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LOCAL AND GLOBAL EXISTENCE OF SMOOTH SOLUTIONS FOR THE STOCHASTIC EULER EQUATIONS WITH MULTIPLICATIVE NOISE

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We establish the local existence of pathwise solutions for the stochastic Euler equations in a three-dimensional bounded domain with slip boundary conditions and a suitable nonlinear multiplicative noise. In the two-dimensional case we obtain the global existence of these solutions with additive or linear-multiplicative noise. Finally, we show that, in the three-dimensional case, the addition of linear multiplicative noise provides a regularizing effect; the global existence of solutions occurs with high probability if the initial data is sufficiently small, or if the noise coefficient is sufficiently large.

CONTENTS

1. Introduction.

In this paper we address the well-posedness of the stochastic incompressible Euler equations with multiplicative noise, in a smooth bounded

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simply-connected domain $\mathcal{D} \subset \mathbb{R}^d$

$$du + (u \cdot \nabla u + \nabla \pi)\, dt = \sigma(u)\, d\mathcal{W}, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

where $d = 2$ or 3, u denotes the velocity vector field, and π the pressure scalar field. Here $\mathcal{W}$ is a cylindrical Brownian motion, and $\sigma(u)\, d\mathcal{W}$ can be written formally in the expansion $\sum_{k \geq 1} \sigma_k(u)\, dW_k$ where W_k are a collection of $1D$ independent Brownian motions. The system (1.1)–(1.2) is supplemented with the classical slip boundary condition

$$u|_{\partial\mathcal{D}} \cdot n = 0, \tag{1.3}$$

where n denotes the outward unit normal to the boundary $\mathcal{D}$. Here $\partial\mathcal{D}$ is taken to be sufficiently smooth. In order to emphasize the stochastic effects and for the simplicity of exposition, we do not include a deterministic forcing f in (1.1), but note that all the results of this paper may be easily modified to include this more general case.

The Euler equations are the classical model for the motion of an inviscid, incompressible, homogeneous fluid. The addition of stochastic terms to the governing equations is commonly used to account for numerical, empirical and physical uncertainties in applications ranging from climatology to turbulence theory. In view of the wide usage of stochastics in fluid dynamics, there is an essential need to improve the mathematical foundations of the stochastic partial differential equations of fluid flow, and in particular to study inviscid models such as the stochastic Euler equations.

Even in the deterministic case, when $d = 3$ the global existence and uniqueness of smooth solutions remains a famously open problem for the Euler equations, and also for their dissipative counterpart, the Navier–Stokes equations. There is a vast literature on the mathematical theory for the deterministic Euler equations; see, for instance, the books [16, 40], the recent surveys [3, 17] and references therein. While the stochastic Navier–Stokes equation has been extensively studied dating back to the seminal works [6, 7] and subsequently in, for example, [5, 10, 12, 15, 18, 20, 23, 24, 27, 32, 43, 44, 51], rather less has been written concerning the stochastic Euler equations. Most of the existing literature on this subject treats only the two-dimensional case; see, for example, [8, 9, 13, 14, 21, 36]. To the best of our knowledge, there are only two works, [37, 45], which consider the local existence of solutions in *dimension three*. Both of these works consider only an *additive noise*, and treat (1.1)–(1.2) on the *full space*, avoiding difficulties which naturally arise in the presence of boundaries, due to the nonlocal nature of the pressure.

In this paper we establish three main results for the system (1.1)–(1.3). The first result addresses the local existence and uniqueness of solutions in both two and three dimensions. From the probabilistic point of view we study *pathwise* solutions, that is, probabilistically *strong* solutions where the driving noise and associated filtration is given in advance, as part of the data. From the PDE standpoint, we consider solutions which evolve continuously in the Sobolev space $W^{m,p}(\mathcal{D})$, for any integer $m > d/p + 1$ and any $p \geq 2$, where $d = 2, 3$.

This local existence result covers a large class of nonlinear multiplicative noise structures in $\sigma(\cdot)$. In particular we can handle Nemytskii operators corresponding to *any* smooth function $g : \mathbb{R}^d \to \mathbb{R}^d$. Here, heuristically speaking,

$$\sigma(u)\, d\mathcal{W}(t,x) = g(u)\dot{\eta}(t,x),$$

where $\dot{\eta}(t,x)$ is formally a Gaussian process with the spatial-temporal correlation structure described by $\mathbb{E}(\dot{\eta}(t,x)\dot{\eta}(s,y)) = \delta_{t-s}K(x,y)$ for any sufficiently smooth correlation kernel K on $\mathcal{D}$. We can also handle functionals of the solution forced by white noise, and of course the classical cases of additive and linear multiplicative noise. See Section 3.2 below for further details on these examples.

As noted above such results appear to be new in dimension three; this seems to be the first work to address (nonlinear) multiplicative noise, or to consider the evolution on a bounded domain. Moreover, our method of proof is quite different from those employed in previous works for a two-dimensional bounded domain. More precisely, we do not approximate solutions of the Euler system by those to the Navier–Stokes equations subject to Navier boundary conditions, and instead construct solutions to the Euler system directly.

In the second part of the paper we address some situations where the global existence of spatially smooth solutions evolving in $W^{m,p}(\mathcal{D})$, with $m > p/d + 1$ can be established. In the case of an additive noise ($\sigma(u) = \sigma$), when $d = 2$ we show that the solutions obtained in the first part of the paper are in fact global in time. To the best of our knowledge such results for smooth solutions were only known in the Hilbert space setting, that is, where $p = 2$; see [9] for a bounded domain and [36, 45] where the evolution is considered over the whole space.

Finally, we turn to the issue of global existence of smooth pathwise solutions with multiplicative noise, in both $d = 2, 3$. Obtaining the global existence of solutions for generic multiplicative noise $\sigma(u)\, d\mathcal{W}$ seems out of reach in view of some open problems that already arise in the deterministic setting for $d = 2$; cf. Remark 4.7 below. However, in the particular case of a *linear multiplicative* stochastic forcing, that is, when $\sigma(u)\, d\mathcal{W} = \alpha u\, dW$, where W is a one-dimensional standard Brownian motion, we show that the noise provides a damping effect on the pathwise behavior of solutions. In the *three-dimensional* case we prove that for any $R \geq 1$,

$$\mathbb{P}(u \text{ is global}) \geq 1 - R^{-1/4} \qquad \text{whenever } \|u_0\|_{W^{m,p}(\mathcal{D})} \leq \kappa(\alpha^2, R),$$

where κ is strictly positive and satisfies

$$\lim_{\alpha^2 \to \infty} \kappa(\alpha^2, R) = \infty$$

for every fixed $R \geq 1$. This may be viewed as a kind of global existence result in the *large noise asymptotic*. Furthermore, in the *two-dimensional* case, we show that solutions are global in time with probability one, for *any* $\alpha \in \mathbb{R}$, and independently of the size of the data. Note that in both cases the linear multiplicative noise allows us to transform (1.1)–(1.3) into an equivalent system for which the presence of an additional damping term becomes evident. We can exploit this random damping by using certain estimates for the exit times of geometric Brownian motion, and hence may establish the improved pathwise behavior of solutions. We note that in the deterministic setting the presence of sufficiently large damping is known to enhance the time of existence of solutions (see, e.g., [47]), but in order to carry over these ideas to the stochastic setting we need to overcome a series of technical difficulties.

The starting point of our analysis of (1.1)–(1.3) is to establish some suitable a priori estimates in the space $L^2(\Omega; L^\infty(0, T; W^{m,p}(\mathcal{D})))$. Here obstacles arise both due to the presence of boundaries and because we have to estimate stochastic integrals taking values in Banach spaces, that is, $L^p(\mathcal{D})$ for $p > 2$. While we handle the convective terms using direct commutator estimates, in order to bound the pressure terms we need to consider the regularity of solutions to an elliptic Neumann problem. At first glance this seems to require bounding expressions involving first order derivatives of the solution on the boundary, that is, $((u \cdot \nabla)u) \cdot n$, which would prevent the estimates from closing. However, by exploiting a geometric insight from [50], one may obtain suitable estimates for the pressure terms in $W^{m,p}(\mathcal{D})$. In order to treat the stochastic elements of the problem we follow the construction of stochastic integrals given in, for example, [38, 42]. Estimates for the resulting stochastic terms are more technically demanding than in the Hibert space setting, and are dealt with by a careful application of the Burkholder–Davis–Gundy inequality. Note also that we obtain bounds on u in $W^{m,p}(\mathcal{D})$ only up to a strictly positive stopping time τ. In contrast to the deterministic setting, quantitative lower bounds on this τ are unavailable. This leads to further difficulties later in establishing the compactness necessary to pass to the limit within a class of approximating solutions of (1.1)–(1.3).

With these a priori estimates in hand, we proceed to the first steps of the rigorous analysis. For this purpose, we introduce a Galerkin approximation scheme directly for (1.1)–(1.3), which we use to construct solutions for the Hilbert space setting $p = 2$. We later employ a density and stability argument to obtain $W^{m,p}(\mathcal{D})$ solutions from the solutions constructed via the Galerkin scheme. We believe that this Galerkin construction is more natural than in the previous works on the stochastic Euler equations on bounded domains [8, 9, 13, 14], which use approximations via the Navier–Stokes equations with Navier boundary conditions, and exploits the vorticity formulation of the equations, a method which is mostly suitable for the two-dimensional case.

As with other nonlinear SPDEs, we face the essential challenge of establishing sufficient compactness in order to be able to pass to the limit in the class of Galerkin approximations; even if a space $\mathcal{X}$ is compactly embedded in another space $\mathcal{Y}$, it is not usually the case that $L^2(\Omega; \mathcal{X})$ is compactly embedded in $L^2(\Omega; \mathcal{Y})$. As such, the standard Aubin or Arzelà–Ascoli type compactness results, which classically make possible the passage to the limit in the nonlinear terms, cannot be directly applied in this stochastic setting. With this in mind, we first establish the existence of martingale solutions following the approach in, for example, [22] and see also [24, 28]. Here the main mathematical tools are the Prokhorov theorem, which is used to obtain compactness in the collection of probability measures associated to the approximate solutions, and the Skorohod embedding theorem, which provides almost sure convergences, but relative to a new underlying stochastic basis.

At this stage there is another difficulty in comparison to previous works, for example, [28], which requires us to consider martingale solutions which are *very smooth* in $x \in \mathcal{D}$, that is, which evolve starting from data in $H^{m'}(\mathcal{D})$, with m' sufficiently large (in particular we may take $m' = m + 5$). The reason for this initially nonsharp range for m' stems from the following complication already alluded to above: the a priori estimates hold only up to a stopping time, so that when we attempt to find uniform estimates, the bounds hold only up to a sequence of times τ_n, which may depend on the order n of the approximation. In contrast to the deterministic case, it is not clear how to bound τ_n from below, uniformly in n. To compensate for this difficulty, we add a smooth cut-off function depending on the size of $\|u\|_{W^{1,\infty}}$ in front of the nonlinear and noise terms in the Galerkin scheme. This cut-off function, however, introduces additional obstacles for inferring uniqueness, which in view of the Yamada–Watanabe theorem is crucial for later arguments that allow us to pass to the case of pathwise solutions. For uniqueness, estimates in the $L^2(\mathcal{D})$ norm give rise to terms involving the $W^{1,\infty}(\mathcal{D})$ norm, which prevents one from closing the estimates in the energy space. On the other hand, if we attempt to prove uniqueness by estimating the difference of solutions in the $H^{m'}$ norm for arbitrary $m' > d/2 + 1$, we encounter problems due to terms which involve an excessive number of derivatives. By momentarily restricting ourselves to sufficiently large values of m', we manage to overcome both difficulties.

Having passed to the limit in the Galerkin scheme, we obtain the existence of very smooth solutions to a modified Euler equation with a cut-off in front of the nonlinearity. We can therefore a posteriori introduce a stopping time and infer the existence of a martingale solution of (1.1)–(1.3). It still remains to deduce the existence of pathwise solutions, that is, solutions of (1.1)–(1.3) defined relative to the initially given stochastic basis $\mathcal{S}$. For this we are guided by the classical Yamada–Watanabe theorem from finite dimensional stochastic analysis. This result tells us that, for finite dimensional systems at least, pathwise solutions exist whenever martingale solutions may be found, and pathwise uniqueness holds; cf. [52, 53]. More recently a different proof of such results was developed in [33] which leans on an elementary characterization of convergence in probability; cf. Lemma 6.10 below. Such an approach can sometimes be used for stochastic partial differential equations; see, for example, [24] in the context of viscous fluids equations.

Notwithstanding previous applications of Lemma 6.10 for the stochastic Navier–Stokes and related systems, the inviscid case studied here presents some new challenges, the most important of which is the difficulty in establishing the uniqueness of pathwise solutions.

With a class of pathwise solutions in very smooth spaces in hand, we next apply a density-stability argument to obtain the existence of solutions evolving in $W^{m,p}(\mathcal{D})$ where the ranges for m, p are now sharp, that is, $m > d/p + 1$ for any $p \geq 2$. Since, for all m' sufficiently large, $H^{m'}(\mathcal{D})$ is densely embedded in $W^{m,p}(\mathcal{D})$, we may smoothen (mollify) the initial data to obtain a sequence of very smooth pathwise approximating solutions u^n which evolve in $H^{m'}(\mathcal{D})$. By estimating these solutions pairwise we are able to show that they form a Cauchy sequence in $W^{m,p}(\mathcal{D})$, up to a strictly positive stopping time. Since almost sure control is needed for the individual solutions which each have their own maximal time of existence, we make use of an abstract lemma from [32, 43]. See also [31] for an application to other SPDEs, and [30] for related results in the deterministic setting.

As above for the uniqueness of solutions, when estimating $u^n - u^m$ we encounter terms involving ∇u^n in the $W^{m,p}$ norm [which is finite since $u^n \in H^{m'}(\mathcal{D})$ and m' is large]. These terms are dealt with using some properties of the mollifier F_ε used to smoothen the initial data (here $\varepsilon = 1/n$). More precisely, the term $\|\nabla u^n\|_{W^{m,p}}$ is of size $1/\varepsilon$, but it is multiplied by $\|u^n - u^m\|_{W^{m-1,p}}$, which converges to 0 when $m \geq n$ and $n \to \infty$, *even when multiplied by* $1/\varepsilon = n$. See [35, 41] for related estimates for the deterministic Euler equation.

In the second part of the manuscript we turn to establish some global existence results for (1.1)–(1.2). We first study the case of additive noise in two spatial dimensions. To address the additive case we apply a classical Beale–Kato–Majda type inequality for $\|u\|_{W^{1,\infty}}$; see, for example, [40]. This shows that if we can control the vorticity of the solution in L^∞ uniformly in time, then the nonlinear terms may be bounded like $\log(\|u\|_{W^{m,p}})\|u\|^p_{W^{m,p}}$. As such our proof relies on suitable estimates for the vorticity $\operatorname{curl} u$ in L^∞, which in this additive case can be achieved via a classical change of variables, and by establishing a suitable stochastic analog of a logarithmic Grönwall lemma.

The case of linear multiplicative noise is more interesting. As noted above, such noise structures evidence a pathwise damping of the solutions of (1.1)–(1.2), which may be seen by analyzing the transformed system (9.4)–(9.5) for a new variable $v(t) = u(t)\exp(-\alpha W_t)$. In order to take advantage of this damping in the three-dimensional case, we need to carefully show that the vortex stretching term is suitably controlled by the damping terms coming from the noise. For a sufficiently large noise coefficient α (or equivalently, for a sufficiently small initial condition), we see that the vorticity must be decaying, at least for some initial period during which $\|u\|_{W^{1,\infty}}$ remains below a certain threshold value. Via the usage of the Beale–Kato–Majda inequality, we see in turn that the growth on $\|u\|_{W^{m,p}}$ is limited by the possible growth of a certain geometric Brownian motion during this initial period. We are therefore able to show that if $\|u_0\|_{W^{m,p}}$ is sufficiently small with respect to a function of α and a given $R > 0$ then, on the event that the geometric Brownian motion never grows to be larger than R, the quantity $\|u\|_{W^{m,p}}$ will remain below a certain bound. In turn, this guarantees that the quantity $\|u\|_{W^{1,\infty}}$ will in fact never reach the critical value that would prevent the decay in vorticity, and we conclude that the solution is in fact global in time on this event that the geometric Brownian motion always stays below the value R. Since we are able to derive probabilistic bounds on this event, which crucially are independent of α, we obtain the desired results.

The manuscript is organized as follows. In Section 2 we review some mathematical background, deterministic and stochastic, needed throughout the rest of the work. We then make precise the conditions that we need to impose on the noise through σ in Section 3. We conclude this section with a detailed discussion of some examples of nonlinear noise structures covered under the given abstract conditions on σ. Section 4 contains the precise definitions of solutions to (1.1)–(1.3), along with statements of our main results. We next carry out some a priori estimates in Section 5. In Section 6 we introduce the Galerkin scheme and establish the existence of very smooth solutions. In Section 7 we establish the existence of solutions in the optimal spaces $W^{m,p}$ for any $m > d/p + 1$. The final two Sections 8 and 9 are devoted to proofs of the global existence results for the cases of additive and linear multiplicative noises, respectively. Appendices gather various additional technical tools used throughout the body of the paper.

2. Preliminaries. Here we recall some deterministic and stochastic ingredients which will be used throughout this paper.

2.1. *Deterministic background.* We begin by defining the main function spaces used throughout the work. For each integer $m \geq 0$ and $p \geq 2$, we let

$$X_{m,p} = \{v \in (W^{m,p}(\mathcal{D}))^d : \nabla \cdot v = 0, v|_{\partial\mathcal{D}} \cdot n = 0\} \tag{2.1}$$

and for simplicity write $X_m = X_{m,2}$; see also [50]. These spaces are endowed with the usual Sobolev norm of order m

$$\|v\|_{W^{m,p}(\mathcal{D})}^p := \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{L^p(\mathcal{D})}^p.$$

As usual, the norm on X_m is denoted by $\|\cdot\|_{H^m}$. We make the convention to write $\|\cdot\|_{W^{m,p}}$ and $\|\cdot\|_{H^m}$ instead of $\|\cdot\|_{W^{m,p}(\mathcal{D})}$ and $\|\cdot\|_{H^m(\mathcal{D})}$, *unless* Sobolev spaces on $\partial\mathcal{D}$ are considered. We let $(\cdot, \cdot)$ denote the usual $L^2(\mathcal{D})$ inner product, which makes $X_0 \subset L^2(\mathcal{D})$ a Hilbert space. The inner product on X_m shall be denoted by $(\cdot, \cdot)_{H^m} = \sum_{|\alpha| \leq m} (\partial^\alpha \cdot, \partial^\alpha \cdot)$.

Throughout the analysis we shall make frequent use of certain classical "calculus inequalities" which can be established directly from the Leibniz rule and the

Gagliardo–Nirenberg inequalities. Whenever $m > d/p$ we have the Moser estimate

$$\|uv\|_{W^{m,p}} \leq C\big(\|u\|_{L^\infty}\|v\|_{W^{m,p}} + \|v\|_{L^\infty}\|u\|_{W^{m,p}}\big) \tag{2.2}$$

for all $u, v \in W^{m,p}(\mathcal{D})$ and some universal constant $C = C(m, p, \mathcal{D}) > 0$. Note that in particular this shows that $W^{m,p}$ is an algebra whenever $m > d/p$. The following commutator estimate will also be used frequently:

$$\begin{aligned} &\sum_{0\leq|\alpha|\leq m} \big\|\partial^\alpha(u\cdot\nabla v) - u\cdot\nabla\partial^\alpha v\big\|_{L^p} \\ &\qquad \leq C\big(\|u\|_{W^{m,p}}\|\nabla v\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|v\|_{W^{m,p}}\big) \end{aligned} \tag{2.3}$$

for some constant $C = C(m, p, \mathcal{D}) > 0$, where $m > 1 + d/p$, $u \in W^{m,p}$ and $v \in W^{m+1,p}$. Note that for what follows we shall assume that $m > 1 + d/p$ and $p \geq 2$, where $d = 2, 3$ is the dimension of $\mathcal{D}$, allowing us to apply (2.2) and (2.3).

In order to treat the pressure term appearing in the Euler equations, we will need to bound the solutions of an elliptic Neumann problem taking the form

$$-\Delta\pi = f \qquad \text{in } \mathcal{D}, \tag{2.4}$$

$$\frac{\partial\pi}{\partial n} = g \qquad \text{on } \partial\mathcal{D} \tag{2.5}$$

for given f and g, sufficiently smooth. For this purpose we recall the result in [2] which gives the bound

$$\|\nabla\pi\|_{W^{m,p}(\mathcal{D})} \leq C\big(\|f\|_{W^{m-1,p}(\mathcal{D})} + \|g\|_{W^{m-1/p,p}(\partial\mathcal{D})}\big), \tag{2.6}$$

where $C = C(m, p, \mathcal{D}) > 0$ is a universal constant. In fact, (2.6) is usually combined with the bound given by the trace theorem: $\|h_{|\partial\mathcal{D}}\|_{W^{m-1/p,p}(\partial\mathcal{D})} \leq C\|h\|_{W^{m,p}(\mathcal{D})}$, which holds for sufficiently smooth h, integers $m \geq 1$ and $p \geq 2$; cf. [1].

Also in relation to the pressure we consider P, the so-called Leray projector, to be the orthogonal projection in $L^2(\mathcal{D})$ onto the closed subspace X_0. Equivalently, for any $v \in L^2(\mathcal{D})$ we have $Pv = (1 - Q)v$ where

$$Qv = -\nabla\pi$$

for any $\pi \in H^1(\mathcal{D})$ which solves the elliptic Neumann problem

$$-\Delta\pi = \nabla\cdot v \qquad \text{in } \mathcal{D}, \tag{2.7}$$

$$\frac{\partial\pi}{\partial n} = v\cdot n \qquad \text{on } \partial\mathcal{D}. \tag{2.8}$$

Moreover, for $v \in W^{m,p}$, observe that $\nabla\cdot v \in W^{m-1,p}(\mathcal{D})$ and $v|_{\partial\mathcal{D}}\cdot n \in W^{m-1/p,p}(\partial\mathcal{D})$. Hence, by applying (2.6) and the trace theorem to (2.7)–(2.8), we infer that

$$\|Pv\|_{W^{m,p}(\mathcal{D})} \leq C\|v\|_{W^{m,p}(\mathcal{D})} \tag{2.9}$$

for any $v \in W^{m,p}(\mathcal{D})$. Thus P is also a bounded linear operator from $W^{m,p}(\mathcal{D})$ into $X_{m,p}$.

We conclude this section with some bounds on the nonlinear terms which involve the Leray projector. These bounds will be used throughout the rest of the work.

LEMMA 2.1 (Bounds on the nonlinear term). *Let $m > d/p + 1$, and $p \geq 2$. The following hold*:

(a) *If $u \in W^{m,p}$ and $v \in W^{m+1,p}$, then $P(u \cdot \nabla v) \in X_{m,p}$, and*

$$\|P(u \cdot \nabla v)\|_{W^{m,p}} \leq C\big(\|u\|_{L^\infty}\|v\|_{W^{m+1,p}} + \|u\|_{W^{m,p}}\|v\|_{W^{1,\infty}}\big). \tag{2.10}$$

(b) *If $u, v \in X_{m,p}$, then $Q(u \cdot \nabla v) \in W^{m,p}(\mathcal{D})$ and*

$$\|Q(u \cdot \nabla v)\|_{W^{m,p}} \leq C\big(\|u\|_{W^{1,\infty}}\|v\|_{W^{m,p}} + \|u\|_{W^{m,p}}\|v\|_{W^{1,\infty}}\big). \tag{2.11}$$

(c) *If $u \in X_{m,p}$ and $v \in X_{m+1,p}$, then*

$$\begin{aligned} &\Big|\sum_{|\alpha|\leq m} \big(\partial^\alpha P(u \cdot \nabla v), \partial^\alpha v|\partial^\alpha v|^{p-2}\big)\Big| \\ &\quad \leq C\big(\|u\|_{W^{1,\infty}}\|v\|_{W^{m,p}} + \|u\|_{W^{m,p}}\|v\|_{W^{1,\infty}}\big)\|v\|_{W^{m,p}}^{p-1}. \end{aligned} \tag{2.12}$$

In (2.10)–(2.12), $C = C(m, p, \mathcal{D})$ is positive universal constant.

PROOF. First, we observe that if $u \in W^{m,p}$ and $v \in W^{m+1,p}$, then by (2.2) we have $u \cdot \nabla v \in W^{m,p}$ and $\|u \cdot \nabla v\|_{W^{m,p}}$ is bounded by the right-hand side of (2.10). Thus (a) follows from (2.9).

The proof of item (b) is due to [50]. If u and v are divergence free, and satisfy the nonpenetrating boundary condition (which occurs when $u, v \in X_{m,p}$), then boundary term $(u \cdot \nabla v) \cdot n$ may be re-written as $u_i v_j \phi_{ij}$, for some smooth functions ϕ_{ij}, independent of u, v which parametrize $\partial\mathcal{D}$ in a suitable way. Also, again due to the divergence-free condition, $\nabla \cdot (u \cdot \nabla v)$ may be re-written as $\partial_i u_j \partial_j v_i$. Hence, neither the boundary condition nor the force have too many derivatives and the elliptic Neumann problem one has to solve for the function π such that $Q(u \cdot \nabla v) = -\nabla\pi$ becomes

$$\begin{aligned} -\Delta\pi &= \partial_i u_j \partial_j v_i, \\ \frac{\partial\pi}{\partial n} &= u_i v_j \phi_{ij}. \end{aligned}$$

The proof of (b) now follows by applying estimate (2.6) to the above system, using the trace theorem and finally (2.2).

