(\frac{nt}{2^k}\right)\right\|_q > \theta, \mathcal{H}^\ell_{k,\chi_k}\right\} > \frac{\varepsilon k^3}{4}\right)$$

$$\leq \frac{1}{k^2}$$

and hence when k is large enough so that $\chi_k < \frac{\varepsilon}{12}$, then

(30)
$$\mathbb{P}\left(\#\left\{\ell \in [k^3] : \max_{1/2<n/2^k\leq 1}\left\|\frac{X^\ell_{\lfloor nt\rfloor}}{n^{1/\alpha}} - \left(\frac{K2^k}{n}\right)^{1/\alpha}\Gamma^\ell\left(\frac{nt}{2^k}\right)\right\|_q > \theta\right\} > \frac{\varepsilon k^3}{3}, \mathcal{H}_{k,\chi_k}\right)$$

$$\leq \frac{1}{k^2}.$$

Now if $(\frac{K2^k}{n})^{1/\alpha}\Gamma^\ell(\frac{n}{2^k}t) \in S^m_i$ and $\|n^{-1/\alpha}X^\ell_{\lfloor nt\rfloor} - (\frac{K2^k}{n})^{1/\alpha}\Gamma^\ell(\frac{n}{2^k}t)\|_{L^q} < \theta$, then by definition, we have that $n^{-1/\alpha}X^\ell_{\lfloor nt\rfloor} \in S^{m+}_i$. It follows by the triangle inequality

that

$$\mathbb{P}\Big(\min_{1/2 < n/2^k \le 1} \mathbb{P}\big(n^{-1/\alpha} X^1_{\lfloor nt \rfloor} \in S_i^{m+} | \omega\big) - \mathbb{P}\big(K^{1/\alpha} \Gamma^1(t) \in S_i^m\big) < -\varepsilon, \mathcal{H}_{k,\chi_k}\Big)$$

$$\le \mathbb{P}\Big(\# \Big\{ \ell \in [k^3] :$$

$$\max_{1/2 < n/2^k \le 1} \Big\| \frac{X^\ell_{\lfloor nt \rfloor}}{n^{1/\alpha}} - \Big(\frac{K2^k}{n}\Big)^{1/\alpha} \Gamma^\ell\Big(\frac{nt}{2^k}\Big) \Big\|_q > \theta, \mathcal{H}^\ell_{k,\chi_k} \Big\} > \frac{\varepsilon k^3}{4} \Big)$$

$$+ \sum_{n=2^{k-1}+1}^{2^k} \mathbb{P}\Big(\Big| \# \Big\{ \ell : \frac{X^\ell_{\lfloor nt \rfloor}}{n^{1/\alpha}} \in S_i^{m+} \Big\} - k^3 \mathbb{P}\Big(\frac{X^1_{\lfloor nt \rfloor}}{n^{1/\alpha}} \in S_i^{m+} \Big| \omega\Big) \Big| > \frac{\varepsilon k^3}{3} \Big)$$

$$+ \sum_{n=2^{k-1}+1}^{2^k} \mathbb{P}\Big(\Big| \# \Big\{ \ell \in [k^3] : \Big(\frac{K2^k}{n}\Big)^{1/\alpha} \Gamma^\ell\Big(\frac{n}{2^k} t\Big) \in S_i^m \Big\}$$

$$- k^3 \mathbb{P}\big(K^{1/\alpha} \Gamma^1(t) \in S_i^m\big) \Big| > \frac{\varepsilon k^3}{3} \Big)$$

$$\le O\big(k^{-2}\big),$$

where we bound line 2 using equation (30), and where each term in the two sums is bounded by e^{-ck^3} by Chernoff bounds for some $c > 0$. Summing over k and using the fact that $\mathbb{P}(0 \in \mathcal{C}^\infty) > 0$, we have that for all large enough k_0,

$$\mathbb{P}_{\mu_0}\Big(\inf_{n > 2^{k_0}} \mathbb{P}\Big(\frac{X^1_{\lfloor nt \rfloor}}{n^{1/\alpha}} \in S_i^{m+} | \omega\Big) - \mathbb{P}\big(K^{1/\alpha} \Gamma^1(t) \in S_i^m\big) < -\varepsilon, \bigcap_{k > k_0} \mathcal{H}_{k,\chi_k}\Big)$$

$$= O\big(k_0^{-1}\big).$$

By Lemma 4.3 we have that $\lim_{k \to \infty} \mathbb{P}_{\mu_0}(\bigcap_{k' > k} \mathcal{H}_{k,\chi_k}) = 1$, and so

$$\liminf_n \mathbb{P}\big(n^{-1/\alpha} X^1_{\lfloor nt \rfloor} \in S_i^{m+} | \omega\big) - \mathbb{P}\big(K^{1/\alpha} \Gamma^1(t) \in S_i^m\big) > -\varepsilon \qquad \mu_0\text{-a.s.,}$$

and the result follows by taking ε to 0. $\square$

Now using the previous lemma, we prove weak convergence of the measure conditioned on the environment establishing the main theorem.

PROOF OF THEOREM 1.1. Fix $\varepsilon > 0$. We will let $X^n = X^n(t)$ denote $n^{-1/\alpha} \sum_{i=1}^{\lfloor nt \rfloor} X^1_i \in L^q([0,1])$ and, for notational convenience, let Z denote $K^{1/\alpha} \Gamma(t) \in L^q([0,1])$ where K is defined in (25). To establish Theorem 1.1 we will show that the law of $(X^n, 0 \le t \le 1)$ converges weakly in $L^q([0,1])$ to the law of $(Z, 0 \le t \le 1)$.

For a bounded continuous functional f on $L^q([0, 1])$ with $\|f\|_\infty \leq 1$, we denote

$$f^\delta(x) = \sup_{y \in L^q : \|x-y\| \leq \delta} f(y).$$

Since f is continuous $f^\delta \to f$ point-wisely and so by the bounded convergence theorem, $\mathbb{E}f^\delta(Z)$ converges to $\mathbb{E}f(Z)$ as $\delta \to 0$. Choose $\delta > 0$ to be small enough so that we have that $\mathbb{E}f^\delta(Z) - \mathbb{E}f(Z) < \varepsilon/4$, and let m be large enough so that $\frac{4}{m} < \varepsilon$ and $\frac{3}{m} < \delta$. With this it follows that

$$\max_{x \in S_i^{m+}, z \in S_i^m} \|x - z\|_{L^q[0,1]} \leq \frac{3}{m} < \delta$$

and hence

(31)
$$\inf_{z \in S_i^m} f^\delta(z) \geq \sup_{x \in S_i^{m+}} f(x).$$

Since $\|f\|_\infty \leq 1$, it follows that

(32) $$\mathbb{E}f(X^n|\omega) \leq \sum_{i=1}^{M(m)} \mathbb{E}(f(X^n)\mathbb{1}\{X^n \in S_i^{m+}\}|\omega) + \mathbb{P}\left(X^n \notin \bigcup_{i=1}^{M(m)} S_i^{m+} \Big| \omega\right).$$

Now by Lemma 7.2 we have that for μ_0 almost every environment $\omega \in \Omega_0$,

(33)
$$\limsup_n \mathbb{P}\left(X^n \notin \bigcup_{i=1}^{M(m)} S_i^{m+} \Big| \omega\right) = \limsup_n 1 - \sum_{i=1}^{M(m)} \mathbb{P}(X^n \in S_i^{m+}|\omega)$$

$$\leq 1 - \sum_{i=1}^{M(m)} \mathbb{P}(Z \in S_i^m) \leq \frac{1}{m} \qquad \mu_0\text{-a.s.}$$

From equation (31), we have that

(34)
$$\mathbb{E}(f(X^n)\mathbb{1}\{X^n \in S_i^{m+}\}|\omega) \leq \sum_{i=1}^{M(m)} \mathbb{E}f^\delta(Z)\mathbb{1}\{Z \in S_i^m\}$$

$$+ 2\sum_{i=1}^{M(m)} |\mathbb{P}(X^n \in S_i^{m+}|\omega) - \mathbb{P}(Z \in S_i^m)|.$$

We bound the first term on the right-hand side by

(35)
$$\sum_{i=1}^{M(m)} \mathbb{E}f^\delta(Z)\mathbb{1}\{Z \in S_i^m\} \leq \mathbb{E}f^\delta(Z) + \mathbb{P}\left(Z \notin \bigcup_{i=1}^{M(m)} S_i^m \Big| \omega\right)$$

$$\leq \mathbb{E}f(Z) + \frac{1}{m} + \frac{\varepsilon}{4}.$$

For the second term by Lemma 7.2, we have that for μ_0 almost every $\omega \in \Omega_0$,

$$\limsup_n \sum_{i=1}^{M(m)} \left| \mathbb{P}(X^n \in S_i^{m+} | \omega) - \mathbb{P}(Z \in S_i^m) \right|$$

