A SIMPLE PATH TO ASYMPTOTICS FOR THE FRONTIER OF A BRANCHING BROWNIAN MOTION

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We give short proofs of two classical results about the position of the extremal particle in a branching Brownian motion, one concerning the median position and another the almost sure behaviour.

1. Introduction and main results. Kolmogorov et al. [13] proved that the extremal particle in a standard branching Brownian motion sits near $\sqrt{2}t$ at time t. Higher order corrections to this result were given by Bramson [5], and then almost sure fluctuations were proved by Hu and Shi [11]. These two remarkable papers, more than thirty years apart, provide results which reflect an extremely deep understanding of the underlying branching structure. This article grew out of a desire to know whether shorter or simpler proofs of these results exist.

We consider a branching Brownian motion (BBM) beginning with one particle at 0, which moves like a standard Brownian motion until an independent exponentially distributed time with parameter 1. At this time it dies and is replaced (in its current position) by two new particles, which—relative to their birth time and position—behave like independent copies of their parent, moving like Brownian motions and branching at rate 1 into two copies of themselves. Let $N(t)$ be the set of all particles alive at time t, and if $v \in N(t)$, then let $X_v(t)$ be the position of v at time t. If $v \in N(t)$ and $s < t$, then let $X_v(s)$ be the position of the unique ancestor of v that was alive at time s. Define $M_t = \max_{v \in N(t)} X_v(t)$.

1.1. *Bramson's result on the distribution of M_t.* Define

$$u(t, x) = \mathbb{P}(M_t \le x).$$

This function u satisfies the Fisher–Kolmogorov–Petrovski–Piscounov equation

$$u_t = \tfrac{1}{2} u_{xx} + u^2 - u$$

(with Heaviside initial condition), which has been studied for many years both analytically and probabilistically; see, for example, Kolmogorov et al. [13], Fisher [6], Skorohod [18], McKean [15], Bramson [4, 5], Neveu [16], Uchiyama [19], Aronson and Weinberger [3], Karpelevich et al. [12], Harris [9], Kyprianou [14], Harris

et al. [8]. In particular (see [13]) u converges to a *travelling wave*: that is, there exist functions m of t and w of x such that $1 - w$ is a probability distribution function, and

$$u(t, m(t) + x) \to w(x)$$

uniformly in x as $t \to \infty$.

We note that m and w are not uniquely determined by this definition; since we shall be concerned with the detailed behaviour of m, to be precise we set $m(t) :=$ $\sup\{x \in \mathbb{R} : \mathbb{P}(M_t \leq x) \leq 1/2\}$. We offer a proof of the following result which is shorter and simpler than the original proof by Bramson [5]:

THEOREM 1 (Bramson [5]).   *The median $m(t)$ satisfies*

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1) \qquad \text{as } t \to \infty.$$

As Bramson notes in [5], "an immediate frontal assault using moment estimates, but ignoring the branching structure of the process, will fail." That is, for $y \geq 0$ define $\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{y}{t}$ and let $G(t)$ be the set of particles near βt at time t. If some particle has large position at time $s < t$, then many particles are likely to have large position at time t, and therefore the moments of $\#G(t)$ are misleading.

To get around this problem we consider a subset $H(t)$ of $G(t)$. A lower bound for $m(t)$ will follow if we can show that the first two moments of $\#H(t)$ are well behaved. Our approach differs from Bramson's only in that our set $H(t)$ is simpler than his, being the set of particles that stay below the straight line $\beta s + 1$ for all $s \leq t$ and end near βt. This drastically reduces the difficulty of the calculations required for bounding the moments and is one reason why our proof is much shorter than the original.

For the upper bound we are forced to return to a more complicated set $\Gamma(t)$ which is the set of particles that stay below a carefully chosen curve $f(s) + y + 1$, $s \leq t$, and end near βt. Calculation of $\mathbb{E}[\#\Gamma(t)]$ is more difficult than that of $\mathbb{E}[\#H(t)]$, but the two quantities turn out to be of roughly the same size. The key observation now is that if a particle reaches $f(s) + y$ for some $s < t$, then it has done the hard work and is likely to have descendants near βt, even if we insist that they stay below $f(r) + y + 1$ for all $r \in [s, t]$. Thus the probability that some particle reached $f(s) + y$ for some $s < t$ cannot be much larger than $\mathbb{E}[\#\Gamma(t)]$.

1.2. *Hu and Shi's result on the paths of M_t.*   Having established Bramson's result on the centring term $m(t)$, we move on to the almost sure behaviour of M_t. We prove the following result, which is the analogue of a result for quite general branching random walks given by Hu and Shi [11].

THEOREM 2.   *The maximum M_t satisfies*

$$(1) \qquad \liminf_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{3}{2\sqrt{2}} \qquad \text{almost surely}$$

and

$$(2) \qquad \limsup_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} = -\frac{1}{2\sqrt{2}} \qquad \text{almost surely}.$$

Thus, although by Theorem 1 the extremal particle looks like $m(t)$ for most times t, occasionally a particle will travel much further from the origin. Technically the theorem as stated here is a new result as Hu and Shi considered only discrete-time branching random walks, but it would not take too much effort to derive it from their work. We proceed instead by directly applying the estimates developed in the proof of Theorem 1. Only the lower bound in (2) requires a significant amount of extra work, and for that we take an approach similar to that of Hu and Shi in [11]. They noticed that although the probability that a particle has position much bigger than $m(t)$ at a fixed time t is very small, the probability that there exists a time s between (say) n and $2n$ such that a particle has position much bigger than $m(s)$ at time s is actually quite large. Here we again simplify the calculations by considering the number of particles staying below a straight line rather than a curve, much as in our lower bound for Theorem 1.

1.3. *Extensions and other models.*   We note that although we consider only the simplest possible BBM, with binary branching at fixed rate 1, our methods can be applied to rather more general models. There is, however, one important necessary condition for the proof of our lower bound, that *the mean and variance of the number of particles born at a branching event must be finite*. This is simply due to the fact that we employ a second moment method.

Addario-Berry and Reed [1] (in their Theorem 3) proved an analogue of Bramson's result (our Theorem 1) for a wide class of branching random walks. We conjecture that the ideas presented in this article could also be used to give a new proof of the Addario-Berry and Reed result, relaxing the conditions on bounded family sizes and independence amongst families. However, this task would require substantial extra technical work. The estimates on Bessel processes used to estimate numbers of particles staying below straight lines can be replaced by small deviations probabilities for random walks conditioned to stay positive (see [20]); but calculating the expected number of particles staying below a curved line, our Lemma 10, becomes much more difficult; see the footnote on page 756 of [11]. Finally, one must make sure that particles do not jump too far beyond this curved line, which can be done with conceptually standard but technically delicate first moment estimates.

In a sense, Bramson [4] improved the $O(1)$ error in Theorem 1, showing that under his definition one could choose $m(t)$ such that the corresponding error

was $o(1)$. A related result for branching random walks was recently given by Aïdékon [2], showing convergence to a specified law for the recentred extremal particle. This extra detail requires new ideas and is beyond the scope of our methods.

