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RANDOM WALKS AT RANDOM TIMES: CONVERGENCE TO ITERATED LÉVY MOTION, FRACTIONAL STABLE MOTIONS, AND OTHER SELF-SIMILAR PROCESSES

BY PAUL JUNG[1] AND GREG MARKOWSKY[2]

University of Alabama at Birmingham and Monash University

For a random walk defined for a doubly infinite sequence of times, we let the time parameter itself be an integer-valued process, and call the orginal process a random walk at random time. We find the scaling limit which generalizes the so-called iterated Brownian motion.

Khoshnevisan and Lewis [*Ann. Appl. Probab.* **9** (1999) 629–667] suggested "the existence of a form of measure-theoretic duality" between iterated Brownian motion and a Brownian motion in random scenery. We show that a random walk at random time can be considered a random walk in "alternating" scenery, thus hinting at a mechanism behind this duality.

Following Cohen and Samorodnitsky [*Ann. Appl. Probab.* **16** (2006) 1432–1461], we also consider alternating random reward schema associated to random walks at random times. Whereas random reward schema scale to local time fractional stable motions, we show that the alternating random reward schema scale to indicator fractional stable motions.

Finally, we show that one may recursively "subordinate" random time processes to get new local time and indicator fractional stable motions and new stable processes in random scenery or at random times. When $\alpha = 2$, the fractional stable motions given by the recursion are fractional Brownian motions with dyadic $H \in (0, 1)$. Also, we see that "un-subordinating" via a time-change allows one to, in some sense, extract Brownian motion from fractional Brownian motions with $H < 1/2$.

1. Introduction. Let $B^{(i)}(t), i = 1, 2, 3$, be three independent Brownian motions, and let a two-sided Brownian motion be defined by

$$\tilde{B}(t) := \begin{cases} B^{(1)}(t), & \text{if } t \geq 0, \\ B^{(2)}(-t), & \text{if } t < 0. \end{cases} \tag{1}$$

In [4], Burdzy studied the process $(\tilde{B}(B^{(3)}(t)))_{t \geq 0}$ which he called an iterated Brownian motion (IBM). It can be thought of as a two-sided Brownian motion

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which is nonmonotonically "subordinated" to another Brownian motion. This process was also used by Deheuvels and Mason [7] to study the Bahadur–Kiefer process. Also, a variant of IBM, where the pure imaginary process $iB^{(2)}(-t)$ was substituted for $t < 0$, was utilized by Funaki [13] to study the PDE

$$\frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^4 u}{\partial x^4}. \tag{2}$$

Recently, more general *processes at random times* called α-time Brownian motions and α-time fractional Brownian motions were introduced in [23, 24]. In these works (along with several references therein), the connection between processes at random times and various PDEs was studied, along with the local time and path properties of the iterated processes. In a different direction, the scaling and asymptotic density of a discretized version of IBM called iterated random walk was analyzed in the physics literature [27].

In this work, we consider generalizations of the iterated random walk which we call *random walks at random times* (RWRT) and *dependent walks at random times* (DWRT) and relate them with a different portion of the probability literature concerning random walks in random scenery. This relation was first noted by Khoshnevisan and Lewis [19] who stated that there was "a surprising connection between the variations (of IBM) and H. Kesten and F. Spitzer's Brownian motion in random scenery." Later, in [20], a form of measure-theoretic duality was shown between the two processes. Here, we present a mechanism on the discrete level which shows a connection between the two processes.

We show that under suitable conditions, the scaling limits of RWRT and DWRT are (H-sssi)-time α-stable Lévy motions, a new class of processes at random times. If $X(t)$ is a two-sided α-stable Lévy motion defined similarly to (1), and Y_t is an independent α-stable Lévy motion, then we call $X(Y_t)$ an *iterated Lévy motion*. If, more generally, Y_t is an independent H-self-similar, stationary-increment process (sssi), then an (H-sssi)-time α-stable Lévy motion is given by $X(Y_t)$. Assuming $0 < H < 1$, we will see that $X(Y_t)$ is an H/α-sssi process with Hurst exponent less than $1/\alpha$. They naturally complement stable processes in random scenery which are the limiting continuous processes of [17] and [28] and which have Hurst exponents greater than $1/\alpha$ (Wang [28] considered only the case $\alpha = 2$, but this was extended to $\alpha < 2$ by Cohen and Dombry [5]).

Random walks in random scenery (RWRS) and their scaling limits, stable processes in random scenery, were first introduced independently in [2, 17]. The purpose of [17] was to introduce a new class of sssi processes given by the scaling limits of RWRS. The scaling limits have integral representations as stable integrals of local time kernels (of a process Y_t). When the random scenery are α-stable laws, they scale to the α-stable random measure against which the local time kernel is integrated. In comparison, there is also an integral representation of (H-sssi)-time α-stable Lévy motions given by the stable integration of random kernels of type $1_{[0,Y_t]}$ against α-stable random measures.

When Y_t is a generic H-sssi process, the stable processes in random scenery discussed above also include the model of [28]. Wang used "dependent walks" to collect the scenery, instead of random walks, leading to a dependent walk in random scenery (DWRS). In particular, the dependent walks he used were discrete-time Gaussian processes known to scale to fractional Brownian motion (fBm).

Random reward schema are sums of independent copies of discrete processes in random scenery. In [5, 6, 10] it was shown that the random reward schema of RWRS and DWRS scale to H-sssi symmetric α-stable (SαS) processes called *local time fractional SαS motions* (with $H > 1/\alpha$). In this work, we show that the scaling limits of random reward schema for RWRT and DWRT are H-sssi SαS processes called *indicator fractional SαS motions* (with $H < 1/\alpha$) which were introduced in [16].

Note that fBm is the only sssi Gaussian process. Thus, when the scenery has finite variance and $\alpha = 2$, local time fractional SαS motions and indicator fractional SαS motions reduce to fBm with $H > 1/2$ and $H < 1/2$, respectively.

As will be seen in Section 2, the mechanism behind the connection between local time and indicator fractional stable motions is the same as the mechanism which connects Brownian motion in random scenery (BMRS) with IBM. In effect, the mechanism shows that the indicator kernels of the latter processes can be thought of as "alternating" versions of the local time kernels of the former.

Together, local time fractional SαS motions and indicator fractional stable motions form a class of fractional stable motions (H-sssi SαS processes) which may be thought of as one of several generalizations of fractional Brownian motion. Their increment processes are stationary and have the ergodic-theoretic property of being null conservative, a concept introduced in [25]. This property distinguishes them from fractional stable motions which have dissipative or positive conservative increment processes. The most well-known examples of fractional stable motions with dissipative or positive conservative increment processes are the linear fractional stable motions and the real harmonizable stable motions, respectively, as can be seen in Figure 1.

We also consider single-scenery random reward schema introduced in [11]. Here we again take sums of identically distributed RWRTs or DWRTs. However, the copies have a dependence structure since they use the same "single scenery."

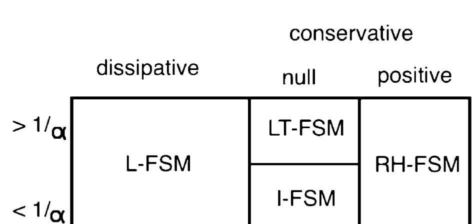

FIG. 1. $\alpha \in (1, 2)$: *LT = local time, I = indicator, L = linear, RH = real harmonizable.*

This dependence will be made more explicit below. The scaling limits of single-scenery random reward schema of RWRS and DWRS no longer have stationary increments; however, they are easily seen to be H-ss SαS processes with $H > 1/\alpha$. Similarly, the scaling limits of single-scenery random reward schema of RWRT and DWRT are H-ss SαS processes with $H < 1/\alpha$.

Finally, we also present a recursive construction of some local time and indicator fractional stable motions. In particular, we show that at each step of the recursion, the local times exist and are in $L^2(\Omega \times \mathbb{R})$. The recursively defined processes give the first examples of local time fractional stable motions for which the processes collecting the scenery are neither fBm nor β-stable Lévy motions. In the case $\alpha = 2$, the processes are given by integrals against Gaussian random measures, and the recursion constructs fBm, of any dyadic Hurst parameter, using one Brownian motion and a countable family of independent random Gaussian measures.

As mentioned above, RWRT and, in particular, its scaling limit are in some sense nonmonotonically subordinated processes. Usually one may not undo a subordination—for example, one can embed a stable process in Brownian motion, but cannot extract Brownian motion from the stable process since the filtration is strictly smaller. However, we will see that when the scaling limit of the random time process, Y_t, is fBm, one can undo the subordination using the time-change $\tau_s = \inf_{t \geq 0}\{t : Y_t = s\}$. Extending such a time-change procedure to the kernels of indicator fractional stable motions when $\alpha = 2$, we find that one can, in some sense, extract Brownian motion from fractional Brownian motions satisfying $H < 1/2$.

The rest of the paper is arranged as follows. In Section 2 we describe RWRTs and RWRSs. We also describe their respective random reward schema and scaling limits. The section ends with a statement describing new scaling limit results. The proofs of the scaling weak convergence results are given in Section 3. In Section 4, we describe the recursive construction mentioned above, and complete the nontrivial task of showing that the recursion produces processes that are well defined. The main component of this task is showing that the local times exist and are in $L^2(\Omega \times \mathbb{R})$. Finally, in Section 5 we explain how to extract Brownian motion from fBm with any Hurst parameter satisfying $H < 1/2$.

2. Discrete and continuous models.

2.1. *Random walks at random times and alternating random reward schema.* We start with a simple description of RWRS. Let $\{\eta_\alpha(k)\}_{k\in\mathbb{Z}}$ be a set of i.i.d. symmetric random variables in the domain of attraction of an SαS law, $\alpha \in (0, 2]$ with scale parameter $\sigma = 1$. The family $\{\eta_\alpha(k)\}$ depicts the *scenery* associated to the vertices of $\mathbb{Z}$. Let

$$W(n) := \sum_{k=1}^{n} \xi_\beta(k) \tag{3}$$

be a symmetric random walk on $\mathbb{Z}$ with steps $\xi_\beta(k)$ in the domain of attraction of an SβS law, $\beta \in (1, 2]$. The random walk roams amidst the scenery $\{\eta_\alpha(k)\}$ which are independent from the steps $\{\xi_\beta(k)\}$.