(36)
$$= \limsup_n \sum_{i=1}^{M(m)} \mathbb{P}(X^n \in S_i^{m+} | \omega) - \mathbb{P}(Z \in S_i^m)$$

$$\leq 1 - \mathbb{P}\left(Z \in \bigcup_{i=1}^{M(m)} S_i^m \right) \leq \frac{1}{m} \qquad \mu_0\text{-a.s.}$$

Combining equations (32), (33), (34), (35) and (36) establishes that for μ_0, almost every $\omega \in \Omega_0$

$$\limsup_n \mathbb{E} f(X^n | \omega) \leq \mathbb{E} f(Z) + \frac{3}{m} + \frac{\varepsilon}{4} \leq \mathbb{E} f(Z) + \varepsilon \qquad \mu_0\text{-a.s.}$$

Since f was arbitrary, the same reasoning applies equally to $-f$ and f. Therefore, we have

$$\liminf_n \mathbb{E} f(X^n | \omega) \geq \mathbb{E} f(Z) - \varepsilon \qquad \mu_0\text{-a.s.}$$

and hence $\mathbb{E} f(X^n | \omega)$ converges to $\mathbb{E} f(Z)$ almost surely which establishes the weak convergence in law. □

8. Ergodic theory. To make sure that the number of new vertices that each walk X_i^ℓ visits in the time interval $[0, 2^k]$ is approximately $2^k C$, we obtain estimates using ergodic theory. We introduce the chain defined by "the environment seen from the particle." This technique only applies to the walks individually. Later (see Lemma 9.10) we will give a quantitative estimate on the number of *vertex* intersections between a pair of walks under the distribution $\mathbb{P}_{\mu_0}$ (and under $\mathbb{P}_\mu$ as well). When combined with the ergodic theory estimates outlined below, we see that with high probability *all* the walks $(X^\ell)_{\ell \in [k^3]}$ visit the same positive density of new vertices in $[0, 2^k]$.

Let $\tau_x : \Omega \to \Omega$ denote the shift operation: for any edge $b \in \mathbb{Z}^d \times \mathbb{Z}^d$ we denote $\tau_x \cdot \omega(b) := \omega(b + x)$. By our assumption of translation invariance of the connection probabilities $p_{i,j}$, the measure μ is clearly translation invariant for all the shifts. The Kolmogorov 0–1 law implies that μ is ergodic with respect to the collection of shifts $\{\tau_x\}_{x \in \mathbb{Z}^d}$; see below for a similar statement for μ_0.

Given an initial environment $\omega \in \Omega$ and a simple random walk trajectory X_i (defined relative to ω), $\tau_{X_i} : \Omega \to \Omega$ defines a (stochastic) map. Let $\omega_i := \tau_{X_i}(\omega)$, with initial environment $\omega_0 = \omega$. It is clear that ω_i is a Markov process with state space Ω, since the underlying random walk is. We let $Q(\omega, d\omega')$ denote the transition kernel for ω_i going from ω to ω'.

Given an environment ω, let $d(\omega) = d^\omega(0)$ denote the degree of ω at 0. Recall $d\nu(\omega) = \frac{d(\omega)}{\mathbb{E}_\mu[d(\omega)]} d\mu(\omega)$, and let us introduce the Hilbert space

$$L^2(\nu) = \left\{ f : \Omega \to \mathbb{R} : \mathbb{E}_\nu(f^2) < \infty \right\}$$

with inner product $\langle f, g \rangle := \int d\nu(\omega) f(\omega) g(\omega)$. Since X_i is reversible under the weighting $d^\omega(x)$, it follows that the operator $A_i f(\omega) := f(\omega_i)$ is self adjoint with respect to $L^2(\nu)$. We are interested in elevating the ergodic properties of $(\mu, \Omega, \mathcal{F}, (\tau_x)_{x \in \mathbb{Z}^d})$ to ergodic properties of the chain ω_i. We briefly indicate how this is done.

Let Ω^{2n+1} denote the space of (finite) two-sided sequences $(\omega_{-n} \cdots \omega_0, \ldots, \omega_n)$, $\omega_j \in \Omega$. On Ω^{2n+1} introduce probability measure induced by ω_i. That is, for a cylinder event $A = A_{-n} \times \cdots \times A_n$,

$$\mathbb{P}_{2n+1}(A) := \int_{A_{-n}} d\nu(\omega_{-n}) \int_{A_{-n+1}} P(\omega_{-n}, d\omega_{-n+1}) \cdots \int_{A_n} P(\omega_{n-1}, d\omega_n).$$

Because ω_i is stationary and reversible under ν, $\{\mathbb{P}_{2n+1}\}_{n \in \mathbb{N}}$ naturally identifies with a consistent family of probability measures on $\Omega^{\mathbb{Z}}$. Equipping $\Omega^{\mathbb{Z}}$ with the Borel σ-field $\mathcal{B}$ defined by the product topology (over time and space), we then have the existence of a probability measure, which by slight abuse of notation, we denote by $\mathbb{P}_\nu$, on $\Omega^{\mathbb{Z}}$ consistent with the family $\mathbb{P}_{2n+1}$. Moreover, if $\mathbf{T}$ denotes the Bernoulli shift [i.e., $(\mathbf{T}\underline{\omega})_i = \underline{\omega}_{i+1}$ for all $\underline{\omega} \in \Omega^{\mathbb{Z}}$], then $\mathbb{P}_\nu$ is stationary with respect to $\mathbf{T}$, and we can study its ergodic components.

Denoting $\Omega_0 = \{\omega : 0 \in \mathcal{C}^\infty(\omega)\}$, we let τ_v act on Ω_0 through the "induced shift" σ_v: for $\omega \in \Omega_0$, define $n_v(\omega) := \inf\{n : \tau_{nv} \omega \in \Omega_0\}$. Then $\sigma_v(\omega) := \tau_{n_v(\omega)v}(\omega)$. Analogously with $(\Omega_0, \mathcal{F}_0, \nu_0)$ denoting the restriction of the probability space $(\Omega, \mathcal{F}, \nu)$ to Ω_0, let $\mathbb{P}_{\nu_0}$ denote the corresponding restriction of $\mathbb{P}_\nu$ on $\Omega_0^{\mathbb{Z}}$ with σ-algebra $\mathcal{B}_0$.

We need the following general result, the proof of which may be found, for example, in [7]. Let $(X, \mathcal{X}, \lambda, T)$ be a probability space with the invertible, measure preserving, ergodic transformation T. Let $A \in \mathcal{X}$ be a set of positive measure. For $x \in X$, let $n(x) = \inf\{k > 0 : T^k(x) \in A\}$. Then the Poincaré recurrence theorem implies μ-a.s. that $n(x) < \infty$. We define $S : X \to A$ by $S(x) = T^{n(x)}(x)$ (which is well defined up to a set of measure zero).

LEMMA 8.1.   *As a map from A to A, S is measure preserving, ergodic and invertible up to a set of λ measure 0.*

As a consequence we have (again see [7]):

LEMMA 8.2.   *Fix $b \in \mathbb{N}$. Let $B \in \mathcal{F}_0$ such that for almost all $\omega \in B$,*

$$\mathbb{P}(\tau_{X_b} \cdot \omega \in B | \omega) = 1.$$

Then it follows that B is a 0–1 event under ν_0.

Finally by a straightforward adaptation of Proposition 3.5 from [7], we have:

LEMMA 8.3.   *The measure space* $(\mathbb{P}_{\nu_0}, \Omega_0^{\mathbb{Z}}, \mathcal{B}_0)$ *is ergodic with respect to the Bernoulli shift* $\mathbf{T}^b$ *for any* $b \in \mathbb{N}$.