Finally, in order to prove (c) one uses the cancellation property $(u \cdot \nabla v, v|v|^{p-2}) = 0$, the definition of P, the bound (2.3), the Hölder inequality and item (b) to obtain

$$\begin{aligned}
&\Big|\sum_{|\alpha|\le m} \big(\partial^\alpha P(u\cdot\nabla v), \partial^\alpha v|\partial^\alpha v|^{p-2}\big)\Big| \\
&\quad \le \sum_{|\alpha|\le m} \big|\big(\partial^\alpha(u\cdot\nabla v), \partial^\alpha v|\partial^\alpha v|^{p-2}\big)\big| + \sum_{|\alpha|\le m} \big|\big(\partial^\alpha Q(u\cdot\nabla v), \partial^\alpha v|\partial^\alpha v|^{p-2}\big)\big| \\
&\quad \le C\bigg(\sum_{|\alpha|\le m} \|\partial^\alpha(u\cdot\nabla v) - u\cdot\nabla\partial^\alpha v\|_{L^p} + \|Q(u\cdot\nabla v)\|_{W^{m,p}}\bigg)\|v\|_{W^{m,p}}^{p-1} \\
&\quad \le C\big(\|u\|_{W^{1,\infty}}\|v\|_{W^{m,p}} + \|u\|_{W^{m,p}}\|v\|_{W^{1,\infty}}\big)\|v\|_{W^{m,p}}^{p-1},
\end{aligned}$$

concluding the proof of item (c). □

2.2. *Background on stochastic analysis*. We next briefly recall some aspects of the theory of the infinite dimensional stochastic analysis which we use below. We refer the reader to [22] for an extended treatment of this subject. For this purpose we start by fixing a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{W})$. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and $\mathcal{W}$ is a cylindrical Brownian motion defined on an auxiliary Hilbert space $\mathfrak{U}$ which is adapted to a complete, right continuous filtration $\{\mathcal{F}_t\}_{t\ge 0}$. By picking a complete orthonormal basis $\{e_k\}_{k\ge 1}$ for $\mathfrak{U}$, $\mathcal{W}$ may be written as the formal sum $\mathcal{W}(t,\omega) = \sum_{k\ge 1} e_k W_k(t,\omega)$ where the elements W_k are a sequence of independent $1D$ standard Brownian motions. Note that $\mathcal{W}(t,\omega) = \sum_{k\ge 1} e_k W_k(t,\omega)$ does not actually converge on $\mathfrak{U}$, and so we will sometimes consider a larger space $\mathfrak{U}_0 \supset \mathfrak{U}$ we define according to

$$\mathfrak{U}_0 := \bigg\{ v = \sum_{k\ge 0} \alpha_k e_k : \sum_k \frac{\alpha_k^2}{k^2} < \infty \bigg\},$$

and endow this family with the norm $\|v\|_{\mathfrak{U}_0}^2 := \sum_k \alpha_k^2 k^{-2}$, for any $v = \sum_k \alpha_k e_k$. Observe that the embedding of $\mathfrak{U} \subset \mathfrak{U}_0$ is Hilbert–Schmidt. Moreover, using standard martingale arguments with the fact that each W_k is almost surely continuous we have that, $\mathcal{W} \in C([0,\infty), \mathfrak{U}_0)$, almost surely. See [22].

Consider now another separable Hilbert space X. We denote the collection of Hilbert–Schmidt operators, the set of all bounded operators G from $\mathfrak{U}$ to X such that $\|G\|_{L_2(\mathfrak{U},X)}^2 := \sum_k |Ge_k|_X^2 < \infty$, by $L_2(\mathfrak{U}, X)$. Whenever $X = \mathbb{R}$, that is, in the case where G is a linear functional, we will denote $L_2(\mathfrak{U}, \mathbb{R})$ by simply L_2. Given an X valued predictable[3] process $G \in L^2(\Omega; L^2_{\mathrm{loc}}([0,\infty), L_2(\mathfrak{U}, X)))$ and

[3] Let $\Phi = \Omega \times [0,\infty)$ and take $\mathcal{G}$ to be the σ-algebra generated by sets of the form

$$(s,t] \times F, \qquad 0 \le s < t < \infty, F \in \mathcal{F}_s; \qquad \{0\} \times F, \qquad F \in \mathcal{F}_0.$$

taking $G_k = Ge_k$, one may define the (Itô) stochastic integral

$$M_t := \int_0^t G\, d\mathcal{W} = \sum_k \int_0^t G_k\, dW_k \tag{2.13}$$

as an element in $\mathcal{M}_X^2$, that is, the space of all X valued square integrable martingales. If we merely assume that the predictable process $G \in L^2_{\mathrm{loc}}([0,\infty), L_2(\mathfrak{U}, X))$ almost surely, that is, without any moment condition, then M_t can still be defined as in (2.13) by a suitable localization procedure. Detailed constructions in both cases may be found in, for example, [22] or [48].

The process $\{M_t\}_{t\geq 0}$ has many desirable properties. Most notably for the analysis here, the Burkholder–Davis–Gundy inequality holds, which in the present context takes the form

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t G\, d\mathcal{W}\right|_X^r\right) \leq C\mathbb{E}\left(\int_0^T |G|^2_{L_2(\mathfrak{U},X)}\, dt\right)^{r/2}, \tag{2.14}$$

valid for any $r \geq 1$, and where C is an absolute constant depending only on r. In the coordinate basis $\{e_k\}$, (2.14) takes the form

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\sum_k \int_0^t G_k\, dW_k\right|_X^r\right) \leq C\mathbb{E}\left(\int_0^T \sum_k |G_k|_X^2\, dt\right)^{r/2}.$$

Since we consider solutions of (1.1)–(1.3) evolving in $X_{m,p}$ for any $p \geq 2$ and $m > d/p + 1$, we will recall some details of the construction of stochastic integrals evolving on $W^{m,p}(\mathcal{D})$. Here we use the approach of [38, 42], to which we refer the reader for further details. See also [11, 46] and containing references for a different, more abstract approach to stochastic integration in the Banach space setting. Suppose that $p \geq 2$, $m \geq 0$, and define

$$\mathbb{W}^{m,p} = \left\{\sigma : \mathcal{D} \to L_2 : \sigma_k(\cdot) = \sigma(\cdot)e_k \in W^{m,p} \text{ and } \sum_{|\alpha|\leq m} \int_{\mathcal{D}} |\partial^\alpha \sigma|^p_{L_2}\, dx < \infty\right\},$$

which is a Banach space according to the norm

$$\|\sigma\|^p_{\mathbb{W}^{m,p}} := \sum_{|\alpha|\leq m} \int_{\mathcal{D}} |\partial^\alpha \sigma|^p_{L_2}\, dx = \sum_{|\alpha|\leq m} \int_{\mathcal{D}} \left(\sum_{k\geq 1} |\partial^\alpha \sigma_k|^2\right)^{p/2} dx. \tag{2.15}$$

Let P be the Leray projection operator defined in Section 2.1. For $\sigma \in \mathbb{W}^{m,p}$ we define $P\sigma$ as an element in $\mathbb{W}^{m,p}$ by taking $(P\sigma)e_k = P(\sigma e_k)$ so that P is a linear continuous operator on $\mathbb{W}^{m,p}$. We take

$$\mathbb{X}_{m,p} = P\mathbb{W}^{m,p} = \{P\sigma : \sigma \in \mathbb{W}^{m,p}\}.$$

Recall that a X valued process U is called predictable (with respect to the stochastic basis $\mathcal{S}$) if it is measurable from $(\Phi, \mathcal{G})$ into $(X, \mathcal{B}(X))$, $\mathcal{B}(X)$ being the family of Borel sets of X.

Note that $\mathbb{X}_{m,2} = L_2(\mathfrak{U}, X_m)$ and in accordance with (2.1), we will denote $\mathbb{X}_{m,2}$ by simply $\mathbb{X}_m$.

Consider any predictable process $G \in L^p(\Omega; L^p_{\mathrm{loc}}([0,\infty), \mathbb{X}_{m,p})$. For such a G we have, for any $T > 0$ and almost every $x \in \mathcal{D}$, that $\mathbb{E}\int_0^T \sum_{|\alpha|\le m} |\partial^\alpha G(x)|^2_{L_2}\,dt < \infty$. We thus obtain from the Hilbert space theory introduced above that M_t as in (2.13) is well defined for almost every $x \in \mathcal{D}$ as a real valued martingale and that for each $|\alpha| \le m$, $\partial^\alpha M_t(x) = \int_0^t \partial^\alpha G(x)\,d\mathcal{W}$. By applying the Burkholder–Davis–Gundy inequality, (2.14) we have that

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]} \|M_t\|^p_{W^{m,p}} &\le C \sum_{|\alpha|\le m} \int_{\mathcal{D}} \mathbb{E}\left(\int_0^T \left|\partial^\alpha G(x)\right|^2_{L_2} dt\right)^{p/2} dx \\ &\le C\mathbb{E}\int_0^T |G|^p_{\mathbb{X}_{m,p}}\,dt.\end{aligned}$$

Finally (cf. [38, 42]), one may show that $M_t \in L^p(\Omega; C([0,\infty); X_{m,p}))$ and is an $X_{m,p}$ valued martingale.

3. Nonlinear multiplicative noise structures and examples. In this section we make precise the conditions that we impose on the noise. While, in abstract form, these conditions appear to be rather involved, they in fact cover a very wide class of physically realistic nonlinear stochastic regimes. We conclude this section by detailing some of these examples.

3.1. *Abstract conditions.* We next describe, in abstract terms, the conditions imposed for σ. Consider any pair of Banach spaces $\mathcal{X}$, $\mathcal{Y}$ with $\mathcal{X} \subset L^\infty(\mathcal{D})$. We denote the space of locally bounded maps

$$\begin{aligned}\mathrm{Bnd}_{u,\mathrm{loc}}(\mathcal{X},\mathcal{Y}) := \bigl\{\sigma \in C\bigl(\mathcal{X}\times[0,\infty);\mathcal{Y}\bigr) : \bigl\|\sigma(x,t)\bigr\|_{\mathcal{Y}} \le \beta\bigl(\|x\|_{L^\infty}\bigr)\bigl(1+\|x\|_{\mathcal{X}}\bigr) \\ \forall x \in \mathcal{X}, t \ge 0\bigr\},\end{aligned}$$

where $\beta(\cdot) \ge 1$ is an increasing function which is locally bounded and is independent of t. In addition we define the space of locally Lipschitz functions,

$$\begin{aligned}\mathrm{Lip}_{u,\mathrm{loc}}(\mathcal{X},\mathcal{Y}) = \bigl\{\sigma \in \mathrm{Bnd}_{u,\mathrm{loc}}(\mathcal{X},\mathcal{Y}) : \\ \bigl\|\sigma(x,t)-\sigma(y,t)\bigr\|_{\mathcal{Y}} \le \beta\bigl(\|x\|_{L^\infty}+\|y\|_{L^\infty}\bigr)\|x-y\|_{\mathcal{X}} \\ \forall x, y \in \mathcal{X}, t \ge 0\bigr\}.\end{aligned}$$

Note that in both cases the subscript u is intended to emphasize the that increasing function β appearing in the above inequalities may be taken to be independent of $t \in [0,\infty)$. Note furthermore that, by considering such locally Lipschitz spaces of functions, we are able to cover stochastic forcing involving Nemytskii operators, that is, smooth functions of the solutions multiplied by spatially correlated white in time Gaussian noise; see Section 3.2 below.

For the main local existence results in the work, Theorem 4.3 below, we fix $p \geq 2$ and an integer $m > d/p + 1$, and suppose that

$$\sigma \in \mathrm{Lip}_{u,\mathrm{loc}}(L^p, \mathbb{W}^{0,p}) \cap \mathrm{Lip}_{u,\mathrm{loc}}(W^{m-1,p}, \mathbb{W}^{m-1,p}) \cap \mathrm{Lip}_{u,\mathrm{loc}}(W^{m,p}, \mathbb{W}^{m,p}). \tag{3.1}$$

Since P is a continuous linear operator on $\mathbb{W}^{k,p}$, for $k \geq 0$ it follows that $P\sigma \in \mathrm{Lip}_{u,\mathrm{loc}}(W^{k,p}, \mathbb{X}_{k,p})$, for $k = m-1, m$. Observe that by (3.1) we have that $\int_0^t P\sigma(u)\, d\mathcal{W} \in C([0,\infty); X_{m,p})$ for each predictable process $u \in C([0,\infty); X_{m,p})$.

We will also impose some additional technical conditions on σ which are required for the proof of local existence of solutions; cf. Theorem 4.3 below. These conditions do no preclude any of the examples we give below. First we suppose that

$$\sigma \in \mathrm{Bnd}_{u,\mathrm{loc}}(W^{m+1,p}, \mathbb{W}^{m+1,p}). \tag{3.2}$$

Fix some m' sufficiently large, such that $H^{m'-2} \subset W^{m+1,p}$, for example, $m' > m + 3 + d(p-2)/(2p)$ by the Sobolev embedding. For simplicity we take an m' which works for all $p \geq 2$, and for the rest of this paper, fix

$$m' = m + 5.$$

We assume that

$$\sigma \in \mathrm{Bnd}_{u,\mathrm{loc}}(H^{m'}, \mathbb{W}^{m',2}). \tag{3.3}$$

Condition (3.2) is used for the density and stability arguments in Section 7, while condition (3.3) seems necessary in order to justify the construction of solutions to the Galerkin system; cf. Section 6.2 below.

In the case of an additive noise when we assume that σ is independent of u (cf. Theorem 4.4), we may alternatively assume that

$$\sigma \in L^p(\Omega, L^p_{\mathrm{loc}}([0,\infty); \mathbb{W}^{m+1,p})) \tag{3.4}$$

and that σ is predictable. Note that while (3.1)–(3.3) covers many additive noise structures, (3.4) is less restrictive and allows for $\omega \in \Omega$ dependence in σ.

3.2. *Examples*. We now describe some examples of stochastic forcing structures for $\sigma(u)\, d\mathcal{W}$ covered under the conditions (3.1)–(3.3) imposed above, or alternatively (3.4) for additive noise.

Nemytskii operators. One important example is stochastic forcing of a smooth function of the solution. Suppose that $g:\mathbb{R}^d \to \mathbb{R}^d$ is C^∞ smooth and consider $\alpha \in \mathbb{W}^{m',2}$, where as above $m' = m + 5$. We then take

$$\sigma_k(u) = \alpha_k(x) g(u), \qquad k \geq 1. \tag{3.5}$$

In this case we have that

$$\sigma(u)\, d\mathcal{W} = \sum_{k\geq 1} \alpha_k(x) g(u)\, dW_k = g(u) \sum_{k\geq 1} \alpha_k(x)\, dW_k = g(u)\alpha\, d\mathcal{W}.$$

Note that $\alpha\, d\mathcal{W}$ is formally a Gaussian process with the spatial-temporal correlation structure

$$\mathbb{E}\big(\alpha\, d\mathcal{W}(x,t)\alpha\, d\mathcal{W}(y,s)\big) = K(x,y)\delta_{t-s} \qquad \text{for all } x, y \in \mathbb{R}^d, t, s \geq 0,$$

with $K(x,y) = \sum_{k\geq 1} \alpha_k(x)\alpha_k(y)$. Observe that if $g(u) \in W^{n,q}$ for $q \geq 2$ and $n \geq d/q$, then

$$\begin{aligned} \|\sigma(u) - \sigma(v)\|_{\mathbb{W}^{n,q}}^q &:= \sum_{|\alpha|\leq n} \int_{\mathcal{D}} \bigg(\sum_{k\geq 1} \big|\partial^\alpha\big(\alpha_k g(u) - \alpha_k g(v)\big)\big|^2\bigg)^{q/2} dx \\ &\leq C\|\alpha\|_{\mathbb{W}^{n,q}}^q \|g(u) - g(v)\|_{W^{n,q}}^q. \end{aligned}$$

We may therefore show that (3.5) satisfies (3.1)–(3.3) by making use of the following general fact about the composition of functions.

LEMMA 3.1 (Locally Lipschitz and bounded). *Fix any $n > d/p$ with $p \geq 2$. Suppose that $g:\mathbb{R}^d \to \mathbb{R}^d$ and that $g \in W^{n+1,\infty}(\mathbb{R}^d)$. Then*

$$\begin{aligned} &\|g(u) - g(v)\|_{W^{n,p}(\mathcal{D})} \leq \beta\big(\|u\|_{L^\infty} + \|u\|_{L^\infty}\big)\|u - v\|_{W^{n,p}(\mathcal{D})} \\ &\qquad\qquad \textit{for every } u, v \in W^{n,p}(\mathcal{D}) \end{aligned} \tag{3.6}$$

holds for some positive, increasing function $\beta(\cdot) \geq 1$.

Note that (3.1) follows from (3.6). Moreover setting $v = 0$ in (3.6) also proves (3.2) and (3.3). The proof of Lemma 3.1 is based on Moser-type estimates [similar to (2.2)], Gagliardo–Nirenberg interpolation inequalities, and the chain rule. See, for example, [49], Chapter 13, Section 3, for further details.

Linear multiplicative noise. One important example covered under this general class of Namytskii operators is a linear multiplicative noise. Here we consider

$$\sigma(u)\, d\mathcal{W} = \alpha u\, dW,$$

where now $\alpha \in \mathbb{R}$ and W is a $1D$ standard Brownian motion. We obtain this special case from the above framework by taking $g = \mathrm{Id}$ and $\alpha_1 \equiv 1$, $\alpha_k = 0$ for $k \geq 2$. We shall treat such noise structures in detail in Section 9; cf. Theorem 4.6.

Stochastic forcing of functionals of the solution. We may also consider functionals (linear and nonlinear) of the solution, forced by independent white noise processes. Suppose that, for $k \geq 1$ we are given $f_k : L^p(\mathcal{D}) \to \mathbb{R}$ such that

$$\left| f_k(u) - f_k(v) \right| \leq C \|u - v\|_{L^p} \qquad \text{for } u, v \in L^p, \tag{3.7}$$

where the constant C is independent of k. We take

$$\sigma_k(u) = f_k(u)\alpha_k(x, t) \tag{3.8}$$

then, for any $n \geq d/q$,

$$\begin{aligned} \|\sigma(u) - \sigma(v)\|^q_{\mathbb{W}^{n,q}} &:= \sum_{|\alpha| \leq n} \int_{\mathcal{D}} \left(\sum_{k \geq 1} \left| f_k(u) - f_k(v) \right|^2 \left| \partial^\alpha \alpha_k \right|^2 \right)^{p/2} dx \\ &\leq \|\alpha\|^p_{\mathbb{W}^{n,q}} \|u - v\|^p_{L^p}. \end{aligned}$$

Thus, under assumption (3.7) if we furthermore assume that $\sup_{t \geq 0} \|\alpha(t)\|_{\mathbb{W}^{m',2}} < \infty$, then σ given by (3.8) satisfies conditions (3.1)–(3.3).

Additive noise. For $\sigma : [0, \infty) \to H^{m'}$, with $\sup_{t \geq 0} \|\sigma(t)\|_{H^{m'}} < \infty$, we may easily observe that σ satisfies (3.1)–(3.3). For such noise $\sigma\, d\mathcal{W}$ may be understood in the formal expansion

$$\sigma\, d\mathcal{W}(t, x, \omega) = \sum_k \sigma_k(t, x)\, dW_k(t, \omega).$$

Note that our results for additive noise in Theorem 4.4 are established under a more general ω-dependent σ, which satisfies (3.4).

4. Main results. With the mathematical preliminaries in hand and having established the noise structures we shall consider, we now make precise the notions of *local*, *maximal* and *global* solutions of the stochastic Euler equation (1.1)–(1.3).

DEFINITION 4.1 (Local pathwise solutions). Suppose that $m > d/p + 1$ with $p \geq 2$ and $d = 2, 3$. Fix a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ and u_0 an $X_{m,p}$ valued $\mathcal{F}_0$ measurable random variable. Suppose that σ satisfies conditions (3.1)–(3.3) [or alternatively (3.4)].

(i) A *local pathwise* $X_{m,p}$ solution of the stochastic Euler equation is a pair (u, τ), with τ a strictly positive stopping time, and $u : [0, \infty) \times \Omega \to X_{m,p}$ is a predictable process satisfying

$$u(\cdot \wedge \tau) \in C\big([0, \infty), X_{m,p}\big)$$

and for every $t \geq 0$,

$$u(t \wedge \tau) + \int_0^{t \wedge \tau} P(u \cdot \nabla u)\, dt = u(0) + \int_0^{t \wedge \tau} P\sigma(u)\, d\mathcal{W}. \tag{4.1}$$

(ii) We say that local pathwise solutions are *unique* (or *indistinguishable*) if, given any pair $(u^{(1)}, \tau^{(1)})$, $(u^{(2)}, \tau^{(2)})$ of local pathwise solutions,

$$\mathbb{P}\big(\mathbb{1}_{u^{(1)}(0)=u^{(2)}(0)}\big(u^{(1)}(t) - u^{(2)}(t)\big) = 0; \forall t \in \big[0, \tau^{(1)} \wedge \tau^{(2)}\big]\big) = 1. \tag{4.2}$$

Given the existence and uniqueness of such local solutions we can quantify the possibility of any finite time blow-up. In some cases we are able to show that such pathwise solutions in fact are global in time.

DEFINITION 4.2 (Maximal and global solutions). Fix a stochastic basis and assume the conditions u_0 and σ are exactly as in Definition (4.1) above. A *maximal* pathwise solution is a triple $(u, \{\tau_n\}_{n\geq 1}, \xi)$ such that each pair (u, τ_n) is a local pathwise solution, τ_n is increasing with $\lim_{n\to\infty} \tau_n = \xi$ and so that

$$\sup_{t\in[0,\tau_n]} \|u(t)\|_{W^{1,\infty}} \geq n \qquad \text{on the set } \{\xi < \infty\}. \tag{4.3}$$

A maximal pathwise solution $(u, \{\tau_n\}_{n\geq 1}, \xi)$ is said to be *global* if $\xi = \infty$ almost surely.[4]

Our primary goal in this work is to study local and global pathwise solutions of the stochastic Euler equation. These type of solutions also fall under the designation of "strong solutions;" we prefer the term "pathwise" since it avoids possible confusion with classical terminology used in deterministic PDEs. In any case one can also establish the existence of "martingale" (or probabilistically "weak" solutions) of (1.1)–(1.3) where the stochastic basis is an unknown in the problem and the initial conditions are only specified in law. Indeed such type of solutions are essentially established as an intermediate step in the analysis which is carried out in Section 6; see Remark 6.6 below.

We now state the main results of this paper. The first result concerns the local existence of solutions, the proof of which is carried out in two steps, in Sections 6 and 7 below.

THEOREM 4.3 (Local existence of pathwise solutions). *Fix a stochastic basis* $\mathcal{S} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$. *Suppose that* $m > d/p + 1$ *with* $p \geq 2$ *and* $d = 2, 3$. *Assume that* u_0 *is an* $X_{m,p}$ *valued*, $\mathcal{F}_0$ *measurable random variable and that* σ *satisfies the conditions* (3.1)–(3.3). *Then there exists a unique maximal pathwise solution* $(u, \{\tau_n\}_{n\geq 1}, \xi)$ *of* (1.1)–(1.3), *in the sense of Definitions* 4.1 *and* 4.2.

In Section 8 we show that in two space dimensions we have, in the case of an *additive noise*, the global existence of solutions. Note that in contrast to the

[4]Under this definition it is clear that, for every $T > 0$, $\sup_{t\in[0,T]} \|u(t)\|_{W^{1,\infty}}$ is almost surely finite on the set $\{\xi = \infty\}$.

situation for the $2D$ Navier–Stokes equations (cf., e.g., [32]), proving the global existence for a general Lipschitz nonlinear multiplicative noise seems to be out of reach with current methods; see Remark 4.7 below for further details.

THEOREM 4.4 (Global existence for additive noise in $2D$). *Fix $m > 2/p + 1$ with $p \geq 2$, a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$, and assume that u_0 is an $X_{m,p}$ valued, $\mathcal{F}_0$ measurable random variable. Assume that σ does not depend on u and* (3.4) [*or* (3.1)–(3.3)] *holds. Then, there exits a unique global pathwise solution of* (1.1)–(1.3), *that is, $\xi = \infty$ almost surely.*

REMARK 4.5. The local existence of pathwise solutions with additive noise follows directly from Theorem 4.3 in the case of a (deterministic) continuous $\sigma : [0, \infty) \to \mathbb{W}^{m',2}$, which satisfies $\sup_{t\geq 0} \|\sigma(t)\|_{\mathbb{X}_{m',2}} < \infty$, where m' is as in (3.3). On the other hand, the proof of local existence for additive noise does not require the involved machinery employed to deal with a general nonlinear multiplicative noise; in this case one can transform (1.1) into a random partial differential equation, which can be treated pathwise, using the classical (deterministic) methods for the Euler equations; cf. [40]. Of course, one has to show that this random transformed system is measurable with respect to the stochastic elements in the problem, but this may be achieved with continuity and stability arguments. These technicalities are essentially contained in [37], to which we refer for further details.

Finally we address the case of a *linear multiplicative noise*. In $2D$ we show that the pathwise solutions are global in time. In $3D$ we go further and prove that the noise is regularizing at the pathwise level. Here we are essentially able to establish that the time of existence converges to $+\infty$ a.s. in the *large noise limit*. More precisely, we have:

THEOREM 4.6 (Global existence for linear multiplicative noise). *Fix a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$[5]. Suppose that $m > d/p + 1$ with $p \geq 2$ and $d = 2, 3$, and assume that u_0 is an $X_{m,p}$ valued, $\mathcal{F}_0$ measurable random variable. For $\alpha \in \mathbb{R}$ we consider* (1.1)–(1.3) *with a linear multiplicative noise*

$$\sigma_k(u) = \sigma(u)e_k = \begin{cases} \alpha u, & \textit{if } k = 1, \\ 0, & \textit{otherwise}. \end{cases}$$

(i) *Suppose $d = 2$. Then for any $\alpha \in \mathbb{R}$ the maximal pathwise solution of guaranteed by Theorem* 4.3 *is in fact global, that is, $\xi = \infty$ almost surely.*

[5]For the noise structure considered here, we need only to have defined a single $1D$ standard Brownian motion.