The cumulative scenery process

$$Z_n = Z_n(\eta_\alpha, W) := \sum_{k=1}^{n} \eta_\alpha\big(W(k)\big) \tag{4}$$

is called a *random walk in random scenery*. The scenery $\{\eta_\alpha(k)\}$ can alternatively be thought of as random *reward* collected by the random walk when it visits vertex k.

We note that some authors call the pair $(W, \eta_\alpha(W))$ a RWRS process (e.g., [8]). Since most of the papers cited in this work refer to (4) as the RWRS, we stick with this notation.

Wang [28] considered a slight modification of RWRS by using a discrete approximation of a Gaussian process instead of a random walk:

$$Z_n = Z_n(\eta_\alpha, G_H) := \sum_{k=1}^{n} \eta_\alpha\big(\lceil G_H(k) \rceil\big). \tag{5}$$

Here $\lceil\cdot\rceil$ is the ceiling function, and $G_H(k)$ is the partial sum of a stationary Gaussian process X_k with correlations $r(j-k) = \mathbf{E}X_j X_k$ satisfying

$$\sum_{j=1}^{n}\sum_{k=1}^{n} r(j-k) \sim n^{2H}, \tag{6}$$

where $0 < H < 1$. In addition to (5), there have been myriad generalizations of (4), and we refer the reader to the introduction of [15] for a nice summary of such generalizations.

We refer to (5) as a *dependent walk in random scenery* (DWRS). In general, we consider $Z_n(\eta_\alpha, W_H)$ for which the *collecting process* $W_H(n)$ has stationary increments and also satisfies the following scaling limit properties:

$$\text{(SLP)}\qquad \begin{cases} \text{(i) } \lim_{n\to\infty} n^{-H} W_H\big(\lfloor nt \rfloor\big) \Rightarrow Y_t, \qquad \text{in } \mathcal{D}\big([0,\infty)\big), \\ \text{(ii) } Y_t \text{ is a nondegenerate } H\text{-sssi process} \\ \qquad\qquad\qquad\qquad (Y_0 = 0 \text{ by self-similarity}), \\ \text{(iii) } \mathbf{E}|Y_t| < \infty, \end{cases}$$

where $\mathcal{D}([0,\infty))$ is equipped with the usual Skorohod topology (also called the J_1-topology).

The condition that Y_t be sssi guarantees that Z_n scales to an sssi process as well, and this was in fact the original motivation of introducing Z_n in [17]. Note that we use the stable parameter $\alpha \in (0, 2]$ for the scenery/reward and consequently the increments of the RWRS/DWRS; however, we reserve the stable parameter

$\beta \in (1, 2]$ for the increments of the collecting process (note that we require $\beta > 1$ in order to guarantee $\mathbf{E}|Y_t| < \infty$).

We introduce a variant of Z_n in which the reward alternate in sign and are associated with edges instead of vertices. In our variant of RWRS, we use symmetric reward $\{\eta_\alpha(e)\}$ together with signs $\{\sigma_e\}$, $\sigma_e \in \{-1, +1\}$, associated to the edge set of $\mathbb{Z}$. At time zero, all signs are plus one, $\sigma_e(0) = +1$; however, $(\sigma_e(n))_{n\geq 0}$ is a process determined by the collecting process in a manner discussed below.

Consider a discrete collecting process $W_H(n)$ satisfying condition (SLP). Note that our definition allows $|W_H(n) - W_H(n-1)|$ to be greater than one. Let $\mathcal{E}_n$ be the set of connected edges traversed on the nth step of $W_H(n)$, that is, the set of edges between $W_H(n-1)$ and $W_H(n)$ [thus $\mathcal{E}_n$ has cardinality $|W_H(n) - W_H(n-1)|$]. At the nth step, the process $W_H(n)$:

- earns the signed reward $\sigma_e(n-1) \cdot \eta_\alpha(e)$ of all edges $e \in \mathcal{E}_n$ and then
- reverses the sign σ_e of each $e \in \mathcal{E}_n$ so that it will receive the exact opposite reward the next time it traverses e.

A (*dependent*) *random walk at random time* (DWRT/RWRT) with a nonmonotonic subordinating *random time process* $W_H(n)$ is a process

$$A_n = A_n(\eta_\alpha, W_H) := \sum_{k=1}^{n} \sum_{e\in\mathcal{E}_k} \sigma_e(k-1) \cdot \eta_\alpha(e), \tag{7}$$

where $\sigma_e(k) \in \{-1, +1\}$ is the sign of e at time k.

To explain the name of the process, consider that in an RWRT, due to cancellation, each reward $\eta_\alpha(e)$ contributes either one or zero net terms to the sum (7). When e is to the right of the origin, the number of net terms is one if and only if $W_H(n)$ is to the right of e, and when e is to the left of the origin, the number of net terms is one if and only if $W_H(n)$ is to the left of e. It follows that

$$A_n = \sum_{e\in[0, W_H(n)]} \eta_\alpha(e), \tag{8}$$

where $e \in [0, x]$ means that e lies between 0 and x regardless of the sign of x. The partial sum of reward $\sum_{e\in[0,n]} \eta_\alpha(e)$ is just a random walk $S_\alpha(n)$. If we let $S_\alpha(0) = 0$ and extend the random walk to negative times in the natural way, then thinking of time being determined by the location of $W_H(n)$, we have

$$A_n = S_\alpha\bigl(W_H(n)\bigr). \tag{9}$$

As an aside, if we take (8) as our initial definition rather than (7), then the reward may equally well be placed on the vertices instead of the edges. The reader may therefore choose to visualize this process in any of several ways according to his or her own aesthetic preference.

The relationship between Z_n and A_n should be clear. In particular, when the collecting process is a simple random walk $W(n)$, a relation is made by using a bijection which assigns to each vertex k either the edge lying to its left whenever the previous step of $W(n)$ was in the positive direction (right), or the edge lying to its right whenever the previous step of $W(n)$ was in the negative direction (left). To extend the relation to other random walks, one must use a modified version of Z_n which, when going from x to y on the nth step, collects a reward not only from y, but all vertices between x and y. In view of this relationship between Z_n and A_n, if $\mathbf{P}_s$ is the measure for the random scenery, and $\mathbf{P}'$ is the measure for W_H, then the processes Z_n and A_n can be defined on the same product space with measure $\mathbf{P}_s \times \mathbf{P}'$.

There is a further relationship between Z_n and the variations of A_n which mirrors the connection between BMRS and the variations of IBM as presented in [20]. In order to explain this relationship, it will be convenient to let the collecting walk $W(n)$ be a simple random walk and to have the reward for both Z_n and A_n be attached to the edges of $\mathbb{Z}$, rather than to the vertices. For $p \in \mathbb{N}$, let the *pth variation* of A_n be defined as

$$V_n^{(p)} := \sum_{i=1}^{n} (A_i - A_{i-1})^p. \tag{10}$$

THEOREM 2.1. *Suppose the i.i.d. reward $\{\eta_\alpha(e)\}$ are symmetric and have finite pth moments. If p is odd, then $V_n^{(p)}$ is another RWRT, while if p is even, then $V_n^{(p)} - n\mathbf{E}[\eta_\alpha^p]$ is a RWRS. In both cases, the reward collected by the processes are given by $\{\zeta^{(p)}(e)\}$ where*

$$\zeta^{(p)}(e) := \eta_\alpha(e)^p - \mathbf{E}\big[\eta_\alpha(e)^p\big].$$

PROOF. For $i \geq 1$ we let $\mathcal{E}_i$ denote the edge between $W(i-1)$ and $W(i)$. We then have

$$Z_n = \sum_{i=1}^{n} \eta_\alpha(\mathcal{E}_i), \qquad A_n = \sum_{e \in [0, W(n)]} \eta_\alpha(e). \tag{11}$$

Note that

$$(A_i - A_{i-1})^p 1_{\{\mathcal{E}_i = e\}} = \big(\sigma_e(i-1)\eta_\alpha(e)\big)^p 1_{\{\mathcal{E}_i = e\}}. \tag{12}$$

If $p = 2q$ is even, then the sign $(\sigma_e(i-1))^{2q}$ in (12) is irrelevant. Therefore,

$$V_n^{(2q)} - n\mathbf{E}\big[\eta_\alpha^{2q}\big] = \left(\sum_{i=1}^{n} (A_i - A_{i-1})^{2q}\right) - n\mathbf{E}\big[\eta_\alpha^{2q}\big] = \sum_{i=1}^{n} \zeta^{(2q)}(\mathcal{E}_i). \tag{13}$$

Comparing with (11) shows this to be a RWRS with reward given by $\{\zeta^{(p)}(e)\}$.

On the other hand, if $p = 2q+1$ is odd, then the sign $(\sigma_{e(k)}(i-1))^{2q+1} = \sigma_{e(k)}(i-1)$ in (12) causes the same cancellation as we have with RWRT, and since

η_α is symmetric, there is no longer a need to subtract the expectation. Thus, (12) yields

$$V_n^{(2q+1)} = \sum_{i=1}^{n} (A_i - A_{i-1})^{2q+1} = \sum_{e \in [0, W_H(n)]} \zeta^{(2q+1)}(e). \tag{14}$$

Comparing again with (11) shows this to be a RWRT with reward given by $\{\zeta^{(p)}(e)\}$. $\square$

We now compare this with the results of [20]. Let I_s denote an IBM, fix an interval $[0, t]$, and let

$$V_n^{(p)}(t) = \sum_{k=1}^{2^n t} \bigl(I(T_{k+1,n}) - I(T_{k,n})\bigr)^p, \tag{15}$$

where $\{T_{k,n} : 1 \le k \le 2^n t\}$ is an induced random partition of the interval $[0, t]$; see [20], Section 1, for details. Among other things, Khoshnevisan and Lewis showed that, when properly renormalized, $V_n^{(p)}(t)$ converges in distribution to IBM when p is odd and BMRS when p is even; see Theorems 3.2, 4.4, 4.5 and the discussion in the middle of page 631. If we consider the natural association between BMRS and RWRS on the one hand and between IBM and RWRT on the other, we see that the simple Theorem 2.1 provides an intuitive backdrop for the much more difficult results concerning the continuous case in [20].