Recall Birkoff's ergodic theorem:

THEOREM 8.4 (Theorem 6.2.1 from [14]).   *Let* $(X, \mathcal{X}, \lambda)$ *be a probability space with measure preserving transformation* T. *Let* $\mathcal{I}$ *be the completion of the* σ-*field of events invariant under* T. *Then for any* $F \in L^1(X, \lambda)$,

$$\frac{1}{N} \sum_{i=1}^{N} F(T^i(\omega)) \to \mathbb{E}[F|\mathcal{I}] \qquad \lambda\text{-}a.s.$$

Our main application of Theorem 8.4 is to the number of new vertices of different "types." Recall that N_i is the indicator of the event that X_i is the first visit to that vertex, and p_v denotes the quenched probability that a random walk started from v will ever return there. For intervals $[a, b] = \mathcal{A} \subset [0, 1]$ and $\mathcal{M} \subset \mathbb{N}$, let

$$N_t^{\mathcal{A}, \mathcal{M}} := \sum_{i=1}^{t} N_i \mathbb{1}\{p_{X_i} \in \mathcal{A}, \mathrm{d}^\omega(X_i) \in \mathcal{M}\}.$$

Let $L(\mathcal{A}, \mathcal{M}) \in \Omega^{\mathbb{Z}}$ denote the event

$$L(\mathcal{A}, \mathcal{M}) := \{\underline{\omega} : \omega_0 \notin \{\omega_i : i \leq -1\}, p(\omega_0) \in \mathcal{A}, \mathrm{d}(\omega_0) \in \mathcal{M}\}.$$

LEMMA 8.5.   *We have that*

$$\sup_{\mathcal{A}, \mathcal{M}} \left| \frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} - \mathbb{P}_{\nu_0}(L(\mathcal{A}, \mathcal{M})) \right| \to 0 \qquad \nu_0\text{-} \text{ or } \mu_0\text{-}a.s.$$

as $t \to \infty$ *where the supremum is over all closed intervals* $\mathcal{A} \subset [0, 1]$ *and* $\mathcal{M} \subset \mathbb{N}$ *and that*

$$\mathbb{P}_{\nu_0}(L([0, 1), \mathbb{N})) > 0.$$

PROOF.   It is enough to prove the converge of $\frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}}$ for an arbitrarary choice of the pair $(\mathcal{A}, \mathcal{M})$ with the extension to uniform convergence over all pairs following by discretising the space.

We cannot immediately apply Theorem 8.4 as the N_i are not a stationary sequence since whether a vertex is new depends on the previous i steps and of course i varies. So we compare it to the number of vertices which are new in the doubly infinite walk $(X_i)_{i \in \mathbb{Z}}$. Defining

$$f(\underline{\omega}) = \mathbb{1}\{\omega_0 \notin \{\omega_j : -\infty < j < 0\}\}\mathbb{1}\{p(\omega_0) \in \mathcal{A}, d(\omega_0) \in \mathcal{M}\},$$

we have that

$$f(\mathbf{T}^i \underline{\omega}) = \mathbb{1}\{\omega_i \notin \{\omega_j : -\infty < j < i\}\} \mathbb{1}\{p(\omega_i) \in \mathcal{A}, \mathrm{d}^\omega(\omega_i) \in \mathcal{M}\}.$$

Thus applying Lemma 8.3 and Theorem 8.4 we have that

(37)
$$\frac{1}{t} \sum_{0 < i \leq t} \mathbb{1}\{\omega_i \notin \{\omega_j : j < i\}\} \mathbb{1}\{p(\omega_i) \in \mathcal{A}, d(\omega_i) \in \mathcal{M}\}$$
$$\to \mathbb{P}_{v_0}\big(L(\mathcal{A}, \mathcal{M})\big) \qquad \text{a.s.}$$

The quantity

$$\frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} - \frac{1}{t} \sum_{0 < i \leq t} \mathbb{1}\{\omega_i \notin \{\omega_j : -\infty < j < i\}\} \mathbb{1}\{p(\omega_i) \in \mathcal{A}, \mathrm{d}^\omega(\omega_i) \in \mathcal{M}\}$$

is positive. By transience and stationarity it converges to 0 in $L^1(\mathbb{P}_{v_0})$. (Note that transience under $\mathbb{P}_{v_0}$ follows immediately from Theorem 9.1. It was originally proved via electrical network methods in [6].) On the other hand if $t = bq + r$,

$$\frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} \leq r/t + q/t \sum_{i \leq b} \frac{1}{q} N_q^{\mathcal{A}, \mathcal{M}} \circ \mathbf{T}^{qi}.$$

For any q fixed, we may apply Lemma 8.3 and Theorem 8.4 (with respect to the transformation $\mathbf{T}^q$) to conclude

$$\limsup_t \frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} \leq \frac{1}{q} \mathbb{E}_{v_0}[N_q^{\mathcal{A}, \mathcal{M}}].$$

Hence we have

$$\mathbb{P}_{v_0}\big(L(\mathcal{A}, \mathcal{M})\big) \leq \liminf_t \frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} \leq \limsup_t \frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} \leq \frac{1}{q} \mathbb{E}_0[N_q^{\mathcal{A}, \mathcal{M}}] \qquad \text{a.s.}$$

The $L^1(\mathbb{P}_{v_0})$ convergence implies $\frac{1}{q} \mathbb{E}_0[N_q^{\mathcal{A}, \mathcal{M}}] \to \mathbb{P}_{v_0}(L(\mathcal{A}, \mathcal{M}))$ as $q \to \infty$. Note that reversibility and transience imply

$$\mathbb{P}_{v_0}\big(L([0, 1), \mathbb{Z})\big) > 0,$$

which completes the lemma. $\square$

Now recalling the definition of $\mathcal{H}_{k, \chi}$ we prove Lemma 4.3.

PROOF OF LEMMA 4.3.   By Lemma 8.5, we have that

$$\lim_{t \to \infty} \sup_{\mathcal{A}, \mathcal{M}} \left| \frac{1}{t} N_t^{\mathcal{A}, \mathcal{M}} - \mathbb{P}_{v_0}\big(L(\mathcal{A}, \mathcal{M})\big) \right| = 0 \qquad \mu_0\text{-a.s.}$$

It follows that for μ_0-almost all random environments $\omega \in \Omega_0$, there exists a finite random variable $k^*(\omega)$, measurable with respect to the σ-algebra $\sigma(\omega)$, such that

$$\mathbb{P}_0\left(\bigcap_{k \geq k^*(\omega)} \mathcal{H}^\ell_{k,\chi} > \chi \,\Big|\, \omega\right) > 1 - \chi/2.$$

Since the walks X^ℓ_i are conditionally independent given ω by applying the strong law of large numbers to the random variables

$$\mathbb{1}\left\{\bigcap_{k \geq k^*(\omega)} \mathcal{H}^\ell_{k,\chi}\right\},$$

we have that for μ_0-almost all random environments $\omega \in \Omega_0$,

$$\liminf_L \frac{1}{L^3}\#\left\{\ell \in [L^3] : \mathbb{1}\left\{\bigcap_{k \geq k^*(\omega)} \mathcal{H}^\ell_{k,\chi}\right\} = 1\right\} \geq 1 - \chi/2 \qquad \text{a.s.}$$

Equation (6) follows by the definition of $\mathcal{H}_{k,\chi}$. Taking a sequence χ_k converging to 0 sufficiently slowly we also have (7). $\square$

9. Bounds on rare events. Let us recall the main result of [11]. It is proved there in the continuous time case, but extends without significant change to the discrete time walk as well (and will be used here in the latter form).

THEOREM 9.1 (Theorem 1 of [11]). *Let us consider $s \in (d, d+2)$ for $d \geq 2$ and $s \in (1, 2)$ for $d = 1$. Assume, for simplicity, that there exists L such that*

$$p_{x,y} = 1 - e^{-\beta \|x-y\|_2^{-s}} \qquad \text{for } \|x-y\|_2 \geq L,$$

and suppose that the $p_{x,y}$ are translation invariant and percolating. Then there exist universal constants $C_1, \zeta > 0$ and a family of random variables $(T_x(\omega))_{x \in \mathbb{Z}^d}$ with the property that $T_x(\omega) < \infty$ whenever $x \in \mathcal{C}^\infty(\omega)$, such that the following holds:

$$P^\omega_t(x, y) \leq C_1 \mathrm{d}^\omega(y) t^{-d/(s-d)} \log^\zeta t$$

for $t \geq T_x(\omega) \vee T_y(\omega)$. Moreover for $x \in \mathcal{C}^\infty(\omega)$ for any $\eta > 0$, there exists $C(\eta) > 0$ so that we have

$$\mu(T_x > k \,|\, x \in \mathcal{C}^\infty(\omega)) \leq C(\eta)k^{-\eta}.$$

Theorem 9.1 will be used to help rule out (in the probabilistic sense) various unwanted dependencies between the random walk trajectories which cause our coupling to fail. Theorem 9.1 applies only to estimates on the law of the k^3 walks under $\mathbb{P}_{\nu_0}$ and $\mathbb{P}_{\mu_0}$. Since the coupling construction relies on $\mathbb{P}_\mu$, it is convenient to extend these estimates to $\mathbb{P}_\mu$. For this purpose the following lemma, whose proof may be found in [11] is useful:

LEMMA 9.2 (Lemmas 2.9 and 2.10 of [11]).   *Let $N > 0$ be fixed, and let $n_1 > n_2 \geq \cdots \geq n_m$ enumerate the cluster sizes inside $[-N, N]^d$ (having sampled all internal edges). Then there exist $c_1, c_2, \zeta > 0$ independent of N, so that*

$$\mu(n_2 > \log^\zeta N) < c_1 e^{-c_2 \log^2 N}.$$

Also, if M denotes the largest internally connected component in $[-N, N]^d$, there is $\varepsilon > 0$ so that

$$\mathbb{P}_\mu(0 \leftrightarrow B_N^c(0) | 0 \notin M) \leq C N^{-\varepsilon}.$$

Let us also recall a (simple) technical lemma from [11] which we will need below. Let $D_t = \max_{0 \leq u \leq t} \|X_u\|_2$ denote the Euclidean diameter of the walk at time t.