1.4. *Notation.* We will often use positive constants $c_1, c_2, \ldots$ that are independent of all other parameters. We shall reset the subscripts at the end of each proof, so the c_1 appearing in the proof of Lemma 4 is not necessarily the same constant as the c_1 appearing in Lemma 5, for example. On the other hand, $C_1, C_2, \ldots$ will be positive constants that are fixed throughout the article.

2. Bessel-3 processes. We begin by recalling some very basic properties of Bessel-3 processes. If W_t, $t \geq 0$, is a Brownian motion in $\mathbb{R}^3$ started from $(x, 0, 0)$, then its modulus $|W_t|$, $t \geq 0$, is called a Bessel-3 process started from x. For aesthetic purposes in this article we shall simply write "Bessel process" when we mean "Bessel-3 process." Suppose that B_t is a Brownian motion in $\mathbb{R}$ started from $B_0 = x$ under a probability measure P_x; then $X_t := x^{-1} B_t \mathbb{1}_{\{B_s > 0 \ \forall s \leq t\}}$ is a nonnegative unit-mean martingale under P_x. We may change measure by X_t, defining a new probability measure $\hat{P}_x$ via

$$\frac{d\hat{P}_x}{dP_x}\bigg|_{\mathcal{F}_t} := X_t$$

(where $\mathcal{F}_t$ is the natural filtration of the Brownian motion B_t) and then B_t, $t \geq 0$, is a Bessel process under $\hat{P}_x$. The density of a Bessel process satisfies

$$\hat{P}_x(B_t \in dz) = \frac{z}{x\sqrt{2\pi t}}\left(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t}\right) dz.$$

This and much more about Bessel processes can be found in many textbooks, for example, Revuz and Yor [17].

LEMMA 3. *Let* $\gamma = 2^{1/2}/\pi^{1/2}$. *For any* $t > 0$ *and* $x, z \geq 0$,

$$\frac{\gamma z^2}{t^{3/2}} e^{-z^2/2t - x^2/2t} \leq \frac{z}{x\sqrt{2\pi t}}\left(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t}\right) \leq \frac{\gamma z^2}{t^{3/2}}.$$

PROOF. The lower bound is trivial since

$$e^{xz/t} - e^{-xz/t} = 2\sinh(xz/t) \geq 2xz/t.$$

For the upper bound, note that

$$\frac{d}{dz}\left(e^{-(z-x)^2/(2t)} - e^{-(z+x)^2/(2t)}\right) = \frac{x}{t}\left(e^{-(z-x)^2/(2t)} + e^{-(z+x)^2/(2t)}\right)$$

$$+ \frac{z}{t}\left(e^{-(z+x)^2/(2t)} - e^{-(z-x)^2/(2t)}\right)$$

$$\leq \frac{2x}{t}$$

so $e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t} \le 2xz/t$. □

The two lemmas that follow do much of the dirty work of Theorem 1 and Proposition 15 (which is the most difficult part of Theorem 2) by calculating the expectation of two functionals of two dependent Bessel-3 processes. These calculations will not be motivated until later in the article, but we include them here as they are facts about Bessel processes that do not contribute a great deal to the main ideas of the proofs. We start with Lemma 4, which will be used in proving the lower bound for Theorem 1.

Suppose that under $\hat{P}$ we have two processes Y_t^1 and Y_t^2, $t \ge 0$, and a time $\tau \in [0, \infty)$ such that:

- $(Y_t^1, t \ge 0)$ is a Bessel process started from 1;
- τ is exponentially distributed with parameter 2, and is independent of $(Y_t^1, t \ge 0)$;
- $Y_t^2 = Y_t^1$ for all $t \le \tau$;
- conditioned on τ and $(Y_t^1, t \le \tau)$, $(Y_{t+\tau}^2, t \ge 0)$ is a Bessel processes started from Y_τ^1 that is independent of $(Y_t^1, t > \tau)$.

It is clear from this description that $(\tau, Y_\tau^1, Y_t^1, Y_t^2)$ has a well-behaved joint density. Note that we continue to use $\hat{P}$ for this setup, as well as for the single Bessel process $(B_t, t \ge 0)$ seen above.

LEMMA 4. *Let*

$$\beta = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} + \frac{y}{t},$$

$$A_1 = \{1 \le Y_t^1 \le 2\} \quad and \quad A_2 = \{1 \le Y_t^2 \le 2\}.$$

There exists a constant C_1 such that for all $y \ge 0$ and large t,

$$\hat{P}\big[Y_\tau^1 e^{2\tau - (3\tau \log t)/(2t) - \beta Y_\tau^1} \mathbb{1}_{A_1 \cap A_2 \cap \{\tau \le t\}}\big] \le C_1 t^{-3}.$$

PROOF. We use the density of τ to rewrite

$$\hat{P}\big[Y_\tau^1 e^{2\tau - (3\tau \log t)/(2t) - \beta Y_\tau^1} \mathbb{1}_{A_1 \cap A_2 \cap \{\tau \le t\}}\big]$$

$$= 2\int_0^t \hat{P}\big[Y_s^1 e^{-(3s \log t)/(2t) - \beta Y_s^1} \mathbb{1}_{A_1 \cap A_2} | \tau = s\big] ds.$$

The idea then is that the probability that a Bessel process is near the origin at time t is approximately $t^{-3/2}$. If s is small, then we have two (almost) independent Bessel processes which must both be near the origin at time t, giving t^{-3}. If s is large, then we effectively have only one Bessel process, giving $t^{-3/2}$, but the $\exp(\frac{3 \log t}{2t} s)$ gives us an extra $t^{-3/2}$. When s is neither large nor small, the above

effects combine so that things turn out nicely. In each case we apply the upper bound from Lemma 3.

We first check the small s case:

$$\int_0^1 \hat{P}\big[Y_s^1 e^{-(3s\log t)/(2t)-\beta Y_s^1} \mathbb{1}_{A_1 \cap A_2} \big| \tau = s\big] ds$$

$$\leq \int_0^1 \hat{P}(A_1 \cap A_2 | \tau = s)\, ds$$

$$\leq \int_0^1 \hat{P}\Big[\int_0^\infty \hat{P}(Y_t^1, Y_t^2 \in [1,2] | \tau = s, Y_s^1 = x) \hat{P}(Y_s^1 \in dx)\Big| \tau = s\Big] ds$$

$$\leq \int_0^1 \hat{P}\Big[\int_0^\infty \Big(\int_1^2 \frac{2z^2}{\sqrt{2\pi}(t-s)^{3/2}}\, dz\Big)^2 \hat{P}(Y_s^1 \in dx)\Big| \tau = s\Big] ds$$

$$\leq c_1 t^{-3},$$

where the third inequality uses Lemma 3. For the large s case,

$$\int_{t-1}^t \hat{P}\big[Y_s^1 e^{-(3s\log t)/(2t)-\beta Y_s^1} \mathbb{1}_{A_1 \cap A_2} \big| \tau = s\big] ds \leq c_2 t^{-3/2} \hat{P}(A_1) \leq c_3 t^{-3},$$