We now return to study of A_n in the general case. We will need processes extended to noninteger times, and we will therefore denote the linear interpolation of A_n as

$$A_t = A_{\lfloor t \rfloor} + \bigl(t - \lfloor t \rfloor\bigr)(A_{\lceil t \rceil} - A_{\lfloor t \rfloor}). \tag{16}$$

Let us now describe the two different *random reward schema* we will use. Let us start with an alternating version of the random reward schema introduced in [6]. Let $\{(W_H^{(i)}(n))_{n\ge 0}\}_{i\in\mathbb{N}}$ be independent copies of $W_H(n)$ which are also independent from independent copies of the reward $\{\{\eta_\alpha^{(i)}(e)\}_{e\in\mathbb{Z}}\}_{i\in\mathbb{N}}$. If (c_n) is a sequence of integers such that $c_n \to \infty$, then

$$\sum_{i=1}^{c_n} A_t\bigl(\eta_\alpha^{(i)}, W_H^{(i)}\bigr) \tag{17}$$

is an *alternating random reward scheme*.

If we instead follow the single-scenery schema of [11] and use the same single copy of reward $\{\eta_\alpha^{(1)}(e)\}_{e\in\mathbb{Z}}$ for each copy of $W_H^{(i)}(n)$, then

$$\sum_{i=1}^{c_n} A_t\bigl(\eta_\alpha^{(1)}, W_H^{(i)}\bigr) \tag{18}$$

is a *single scenery* alternating random reward scheme.

2.2. *Scaling limits of random reward schema.* In this section we state some known results concerning the scalings of RWRS and DWRS to stable integral representations. These will motivate our results concerning the scalings of RWRT and DWRT.

Let us first recall an important definition. Suppose m is a σ-finite measure on a measurable space $(E,\mathcal{B})$, and that

$$\mathcal{B}_0 = \{A \in \mathcal{B} : m(A) < \infty\}.$$

DEFINITION 2.2. A *SαS random measure M* with *control measure m* is a σ-additive set function on $\mathcal{B}_0$ such that for all $A_i \in \mathcal{B}_0$:

(1) $M(A_1) \sim \mathcal{S}_\alpha(m(A_1)^{1/\alpha})$;
(2) $M(A_1)$ and $M(A_2)$ are independent whenever $A_1 \cap A_2 = \varnothing$,

where $\mathcal{S}_\alpha(\sigma)$ is an SαS random variable.

In particular, if $f \in L^\alpha(E,\mathcal{B},m)$, then

$$\int_E f(x)M(dx) \sim \mathcal{S}_\alpha(\|f(x)\|_{L^\alpha}). \tag{19}$$

Section 3.3 of [26] contains an introduction to this topic. The immediate importance to us is that the scaling limits of RWRS and DWRS are integrals with respect to stable random measures, where the integral kernel is the local time of a properly scaled collecting process $\tilde{W}_{H'}$ (linearly interpolated) which is either $G_{H'}$ or S_β with $\beta \in (1,2]$. The process $\frac{1}{n^{H'}}\tilde{W}_{H'}(nt)$ converges weakly to a scaling limit, denoted by $\tilde{Y}_t$, which is, respectively, fBm-H' in $\mathcal{C}([0,\infty))$ or a β-stable Lévy motion in $\mathcal{D}([0,\infty))$. Let $(\Omega',\mathcal{F}',\mathbf{P}')$ be the probability space of $\tilde{Y}_t$. It is known that $\tilde{Y}_t$ has a jointly continuous local time $\ell_{\tilde{Y}}(t,x)$; this was shown for β-stable Lévy motions in [3] and for fBm in [1]. Moreover, for all $t \ge 0$ and all $\alpha \in (0,2]$, $\tilde{Y}_t$ satisfies

$$\mathbf{E}' \int_{\mathbb{R}} |\ell_{\tilde{Y}}(t,x)|^\alpha \, dx < \infty \tag{20}$$

by Theorem 3.1 in [6] and Lemma 2.1 in [10]. Here we interpret $\ell_{\tilde{Y}}(t)$ as the increasing family of random functions which satisfy the occupation time formula

$$\int_0^t 1_A(\tilde{Y}_s)\,ds = \int_A \ell_{\tilde{Y}}(t,x)\,dx \tag{21}$$

for any Borel set A.

Let $M_0(dx)$ be an SαS random measure with Lebesgue control measure which is independent from $\tilde{Y}_t$. Throughout this subsection we will let

$$H = 1 - H' + H'/\alpha. \tag{22}$$

A *stable process in random scenery* is an H-sssi SαS process given by

$$\Delta_t^H(M_0, \tilde{Y}) := \int_{\mathbb{R}} \ell_{\tilde{Y}}(t,x) M_0(dx), \qquad t \ge 0, \tag{23}$$

which is well defined by (20); see Chapter 3 of [26]. Recall that $\eta_\alpha(k)$ is in the domain of attraction of an SαS law. It was shown in [5, 17, 28] that the following weak convergence holds in $\mathcal{C}([0,\infty))$:

$$\frac{1}{n^H} Z_{nt}(\eta_\alpha, \tilde{W}_{H'}) \Rightarrow \Delta_t^H(M_0, \tilde{Y}). \tag{24}$$

Henceforth we will use H' for the Hurst parameter of the collecting process and H for the Hurst parameter of the resulting stable process in random scenery.

The Hurst exponent $H = 1 - H' + H'/\alpha$ can be explained by using the local time scaling relation

$$\big(\ell_{\tilde{Y}}(ct,x), x \in \mathbb{R}, t \ge 0\big) \stackrel{d}{=} \big(c^{1-H'} \ell_{\tilde{Y}}(t, x/c^{H'}), x \in \mathbb{R}, t \ge 0\big). \tag{25}$$

In [6], weak convergence in $\mathcal{C}([0,\infty))$ was shown for a properly normalized random reward scheme

$$c_n^{-1/\alpha} \sum_{i=1}^{c_n} n^{-(\alpha+1)/(2\alpha)} Z_{nt}\big(\eta_\alpha^{(i)}, S_2^{(i)}\big),$$

where Z_t is the linear interpolation of Z_n in the same manner as (16). The

$$\big\{S_2^{(i)}(n)\big\}_{i \in \mathbb{N}}$$

are independent copies of mean zero, finite variance ($\beta = 2$) random walks which have $H' = 1/2$ explaining the exponent $H = \frac{\alpha+1}{2\alpha}$. They collect independent copies of i.i.d. reward $\{\eta_\alpha^{(i)}(k)\}_{i \in \mathbb{N}}$ which are also independent from the random walks. Cohen and Samorodnitsky called the limiting process an fBm-1/2 local time fractional stable motion. In [5], the discrete collecting process was generalized to $G_{H'}$ and convergence to fBm-H' local time fractional stable motions for any $H' \in (0,1)$ was proved. In [10], a collecting process scaling to β-stable Lévy motion ($\beta > 1$) was used, and consequently, other local time fractional stable motions were obtained in the limit. Let us now explicitly state these collective results.

Recall that $(\Omega', \mathcal{F}', \mathbf{P}')$ is the probability space of $\tilde{Y}_t$. Suppose $M_1(d\omega', dx)$ is an SαS random measure that has control measure $\mathbf{P}' \times$ Lebesgue, but lives on some other probability space $(\Omega, \mathcal{F}, \mathbf{P})$. As above, $\tilde{W}_{H'}$ is either $G_{H'}$ or S_β with $\beta \in (1,2]$. Letting H be as in (22), in light of (20) we define a *local time fractional stable motion* as the process

$$\Gamma_t^H(M_1, \tilde{Y}) := \int_{\Omega' \times \mathbb{R}} \ell_{\tilde{Y}}(t, x; \omega') M_1(d\omega', dx), \qquad t \ge 0. \tag{26}$$

Let (c_n) be an integer sequence with $c_n \to \infty$, and let $\{\eta_\alpha^{(i)}(k)\}$ be independent copies of i.i.d. reward in the domain of attraction of an SαS law. The following weak convergence holds in $\mathcal{C}([0,\infty))$ as $n \to \infty$:

$$c_n^{-1/\alpha} \sum_{i=1}^{c_n} \frac{1}{n^H} Z_{nt}(\eta_\alpha^{(i)}, \tilde{W}_{H'}^{(i)}) \Rightarrow \Gamma_t^H(M_1, \tilde{Y}) \qquad \text{(independent scenery)}. \tag{27}$$

Let M_2 be a stable random measure with Lebesgue control measure with the restriction that $\alpha \in (1,2]$, and again let H be as in (22). We may use (20) and Hölder's inequality to define

$$\Lambda_t^H(M_2, \tilde{Y}) := \int_{\mathbb{R}} \mathbf{E}' \ell_{\tilde{Y}}(t, x; \omega') M_2(dx), \qquad t \geq 0. \tag{28}$$

Note that the scale parameter at time t for (28) is

$$\sigma = \| \mathbf{E}' \ell_{\tilde{Y}}(t, x; \omega') \|_{L^\alpha(\mathbb{R})} \tag{29}$$

versus $\sigma = \| \ell_{\tilde{Y}}(t, x; \omega') \|_{L^\alpha(\Omega' \times \mathbb{R})}$ for (26). For $\alpha \in (1,2]$, a convergence result (in finite-dimensional distributions) with respect to the single scenery case was given in Theorem 4.2 of [11]:

$$c_n^{-1} \sum_{i=1}^{c_n} \frac{1}{n^H} Z_{nt}(\eta_\alpha^{(1)}, \tilde{W}_{H'}^{(i)}) \overset{\text{f.d.d.}}{\Longrightarrow} \Lambda_t^H(M_2, \tilde{Y}) \qquad \text{(single scenery)}. \tag{30}$$

As stated earlier, the process on the right-hand side is H-ss, but using (29) one can see that this process does not in general have stationary increments.

It is convenient to write (26) and (28) as renormalized sums of (23) which appeal to the stable central limit theorem and the law of large numbers, respectively; see [5, 10, 11]. The former renormalization is applied to the entire integral in (23), and the convergence is in $\mathcal{C}([0,\infty))$ whereas the latter renormalization applies only to the integral kernel

$$n^{-1/\alpha} \sum_{i=1}^{n} (\Delta_t^H)^{(i)} \;\Rightarrow\; \Gamma_t^H, \tag{31}$$

$$\int_{\mathbb{R}} \left(n^{-1} \sum_{i=1}^{n} \ell_{\tilde{Y}}^{(i)}(t, x) \right) M_2(dx) \overset{\text{f.d.d.}}{\Longrightarrow} \Lambda_t^H. \tag{32}$$

2.3. *Scaling limits of alternating random reward schema.* We are now ready to state our results concerning the scaling limits of $A(t)$ and its associated random reward schema (17) and (18).