LEMMA 9.3 (Lemma 4.1 of [11]).   *Let $1/(s - d) < p$. Then for either the discrete or continuous time process, there exists a constant $c > 0$ so that for any $x \in \mathbb{Z}^d$,*

$$P_x^\omega(D_t > ct^{p+1} \text{ infinitely often}) = 0$$

μ-a.s. Moreover, there exist constants $c_1, c_2, c_3 > 0$ such that for any $T, \lambda, p, r > 0$ with p as above and $r < s - d$,

$$\mathbb{P}(\{P_0^\omega(\exists t \leq T : D_t \geq c_1 t^{p+1}) > c_2/T^\lambda\}) \leq c_3 T^{\lambda + 1 - pr}.$$

The implicit consequence of the previous two lemmas is the following: if we observe the process of k^3 walks under $\mathbb{P}_\mu$ up to time 2^k and find that more than $k^{\delta_1 + \varepsilon}$ vertices have been uncovered in the exploration process, then with very high probability, $0 \in \mathcal{C}^\infty(\omega)$.

In what follows, we formulate technical lemmas controlling the behavior of the walks $(X_i^\ell)_{i \in [2^k]}^{\ell \in [k^3]}$. As we have indicated above, it is ultimately important to us that these estimates hold for the measure $\mathbb{P}_\mu$. Most often, after scaling the walk by $n^{-\alpha}$, the nontrivial statements only concern $\mathbb{P}_{\mu_0}$. Further, it is often convenient for us to use stationarity, only available under $\mathbb{P}_\nu, \mathbb{P}_{\nu_0}$. For these reasons, we give the following lemma which allows us to transfer bounds from one of these measures to another:

LEMMA 9.4.   *Let A be an event defined on the sample space $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}^{[k^3]}}$. Then, there exist constants $C_1, \ldots, C_4 < \infty$ so that we have*

$$\mathbb{P}_{\mu_0}(A) \leq C_1 \mathbb{P}_\mu(A),$$

$$\mathbb{P}_{\nu_0}(A) \leq C_2 \mathbb{P}_\nu(A),$$

$$C_3^{-1} \frac{\mathbb{P}_{\nu_0}(A)}{-\log \mathbb{P}_{\nu_0}(A)} \leq \mathbb{P}_{\mu_0}(A) \leq C_3 \mathbb{P}_{\nu_0}(A),$$

$$C_4^{-1} \frac{\mathbb{P}_\nu(\mathrm{d}^\omega(0) \geq 1, A)}{-\log \mathbb{P}_\nu(\mathrm{d}^\omega(0) \geq 1, A)} \leq \mathbb{P}_\mu(\mathrm{d}^\omega(0) \geq 1, A) \leq C_4 \mathbb{P}_\nu(\mathrm{d}^\omega(0) \geq 1, A).$$

PROOF.   Since $s > d$, and we have assumed that μ is supercritical, the first two inequalities, as well as the upper bounds in the third and fourth, are obvious.

On the other hand, an easy calculation gives $\mathbb{P}_\mu(\mathrm{d}^\omega(0) > x) \leq B_1 \exp(-B_2 x)$ for some $B_1, B_2 > 0$. Thus we have

$$\mathbb{P}_{\mu_0}(A) \geq \mathbb{P}_{\mu_0}\big(\omega \in A, \mathrm{d}^\omega(0) < x\big)$$

(38)
$$\geq x^{-1} C^{-1} \mathbb{P}_{\nu_0}\big(\omega \in A, \mathrm{d}^\omega(0) < x\big)$$

$$\geq x^{-1} C^{-1}\big[\mathbb{P}_{\nu_0}(A) - \mathbb{P}_{\nu_0}\big(\mathrm{d}^\omega(0) \geq x\big)\big].$$

Using the tail bound, let us take $x = -\frac{2}{B_2} \log \mathbb{P}_{\nu_0}(A)$. We get that

$$\mathbb{P}_{\mu_0}(A) \geq C' \frac{\mathbb{P}_{\nu_0}(A)}{-\log \mathbb{P}_{\nu_0}(A)}.$$

The last bound follows similarly.   □

Recall that $p(\omega)$ denotes the return probability of the origin in the environment ω, $(\mathcal{P}, \mathcal{D})$ denotes joint distribution of $(p(\omega), \mathrm{d}(\omega))$ under $\mathbb{P}_\mu$.

We now recall the definitions made in Section 4 and prove a series of lemmas establishing Proposition 4.2 which provides the quantitative estimates needed for the proof of our main theorem. For $\gamma, \delta > 0$, let

$$A(\rho) = \big\{\exists v \in \mathbb{Z}^d, \omega_{0,v} = 1, |v| > \rho\big\},$$

$$B(\rho, \gamma, k) = \big\{\forall v \in \mathbb{Z}^d \text{ if } \omega_{0,v} = 1, |v| > \rho, \text{ then } v \notin \{X_1, \ldots, X_{2^{\gamma k+1}}\}\big\},$$

$$C(\delta, \gamma, k) = \Big\{\max_{0 \leq t \leq 2^{\gamma k}} |X_t| > 2^{\delta k}\Big\}.$$

Denote the event G as

$$G(\rho, \delta, \gamma, k) = A(\rho) \cap B(\rho, \gamma, k) \cap C(\delta, \gamma, k),$$

which which we bound in the following lemma.

LEMMA 9.5 (No quick escapes of local neighborhoods without using long edges).   *Let* $\rho > 2^{k/(2(s-d))}$. *For any* $\delta \in (0, 1)$ *there exists* $\gamma, \varepsilon > 0$ *and* c *depending on* γ, ε *such that*

$$\mathbb{P}_\mu\big(G(\rho, \gamma, \delta, k)\big) < c2^{-\varepsilon k} \rho^{-\alpha(1-o(1))}.$$

PROOF.   The lemma will be proved if we can show there exists $\varepsilon > 0$, not necessarily the same as in the statement of the lemma, so that

(39) $$\mathbb{P}_\mu\big(G(k, \rho, \gamma, \delta)|A(\rho)\big) < c2^{-\varepsilon k}$$

since it is easy to see that

$$\mathbb{P}_\mu\big(A(\rho)\big) \leq c_1(\varepsilon_1)\rho^{-\alpha(1-\varepsilon_1)}$$

for any $\varepsilon_1 > 0$.

Let $A^2(\rho)$ denote the event that there are two or more edges of lengths at least ρ connected to 0. We have that

$$\mathbb{P}_\mu\big(P_0^\omega(C(\delta, \gamma, k) \cap B(\rho, \gamma, k))|A(\rho)\big)$$
$$\leq \mathbb{P}_\mu\big(C(\delta, \gamma, k)|A(\rho)^c\big) + \mathbb{P}_\mu\big(A^2(\rho)|A(\rho)\big).$$

Now

(40) $$\mathbb{P}_\mu\big(A^2(\rho)|A(\rho)\big) \leq C\rho^{-\alpha(1-o(1))}.$$

But for all k sufficiently large,

$$\mathbb{P}_\mu\big(C(\delta, \gamma, k)|A(\rho)^c\big) \leq 2\mathbb{P}_\mu\big(C(\delta, \gamma, k)\big).$$

We apply Lemma 9.3. It suffices to choose the parameters γ, λ, p, r so that $\varepsilon' = 1/\gamma(pr - 1 - \lambda) > 0$, $\gamma(p+1) < \delta$. This can be done by first taking pr sufficiently large depending on λ, and then taking γ sufficiently small depending on $\delta/(p+1)$. Then we have

$$\mathbb{P}\big(C(\delta, \gamma, k)\big) \leq c_2 2^{-\gamma\lambda k}.$$

Letting $\varepsilon = \min(\gamma\lambda, \varepsilon')$ proves (39). $\quad\square$

For the statement of the next lemma, recall that

$$D(\rho, k) = \{\exists v \in \mathbb{Z}^d, \omega_{0,v} = 1, |v| > \rho,$$
$$\exists J \in [2^k] \text{ s.t. } X_J = v, (0, v) \notin \{(X_i, X_{i+1})\}_{i \leq J}\}.$$