(ii) *Suppose* $d = 3$. *Let* $R \geq 1$ *and* $\alpha \neq 0$ *be arbitrary parameters*. *Then there exists a positive deterministic function* $\kappa(R, \alpha)$ *which satisfies*

$$\lim_{\alpha^2 \to \infty} \kappa(R, \alpha) = \infty$$

for every fixed $R \geq 1$, *such that whenever*

$$\|u_0\|_{W^{m,p}} \leq \kappa(R, \alpha), \qquad a.s. \tag{4.4}$$

then

$$\mathbb{P}(\xi = \infty) \geq 1 - \frac{1}{R^{1/4}}.$$

In particular, *for every* $\varepsilon > 0$ *and any given deterministic initial condition*, *the probability that solutions corresponding to sufficiently large* $|\alpha|$ *never blow up*, *is greater than* $1 - \varepsilon$.

REMARK 4.7 (Lack of global well-posedness in two dimensions with generic multiplicative noise). We emphasize that even in the two-dimensional setting, and even for $\mathcal{D} = \mathbb{R}^2$, the global existence of smooth solutions to (1.1)–(1.3) for a *general* Lipschitz multiplicative noise appears to be out of reach. In fact, the analogous result remains open even in the deterministic setting unless the forcing is linear. Indeed, let us consider the Euler equations with a *solution-dependent forcing*

$$\partial_t u + u \cdot \nabla u + \nabla \pi = f(u), \qquad \nabla \cdot u = 0, \tag{4.5}$$

where f is a smooth function mapping $\mathbb{R}^2 \to \mathbb{R}^2$, which decays sufficiently fast at infinity. In order to obtain the global in time regularity of (4.5) one must have an a priori global in time bound for the supremum of the vorticity $w = \nabla^\perp \cdot u$ (or at least a bound in a Besov space "sufficiently close" to L^∞). However, using the Biot–Savart law, the evolution of w is governed by

$$\begin{aligned} \partial_t w + u \cdot \nabla w = -\partial_1 f_1(u) w - \big(\partial_1 f_2(u) + \partial_2 f_1(u)\big)\mathcal{R}_{12} w \\ + \big(\partial_2 f_2(u) - \partial_1 f_1(u)\big)\mathcal{R}_{11} w, \end{aligned} \tag{4.6}$$

where $\mathcal{R}_{ij}$ are the Riesz transforms $\partial_i \partial_j (-\Delta)^{-1}$, and $f(u) = (f_1(u), f_2(u))$. While the first term on the right side of (4.6) is harmless for L^∞ estimates on w, *unless* f is such that $\partial_1 f_2 + \partial_2 f_1 = \partial_2 f_2 - \partial_1 f_1 = 0$ identically (which is true for $f(u) = u$, i.e., $f_1(x, y) = x$ and $f_2(x, y) = y$), the remaining two terms prevent one from obtaining a bound on $\|w\|_{L^\infty}$ using classical methods, since Calderón–Zygmund operators are not bounded on L^∞. Recently it was proven in [19] that if one adds an *arbitrary* amount of dissipation, in the form of a positive power of $-\Delta$, or even dissipation as mild as $\log(1 - \Delta)$, to the left-hand side of (4.5), then the equations have global in time smooth solutions. The global well-posedness of (4.5) with no dissipation remains open for generic smooth forcing f.

5. A priori estimates. In this section we carry out a priori estimates for solutions evolving in $X_{m,p}$ of (1.1)–(1.3) with $m > d/p + 1$, $p \geq 2$. The bounds established in this section will be used extensively throughout the rest of the work. We begin with the bounds in the Hilbert space case, namely for solutions in X_m. These estimates will be used in Section 6 in the context of a Galerkin scheme.

5.1. *L^2-based estimates.* We start with estimates in $H^m(\mathcal{D})$, where $m > d/2 + 1$. Let u be a solution of (1.1)–(1.2), which lies in $H^{m+1}(\mathcal{D})$ and is defined up to a (possibly infinite) maximal stopping time of existence $\xi > 0$. Note, however, that the a priori estimates (5.4)–(5.8) involve only the H^m norm of the solution u.

Let $\alpha \in \mathbb{N}^d$ be a multi-index with $|\alpha| \leq m$. Applying the Leray projector P and then ∂^α to (1.1) we obtain

$$d(\partial^\alpha u) + \partial^\alpha P(u \cdot \nabla u)\, dt = \partial^\alpha P\sigma(u)\, d\mathcal{W}. \tag{5.1}$$

By the Itô lemma we find

$$\begin{aligned} d\|\partial^\alpha u\|_{L^2}^2 &= -2(\partial^\alpha u, \partial^\alpha P(u \cdot \nabla u))\, dt + \|\partial^\alpha P\sigma(u)\|_{\mathbb{X}_0}^2\, dt \\ &\quad + 2(\partial^\alpha u, \partial^\alpha P\sigma(u))\, d\mathcal{W} \\ &= (J_1^\alpha + J_3^\alpha)\, dt + J_3^\alpha\, d\mathcal{W}. \end{aligned} \tag{5.2}$$

Fix $T > 0$ and any stopping time $\tau \leq \xi \wedge T$. We find that for every $s \in [0, \tau]$,

$$\|\partial^\alpha u(s)\|_{L^2}^2 \leq \|\partial^\alpha u_0\|_{L^2}^2 + \int_0^s (|J_1^\alpha| + |J_2^\alpha|)\, ds' + \left|\int_0^s J_3^\alpha\, d\mathcal{W}\right|.$$

Hence, summing over all $|\alpha| \leq m$, taking a supremum over $s \in [0, \tau]$ and then taking the expected value we get

$$\begin{aligned} \mathbb{E} \sup_{s \in [0,\tau]} \|u(s)\|_{H^m}^2 &\leq \mathbb{E}\|u_0\|_{H^m}^2 + \mathbb{E} \sum_{|\alpha| \leq m} \int_0^\tau (|J_1^\alpha| + |J_2^\alpha|)\, dt' \\ &\quad + \sum_{|\alpha| \leq m} \mathbb{E}\left(\sup_{s \in [0,\tau]} \left| \int_0^s J_3^\alpha\, d\mathcal{W} \right| \right). \end{aligned} \tag{5.3}$$

We first treat the drift terms J_1^α and J_2^α which may be estimated pointwise in time. We bound the nonlinear term J_1^α by setting $p = 2$ and $v = u$ in (2.12) to obtain

$$\sum_{|\alpha| \leq m} |J_1^\alpha| \leq C\|u\|_{W^{1,\infty}} \|u\|_{H^m}^2 \tag{5.4}$$

for some positive constant $C = C(m, \mathcal{D})$. In view of assumption (3.1) the J_2^α term is direct.

$$\sum_{|\alpha| \leq m} |J_2^\alpha| \leq \beta(\|u\|_{L^\infty})^2 (1 + \|u\|_{H^m}^2), \tag{5.5}$$

where β is the increasing function given in (3.1).

We handle the stochastic term, involving J_3^α, using the Burkholder–Davis–Gundy inequality (2.14) and assumption (3.1):

$$\mathbb{E}\left(\sup_{s\in[0,\tau]}\left|\int_0^s J_3^\alpha\, d\mathcal{W}\right|\right) \leq C\mathbb{E}\left(\int_0^\tau \|\partial^\alpha u\|_{L^2}^2 \|P\sigma(u)\|_{\mathbb{W}^{m,2}}^2\, dt\right)^{1/2}$$
$$\leq C\mathbb{E}\left(\int_0^\tau \|\partial^\alpha u\|_{L^2}^2 \beta\big(\|u\|_{L^\infty}\big)^2\big(1+\|u\|_{H^m}^2\big)\, dt\right)^{1/2}.$$

Now, summing over $|\alpha| \leq m$, we infer

$$\sum_{|\alpha|\leq m}\mathbb{E}\left(\sup_{s\in[0,\tau]}\left|\int_0^s J_3^\alpha\, d\mathcal{W}\right|\right) \leq \frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau]}\|u\|_{H^m}^2 + C\mathbb{E}\int_0^\tau \beta\big(\|u\|_{L^\infty}\big)^2\big(1+\|u\|_{H^m}^2\big)\, dt. \tag{5.6}$$

In view of estimate (5.4) for the nonlinear term, we now define the stopping time

$$\xi_R = \inf\{t \geq 0 : \|u(t)\|_{W^{1,\infty}} \geq R\}. \tag{5.7}$$

Combining estimates (5.4)–(5.6), we find that for any $t > 0$, by taking $\tau = t \wedge \xi_R$,

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,\xi_R\wedge t]}\|u\|_{H^m}^2 \\
&\quad\leq 2\mathbb{E}\|u_0\|_{H^m}^2 + C\mathbb{E}\int_0^{\xi_R\wedge t}\big(\|u\|_{W^{1,\infty}} + \beta\big(\|u\|_{L^\infty}\big)^2\big)\big(1+\|u\|_{H^m}^2\big)\, ds \\
&\quad\leq 2\mathbb{E}\|u_0\|_{H^m}^2 + C\int_0^t\Big(1+\mathbb{E}\sup_{r\in[0,\xi_R\wedge s]}\|u\|_{H^m}^2\Big)\, ds,
\end{aligned}$$

where the final constant C depends on R through $R + \beta(R)^2$. From the classical Grönwall inequality we infer

$$\mathbb{E}\sup_{s\in[0,\xi_R\wedge T]}\|u\|_{H^m}^2 \leq C\big(1+\mathbb{E}\|u_0\|_{H^m}^2\big), \tag{5.8}$$

where $C = C(m, d, \mathcal{D}, T, R, \beta)$.

Of course estimate (5.8) does not prevent $\|u\|_{W^{1,\infty}}$ from blowing up before T; bound (5.8) grows exponentially in R, and hence we do not a priori know that $\xi_R \to \infty$ as $R \to \infty$. Note also that, in contrast to the case of the full space (or in the periodic setting), when $\mathcal{D}$ is a smooth simply-connected bounded domain, the nonblow-up of solutions is controlled by $\|u\|_{W^{1,\infty}}$, rather than the classical $\|\nabla u\|_{L^\infty}$. This is due to the nonlocal nature of the pressure. In bound (5.8) this is inherently expressed through the definition of the stopping time ξ_R. Of course, the L^∞ bound on u is also needed to control the terms involving σ.

5.2. *L^p-based estimates*, $p > 2$. We now return to (5.1) again for any α, $|\alpha| \leq m$. We apply the Itô formula, pointwise in x, for the function $\phi(v) = |v|^p = (|v|^2)^{p/2}$. After integrating in x and using the stochastic Fubini theorem (see [22]), we obtain

$$
\begin{aligned}
d\|\partial^\alpha u\|_{L^p}^p = &-p\int_{\mathcal{D}} \partial^\alpha u \cdot \partial^\alpha P(u\cdot\nabla u)|\partial^\alpha u|^{p-2}\,dx\,dt \\
&+ \sum_{k\geq 1}\int_{\mathcal{D}}\biggl(\frac{p}{2}|\partial^\alpha P\sigma_k(u)|^2|\partial^\alpha u|^{p-2} \\
&+ \frac{p(p-2)}{2}(\partial^\alpha u\cdot\partial^\alpha P\sigma_k(u))^2|\partial^\alpha u|^{p-4}\biggr)\,dx\,dt \qquad (5.9)\\
&+ p\sum_{k\geq 1}\biggl(\int_{\mathcal{D}}\partial^\alpha u\cdot\partial^\alpha P\sigma_k(u)|\partial^\alpha u|^{p-2}\,dx\biggr)\,dW_k \\
:= &\; I_1^\alpha\,dt + I_2^\alpha\,dt + I_3^\alpha\,d\mathcal{W}.
\end{aligned}
$$

By letting $v = u$ in (2.12) we bound

$$
|I_1^\alpha| \leq C\|u\|_{W^{1,\infty}}\|u\|_{W^{m,p}}^p. \qquad (5.10)
$$

We turn now to estimate the terms specific to the stochastic case. For I_2^α, using (3.1) we have

$$
\begin{aligned}
|I_2^\alpha| &\leq C\int_{\mathcal{D}}\sum_{k\geq 1}|\partial^\alpha P\sigma_k(u)|^2|\partial^\alpha u|^{p-2}\,dx \\
&\leq C\|P\sigma(u)\|_{\mathbb{W}^{m,p}}^2\|u\|_{W^{m,p}}^{p-2} \qquad (5.11)\\
&\leq C\beta(\|u\|_{L^\infty})^2(1+\|u\|_{W^{m,p}}^p).
\end{aligned}
$$

To estimate the stochastic integral terms involving I_3, we apply the Burkholder–Davis–Gundy inequality (2.14) the Minkowski inequality for integrals, and use (3.1). We obtain, for any stopping time $\tau \leq T\wedge\xi$,

$$
\begin{aligned}
&\mathbb{E}\biggl(\sup_{s\in[0,\tau]}\biggl|\int_0^s I_3^\alpha\,d\mathcal{W}\biggr|\biggr) \\
&\quad\leq C\mathbb{E}\biggl(\int_0^\tau\sum_{k\geq 1}\biggl(\int_{\mathcal{D}}\partial^\alpha u\cdot\partial^\alpha P\sigma_k(u)|\partial^\alpha u|^{p-2}\,dx\biggr)^2 ds\biggr)^{1/2} \\
&\quad\leq C\mathbb{E}\biggl(\int_0^\tau\biggl(\int_{\mathcal{D}}\biggl(\sum_{k\geq 1}|\partial^\alpha P\sigma_k(u)|^2|\partial^\alpha u|^{2(p-1)}\biggr)^{1/2} dx\biggr)^2 ds\biggr)^{1/2} \qquad (5.12)\\
&\quad\leq C\mathbb{E}\biggl(\int_0^\tau\|\partial^\alpha u\|_{L^p}^{2(p-1)}\biggl(\int_{\mathcal{D}}\biggl(\sum_{k\geq 1}|\partial^\alpha P\sigma_k(u)|^2\biggr)^{p/2} dx\biggr)^{2/p} ds\biggr)^{1/2}
\end{aligned}
$$

$$\leq C\mathbb{E}\left(\sup_{s\in[0,\tau]}\|\partial^\alpha u\|_{L^p}^{p/2}\left(\int_0^\tau \|u\|_{W^{m,p}}^{p-2}\beta(\|u\|_{L^\infty})^2(1+\|u\|_{W^{m,p}}^2)\,ds\right)^{1/2}\right)$$
$$\leq \frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau]}\|\partial^\alpha u\|_{L^p}^p + C\mathbb{E}\int_0^\tau \beta(\|u\|_{L^\infty})^2(1+\|u\|_{W^{m,p}}^p)\,ds.$$

Combining the L^p Itō formula (5.9) with estimates (5.10)–(5.12), and making use of the stopping time ξ_R defined in (5.7), we may obtain, as in the Hilbert case,

$$\mathbb{E}\sup_{s\in[0,\xi_R\wedge T]}\|u\|_{W^{m,p}}^p \leq C\bigl(1+\mathbb{E}\|u_0\|_{W^{m,p}}^p\bigr), \tag{5.13}$$

where $C = C(m, d, \mathcal{D}, T, R, \beta)$.

REMARK 5.1 (From a priori estimates to the construction of solutions). Having completed the a priori estimates in $W^{m,p}$, we observe that, even for the deterministic Euler equations on a bounded domain, the construction of solutions is nontrivial and requires a delicate treatment of the coupled elliptic/degenerate-hyperbolic system; see, for example, [35, 50]. In addition, the stochastic nature of the equations introduces a number of additional difficulties, such as the lack of compactness in the ω variable. We overcome these difficulties in Sections 6 and 7 below, by first constructing a sequence of very smooth approximate solutions evolving from mollified initial data, and then passing to a limit using a Cauchy-type argument.

6. Compactness methods and the existence of very smooth solutions. Leet $p \geq 2$ and $m > d/p + 1$ be as in the statement of Theorem 4.3. In this section we establish the existence of "very smooth" solutions of (1.1)–(1.3), that is, solutions in $H^{m'}$, where $m' = m + 5$ [so that $m' > m + 3 + d(p-2)/(2p)$ for any $d = 2, 3$ and $p \geq 2$]. We fix this m' throughout the rest of the paper. In particular we shall use that $H^{m'-2} \subset W^{m+1,p}$ and $m' > d/2 + 3$. Note that the initial data in the statement of our main theorem only lies in $W^{m,p}$, not necessarily in $H^{m'}$, but we will apply the results in this section to a sequence of mollified initial data [cf. (7.1) below], and then use a limiting argument in order to obtain the local existence of pathwise solutions for all data in $W^{m,p}$; see Section 7.

We begin by introducing a Galerkin scheme with cut-offs in front of both the nonlinear drift and diffusion terms. Crucially, these cut-offs allow us to obtain uniform estimates in the Galerkin approximations globally in time; see Remark 6.1 below. We then exhibit the relevant uniform estimates for these systems which partially follow from the a priori estimates in Section 5. We next turn to establish compactness with a variation on the Arzela–Ascoli theorem, tightness arguments and the Skorohod embedding theorem. In this manner we initial infer the existence of martingale solutions to a cutoff stochastic Euler system [cf. (6.17) below] in a very smooth spaces. We finally turn to prove the existence of pathwise solutions by establishing the uniqueness for this cutoff system and applying the Gyöngy–Krylov convergence criteria, as recalled in Lemma 6.10 below.

6.1. *Finite dimensional spaces and the Galerkin scheme.* For each $u \in X_0$, by the Lax–Milgram theorem, there exists a unique $\Phi(u) \in X_{m'}$ solving the variational problem

$$\big(\Phi(u), v\big)_{H^{m'}} = (u, v) \qquad \text{for all } v \in X_{m'}.$$

Actually, the regularity of $\Phi(u)$ is expected to be better. In [29] it is shown that in fact the maximal regularity $\Phi(u) \in X_{2m'}$ holds. We let $\{\phi_k\}_{k=1}^{\infty}$ be the complete orthonormal system (in X_0) of eigenfunctions for the linear map $u \mapsto \Phi(u)$, which is compact, injective and self-adjoint on X_0. Therefore, $(\phi_k, v)_{H^{m'}} = \lambda_k(\phi_k, v)$ for all $v \in X_{m'}$, where $\lambda_k^{-1} > 0$ is the eigenvalue associated to ϕ_k, and by [29] we know ϕ_k lies in $X_{2m'}$ for all $k \geq 1$.

For all $n \geq 1$, consider the orthogonal projection operator P_n, mapping X_0 onto $\operatorname{span}\{\phi_1, \dots, \phi_n\}$, given explicitly by

$$P_n v = \sum_{j=1}^{n} (v, \phi_j)\phi_j \qquad \text{for all } v \in X_0.$$

Note that these P_n are also uniformly bounded in n on $X_{m'}$, $X_{m'-1}$, etc. See, for example, [39] for further details.

Fix $R > 0$ to be determined, choose a C^{∞}-smooth nonincreasing function $\theta_R : [0, \infty) \mapsto [0, 1]$ such that

$$\theta_R(x) = \begin{cases} 1, & \text{for } |x| < R, \\ 0, & \text{for } |x| > 2R. \end{cases}$$

We consider the following Galerkin approximation scheme for (1.1):

$$du^n + \theta_R\big(\|u^n\|_{W^{1,\infty}}\big) P_n P\big(u^n \cdot \nabla u^n\big)\, dt = \theta_R\big(\|u^n\|_{W^{1,\infty}}\big) P_n P\sigma\big(u^n\big)\, d\mathcal{W}, \tag{6.1}$$

$$u^n(0) = P_n u_0. \tag{6.2}$$

The system (6.1)–(6.2) may be considered as an SDE in n dimensions, with locally Lipschitz drift (cf. Proposition 6.8 below) and globally Lipschitz diffusion; cf. (3.1). Since we also have the additional cancelation property $(P_n P(u \cdot \nabla u), u)_{L^2} = 0$ for all $u \in P_n X_{m'}$ we may infer that there exists a unique global in time solution u^n to (6.1)–(6.2), evolving continuously on $P_n X_{m'}$. See, for example, [27] for further details.

REMARK 6.1. The cutoff functions in (6.1) allow us to obtain uniform estimates for u^n in $L^{\infty}([0, T], X_{m'})$ for *any fixed, deterministic* $T > 0$. Without this cutoff function we are only able to obtain uniform estimates up to a sequence of stopping times τ^n, depending on n. In contrast to the deterministic case it is unclear if, for example, $\inf_{n \geq 1} \tau^n > 0$ almost surely. Note, however, that the presence of this cut-off causes additional difficulties in the passage to the limit of martingale solutions (see Remark 6.6), and in order to establish the uniqueness of solutions associated to the related to the limit cut-off system, see (6.17), Proposition 6.8 and Remark 6.9 below.

6.2. *Uniform estimates*. Applying the Itô formula to (6.1), and using that P_n is self-adjoint on $X_{m'}$, similar to (5.2) we obtain

$$
\begin{aligned}
d\|u^n\|_{H^{m'}}^2 = &-2\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)\big(u^n, P\big(u^n\cdot\nabla u^n\big)\big)_{H^{m'}}\,dt \\
&+\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)^2\|P_nP\sigma(u^n)\|_{\mathbb{X}_{m'}}^2\,dt \\
&+2\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)\big(u^n, P\sigma(u^n)\big)_{H^{m'}}\,d\mathcal{W}.
\end{aligned}
$$

Further on, in order to establish the needed compactness in the probability distributions associated to u^n, we need uniform estimates on higher moments of $\|u^n\|_{H^{m'}}^2$. For this purpose we fix any $r\geq 2$ and compute $d(\|u^n\|_{H^{m'}}^2)^{r/2}$ from the Itô formula and the evolution of $\|u^n\|_{H^{m'}}^2$. We find

$$
\begin{aligned}
d\|u^n\|_{H^{m'}}^r = &-r\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)\|u^n\|_{H^{m'}}^{r-2}\big(u^n, P\big(u^n\cdot\nabla u^n\big)\big)_{H^{m'}}\,dt \\
&+\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)^2\bigg(\frac{r}{2}\|u^n\|_{H^{m'}}^{r-2}\|P_nP\sigma(u^n)\|_{\mathbb{X}_{m'}}^2 \\
&\qquad+\frac{r(r-2)}{2}\|u^n\|_{H^{m'}}^{r-4}\big(u^n, P\sigma(u^n)\big)_{H^{m'}}^2\bigg)\,dt \\
&+r\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)\|u^n\|_{H^{m'}}^{r-2}\big(u^n, P\sigma(u^n)\big)_{H^{m'}}\,d\mathcal{W}.
\end{aligned}
\tag{6.3}
$$

Let us introduce the stopping time

$$
\tau_K := \inf\Big\{t\geq 0 : \sup_{s\in[0,t]}\|u^n\|_{H^{m'}}\geq K\Big\}\qquad\text{for any } K>0.
$$

Using bounds similar to the a priori estimates of Section 5, we obtain the estimate

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_K]}\|u^n\|_{H^{m'}}^r\Big) \\
&\quad\leq \mathbb{E}\|P_nu_0\|_{H^{m'}}^r + C\mathbb{E}\int_0^{t\wedge\tau_K}\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)\big(\beta\big(\|u^n\|_{L^\infty}\big)^2+\|u^n\|_{W^{1,\infty}}\big) \\
&\qquad\qquad\times\big(1+\|u^n(s)\|_{H^{m'}}^r\big)\,ds \\
&\qquad+C\mathbb{E}\bigg(\int_0^{t\wedge\tau_K}\theta_R\big(\|u^n\|_{W^{1,\infty}}\big)^2\beta\big(\|u^n\|_{L^\infty}\big)^2\|u^n\|_{H^{m'}}^r\big(1+\|u^n\|_{H^{m'}}^r\big)\,ds\bigg)^{1/2} \\
&\quad\leq \mathbb{E}\|u_0\|_{H^{m'}}^r + C\int_0^t 1+\mathbb{E}\Big(\sup_{s'\in[0,s\wedge\tau_K]}\|u^n(r)\|_{H^{m'}}^r\Big)\,ds \\
&\qquad+\frac{1}{2}\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_K]}\|u^n(r)\|_{H^{m'}}^r\Big),
\end{aligned}
$$

where C is a constant independent of n and K but depends on $\mathcal{D}$, m', r, and R (through θ_R and β). Therefore, rearranging and applying the standard Grönwall inequality, we obtain that, for any $T > 0$

$$\mathbb{E} \sup_{s\in[0,T\wedge\tau_K]} \|u^n\|_{H^{m'}}^r \le C < \infty$$

for some positive finite constant $C = C(T, R, r, \beta, \mathbb{E}\|u_0\|_{H^{m'}}^r)$ which is independent of n and K. Since $\tau_K \to \infty$ as $K \to \infty$, with the monotone convergence theorem we conclude

$$\sup_{n\ge1} \mathbb{E} \sup_{s\in[0,T]} \|u^n\|_{H^{m'}}^r \le C < \infty. \tag{6.4}$$

In order to obtain the compactness needed to pass to the limit in u^n we also would like to have uniform estimates on the time derivatives of u^n. Since in the stochastic case we do not expect u^n to be differentiable in time, we have to content ourselves instead with estimates on fractional time derivatives of order strictly less than $1/2$. In order to carry out such estimates we shall also make use of a variation on the Burkholder–Davis–Gundy inequality (2.14), as derived in [28].

For this purpose, let us recall a particular characterization of the Sobolev spaces $W^{\alpha,q}([0, T], X)$ where X may be any separable Hilbert space. See, for example, [22] for further details. For $q > 1$ and $\alpha \in (0, 1)$ we define

$$W^{\alpha,q}([0, T]; X) := \left\{v \in L^q([0, T]; X); \int_0^T \int_0^T \frac{\|v(t') - v(t'')\|_X^q}{|t' - t''|^{1+\alpha q}}\, dt'\, dt'' < \infty\right\},$$

which is endowed with the norm

$$\|v\|_{W^{\alpha,p}([0,T];X)}^q := \int_0^T \|v(t')\|_X^q\, dt' + \int_0^T \int_0^T \frac{\|v(t') - v(t'')\|_X^q}{|t' - t''|^{1+\alpha q}}\, dt'\, dt''.$$

Note that for $\alpha \in (0, 1)$, $W^{1,q}([0, T]; X) \subset W^{\alpha,q}([0, T]; X)$ with $\|v\|_{W^{\alpha,q}([0,T];X)} \le C\|v\|_{W^{1,q}([0,T];X)}$. As in [28] one can show from (2.14) that for any $q \ge 2$ and any $\alpha \in [0, 1/2)$

$$\mathbb{E}\left(\left\|\int_0^t G\, dW\right\|_{W^{\alpha,q}([0,T];X)}^q\right) \le C\mathbb{E}\left(\int_0^T \|G\|_{L_2(\mathfrak{U},X)}^q\, dt\right), \tag{6.5}$$

over all X valued predictable $G \in L^q(\Omega; L^q_{\mathrm{loc}}([0, \infty), L_2(\mathfrak{U}, X)))$ and where $C = C(\alpha, q, T)$.