Throughout this subsection we assume that the discrete collecting process $W_{H'}(n)$ is extended to continuous time by linear interpolation and that it has the scaling limit Y_t as given in condition (SLP). Independent copies of i.i.d. reward

$\{\eta_\alpha^{(i)}(k)\}_{i\in\mathbb{N}}$ are, as usual, in the domain of attraction of an SαS law (scale parameter $\sigma = 1$) and independent from the random walks. The space $(\Omega', \mathcal{F}', \mathbf{P}')$ supports Y_t, and the SαS random measures M_i are as in the previous subsection. Define the processes

$$\Delta_H(t) = \Delta_H(t; M_0, Y) := \int_{\mathbb{R}} 1_{[0,Y_t(\omega')]}(x) M_0(dx), \qquad t \ge 0, \tag{33}$$

$$\Gamma_H(t) = \Gamma_H(t; M_1, Y) := \int_{\Omega'\times\mathbb{R}} 1_{[0,Y_t(\omega')]}(x) M_1(d\omega', dx), \qquad t \ge 0, \tag{34}$$

$$\Lambda_H(t) = \Lambda_H(t; M_2, Y) := \int_{\mathbb{R}} \mathbf{E}' 1_{[0,Y_t(\omega')]}(x) M_2(dx), \qquad t \ge 0, \tag{35}$$

which are analogous to (23), (26) and (28).

The above are all self-similar with common index $H = H'/\alpha$, and (34) and (35) are SαS processes. One can also observe (see Theorem 2.2 in [16]) that both (33) and (34) have stationary increments. We call (33) an *(H'-sssi)-time α-stable Lévy motion* or more generally a *stable process at random time*. If $X(t)$ is a two-sided α-stable Lévy motion, then we may also write (33) as $X(Y_t)$. The process (34) is an *indicator fractional stable motion* as introduced in [16]. The process (35) is the alternating analog of the scaling limit of a single scenery random reward scheme introduced in [11].

THEOREM 2.3. *Let $H = H'/\alpha$, and let $c_n \to \infty$ as $n \to \infty$.*

- *The following convergence holds in f.d.d.*:

$$n^{-H} S_\alpha\big(W_{H'}(nt)\big) \Rightarrow \Delta_H(t; M_0, Y). \tag{36}$$

  If the reward are symmetric with finite variance ($\alpha = 2$), and $n^{-H'} W_{H'}(nt)$ converges weakly in $\mathcal{D}([0,\infty))$ ($\mathcal{C}([0,\infty))$), then (36) also holds weakly in $\mathcal{D}([0,\infty))$ ($\mathcal{C}([0,\infty))$, resp.).
- *If $n^{-H}|W_H(\lfloor nt\rfloor)|$ is uniformly integrable, then*

$$\begin{aligned} &c_n^{-1/\alpha} \sum_{i=1}^{c_n} n^{-H} A_{nt}\big(\eta_\alpha^{(i)}, W_{H'}^{(i)}\big) \\ &\quad \stackrel{f.d.d.}{\Longrightarrow} \Gamma_H(t; M_1, Y) \qquad (\textit{independent scenery}). \end{aligned} \tag{37}$$

- *If $\alpha > 1$, then*

$$c_n^{-1} \sum_{i=1}^{c_n} n^{-H} A_{nt}\big(\eta_\alpha^{(1)}, W_{H'}^{(i)}\big) \stackrel{f.d.d.}{\Longrightarrow} \Lambda_H(t; M_2, Y) \qquad (\textit{single scenery}). \tag{38}$$

The interest of the first convergence result [to (H'-sssi)-time α-stable Lévy motion] lies in the fact that this seems to be the first such Donsker-type theorem for iterated processes where the random time process is not a subordinator, that is, not an increasing Lévy process. In the case where the random time process is a subordinator, similar convergence results are well known. In fact, in Section 2.2 of [24], such results are extended to the case where the scenery have a certain dependence structure. Their Donsker-type theorem shows convergence to an α-time fractional Brownian motion.

It is not hard to see that $\Delta_H(t)$, $\Gamma_H(t)$, and $\Lambda_H(t)$ are all continuous in probability. However, by Theorem 10.3.1 in [26], when $\alpha < 2$, $\Delta_H(t)$ and $\Gamma_H(t)$ are not sample continuous. In those cases, the best we can hope for is weak convergence in $\mathcal{D}([0,\infty))$. We will see in the remark at the end of Section 3, that even this is a lot to ask. In that remark, it is argued that even in the simplest cases, $\Delta_H(t)$ is not even in $\mathcal{D}([0,\infty))$. In particular, the weak convergence in $\mathcal{C}([0,\infty))$ and $\mathcal{D}([0,\infty))$ given in the first part of Theorem 2.3 depends heavily on the fact that $\alpha = 2$. In this case, the scaling limit of S_α is continuous since it is simply Brownian motion.

The condition that $n^{-H'} W_{H'}(\lfloor nt \rfloor)$ is uniformly integrable holds when $W_{H'}$ is either $G_{H'}$ or S_β, $\beta > 1$. The former follows from a Gaussian concentration inequality which bounds $n^{-H'} W_{H'}(\lfloor nt \rfloor)$ in L^p for all $p \geq 1$ (see [22], page 60), and the latter follows from equation (5.s) in [21] and the bound $\mathbf{E}(|X| 1_A) \leq \|X\|_p (\mathbf{P}(A))^{1/q}$.

3. Proof of Theorem 2.3. A convenient tool in proving convergence of the finite-dimensional distributions is a diagonal convergence theorem of [9]. In order to state this theorem, we require some definitions.

As usual $\eta_\alpha(k)$ is in the domain of attraction of the SαS law with scale parameter $\sigma = 1$, and it is the reward on the edge between k and $k+1$. For fixed positive h, define μ_h to be the random signed measure on $\mathbb{R}$ which is a.s. absolutely continuous with respect to Lebesgue measure and whose random density is given by

$$\frac{d\mu_h}{dx}(x) = h^{-1+1/\alpha} \sum_{k \in \mathbb{Z}} \eta_\alpha(k) 1_{(hk, h(k+1)]}(x). \tag{39}$$

For a locally integrable function $f \in L^1_{\text{loc}}$, define

$$\mu_h[f] = \int f \, d\mu_h := \sum_{k \in \mathbb{Z}} \eta_\alpha(k) h^{-1+1/\alpha} \int_{hk}^{h(k+1)} f(x)\, dx. \tag{40}$$

For $0 < \alpha < 1$, we will say that $(f_n)_{n \in \mathbb{N}}$ converges to f in $\mathcal{D}^\alpha$ if the following two conditions hold:

- for any compact $K \subset \mathbb{R}$, $f_n \mathbf{1}_K$ converges to $f \mathbf{1}_K$ in $L^1(\mathbb{R})$;
- there is some $\eta > \alpha^{-1}$ such that $f_n(x) = o(|x|^{-\eta})$ and $f(x) = o(|x|^{-\eta})$ as $x \to \infty$.

Let $\mathcal{F}^\alpha = L^\alpha(\mathbb{R})$ if $1 \le \alpha \le 2$ and $\mathcal{F}^\alpha = \mathcal{D}^\alpha$ if $0 < \alpha < 1$. The following diagonal convergence is shown in Proposition 3.1 of [9]; see also Proposition 3.1 of [11].

PROPOSITION 3.1 (Dombry). *Suppose $M_0(dx)$ is an α-stable random measure, $\alpha \in (0, 2]$ and $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{F}^\alpha$. If $h_n \to 0$ as $n \to \infty$, then the random variables $\mu_{h_n}[f_n]$ converge weakly as $n \to \infty$ and in particular,*

$$\mu_{h_n}[f_n] \Rightarrow \int_{\mathbb{R}} f M_0(dx). \tag{41}$$

We now start by showing convergence in f.d.d. for Theorem 2.3. However, to reduce notation and simplify the presentation, we only prove convergence of the one-dimensional distributions for some fixed $t > 0$. The extension to f.d.d. in all three cases follows easily using the Cramér–Wold device; see, e.g., Theorem 3.9.5 in [12].

Also without loss of generality we use $n^{-H} A_{\lfloor nt \rfloor}$ instead of the linear interpolation $n^{-H} A_{nt}$ since they differ by at most $n^{-H}\eta_\alpha(k)$ which goes a.s. to 0 as $n \to \infty$.

Convergence in f.d.d. for (36). Fix $t \in [0, \infty)$. Let $X_n(t) = \frac{1}{n^{H'}} W_{H'}(\lfloor nt \rfloor)$. According to assumption (SLP), $X_n(t) \Rightarrow Y(t)$. By Skorohod's representation theorem, there is a common probability space on which $\bar{X}_n \overset{d}{=} X_n(t)$, $\bar{Y} \overset{d}{=} Y(t)$ live and such that $\bar{X}_n(\bar{\omega}) \to \bar{Y}(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$ (note that the bar includes the dependence on t).

Fix an $\bar{\omega}$ and recall that $H = H'/\alpha$ and that for $a < 0$, we let $[0, a] := [a, 0]$. We have

$$\begin{aligned}
&\mu_{n^{-H'}}[1_{[0,\bar{X}_n(\bar{\omega})]}] \\
&\quad = n^{H'-H'/\alpha} \sum_{k\in\mathbb{Z}} \eta_\alpha(k) \int_{kn^{-H'}}^{(k+1)n^{-H'}} 1_{[0,\bar{X}_n(\bar{\omega})]}(x)\, dx \\
&\quad = n^{H'-H'/\alpha} \sum_{k\in\mathbb{Z}} \eta_\alpha(k) n^{-H'} 1_{\{\bar{W}_{H'}(\lfloor nt \rfloor) > k \ge 0\} \cup \{\bar{W}_{H'}(\lfloor nt \rfloor) \le k < 0\}}(\bar{\omega}) \\
&\quad = n^{-H} A_{\lfloor nt \rfloor}(\eta_\alpha, \bar{W}_{H'}(\bar{\omega})).
\end{aligned} \tag{42}$$

By Proposition 3.1 and the fact that $1_{[0,\bar{X}_n(\bar{\omega})]} \to 1_{[0,\bar{Y}(\bar{\omega})]}$ in $\mathcal{F}^\alpha$, we have that the one-dimensional distributions of $n^{-H} A_{\lfloor nt \rfloor}(\eta_\alpha, W_{H'})$ converge to those of $\Delta_H(t; M_0, Y)$.