LEMMA 9.6 (Visiting endpoints of long edges without using the edge is unlikely). *Let $\rho > 2^{k/(2(s-d))}$. There exists $c > 0$ such that*

$$\mathbb{P}_\mu\big(D(\rho, k)\big) < c2^k \rho^{-s}.$$

PROOF. Consider $\mathbb{P}_\mu(D(\rho, k))$ and recall the definition of $A^2(\rho)$ from in Lemma 9.5. Since the event D requires reaching the other end of a long edge without traversing it first, we have that

$$\mathbb{P}_\mu\big(D(\rho, k)\big) \leq \mathbb{P}_\mu\big(A^2(\rho)\big) + \sum_{t=1}^{2^k} \sum_{v:|v| \geq \rho} \mathbb{P}_\mu(X_t = v)\mathbb{P}_\mu(\omega_{0,v} = 1)$$
$$\leq C\rho^{-2\alpha(1-o(1))} + 2^k \mathrm{P}(\rho)$$
$$= O\big(2^k \rho^{-s}\big). \qquad\qquad\qquad\qquad\square$$

Finally, let us consider the pair of related events

$$F(\rho, \gamma, k) = \{\exists i, 2^{\gamma k+1} \leq i \leq 2^k,$$

$$\exists v \in \mathbb{Z}^d \text{ s.t. } \omega_{0,v} = 1 \text{ and } \|v\|_\infty \geq \rho, X_i \in \{0, v\}\}$$

$$F^*(\rho, k) = \{\exists x, v \in \mathbb{Z}^d, i, j \in [2^k], \omega_{x,v} = 1, \|v\|_\infty \geq \rho, X_i^1 \text{ and } X_j^2 \in \{x, v\}\}.$$

LEMMA 9.7 (Returning to long edges after a transient time is unlikely).   *For any $0 < \gamma < 1$, there exists $\varepsilon > 0$ and a constant $c = c(\varepsilon)$ such that*

$$\mathbb{P}_{\mu_0}\big(F(\rho, \gamma, k)\big) < c 2^{-\varepsilon k} \rho^{-\alpha(1-o(1))}.$$

PROOF.   The lemma follows from a straightforward application of Theorem 9.1 and the fact that $\mathbb{P}_{\mu_0}(A(\rho)) = C\rho^{-\alpha(1-o(1))}$.   □

COROLLARY 9.8 (Pairs of walks do not intersect at long edges).   *Let $\rho > 2^{k/(2(s-d))}$. There exists $\varepsilon > 0$ independent of ρ and a constant $c = c(\varepsilon)$ such that*

$$\mathbb{P}_\mu\big(F^*(\rho, k)\big) < c\big[\rho^{-\varepsilon} + 2^{-\varepsilon k} 2^k \rho^{-\alpha(1-o(1))}\big].$$

PROOF.   Let us first reduce the estimate to the event $F^*(\rho, k) \cap \{0 \in C^\infty(\omega)\}$. By Lemma 9.2,

$$\mathbb{P}_\mu\big(F^*(\rho, k) \cap \{0 \in C^\infty(\omega)\}\big) \leq c\rho^{-\varepsilon}.$$

Thus we may work under $\mathbb{P}_{\mu_0}$ and hence, by Lemma 9.4, under $\mathbb{P}_{\nu_0}$. Since the paths X^1, X^2 are reversible under $\mathbb{P}_\nu$, the walk

$$Y_t := \begin{cases} X_{2^k-t}^1, & \text{for } t \leq 2^k, \\ X_{t-2^k}^2, & \text{for } t \geq 2^k, \end{cases}$$

is distributed as (again, under $\mathbb{P}_\nu$) $(X_t^1)_{t \in [2^{k+1}]}$. We now employ Lemma 9.7,

$$\mathbb{P}_{\nu_0}\big(F^*(\rho, k)\big) \leq c 2^{-\varepsilon k} + \mathbb{P}_{\nu_0}(\exists v \in \mathbb{Z}^d, t \in [2^{\gamma k}] : \|X_t^1 - v\| \geq \rho, \omega_{X_t^1, v} = 1),$$

where we used Lemma 9.4 to translate our bounds between the measures μ_0 and ν_0.

The proof is finished by noting that

$$\mathbb{P}_{\nu_0}(\exists v \in \mathbb{Z}^d, t \in [2^{\gamma k}] : \|X_t^1 - v\| \geq \rho, \omega_{X_t^1, v} = 1) \leq c 2^{\gamma k} \rho^{-\alpha(1-o(1))}$$

and applying Lemma 9.4.   □

Recalling our definitions from Section 4 for any event $E \subset \Omega^{\mathbb{Z}}$, let $\mathbf{T}^{-i} \cdot E = \{\underline{\omega} : T^i \cdot \underline{\omega} \in E\}$. Then we denote

$$\mathscr{G}(\rho, \gamma, \delta, k) = \bigcup_{i=0}^{2^k} \mathbf{T}^{-i} \cdot G(\rho\gamma, \delta, k),$$

$$\mathscr{D}(\rho, k) = \bigcup_{i=0}^{2^k} \mathbf{T}^{-i} D(\rho, k),$$

$$\mathscr{F}(\rho, \gamma, k) = \bigcup_{i=0}^{2^k} \mathbf{T}^{-i} \cdot F(\rho, \gamma, k).$$

The following corollary is the content of Proposition 4.2.

COROLLARY 9.9. *Let $\delta > 0$ be fixed. Then there exist $\gamma_1, \varepsilon > 0$ and $c = c(\gamma_1, \varepsilon)$ such that*

$$\mathbb{P}_\mu\big(\mathscr{G}(\rho, \gamma, \delta, k) \cup \mathscr{D}(\rho, k) \cup \mathscr{F}(\rho, \gamma, k)\big) < c\big[\rho^{-\varepsilon} + 2^{-\varepsilon k} 2^k \rho^{-\alpha(1-o(1))}\big]$$

for all $0 < \gamma < \gamma_1$.

PROOF. Application of Lemma 9.4 allows us to write

$$\mathbb{P}_\mu\big(\mathscr{G}(\rho, \gamma, \delta, k) \cup \mathscr{D}(\rho, k) \cup \mathscr{F}(\rho, \gamma, k)\big)$$
$$\leq C\big[\mathbb{P}_\nu\big(\mathscr{G}(\rho, \gamma, \delta, k) \cup \mathscr{D}(\rho, k) \cup \mathscr{F}(\rho, \gamma, k)\big)\big].$$

For the first term on the right-hand side, we use stationarity under $\mathbb{P}_\nu$ along with Lemmas 9.5, 9.6 and finally Lemma 9.4 to translate back into $\mathbb{P}_\mu$. For the second term on the right-hand side, we must first separate the event according to whether $\{0 \in C^\infty(\omega)\}$ occurs, or not. As in Corollary 9.8, Lemmas 9.2 and 9.4 imply

$$\mathbb{P}_\nu\big(\mathscr{F}(\rho, \gamma, k) \cap \{0 \notin C^\infty(\omega)\}\big) \leq c\rho^{-\varepsilon}.$$

On the other hand, Lemmas 9.4, 9.7 and stationarity imply

$$\mathbb{P}_{\nu_0}\big(\mathscr{F}(\rho, \gamma, k) \cap \{0 \in C^\infty(\omega)\}\big) \leq c2^{k(1-\varepsilon)} \rho^{-\alpha(1-o(1))}.$$

A final application of Lemma 9.4 finishes the proof. □

LEMMA 9.10 (Not too many intersections). *Let us suppose that $s \in (d, d+1)$. There exists $\varepsilon > 0$ and a constant $c > 0$ such that, for $\sigma \in \{\mu_0, \mu\}$,*

$$\mathbb{E}_\sigma\big[|\{X_i^1 : 0 \leq i \leq 2^k\} \cap \{X_i^2 : 0 \leq i \leq \infty\}|\big] \leq c2^{(1-\varepsilon)k}.$$

PROOF. For any $L > 0$, let $B_L(0) = [-L, L]^d \cap \mathbb{Z}^d$. We begin by considering the claim of the lemma under $\mathbb{P}_{\mu_0}$. We denote

(41) $$F(m, n) = \mathbb{E}_{\mu_0}\big[|\{X_i^1 : 0 \leq i \leq n\} \cap \{X_i^2 : 0 \leq i \leq m\}|\big].$$