With these definitions and (6.5) in hand we return to (6.3). For any $0 < \alpha < 1/2$, we have

$$\begin{aligned} &\mathbb{E}\|u^n\|_{W^{\alpha,r}([0,T],H^{m'-1})}^r \\ &\quad\le C\mathbb{E}\left\|P_n u_0 + \int_0^t \theta_R(\|u^n\|_{W^{1,\infty}}) P_n P(u^n \cdot \nabla u^n)\, ds\right\|_{W^{1,r}([0,T],H^{m'-1})}^r \\ &\qquad + C\mathbb{E}\left\|\int_0^t \theta_R(\|u^n\|_{W^{1,\infty}}) P_n P\sigma(u^n)\, d\mathcal{W}\right\|_{W^{\alpha,r}([0,T],H^{m'-1})}^r \end{aligned} \tag{6.6}$$

for some positive constant $C = C(T)$, independent of n. Since $P_n P$ is uniformly bounded in $X_{m'-1}$ independently of n, using (2.2) and (6.4) we bound the first term on the right-hand side of (6.6) as

$$
\begin{aligned}
&\mathbb{E}\left\| P_n u_0 + \int_0^t \theta_R(\|u^n\|_{W^{1,\infty}}) P_n P(u^n \cdot \nabla u^n)\, ds \right\|^r_{W^{1,r}([0,T],H^{m'-1})} \\
&\quad \le C\mathbb{E}\|u_0\|^r_{H^{m'}} + C\mathbb{E}\int_0^T \theta_R(\|u^n\|_{W^{1,\infty}}) \|u^n \cdot \nabla u^n\|^r_{H^{m'-1}}\, dt \\
&\quad \le C\mathbb{E}\|u_0\|^r_{H^{m'}} + C\mathbb{E}\int_0^T \theta_R(\|u^n\|_{W^{1,\infty}}) \|u^n\|^r_{W^{1,\infty}} \|u^n\|^r_{H^{m'}}\, dt \\
&\quad \le C\mathbb{E}\Big(\sup_{t\in[0,T]} \|u^n(t)\|^r_{H^{m'}}\Big) \le C,
\end{aligned} \tag{6.7}
$$

where the final constant $C = C(T, R, r, \mathbb{E}\|u_0\|^r_{H^{m'}})$ does not depend on n. For the second term on the left-hand side of (6.6) we make use of (6.5) with $q = r$ and $\alpha \in (0, 1/2)$, then (3.1) and (6.4) to estimate

$$
\begin{aligned}
&\mathbb{E}\left\| \int_0^t \theta_R(\|u^n\|_{W^{1,\infty}}) P_n P\sigma(u^n)\, d\mathcal{W} \right\|^r_{W^{\alpha,r}([0,T],H^{m'-1})} \\
&\quad \le C\mathbb{E}\Big(\int_0^T \theta_R(\|u^n\|_{W^{1,\infty}})^r \|P_n P\sigma(u^n)\|^r_{\mathbb{X}_{m'-1}}\, dt\Big) \\
&\quad \le C\mathbb{E}\int_0^T \theta_R(\|u^n\|_{W^{1,\infty}})^r \beta(\|u^n\|_{L^\infty})^r (1 + \|u^n\|^r_{H^{m'}})\, dt \\
&\quad \le C\mathbb{E}\Big(1 + \sup_{t\in[0,T]} \|u^n(t)\|^r_{H^{m'}}\Big) \le C,
\end{aligned} \tag{6.8}
$$

where in the final constant $C = C(T, R, r, \beta, \mathbb{E}\|u_0\|^r_{H^{m'}})$ is a sufficiently large constant independent on n. Combining (6.6)–(6.8) we have now shown that

$$
\sup_{n\ge 1} \mathbb{E}\|u^n\|^r_{W^{\alpha,r}([0,T],H^{m'-1})} \le C \tag{6.9}
$$

for some positive finite constant $C = C(T, R, r, \mathbb{E}\|u_0\|^r_{H^{m'}}, \alpha)$. In summary, we have proven:

PROPOSITION 6.2. *Fix $m > d/2 + 1$, $m' = m + 5$, $\alpha \in (0, 1/2)$, $r \ge 2$ and suppose that σ satisfies conditions* (3.1)–(3.3). *Given $u_0 \in L^r(\Omega; X_{m'})$, $\mathcal{F}_0$ measurable, consider the associated sequence of solutions $\{u_n\}_{n\ge 1}$ of the Galerkin system* (6.1)–(6.2). *Then the sequence $\{u^n\}_{n\ge 1}$ is uniformly bounded in*

$$
L^r(\Omega; L^\infty([0, T], X_{m'}) \cap W^{\alpha,r}(0, T; X_{m'-1}))
$$

for any $T > 0$. *Moreover*, *under the given conditions*, *we have*

$$\sup_{n\geq 1}\mathbb{E}\left\|\int_0^t \theta_R(\|u^n\|_{W^{1,\infty}})P_n P\sigma(u^n)\,d\mathcal{W}\right\|^r_{W^{\alpha,r}([0,T],H^{m'-1})} < \infty, \tag{6.10}$$

$$\sup_{n\geq 1}\mathbb{E}\left\|u^n(t) - \int_0^t \theta_R(\|u^n\|_{W^{1,\infty}})P_n P\sigma(u^n)\,d\mathcal{W}\right\|^r_{W^{1,r}([0,T],H^{m'-1})} < \infty. \tag{6.11}$$

6.3. *Tightness*, *compactness and the existence of martingale solutions*. For a given initial distribution μ_0 on $X_{m'}$ we fix a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ upon which is defined an $\mathcal{F}_0$ measurable random element u_0 with distribution μ_0. As described above, we define the sequence of Galerkin approximations $\{u^n\}_{n\geq 1}$ solving (6.1)–(6.2) relative to this basis and initial condition.

To define a sequence of measures associated with $\{(u^n, \mathcal{W})\}_{n\geq 1}$ we consider the phase space

$$\begin{aligned} &\mathcal{X} = \mathcal{X}_S \times \mathcal{X}_W \\ &\text{where } \mathcal{X}_S = C([0,T], X_{m'-2}),\ \mathcal{X}_W = C([0,T], \mathfrak{U}_0). \end{aligned} \tag{6.12}$$

We may think of the first component, $\mathcal{X}_S \supset C([0,T], X_{m'})$, as the space where the u^n lives, and the second component, $\mathcal{X}_W$, as being the space on which the driving Brownian motions are defined. On $\mathcal{X}$ we define the probability measures

$$\mu^n = \mu^n_S \times \mu_W \qquad \text{where } \mu^n_S(\cdot) = \mathbb{P}(u^n \in \cdot),\ \mu_W(\cdot) = \mathbb{P}(\mathcal{W} \in \cdot). \tag{6.13}$$

We next show that the collection $\{\mu^n\}_{n\geq 1}$ is in fact *weakly compact*. Let $\mathrm{Pr}(\mathcal{X})$ be the collection of Borel probability measures on $\mathcal{X}$. Recall that a sequence $\{\nu_n\}_{n\geq 0} \subset \mathrm{Pr}(\mathcal{X})$ is said to *converge weakly* to an element $\nu \in \mathrm{Pr}(\mathcal{X})$ if $\int f\,d\nu_n \to \int f\,d\nu$ for all continuous bounded f on $\mathcal{X}$. As such, we say that a set $\Lambda \subset \mathrm{Pr}(\mathcal{X})$ is weakly compact if every sequence $\{\nu_n\} \subset \Lambda$ possesses a weakly convergent subsequence. On the other hand we say that a collection $\Lambda \subset \mathrm{Pr}(\mathcal{X})$ is *tight* if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathcal{X}$ such that, $\mu(K_\varepsilon) \geq 1 - \varepsilon$ for all $\mu \in \Lambda$. The classical result of Prohorov (see, e.g., [22]) asserts that weak compactness and tightness are in fact equivalent conditions for collections $\Lambda \subset \mathrm{Pr}(\mathcal{X})$. We have:

LEMMA 6.3 (Tightness of measures for the Galerkin scheme). *Let* $m > d/2 + 1$, $m' = m + 5$, $r > 2$, *assume that* σ *satisfies conditions* (3.1)–(3.3) *and consider any* $\mu_0 \in \mathrm{Pr}(X_{m'})$ *with* $\int_{X_{m'}} |u|^r\,d\mu_0(u) < \infty$. *Fix any stochastic basis* $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ *upon which is defined an* $\mathcal{F}_0$ *measurable random element* u_0 *with this distribution* μ_0 *and take* $\{u^n\}_{n\geq 1}$ *to be the sequence solving* (6.1), (6.2) *relative to this basis and initial condition. Define the sequence* $\{\mu^n\}_{n\geq 1}$ *according to* (6.13) *using the sequence* $\{u^n\}_{n\geq 1}$. *Then* $\{\mu^n\}_{n\geq 1} \subset \mathrm{Pr}(\mathcal{X})$ *is tight and hence weakly compact*.

In order to obtain the compact sets used to show that the sequence $\{\mu^n\}_{n\geq 1}$ is tight, we use the following variation on the classical Arzela–Ascoli compactness theorem from [28].

LEMMA 6.4. *Suppose that $Y^{(0)} \supset Y$ are Banach spaces with Y compactly embedded in $Y^{(0)}$. Let $\alpha \in (0,1]$ and $q \in (1,\infty)$ be such that $\alpha q > 1$; then*

$$W^{\alpha,q}([0,T];Y) \subset\subset C([0,T],Y^{(0)}) \tag{6.14}$$

and the embedding is compact.

With this result in hand we proceed to the proof of Lemma 6.3:

PROOF OF LEMMA 6.3. Fix any $\alpha \in (0,1/2)$ such that $\alpha r > 1$. According to Lemma 6.4 we have that both $W^{1,2}([0,T];X_{m'-1})$, $W^{r,\alpha}([0,T];X_{m'})$ are compactly embedded in $\mathcal{X}_S$. Therefore, for $s > 0$, the sets

$$\begin{aligned}B_s^2 &:= \{u \in W^{1,2}([0,T];X_{m'-1}) : \|u\|_{W^{1,2}([0,T];H^{m'-1})} \leq s\}\\ &\quad + \{u \in W^{\alpha,r}([0,T];X_{m'}) : \|u\|_{W^{\alpha,r}([0,T];H^{m'-1})} \leq s\}\end{aligned}$$

are pre-compact in $\mathcal{X}_S$. Since $\{u^n \in B_s^2\}$ contains

$$\begin{aligned}&\left\{\left\|u^n(t) - \int_0^t \beta(\|u^n\|_{W^{m,p}}) P_n P\sigma(u^n)\,d\mathcal{W}\right\|_{W^{1,2}([0,T];H^{m'-1})} \leq s\right\}\\ &\quad\cap \left\{\left\|\int_0^t \beta(\|u^n\|_{W^{m,p}}) P_n P\sigma(u^n)\,d\mathcal{W}\right\|_{W^{\alpha,r}([0,T];H^{m'-1})} \leq s\right\},\end{aligned}$$

and using Proposition 6.2, estimates (6.10)–(6.11) and the Chebyshev inequality, we bound

$$\begin{aligned}\mu_S^n((B_s^2)^C) &\leq \mathbb{P}\left(\left\|u^n(t) - \int_0^t \theta_R(\|u^n\|_{W^{1,\infty}}) P_n P\sigma(u^n)\,d\mathcal{W}\right\|_{W^{1,2}([0,T];H^{m'-1})} > s\right)\\ &\quad + \mathbb{P}\left(\left\|\int_0^t \theta_R(\|u^n\|_{W^{1,\infty}}) P_n P\sigma(u^n)\,d\mathcal{W}\right\|_{W^{\alpha,r}([0,T];H^{m'-1})} > s\right)\\ &\leq \frac{C}{s},\end{aligned}$$

where C is a universal constant independent of s and n. We infer that μ_S^n is a tight sequence on $\mathcal{X}$. Now, since the sequence $\{\mu_W\}$ is constant, it is trivially weakly compact and hence tight. We may thus finally infer that the $\{\mu^n\}$ is tight, completing the proof. □

With this weak compactness in hand we next apply the Skorokhod embedding theorem (cf. [22]) to a weakly convergent subsequence of $\{\mu^n\}_{n\geq 1}$. We obtain a new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ on which we have a sequence of random elements $\{(\widetilde{u}^n, \widetilde{\mathcal{W}}^n)\}_{n\geq 1}$ converging almost surely in $\mathcal{X}$ to an element $(\widetilde{u}, \widetilde{\mathcal{W}})$, that is,

$$\widetilde{u}^n \to \widetilde{u} \qquad \text{in } C\big([0,T], X_{m'-2}\big) \text{ almost surely} \tag{6.15}$$

and

$$\widetilde{\mathcal{W}}^n \to \widetilde{\mathcal{W}} \qquad \text{in } C\big([0,T], \mathfrak{U}_0\big) \text{ almost surely.} \tag{6.16}$$

One may verify as in [4] that $(\widetilde{u}^n, \widetilde{\mathcal{W}}^n)$ satisfies the nth order Galerkin approximation (6.1)–(6.2) relative to the stochastic basis $\mathcal{S}^n := (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \{\widetilde{\mathcal{F}}_t^n\}, \widetilde{\mathcal{W}}^n)$ with $\widetilde{\mathcal{F}}_t^n$ the completion of the σ-algebra generated by $\{(u^n(s), \mathcal{W}^n(s)) : s \leq t\}$. Using the uniform bound (6.4) and the almost sure convergences (6.15)–(6.16), we may now show that $(\widetilde{u}, \widetilde{W})$ solves the cut-off system

$$d\widetilde{u} + \theta_R\big(\|\widetilde{u}\|_{W^{1,\infty}}\big) P(\widetilde{u} \cdot \nabla \widetilde{u})\, dt = \theta_R\big(\|\widetilde{u}\|_{W^{1,\infty}}\big) P\sigma(\widetilde{u})\, d\widetilde{\mathcal{W}}. \tag{6.17}$$

For the technical details of this passage to the limit we refer to, for example, [24] where this analysis is carried out for the primitive equations. Applying these arguments to the Euler equations introduces no additional difficulties, so we omit further details. More precisely we have established the following:

PROPOSITION 6.5. *Fix any $m' > d/2 + 3$, $r > 2$ and $R > 0$. Suppose that $\mu_0 \in \Pr(X_{m'})$ is given such that $\int_{X_{m'}} \|u\|_{H^{m'}}^r \, d\mu_0(u) < \infty$. Then there exists a stochastic basis $\mathcal{S} := (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathcal{W}})$ and an $X_{m'}$ valued, predictable process*

$$\widetilde{u} \in L^2\big(\Omega; L^\infty_{\text{loc}}([0,\infty); X_{m'})\big) \cap L^2\big(\Omega; C\big([0,\infty), X_{m'-2}\big)\big)$$

with $\widetilde{\mathbb{P}}(\widetilde{u}(0) \in \cdot) = \mathbb{P}(u_0 \in \cdot)$ such that

$$\widetilde{u}(t) + \int_0^t \theta_R\big(\|\widetilde{u}\|_{W^{1,\infty}}\big) P(\widetilde{u} \cdot \nabla \widetilde{u})\, dt = \widetilde{u}(0) + \int_0^t \theta_R\big(\|\widetilde{u}\|_{W^{1,\infty}}\big) P\sigma(\widetilde{u})\, d\widetilde{\mathcal{W}}$$

for every $t \geq 0$.

REMARK 6.6. The assumption $m' > d/2 + 3$ is needed facilitate the passage from (6.1) to (6.17). Indeed, when passing to the limit we need to handle some stray terms arising due to the cut-off terms involving the $W^{1,\infty}$ norm of the solution. These stray terms are of higher order than the other terms in the estimates, and in order to deal with them we need to have compactness in sufficiently regular spaces. In the analysis above this compactness is provided by the Arzela–Ascoli type result, Lemma 6.4. In order to apply this lemma we need estimates on (fractional) time derivatives of u^n, which in view of (6.7) must be made in $X_{m'-1}$. An additional degree of regularity is then lost in order to obtain a compact embedding in $X_{m'-1}$, as required by Lemma 6.4, and we therefore arrive at the condition $m' > d/2 + 3$.

We also observe that Proposition 6.5 immediately yields new results on the existence of martingale solutions of the stochastic Euler equation.

REMARK 6.7 (Existence of martingale solutions). We may show that the pair $(\widetilde{u}, \widetilde{\mathcal{S}})$, obtained from Proposition 6.5 is a local martingale solution of (1.1)–(1.3) by introducing the stopping time

$$\tau = \inf\{t \geq 0 : \|\widetilde{u}\|_{W^{1,\infty}} \geq R\}.$$

Of course, unless $\|\widetilde{u}(0)\|_{W^{1,\infty}} < R$, that is, unless $\mu_0(\{u_0 \in X_{m'} : \|u_0\|_{W^{1,\infty}} < R\}) = 1$, we have $\widetilde{P}(\tau = 0) > 0$. Such stopping times τ will also be used further on to infer the existence of solutions in the pathwise case. Note, however, that in this case the $L^{\infty}(\Omega)$ condition may be subsequently removed with a cutting argument; cf. (6.26)–(6.27) below.

6.4. *Uniqueness, the Gyöngy–Krylov lemma and the existence of strong solutions*. Having now established Proposition 6.5, and guided by the classical Yamada–Wannabe theorem (see [52, 53]), we would now expect pathwise solutions to exist once we establish that solutions are "pathwise unique."

PROPOSITION 6.8 (Pathwise uniqueness). *Fix any $r > 2$, $R > 0$ and $m' = m + 5$, where $p \geq 2$ and $m > d/p + 1$. Assume that σ satisfies (3.1)–(3.3), and suppose $(\mathcal{S}, u^{(1)})$ and $(\mathcal{S}, u^{(2)})$ are two global solutions of (6.17) in the sense of Proposition 6.5, relative to the same stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$. If $u^{(1)}(0) = u^{(2)}(0) = u_0$, a.s, with $\mathbb{E}\|u_0\|^r_{H^{m'}} < \infty$, then $u^{(1)}$ and $u^{(2)}$ are indistinguishable, that is,*

$$\mathbb{P}\big(u^{(1)}(t) = u^{(2)}(t); \forall t \geq 0\big) = 1. \tag{6.18}$$

PROOF. By the assumption on u_0 and Proposition 6.5, we have for every $T > 0$

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big(\|u^{(1)}\|^2_{H^{m'}} + \|u^{(2)}\|^2_{H^{m'}}\big)\Big) \leq C < \infty, \tag{6.19}$$

where C is a universal constant depending only on $\mathbb{E}\|u_0\|^2_{H^{m'}}$, R, β and T. However, continuity in time is only guaranteed for the $H^{m'-2}$ norms of $u^{(1)}$ and $u^{(2)}$, and so, in view of the choice of m', we may define the collection of stopping times

$$\xi^K := \inf\{t \geq 0 : \|u^{(1)}\|^2_{W^{m+1,p}} + \|u^{(2)}\|^2_{W^{m+1,p}} > K\}, \qquad K > 0.$$

Observe that due to (6.19) we have $\xi^K \to \infty$ almost surely as $K \to \infty$.

Take $v = u^{(1)} - u^{(2)}$. We have

$$\begin{aligned} dv &+ \theta_R\big(\|u^{(1)}\|_{W^{1,\infty}}\big) P\big(u^{(1)} \cdot \nabla u^{(1)}\big)\,dt - \theta_R\big(\|u^{(2)}\|_{W^{1,\infty}}\big) P\big(u^{(2)} \cdot \nabla u^{(2)}\big)\,dt \\ &= \big(\theta_R\big(\|u^{(1)}\|_{W^{1,\infty}}\big) P\sigma\big(u^{(1)}\big) - \theta_R\big(\|u^{(2)}\|_{W^{1,\infty}}\big) P\sigma\big(u^{(2)}\big)\big)\,d\mathcal{W}. \end{aligned}$$

We now estimate v in $W^{m,p}$. For any multi-index $|\alpha| \leq m$ we apply ∂^α to the equation for v. With the Itô lemma in L^p we find

$$
\begin{aligned}
&d\|\partial^\alpha v\|_{L^p}^p \\
&\quad = -p \int_{\mathcal{D}} \partial^\alpha v \cdot \bigl(\theta_R(\|u^{(1)}\|_{W^{1,\infty}}) \partial^\alpha P(u^{(1)} \cdot \nabla u^{(1)}) \\
&\qquad\qquad - \theta_R(\|u^{(2)}\|_{W^{1,\infty}}) \partial^\alpha P(u^{(2)} \cdot \nabla u^{(2)})\bigr) |\partial^\alpha v|^{p-2}\, dx\, dt \\
&\qquad + \sum_{k\geq 1} \int_{\mathcal{D}} \biggl(\frac{p}{2} \bigl|\partial^\alpha P\bigl(\theta_R(\|u^{(1)}\|_{W^{m,p}}) \sigma_k(u^{(1)}) \\
&\qquad\qquad - \theta_R(\|u^{(2)}\|_{W^{m,p}}) \sigma_k(u^{(2)})\bigr)\bigr|^2 |\partial^\alpha v|^{p-2} \\
&\qquad\qquad + \frac{p(p-2)}{2} \\
&\qquad\qquad \times \bigl(\partial^\alpha v \cdot P\bigl(\theta_R(\|u^{(1)}\|_{W^{m,p}}) \sigma_k(u^{(1)}) \\
&\qquad\qquad\qquad - \theta_R(\|u^{(2)}\|_{W^{m,p}}) \sigma_k(u^{(2)})\bigr)\bigr)^2 |\partial^\alpha v|^{p-4} \biggr)\, dx\, dt \\
&\qquad + p \sum_{k\geq 1} \biggl(\int_{\mathcal{D}} \partial^\alpha v \cdot \partial^\alpha P\bigl(\theta_R(\|u^{(1)}\|_{W^{1,\infty}}) \sigma_k(u^{(1)}) \\
&\qquad\qquad - \theta_R(\|u^{(2)}\|_{W^{1,\infty}}) \sigma_k(u^{(2)})\bigr) |\partial^\alpha v|^{p-2}\, dx \biggr)\, dW_k \\
&\quad := (J_1^\alpha + J_2^\alpha)\, dt + J_3^\alpha\, d\mathcal{W}.
\end{aligned}
$$

Using the mean value theorem for θ_R, the embedding $W^{1,\infty} \subset W^{m,p}$ and Lemma 2.1 we estimate J_1^α as

$$
\begin{aligned}
|J_1^\alpha| &\leq C |\theta_R(\|u^{(1)}\|_{W^{1,\infty}}) - \theta_R(\|u^{(2)}\|_{W^{1,\infty}})| \\
&\qquad \times |(\partial^\alpha P(u^{(1)} \cdot \nabla u^{(1)}), \partial^\alpha v |\partial^\alpha v|^{p-2})| \\
&\qquad + C |(\partial^\alpha P(u^{(1)} \cdot \nabla u^{(1)}) - \partial^\alpha P(u^{(2)} \cdot \nabla u^{(2)}), \partial^\alpha v |\partial^\alpha v|^{p-2})| \\
&\leq C \bigl| \|u^{(1)}\|_{W^{1,\infty}} - \|u^{(2)}\|_{W^{1,\infty}} \bigr| \|P(u^{(1)} \cdot \nabla u^{(1)})\|_{W^{m,p}} \|v\|_{W^{m,p}}^{p-1} \\
&\qquad + C |(\partial^\alpha P(v \cdot \nabla u^{(1)}), \partial^\alpha v |\partial^\alpha v|^{p-2})| \\
&\qquad + C |(\partial^\alpha P(u^{(2)} \cdot \nabla v), \partial^\alpha v |\partial^\alpha v|^{p-2})| \\
&\leq C \|v\|_{W^{m,p}}^p \|u^{(1)}\|_{W^{m,p}} \|u^{(1)}\|_{W^{m+1,p}} \\
&\qquad + \|v\|_{W^{m,p}}^{p-1} (\|v\|_{L^\infty} \|u^{(1)}\|_{W^{m+1,p}} + \|u^{(1)}\|_{W^{1,\infty}} \|v\|_{W^{m,p}}) \\
&\qquad + C \|v\|_{W^{m,p}}^{p-1} (\|u^{(2)}\|_{W^{m,p}} \|v\|_{W^{1,\infty}} + \|u^{(2)}\|_{W^{1,\infty}} \|v\|_{W^{m,p}}) \\
&\leq C \|v\|_{W^{m,p}}^p \bigl((1 + \|u^{(1)}\|_{W^{m,p}}) \|u^{(1)}\|_{W^{m+1,p}} + \|u^{(2)}\|_{W^{m,p}} \bigr).
\end{aligned} \tag{6.20}
$$

Using the local Lipschitz condition on σ, that is, (3.1), we have

$$
\begin{aligned}
|J_2^\alpha| &\le C\|v\|_{W^{m,p}}^{p-2}\|\theta_R(\|u^{(1)}\|_{W^{1,\infty}})\sigma(u^{(1)}) - \theta_R(\|u^{(2)}\|_{W^{1,\infty}})\sigma(u^{(2)})\|_{\mathbb{W}^{m,p}}^2 \\
&\le C\|v\|_{W^{m,p}}^{p-2}\big(\theta_R(\|u^{(1)}\|_{W^{1,\infty}})^2\|\sigma(u^{(1)}) - \sigma(u^{(2)})\|_{\mathbb{W}^{m,p}}^2 \\
&\qquad\qquad + |\theta_R(\|u^{(1)}\|_{W^{1,\infty}}) - \theta_R(\|u^{(2)}\|_{W^{1,\infty}})|^2\|\sigma(u^{(2)})\|_{\mathbb{W}^{m,p}}^2\big) \\
&\le C\beta(\|u^{(1)}\|_{L^\infty} + \|u^{(2)}\|_{L^\infty})^2(1 + \|u^{(2)}\|_{W^{m,p}}^2)\|v\|_{W^{m,p}}^p.
\end{aligned}
\tag{6.21}
$$