Convergence in f.d.d. for (38). For multiple independent walkers in the same scenery, we follow the arguments of Proposition 2.4 in [11]. Fix $t \in [0, \infty)$. As in the proof of (36), using Skorohod's representation theorem and Proposition 3.1, we have for $\alpha \in (1, 2]$,

$$\frac{1}{c_n}\sum_{i=1}^{c_n} n^{-H'/\alpha} A_{\lfloor nt \rfloor}(\eta_\alpha, \bar{W}_{H'}^{(i)}) = \mu_{n^{-H'}}\left[c_n^{-1}\sum_{i=1}^{c_n} 1_{[0,\bar{X}_n^{(i)}(\bar{\omega})]}\right],$$

where $\bar{X}_n^{(i)}(\bar{\omega}) \to \bar{Y}^{(i)}(\bar{\omega})$ for each $i \in \mathbb{N}$ and for all $\bar{\omega} \in \bar{\Omega}$ (the bar includes the dependence on t).

We need only show the following converges in probability to zero as $n \to \infty$:

$$\left\| \frac{1}{c_n}\sum_{i=1}^{c_n}(1_{[0,\bar{X}_n^{(i)}(\bar{\omega})]}) - 1_{[0,\bar{Y}^{(i)}(\bar{\omega})]} + 1_{[0,\bar{Y}^{(i)}(\bar{\omega})]} - \mathbf{E}' 1_{[0,Y_t]} \right\|_{L^\alpha(\mathbb{R})}, \tag{43}$$

where for fixed n, the random variables $\bar{X}_n^{(i)}$, $1 \le i \le c_n$ are i.i.d. Also, for each fixed i, $\bar{X}_n^{(i)}$ converges a.s. to $\bar{Y}^{(i)}$. We first show that as $n \to \infty$,

$$\begin{aligned} &\left\| \frac{1}{c_n}\sum_{i=1}^{c_n}(1_{[0,\bar{X}_n^{(i)}(\bar{\omega})]} - 1_{[0,\bar{Y}^{(i)}(\bar{\omega})]}) \right\|_{L^\alpha(\mathbb{R})} \\ &\qquad \le \frac{1}{c_n}\sum_{i=1}^{c_n} \|1_{[0,\bar{X}_n^{(i)}(\bar{\omega})]} - 1_{[0,\bar{Y}^{(i)}(\bar{\omega})]}\|_{L^\alpha(\mathbb{R})} \xrightarrow{p} 0. \end{aligned} \tag{44}$$

Consider a triangular array such that for each fixed n, there are c_n i.i.d. random variables

$$(U_i^{(n)})_{1 \le i \le c_n} := \left(\|1_{[0,\bar{X}_n^{(i)}(\bar{\omega})]} - 1_{[0,\bar{Y}^{(i)}(\bar{\omega})]}\|_{L^\alpha(\mathbb{R})}\right)_{1 \le i \le c_n}$$

in each row, and for each fixed i, the column of random variables $(U_i^{(n)})_{n \in \mathbb{N}}$ converges weakly to zero. For such triangular arrays, the following weak law holds (see Proposition 2.4 in [11]):

$$\frac{1}{c_n}\sum_{i=1}^{c_n} U_i^{(n)} \xrightarrow{p} 0 \qquad \text{as } n \to \infty, \tag{45}$$

thus proving (44).

Since $\mathbf{E}\|1_{[0,Y_t]}\|_{L^\alpha(\mathbb{R})} < \infty$, the strong law of large numbers for Banach space valued random variables implies that the following converges a.s. in $L^\alpha(\mathbb{R})$:

$$\frac{1}{c_n}\sum_{i=1}^{c_n} 1_{[0,\bar{Y}^{(i)}(\bar{\omega})]} \to \mathbf{E}' 1_{[0,Y_t]}, \tag{46}$$

thus completing the proof of one-dimensional weak convergence for (38).

Convergence in f.d.d. for (37). We will mimic the arguments of [5, 10, 17]. Let

$$1_{\{W_{H'}(nt;k)\}}(\omega') := 1_{\{W_{H'}(\lfloor nt \rfloor) > k \ge 0\} \cup \{W_{H'}(\lfloor nt \rfloor) \le k < 0\}}(\omega').$$

Using the last equality in (42), we have

$$\begin{aligned}
&\mathbf{E}\exp\Biggl(\sum_{\ell=1}^{c_n} i\theta c_n^{-1/\alpha} n^{-H} A_{\lfloor nt\rfloor}\bigl(\eta_\alpha^{(\ell)}, W_{H'}^{(\ell)}\bigr)\Biggr)\\
&\quad = \mathbf{E}\exp\Biggl(\sum_{\ell=1}^{c_n} i\theta c_n^{-1/\alpha} n^{-H}\sum_{k\in\mathbb{Z}}\eta_\alpha^{(\ell)}(k)1_{\{W_{H'}^{(\ell)}(nt;k)\}}(\omega')\Biggr) \qquad (47)\\
&\quad = \biggl(\mathbf{E}'\biggl[\prod_{k\in\mathbb{Z}}\phi_{\eta_\alpha}\bigl(v_n(\omega',k)\bigr)\biggr]\biggr)^{c_n},
\end{aligned}$$

where ϕ_{η_α} is the real-valued characteristic function of a symmetric reward η_α and

$$v_n(\omega',k) = \theta c_n^{-1/\alpha} n^{-H} 1_{\{W_{H'}(nt;k)\}}(\omega'), \qquad k\in\mathbb{Z}. \tag{48}$$

Suppose $\phi_\alpha(v) = \exp(-|v|^\alpha)$ is the characteristic function of the SαS law of scale parameter $\sigma = 1$. We show that the following asymptotic holds as $n\to\infty$:

$$\mathbf{E}'\biggl[\prod_{k\in\mathbb{Z}}\phi_{\eta_\alpha}\bigl(v_n(\omega',k)\bigr)\biggr] = \mathbf{E}'\biggl[\prod_{k\in\mathbb{Z}}\phi_\alpha\bigl(v_n(\omega',k)\bigr)\biggr] + o\bigl(c_n^{-1}\bigr). \tag{49}$$

If $(x_i)_{i\in\mathbb{Z}}$ and $(x_i')_{i\in\mathbb{Z}}$ are sequences in $[-1,1]$ with only finitely many terms not equal to one, then

$$\biggl|\prod_{i\in\mathbb{Z}} x_i' - \prod_{i\in\mathbb{Z}} x_i\biggr| \le \sum_{i\in\mathbb{Z}}|x_i' - x_i|. \tag{50}$$

Letting

$$g(y) = \sup_{|x|\le y}|x|^{-\alpha}\bigl|\phi_{\eta_\alpha}(x) - \phi_\alpha(x)\bigr|, \qquad x\neq 0, \tag{51}$$

we have

$$\begin{aligned}
&c_n\biggl|\prod_{k\in\mathbb{Z}}\phi_{\eta_\alpha}\bigl(v_n(\omega',k)\bigr) - \prod_{k\in\mathbb{Z}}\phi_\alpha\bigl(v_n(\omega',k)\bigr)\biggr|\\
&\quad \le c_n\sum_{k\in\mathbb{Z}}\bigl|\phi_{\eta_\alpha}\bigl(v_n(\omega',k)\bigr) - \phi_\alpha\bigl(v_n(\omega',k)\bigr)\bigr|\\
&\quad \le g\Bigl(\sup_{k\in\mathbb{Z}}\bigl|v_n(\omega',k)\bigr|\Bigr)\sum_{k\in\mathbb{Z}} c_n\bigl|v_n(\omega',k)\bigr|^\alpha \qquad (52)\\
&\quad = g\bigl(\theta c_n^{-1/\alpha} n^{-H}\bigr)\sum_{k\in\mathbb{Z}}\bigl|n^{-H}\theta 1_{\{W_{H'}(nt;k)\}}(\omega')\bigr|^\alpha\\
&\quad = g\bigl(\theta c_n^{-1/\alpha} n^{-H}\bigr)|\theta|^\alpha n^{-H'}\bigl|W_{H'}\bigl(\lfloor nt\rfloor;\omega'\bigr)\bigr|.
\end{aligned}$$

By assumption, $n^{-H'}|W_{H'}(\lfloor nt\rfloor;\omega')|$ converges weakly and is bounded in L^1, so to prove (49) we need only show that $g(\theta c_n^{-1/\alpha} n^{-H})$ is bounded and converges in probability to 0. Since η_α is in the domain of attraction of the SαS law with $\sigma = 1$, by the stable central limit theorem, we have that as $v \to 0$,

$$\phi_{\eta_\alpha}(v) = \phi_\alpha(v) + o(|v|^\alpha).$$

Thus g is bounded, continuous and vanishes at 0. Equation (49) follows since $\theta c_n^{-1/\alpha} n^{-H}$ goes to zero.

Let $\{(\zeta_\alpha^{(\ell)}(k))_{k\in\mathbb{Z}}\}_{\ell\in\mathbb{N}}$ be independent copies of i.i.d. reward such that $\zeta_\alpha^{(0)}(1)$ has an SαS law with scale parameter $\sigma = 1$. Using (49), the c_nth root of (47) is equal to

$$\begin{aligned}
&\mathbf{E}\exp\bigl(i\theta c_n^{-1/\alpha} n^{-H} A_{\lfloor nt\rfloor}(\eta_\alpha^{(1)}, W_{H'}^{(1)})\bigr)\\
&\quad = \mathbf{E}\exp\bigl(i\theta c_n^{-1/\alpha} n^{-H} A_{\lfloor nt\rfloor}(\zeta_\alpha^{(1)}, W_{H'}^{(1)})\bigr) + o(c_n^{-1})\\
&\quad = \mathbf{E}'\exp\Bigl(-c_n^{-1} n^{-H'} \sum_{k\in\mathbb{Z}} \bigl(\theta 1_{\{W_{H'}(nt;k)\}}(\omega')\bigr)^\alpha\Bigr) + o(c_n^{-1}) \qquad (53)\\
&\quad = \mathbf{E}'\exp\bigl(-c_n^{-1} n^{-H'} \bigl|W_{H'}(\lfloor nt\rfloor;\omega')\bigr|\theta^\alpha\bigr) + o(c_n^{-1})\\
&\quad = \mathbf{E}'\bigl(1 - c_n^{-1} n^{-H'} \bigl|W_{H'}(\lfloor nt\rfloor;\omega')\bigr|\theta^\alpha + o(c_n^{-1})\bigr).
\end{aligned}$$

If b_n is such that $c_n b_n \to \lambda$, then $(1 + b_n)^{c_n} \to e^\lambda$. Letting

$$b_n = -c_n^{-1}\mathbf{E}'\bigl(n^{-H'}\bigl|W_H(\lfloor nt\rfloor)\bigr|\bigr)\theta^\alpha$$

and using the assumption of uniform integrability, we have that

$$\bigl(1 - c_n^{-1}\mathbf{E}'\bigl(n^{-H'}\bigl|W_H(\lfloor nt\rfloor)\bigr|\bigr)\theta^\alpha + o(c_n^{-1})\bigr)^{c_n} \to e^{-|\theta|^\alpha \mathbf{E}'|Y_t|} \qquad (54)$$

as required.