where we have used the fact that, since $\beta \geq 1$, $xe^{-\beta x} \leq 1$. (We will use the fact that $\beta \geq 1$ throughout the article without further mention.) Finally the main case, for $s \in [1, t-1]$,

$$\int_1^{t-1} \hat{P}\big[Y_s^1 e^{-(3s\log t)/(2t)-\beta Y_s^1} \mathbb{1}_{A_1 \cap A_2} \big| \tau = s\big] ds$$

$$\leq \int_1^{t-1} \int_0^\infty \frac{z^3}{s^{3/2}} e^{-\beta z - (3s\log t)/(2t)} \Big(\int_1^2 \frac{2x^2}{\sqrt{2\pi}(t-s)^3}\, dx\Big)^2 dz\, ds$$

$$\leq c_4 \int_1^{t-1} \frac{e^{-(3s\log t)/(2t)}}{s^{3/2}(t-s)^3} \int_0^\infty z^3 e^{-z}\, dz\, ds$$

$$\leq c_5 \int_1^{t-1} \frac{e^{-(3s\log t)/(2t)}}{s^{3/2}(t-s)^3}\, ds,$$

where for the first inequality we applied Lemma 3. It is a simple task to bound the last integral above by t^{-3} times a constant

$$\int_1^{t-1} \frac{e^{-(3s\log t)/(2t)}}{s^{3/2}(t-s)^3}\, ds \leq \frac{c_6}{t^3} \int_1^{2t/3} \frac{1}{s^{3/2}}\, ds + \frac{c_7}{t^3} \int_{2t/3}^{t-\sqrt{t}} e^{-\log t}\, ds$$

$$+ \frac{c_8}{t^3} \int_{t-\sqrt{t}}^{t-1} \frac{e^{(3\log t)/(2\sqrt{t})}}{(t-s)^3}\, ds$$

$$\leq c_9 t^{-3},$$

which completes the proof. $\quad\square$

Our next lemma is very similar; it estimates a slightly different functional, which will appear in Proposition 15 (the most difficult part of Theorem 2).

LEMMA 5. *Let* $\beta_t = \sqrt{2} - \frac{1}{2\sqrt{2}}\frac{\log t}{t}$ *and* $a_{s,t} = \frac{1}{2\sqrt{2}}\log s - \frac{1}{2\sqrt{2}}\frac{\log t}{t}s$. *If* $e \leq s \leq t \leq 2s$, *then*

$$\hat{P}\big[Y_\tau^1 e^{2\tau - (\tau \log t)/(2t) - \beta_t Y_\tau^1} \mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1 \leq Y_t^2 \leq 2\}} \mathbb{1}_{\{\tau \leq s\}}\big]$$

$$\leq C_2 e^{-(s \log t)/(2t)} \left(\frac{1}{t^{5/2}} + \frac{1}{t^{3/2}(t-s+1)^{3/2}} \right)$$

for some constant C_2 *not depending on* s *or* t.

PROOF. Just as in the proof of Lemma 4, we use the density of τ to rewrite

$$\hat{P}\big[Y_\tau^1 e^{2\tau - (\tau \log t)/(2t) - \beta_t Y_\tau^1} \mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1 \leq Y_t^2 \leq 2\}} \mathbb{1}_{\{\tau \leq s\}}\big]$$

$$= 2 \int_0^s e^{-(r \log t)/(2t)} \hat{P}\big[Y_r^1 e^{-\beta_t Y_r^1} \mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1 \leq Y_t^2 \leq 2\}} | \tau = r\big] dr$$

and then approximate the integral. Essentially the $e^{-\beta_t Y_r^1}$ term means our initial Bessel process must be near the origin at time r; then two independent Bessel processes started from time r must be near the origin at times s and t, respectively. If $r \in [1, s-1]$, then integrating out over Y_r^1, applying Lemma 3 three times and using the fact that $\int_0^\infty z^3 e^{-\beta_t z} dz < \infty$,

$$\hat{P}\big[Y_r^1 e^{-\beta_t Y_r^1} \mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1 \leq Y_t^2 \leq 2\}} | \tau = r\big]$$

$$\leq c_1 \int_0^\infty z e^{-\beta_t z} \frac{z^2}{r^{3/2}} \cdot \frac{1}{(s-r)^{3/2}} \cdot \frac{1}{(t-r)^{3/2}} dz$$

$$\leq c_2 r^{-3/2}(s-r)^{-3/2}(t-r)^{-3/2}.$$

For $r \leq 1$ we are effectively asking two independent Bessel processes to be near the origin at times s and t, giving $s^{-3/2}t^{-3/2}$, and for $r \geq s-1$ we have just one Bessel process which must be near the origin at times s and t, giving $s^{-3/2}(t-s+1)^{-3/2}$. These two simple calculations follow as in Lemma 4. Thus

$$\int_0^s e^{-(r \log t)/(2t)} \hat{P}\big[Y_r^1 e^{-\beta_t Y_r^1} \mathbb{1}_{\{a_{s,t}+1 \leq Y_s^1 \leq a_{s,t}+2\}} \mathbb{1}_{\{1 \leq Y_t^2 \leq 2\}} | \tau = r\big] dr$$

$$\leq \frac{c_3}{s^{3/2}t^{3/2}} + c_4 \int_1^{s-1} \frac{e^{-(r \log t)/(2t)}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}} dr + \frac{c_5 e^{-(s \log t)/(2t)}}{s^{3/2}(t-s+1)^{3/2}}.$$

Since s and t are of the same order, and $\log s \geq \frac{\log t}{t} s$ provided $s, t \geq e$, it remains to estimate the integral in the last line above. We proceed again just as in Lemma 4. First the small r case,

$$\int_1^{s/2} \frac{e^{-(r \log t)/(2t)}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}} \, dr \leq c_6 \int_1^{s/2} \frac{1}{r^{3/2} s^{3/2} t^{3/2}} \, dr$$

$$\leq \frac{c_7}{s^{3/2} t^{3/2}},$$

the large r case,

$$\int_{s-s/t^{1/4}}^{s-1} \frac{e^{-(r \log t)/(2t)}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}} \, dr \leq \frac{c_8 e^{-(s \log t)/(2t)}}{s^{3/2}(t-s+1)^{3/2}}$$

and finally the intermediate r case,

$$\int_{s/2}^{s-s/t^{1/4}} \frac{e^{-(r \log t)/(2t)}}{r^{3/2}(s-r)^{3/2}(t-r)^{3/2}} \, dr \leq c_9 \frac{t^{3/4}}{s^{9/2}} \int_{s/2}^{s-s/t^{1/4}} e^{-(r \log t)/(2t)} \, dr$$

$$\leq c_{10} \frac{t^{7/4}}{s^{9/2}} e^{-(s \log t)/(4t)}$$

$$\leq \frac{c_{11}}{t^{5/2}} e^{-(s \log t)/(2t)},$$

where we have again used $\log s \geq \frac{\log t}{t} s$ and $s \leq t \leq 2s$. □

3. The many-to-one and many-to-two lemmas. We mentioned in the Introduction that we will attempt to count the number of particles remaining below certain lines and ending near βt. To do this we will need to calculate the first two moments of the number of such particles. In this section we state results for doing so in the form that will be most useful to us. These are standard first and second moment bounds for branching processes combined with one- and two-particle changes of measure.