For the terms involving J_3^α we make use of the Burkholder–Davis–Gundy inequality in a similar way to (5.12) and then argue as in (6.21) in order to finally estimate, that for every $t \ge 0$,

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^{s\wedge\xi^K} J_3^\alpha\, d\mathcal{W}\right| \\
&\quad\le C\mathbb{E}\bigg(\int_0^{t\wedge\xi^K}\sum_{k\ge1}\bigg(\int_{\mathcal{D}}\partial^\alpha v\cdot\partial^\alpha P\big(\theta_R(\|u^{(1)}\|_{W^{1,\infty}})\sigma_k(u^{(1)}) \\
&\qquad\qquad - \theta_R(\|u^{(2)}\|_{W^{1,\infty}})\sigma_k(u^{(2)})\big) \\
&\qquad\qquad \times|\partial^\alpha v|^{p-2}\,dx\bigg)^2 ds\bigg)^{1/2} \\
&\quad\le C\mathbb{E}\bigg(\int_0^{t\wedge\xi^K}\|\partial^\alpha v\|_{L^p}^{2(p-1)}\|\theta_R(\|u^{(1)}\|_{W^{1,\infty}})\sigma(u^{(1)}) \\
&\qquad\qquad - \theta_R(\|u^{(2)}\|_{W^{1,\infty}})\sigma(u^{(2)})\|_{\mathbb{W}^{m,p}}^2\,ds\bigg)^{1/2} \\
&\quad\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\xi^K]}\|\partial^\alpha v\|_{L^p}^p + C\mathbb{E}\int_0^{t\wedge\xi^K}\|v\|_{W^{m,p}}^p\,ds.
\end{aligned}
\tag{6.22}
$$

We now combine the estimates obtained in (6.20)–(6.22) and sum over all α with $|\alpha| \le m$. We find that for any fixed $K > 0$,

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\xi^K]}\|v\|_{W^{m,p}}^p \\
&\quad\le C\mathbb{E}\int_0^{t\wedge\xi^K}\|v\|_{W^{m,p}}^p\big(\beta(\|u^{(1)}\|_{L^\infty} + \|u^{(2)}\|_{L^\infty})^2 + 1\big) \\
&\qquad\qquad \times\big(1 + \|u^{(1)}\|_{W^{m+1,p}}^2 + \|u^{(2)}\|_{W^{m,p}}^2\big)\,ds \\
&\quad\le C\int_0^t\mathbb{E}\sup_{r\in[0,s\wedge\xi^K]}\|v\|_{W^{m,p}}^p\,ds,
\end{aligned}
$$

where the constant C may depend on K via the definition of the stopping time ξ_K. By a classical version of the Grönwall lemma, the monotone convergence theorem and the fact that $\xi^K \to \infty$ as $K \to \infty$, we infer that, for every $T \geq 0$,

$$\mathbb{E} \sup_{t \in [0,T]} \|v\|_{W^{m,p}}^p = 0.$$

Since T is arbitrary, (6.18) follows, and the proof of uniqueness is therefore complete. $\square$

REMARK 6.9. With obvious modifications the above proof can be used to show that if $(u^{(1)}, \tau^{(1)})$ and $(u^{(2)}, \tau^{(2)})$ are local pathwise solutions of (1.1)–(1.2), then

$$\mathbb{P}\big(\mathbb{1}_{u^{(1)}(0)=u^{(2)}(0)}\big(u^{(1)}(t) - u^{(2)}(t)\big) = 0; \forall t \in \big[0, \tau^{(1)} \wedge \tau^{(2)}\big]\big) = 1. \tag{6.23}$$

With uniqueness for (6.17) in hand, in order to establish the existence of pathwise solution, we shall use the following criteria from [33].

LEMMA 6.10. *Let X be a complete separable metric space and consider a sequence of X valued random variables $\{Y_j\}_{j\geq 0}$. We denote the collection of joint laws of $\{Y_j\}_{j\geq 1}$ by $\{\nu_{j,l}\}_{j,l\geq 1}$; that is, we take*

$$\nu_{j,l}(E) := \mathbb{P}\big((Y_j, Y_l) \in E\big), \qquad E \in \mathcal{B}(X \times X).$$

Then $\{Y_j\}_{j\geq 1}$ converges in probability if and only if for every subsequence of joint probabilities laws, $\{\nu_{j_k,l_k}\}_{k\geq 0}$, there exists a further subsequence which converges weakly to a probability measure ν such that

$$\nu\big(\{(u,v) \in X \times X : u = v\}\big) = 1. \tag{6.24}$$

With this result in mind let us now return again to the sequence of solutions u^j to the system (6.1) relative to some stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ which we fix in advance. We define sequences of measures $\nu_{j,l}(\cdot) = \mathbb{P}((u^j, u^l) \in \cdot)$ and $\mu_{j,l}(\cdot) = \mathbb{P}((u^j, u^l, \mathcal{W}) \in \cdot)$ on the phase spaces $\mathcal{X}_J = \mathcal{X}_S \times \mathcal{X}_S = C([0,T], X_{m'-2}) \times C([0,T], X_{m'-2})$, $\mathcal{X}_T = \mathcal{X}_J \times C([0,T], \mathfrak{U}_0)$, respectively. With only minor modifications to the arguments in Lemma 6.3, we see that the collection $\{\mu_{j,l}\}_{j,l\geq 1}$ is weakly compact. Extracting a convergent subsequence $\mu_{j,l} \rightharpoonup \mu$ and invoking the Skorokhod theorem we infer the existence of a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ on which there are defined random elements $(\widetilde{u}^l, \widetilde{u}^l, \widetilde{\mathcal{W}}^{j,l})$ equal in law to $\mu_{j,l}$ and so that

$$(\widetilde{u}^j, \widetilde{u}^l, \widetilde{\mathcal{W}}^{j,l}) \to (\widetilde{u}, \widetilde{u}^*, \widetilde{\mathcal{W}}), \tag{6.25}$$

where the convergence occurs $\widetilde{\Omega}$ almost surely in $\mathcal{X}_T$. As above we infer that each of $(\widetilde{u}, \widetilde{\mathcal{W}})$ and $(\widetilde{u}^*, \widetilde{\mathcal{W}})$ are solutions of (6.17) relative to the *same* stochastic basis $\mathcal{S} := (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathcal{W}})$ with $\widetilde{\mathcal{F}}_t$ the completion of σ algebra generated by $\{\widetilde{u}(s), \widetilde{u}^*(s), \widetilde{\mathcal{W}}(s)) : s \leq t\}$. Define $\nu(\cdot) = \widetilde{\mathbb{P}}((\widetilde{u}, \widetilde{u}^*) \in \cdot)$ and observe that, due to (6.25) $\nu^{j,l} \to \nu$, weakly. Now Proposition 6.8 implies that $\nu(\{(u, u^*) \in \mathcal{X}_J : u = u^*\}) = 1$. Here we use that $H^{m'-2} \subset W^{m,p}$, and so uniqueness in $W^{m,p}$ (which is proven in Proposition 6.8) implies uniqueness everywhere, and hence in $H^{m'-2}$. We may therefore infer (passing if needed to a subsequence) that $u^j \to u$ in $\mathcal{X}_S$ almost surely, and *on the original probability space*. Having obtained this convergence and referring again to (6.4), we may thus show that u is a pathwise solution of (6.17). We finally define the stopping time

$$\tau = \inf\{t \geq 0 : \|u\|_{W^{m,p}} > R\}.$$

Note that this stopping time is well defined since $u \in C([0,\infty), X_{m'-2}) \subset C([0,\infty), X_{m,p})$ for $m' = m + 5$. Hence, relative to the initial fixed stochastic basis $\mathcal{S}$, (u, τ) is a local pathwise solution of the stochastic Euler equation (1.1)–(1.2), in the sense that $u(\cdot \wedge \tau) \in L^\infty_{\mathrm{loc}}([0,\infty); X_{m'}) \cap C([0,\infty); X_{m'-2})$, and (4.1) holds for every $t \geq 0$.

In order to show that $\tau > 0$ we initially assume $\|u_0\|_{H^{m'}} \leq M$ for some *deterministic* $M > 0$, and choose $R > \bar{C}M$, where $\bar{C} \geq 1$ is the constant such that $\|u\|_{W^{1,\infty}} \leq \bar{C}\|u\|_{H^{m'}}$, in the cut-off function in (6.1). To pass to the general case $\|u_0\|_{H^{m'}} < \infty$ almost surely, we proceed as follows; see, for example, [32], Section 4.2. For $k \geq 0$ we define $u_0^k = u_0 \mathbb{1}_{k \leq \|u_0\|_{H^{m'}} < k+1}$ and obtain a corresponding local pathwise solution (u_k, τ_k) by applying the above construction with any $R > \bar{C}(k+1)$ in the cut-off function θ_R. We then define

$$u = \sum_{k \geq 0} u_k \mathbb{1}_{k \leq \|u_0\|_{H^{m'}} < k+1}, \tag{6.26}$$

$$\tau = \sum_{k \geq 0} \tau_k \mathbb{1}_{k \leq \|u_0\|_{H^{m'}} < k+1} \tag{6.27}$$

and find that (u, τ) is in fact the local pathwise solution corresponding to the initial condition u_0.

For any fixed $u_0 \in X_{m'}$ we next extend the solution (u, τ) to a maximal time of existence ξ; cf. [32, 34, 43]. Take $\mathcal{E}$ to be the set of all stopping times σ corresponding to a local pathwise solution of (1.1)–(1.2) with initial condition u_0. Let $\xi = \sup \mathcal{E}$, and consider a sequence $\sigma_k \in \mathcal{E}$ increasing to ξ. Due to the local uniqueness of pathwise solutions we obtain a process u defined on $[0, \xi)$ such that (u, σ_k) are local pathwise solutions. For each $r > 0$ we now take

$$\rho_r = \inf\{t \geq 0 : \|u(t)\|_{W^{1,\infty}} > r\} \wedge \xi.$$

Note that u is continuous on $W^{1,\infty}$ and so ρ_r is a well-defined stopping time. By continuity and uniqueness arguments we may infer that (u, ρ_r) is a local pathwise solution for each $r > 0$.[6] Suppose toward a contradiction that, for some $T, r > 0$

[6]Note that, for a given $r > 0$, we may have $\mathbb{P}(\rho_r = 0) \neq 0$. However, for almost every $\omega \in \Omega$, there exists $r > 0$ such that, $\rho_r(\omega) > 0$.

we have $\mathbb{P}(\xi = \rho_r \wedge T) > 0$. Since $(u, \rho_r \wedge T)$ is a local pathwise solution then there exists, another stopping time $\zeta > \rho_r \wedge T$ and a process u^* such that (u^*, ζ) is a local pathwise solution corresponding to u_0, contradicting the maximality of ξ. Hence we have proven that for every $T, r > 0$, we have $\mathbb{P}(\xi = \rho_r \wedge T) = 0$. Observe that on the set $\{\xi < \infty\}$, by suitably choosing T, we obtain that $\rho_r < \xi$ for every $r > 0$. On this set we hence have $\sup_{t\in[0,\rho_r]} \|u(t)\|_{W^{1,\infty}} = r$ for all $r > 0$, which gives

$$\sup_{t\in[0,\xi)} \|u(t)\|_{W^{1,\infty}} = \infty \qquad \text{on the set } \{\xi < \infty\}. \tag{6.28}$$

In summary in this section we have so far constructed maximal local pathwise $H^{m'}$ solutions, but only for the nonsharp smoothness regime $m' = m + 5$, with the solution guaranteed to evolve continuously only in $X_{m'-2}$, and which remains bounded in $X_{m'}$. In the next section we shall use these very smooth solutions to construct local (maximal) pathwise $W^{m,p}$ solutions for all $m > d/p + 1$, and for all $p \geq 2$, which will then prove Theorem 4.3.

7. Construction of $W^{m,p}$ solutions. For $m > d/p + 1$ with $p \geq 2$, we now establish the local existence of solutions for any initial data $u_0 \in X_{m,p}$, which is $\mathcal{F}_0$ measurable, which concludes the proof of Theorem 4.3. For this purpose we will adapt a density and stability argument from [35, 41], which makes use of the very smooth solutions constructed in Section 6, as approximating solutions. Indeed, when the initial data lies in $X_{m'}$, where $m' = m + 5$, we obtained in Section 6 maximal pathwise solutions in the sense of Definition 4.1. In order to make use of these smooth solutions we define a sequence of regularized initial data

$$u_0^j = F_{j^{-1}} u^0, \tag{7.1}$$

where the smoothing operators $F_{j^{-1}}$ and their properties are recalled in Appendix A below; see also [35]. For technical reasons we assume initially, that $\|u_0\|_{W^{m,p}} \leq M$ for some deterministic fixed constant M. As in Section 6, once we obtain the local existence of solutions for each fixed M, this assumption can be relaxed to the general case via a cutting argument as given in (6.26)–(6.27). Note that in view of Lemma A.1, estimate (A.2)

$$\sup_{j\geq 1} \|u_0^j\|_{W^{m,p}} \leq C\|u_0\|_{W^{m,p}} \leq CM, \tag{7.2}$$

where $C = C(m, p, \mathcal{D})$ is a universal constant. Bound (7.2) will be used in a crucial way in the forthcoming estimates. Since $F_{j^{-1}}$ is smoothing, $\{u_0^j\}_{j\geq 1} \subset X_{m'}$, and we obtain from the results in Section 6 a sequence (u^j, ξ^j) of maximal, pathwise solutions evolving continuously in $X_{m'-2}$ which are bounded in $X_{m'}$. In order to show that this sequence converges to a local $X_{m,p}$ solution corresponding to the initial condition u_0 we show that, up to some stopping time $\tau > 0$ the sequence $\{u^j\}_{j\geq 1}$ is Cauchy and hence convergent in $C([0, \tau); X_{m,p})$.

To obtain this convergence (along with an associated stopping time τ), we apply an abstract result from [32]; see also [43]. For this purpose pick fix any $T > 0$ and define the sequence of stopping times

$$\tau_j^T := \inf\{t \geq 0 : \|u^j(t)\|_{W^{m,p}} \geq 2 + \|u_0^j\|_{W^{m,p}}\} \wedge T, \tag{7.3}$$

and let

$$\tau_{j,k}^T = \tau_j^T \wedge \tau_k^T \tag{7.4}$$

for $j, k \geq 1$. Since $W^{m,p}$ is continuously embedded in $W^{1,\infty}$ it is clear that $\tau_j^T < \xi^j$, where as usual ξ^j is the maximal (stopping) time of existence of u^j, that is,

$$\sup_{t \in [0,\xi^j]} \|u^j(t)\|_{W^{m,p}} = \infty \qquad \text{on the set } \{\xi^j < \infty\}. \tag{7.5}$$

From [32], Lemma 5.1, we recall:

LEMMA 7.1 (Abstract Cauchy lemma). *For $T > 0$ and $\tau_{j,k}^T$ as defined in (7.4), suppose that we have*

$$\lim_{k\to\infty} \sup_{j\geq k} \mathbb{E} \sup_{t\in[0,\tau_{j,k}^T]} \|u^j(t) - u^k(t)\|_{W^{m,p}} = 0 \tag{7.6}$$

and

$$\lim_{S\to 0} \sup_{j\geq 1} \mathbb{P}\Big[\sup_{t\in[0,\tau_j^T\wedge S]} \|u^j(t)\|_{W^{m,p}} > \|u_0^j\|_{W^{m,p}} + 1\Big] = 0. \tag{7.7}$$

Then, there exists a stopping time τ with

$$\mathbb{P}(0 < \tau \leq T) = 1, \tag{7.8}$$

and a process predictable process $u(\cdot) = u(\cdot \wedge \tau) \in C([0, \tau], X_{m,p})$ such that

$$\sup_{t\in[0,\tau]} \|u^{j_l} - u\|_{W^{m,p}} \to 0 \qquad a.s. \tag{7.9}$$

for some subsequence $j_l \to \infty$. Moreover, the bound

$$\|u(t)\|_{W^{m,p}} \leq 2 + \sup_j \|u_0^j\|_{W^{m,p}} \quad a.s. \tag{7.10}$$

holds uniformly for $t \in [0, \tau]$.

In view of Lemma 7.1, we may now establish the essential convergence needed for Theorem 4.3 in the general case by verifying (7.6) and (7.7). To prove (7.6) we fix arbitrary $j, k \geq 1$ and denote $v = u^k - u^j$ where $v_0 = u_0^k - u_0^j$. We have

$$dv + P(v \cdot \nabla u^k + u^j \cdot \nabla v)\, dt = P(\sigma(u^j) - \sigma(u^k))\, d\mathcal{W}.$$

Applying ∂^α to this system and then the Itô lemma in L^p we obtain

$$
\begin{aligned}
d\|\partial^\alpha v\|_{L^p}^p
&= -p\int_{\mathcal{D}} \partial^\alpha v\cdot \partial^\alpha P\big(v\cdot\nabla u^k + u^j\cdot\nabla v\big)|\partial^\alpha v|^{p-2}\,dx\,dt\\
&\quad + \sum_{l\geq 1}\int_{\mathcal{D}}\bigg(\frac{p}{2}\big|\partial^\alpha P\big(\sigma_l(u^j)-\sigma_l(u^k)\big)\big|^2|\partial^\alpha v|^{p-2}\\
&\qquad\qquad + \frac{p(p-2)}{2}\big(\partial^\alpha v\cdot P\big(\sigma_l(u^j)-\sigma_l(u^k)\big)\big)^2|\partial^\alpha v|^{p-4}\bigg)\,dx\,dt\\
&\quad + p\sum_{l\geq 1}\bigg(\int_{\mathcal{D}}\partial^\alpha v\cdot\partial^\alpha P\big(\sigma_l(u^j)-\sigma_l(u^k)\big)|\partial^\alpha v|^{p-2}\,dx\bigg)\,dW_l\\
&:= \big(J_1^\alpha + J_2^\alpha\big)\,dt + J_3^\alpha\,d\mathcal{W}.
\end{aligned}
\tag{7.11}
$$

Using (7.11), we now estimate v in $W^{m,p}$. For the nonlinear terms we use Lemma 2.1 and infer

$$
\begin{aligned}
\sum_{\alpha\leq m}|J_1^\alpha| &\leq C\|P(v\cdot\nabla u^k)\|_{W^{m,p}}\|v\|_{W^{m,p}}^{p-1}\\
&\qquad + \sum_{\alpha\leq m}\big|\big(\partial^\alpha P(u^j\cdot\nabla v), \partial^\alpha v|\partial^\alpha v|^{p-2}\big)\big|\\
&\leq C\|v\|_{W^{m,p}}^{p-1}\big(\|v\|_{L^\infty}\|u^k\|_{W^{m+1,p}} + \|v\|_{W^{m,p}}\|\nabla u^k\|_{L^\infty}\big)\\
&\qquad + C\|v\|_{W^{m,p}}^{p-1}\big(\|u^j\|_{W^{1,\infty}}\|v\|_{W^{m,p}} + \|v\|_{W^{1,\infty}}\|u^j\|_{W^{m,p}}\big)\\
&\leq C\|u^k\|_{W^{m+1,p}}\|v\|_{W^{m-1,p}}\|v\|_{W^{m,p}}^{p-1}\\
&\qquad + C\big(\|u^k\|_{W^{m,p}} + \|u^j\|_{W^{m,p}}\big)\|v\|_{W^{m,p}}^p\\
&\leq C\|u^k\|_{W^{m+1,p}}^p\|v\|_{W^{m-1,p}}^p + C\big(\|u^k\|_{W^{m,p}} + \|u^j\|_{W^{m,p}} + 1\big)\|v\|_{W^{m,p}}^p.
\end{aligned}
\tag{7.12}
$$

Note that the first term in the final inequality prevents one from directly closing the estimates for v in $W^{m,p}$. We will therefore need to make further estimates for u^k in $W^{m+1,p}$ and v in $W^{m-1,p}$; cf. (7.16)–(7.17) below. For the terms involving J_2^α we use the local Lipschitz condition (3.1) and obtain

$$
\begin{aligned}
\sum_{\alpha\leq m}|J_2^\alpha| &\leq C\|v\|_{W^{m,p}}^{p-2}\bigg(\sum_{\alpha\leq m}\int_{\mathcal{D}}\bigg(\sum_{l\geq 1}\big|\partial^\alpha P\big(\sigma_l(u^j)-\sigma_l(u^k)\big)\big|^2\bigg)^{p/2}\bigg)^{2/p}\\
&\leq C\|v\|_{W^{m,p}}^{p-2}\big\|P\big(\sigma(u^j)-\sigma(u^k)\big)\big\|_{\mathbb{W}^{m,p}}^2\\
&\leq C\beta\big(\|u^k\|_{L^\infty} + \|u^j\|_{L^\infty}\big)^2\|v\|_{W^{m,p}}^p.
\end{aligned}
\tag{7.13}
$$

Finally, estimating in a similar manner to (5.12), we find that for any stopping time τ,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,\tau]}\Big|\int_0^s J_3^\alpha\,dW\Big|\Big)\\
&\quad\le C\mathbb{E}\bigg(\int_0^\tau \sum_{k\ge 1}\Big(\int_{\mathcal{D}} \partial^\alpha v\cdot\partial^\alpha P\big(\sigma_l(u^j)-\sigma_l(u^k)\big)|\partial^\alpha v|^{p-2}\,dx\Big)^2\,ds\bigg)^{1/2} \qquad (7.14)\\
&\quad\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau]}\|\partial^\alpha v\|_{L^p}^p + C\mathbb{E}\int_0^\tau \beta\big(\|u^k\|_{L^\infty}+\|u^j\|_{L^\infty}\big)^2\|v\|_{W^{m,p}}^p\,ds.
\end{aligned}
$$

Combining the estimates obtained in (7.12)–(7.14) and recalling the definition of $\tau_{j,k}^T$ in (7.4) we find that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{[0,\tau_{j,k}^T\wedge t]}\|v\|_{W^{m,p}}^p\Big)\\
&\quad\le 2\mathbb{E}\|v_0\|_{W^{m,p}}^p + C\mathbb{E}\int_0^{\tau_{j,k}^T\wedge t}\big(\|u^k\|_{W^{m+1,p}}^p\|v\|_{W^{m-1,p}}^p\big)\,ds\\
&\qquad + C\mathbb{E}\int_0^{\tau_{j,k}^T\wedge t}\big(\|u^j\|_{W^{m,p}}+\|u^k\|_{W^{m,p}}\\
&\qquad\qquad + \beta\big(\|u^k\|_{L^\infty}+\|u^j\|_{L^\infty}\big)^2\big)\|v\|_{W^{m,p}}^p\,ds\\
&\quad\le 2\mathbb{E}\|v_0\|_{W^{m,p}}^p\\
&\qquad + C\int_0^t\Big(\mathbb{E}\sup_{[0,\tau_{j,k}^T\wedge s]}\|v\|_{W^{m,p}}^p + \mathbb{E}\sup_{[0,\tau_{j,k}^T\wedge s]}\big(\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p\big)\Big)\,ds,
\end{aligned}
$$

where C is a positive constant that depends on M and β but is independent of j,k. By an application of the classical Grönwall lemma, we obtain that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{[0,\tau_{j,k}^T]}\|u^k-u^j\|_{W^{m,p}}^p\Big)\\
&\quad = \mathbb{E}\Big(\sup_{[0,\tau_{j,k}^T]}\|v\|_{W^{m,p}}^p\Big)\\
&\quad \le C\Big(\mathbb{E}\|u_0^k-u_0^j\|_{W^{m,p}}^p + \mathbb{E}\sup_{[0,\tau_{j,k}^T]}\big(\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p\big)\Big),
\end{aligned}
$$

where $C=C(m,p,\mathcal{D},M,T)$ is *independent* of both j,k. Observe that, in view of Lemma A.1, estimate (A.5), and applying the dominated convergence theorem we conclude that $\sup_{j\geq k}\mathbb{E}\|u_0^k-u_0^j\|^p_{W^{m,p}}$ goes to zero as $k\to\infty$. As such, (7.6) will follow once we show that

$$\lim_{k\to\infty}\sup_{j\geq k}\mathbb{E}\sup_{[0,\tau^T_{j,k}]}\big(\|v\|^p_{W^{m-1,p}}\|u^k\|^p_{W^{m+1,p}}\big)=0. \tag{7.15}$$

With this goal of establishing (7.15) in mind, let us determine $d(\|v\|^p_{W^{m-1,p}}\times\|u^k\|^p_{W^{m+1,p}})$. We have [cf. (5.9) and (7.11)] that

$$d\|u^k\|^p_{W^{m+1,p}}=(I_1+I_2)\,dt+I_3\,d\mathcal{W}, \tag{7.16}$$

$$d\|v\|^p_{W^{m-1,p}}=(J_1+J_2)\,dt+J_3\,d\mathcal{W}, \tag{7.17}$$

where, to make the notation less cumbersome, we take

$$I_l=\sum_{|\alpha|\leq m+1}I_l^\alpha\quad\text{and}\quad J_l=\sum_{|\alpha|\leq m-1}J_l^\alpha\qquad\text{for } l=1,2,3.$$