Tightness in $\mathcal{D}([0,\infty))$ *and* $\mathcal{C}([0,\infty))$ *for* (36). Suppose that $\alpha = 2$ so that $n^{-1/\alpha} S_\alpha(n^{1/\alpha} t)$ converges weakly in $\mathcal{C}(\mathbb{R})$ to a two-sided Brownian motion $\tilde{B}_t$. By (SLP) and the independence of $S_\alpha(t)$ and $W_{H'}(t)$, the joint process

$$\bigl(n^{-H'} S_\alpha(n^{H'} t), n^{-H'} W_{H'}(nt)\bigr)$$

converges weakly to $(\tilde{B}_t, Y_t)$ in $\mathcal{C}(\mathbb{R}) \times \mathcal{D}([0,\infty))$. The weak convergence of $n^{-H'} S_\alpha(W_{H'}(nt))$ in $\mathcal{D}([0,\infty))$ therefore follows from the continuous mapping theorem, provided that $(x, y) \longrightarrow x \circ y$ is continuous from $\mathcal{C}(\mathbb{R}) \times \mathcal{D}([0,\infty))$ to $\mathcal{D}([0,\infty))$.

The topologies on $\mathcal{C}(\mathbb{R})$ and $\mathcal{D}([0,\infty))$ are first countable, so proving sequential continuity suffices. Suppose $x_n \longrightarrow x$ in $\mathcal{C}(\mathbb{R})$ and $y_n \longrightarrow y$ in $\mathcal{D}([0,\infty))$, and let T be a continuity point of y. By the definition of convergence on $\mathcal{D}([0,\infty))$, we must show that there is a sequence of homeomorphisms λ_n^T from $[0, T]$ onto $[0, T]$ such that λ_n^T converges uniformly to the identity and $x_n \circ y_n \circ \lambda_n^T$ converges uniformly to $x \circ y$.

Let $\varepsilon > 0$ be given. Since $y_n \to y$ in $\mathcal{D}([0,\infty))$, there are homeomorphisms λ_n^T from $[0, T]$ onto $[0, T]$ such that λ_n^T converges uniformly to the identity, and $y_n \circ \lambda_n^T$ converges uniformly to y. The set $\mathcal{A} = \bigcup_n y_n([0, T])$ is bounded, so x_n converges uniformly to x on $\bar{\mathcal{A}}$. Thus, x is uniformly continuous on $\bar{\mathcal{A}}$, and thus on $\mathcal{A}$.

Choose $\delta > 0$ such that $|x(y_1) - x(y_2)| < \frac{\varepsilon}{2}$ for $y_1, y_2 \in \mathcal{A}$ and $|y_1 - y_2| < \delta$. Next, find $M_1 > 0$ such that $\sup_{t\in[0,T]} |y_n \circ \lambda_n^H(t) - y(t)| < \delta$ whenever $n > M_1$, and find $M_2 > 0$ such that $\sup_{y\in\mathcal{A}} |x_n(y) - x(y)| < \frac{\varepsilon}{2}$ whenever $n > M_2$. Then, whenever $n > \max(M_1, M_2)$, we have

$$
\begin{aligned}
&\left|x_n \circ y_n \circ \lambda_n^H(t) - x \circ y(t)\right| \\
&\quad \le \left|x_n \circ y_n \circ \lambda_n^H(t) - x \circ y_n \circ \lambda_n^H(t)\right| + \left|x \circ y_n \circ \lambda_n^H(t) - x \circ y(t)\right| \qquad (55)\\
&\quad \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

for all $t \in [0, T]$. Thus, $x_n \circ y_n \circ \lambda_n^H$ converges uniformly to $x \circ y$ showing continuity of the composition map. The same argument holds if $n^{-H'} W_{H'}(nt)$ converges weakly in $\mathcal{C}([0,\infty))$, except that proving the continuity of the composition map on $\mathcal{C}(\mathbb{R}) \times \mathcal{C}([0,\infty))$ is even simpler.

REMARK. We thank an anonymous referee for the above tightness proof which simplifies our original proof. The referee also noticed the following informative observation. If $\alpha < 2$, then S_α scales to an α-stable Lévy motion, $X(t)$. Fix $\varepsilon > 0$ and let $\tau_\varepsilon > 0$ be the first positive time such that $|X(\tau_\varepsilon) - \lim_{t\to\tau_\varepsilon^-} X(t)| > \varepsilon$. Consider the simple case where $W_{H'}$ scales to a Brownian motion, B_t. Let τ be the first time $B_t - \tau_\varepsilon$ hits 0. As is well known, $B_t - \tau_\varepsilon$ oscillates around 0 immediately, thus $\lim_{t\to\tau^+} X(B_t)$ does not exist a.s. This argument, which can be made rigorous, shows that even in the elementary case where the collecting process scales to Brownian motion, the process $X(B_t)$ is not cadlag.

4. A recursive construction of some fractional stable motions.

Throughout this section we will suppose that $\alpha \in (1, 2]$. We present two related recursive constructions of some H-sssi processes. The first recursion produces stable processes in random scenery, while the second recursion produces local time and indicator fractional stable motions. Note that only the second recursion leads to SαS processes. Since fBm is the only sssi Gaussian process, when $\alpha = 2$ the second construction gives us fBm. In particular, if on the first step of the recursion we use Brownian motion as the collecting process (or random time process), then we obtain fBm of any dyadic Hurst parameter.

Although the first construction does not in general lead to α-stable processes, we will see that the finite-dimensional distributions of the processes have finite α moments, and thus one can appeal to the stable central limit theorem and normalize partial sums of independent copies of the stable processes in random scenery in order to get honest stable processes [in a manner similar to (31)].

Let $Y_t^{\varnothing}$ be an H-sssi process satisfying the four conditions of Theorem 4.1 below. Consider the vector $v = (v_1, \ldots, v_n)$ with coordinates $v_j \in \{+, -\}$. Let us use the notation $\hat{v}$ to denote v truncated by removing the last element, that is, $\hat{v} = (v_1, \ldots, v_{n-1})$. The empty set $\varnothing$ will denote the empty vector.

We define the process Y_t^v recursively from $Y_t^{\hat{v}}$ and an α-stable random measure $M_0(dx)$, with $\alpha \in (1, 2]$, assumed to be independent from $Y_t^{\hat{v}}$. If $v_n = (+)$, we let

$$Y_t^v := \int_{\mathbb{R}} \ell_{Y^{\hat{v}}}(t, x) M_0(dx), \tag{56}$$

and if $v_n = (-)$, we let

$$Y_t^v := \int_{\mathbb{R}} 1_{[0, Y_t^{\hat{v}}]}(x) M_0(dx). \tag{57}$$

The second recursive procedure is defined similarly. We again use vectors, now denoted $w = (w_1, \ldots, w_n)$, with coordinates taking one of two different values. However, in order to distinguish between the two procedures, we let $w_j \in \{*, \times\}$. As before, we let $\hat{w} = (w_1, \ldots, w_{n-1})$.

Once again Y_t^w is defined recursively from $Y_t^{\hat{w}}$ and an α-stable random measure M_1 with $\alpha \in (1, 2]$; however, the control measure of M_1 is no longer Lebesgue measure as it was in the case of M_0. Suppose that $(\Omega', \mathcal{F}', \mathbf{P}')$ is the probability space of $Y_t^{\hat{w}}$. Then, just as in (26), $M_1(d\omega' \times dx)$ has control measure $\mathbf{P}' \times$ Lebesgue and lives on some other probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If $w_n = (*)$, we let

$$Y_t^w := \int_{\Omega' \times \mathbb{R}} \ell_{Y^{\hat{w}}}(t, x)(\omega') M_1(d\omega' \times dx), \tag{58}$$

and if $w_n = (\times)$, we let

$$Y_t^w := \int_{\Omega' \times \mathbb{R}} 1_{[0, Y_t^{\hat{w}}(\omega')]}(x) M_1(d\omega' \times dx). \tag{59}$$

We must show that the above recursions makes sense, that is, that the integrals are well defined. In general, it is known that H-sssi SαS processes have $L^2(\mathbb{R})$ local times almost surely. This almost gets us to where we want to be; however, there are two separate issues with which we must deal.

According to (19) we need that the integral kernels of (56) and (57) are in $L^\alpha(\mathbb{R})$ [which easily follows if they are in $L^2(\mathbb{R})$], but (56) and (57) are not in general SαS processes, and thus we need an extra argument to show that they have $L^2(\mathbb{R})$ local times almost surely.

The second issue concerns (58) and (59) which are SαS processes, but are well defined only if the local times are in $L^\alpha(\Omega' \times \mathbb{R})$. In other words, we will need the αth moment of the local times to be integrable. To solve these two issues, we use the following result.

THEOREM 4.1. *Suppose $Y_t = Y_t(\omega')$ is an H'-sssi process which satisfies*:

(a) $0 < \mathbf{E}|Y_1| < \infty$.
(b) *Y_t has a local time satisfying* $0 < \mathbf{E}\int_{\mathbb{R}} \ell_Y(t,x)^2\,dx < \infty$.
(c) *Y_1 has a bounded continuous density.*
(d) $\mathbf{E}[\sup_{t\in[0,1]} |Y_t|] < \infty$.