3.1. *The many-to-one lemma.* The many-to-one lemma is a simple and well-known tool in branching processes. It essentially says that the expected number of particles with a certain property equals the expected number of particles times the probability that one particle has that property. To be more precise, let $g_t(v)$ be a measurable functional of t and the path of a particle v up to time t; so, for example, we might take

$$g_t(v) = \mathbb{1}_{\{X_v(t) \geq x\}} \quad \text{or} \quad g_t(v) = t^2 e^{\int_0^t X_v(s) \, ds}.$$

Then the standard many-to-one lemma says

$$\mathbb{E}\left[\sum_{v \in N(t)} g_t(v) \right] = e^t E[g_t(\xi)],$$

where ξ_t, $t \geq 0$, is just a standard Brownian motion under P.

Now, sometimes it will be easiest to calculate $E[g_t(\xi)]$ by using a change of measure. Fixing $\alpha > 0$ and $f : [0, \infty) \to \mathbb{R}$ such that $f \in C^2$, and defining

$$\zeta(t) = \frac{1}{\alpha}(\alpha + f(t) - \xi_t)e^{\int_0^t f'(s)\,d\xi_s - (\int_0^t f'(s)^2\,ds)/2}\mathbb{1}_{\{\xi_s < \alpha + f(s)\ \forall s \leq t\}},$$

the following lemma is a result of Girsanov's theorem and the knowledge of Bessel processes at the start of Section 2. It will be useful for counting the number of particles near βt that have remained below $\alpha + f(s)$ for all $s \leq t$. For a proof see Theorem 8.5 of [7].

LEMMA 6 (Many-to-one lemma).

$$\mathbb{E}\left[\sum_{v \in N(t)} g_t(v)\mathbb{1}_{\{X_v(s) < \alpha + f(s)\ \forall s \leq t\}}\right] = e^t\mathbb{Q}\left[\frac{1}{\zeta(t)}g_t(\xi)\right],$$

where under $\mathbb{Q}$, $\alpha + f(t) - \xi_t$, $t \geq 0$, is a Bessel process.

3.2. *The many-to-two lemma.* We also use a many-to-two lemma, which—just as the many-to-one lemma reduces calculating first moments to consideration of just one particle—will reduce calculating second moments to functionals of two, necessarily dependent, particles. This is a natural idea, and Bramson uses a basic many-to-two lemma in [5]. Again we will combine this idea with a change of measure. [Note, however, that while we used a general C^2 function f in our many-to-one lemma, we will need only $f(s) = \beta s$ here.] We do not prove Lemma 7—as Bramson says, "a rigorous verification is quite messy"—and refer to Lemma 3 of [10] which gives a quite general formulation.

Suppose that under $\mathbb{Q}$, as well as the process ξ_t seen in Section 3.1, we have two processes ξ_t^1 and ξ_t^2, $t \geq 0$, and a time $T \in [0, \infty)$ such that:

- $(1 + \beta t - \xi_t^1, t \geq 0)$ is a Bessel processes started from 1;
- T is exponentially distributed with parameter 2, and is independent of $(\xi_t^1, t \geq 0)$;
- $\xi_t^2 = \xi_t^1$ for all $t \leq T$;
- conditioned on T and $(\xi_t^1, t \leq T)$, $(\beta(T+s) - \xi_{T+s}^2, s \geq 0)$ is a Bessel processes started from $\beta T - \xi_T^1$ that is independent of $(\xi_t^1, t > T)$.

Define

$$\zeta^i(t) = (1 + \beta t - \xi_t^i)e^{\beta \xi_t^i - \beta^2 t/2}\mathbb{1}_{\{\xi_s^i < 1 + \beta s\ \forall s \leq t\}}$$

for $i = 1, 2$ and $t \geq 0$.

LEMMA 7 (Many-to-two lemma). *Let $g_t(\cdot)$ and $h_t(\cdot)$ be measurable functionals of t and the path of a particle up to time t, as in Section* 3.1. *Then*

$$\mathbb{E}\left[\sum_{u,v\in N(t)} g_t(u)h_t(v)\mathbb{1}_{\{X_u(s)<1+\beta s\ \forall s\le t, X_v(s)<1+\beta s\ \forall s\le t\}}\right]$$

$$= \mathbb{Q}\left[e^{2t+T\wedge t}\frac{\zeta^1(T\wedge t)}{\zeta^1(t)\zeta^2(t)}g_t(\xi^1)h_t(\xi^2)\right]$$

$$= e^{3t}\mathbb{Q}\left[\frac{1}{\zeta^1(t)}\mathbb{1}_{\{T>t\}}g_t(\xi^1)h_t(\xi^1)\right]$$

$$+ e^{2t}\mathbb{Q}\left[e^T\frac{\zeta^1(T)}{\zeta^1(t)\zeta^2(t)}\mathbb{1}_{\{T\le t\}}g_t(\xi^1)h_t(\xi^2)\right].$$

The dependence between the two Bessel processes reflects the dependence structure of the BBM: any pair of particles (u, v) in the BBM are dependent up until their most recent common ancestor. The first term on the right-hand side above takes account of the possibility that the Bessel processes have not yet split (which corresponds to the event that u and v are in fact the same particle) and otherwise the second term incorporates the split time T of the two Bessel processes (which corresponds to the last time at which the most recent common ancestor of u and v was alive).

4. Proof of Theorem 1.

4.1. *The lower bound for Theorem* 1. Fix $t > 0$ and set (as in Section 2)

$$\beta = \sqrt{2} - \frac{3}{2\sqrt{2}}\frac{\log t}{t} + \frac{y}{t}.$$

Now define

$$H(y,t) = \#\{u \in N(t) : X_u(s) \le \beta s + 1 \ \forall s \le t, \beta t - 1 \le X_u(t) \le \beta t\}.$$

We shall show that the first two moments of $H(y,t)$ give an accurate picture of the probability that there is a particle near βt at time t. We write $g(y,t) \asymp h(y,t)$ if $c_1 g \le h \le c_2 g$ for some strictly positive constants c_1 and c_2 not depending on t or y.