The elements I_l^α are defined as in (5.9) (with u replaced with u^k throughout) and J_l^α are as in (7.11). By an application of the Itô product rule we find that

$$\begin{aligned}
&d\big(\|v\|^p_{W^{m-1,p}}\|u^k\|^p_{W^{m+1,p}}\big)\\
&\quad=\|v\|^p_{W^{m-1,p}}d\|u^k\|^p_{W^{m+1,p}}+\|u^k\|^p_{W^{m+1,p}}d\|v\|^p_{W^{m-1,p}}\\
&\qquad+d\|v\|^p_{W^{m-1,p}}d\|u^k\|^p_{W^{m+1,p}}\\
&\quad=\big(\|v\|^p_{W^{m-1,p}}(I_1+I_2)+\|u^k\|^p_{W^{m+1,p}}(J_1+J_2)+K\big)\,dt\\
&\qquad+\big(\|v\|^p_{W^{m-1,p}}I_3+\|u^k\|^p_{W^{m+1,p}}J_3\big)\,d\mathcal{W},
\end{aligned}$$

where K is the term arising from $I_3\,d\mathcal{W}J_3\,d\mathcal{W}$ and is given by

$$\begin{aligned}
K=p^2\sum_{l\geq1}&\bigg(\sum_{|\alpha|\leq m+1}\int_{\mathcal{D}}\partial^\alpha u^k\cdot\partial^\alpha P\sigma_l(u^k)|\partial^\alpha u^k|^{p-2}\,dx\bigg)\\
&\times\bigg(\sum_{|\alpha|\leq m-1}\int_{\mathcal{D}}\partial^\alpha v\cdot\partial^\alpha P\big(\sigma_l(u^j)-\sigma_l(u^k)\big)|\partial^\alpha v|^{p-2}\,dx\bigg).
\end{aligned}$$

In view of the estimates carried out in Section 5 [cf. (5.10)–(5.11)] and making use of the assumption (3.2), we immediately infer that

$$\begin{aligned}
&\big|\|v\|^p_{W^{m-1,p}}(I_1+I_2)\big|\\
&\quad\leq C\beta\big(\|u^k\|_{L^\infty}\big)^2\|v\|^p_{W^{m-1,p}}\\
&\qquad+C\big(\beta\big(\|u^k\|_{L^\infty}\big)^2+\|u^k\|_{W^{1,\infty}}\big)\|u^k\|^p_{W^{m+1,p}}\|v\|^p_{W^{m-1,p}}.
\end{aligned} \tag{7.18}$$

We next treat the drift terms in (7.17). For J_1, recalling that $P = I - Q$ we write

$$
\begin{aligned}
|J_1| &\le p \sum_{|\alpha|\le m-1} \left| \int_{\mathcal{D}} \partial^\alpha v \cdot \partial^\alpha P\big(v\cdot\nabla u^k + u^j\cdot\nabla v\big) |\partial^\alpha v|^{p-2}\,dx \right| \\
&\le C\|v\|_{W^{m-1,p}}^{p-1} \big\|P\big(v\cdot\nabla u^k\big)\big\|_{W^{m-1,p}} \\
&\qquad + C\sum_{|\alpha|\le m-1} \left| \int_{\mathcal{D}} \partial^\alpha v\cdot\partial^\alpha\big(u^j\cdot\nabla v\big)|\partial^\alpha v|^{p-2}\,dx\right| \\
&\qquad + C\sum_{|\alpha|\le m-1} \left| \int_{\mathcal{D}} \partial^\alpha v\cdot\partial^\alpha Q\big(u^j\cdot\nabla v\big)|\partial^\alpha v|^{p-2}\,dx\right| \\
&= J_{1,1} + J_{1,2} + J_{1,3}.
\end{aligned}
\tag{7.19}
$$

The right-hand side of the above estimate may be bounded as follows. To bound $J_{1,1}$ we use Lemma 2.1 and obtain

$$
\begin{aligned}
J_{1,1} &\le C\|v\|_{W^{m-1,p}}^{p-1}\big(\|v\|_{L^\infty}\|u^k\|_{W^{m,p}} + \|v\|_{W^{m-1,p}}\|u^k\|_{W^{1,\infty}}\big) \\
&\le C\|v\|_{W^{m-1,p}}^{p}\|u^k\|_{W^{m,p}}.
\end{aligned}
\tag{7.20}
$$

For the other two terms on the right-hand side of (7.19) we cannot estimate as in Lemma 2.1 directly; we would obtain bound of the type $\|u^j\|_{W^{m-1,p}}\|v\|_{W^{m,p}} \times \|v\|_{W^{m-1,p}}^{p-1}$, which would prevent us from closing the estimates involving $\|v\|_{W^{m-1,p}}^{p}$. To bound $J_{1,2}$ we use the Leibniz rule, the Hölder inequality and Gagliardo–Nirenberg inequality. There is only one nonstandard term $\|\partial^\alpha u^j \cdot \nabla v\|_{L^p}$, which is bounded as

$$
\sum_{|\alpha|\le m-1} \big\|\partial^\alpha u^j\cdot\nabla v\big\|_{L^p} \le C\big\|u^j\big\|_{W^{m-1,q}}\|\nabla v\|_{L^r} \le \big\|u^j\big\|_{W^{m,p}}\|v\|_{W^{m-1,p}},
$$

where $q = pd/(d-p)$, $r = pq/(q-p) = d$ if $p < d$, and $q = \infty$, $r = p$ if $p \ge d$. The other terms are bounded as in Lemma 2.1, and we obtain

$$
J_{1,2} \le C\|v\|_{W^{m-1,p}}^{p}\big\|u^j\big\|_{W^{m,p}}.
\tag{7.21}
$$

Finally, the "pressure term" $J_{1,3}$ is estimated using the Hölder inequality, the Agmon–Douglis–Nirenberg estimate (2.6), and the Gagliardo–Nirenberg inequality as

$$
\begin{aligned}
J_{1,3} &\le \big\|Q\big(u^j\cdot\nabla v\big)\big\|_{W^{m-1,p}}\|v\|_{W^{m-1,p}}^{p-1} \\
&\le C\big(\big\|\partial_i u_l^j\,\partial_l v_i\big\|_{W^{m-2,p}} + \big\|u^j v\big\|_{W^{m-1,p}}\big)\|v\|_{W^{m-1,p}}^{p-1} \\
&\le C\|v\|_{W^{m-1,p}}^{p}\big\|u^j\big\|_{W^{m,p}}.
\end{aligned}
\tag{7.22}
$$

Combining (7.20)–(7.22) we conclude

$$
|J_1| \le C\|v\|_{W^{m-1,p}}^{p}\big(\big\|u^j\big\|_{W^{m,p}} + \big\|u^k\big\|_{W^{m,p}}\big).
\tag{7.23}
$$

For J_2 we find, as above in (7.13) that

$$
\begin{aligned}
|J_2| &\le C\|v\|_{W^{m-1,p}}^{p-2}\|P(\sigma(u^j)-\sigma(u^k))\|_{\mathbb{W}^{m-1,p}}^2 \\
&\le C\beta(\|u^k\|_{L^\infty}+\|u^j\|_{L^\infty})^2\|v\|_{W^{m-1,p}}^p.
\end{aligned}
\tag{7.24}
$$

Combining (7.23)–(7.24) we find

$$
\begin{aligned}
&\big|\|u^k\|_{W^{m+1,p}}^p(J_1+J_2)\big| \\
&\quad\le C\big(\beta(\|u^k\|_{L^\infty}+\|u^j\|_{L^\infty})^2+\|u^j\|_{W^{m,p}}+\|u^k\|_{W^{m,p}}\big) \\
&\qquad\times\|u^k\|_{W^{m+1,p}}^p\|v\|_{W^{m-1,p}}^p.
\end{aligned}
\tag{7.25}
$$

The term K is estimated using the Hölder and Minkowski inequalities followed by the standing assumption on σ (3.1),

$$
\begin{aligned}
|K| &\le C\bigg(\sum_{l\ge1}\bigg(\sum_{|\alpha|\le m+1}\int_{\mathcal{D}}|\partial^\alpha P\sigma_l(u^k)||\partial^\alpha u^k|^{p-1}\,dx\bigg)^2\bigg)^{1/2} \\
&\quad\times\bigg(\sum_{l\ge1}\bigg(\sum_{|\alpha|\le m-1}\int_{\mathcal{D}}|\partial^\alpha P(\sigma_l(u^j)-\sigma_l(u^k))||\partial^\alpha v|^{p-1}\,dx\bigg)^2\bigg)^{1/2} \\
&\le C\sum_{|\alpha|\le m+1}\int_{\mathcal{D}}\bigg(\sum_{l\ge1}|\partial^\alpha P\sigma_l(u^k)|^2\bigg)^{1/2}|\partial^\alpha u^k|^{p-1}\,dx \\
&\quad\times\sum_{|\alpha|\le m-1}\int_{\mathcal{D}}\bigg(\sum_{l\ge1}|\partial^\alpha P(\sigma_l(u^j)-\sigma_l(u^k))|^2\bigg)^{1/2}|\partial^\alpha v|^{p-1}\,dx \\
&\le C\|u^k\|_{W^{m+1,p}}^{p-1}\bigg(\sum_{|\alpha|\le m+1}\int_{\mathcal{D}}\bigg(\sum_{l\ge1}|\partial^\alpha P\sigma_l(u^k)|^2\bigg)^{p/2}dx\bigg)^{1/p} \\
&\quad\times\|v\|_{W^{m-1,p}}^{p-1}\bigg(\sum_{|\alpha|\le m-1}\int_{\mathcal{D}}\bigg(\sum_{l\ge1}|\partial^\alpha P(\sigma_l(u^j)-\sigma_l(u^k))|^2\bigg)^{p/2}dx\bigg)^{1/p} \\
&\le C\|u^k\|_{W^{m+1,p}}^{p-1}\|P\sigma(u^k)\|_{\mathbb{W}^{m+1,p}}\|v\|_{W^{m-1,p}}^{p-1}\|P(\sigma(u^j)-\sigma(u^k))\|_{\mathbb{W}^{m-1,p}} \\
&\le C\beta(\|u^k\|_{L^\infty}+\|u^j\|_{L^\infty})^2(\|u^k\|_{W^{m+1,p}}^p\|v\|_{W^{m-1,p}}^p+\|v\|_{W^{m-1,p}}^p).
\end{aligned}
\tag{7.26}
$$

To treat the stochastic terms we proceed similar to (5.12) and find that for any stopping time τ,

$$
\begin{aligned}
&\mathbb{E}\bigg(\sup_{s\in[0,\tau]}\bigg|\int_0^s\|u^k\|_{W^{m+1,p}}^p J_3\,d\mathcal{W}\bigg|\bigg) \\
&\quad\le C\mathbb{E}\bigg(\int_0^\tau\|u^k\|_{W^{m+1,p}}^{2p}
\end{aligned}
$$

$$\times \sum_{l\geq 1}\Big(\sum_{|\alpha|\leq m-1}\int_{\mathcal{D}}|\partial^{\alpha}P(\sigma_l(u^j)-\sigma_l(u^k))||\partial^{\alpha}v|^{p-1}\,dx\Big)^2\,ds\Big)^{1/2}$$

$$\begin{aligned}
&\leq C\mathbb{E}\Big(\int_0^{\tau}\|u^k\|_{W^{m+1,p}}^{2p} \\
&\qquad \times\Big(\sum_{|\alpha|\leq m-1}\int_{\mathcal{D}}\Big(\sum_{l\geq 1}|\partial^{\alpha}P(\sigma_l(u^j)-\sigma_l(u^k))|^2\Big)^{1/2} \\
&\qquad\qquad \times|\partial^{\alpha}v|^{p-1}\,dx\Big)^2\,ds\Big)^{1/2} \\
&\leq C\mathbb{E}\Big(\int_0^{\tau}\|u^k\|_{W^{m+1,p}}^{2p}\|P(\sigma(u^j)-\sigma(u^k))\|_{\mathbb{W}^{m-1,p}}^2\|v\|_{W^{m-1,p}}^{2(p-1)}\,ds\Big)^{1/2} \\
&\leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,\tau]}(\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p) \\
&\quad + C\mathbb{E}\int_0^{\tau}\beta(\|u^k\|_{L^\infty}+\|u^j\|_{L^\infty})^2\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p\,ds.
\end{aligned}\tag{7.27}$$

Similar to (7.27) above, we also obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[0,\tau]}\Big|\int_0^s\|v\|_{W^{m-1,p}}^pI_3\,d\mathcal{W}\Big|\Big) \\
&\quad\leq C\mathbb{E}\Big(\int_0^{\tau}\|v\|_{W^{m-1,p}}^{2p}\|P(\sigma(u^k))\|_{\mathbb{W}^{m+1,p}}^2\|u^k\|_{W^{m+1,p}}^{2(p-1)}\,ds\Big)^{1/2} \\
&\quad\leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,\tau]}(\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p) \\
&\qquad + C\mathbb{E}\int_0^{\tau}\beta(\|u^k\|_{L^\infty})^2\|v\|_{W^{m-1,p}}^p(\|u^k\|_{W^{m+1,p}}^p+1)\,ds.
\end{aligned}\tag{7.28}$$

Summarizing, from estimates (7.18), (7.25)–(7.28), and the definition of $\tau_{n,m}^T$ in (7.3), we find that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,\tau_{j,k}^T\wedge t]}\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p\Big) \\
&\quad\leq 2\mathbb{E}(\|v_0\|_{W^{m-1,p}}^p\|u_0^k\|_{W^{m+1,p}}^p) \\
&\qquad + C\mathbb{E}\int_0^t\Big(\sup_{t\in[0,\tau_{j,k}^T\wedge s]}(\|v\|_{W^{m-1,p}}^p\|u^k\|_{W^{m+1,p}}^p)+\sup_{t\in[0,\tau_{j,k}^T\wedge s]}\|v\|_{W^{m-1,p}}^p\Big)\,ds
\end{aligned}$$

for any $t > 0$ where the constant C depends on M, β and the data but not on j, k. Thus, again invoking the Grönwall lemma, we finally conclude that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,\tau^T_{j,k}]} \|u^k\|^p_{W^{m+1,p}}\|v\|^p_{W^{m-1,p}}\Big)\\
&\quad\le C\mathbb{E}\big(\|u^k_0\|^p_{W^{m+1,p}}\|u^k_0 - u^j_0\|^p_{W^{m-1,p}}\big)\\
&\qquad + C\mathbb{E}\Big(\sup_{t\in[0,\tau^T_{j,k}]} \|u^k - u^j\|^p_{W^{m-1,p}}\Big),
\end{aligned} \tag{7.29}
$$

where the constant C is independent of j, k. By the dominated convergence theorem [for $(\Omega, \mathcal{F}, \mathbb{P})$] and making use of the properties of the smoothing operator F_ε [cf. (A.3) and (A.6)], we find

$$
\begin{aligned}
&\lim_{k\to\infty}\sup_{j\ge k}\mathbb{E}\big(\|u^k_0\|^p_{W^{m+1,p}}\|u^k_0 - u^j_0\|^p_{W^{m-1,p}}\big)\\
&\quad\le C\lim_{k\to\infty}\sup_{j\ge k}\mathbb{E}\big(\|u_0\|^p_{W^{m,p}}k^p\|u^k_0 - u^j_0\|^p_{W^{m-1,p}}\big) = 0.
\end{aligned}
$$

To handle the second term in (7.29), we refer back to (7.17) and the estimates in (7.23)–(7.24). The stochastic terms involving J_3 are handled similar to (7.27) [and also cf. (5.12) above]. Combining these observation, using the Grönwall inequality and the properties of F_ε we finally infer

$$
\lim_{k\to\infty}\sup_{j\ge k}\mathbb{E}\Big(\sup_{t\in[0,\tau^T_{j,k}]}\|u^j - u^k\|^p_{W^{m-1,p}}\Big) = 0. \tag{7.30}
$$

With this final observation in place we have now established (7.15) and hence the first requirement (7.6) of Lemma 7.1.

To establish the second condition (7.7) required by Lemma 7.1, we return to (5.9). We find that for any $k \ge 1$ and $S > 0$,

$$
\begin{aligned}
\sup_{t\le[0,\tau^T_k\wedge S]}\|u^k(t)\|^p_{W^{m,p}} &\le \|u^k_0\|^p_{W^{m,p}} + \sum_{|\alpha|\le m}\int_0^{\tau^T_k\wedge S}|I^\alpha_1 + I^\alpha_2|\,dt\\
&\quad + \sup_{t\le[0,\tau^T_k\wedge S]}\Big|\int_0^t\sum_{|\alpha|\le m} I^\alpha_3\,d\mathcal{W}\Big|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{t\le[0,\tau^T_k\wedge S]}\|u^k(t)\|^p_{W^{m,p}} > \|u^k_0\|^p_{W^{m,p}} + 1\Big)\\
&\quad\le \mathbb{P}\Big(\sum_{|\alpha|\le m}\int_0^{\tau^T_k\wedge S}|I^\alpha_1 + I^\alpha_2|\,dt > \frac{1}{2}\Big)\\
&\qquad + \mathbb{P}\Big(\sup_{t\le[0,\tau^T_k\wedge S]}\Big|\int_0^t\sum_{|\alpha|\le m} I^\alpha_3\,dW\Big| > \frac{1}{2}\Big).
\end{aligned} \tag{7.31}
$$

For the first term on the right-hand side of (7.31), we apply the estimates in (5.10)–(5.11), and then the Chebyshev inequality, and find

$$
\begin{aligned}
&\mathbb{P}\biggl(\sum_{|\alpha|\le m}\int_0^{\tau_k^T\wedge S}\bigl|I_1^\alpha+I_2^\alpha\bigr|\,dt>\frac{1}{2}\biggr)\\
&\quad\le\mathbb{P}\biggl(\int_0^{\tau_k^T\wedge S}C\bigl(\beta(\|u^k\|_{L^\infty})^2+\|u^k\|_{W^{m,p}}\bigr)\|u^k\|_{W^{m,p}}^p\,dt>\frac{1}{2}\biggr)\\
&\quad\le C\mathbb{E}\int_0^{\tau_k^T\wedge S}\bigl(\beta(\|u^k\|_{L^\infty})^2+\|u^k\|_{W^{m,p}}\bigr)\|u^k\|_{W^{m,p}}^p\,dt\le CS,
\end{aligned}
\tag{7.32}
$$

where the constant $C=C(m,p,M,\beta,\mathcal{D})$ is independent of k and S. With Doob's inequality, the Itô isometry and the integral Minkowski inequality, we estimate the second term,

$$
\begin{aligned}
&\mathbb{P}\biggl(\sup_{t\in[0,\tau_k^T\wedge S]}\biggl|\int_0^t\sum_{|\alpha|\le m}I_3^\alpha\,d\mathcal{W}\biggr|>\frac{1}{2}\biggr)\\
&\quad\le 4\mathbb{E}\biggl(\int_0^{\tau_k^T\wedge S}\sum_{|\alpha|\le m}I_3^\alpha\,d\mathcal{W}\biggr)^2\\
&\quad\le C\mathbb{E}\int_0^{\tau_k^T\wedge S}\sum_{|\alpha|\le m}\sum_{l\ge 1}\biggl(\int_{\mathcal{D}}\partial^\alpha u^k\cdot\partial^\alpha P\sigma_l(u^k)\bigl|\partial^\alpha u^k\bigr|^{p-2}\,dx\biggr)^2\,dt\\
&\quad\le C\mathbb{E}\int_0^{\tau_k^T\wedge S}\beta(\|u^k\|_{L^\infty})^2\bigl(1+\|u^k\|_{W^{m,p}}^{2p}\bigr)\,dt\le CS,
\end{aligned}
\tag{7.33}
$$

where again the constant C is independent of S and k. With (7.31)–(7.33) we now conclude the proof of the second item in Lemma 7.1, that is, (7.7).

Having finally established both (7.6) and (7.7), we apply Lemma 7.1 to infer the existence of a strictly positive stopping time τ, a subsequence $\{u^{j_l}\}_{l\ge 1}$ of $\{u^j\}_{j\ge 1}$ and a predictable process u such that, up to a set of measure zero, u^{j_k} converges to u in $C(0,\tau;X_{m,p})$ and $\sup_{t\in[0,\tau]}\|u\|_{W^{m,p}}\le C<\infty$. We may infer that (u,τ) is a local pathwise solution of (1.1)–(1.3) in the sense of Definition 4.1. Note that, in order to initially obtain this u, we had to impose the almost sure bound on the initial data, u_0, in (7.2). This restriction is easily removed with the cutting argument as employed in Section 6; cf. (6.26)–(6.27). We may pass from the case of local to maximal pathwise solutions as given in Definition 4.2 via maximality arguments similar to those at the end of Section 6, in (6.28). Recall that this maximality argument involves considering the set of all stopping times up to which the solution exists. We then show by contradiction that the supremum of all these stopping times yields the maximal time of existence of the solution; see Section 6 for further details. The proof of Theorem 4.3 is now complete.

8. Global existence in the two-dimensional case for additive noise. In this section we establish the global existence of solutions to (1.1)–(1.3) in dimension two, forced by an additive noise. Note that, while the local existence of solutions for (1.1)–(1.3) in the case of a general ω dependent additive noise [cf. (3.4) above], is not covered under the proof of local existence given here, equations with additive noise can be treated "pathwise" via a simple change of variables. In this way the local existence follows from more classical arguments. See Remark 4.5 above and the proof of Lemma 8.1 below.

Recalling the a priori estimates in Section 5, we have that, for any $m > d/p+1$,

$$d\|u\|_{W^{m,p}}^p = X\,dt + Z\,d\mathcal{W}, \tag{8.1}$$

where X and Z are defined according to (5.9). Making use of the estimates in (5.10)–(5.11), we have

$$|X| \le C\big(1 + \|u\|_{W^{1,\infty}}\big)\|u\|_{W^{m,p}}^p + C\|\sigma\|_{\mathbb{W}^{m,p}}^p \tag{8.2}$$

for some universal constant $C = C(m, d, \mathcal{D})$. For Z we observe with similar estimate to (5.12) that

$$\begin{aligned} \|Z\|_{L_2} &\le \left(\sum_{k\ge 1}\left(\int_{\mathcal{D}} \partial^\alpha u \cdot \partial^\alpha P\sigma_k |\partial^\alpha u|^{p-2}\,dx\right)^2\right)^{1/2} \\ &\le C\|\sigma\|_{\mathbb{W}^{m,p}}\|u\|_{W^{m,p}}^{p-1}. \end{aligned} \tag{8.3}$$

Thus, in view of (8.2)–(8.3), to close the estimates for (8.1) we make use of the Beale–Kato–Majda type inequality,

$$\|u\|_{W^{1,\infty}} \le C_2\|u\|_{L^2} + C_2\|\operatorname{curl} u\|_{L^\infty}\left(1 + \log^+\left(\frac{\|u\|_{W^{m,p}}}{\|\operatorname{curl} u\|_{L^\infty}}\right)\right), \tag{8.4}$$

where C_2 is a universal constant depending only on $\mathcal{D}$, m, p. See, for example, [26] for the simply-connected bounded domain case. As such the proof of global existence requires us to obtain uniform bound on the vorticity of the solution in L^∞ and also for $\|u\|_{L^2}$ and to establish a stochastic analog of the log-Grönwall lemma. The latter is developed in Appendix C below; see also related results in [25].

In order to carry out suitable estimates for $w = \operatorname{curl} u$ we apply $\nabla^\perp = (\partial_2, -\partial_1)$ to (4.1) and obtain the evolution

$$dw + u \cdot \nabla w\,dt = \rho\,d\mathcal{W}, \tag{8.5}$$

$$w = \nabla^\perp \cdot u, \qquad \nabla \cdot u = 0, \tag{8.6}$$

where for ease of notation we denoted $\rho = \nabla^\perp \cdot \sigma$. Note that crucially, in contrast to the three-dimensional case, no vortex stretching term $w \cdot \nabla u$ appears in (8.5). For w we now establish the following result:

LEMMA 8.1 (Nonblow-up of the energy and the supremum of vorticity). *Fix $m > 2/p + 1$, consider any σ that satisfies (3.4), and any $u_0 \in X_{m,p}$. Take (u, ξ) be the maximal solution corresponding to this σ and u_0. Then we have*

$$\sup_{t\in[0,T\wedge\xi]} \|u\|_{L^2}^2 + \sup_{t\in[0,T\wedge\xi]} \|w\|_{L^\infty} < \infty, \tag{8.7}$$

almost surely, for each $T > 0$.

PROOF. The bound for $\|u\|_{L^2}$ required in (8.7) follows directly in view of the cancelation $(P(u\cdot\nabla u), u)_{L^2} = 0$; cf. Section 5.1.

We turn to estimate the vorticity term (8.7). Since (8.5) is forced with an the additive noise we have the option to introduce the stochastic process

$$dz = \rho\, d\mathcal{W}, \qquad z(0) = 0 \tag{8.8}$$

and then consider the evolution of $\widetilde{w} := w - z$. The equation for $\widetilde{w}$ is the random partial differential equation

$$\partial_t \widetilde{w} + u\cdot\nabla\widetilde{w} + u\cdot\nabla z = 0, \tag{8.9}$$

$$\widetilde{w} = \nabla^\perp\cdot u - z, \qquad \nabla\cdot u = 0, \tag{8.10}$$

$$\widetilde{w}(0) = w_0. \tag{8.11}$$

This system can be treated pathwise with the methods of ordinary calculus. Multiplying (8.9) by $\widetilde{w}|\widetilde{w}|^{p-2}$ and integrating over $\mathcal{D}$ we obtain

$$\frac{d}{dt}\|\widetilde{w}\|_{L^p} \le \|u\|_{L^p}\|\nabla z\|_{L^\infty},$$

where we have used the divergence-free nature of u. Integrating in time and sending p to ∞, the above estimate gives

$$\|\widetilde{w}(t)\|_{L^\infty} \le \|w_0\|_{L^\infty} + \int_0^t \|u(s)\|_{L^\infty}\|\nabla z(s)\|_{L^\infty}\, ds. \tag{8.12}$$

We can use the two-dimensional Sobolev embedding and the Biot–Savart law to bound

$$\|u\|_{L^\infty} \le C\|\nabla u\|_{L^4} + C\|u\|_{L^2} \le C\|w\|_{L^4} + C\|u\|_{L^2}, \tag{8.13}$$

where $C = C(\mathcal{D})$. Thus, in view of (8.12)–(8.13) and the fact that $w = \widetilde{w} + z$, the proof will be complete once we obtain suitable bounds for the quantities $\|w\|_{L^4}$ and $\|\nabla z\|_{L^\infty}$.