Then the processes

$$Y_t^{(+)} = \int_{\mathbb{R}} \ell_Y(t,x) M_0(dx), \tag{60}$$

$$Y_t^{(-)} = \int_{\mathbb{R}} 1_{[0,Y_t]}(x) M_0(dx), \tag{61}$$

$$Y_t^{(*)} = \int_{\Omega'\times\mathbb{R}} \ell_Y(t,x)(\omega') M_1(d\omega' \times dx), \tag{62}$$

$$Y_t^{(\times)} = \int_{\Omega'\times\mathbb{R}} 1_{[0,Y_t(\omega')]}(x) M_1(d\omega' \times dx) \tag{63}$$

are well-defined H-sssi processes satisfying (b)–(d), *where $H = 1 - H' + H'/\alpha$ for $Y^{(+)}$ and $Y^{(*)}$ and $H = H'/\alpha$ for $Y^{(-)}$ and $Y^{(\times)}$. Moreover, all four processes have finite α moments which implies they also satisfy* (a).

REMARKS. (1) For the proof, we need that ℓ_Y satisfies the occupation time formula

$$\int_0^t 1_A(Y_s)\,ds = \int_A \ell_Y(t,x)\,dx$$

for any Borel set A. This follows from definition (21).

(2) The processes $Y_t^{(+)}$ and $Y_t^{(-)}$ are not generally stable. However, as mentioned above, when they have finite α moments, one can use the stable central limit theorem and normalize partial sums of independent copies of these processes to get stable processes.

PROOF OF THEOREM 4.1. *Well-defined H-sssi processes with finite αth moments.* To see that the $Y_t^{\bullet}$ are well defined and satisfy (a), we have

$$\mathbf{E}\big[\big(Y_1^{(-)}\big)^\alpha\big] = \mathbf{E}\big[\big(Y_1^{(\times)}\big)^\alpha\big] = \mathbf{E}' \int_{\mathbb{R}} \big|1_{[0,Y_1]}(x)\big|^\alpha\,dx = \mathbf{E}'|Y_1|, \tag{64}$$

which is positive and finite since Y_t satisfies (a). Also,

$$\mathbf{E}\big[\big(Y_1^{(+)}\big)^\alpha\big] = \mathbf{E}\big[\big(Y_1^{(*)}\big)^\alpha\big] = \mathbf{E}' \int_{\mathbb{R}} \ell_Y(1,x)^\alpha\,dx. \tag{65}$$

To see that (65) is finite and nonzero, note that $\mathbf{E}'\int_{\mathbb{R}} \ell_Y(1,x)\,dx = 1$ by the occupation time formula and $\mathbf{E}'\int_{\mathbb{R}} \ell_Y(1,x)^2\,dx < \infty$ by (b), thus $\ell_Y \in L^\alpha(\Omega'\times\mathbb{R})$ for $\alpha \in (1,2]$.

To see that the $Y^{\bullet}$ are H-sssi, we refer the reader to Theorem 3.1 in [6] and Theorem 2.2 in [16].

Property (b). Next we use Theorem 21.9 of [14] which implies condition (b) under the assumption that

$$\int_{\mathbb{R}} \int_0^T \int_0^T \mathbf{E}\big[e^{i\theta(Y_t^{\bullet}-Y_s^{\bullet})}\big]\,ds\,dt\,d\theta < \infty. \tag{66}$$

Let us show (b) for $Y_t^{(*)}$. We have

$$\mathbf{E}\big[e^{i\theta(Y_t^{(*)}-Y_s^{(*)})}\big] = \exp\biggl(-\theta^{\alpha}\mathbf{E}' \int_{\mathbb{R}} \big|\ell_Y(t,x)-\ell_Y(s,x)\big|^{\alpha}\,dx\biggr). \tag{67}$$

Using $\ell_Y(t,x-B_s)-\ell_Y(s,x-B_s) \stackrel{d}{=} \ell_Y(t-s,x)$ and

$$\ell_Y\big(ct,c^{H'}x\big) \stackrel{d}{=} c^{1-H'}\ell_Y(t,x) \tag{68}$$

we see that

$$\begin{aligned} \int_{\mathbb{R}} \big|\ell_Y(t,x)-\ell_Y(s,x)\big|^{\alpha}\,dx &\stackrel{d}{=} \int_{\mathbb{R}} \ell_Y(t-s,x)^{\alpha}\,dx \\ &\stackrel{d}{=} \int_{\mathbb{R}} |t-s|^{\alpha(1-H'+H'/\alpha)}\ell_Y(1,u)^{\alpha}\,du. \end{aligned} \tag{69}$$

Substituting $v=\theta\cdot|t-s|^{1-H'+H'/\alpha}(\mathbf{E}'\int_{\mathbb{R}}\ell_Y(1,u)^{\alpha}\,du)^{1/\alpha}$ we get that (66) equals

$$\biggl(\mathbf{E}'\int_{\mathbb{R}}\ell_Y(1,u)^{\alpha}\,du\biggr)^{-1/\alpha}\int_{\mathbb{R}} e^{-v^{\alpha}}\int_0^T\int_0^T |t-s|^{-1+H'-H'/\alpha}\,ds\,dt\,dv, \tag{70}$$

which is finite since $\mathbf{E}'\int_{\mathbb{R}}\ell_Y(1,u)^{\alpha}\,du>0$ by the occupation time formula.

To show (b) for $Y_t^{(+)}$, write

$$\begin{aligned} \mathbf{E}\big[e^{i\theta(Y_t^{(+)}-Y_s^{(+)})}\big] &= \int_{\Omega'}\int_{\Omega}\exp\biggl(i\theta\int_{\mathbb{R}}\big(\ell_Y(t,x)-\ell_Y(s,x)\big)M_0(dx)\biggr)\,d\omega\,d\omega' \\ &= \mathbf{E}'\exp\biggl(-\theta^{\alpha}\int_{\mathbb{R}}\big|\ell_Y(t,x)-\ell_Y(s,x)\big|^{\alpha}\,dx\biggr). \end{aligned} \tag{71}$$

Using (69) and (71), we have that, in this case, (66) is

$$\int_0^T\int_0^T \mathbf{E}'\biggl[\int_{\mathbb{R}}\exp\biggl(-\theta^{\alpha}\int_{\mathbb{R}}|t-s|^{\alpha(1-H'+H'/\alpha)}\ell_Y(1,u)^{\alpha}\,du\biggr)\,d\theta\biggr]\,ds\,dt.$$

Substituting $v=\theta\cdot|t-s|^{1-H'+H'/\alpha}(\int_{\mathbb{R}}\ell_Y(1,u)^{\alpha}\,du)^{1/\alpha}$ and integrating we obtain, for some constant $c>0$,

$$c\mathbf{E}'\biggl[\biggl(\int_{\mathbb{R}}\ell_Y(1,u)^{\alpha}\,du\biggr)^{-1/\alpha}\biggr]\int_0^T\int_0^T\frac{ds\,dt}{|t-s|^{1-H'+H'/\alpha}}. \tag{72}$$

To show that this is finite we need only show that $\mathbf{E}'[(\int_{\mathbb{R}} \ell_Y(1,u)^\alpha\, du)^{-1/\alpha}] < \infty$. We have $\ell_Y(1,x)(\omega') = 0$ for

$$|x| > A(\omega') := \sup_{t\in[0,1]} |Y_t(\omega')|, \tag{73}$$

so by Holdër's inequality,

$$\begin{aligned}\int_{-\infty}^{\infty} \ell_Y(1,x)(\omega')\,dx &= \int_{-A(\omega')}^{A(\omega')} \ell_Y(1,x)(\omega')\,dx \\ &\le (2A(\omega'))^{(\alpha-1)/\alpha}\left(\int_{-A(\omega')}^{A(\omega')} (\ell_Y(1,x)(\omega'))^\alpha\,dx\right)^{1/\alpha}.\end{aligned} \tag{74}$$

By the occupation time formula, the left-hand side of (74) equals 1 a.s. so that

$$\mathbf{E}'\left(\int_{\mathbb{R}} \ell_Y(1,x)^\alpha\,dx\right)^{-1/\alpha} \le \mathbf{E}'(2A)^{(\alpha-1)/\alpha}. \tag{75}$$

Property (d) of Y_t completes the proof of (b) for $Y^{(+)}$.

Moving on to $Y_t^{(\times)}$, we have

$$\begin{aligned}\mathbf{E}[e^{i\theta(Y_t^{(\times)}-Y_s^{(\times)})}] &= \exp\left(-\theta^\alpha \mathbf{E}'\int_{\mathbb{R}} 1_{[Y_s,Y_t]}\,dx\right) \\ &= \exp(-\theta^\alpha \mathbf{E}'|Y_{t-s}|) = \exp(-\theta^\alpha |t-s|^{H'}\mathbf{E}'|Y_1|).\end{aligned} \tag{76}$$

Thus (66) reduces to

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^T\int_0^T \exp(-\theta^\alpha|t-s|^{H'}\mathbf{E}'|Y_1|)\,dt\,ds\,d\theta \\ &\quad = C\int_0^T\int_0^T |t-s|^{-H'/\alpha}\,dt\,ds < \infty,\end{aligned} \tag{77}$$

where $C = \int_{\mathbb{R}} \exp(-u^\alpha \mathbf{E}'|Y_1|)\,du$.

Finally, let us consider $Y^{(-)}$. We may mimic steps (67) through (72) in order to reduce (66) to showing

$$\mathbf{E}'\left[\left(\int_{\mathbb{R}} 1_{[0,Y_1]}(x)\,dx\right)^{-1/\alpha}\right] = \mathbf{E}'[|Y_1|^{-1/\alpha}] < \infty. \tag{78}$$

But this follows from assumption (c) on Y_t, since we may simply integrate $|x|^{-1/\alpha}$ against the bounded continuous density of Y_1 which will give a finite value. This establishes (b) for $Y^{(-)}$.

Property (c). In the course of showing property (b) for $Y_t^\bullet$, we showed that in all cases $Y_t^\bullet$ possesses a nonnegative and integrable characteristic function, and thus (c) follows from Theorem 3.3.5 in [12].

Property (d). Consider first $\alpha = 2$. Property (d) is known for $Y_t^{(*)}$ and $Y_t^{(\times)}$ since they are sssi Gaussian processes, that is, fractional Brownian motions.