LEMMA 8. *For $t \ge 1$ and $y \in [0, \sqrt{t}]$,*

$$\mathbb{E}[H(y,t)] \asymp e^{-\sqrt{2}y}.$$

PROOF.   We apply the many-to-one lemma with $f(t) = \beta t$ and $\alpha = 1$.

$$\mathbb{E}[H(y, t)] = e^t \mathbb{Q}\left[\frac{1}{\zeta(t)} \mathbb{1}_{\{\beta t - 1 \le \xi_t \le \beta t\}}\right]$$

$$= e^t \mathbb{Q}\left[\frac{e^{-\beta \xi_t + \beta^2 t/2}}{\beta t + 1 - \xi_t} \mathbb{1}_{\{\beta t - 1 \le \xi_t \le \beta t\}}\right]$$

$$\asymp e^{t - \beta^2 t/2} \mathbb{Q}(\beta t - 1 \le \xi_t \le \beta t)$$

$$\asymp t^{3/2} e^{-\sqrt{2}y} \mathbb{Q}(1 \le \beta t + 1 - \xi_t \le 2).$$

Now, $\beta t + 1 - \xi_t$ is a Bessel process started from 1 under $\mathbb{Q}$, so by Lemma 3,

$$\mathbb{Q}(1 \le \beta t + 1 - \xi_t \le 2) \asymp \int_1^2 \frac{z^2}{t^{3/2}} \, dz \asymp t^{-3/2}. \qquad \square$$

We now use the second moment of $H(y, t)$ to get a lower bound for $m(t)$.

PROPOSITION 9.   *There exists a constant $C_3 > 0$ such that for $t \ge 1$ and $y \in [0, \sqrt{t}]$,*

$$\mathbb{P}(\exists u \in N(t) : X_u(t) \ge \sqrt{2}t - \tfrac{3}{2\sqrt{2}} \log t + y) \ge C_3 e^{-\sqrt{2}y}.$$

PROOF.   By reducing C_3 if necessary, it suffices to establish the claim for all large t. For $i = 1, 2$ let $A_i' = \{\beta t - 1 \le \xi_t^i \le \beta t\}$. By the many-to-two lemma,

$$\mathbb{E}[H(y, t)^2] = e^{3t} \mathbb{Q}\left[\frac{\mathbb{1}_{\{T > t\}}}{\zeta^1(t)} \mathbb{1}_{A_1'}\right] + e^{2t} \mathbb{Q}\left[\frac{e^T \zeta^1(T)}{\zeta^1(t)\zeta^2(t)} \mathbb{1}_{A_1' \cap A_2' \cap \{T \le t\}}\right]$$

$$= e^t \mathbb{Q}\left[\frac{1}{\zeta^1(t)} \mathbb{1}_{A_1'}\right]$$

$$\quad + e^{2t} \mathbb{Q}\left[\frac{e^T(\beta T + 1 - \xi_T^1)e^{\beta \xi_T^1 - \beta^2 T/2}}{(\beta t + 1 - \xi_t^1)(\beta t + 1 - \xi_t^2)e^{\beta \xi_t^1 + \beta \xi_t^2 - \beta^2 t}} \mathbb{1}_{A_1' \cap A_2' \cap \{T \le t\}}\right]$$

$$\le \mathbb{E}[H(y, t)]$$

$$\quad + e^{2t - \beta^2 t + 2\beta} \mathbb{Q}[e^T(\beta T + 1 - \xi_T^1)e^{\beta \xi_T^1 - \beta^2 T/2} \mathbb{1}_{A_1' \cap A_2' \cap \{T \le t\}}]$$

$$\le \mathbb{E}[H(y, t)]$$

$$\quad + c_1 t^3 e^{-\sqrt{2}y} \mathbb{Q}[(\beta T + 1 - \xi_T^1)$$

$$\qquad \times e^{2T - (3T \log t)/(2t) - \beta(\beta T + 1 - \xi_T^1)} \mathbb{1}_{A_1' \cap A_2' \cap \{T \le t\}}],$$

where for the second equality we used that T is an exponential random variable of parameter 2 independent of the path of ξ^1, and for the final inequality we used that

if $y \in [0, \sqrt{t}]$, then

$$\beta^2 T = 2T - 3\frac{\log t}{t}T + \frac{2\sqrt{2}y}{t}T + O(1).$$

Under $\mathbb{Q}$, $(\beta s + 1 - \xi_s^1, s \geq 0)$ and $(\beta s + 1 - \xi_s^2, s \geq 0)$ are Bessel processes starting from 1 that are equal up to T and independent (given T and ξ_T^1) after T. Thus, taking notation from Lemma 4, we have

$$\mathbb{E}[H(y,t)^2] \leq \mathbb{E}[H(y,t)] + c_1 t^3 e^{-\sqrt{2}y} \hat{P}\big[Y_\tau^1 e^{2\tau - (3\tau \log t)/(2t) - \beta Y_\tau^1} \mathbb{1}_{A_1 \cap A_2 \cap \{\tau \leq t\}}\big].$$

Lemma 4 tells us that the $\hat{P}$-expectation is at most a constant times t^{-3}, so for large t and $y \geq 0$,

$$\mathbb{E}[H(y,t)^2] \leq c_2 \mathbb{E}[H(y,t)]$$

for some constant c_2 not depending on y or t. Using Lemma 8 we deduce that

$$\mathbb{P}(H(y,t) \neq 0) \geq \frac{\mathbb{E}[H(y,t)]^2}{\mathbb{E}[H(y,t)^2]} \geq c_3 e^{-\sqrt{2}y}. \qquad \square$$

4.2. *The upper bound for Theorem* 1. We use a first moment method for an object similar to $H(y,t)$ together with an estimate of the probability that a particle ever crosses a carefully chosen line. Again fix t, and define

$$l(s) = \begin{cases} \frac{3}{2\sqrt{2}} \log(s+1), & \text{if } 0 \leq s \leq t/2, \\ \frac{3}{2\sqrt{2}} \log(t - s + 1), & \text{if } t/2 \leq s \leq t. \end{cases}$$

Unfortunately l is not differentiable at $t/2$, so we now choose a twice continuously differentiable function $L : [0, t] \to \mathbb{R}$ such that:

- $L(s) = l(s)$ for all $s \notin [t/2 - 1, t/2 + 1]$;
- $L(s) = L(t - s)$ for all $s \in [0, t]$;
- $L''(s) \in [-10/t, 0]$ for all $s \in [t/2 - 1, t/2 + 1]$.