In order to obtain bounds on $\|w\|_{L^4}$ we apply the Itô formula in L^4 to (8.5) and obtain

$$\begin{aligned} d\|w\|_{L^4}^4 &= \int_{\mathcal{D}}\Big(2|w|^2\sum_{k\ge1}|\rho_k|^2 + 4\sum_{k\ge1}(w\rho_k)^2\Big)\,dx\,dt \\ &\quad + 4\sum_{k\ge1}\Big(\int_{\mathcal{D}}|w|^2 w\rho_k\,dx\Big)\,dW_k, \end{aligned} \tag{8.14}$$

where we have used the cancellation $(u \cdot \nabla w, w|w|^2)_{L^2} = 0$. Let

$$\sigma_R = \inf\{t \geq 0 : \|w(t)\|_{L^4} > R\} \wedge \inf\left\{t \geq 0 : \int_0^t \|\rho\|_{\mathbb{W}^{0,4}}^2 \, ds > R\right\} \wedge \xi. \tag{8.15}$$

From (3.4) and the definition of ξ as the maximal time of existence, it follows that $\sigma_R \to \xi$ almost surely as $R \to \infty$. In addition, for every $T > 0$ and a.s. ω, if R is sufficiently large we have that $\sigma_R \wedge T = \xi \wedge T$.

Upon taking a supremum in time in (8.14), and applying the Hölder inequality in the last term, we obtain on the set $\{\sigma_R > 0\}$

$$\begin{aligned}
\sup_{t \in [0, \sigma_R \wedge T]} \|w(t)\|_{L^4}^4 &\leq \|w_0\|_{L^4}^4 + 4 \sup_{t \in [0, \sigma_R \wedge T]} \left| \sum_{k \geq 1} \int_0^t \int_{\mathcal{D}} |w|^2 w \cdot \rho_k \, dx \, dW_k \right| \\
&\quad + 4 \int_0^{\sigma_R \wedge T} \|w(t)\|_{L^4}^2 \|\rho\|_{\mathbb{W}^{0,4}}^2 \, dt \\
&\leq \|w_0\|_{L^4}^4 + 4 \sup_{t \in [0, \sigma_R \wedge T]} \left| \sum_{k \geq 1} \int_0^t \int_{\mathcal{D}} |w|^2 w \cdot \rho_k \, dx \, dW_k \right| \\
&\quad + \frac{1}{4} \sup_{t \in [0, \sigma_R \wedge T]} \|w(t)\|_{L^4}^4 \\
&\quad + C \left(\int_0^{\sigma_R \wedge T} \|\rho\|_{\mathbb{W}^{0,4}}^2 \, dt \right)^2.
\end{aligned}$$

To estimate the stochastic integral terms we find with the Burkholder–Davis–Gundy inequality, (2.14) that

$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0, \sigma_R \wedge T]} \left| \mathbb{1}_{\sigma_R > 0} \sum_{k \geq 1} \int_0^t \int_{\mathcal{D}} |w|^2 w \cdot \rho_k \, dx \, dW_k \right| \\
&\quad \leq C \mathbb{E} \left(\mathbb{1}_{\sigma_R > 0} \int_0^{\sigma_R \wedge T} \sum_{k \geq 1} \left(\int_{\mathcal{D}} |w|^3 |\rho_k| \, dx \right)^2 dt \right)^{1/2} \\
&\quad \leq C \mathbb{E} \left(\mathbb{1}_{\sigma_R > 0} \int_0^{\sigma_R \wedge T} \left(\int_{\mathcal{D}} |w|^3 \left(\sum_{k \geq 1} |\rho_k|^2 \right)^{1/2} dx \right)^2 dt \right)^{1/2} \\
&\quad \leq C \mathbb{E} \left(\mathbb{1}_{\sigma_R > 0} \int_0^{\sigma_R \wedge T} \|w\|_{L^4}^6 \|\rho\|_{\mathbb{W}^{0,4}}^2 \, dt \right)^{1/2} \\
&\quad \leq \frac{1}{4} \mathbb{E} \left(\mathbb{1}_{\sigma_R > 0} \sup_{t \in [0, \sigma_R \wedge T]} \|w\|_{L^4}^4 \right) + C \mathbb{E} \left(\mathbb{1}_{\sigma_R > 0} \int_0^{\sigma_R \wedge T} \|\rho\|_{\mathbb{W}^{0,4}}^2 \, dt \right)^2.
\end{aligned}$$

Combining the above observations we find $\mathbb{E}(\mathbb{1}_{\sigma_R > 0} \sup_{t \in [0, \sigma_R \wedge T]} \|w\|_{L^4}^4) \leq C$, by recalling the definition of σ_R [cf. (8.15)], for some $C > 0$ which depends on R.

Since $\|w_0\|_{L^4} < \infty$ almost surely we conclude that $\sup_{t\in[0,\sigma_R\wedge T]} \|w\|_{L^4}^4 < \infty$ almost surely for all $R > 0$. Thus we finally conclude that for almost every ω that

$$\sup_{t\in[0,\xi\wedge T]} \|w\|_{L^4}^4 < \infty. \tag{8.16}$$

We now turn to make estimates for z. In view of the Sobolev embedding $W^{1,\infty} \subset W^{m,p}$ and the definition of z, given in (8.8), we estimate using (2.14),

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t \rho \, d\mathcal{W} \right\|_{W^{m,p}}^p &\leq \sum_{|\alpha|\leq m} \int_{\mathcal{D}} \mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \partial^\alpha \rho \, d\mathcal{W} \right|^p dx \\
&\leq C \sum_{|\alpha|\leq m} \int_{\mathcal{D}} \mathbb{E} \left(\int_0^T |\partial^\alpha \rho|_{L_2}^2 \, dt \right)^{p/2} dx \\
&\leq C \mathbb{E} \int_0^T \|\rho\|_{\mathbb{W}^{m,p}}^p \, dt.
\end{aligned}$$

We therefore infer that

$$\mathbb{E} \sup_{t\in[0,T]} \|z(t)\|_{W^{1,\infty}}^2 \leq C \left(\mathbb{E} \sup_{t\in[0,T]} \left\| \int_0^t \rho \, d\mathcal{W} \right\|_{W^{m,p}}^p \right)^{p/2} < \infty. \tag{8.17}$$

Taking the supremum in time over $[0, T \wedge \xi]$ for (8.12), and applying (8.13), we obtain for almost every ω that

$$\begin{aligned}
&\sup_{t\in[0,T\wedge\xi]} \|\widetilde{w}(t)\|_{L^\infty} \\
&\quad \leq \|w_0\|_{L^\infty} + C \left(\sup_{t\in[0,T\wedge\xi]} \|u(t)\|_{L^2} \int_0^T \|\nabla z(t)\|_{L^\infty} \, dt \right) \\
&\qquad + C \left(\sup_{t\in[0,T\wedge\xi]} \|w(t)\|_{L^4} \int_0^T \|\nabla z(t)\|_{L^\infty} \, dt \right) \\
&\quad \leq \|w_0\|_{L^\infty} \\
&\qquad + C \left(\sup_{t\in[0,T\wedge\xi]} \|u(t)\|_{L^2}^2 + \sup_{t\in[0,T\wedge\xi]} \|w(t)\|_{L^4}^2 + \sup_{t\in[0,T]} \|z(t)\|_{W^{1,\infty}}^2 \right),
\end{aligned} \tag{8.18}$$

where C may depend on T. Given the bounds established in (8.16)–(8.17), and since by construction $w = \widetilde{w} + z$, referring once more to (8.17), the proof of the lemma is now complete. $\square$

With the estimates in Lemma 8.1 in hand we apply the results established in Appendix C below, to show that (u, ξ) is a *global* pathwise solution.

PROOF OF THEOREM 4.4. We need to verify that the conditions in Lemma C.1 are satisfied. In what follows we will assume, without loss of generality that

$\|u_0\|_{W^{m,p}} \leq M$, for some deterministic constant $M > 0$. Indeed, after we obtain global existence in this special case, the general case, $u_0 \in X_{m,p}$ a.s, follows from a cutting argument as in Section 6; see (6.26)–(6.27).

Define the collection of stopping times

$$\tau_R := \inf\{t \geq 0 : \|u(t)\|_{L^2}^2 + \|w(t)\|_{L^\infty} > R\} \wedge \xi, \tag{8.19}$$

where we recall that $w = \operatorname{curl} u$. Obviously, τ_R is increasing in R, almost surely. We need to verify that (C.3) is satisfied. In other words, we need to show

$$\mathbb{P}\Big(\bigcap_R \{\tau_R < T \wedge \xi\}\Big) = 0 \tag{8.20}$$

for every $T > 0$. For this purpose we make use of the conclusions of Lemma 8.1. Owing to the fact that τ_R is increasing in R and (8.7), we infer

$$\begin{aligned}
\mathbb{P}\Big(\bigcap_{R>0} \{\tau_R < T \wedge \xi\}\Big) &= \lim_{R^* \to \infty} \mathbb{P}\Big(\bigcap_{0<R\leq R*} \{\tau_R < T \wedge \xi\}\Big) \\
&= \lim_{R^* \to \infty} \mathbb{P}(\tau_{R^*} < T \wedge \xi) \\
&\leq \lim_{R^* \to \infty} \mathbb{P}\Big(\sup_{t\in[0,T\wedge\xi]} \big(\|u\|_{L^2}^2 + \|w\|_{L^\infty}\big) > R^*\Big) \\
&\leq \mathbb{P}\Big(\bigcap_{R^*>0} \Big\{\sup_{t\in[0,T\wedge\xi]} \big(\|u\|_{L^2}^2 + \|w\|_{L^\infty}\big) > R^*\Big\}\Big) = 0
\end{aligned}$$

for every $T > 0$.

Returning to the a priori estimates (8.1)–(8.3), we now define the quantities

$$Y = 1 + \|u\|_{W^{m,p}}^p, \qquad \eta = \big(1 + \|\sigma\|_{\mathbb{W}^{m,p}}\big)^p. \tag{8.21}$$

Of course, Y satisfies $dY = X\,dt + Z\,d\mathcal{W}$. Combining (8.2), (8.4) and the definition of τ_R, we find that for each R there exists a deterministic constant K_R such that on $[0, \tau_R]$ we have

$$\begin{aligned}
|X| &\leq C\bigg(1 + \|u\|_{L^2} + \|w\|_{L^\infty}\bigg(1 + \log^+\bigg(\frac{\|u\|_{W^{m,p}}}{\|w\|_{L^\infty}}\bigg)\bigg)\bigg)\|u\|_{W^{m,p}}^p \\
&\quad + C\|\sigma\|_{\mathbb{W}^{m,p}}^p \\
&\leq C\big(2 + R^{1/2} + R + \|w\|_{L^\infty} \log^+ \|u\|_{W^{m,p}}\big)Y + C\|\sigma\|_{\mathbb{W}^{m,p}}^p \\
&\leq K_R(1 + \log Y)Y + C\big(1 + \|\sigma\|_{\mathbb{W}^{m,p}}\big)^p,
\end{aligned} \tag{8.22}$$

and from (8.3) we in addition obtain

$$\|Z\|_{L_2} \leq C\|\sigma\|_{\mathbb{W}^{m,p}}\|u\|_{W^{m,p}}^{p-1} \leq C\big(1 + \|\sigma\|_{\mathbb{W}^{m,p}}\big)Y^{(p-1)/p}. \tag{8.23}$$

We now have all the ingredients need to apply Lemma C.1. More precisely we take Y and η according to (8.21), $r = 1/p$, ξ as the maximal time of existence of u and τ_R according to (8.19). Having established (8.20)–(8.23) and recalling the standing assumption (3.4) we infer from Lemma C.1 that indeed $\xi = \infty$. The proof of Theorem 4.4 is therefore complete. $\square$

9. Global existence for linear multiplicative noise. Here we consider the stochastic Euler equations in two and three dimensions, with *linear multiplicative noise*

$$du + P(u \cdot \nabla u)\,dt = \alpha u\,dW, \tag{9.1}$$

where in this case $\alpha \in \mathbb{R}$, and W is a *single* $1D$ *Brownian motion*. This forcing regime is covered under the theory developed in the previous sections, so we are guaranteed the existence of a local pathwise solution in the sense of Definition 4.1; cf. Theorem 4.3.

As in the case of an additive noise above we may transform (9.1) to an random PDE. To this end consider the (real valued) stochastic process

$$\gamma(t) = e^{-\alpha W_t}. \tag{9.2}$$

Due to the Itô formula we find the γ satisfies

$$d\gamma = -\alpha\gamma\,dW + \tfrac{1}{2}\alpha^2\gamma\,dt, \qquad \gamma(0) = 1.$$

By apply the Itô product rule we therefore find that

$$\begin{aligned} d(\gamma u) &= \gamma\,du + u\,d\gamma + d\gamma\,du \\ &= -\gamma P(u \cdot \nabla u)\,dt + \alpha\gamma u\,dW - \alpha\gamma u\,dW + \tfrac{1}{2}\alpha^2\gamma u\,dt - \alpha^2\gamma u\,dt \\ &= -\gamma P(u \cdot \nabla u)\,dt - \tfrac{1}{2}\alpha^2(\gamma u)\,dt. \end{aligned} \tag{9.3}$$

By defining $v = \gamma u$ we therefore obtain the system

$$\partial_t v + \frac{\alpha^2}{2} v + \gamma^{-1} P(v \cdot \nabla v) = 0, \tag{9.4}$$

$$v(0) = u_0. \tag{9.5}$$

Fix $p \geq 2$, and $m > d/p + 1$ throughout the rest of this section. First, using the standard estimates on the nonlinear term [cf. (5.4) for $p = 2$, or (5.10) for $p > 2$], we may obtain

$$\frac{d}{dt}\|v\|_{W^{m,p}} + \frac{\alpha^2}{2}\|v\|_{W^{m,p}} \leq C_1\gamma^{-1}\|v\|_{W^{1,\infty}}\|v\|_{W^{m,p}} \tag{9.6}$$

for a positive constant $C_1 = C_1(m, p, \mathcal{D})$. In order to bound the right-hand side of (9.6) we recall the Beale–Kato–Majda-type inequality [cf. (8.4)]

$$\|v\|_{W^{1,\infty}} \leq C_2\|v\|_{L^2} + C_2\|w\|_{L^\infty}\left(1 + \log^+\left(\frac{\|v\|_{W^{m,p}}}{\|w\|_{L^\infty}}\right)\right), \tag{9.7}$$

where the constant $C_2 = C_2(m, p, \mathcal{D})$ is fixed, and as usual $w = \operatorname{curl} v$. Due to the cancellation property $(P(v \cdot \nabla v), v) = 0$, it follows directly from (9.4) that

$$\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} e^{-\alpha^2 t/2} \tag{9.8}$$

for all $t \geq 0$. On the other hand, obtaining an a priori estimate on $\|w(t)\|_{L^\infty}$ is more delicate. For this purpose, we return to (9.4) and consider the equation satisfied by $w = \operatorname{curl} v$, that is,

$$\partial_t w + \frac{\alpha^2}{2} w + \gamma^{-1} v \cdot \nabla w = \begin{cases} 0, & \text{for } d = 2, \\ \gamma^{-1} w \cdot \nabla v, & \text{for } d = 3. \end{cases} \tag{9.9}$$

Multiplying (9.9) by $w|w|^{p-2}$, integrating in x and making use of the divergence-free nature of v, we obtain

$$\frac{1}{p} \frac{d}{dt} \|w\|_{L^p}^p + \frac{\alpha^2}{2} \|w\|_{L^p}^p \leq \begin{cases} 0, & \text{for } d = 2, \\ \gamma^{-1} \|v\|_{W^{1,\infty}} \|w\|_{L^p}^p, & \text{for } d = 3. \end{cases}$$

Upon canceling $\|w\|_{L^p}^{p-1}$ and sending $p \to \infty$ in the above estimate, we have

$$\frac{d}{dt} \|w\|_{L^\infty} + \frac{\alpha^2}{2} \|w\|_{L^\infty} \leq \begin{cases} 0, & \text{for } d = 2, \\ \gamma^{-1} \|v\|_{W^{1,\infty}} \|w\|_{L^\infty}, & \text{for } d = 3. \end{cases} \tag{9.10}$$

In view of the different bounds obtained in (9.10) in $2D$ versus $3D$, we now treat the two cases separately. For this purpose it is convenient to first fix the Sobolev embedding constant $C_3 = C_3(m, p, \mathcal{D})$ such that

$$\|v\|_{L^2} + \|v\|_{W^{1,\infty}} \leq C_3 \|v\|_{W^{m,p}} \tag{9.11}$$

and to let $\bar{C} = C_1 C_2 + C_3 + 1$.

9.1. *The two-dimensional case.* In two dimensions we prove the global in time existence of smooth pathwise solutions, as stated in Theorem 4.6. From (9.10) we immediately obtain that the function

$$z(t) = \|w(t)\|_{L^\infty} \exp\left(\frac{\alpha^2 t}{2}\right)$$

is such that

$$z(t) \leq z(0) = \|w_0\|_{L^\infty} \tag{9.12}$$

for all $t \geq 0$. Therefore, letting

$$y(t) = \|v(t)\|_{W^{m,p}} \exp\left(\frac{\alpha^2 t}{2}\right)$$

we obtain from (9.6)–(9.8) and (9.12) that

$$\begin{aligned}\frac{dy}{dt} &\le \bar{C}\gamma^{-1}y\biggl(\|v(t)\|_{L^2}\\ &\qquad + \|w(t)\|_{L^\infty}\biggl(1+\log^+\biggl(\frac{y(t)}{\|w(t)\|_{L^\infty}\exp(\alpha^2 t/2)}\biggr)\biggr)\biggr)\\ &\le \bar{C}\gamma^{-1}\exp\biggl(-\frac{\alpha^2 t}{2}\biggr)y\biggl(\|v_0\|_{L^2}+\|w_0\|_{L^\infty}+z\log^+\biggl(\frac{y}{z}\biggr)\biggr).\end{aligned}\tag{9.13}$$

A short computation reveals that $z\log^+(y/z) \le 1/e + z\log^+(y)$. In view of (9.12), and defining $\rho_\alpha(t) = \exp(\alpha W_t - \alpha^2 t/2)$ estimate (9.13) gives

$$\frac{dy}{dt} \le \bar{C}\rho_\alpha y\bigl(\|v_0\|_{L^2} + \|w_0\|_{L^\infty} + 1 + \|w_0\|_{L^\infty}\log^+(y)\bigr).\tag{9.14}$$

By the law of iterated logarithms we have $\sup_{t\ge 0}\rho_\alpha < \infty$ a.s. for every $\alpha > 0$. Hence, (9.14) implies

$$\frac{dy}{dt} \le Ay\bigl(1+\log^+(y)\bigr),\tag{9.15}$$

where

$$A = \bar{C}\Bigl(\sup_{t\ge 0}\rho_\alpha\Bigr)\bigl(\|v_0\|_{L^2} + \|w_0\|_{L^\infty} + 1\bigr).\tag{9.16}$$

Let $Y(t) = \log(1+y(t))$. We obtain from (9.15) that

$$\frac{dY}{dt} \le A\bigl(1+Y(t)\bigr)$$

for all $t \ge 0$. This gives $Y(t) \le Y(0)\exp(tA) + tA\exp(tA)$, and hence

$$y(t) \le (1+y_0)^{\exp(tA)}\exp\bigl(tA\exp(tA)\bigr).\tag{9.17}$$

Recalling the definition of $y(t)$, we note that $\|u(t)\|_{W^{m,p}} = \gamma^{-1}(t)y(t) \times \exp(-\alpha^2 t/2) = \rho_\alpha(t)y(t)$. Thus, estimate (9.17) shows that

$$\|u(t)\|_{W^{m,p}} \le \rho_\alpha(t)\bigl(1+\|u_0\|_{W^{m,p}}\bigr)^{\exp(tA)}\exp\bigl(tA\exp(tA)\bigr)$$

with A as defined in (9.16). Therefore, for all $T > 0$ we have proven

$$\sup_{t\in[0,T\wedge\xi]}\|u\|_{W^{1,\infty}} < \infty \qquad \text{a.s.}$$

So that necessarily (u,ξ) is a global pathwise solution, that is, we have $\xi = \infty$; cf. Definition 4.2. We have thus now established part (i) of Theorem 4.6.

9.2. *The three-dimensional case.* Fix $\alpha > 0$. Let (u, ξ) be the maximal strong solution of (9.1). As in the two-dimensional case, the key ingredient to global regularity is an a priori bound on $\|w\|_{L^\infty}$. However, due to the presence of the vortex stretching term, in the three-dimensional case we have [cf. (9.10) above]

$$\frac{d}{dt}\|w\|_{L^\infty} + \frac{\alpha^2}{2}\|w\|_{L^\infty} \leq \gamma^{-1}\|v\|_{W^{1,\infty}}\|w\|_{L^\infty}. \tag{9.18}$$

To exploit the damping in (9.18), we now define the stopping time

$$\sigma = \inf_{t\geq 0}\left\{t : \gamma^{-1}(t)\|v(t)\|_{W^{m,p}} \geq \frac{\alpha^2}{4\bar{C}}\right\} = \inf_{t\geq 0}\left\{t : \|u(t)\|_{W^{m,p}} \geq \frac{\alpha^2}{4\bar{C}}\right\}, \tag{9.19}$$

where $\bar{C} \geq 1$ is the constant defined above (9.11). Note that $\sigma < \xi$ on the set $\{\xi < \infty\}$; cf. (4.3) and the Sobolev embedding. In order to ensure that $\sigma > 0$ a.s. we will at least need to impose the condition

$$\|u_0\|_{W^{m,p}} < \frac{\alpha^2}{4\bar{C}}. \tag{9.20}$$

In fact, in order to close the estimates we shall impose additional assumptions on u_0; cf. (9.31) below.

Due to the Sobolev embedding, on $[0, \sigma]$ we have

$$\gamma^{-1}\|w\|_{L^\infty} \leq \gamma^{-1}\|v\|_{W^{1,\infty}} \leq \frac{\alpha^2}{4}. \tag{9.21}$$

Hence, by (9.18) and (9.21) we obtain

$$\frac{d}{dt}\|w\|_{L^\infty} + \frac{\alpha^2}{4}\|w\|_{L^\infty} \leq 0 \tag{9.22}$$

on $[0, \sigma)$. Therefore, letting

$$z(t) = \|w(t)\|_{L^\infty}\exp\left(\frac{\alpha^2 t}{4}\right),$$

we find from (9.21) and (9.22) that

$$z(t) \leq z(0) = \|w_0\|_{L^\infty} \leq \frac{\alpha^2}{4}, \tag{9.23}$$

where we also used that $\gamma(0) = 1$. Similar to above, we now let

$$y(t) = \|v(t)\|_{W^{m,p}}\exp\left(\frac{\alpha^2 t}{4}\right). \tag{9.24}$$

By (9.6) and (9.7) we obtain

$$\frac{dy}{dt} \leq \bar{C}\gamma^{-1}y\left(\|v\|_{L^2} + \|w\|_{L^\infty}\left(1 + \log^+\left(\frac{y}{z}\right)\right)\right).$$

Using the decay of $\|v(t)\|_{L^2}$ obtained in (9.8), and assumption (9.20), the above estimate implies

$$
\begin{aligned}
\frac{dy}{dt} &\le \bar{C}\gamma^{-1}\exp\Big(-\frac{\alpha^2 t}{4}\Big)y\Big(\|u_0\|_{L^2}+z\Big(1+\log^+\Big(\frac{y}{z}\Big)\Big)\Big)\\
&\le \bar{C}\rho_\alpha\exp\Big(-\frac{\alpha^2 t}{8}\Big)y\Big(\frac{\alpha^2}{4}+z+z\log^+\Big(\frac{y}{z}\Big)\Big),
\end{aligned}
\tag{9.25}
$$

where we now denote

$$
\rho_\alpha(t)=\gamma^{-1}(t)\exp\Big(-\frac{\alpha^2 t}{8}\Big)=\exp\Big(\alpha W_t-\frac{\alpha^2 t}{8}\Big). \tag{9.26}
$$

To simplify the right-hand side of (9.25), it is convenient to observe that

$$
\frac{\alpha^2}{4}+z+z\log^+\Big(\frac{y}{z}\Big)\le \bar{C}+\alpha^2+z\log y \tag{9.27}
$$

holds whenever $0<z\le\alpha^2/4$, and $z\le\bar{C}y$ [note that we indeed have these a priori bounds on z, due to (9.11) and (9.23)]. In order to prove (9.27) we distinguish two cases: $z<y$ and $z/\bar{C}\le y\le z$. If $z<y$, then $\log^+(y/z)=\log(y/z)=\log(y)-\log(z)$. Hence the left-hand side of (9.27) is bounded by

$$
\frac{\alpha^2}{2}+z\log(y)-\mathbb{1}_{z\in(0,1]}z\log(z)\le\alpha^2+z\log y+\bar{C},
$$

where we have used the fact that $0\le -z\log(z)\le 1/e\le\bar{C}$ for all $z\in(0,1]$. This concludes the proof of (9.27) for $y>z$. On the other hand, if $y\le z$, then $\log^+(y/z)=0$, and hence we need to prove that $\alpha^2/4+z$ is less than the left-hand side of (9.27). For this purpose, it is sufficient to prove that

$$
\bar{C}+z\log y\ge 0
$$

for all $y\in[z/\bar{C},z]$ and all $z>0$. Indeed, the right-hand side of the above inequality is monotone increasing in y, so the minimum is attained at $y=z/\bar{C}$, and it equals $\bar{C}+z\log(z/\bar{C})$. A simple calculation shows that $\bar{C}+z\log(z/\bar{C})\ge\bar{C}-\bar{C}/e>0$, for all $z\ge 0$, concluding the proof of (9.27).