For $Y_t^{(+)}$, let $\tilde{B}_t$ be a two-sided Brownian motion. We use Proposition 2.2 in [18] which is essentially a corollary of Slepian's lemma. It implies that for each fixed ω',

$$\mathbf{P}\Big(\sup_{t\in[0,1]}\int_{\mathbb{R}} 1_{[0,Y_t(\omega')]}(s)\,d\tilde{B}_s > y\Big) \le 2\mathbf{P}\Big(\int_{\mathbb{R}} 1_{[0,Y_1(\omega')]}(s)\,d\tilde{B}_s > y\Big). \tag{79}$$

Integrating over Ω', property (d) for $Y_t^{(+)}$ follows from property (d) for Y_t.

For $Y_t^{(-)}$, let $Y^* := \sup_{t\in[0,1]} Y_t$, and $Y_* := \inf_{t\in[0,1]} Y_t$. We have

$$\begin{aligned}
\mathbf{E}\Big[\sup_{t\in[0,1]}\big|Y_t^{(-)}\big|\Big] &\le \mathbf{E}\Big[\sup_{t\in[0,1]} Y_t^{(-)} + \sup_{t\in[0,1]}\big(-Y_t^{(-)}\big)\Big] \\
&\le 2\mathbf{E}'\Big[\sup_{t\in[0,1]}\int_{\mathbb{R}} 1_{[0,Y_t]}(s)\,d\tilde{B}_s\Big] \\
&\le 2\mathbf{E}'\Big[\sup_{T\in[Y_*,Y^*]}\int_{\mathbb{R}} 1_{[0,T]}(s)\,d\tilde{B}_s\Big] \\
&\le 8\mathbf{E}'\big(Y^*\big).
\end{aligned} \tag{80}$$

The last inequality follows since the integral in the second to last line is just a two-sided Brownian motion at time T and $\mathbf{E}'(Y^*) = \mathbf{E}'(-Y_*) < \infty$. We thus get property (d) for $Y_t^{(-)}$ since property (d) holds for Y_t.

Let us now suppose that $1 < \alpha < 2$. Theorem 10.5.1 of [26] states that if

$$Y_t = \int_E f_t(x) M(dx) \tag{81}$$

for some family of $L^\alpha(E, m)$ functions $\{f_t(x)\}_{t\ge 0}$, where m is the control measure of M, then there is a constant C such that

$$\mathbf{P}\Big(\sup_{t\in[0,1]}|Y_t| > y\Big) \le \frac{C}{y^\alpha}\int_E \sup_{t\in[0,1]}\big|f_t(x)\big|^\alpha m(dx) \tag{82}$$

for any $y > 0$.

We can therefore obtain (d) for $Y_t^\bullet$ by showing that

$$\mathbf{E}'\int_{\mathbb{R}}\sup_{t\in[0,1]}\big(\ell_Y(t,x)^\alpha\big)\,dx = \mathbf{E}'\int_{\mathbb{R}}\ell_Y(1,x)^\alpha\,dx \tag{83}$$

and

$$\mathbf{E}'\int_{\mathbb{R}}\sup_{t\in[0,1]}\big(1_{[0,Y_t]}(x)^\alpha\big)\,dx = 2\mathbf{E}'\Big(\sup_{t\in[0,1]} Y_t\Big) \tag{84}$$

are both finite. As seen in (65) and the argument thereafter, (83) is finite since Y_t satisfies (b). Also (84) is finite since Y_t satisfies (d). □

For fixed $\alpha \in (1, 2]$, define

$$\phi_+(x) := 1 - x + x/\alpha \quad \text{and} \quad \phi_-(x) := x/\alpha. \tag{85}$$

Applying Theorem 4.1 recursively, we have the following corollary:

COROLLARY 4.2. *If $Y_t^{\varnothing}$ is an H'-sssi process satisfying* (a)–(d) *of Theorem* 4.1, *then $Y_t^{(v_1,\dots,v_n)}$ and $Y_t^{(w_1,\dots,w_n)}$* [*as defined in* (56)–(59)] *are H-sssi processes with*

$$H = \phi_{v_n} \circ \cdots \circ \phi_{v_1}(H').$$

Moreover, $Y_t^{(w_1,\dots,w_n)}$ is an $S\alpha S$ process.

5. Brownian motion extracted from fBm, $H < 1/2$. Suppose $\alpha = 2$. Then the family of stochastic integrals, $(Y_t^{(\times)})_{t\ge 0}$, is an H'-sssi Gaussian process, thus it is precisely fBm with Hurst exponent $H' < 1/2$. In this section, we show that Brownian motion can be extracted from $Y_t^{(\times)}$ by time-changing its integral kernels. In order to motivate our time-changed kernels, we first show that Brownian motion can also be extracted from a stable process at random time, $Y_t^{(-)}$, using a time-change.

To keep things simple, we assume in this section that the random time process Y_t is itself an fBm. Thus it is a.s. continuous and satisfies the property that for each $s > 0$,

$$\tau_s = \inf_{t\ge 0}\{t : Y_t = s\} < \infty \qquad \text{a.s.} \tag{86}$$

Heuristically, time-changing the kernel of $Y_t^{\bullet}$ undoes the subordination of $Y_t^{\bullet}$ to the process Y_t, leaving us with a process $(M(A_t))_{t\ge 0}$. We then observe that $A_s \subset A_t$ for $s < t$, and that $m(A_t)$ is linearly increasing (here m is the control measure). One need only check that such a procedure gives us what we want, by looking at the finite-dimensional distributions. Since our interest is in the case $\alpha = 2$, we have that M_0, M_1 are Gaussian random measures on $\mathbb{R}$ and $\Omega' \times \mathbb{R}$, respectively, and we in fact need only check covariances.

Let us start by presenting the time-change of $Y_t^{(-)}$.

PROPOSITION 5.1. *Let the random time process Y_t be a fractional Brownian motion. If $Y_t^{(-)}$ is defined as in* (61) *with $\alpha = 2$, then $Y_{\tau_t}^{(-)}$ is a Brownian motion.*

PROOF. We have

$$Y_{\tau_t}^{(-)} = \int_{\mathbb{R}} 1_{[0,Y_{\tau_t}]}(x) M_0(dx) = \int_{\mathbb{R}} 1_{[0,t]}(s)\, d\tilde{B}_s = \tilde{B}_t, \tag{87}$$

where $\tilde{B}_t$ is a two-sided Brownian motion. For the covariances, if $s < t$, we have

$$\begin{aligned}\mathbf{E}\big(Y_{\tau_s}^{(-)}Y_{\tau_t}^{(-)}\big) &= \mathbf{E}\bigg(\int_{\mathbb{R}} 1_{[0,s]}(r)\,d\tilde{B}_r \cdot \int_{\mathbb{R}} 1_{[0,t]}(r)\,d\tilde{B}_r\bigg)\\ &= \int_{\mathbb{R}} \big(1_{[0,s]}(r)\big)^2\,dr = s.\end{aligned} \tag{88}$$

□

In the case of

$$Y_t^{(\times)} = \int_{\Omega'\times\mathbb{R}} 1_{[0,Y_t(\omega')]}(x) M_1\big(d\omega' \times dx\big),$$

we cannot look at "$Y_{\tau_t}^{(\times)}$" since τ_t lives on the same probability space as M_1. We address this issue by instead time-changing the kernel $1_{[0,Y_t]}$. Let us define

$$Y_t^{(\times)_\tau} := \int_{\Omega'\times\mathbb{R}} 1_{[0,Y_{\tau_t}(\omega')]}(x) M_1\big(d\omega' \times dx\big). \tag{89}$$

A good way to think about the above integral is in terms of a central limit theorem similar to (31):

$$n^{-1/2}\sum_{i=1}^{n} \Delta_H\big(\tau_t^{(i)}\big)^{(i)} \stackrel{\text{f.d.d.}}{\Longrightarrow} \int_{\Omega'\times\mathbb{R}} 1_{[0,Y_{\tau_t}(\omega')]}(x) M_1\big(d\omega' \times dx\big). \tag{90}$$

Here, $\tau^{(i)}$ is measurable with respect to the σ-field of $\Delta_H^{(i)}$. By Proposition 5.1, the $\Delta_H(\tau_t^{(i)})^{(i)}$ are independent Brownian motions. The next proposition shows that the right-hand side is also a Brownian motion thus proving (90).

PROPOSITION 5.2. *Let the random time process Y_t be a fractional Brownian motion. If $Y_t^{(\times)}$ is defined as in* (63) *with $\alpha = 2$, then $Y_t^{(\times)_\tau}$ is a Brownian motion.*

PROOF. We have

$$\begin{aligned}Y_t^{(\times)_\tau} &= \int_{\Omega'\times\mathbb{R}} 1_{[0,Y_{\tau_t}]}(x) M_1\big(d\omega' \times dx\big)\\ &= \int_{\Omega\times\mathbb{R}} 1_{[0,t]}(x) M_1\big(d\omega' \times dx\big) = M_1\big(\Omega' \times [0,t]\big),\end{aligned} \tag{91}$$

which is a Gaussian random variable with variance $\mathbf{P}' \times \mathrm{Leb}(\Omega' \times [0,t]) = t$. For the covariances we analyze second moments. If $s < t$, we have

$$\begin{aligned}\mathbf{E}\big(Y_s^{(\times)_\tau} + Y_t^{(\times)_\tau}\big)^2 &= \mathbf{E}\bigg(\int_{\Omega'\times\mathbb{R}} (2\cdot 1_{[0,s]} + 1_{[s,t]}) M_1\big(d\omega' \times dx\big)\bigg)^2\\ &= \int_{\Omega'\times\mathbb{R}} (2\cdot 1_{[0,s]} + 1_{[s,t]})^2 \mathbf{P}' \times \mathrm{Leb}\big(d\omega' \times dx\big)\\ &= \int_{\mathbb{R}} (4\cdot 1_{[0,s]} + 1_{[s,t]})\,dx\\ &= 3s + t\end{aligned} \tag{92}$$

as required. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA AT BIRMINGHAM
CH 452
1720 2ND AVE. SOUTH
BIRMINGHAM, ALABAMA 35294
USA

DEPARTMENT OF MATHEMATICAL SCIENCES
MONASH UNIVERSITY
CLAYTON, VICTORIA 3800
AUSTRALIA