Let $f(s) = \beta s + L(s)$ for $s \in [0, t]$, and define

$$\Gamma = \#\{u \in N(t) : X_u(s) < f(s) + y + 1 \; \forall s \leq t, \; \beta t - 1 \leq X_u(t) \leq \beta t + y\}.$$

LEMMA 10. *There exists C_4 such that for all $t \geq 1$ and $y \in [0, \sqrt{t}]$,*

$$\mathbb{E}[\Gamma] \leq C_4 (y + 2)^4 e^{-\sqrt{2}y}.$$

PROOF. By the many-to-one lemma with $\alpha = y + 1$, we have

$$\mathbb{E}[\Gamma] = e^t \mathbb{Q}\bigg[\frac{y+1}{y+1+f(t)-\xi_t} e^{-\int_0^t f'(s)\,d\xi_s + (\int_0^t f'(s)^2\,ds)/2} \mathbb{1}_{\{\beta t - 1 \leq \xi_t \leq \beta t + y\}}\bigg],$$

where under $\mathbb{Q}$ the process $y + 1 + f(s) - \xi_s$, $s \geq 0$, is a Bessel process. Using the fact that

$$f'(t)\xi_t = \int_0^t f'(s)\,d\xi_s + \int_0^t f''(s)\xi_s\,ds,$$

which follows from integration by parts, we obtain

$$\mathbb{E}[\Gamma] \leq (y+1)e^t \mathbb{Q}\big[e^{-f'(t)\xi_t + \int_0^t f''(s)\xi_s\,ds + (\int_0^t f'(s)^2\,ds)/2}\mathbb{1}_{\{\xi_t \geq \beta t - 1\}}\big]$$

$$\leq (y+1)e^t \hat{P}_{y+1}\bigg[\exp\bigg(-f'(t)\beta t + \int_0^t f''(s)f(s)\,ds$$

$$+ (y+1)\int_0^t f''(s)\,ds$$

$$- \int_0^t f''(s)B_s\,ds + \frac{1}{2}\int_0^t f'(s)^2\,ds\bigg)\mathbb{1}_{\{B_t \leq y+2\}}\bigg]$$

$$= (y+1)e^{t - \beta^2 t/2 - (\int_0^t L'(s)^2\,ds)/2}\hat{P}_{y+1}\big[e^{\int_0^t L''(s)(y+1-B_s)\,ds}\mathbb{1}_{\{B_t \leq y+2\}}\big],$$

where $(B_s, s \geq 0)$ is a Bessel process under $\hat{P}$. Note that $t - \frac{1}{2}\beta^2 t = \frac{3}{2}\log t - \sqrt{2}y + O(1)$, so

$$\mathbb{E}[\Gamma] \leq c_1(y+1)t^{3/2}e^{-\sqrt{2}y}\hat{P}_{y+1}\big[e^{\int_0^t L''(s)(y+1-B_s)\,ds}\mathbb{1}_{\{B_t \leq y+2\}}\big].$$

Now, let

$$\kappa(s) = \begin{cases} (s+1)^{2/3}, & \text{if } s \leq t/2, \\ (t-s+1)^{2/3}, & \text{if } s > t/2; \end{cases}$$

then $-\int_0^t L''(s)\kappa(s)\,ds \uparrow \kappa$ for some $\kappa \in (0, \infty)$. We know that on the event $\{B_t \leq y + 2\}$, $B_s - (y + 1)$ will stay well below the curve $\kappa(s)$ with exceedingly high probability, so the $\hat{P}_{y+1}$-expectation above should look like a constant times $(y + 2)^3 t^{-3/2}$. The following calculations verify this fact. We split the event that $B_s - (y + 1)$ goes above $\kappa(s)$ into four possibilities. Either there is a sharp increase over a small time interval, or $B_s - (y + 1)$ is large at some time of the form j/t for $j \in \mathbb{N}$; in the latter case, either $(y + 1)t^{4/3} \leq j \leq t - (y + 1)t^{4/3}$, which is so unlikely that we can forget about insisting that $B_t \leq y + 2$, or j is close to 0 or t^2, and we may apply the Markov property at time j/t. Indeed, letting $\tilde{B}_s = (B_s - y - 1)/\kappa(s)$,

$$\hat{P}_{y+1}\big[e^{\int_0^t L''(s)(y+1-B_s)\,ds}\mathbb{1}_{\{B_t \leq y+2\}}\big]$$

$$\leq e^\kappa \hat{P}_{y+1}(B_t \leq y+2)$$

$$+ \sum_{k=1}^\infty e^{(k+1)\kappa}\hat{P}_{y+1}\Big(\sup_{s \in [0,t]} \tilde{B}_s \in [k, k+1), B_t \leq y+2\Big)$$

$$\le e^{\kappa}(y+2)^3 t^{-3/2}$$

$$+ \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=0}^{\lceil t^2 \rceil} \hat{P}_{y+1}\Big(\sup_{s\in[j/t,(j+1)/t]} \tilde{B}_s \ge (\tilde{B}_{j/t} \vee \tilde{B}_{(j+1)/t}) + \frac{k}{2},$$

$$B_t \le y+2 \Big)$$

$$+ \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=1}^{\lceil (y+1)t^{4/3} \rceil} \hat{P}_{y+1}(\tilde{B}_{j/t} \ge k/2, B_t \le y+2)$$

$$+ \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil (y+1)t^{4/3} \rceil+1}^{\lceil t^2-(y+1)t^{4/3} \rceil} \hat{P}_{y+1}(\tilde{B}_{j/t} \ge k/2)$$

$$+ \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil t^2-(y+1)t^{4/3} \rceil+1}^{\lceil t^2 \rceil-1} \hat{P}_{y+1}(\tilde{B}_{j/t} \ge k/2, B_t \le y+2).$$

The first double sum is bounded above by

$$\sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=0}^{\lceil t^2 \rceil} \frac{y+2}{y+1} \mathbb{P}_{y+1}\Big(\sup_{s\in[j/t,(j+1)/t]} \tilde{B}_s \ge (\tilde{B}_{j/t} \vee \tilde{B}_{(j+1)/t}) + k/2 \Big)$$

$$\le \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=0}^{\lceil t^2 \rceil} 2\mathbb{P}_0\Big(\sup_{s\in[0,1/t]} B_s \ge k/2 \Big) \le c_2 t^2 \sum_{k=1}^{\infty} e^{(k+1)\kappa - k^2 t/8}.$$

Writing out the Bessel density and applying the Markov property and then Lemma 3, and using that $z + y + 1 \le z(y+2)$ for all $z \ge 1$, the second double sum is bounded above by

$$\sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=1}^{\lceil (y+1)t^{4/3} \rceil} \int_{k(j/t+1)^{2/3}/2+y+1}^{\infty} \frac{z e^{-(z-y-1)^2 t/2j}}{(y+1)\sqrt{2\pi j/t}} \hat{P}_z(B_{t-j/t} \le y+2)\, dz$$

$$\le \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=1}^{\lceil (y+1)t^{4/3} \rceil} \int_{k(j/t+1)^{2/3}/2}^{\infty} \frac{z e^{-z^2 t/2j}}{\sqrt{2\pi j/t}} \cdot \frac{\gamma(y+2)^3}{(t-j/t)^{3/2}}\, dz$$

$$\le c_3 \frac{(y+2)^3}{t^{3/2}} \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=1}^{\lceil (y+1)t^{4/3} \rceil} \int_{k(j/t)^{1/6}/2}^{\infty} (j/t)^{1/2} z e^{-z^2/2}\, dz$$

$$\le c_3 \frac{(y+2)^3}{t^{3/2}} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k(j/t)^{2/3} e^{(k+1)\kappa - k^2 j^{1/3}/8t^{1/3}}.$$