Large deviations estimates imply that

$$\mu_0\Big(\max_{x \in B_L(0)} d^\omega(x) \geq \log^2 L\Big) \leq Ce^{-\log^2 L},$$

while Lemma 9.3 implies

$$\mathbb{P}_{\mu_0}\big(\{P_0^\omega(\exists t \le T : D_t \ge c_1 t^{p+1}) > c_2/T^\eta\}\big) \le c_3 T^{\eta+1-pr},$$

so that for any $\varepsilon_0 > 0$, we can find an exponent $\lambda > 0$ so that

$$\mathbb{E}_{\mu_0}\big(P_L^\omega\big(0, B_{L^\lambda}^c(0)\big)\big) < L^{-(1+\varepsilon_0)}.$$

Let us apply these estimates to (41). We have

$$F(m,n) \le \mathbb{E}_{\mu_0}\Big[\sum_{i \le n, j \le m}\sum_{y \in \mathbb{Z}^d} P_i^\omega(0, y) P_j^\omega(0, y)\Big].$$

Reversibility of the quenched walk implies $d^\omega(0)P_k^\omega(0, x) = d^\omega(x)P_k^\omega(x, 0)$. Thus, there exist $\eta', C_2 > 0$ so that

$$F(m,n) \le C_1 \mathbb{E}_{\mu_0}\Big[\sum_{k \le n, l \le m} \log^{\eta'}(k) \log^{\eta'}(l) P_{k+l}^\omega(0, 0)\Big] + C_2.$$

Next we apply Theorem 9.1: there exists a random variable $T(\omega)$ with $P(T > N) \le C(\eta''')N^{-\eta'''}$ for any $\eta''' > 0$, so that for $t \ge T(\omega)$,

$$P_t^\omega(0, 0) \le t^{-d/(s-d)} \log^\delta(t).$$

We have, for $\eta_0 > 0$ large enough,

$$\mathbb{E}_{\mu_0}\Big[\sum_{k \le n, l \le m} \log^{\eta'}(k) \log^{\eta'}(l) P_{k+l}^\omega(0, 0)\Big]$$

$$\le \mathbb{E}_{\mu_0}\Big[\sum_{T \le k \le n, T \le l \le m} \frac{\log^{\eta_0}(k+l)}{(k+l)^{d/(s-d)}}\Big] + \mathbb{E}_{\mu_0}\Big[T \sum_{T \le l \le m} \frac{\log^{\eta_0}(T+l)}{(l)^{d/(s-d)}}\Big]$$

$$+ \mathbb{E}_{\mu_0}\Big[T \sum_{T \le k \le n} \frac{\log^{\eta_0}(T+k)}{(k)^{d/(s-d)}}\Big] + \mathbb{E}_{\mu_0}\big[T^2 \log^{\eta_0}(T)\big].$$

By the tail bound for T, and since $d/(s-d) > 1$, the latter three terms on the right-hand side are all uniformly bounded in m, n.

For the first term, we have

$$\mathbb{E}_{\mu_0}\Big[\sum_{T \le k \le n, T \le l \le m} \frac{\log^{\eta'}(k+l)}{(k+l)^{d/(s-d)}}\Big] \le C(\varepsilon_1)\mathbb{E}_{\mu_0}\Big[\sum_{T \le k \le n} k^{1-d/(s-d)+\varepsilon}\Big]$$

$$\le C(\varepsilon)n^{2-d/(s-d)+\varepsilon_1}$$

for any $\varepsilon_1 > 0$. Let $m \to \infty$. Again, since $d/(s-d) > 1$, the lemma is immediate.

The statement for

$$\mathbb{E}_\mu\big[|\{X_i^1 : 0 \le i \le 2^k\} \cap \{X_i^2 : 0 \le i \le \infty\}| \,|\, 0 \notin \mathcal{C}^\infty(\omega)\big]$$

follows immediately from Lemma 9.2. $\square$

LEMMA 9.11 (Small jumps give small contribution).   *Let us suppose that $s \in (d, d+1)$. For large $k \in \mathbb{N}$,*

$$\mathbb{P}_\mu \left(\sum_{i=1}^{2^k} |X_i - X_{i-1}| \mathbb{1}\{|X_i - X_{i-1}| \le \rho\} > \frac{1}{2k} 2^{k/(s-d)} \right) \le o(k^{-100})$$

for some constant $C > 0$.

PROOF.   This is an immediate consequence of Lemma 9.4 and stationarity of the environment process. By stationarity, we have

$$\mathbb{E}_\nu \left[\sum_{i=1}^{2^k} |X_i - X_{i-1}| \mathbb{1}\{|X_i - X_{i-1}| \le \rho\} \right]$$
$$= 2^k \mathbb{E}_\nu \left[|X_1 - X_0| \mathbb{1}\{|X_1 - X_0| \le \rho\} \right].$$

Now

$$\mathbb{E}_\nu \left[|X_1 - X_0| \mathbb{1}\{|X_1 - X_0| \le \rho\} \right] = O(\rho^{1-\alpha}) = o(k^{-101} 2^{-k} 2^{k/(s-d)}).$$

Thus Markov's inequality implies

$$\mathbb{P}_\nu \left(\sum_{i=1}^{2^k} |X_i - X_{i-1}| \mathbb{1}\{|X_i - X_{i-1}| \le \rho\} > \frac{1}{2k} 2^{k/(s-d)} \right) \le o(k^{-100}).$$

Applying Lemma 9.4 finishes the job.   □

Given $\delta, \gamma > 0$, for a vertex v recall that $\tilde{p}_v = \tilde{p}_v(k)$ denotes the probability that a walk started from v and conditioned to stay in the set $\{u : |v - u| < 2^{k\delta}\}$ returns to v before time $2^{k\gamma}$ and set to 1 if v has no neighbors within distance $2^{k\delta}$. Recall that $\tilde{d}^\omega(v) := \#\{u : \|x - u\|_2 \le \rho\}$.

LEMMA 9.12.   *For all $\delta \in (0, 1)$, there exists $\gamma, \varepsilon > 0$ so that for $1 \le i \le 2^k$,*

$$\mathbb{P}_\mu \left(|p_{X_i^\ell} - \tilde{p}_{X_i^\ell}| > \frac{1}{k} \right) \le 2^{-2k\varepsilon}$$

and

$$\mathbb{P}_\mu \left(\tilde{d}^\omega(X_i^\ell) \ne d^\omega(X_i^\ell) \right) \le 2^{-2k\varepsilon}$$

and hence

$$\mathbb{P}_\mu \left(\#\left\{ i : |p_{X_i^\ell} - \tilde{p}_{X_i^\ell}| > \frac{1}{k} \right\} + \#\{i : \tilde{d}^\omega(X_i^\ell) \ne d^\omega(X_i^\ell)\} > \frac{1}{k} 2^k \right) \le 2^{-k\varepsilon}.$$

PROOF.   By Lemma 9.4 it is enough to prove the results under the measure $\mathbb{P}_\nu$. Using the stationarity of $\mathbb{P}_\nu$, the second bound follows by a union bound over the connection probabilities.

For the first bound, we prove the result under $\mathbb{P}_{\nu_0}$ and then for the other measures. Using stationarity, it is enough to consider $i = 0$. At a quenched level, given ω such that $0 \in \mathcal{C}^\infty(\omega)$, we may couple the conditioned walk (which gives rise to $\tilde{p}_0$), denoted by Y_t, to an unconditioned one, denoted by Z_t, until the first time the unconditioned one leaves $B_{2^{k\delta}}(0)$, and hence we have that

$$\left| p_0(\omega) - \tilde{p}_0(\omega) \right| \leq P_0^\omega(E),$$

where

$$E = \left\{ Z_t \text{ exits } B_{2^{k\delta}}(0) \text{ before time } 2^{\gamma k} \right\}.$$

Combining the proofs of Lemmas 9.5, 9.7, $\mathbb{P}_{\mu_0}(E) \leq o(2^{-2\varepsilon k})$ and hence

$$\mathbb{P}_{\mu_0}\left(|p_{X_i^\ell} - \tilde{p}_{X_i^\ell}| > \frac{1}{k} \right) \leq 2^{-2k\varepsilon}.$$

It remains to consider the event $\{0 \notin \mathcal{C}^\infty(\omega)\}$. Let us denote by $\mathcal{C}(0)$ the connected cluster of the origin. Clearly the bound is trivial if 0 has no nearest neighbors. Consider

$$\mathbb{P}_\nu\left(E, \mathrm{d}^\omega(0) \geq 1 | 0 \notin \mathcal{C}^\infty(\omega) \right).$$

To control this quantity, we bound

$$\mathbb{P}_\nu\left(0 \leftrightarrow B_{2^{k\delta}}^c(0) | 0 \notin \mathcal{C}^\infty(\omega) \right).$$

Recalling Lemma 9.2, if M denotes the largest component of $B_{2^{k\delta}}(0)$, then

$$\mathbb{P}_\nu\left(0 \in M | 0 \notin \mathcal{C}^\infty(\omega) \right) \leq C2^{-\varepsilon k}$$

and

$$\mathbb{P}_\nu\left(0 \leftrightarrow B_{2^{k\delta}}^c(0), 0 \notin M | 0 \notin \mathcal{C}^\infty(\omega) \right) \leq C2^{-\varepsilon k}$$

with $\varepsilon = \varepsilon(\delta)$.