Therefore, by (9.25) and (9.27) we have

$$
\frac{dy}{dt}\le\bar{C}\rho_\alpha\exp\Big(-\frac{\alpha^2 t}{8}\Big)y(\bar{C}+\alpha^2+z\log y). \tag{9.28}
$$

Fix any $R\ge 1$ and define the stopping time

$$
\tau_R=\inf\{t\ge 0:\rho_\alpha(t)\ge R\}. \tag{9.29}
$$

From (9.28) we obtain the bound

$$
\frac{dy}{dt}\le\bar{C}R\exp\Big(-\frac{\alpha^2 t}{8}\Big)y(\bar{C}+\alpha^2+z\log y) \tag{9.30}
$$

for all $t \in [0, \tau_R \wedge \sigma]$. We now may apply Lemma B.1, which is a suitable version of the logarithmic Grönwall inequality. Lemma B.1 guarantees the existence of a positive deterministic function $\kappa(R,\alpha)$ with the following properties:

$$\begin{aligned} \kappa(R,\alpha) &\leq \frac{\alpha^2}{8\bar{C}} && \text{for every } R \geq 1; \\ \lim_{R\to\infty} \kappa(R,\alpha) &= 0 && \text{for every fixed } \alpha \neq 0; \\ \lim_{\alpha^2\to\infty} \kappa(R,\alpha) &= \infty && \text{for every fixed } R \geq 1; \\ \lim_{\alpha^2\to 0} \kappa(R,\alpha) &= 0 && \text{for every fixed } R \geq 1; \end{aligned}$$

such that if the initial data satifies

$$\|u_0\|_{W^{m,p}} = y(0) \leq \kappa(R,\alpha), \tag{9.31}$$

then a smooth solution of (9.30) satisfies

$$y(t) \leq \frac{\alpha^2}{8R\bar{C}} \tag{9.32}$$

for all $t \in [0, \tau_R \wedge \sigma]$. For clarity of the presentation, we postpone the precise formula for the function $\kappa(R,\alpha)$ and the proof that (9.31) implies (9.32) to Appendix B below.

Note that the condition (9.31) imposed on the initial data automatically implies (9.20), and hence $\sigma > 0$. Recalling the definition of $y(t)$ and $\rho_\alpha(t)$ in (9.24) and (9.26) we obtain from (9.32) that for every t in the interval $[0, \sigma \wedge \tau_R]$

$$\begin{aligned} \|u(t)\|_{W^{m,p}} &= \gamma^{-1}(t)\|v(t)\|_{W^{m,p}} = \exp\left(-\frac{\alpha^2 t}{8}\right)\rho_\alpha(t) y(t) \leq R\frac{\alpha^2}{8R\bar{C}} \\ &= \frac{\alpha^2}{8\bar{C}}. \end{aligned} \tag{9.33}$$

Hence, due to the definition of σ [cf. (9.19)], bound (9.33) shows that $\sigma \wedge \tau_R = \tau_R$. Therefore

$$\sup_{t\in[0,\tau_R]} \|u(t)\|_{W^{1,\infty}} \leq C_3 \sup_{t\in[0,\tau_R]} \|u(t)\|_{W^{m,p}} \leq \frac{\alpha^2}{8},$$

which implies that $\xi \geq \tau_R$. Therefore, the maximal pathwise solution (u, ξ) of (9.1) is global in time on the set $\{\tau_R = \infty\}$, that is, on the set where $\rho_\alpha(t)$ always stays below R; cf. (9.29). We now claim that

$$\mathbb{P}(\tau_R = \infty) \geq 1 - \frac{1}{R^{1/4}} \tag{9.34}$$

holds, for any $R > 1$. Note carefully that this lower bound in (9.34) is independent of α. Thus if we wish to obtain that the local pathwise solution is global in time with high probability, that is,

$$\mathbb{P}(\xi = \infty) = 1 - \varepsilon$$

for some $\varepsilon \in (0, 1)$, it is sufficient to choose R so that

$$\frac{1}{\varepsilon^4} \le R \tag{9.35}$$

and for this fixed R, consider an initial data u_0 which satisfies $\|u_0\|_{W^{m,p}} \le \kappa(R, \alpha)$. Alternatively for this R and a *given* (deterministic) initial data $\|u_0\|_{W^{m,p}}$ we may choose α^2 sufficiently large so that $\|u_0\|_{W^{m,p}} \le \kappa(R, \alpha)$ to guarantee that the associated (u, ξ) is global with probability $1 - \varepsilon$. The proof of Theorem 4.6(ii), is now complete, modulo a proof of (9.34), which we give next.

In order to estimate $\mathbb{P}(\tau_R = \infty)$, letting $\mu = \frac{3\alpha^2}{8}$ we observe that

$$\rho_\alpha(t) = \exp\left(\left(\mu - \frac{\alpha^2}{2}\right)t + \alpha W_t\right)$$

is a geometric Brownian motion, the solution of

$$dx = \mu x\, dt + \alpha x\, dW, \qquad x(0) = 1, \tag{9.36}$$

where W is a standard $1 - D$ Brownian motion. The following lemma, with $\mu = \frac{3\alpha^2}{8}$, proves estimate (9.34), and by the above discussion it concludes the proof of Theorem 4.6.

LEMMA 9.1 (Estimates for the exit times of geometric Brownian motion). *Suppose that $\mu < \frac{\alpha^2}{2}$ and $x_0 > 0$ and is deterministic. Let $x(t)$ be the solution of (9.36) and for $R > 1$ define τ_R as*

$$\tau_R = \inf\{t \ge 0 : x(t) > R\}. \tag{9.37}$$

Then we have

$$\mathbb{P}(\tau_R = \infty) \ge 1 - \left(\frac{1}{R}\right)^{1 - 2\mu/\alpha^2}. \tag{9.38}$$

PROOF OF LEMMA 9.1. For $\lambda > 0$ we apply the Itô formula for $f(x) = x^\lambda$ and obtain that

$$dx^\lambda = \lambda x^{\lambda-1}\, dx + \frac{\lambda(\lambda-1)}{2} x^{\lambda-2}\, dx\, dx = \left(\mu\lambda + \frac{\alpha^2\lambda(\lambda-1)}{2}\right)x^\lambda\, dt + \alpha\lambda x^\lambda\, dW.$$

Integrating up to any time $t \wedge \tau_R$ and taking an expected value, we find that

$$\mathbb{E}x^\lambda(t \wedge \tau_R) = 1 + \mathbb{E}\int_0^{t\wedge\tau_R} \left(\mu\lambda + \frac{\alpha^2\lambda(\lambda-1)}{2}\right)x^\lambda\, ds.$$

Taking $\lambda = \lambda_c = 1 - \frac{2\mu}{\alpha^2}$ in the above expression, we find that

$$\mathbb{E}x^{\lambda_c}(t \wedge \tau_R) = 1.$$

Now, using that τ_R is increasing in R and the continuity of measures, we get

$$\begin{aligned}\mathbb{P}(\tau_R = \infty) &= \mathbb{P}\Big(\bigcap_n \{\tau_R > n\}\Big)\\ &= \lim_{N\to\infty} \mathbb{P}(\tau_R > N) = \lim_{N\to\infty} \mathbb{P}\big(x^{\lambda_c}(N \wedge \tau_R) < R^{\lambda_c}\big)\\ &\geq \lim_{N\to\infty}\Big(1 - \frac{\mathbb{E}x^{\lambda_c}(N \wedge \tau_R)}{R^{\lambda_c}}\Big) = 1 - \frac{1}{R^{\lambda_c}},\end{aligned}$$

which concludes the proof of the lemma. $\square$

APPENDIX A: THE SMOOTHING OPERATOR AND ASSOCIATED PROPERTIES

In this Appendix we define and review some basic properties of a class of smoothing operators F_ε as used in [35]. These mollifiers are used to construct solutions in $W^{m,p}$ in Section 7 above.

For every $\varepsilon > 0$, let $\widetilde{F}_\varepsilon$ be a standard mollifier on $\mathbb{R}^d$; for instance, consider $\widetilde{F}_\varepsilon$ to be the convolution against the inverse Fourier transform of $\exp(-\varepsilon|\xi|^2)$. Assuming $\partial\mathcal{D}$ is sufficiently smooth, there exists (see, e.g., [1], Chapter 5) a linear extension operator E from $\mathcal{D}$ to $\mathbb{R}^d$, that is, $Eu(x) = u(x)$ a.e. in $\mathcal{D}$, and $\|Eu\|_{W^{m,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{m,p}(\mathcal{D})}$ for $m \geq 0$, and all $2 \leq p < \infty$. We also take R to be a restriction operator, which is bounded from $W^{m,p}(\mathbb{R}^d)$ into $W^{m,p}(\mathcal{D})$ for $m \geq 0$ and all $p \geq 2$. We let P be the Leray projection operator as defined in Section 2. We finally define the smoothing operators F_ε by

$$F_\varepsilon = PR\widetilde{F}_\varepsilon E \tag{A.1}$$

for every $\varepsilon > 0$. We have the following basic properties for F_ε.

LEMMA A.1 (Properties of the smoothing operator). *Suppose that $m \geq 0$, and $p \geq 2$. For every $\varepsilon > 0$ the operator F_ε maps $X_{m,p}$ into $X_{m'}$, where $m' = m + 5$. Moreover the following properties hold*:

(i) *The collection F_ε is uniformly bounded on $X_{m,p}$ independently of ε*

$$\|F_\varepsilon u\|_{W^{m,p}} \leq C\|u\|_{W^{m,p}}, \qquad u \in X_{m,p}, \tag{A.2}$$

where $C = C(m, p, \mathcal{D})$ is a universal constant independent of $\varepsilon > 0$.

(ii) *For every* $\varepsilon > 0$, *when* $m \geq 1$ *we have*

$$\|F_\varepsilon u\|_{W^{m,p}} \leq \frac{C}{\varepsilon}\|u\|_{W^{m-1,p}}, \qquad u \in X_{m,p} \tag{A.3}$$

and

$$\|F_\varepsilon u - u\|_{W^{m-1,p}} \leq C\varepsilon\|u\|_{W^{m,p}}, \qquad u \in X_{m,p}, \tag{A.4}$$

where $C = C(m, p, \mathcal{D})$ *is a universal constant independent of* $\varepsilon > 0$.

(iii) *The sequence of mollifications* $F_\varepsilon u$ *converge to* u, *for every* u *in* $X_{m,p}$, *that is*,

$$\lim_{\varepsilon \to 0} \|F_\varepsilon u - u\|_{W^{m,p}} = 0, \tag{A.5}$$

and when $m \geq 1$ *we also have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\|F_\varepsilon u - u\|_{W^{m-1,p}} = 0. \tag{A.6}$$

(iv) *The convergence of* $F_\varepsilon u$ *to* u *is uniform over compact subsets of* $X_{m,p}$. *In particular if* $\{u^k\}_{k\geq 1}$ *is a sequence of functions in* $X_{m,p}$ *which converge in* $X_{m,p}$, *then we have*

$$\lim_{\varepsilon \to 0} \sup_{k \geq 1} \|F_\varepsilon u^k - u^k\|_{W^{m,p}} = 0 \tag{A.7}$$

and

$$\lim_{\varepsilon \to 0} \sup_{k \geq 1} \frac{1}{\varepsilon}\|F_\varepsilon u^k - u^k\|_{W^{m-1,p}} = 0, \tag{A.8}$$

when $m \geq 1$.

The above properties hold for F_ε, since they hold for the standard mollifier $\widetilde{F}_\varepsilon$ on $\mathbb{R}^d$, we have that R and E are bounded maps between the relevant Sobolev spaces and $RE = \mathrm{Id}_{\mathcal{D}}$ a.e. For further details, see, for instance, [1, 35].

APPENDIX B: A TECHNICAL LEMMA ABOUT ODES

In this Appendix we give the proof of a technical lemma which was used in proving the $3D$ case of Theorem 4.6, in Section 9.2 above. The *raison d'être* of the below lemma is to very carefully keep track of the dependence on α for all constants involved. This enables us to control the quantities involved as the parameter α is sent to either 0 or ∞.

LEMMA B.1. *Let* $\bar{C} \geq 1$ *be a universal constant. Fix the parameters* $R \geq 1$, $\alpha \neq 0$ *and* $T > 0$. *For* $y_0 > 0$, *let* $y(t)$ *be a positive smooth function satisfying*

$$\frac{dy}{dt}(t) \leq \bar{C}R\exp\left(-\frac{\alpha^2 t}{8}\right)y(t)\big(\bar{C} + \alpha^2 + z(t)\log y(t)\big), \tag{B.1}$$

$$y(0) = y_0, \tag{B.2}$$

where $z(t)$ is a given continuous function such that $0 < z(t) \leq \alpha^2/4$ for all $t \in [0, T]$. There exits a positive function $K(R, \alpha) \geq 2$ such that if

$$y_0 \leq \frac{\alpha^2}{4\bar{C}K(R,\alpha)}, \tag{B.3}$$

then we have

$$y(t) \leq \frac{K(R,\alpha)}{2R} y_0 \leq \frac{\alpha^2}{8R\bar{C}} \tag{B.4}$$

for all $t \in [0, T)$. This function $K(R, \alpha)$ may be chosen explicitly as

$$K(R,\alpha) = 2R\left(1 + \left(\frac{\alpha^2}{8\bar{C}}\right)^{(1-1/(8(D_R-1)))}\right) \exp\left(\frac{8\bar{C}RD_R(\bar{C}+\alpha^2)}{\alpha^2}\right), \tag{B.5}$$

where we have denoted $D_R = \exp(4\bar{C}R)$. Additionally, for every fixed $R \geq 1$ we obtain the asymptotic behavior for the function

$$\kappa(R,\alpha) = \frac{\alpha^2}{2\bar{C}K(R,\alpha)}$$

to be

$$\lim_{\alpha^2\to\infty} \kappa(R,\alpha) = \lim_{\alpha^2\to\infty} \frac{\alpha^2}{K(R,\alpha)} = \infty, \tag{B.6}$$

$$\lim_{\alpha^2\to 0} \kappa(R,\alpha) = \lim_{\alpha^2\to 0} \frac{\alpha^2}{K(R,\alpha)} = 0. \tag{B.7}$$

PROOF OF LEMMA B.1. For ease of notation, let $a(t) = \bar{C}R\exp(-\alpha^2 t/8)$. After letting $Y(t) = \log y(t)$, inequality (B.1) reads

$$\frac{dY(t)}{dt} \leq a(t)\big((\bar{C}+\alpha^2) + z(t)Y(t)\big) \tag{B.8}$$

with initial condition $Y(0) = \log y_0$. The initial value problem associated to (B.8) leads to the bound

$$\begin{aligned} Y(t) &\leq Y(0)\exp\left(\int_0^t a(s)z(s)\,ds\right) \\ &\quad + (\bar{C}+\alpha^2)\int_0^t a(s)\exp\left(\int_s^t a(s')z(s')\,ds'\right)ds \\ &\leq Y(0)\exp\left(\int_0^t a(s)z(s)\,ds\right) + (\bar{C}+\alpha^2)\exp(2\bar{C}R)\int_0^t a(s)\,ds \\ &\leq Y(0)\exp\left(\int_0^t a(s)z(s)\,ds\right) + \frac{8\bar{C}R(\bar{C}+\alpha^2)\exp(2\bar{C}R)}{\alpha^2}, \end{aligned} \tag{B.9}$$

where we used the a priori bound $z \le \alpha^2/4$ and the identity $\int_0^\infty a(t)\,dt = 8\bar{C}R/\alpha^2$. By exponentiation it follows that

$$y(t) \le y_0^{\exp(\int_0^t a(s)z(s)\,ds)} \exp\left(\frac{8\bar{C}R(\bar{C}+\alpha^2)\exp(2\bar{C}R)}{\alpha^2}\right). \tag{B.10}$$

We note that if $y_0 \le 1$, since $\exp(\int_0^t a(s)z(s)\,ds) \ge 1$, we have

$$y_0^{\exp(\int_0^t a(s)z(s)\,ds)} \le y_0. \tag{B.11}$$

On the other hand, if $y_0 > 1$, due to (B.3) we may bound

$$y_0 \le \frac{\alpha^2}{M}, \tag{B.12}$$

whenever $M \le 4\bar{C}K$. Hence, recalling the a priori bound on $z(t)$ and integrating $a(t)$ from 0 to ∞, we obtain from (B.10) and (B.12) that

$$y_0^{\exp(\int_0^t a(s)z(s)\,ds)} \le y_0^{D_R} \le y_0 y_0^{D_R-1} \le y_0\left(\frac{\alpha^2}{M}\right)^{D_R-1} \tag{B.13}$$

for $y_0 > 1$, since $D_R = \exp(2\bar{C}R) \ge 3$. Hence, we obtain from (B.10), (B.11) and (B.13) that

$$\begin{aligned} y(t) &\le y_0 \frac{1}{2R}\left(2R\left(1+\left(\frac{\alpha^2}{M}\right)^{D_R-1}\right)\exp\left(\frac{8\bar{C}C_* D_R(\bar{C}+\alpha^2)}{\alpha^2}\right)\right) \\ &=: y_0 \frac{1}{2R}\bar{K}(M). \end{aligned} \tag{B.14}$$

The proof of (B.4) is complete if we show that $\bar{K}(M) \le K$ for all $\alpha > 0$, for some M is chosen such that $M \le 4\bar{C}K$. We now let

$$M = 8\bar{C}\mathbb{1}_{\alpha^2 \le 8\bar{C}} + \mathbb{1}_{\alpha^2 > 8\bar{C}}(8\bar{C})^{1/(2(D_R-1))}\alpha^{(2-1/(D_R-1))} \tag{B.15}$$

and define

$$K(R,\alpha) = 2R\left(1+\left(\frac{\alpha^2}{8\bar{C}}\right)^{(1-1/(8(D_R-1)))}\right)\exp\left(\frac{8\bar{C}RD_R(\bar{C}+\alpha^2)}{\alpha^2}\right).$$

Indeed, it is not hard to verify that for $R \ge 1$, and $\bar{C} \ge 1$, we have $4\bar{C}K \ge M$ for all $\alpha > 0$. Finally, to verify that the above defined K indeed is larger than $\bar{K}(M)$ [which was defined in (B.14)], it is sufficient to check that

$$\left(\frac{\alpha^2}{M}\right)^{D_R-1} \le \left(\frac{\alpha^2}{8\bar{C}}\right)^{(1-1/(8(D_R-1)))} \tag{B.16}$$

for all $\alpha > 0$. Indeed, (B.16) may be checked by a direct computation using (B.15) and $D_R \ge 3$.

Finally, one may directly check that for any fixed $R \geq 1$, as $\alpha \to \infty$ we have $K(R,\alpha) = O(\alpha^{2-1/(4(D_R-1))})$, and therefore $\alpha^2/K(R,\alpha) \to \infty$, as $\alpha \to \infty$, which concludes the proof of (B.6). To conclude, it is clear from the definition of $K(R\alpha)$ that it is larger than 2, and hence $\alpha^2/K(R,\alpha) \to 0$ as $\alpha \to 0$, which completes the proof of the lemma. □

APPENDIX C: A NONBLOWUP CONDITION FOR SDES WITH LINEAR-LOGARITHMIC GROWTH IN THE DRIFT

In this Appendix we state and prove a condition for the nonblow-up of solutions to SODEs via a logarithmic Grönwall-type argument; see, for example, [25] for related results.

LEMMA C.1. *Fix a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$. Suppose that on $\mathcal{S}$ we have defined Y a real valued, predictable process defined up to a blow-up time $\xi > 0$, that is, for all bounded stopping times $\tau < \xi$, $\sup_{t\in[0,\tau]} Y < \infty$ a.s. and*

$$\sup_{t\in[0,\xi)} Y = \infty \qquad \text{on the set } \{\xi < \infty\}.$$

Assume that $Y \geq 1$ and that on $[0,\xi)$, Y satisfies the Itô stochastic differential

$$dY = X\,dt + Z\,d\mathcal{W}, \qquad Y(0) = Y_0, \tag{C.1}$$

where on $[0,\xi)$, X, Z are, respectively, real valued and L_2 valued predictable processes, and Y_0 is $\mathcal{F}_0$ and bounded above by a deterministic constant $M > 0$.[7] *Suppose there exists a stochastic process*

$$\eta \in L^1(\Omega; L^1_{\mathrm{loc}}[0,\infty)) \tag{C.2}$$

with $\eta \geq 1$ for almost every (ω, t) and an increasing collection of stopping times τ_R with $\tau_R \leq \xi$ and such that

$$\mathbb{P}\biggl(\bigcap_{R>0} \{\tau_R < \xi \wedge T\}\biggr) = 0. \tag{C.3}$$

We further assume that for every fixed $R > 0$, there exists a deterministic constant K_R depending only on R (independent of t), and a number $r \in [0, 1/2]$ such that

$$|X| \leq K_R\bigl((1+\log Y)Y + \eta\bigr), \qquad \|Z\|_{L_2} \leq K_R Y^{1-r}\eta^r,$$

which holds over $[0,\tau_R]$. Then $\xi = \infty$ and in particular, $\sup_{t\in[0,T]} Y < \infty$, a.s. for every $T > 0$.

[7] This condition is not essential; we may merely assume that $Y_0 < \infty$, almost surely; see Remark C.2 below.

PROOF. As in [25], we introduce the functions

$$\zeta(x) = (1 + \ln x), \qquad \Psi(x) = \int_0^x \frac{1}{r\zeta(r) + 1}\,dr,$$
$$\Phi(x) = \exp(\Psi(x)). \tag{C.4}$$

By direct computation we find that

$$\Phi'(x) = \frac{\Phi(x)}{x\zeta(x) + 1}, \qquad \Phi''(x) = -\frac{\Phi(x)\zeta(x)}{(x\zeta(x) + 1)^2}.$$

Thus, by an application of the Itô lemma, we have

$$\begin{aligned} d\Phi(Y) &= \Phi'(Y)\,dY + \frac{1}{2}\,dY\,dY = \frac{\Phi(Y)}{Y\zeta(Y) + 1}X\,dt - \frac{1}{2}\frac{\Phi(Y)\zeta(Y)}{(Y\zeta(Y) + 1)^2}\|Z\|_{L_2}^2\,dt \\ &\quad + \frac{\Phi(Y)}{Y\zeta(Y) + 1}Z\,d\mathcal{W}. \end{aligned}$$

For $S > 0$ we define the stopping times

$$\zeta_S := \inf\{t \geq 0 : Y(t) > S\} \wedge \tau_R, \qquad \rho_S := \inf\left\{t \geq 0 : \int_0^t \eta\,ds > S\right\}.$$

In view of the definition of ξ, we have that $\lim_{S\to\infty} \zeta_S = \tau_R \wedge \xi$. Due to (C.2) we also have that $\lim_{S\to\infty} \rho_S = \infty$. Fix T, S_1, $S_2 > 0$. We estimate and any stopping times $0 \leq \tau_a \leq \tau_b \leq \zeta_{S_1} \wedge \rho_{S_2} \wedge T$

$$\begin{aligned} &\mathbb{E} \sup_{t\in[\tau_a,\tau_b]} \Phi(Y) \\ &\quad \leq \mathbb{E}\Phi(Y(\tau_a)) + \mathbb{E}\int_{\tau_a}^{\tau_b} \Phi(Y)\left(\frac{|X|}{Y\zeta(Y) + 1} + \frac{1}{2}\left|\frac{\zeta(Y)\|Z\|_{L_2}^2}{(Y\zeta(Y) + 1)^2}\right|\right)dt \\ &\qquad + \mathbb{E} \sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^t \frac{\Phi(Y)}{Y\zeta(Y) + 1}Z\,d\mathcal{W}\right| \\ &\quad \leq \mathbb{E}\Phi(Y(\tau_a)) + C\mathbb{E}\int_{\tau_a}^{\tau_b} \Phi(Y)(1 + \eta)\,dt \\ &\qquad + C\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\left(\frac{\Phi(Y)}{Y\zeta(Y) + 1}\right)^2\|Z\|_{L_2}^2\,dt\right)^{1/2} \\ &\quad \leq \mathbb{E}\Phi(Y(\tau_a)) + C\mathbb{E}\int_{\tau_a}^{\tau_b} \Phi(Y)(1 + \eta)\,dt + C\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \Phi(Y)^2\eta\,dt\right)^{1/2} \\ &\quad \leq \mathbb{E}\Phi(Y(\tau_a)) + C\mathbb{E}\int_{\tau_a}^{\tau_b} \Phi(Y)(1 + \eta)\,dt + \frac{1}{2}\mathbb{E} \sup_{t\in[\tau_a,\tau_b]} \Phi(Y), \end{aligned}$$

where C, depends on R through K_R and is independent of T, S_1, ξ, τ_a and τ_b. Rearranging and applying a stochastic version of the Grönwall lemma given in [32], Lemma 5.3, we find

$$\mathbb{E} \sup_{t\in[0,\sigma_{S_1}\wedge\rho_{S_2}\wedge T]} \Phi(Y) \leq C,$$

where here $C = C(R, T, S_2, M)$ and is independent of S_1 and ξ. Thus, sending $S_1 \to \infty$ and applying the monotone convergence theorem,

$$\mathbb{E} \sup_{t\in[0,\rho_{S_2}\wedge\tau_R\wedge T]} \Phi(Y) \leq C. \tag{C.5}$$

Thus, by the properties of Φ [cf. (C.4)] we infer

$$\sup_{t\in[0,\rho_{S_2}\wedge\tau_R\wedge T]} Y < \infty \qquad \text{for each } R, S_2 > 0,$$

on a set of full measure. Thus, since $\lim_{S_2\to\infty} \rho_{S_2} = \infty$ we infer that, for each $R > 0$, $\sup_{t\in[0,\tau_R\wedge T]} Y < \infty$, almost surely. In view of condition (C.3) imposed on the stopping times τ_R, this in turn implies $\sup_{t\in[0,\xi\wedge T]} Y < \infty$. Since T was also arbitrary to begin with, we have perforce $\xi = \infty$, almost surely. The proof is therefore complete. $\square$

REMARK C.2. In Lemma C.1 we may actually just assume that Y_0 is finite almost surely. Indeed if we define the sets $\Omega_M := \{Y_0 \leq M\}$ we infer, arguing similar to above that

$$\mathbb{E}\Big(\mathbb{1}_{\Omega_M} \sup_{t\in[0,\zeta_{S_1}\wedge\rho_{S_2}\wedge T]} \Phi(Y)\Big) \leq C_M.$$

We thus find that $\xi = \infty$ for almost every ω in $\bigcap_M \Omega_M$. Since this latter set is clearly of full measure, we may thus establish the proof of Lemma C.1 in this more general situation.

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