Writing out the Bessel density and again using that $z + y + 1 \le z(y+2)$ for all $z \ge 1$, we see that the third double sum is bounded above by

$$\sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil (y+1)t^{4/3}\rceil+1}^{\lceil t^2-(y+1)t^{4/3}\rceil} \int_{k\kappa(j/t)/2+y+1}^{\infty} \frac{ze^{-(z-y-1)^2 t/2j}}{(y+1)\sqrt{2\pi j/t}} dz$$

$$\le \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil (y+1)t^{4/3}\rceil+1}^{\lceil t^2-(y+1)t^{4/3}\rceil} \int_{k\kappa(j/t)t^{1/2}/2j^{1/2}}^{\infty} (j/t)^{1/2} z e^{-z^2/2} dz$$

$$\le \sum_{k=1}^{\infty} c_4 k t^{2/3} e^{(k+1)\kappa - k^2(y+1)^{1/3} t^{1/9}/8}.$$

Finally, the fourth double sum is essentially the time reversal of the second double sum: applying Lemma 3 and the Markov property, and then writing out the Bessel density, we see that the fourth double sum is bounded above by

$$\sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil t^2-(y+1)t^{4/3}\rceil}^{\lceil t^2\rceil-1} \int_{y+1+(t-j/t+1)^{2/3}k/2}^{\infty} \frac{\gamma z^2}{(j/t)^{3/2}} \hat{P}_z(B_{t-j/t} \le y+2) dz$$

$$\le \sum_{k=1}^{\infty} e^{(k+1)\kappa}$$

$$\times \sum_{j=\lceil t^2-(y+1)t^{4/3}\rceil}^{\lceil t^2\rceil-1} \int_{y+1+(t-j/t+1)^{2/3}k/2}^{\infty} \int_0^{y+2} \frac{\gamma z w e^{-(w-z)^2/2(t-j/t)}}{(j/t)^{3/2}\sqrt{2\pi(t-j/t)}} dw\, dz$$

$$\le \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil t^2-(y+1)t^{4/3}\rceil}^{\lceil t^2\rceil-1} c_5 \frac{(y+2)^2}{t^{3/2}}$$

$$\times \int_{y+1+(t-j/t+1)^{2/3}k/2}^{\infty} \frac{z}{\sqrt{t-j/t}} e^{-(z-y-1)^2/2(t-j/t)} dz$$

$$\le \sum_{k=1}^{\infty} e^{(k+1)\kappa} \sum_{j=\lceil t^2-(y+1)t^{4/3}\rceil}^{\lceil t^2\rceil-1} c_6 \frac{(y+2)^3}{t^{3/2}} \int_{(t-j/t)^{1/6}k/2}^{\infty} (t-j/t)^{1/2} z e^{-z^2/2} dz$$

$$\le c_7 \frac{(y+2)^3}{t^{3/2}} \sum_{k=1}^{\infty} \sum_{j=\lceil t^2-(y+1)t^{4/3}\rceil}^{\lceil t^2\rceil-1} k(t-j/t)^{2/3} e^{(k+1)\kappa - k^2(t-j/t)^{1/3}/8}.$$

For $t \ge 1$ each of these terms is smaller than a constant times $(y+2)^3 t^{-3/2}$, as required. $\square$

PROPOSITION 11. *There exists a constant C_5 such that*

$$\mathbb{P}\big(\exists u \in N(t) : X_u(t) \geq \sqrt{2}t - \tfrac{3}{2\sqrt{2}}\log t + y\big) \leq C_5(y+2)^4 e^{-\sqrt{2}y},$$

whenever $t \geq 1$ and $y \in [0, \sqrt{t}]$.

PROOF. We need to check that with high probability no particles ever go above $\beta s + L(s) + y$ for $s \in [0, t]$. To this end define

$$\tau = \inf\{s \in [0, t] : \exists u \in N(s) \text{ with } X_u(s) > \beta s + L(s) + y\}.$$

We claim that

$$\mathbb{E}[\Gamma \,|\, \tau < t] \geq c_1$$

for some constant $c_1 > 0$ not depending on t or y; essentially if a particle has already reached $\beta s + L(s) + y$, then it has done the hard work, and the usual cost $e^{-\sqrt{2}y}$ of reaching βt disappears. Choose $s < t$. On the event $\tau = s$, let v be the particle at position $\beta s + L(s) + y$ at time s and define $N_v(r)$ to be the set of descendants of particle v at time r, for $r \geq s$. Then on the event $\tau = s$

$$\Gamma \geq \#\big\{u \in N_v(t) : X_u(r) - X_u(s) \leq \beta_s(r - s) + 1 \,\forall r \in [s, t],$$

$$\beta_s(t - s) - 1 \leq X_u(t) - X_u(s) \leq \beta_s(t - s)\big\},$$

where $\beta_s = (\beta - \frac{L(s)+y}{t-s}) \wedge 0$. It is easy to check that $\beta_s \leq \sqrt{2} - \frac{3}{2\sqrt{2}}\frac{\log(t-s)}{t-s}$. Thus $\mathbb{E}[\Gamma \,|\, \tau = s] \geq \mathbb{E}[H(1, t-s)]$, and by Lemma 8, if $s \leq t - 1$, then

$$\mathbb{E}[\Gamma \,|\, \tau = s] \geq c_2.$$

If $s > t - 1$, then $\mathbb{E}[\Gamma \,|\, \tau = s]$ is at least the probability that a single Brownian motion $B_r, r \geq 0$, remains within $[-1, 1]$ for all $r \in [0, 1]$, and satisfies $B_1 \in [-1, 0]$. This establishes our claim, so

$$\mathbb{E}[\Gamma \,|\, \tau < t] \geq c_1 \quad \text{and} \quad \mathbb{E}[\Gamma] \leq C_4(y+2)^4 e^{-\sqrt{2}y}.$$

But then

$$\mathbb{P}(\tau < t) \leq \frac{\mathbb{E}[\Gamma]\mathbb{P}(\tau < t)}{\mathbb{E}[\Gamma \mathbb{1}_{\{\tau < t\}}]} = \frac{\mathbb{E}[\Gamma]}{\mathbb{E}[\Gamma \,|\, \tau < t]} \leq \frac{C_4}{c_1}(y+2)^4 e^{-\sqrt{2}y}.$$

Applying Markov's inequality, we have

$$\mathbb{P}\big(\exists u \in N(t) : X_u(t) \geq \sqrt{2}t - \tfrac{3}{2\sqrt{2}}\log t + y\big) \leq \mathbb{P}(\Gamma \geq 1) + \mathbb{P}(\tau < t)$$

$$\leq c_3(y+2)^4 e^{-\sqrt{2}y}$$

as required. $\square$

PROOF OF THEOREM 1.   We have shown that for $t \geq 1$ and $y \in [0, \sqrt{t}]$, for some constants $C_3, C_5 \in (0, \infty)$,

$$(3) \qquad C_3 e^{-\sqrt{2}y} \leq \mathbb{P}\big(M_t > \sqrt{2}t - \tfrac{3}{2\sqrt{2}} \log t + y\big) \leq C_5(y+2)^4 e^{-\sqrt{2}y}.$$