We conclude $\mathbb{P}_\nu(E | 0 \notin \mathcal{C}^\infty(\omega)) \leq C2^{-\varepsilon k}$. Gathering estimates together and applying Lemma 9.4 proves the result. $\square$

10. Proof of Theorem 1.2.   In this section we show that when $d = 1$ and $s > 2$ that the scaling limit of the walk is Brownian motion. Recall the hypotheses of Theorem 1.2 where we assume that

$$\mathrm{P}(1) = 1, \qquad \mathrm{P}(r) = 1 - e^{-\beta r^{-s}} \qquad \text{for } r \geq 2,$$

since we clearly need there to be an infinite component.

10.1. *Geometry of the random graph and ergodic theory.* The notion of a cut-point of the graph plays a key role in our analysis.

DEFINITION 1.   Given an environment ω and $x \in \mathbb{Z}$, say that x is a *cutpoint* for $\omega \in \Omega$ if

$$\omega_{a,b} = 0 \qquad \text{for all } a \leq x, b \geq x \text{ so that } b - a \geq 2.$$

Let Ξ denote the set of cupoints.

Note that if the walk passes from the left of the cutpoint to the right, it must pass through it and that it is only connected to its nearest neighbors. By direct calculation, for $s > 2, d = 1$, we have that

$$\mu(0 \in \Xi) > 0.$$

Note that this does not hold when $s \leq 2$ which results in different scaling limits. Given an interval $[a, b]$, let $\mathfrak{C}_{[a,b]}$ denote the number of cutpoints $[a, b]$. Then ergodic considerations imply

LEMMA 10.1.   *For any $a, b \in \mathbb{R}, a < b$,*

$$\frac{1}{(b-a)N}\mathfrak{C}_{[aN,bN]} \to \mu(0 \in \Xi) \qquad \mu\text{-a.s.}$$

as $N \to \infty$.

Herein, it will be convenient to assume that the origin is a cutpoint. Suppose we show that for μ-almost every environment, conditioned on the origin being a cutpoint, the scaling limit of the walk started at the origin is Brownian motion. As the distribution is invariant under shifts, this implies that the walk started at any cutpoint has scaling limit Brownian motion. Finally, since the walk reaches a cutpoint in a finite amount of time, this implies a scaling limit of Brownian motion from any starting point. This justifies conditioning the origin to be a cutpoint. Let Ω_c denote the environments with $0 \in \Xi$, and let μ_c be the induced measure on Ω_c.

Let us define $c_i, i \in \mathbb{Z}$ as the ith cutpoint from 0 with i negative to the left, i positive to the right and $c_0 = 0$. Note that the gaps between the cutpoints $(c_j - c_{j-1})_{j \in \mathbb{Z}}$ are independent and identically distributed. This can be seen from the fact that given c_i is a cutpoint, there are no edges from the left of c_i to the right and the edge on the left side of c_i and on the right are independent. Similarly, the gap environments $([c_{j-1}, c_j])_{j \in \mathbb{Z}}$ also form an independent identically distributed sequence.

LEMMA 10.2.   *The expected gap size is finite,*

$$c_j - c_{j-1} \text{ is in } L^1(\mu)$$

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and its mean is given by

(42)
$$\mathbb{E}[c_j - c_{j-1}] \le \frac{1}{\mu(0 \in \Xi)}.$$

PROOF. By the strong law of large numbers,

$$\frac{c_n}{n} = \frac{1}{n}\left[\sum_{i=1}^{n} c_i - c_{i-1}\right] \to \mathbb{E}[c_i - c_{i-1}] \qquad \mu\text{-a.s.}$$

as $n \to \infty$ while by Lemma 10.1, we have that

$$\frac{1}{n}\mathfrak{C}_{[0,N]} \to \mu_c(0 \in \Xi) \qquad \mu\text{-a.s.}$$

as $n \to \infty$. Combining these obsevations yields equation (42). □

Given $\omega \in \Omega_c$, let X_i denote a simple random walk on ω started from $y \in \mathbb{Z}$. The initial state is specified by P_y^ω. Let us define

$$Q^\omega(j, j) = P_{c_j}^\omega(X_i \text{ returns to } c_j \text{ before hitting } \{c_{j-1}, c_{j+1}\}),$$

$$Q^\omega(j, j+1) = P_{c_j}^\omega(X_j \text{ hits } c_{j+1} \text{ before hitting } \{c_{j-1}, c_j\}),$$

$$Q^\omega(j, j-1) = P_{c_j}^\omega(X_i \text{ hits } c_{j-1} \text{ before hitting } \{c_j, c_{j+1}\}),$$

$$Q^\omega(k, \ell) = 0 \qquad \text{otherwise.}$$

These give us the transition probabilities of the walk restricted to the times at which it is at a cutpoint.

PROPOSITION 10.3. *We have*

$$Q^\omega(j, j+1) = Q^\omega(j+1, j)$$

and

$$\frac{1}{Q^\omega(j, j+1)} \le 2(c_{j+1} - c_j).$$

Hence $\frac{1}{Q^\omega(j,j+1)} \in L^1(\mu)$ *and* $\mathbb{E}_{\mu_c}\frac{1}{Q^\omega(j,j+1)} \le 2\mathbb{E}_{\mu_c}[c_{j+1} - c_j]$.

PROOF. The first statement holds by construction, while the second is a simple application of the electrical network interpretation of escape probabilities on graphs [13]. □

10.2. *Quenched functional central limit theorem for the simple random walk.*
The next step is to define a modified walk, which we will denote by Z_j, which is
manifestly a square integrable martingale. To do this, we first define a sequence
$(p_j)_{j \in \mathbb{Z}}$ by

$$p_0 = c_0 = 0,$$

$$p_j - p_{j-1} = \frac{1}{Q(j, j+1)} \qquad \text{otherwise.}$$

Let Z_i denote the (quenched) walk on $(p_j)_{j \in \mathbb{Z}}$ with transition probabilities
(slight abuse of notation here)

$$Q(p_j, p_k) := Q(j, k).$$

By fiat, the walk Z_i always starts from $p_0 = c_0$. It is then easy to check that Z_i is
a martingale, and the quadratic variation increments

$$E^\omega \big[(Z_j - Z_{j-1})^2 | \mathcal{F}_{j-1} \big] = \frac{1}{Q(Z_{j-1}, Z_{j-1} + 1)} + \frac{1}{Q(Z_{j-1}, Z_{j-1} - 1)}.$$

LEMMA 10.4. *For μ_c-a.e. ω, the law of the process*

$$M_t^{\omega, n} := \frac{1}{\sqrt{n}} \big(Z_{\lfloor nt \rfloor} + (tn - \lfloor tn \rfloor)(Z_{\lfloor nt \rfloor + 1} - Z_{\lfloor nt \rfloor}) \big)$$

*converges weakly in $C[0, 1]$ with the uniform norm to the law of $\sqrt{K} B_t$ where B_t
is a standard one-dimensional Brownian motion. Here, the diffusion constant K is
given by*

$$K = 2\mathbb{E}_{\mu_c} \bigg[\frac{1}{Q^\omega(0, 1)} \bigg].$$

PROOF.   The walk is stationary with respect to the uniform distribution (p_j).
Hence by the ergodic theorem and Proposition 10.3,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \mathbb{E}_{\mu_c} \big[(Z_j - Z_{j-1})^2 | \mathcal{F}_{j-1} \big] \to 2\mathbb{E}_{\mu_c} \bigg[\frac{1}{Q^\omega(0, 1)} \bigg] \qquad \mu_c\text{-a.s.,}$$

which shows convergence of the quadratic variation. The result then follows by an
application of martingale central limit theorem; see, for example, [14].   □

Let us define Y_i as the walk on $(c_j)_{j \in \mathbb{Z}}$ with transition probabilities given by
$Q(j, k)$. Observe that we have a natural mapping between Y_i and Z_i. The following
is a consequence of Proposition 10.3 and the strong law of large numbers.

LEMMA 10.5.   *For $n \neq 0$, we have that*

$$|p_n| \leq 2|c_n|$$

and

$$\lim \frac{1}{|j|}\left[\frac{p_j}{\mathbb{E}_{\mu_c}[p_1 - p_0]} - \frac{c_j}{\mathbb{E}_{\mu_c}[c_1 - c_0]}\right] \rightarrow 0 \qquad \mu_c\text{-a.s.}$$

as $|j| \rightarrow \infty$.

Lemma 10.5 establishes the scaling of the (p_j) compared to the position of the cutpoints (c_j). The next corollary then follows from Lemmas 10.4 and 10.5.

COROLLARY 10.6.   *For μ_c a.e. environment ω, the law of the process*

$$Y_t^{\omega,n} := \frac{1}{\sqrt{n}}(Y_{\lfloor nt \rfloor} + (tn - \lfloor tn \rfloor)(Y_{\lfloor nt \rfloor+1} - Y_{\lfloor nt \rfloor}))$$

converges weakly (under Q^ω) in $C[0, 1]$ with the uniform norm to the law of $\sqrt{K^*}B_t$ where B_t a standard one-dimensional Brownian motion. The constant K^* is given by

$$K^* = \frac{2(\mathbb{E}_{\mu_c}[c_1 - c_0])^2}{\mathbb{E}_{\mu_c}[p_1 - p_0]}.$$

Finally, let us return to the SRW X_i. Note that there is a natural coupling between X_i and Y_i, both the origin, so that for all i,

$$(43) \qquad\qquad\qquad Y_i = X_{\tau_i},$$

where τ_i denotes the ith visit to a cutpoint by X. Another application of the ergodic theorem implies that

$$(44) \qquad\qquad\qquad \lim_n \frac{1}{n}\tau_n \rightarrow \mu(0 \in \Xi).$$

PROOF OF THEOREM 1.2.   The theorem follows directly from Lemma 10.1, Corollary 10.6 and equations (43) and (44). □

11. List of symbols.   For the convenience of the reader we give a list of the symbols used in the text.

General

α	$s - d$ (Section 1)
$\mathcal{B}$	σ-algebra on Ω (Section 1)
$B_r(x), V_x$	$\{y : \|y - x\|_\infty < r\}$, respectively, $B_{2^{\delta k}}(x)$ (Sections 2 and 2.1)
$C^\infty(\omega)$	The infinite component of the graph induced by ω (Section 2)
$C_{q_{j-1},q_j,m}$	Arrival rate of new vertices of type (j, m) (Section 4.1)
$d^\omega(x), \mathcal{D}$	Degree of x in ω, respectively, marginal under μ (Sections 2 and 5)
$\tilde{d}^\omega(x), \mathcal{D}(k)$	Local degree of x, respectively, marginal under μ (Section 5)
$\mathcal{F}_{i,\ell}$	Filtration of "environment exploration process" (Section 2.1)
$\mathcal{F}_{i,\ell}^-$	Reduced filtration of exploration process (Section 2.1)
$\mathcal{H}_{k,\chi}^\ell$	Rate of new vertices for X^ℓ is χ close to ave. (Section 4.1)
$\mathcal{H}_{k,\chi}$	$\mathcal{H}_{k,\ell}$ holds on $1 - \chi$ fractions of paths (Section 4.1)
i, t	Time; i always discrete time (Section 1)
ℓ	Index for copy of i.i.d. family of SRWs (Section 2.1)
μ	Probability measure on Ω (Section 1)
μ_0	μ conditioned on $\{0 \in C^\infty(\omega)\}$ (Section 1)
ν	μ biased by degree of 0 (Section 2)
ν_0	ν conditioned on $\{0 \in C^\infty(\omega)\}$ (Section 2)
$\Gamma_\alpha(t), \Gamma(t)$	Isotropic α-stable Lévy process (Section 1)
$\omega_{x,y}$	State of edge indexed by $x, y \in \mathbb{Z}^d$ (Section 2)
Ω	Sample space for $\omega_{x,y}$'s (Section 1)
Ω_0	$\{0 \in C^\infty(\omega)\} \subset \Omega$ (Section 1)
$p_v(\omega), \mathcal{P}$	Escape probability of SRW, respectively, marginal under μ (Section 5)
$\tilde{p}_v(\omega), \mathcal{P}(k)$	Local "escape" probability, respectively, marginal under μ (Section 5)
ψ_J	Mesh size for $(q_j)_{j \leq J}$; (Section 5)
$\mathbb{P}_\lambda$	Joint law of $(\omega, (X^\ell)_{\ell \in \mathbb{N}})$ with $\lambda \in \{\mu, \mu_0, \nu, \nu_0\}$ (Section 2)
$(q_j)_{j \leq J}$	Discretization of law of $\mathcal{P}$ away from atoms (Section 5)
ρ, ρ_k	Cutoff for jump size (Section 2.1)
v, x, y	Elements of $\mathbb{Z}^d$ (Section 1)
$\mathcal{W}_{i,\ell}$	Vertices visited by time i of ℓth walk and by first 2^k steps of previous walks (Section 2.1)
$\mathcal{W}_{i,\ell}^+$	$\mathcal{W}_{i,\ell} \cup \{y : \exists x \in W_{i,\ell}, \omega_{x,y} = 1, \|x - y\|_\infty \geq \rho\}$ (Section 2.1)
X_i	SRW on graph induced by ω (Section 1)
X_i^ℓ	ℓth independent sample of SRW on ω (Section 2)
$X_n(t)$	$n^{1/\alpha} X_{\lfloor nt \rfloor}$ (Section 1)
$\hat{X}_i, \hat{X}_i^\ell$	First approximation to X_i; space independent increments (Section 6.1)
$\mathfrak{X}_i, \mathfrak{X}_i^\ell$	Second approximation to X_i; increments time independent (Section 6.2)

Coupling construction

$\mathcal{A}_{i,\ell}$	Indicator of new long edge at X_i^ℓ (Section 5.2)
$\mathcal{A}_{i,\ell}^*$	Indicator of coupling phase; special phase $= 1$ (Section 5.2)
$\mathcal{B}_{i,\ell}$	$\{0,\dots,6\}$-valued; nonzero value means coupling failed, value determines manner of failing (Section 5.2)
$\mathfrak{d}_i^{\ell,j,m}$	i.i.d. distributed as $\mathcal{D}(k)$ (Section 5.1)
$\iota,\phi_i^{\ell,j,m}$	$\sum_{i'=0}^{i-1}\mathcal{N}_{i'}^{\ell,j,m}$ (Section 5.2)
$\mathcal{N}_i^{\ell,j,m}$	Indicator "type" (j,m) seen by X^ℓ at i (Section 5.2)
$r_i^{\ell,j,m}$	i.i.d. distributed as $\mathcal{P}(k)$ (Section 5.1)
$R(t)$	Decreasing process, marginals geometric(t) (Section 5.1)
$R_t^{\ell,j,m},\tilde{R}_t^{\ell,j,m}$	i.i.d. with distribution $R(t)$ (Section 5.1)
$w_i^{\ell,j,m}(x)$	i.i.d. distributed as $\mathrm{Ber}(P(\|x\|_2))$ (Section 5.1)

Bounds on rare events

$A(\rho)$	$\{\exists v\in\mathbb{Z}^d,\omega_{0,v}=1,|v|>\rho\}$ (Section 9)
$B(\rho,\gamma,k)$	Walk does not visit vertices' edges connected to origin before time $2^{\gamma k+1}$ (Section 9)
$C(\delta,\gamma,k)$	$\{\max_{0\le t\le 2^{\gamma k}}|X_t|>2^{\delta k}\}$ (Section 9)
$D(\rho,k)$	$\{\exists v\in\mathbb{Z}^d, J\in[2^k]:\omega_{0,v}=1,|v|>\rho,$ $X_J=v,(0,v)\notin\{(X_i,X_{i+1})\}_{i\le J}\}$ (Section 9)
$E(\rho,\delta,k)$	Long edge at origin connected to long edge
$F(\rho,\gamma,k)$	Walk returns to long jumps at origin (Section 9)
$G(\rho,\delta,\gamma,k)$	$A(\rho)\cap B(\rho,\gamma,k)\cap C(\delta,\gamma,k)$ (Section 9)
$\mathscr{D}(\rho,k)$	$\bigcup_{i=0}^{2^k}\mathbf{T}^{-i}D(\rho,k)$ (Section 9)
$\mathscr{E}(\rho,k)$	$\bigcup_{i=0}^{2^k}\mathbf{T}^{-i}E(\rho,k)$ (Section 9)
$\mathscr{F}(\rho,\gamma,k)$	$\bigcup_{i=0}^{2^k}\mathbf{T}^{-i}\cdot F(\rho,\gamma,k)$ (Section 9)
$\mathscr{G}(\rho,\gamma,\delta,k)$	$\bigcup_{i=0}^{2^k}\mathbf{T}^{-i}\cdot G(\rho\gamma,\delta,k)$ (Section 9)

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