Thus there exists $\delta > 0$ such that if we define $\tilde{m}(t) := \sup\{x \in \mathbb{R} : \mathbb{P}(M_t \leq x) \leq 1 - \delta\}$, then

$$\tilde{m}(t) = \sqrt{2}t - \tfrac{3}{2\sqrt{2}} \log t + O(1).$$

Fix $\varepsilon > 0$. Choose L such that $\mathbb{E}[(1-\delta)^{-|N(L)|}] < \varepsilon/2$, and then a such that $\mathbb{P}(M_L^- < -a) < \varepsilon/2$ where $M_t^- = \min_{u \in N(t)} X_u(t)$ is the minimum at time t. For a particle $u \in N(L)$ and $t > L$, we let $M_t^{(u)} = \max_{v \in N(t) : u \leq v} X_v(t)$ be the maximum position among descendants of u at time t. Then for $t > L$,

$$\mathbb{P}\big(M_t < \tilde{m}(t-L) - a\big)$$
$$\leq \mathbb{P}(M_L^- < -a) + \mathbb{P}\Big(M_L^- \geq -a, \max_{u \in N(L)} M_t^{(u)} < \tilde{m}(t-L) - a\Big)$$
$$\leq \mathbb{P}(M_L^- < -a) + \mathbb{E}\big[\mathbb{P}(M_{t-L} < \tilde{m}(t-L))^{|N(L)|}\big]$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $M_t - \tilde{m}(t)$ is tight, and we deduce that also

$$m(t) = \sqrt{2}t - \tfrac{3}{2\sqrt{2}} \log t + O(1). \qquad \square$$

REMARK.   It may be helpful to note that Bessel processes are not a *necessary* ingredient in our proof. One may instead replace every appearance of a Bessel change of measure with a calculation of the probability for a Brownian motion to stay positive, using the reflection principle. Indeed the Bessel density can be derived directly in this way, giving an indication that the two approaches are interchangeable. Using the Bessel change of measure, however, conforms with a method that works with a variety of similar problems. The general principle is that if one wishes to calculate the number of particles in a certain set, then one finds the martingale that forces one particle (the *spine*) to stay within that set, and studies the corresponding measure change.

5. Proof of Theorem 2.   For Theorem 2 we proceed via a series of four results, each proving one of the upper or lower bounds in one of the statements (1) or (2).

LEMMA 12.   *The upper bound in* (1) *holds*

$$\liminf_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} \leq -\frac{3}{2\sqrt{2}} \qquad \textit{almost surely.}$$

PROOF. To rephrase the statement of the lemma, we show that for any $\varepsilon > 0$, there are arbitrarily large times such that there are no particles above $\sqrt{2}t - (3/2\sqrt{2} - \varepsilon) \log t$. Choose $R > 2/\varepsilon$, let $t_1 = 1$ and for $n > 1$ let $t_n = e^{Rt_{n-1}}$. Define

$$E_n = \{\exists u \in N(t_n) : X_u(t_n) > \sqrt{2}t_n - (\tfrac{3}{2\sqrt{2}} - \varepsilon) \log t_n\}$$

and

$$F_n = \{|N(t_n)| \leq e^{2t_n}, |X_u(t_n)| \leq \sqrt{2}t_n \; \forall u \in N(t_n)\}.$$

We know that F_n happens for all large n, so it suffices to show that

$$\mathbb{P}\left(\bigcap_{k \geq n}(E_k \cap F_k)\right) = \lim_{N \to \infty} \prod_{k=n}^{N} \mathbb{P}\left(E_k \cap F_k \Big| \bigcap_{j=n}^{k-1}(E_j \cap F_j)\right) \to 0 \qquad \text{as } n \to \infty.$$

For a particle u, let E_n^u be the event that some descendant of u at time t_n has position larger than $\sqrt{2}t_n - (\tfrac{3}{2\sqrt{2}} - \varepsilon) \log t_n$. Also let $s_n = t_n - t_{n-1}$ and

$$G_n = \Big\{\exists u \in N(s_n) :$$

$$X_u(s_n) > \sqrt{2}s_n - \frac{3}{2\sqrt{2}} \log s_n + \frac{3}{2\sqrt{2}} \log\left(\frac{t_n - t_{n-1}}{t_n}\right) + \varepsilon \log t_n\Big\}.$$

Then

$$\mathbb{P}\left(E_k \cap F_k \Big| \bigcap_{j=n}^{k-1}(E_j \cap F_j)\right) \leq \mathbb{P}\left(E_k \Big| \bigcap_{j=n}^{k-1}(E_j \cap F_j)\right)$$

$$\leq \mathbb{P}\left(\bigcup_{u \in N(t_{k-1})} E_k^u \Big| \bigcap_{j=n}^{k-1}(E_j \cap F_j)\right)$$

$$\leq e^{2t_{k-1}} \mathbb{P}(G_k)$$

$$\leq C_5 (\log t_k + 2)^4 t_k^{2/R} \left(1 - \frac{t_{k-1}}{t_k}\right)^{-3/2} t_k^{-\varepsilon},$$

where the last inequality used Proposition 11. Since we chose $R > 2/\varepsilon$, this is much smaller than 1 when k is large, as required. $\square$

LEMMA 13. *The upper bound in* (2) *holds*

$$\limsup_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} \leq -\frac{1}{2\sqrt{2}} \qquad \text{almost surely.}$$

PROOF.   We show that for large t and any $\varepsilon > 0$, there are no particles above $\sqrt{2}t - (1/2\sqrt{2} - 2\varepsilon) \log t$. By Proposition 11,

$$\mathbb{P}\big(\exists u \in N(t) : X_u(t) > \sqrt{2}t - (\tfrac{1}{2\sqrt{2}} - \varepsilon) \log t\big) \le C_5 (\log t + 2)^4 t^{-1-\varepsilon\sqrt{2}}.$$

Thus for any lattice times $t_n \to \infty$, by Borel–Cantelli,

$$\mathbb{P}\big(\exists u \in N(t_n) : X_u(t_n) > \sqrt{2}t_n - (\tfrac{1}{2\sqrt{2}} - \varepsilon) \log t_n \text{ for infinitely many } n\big) = 0.$$

It is now a simple exercise using the exponential tightness of Brownian motion and the fact that we may choose $t_n - t_{n-1}$ arbitrarily small to ensure that no particle goes above $\sqrt{2}t - (\tfrac{1}{2\sqrt{2}} - 2\varepsilon) \log t$ for any time t.   □

LEMMA 14.   *The lower bound in* (1) *holds*:

$$\liminf_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} \ge -\frac{3}{2\sqrt{2}} \qquad \textit{almost surely.}$$

PROOF.   We show that for large t and any $\varepsilon > 0$, there are always particles above $\sqrt{2}t - (\tfrac{3}{2\sqrt{2}} + 3\varepsilon) \log t$. Let

$$A_t = \big\{ \nexists u \in N(t) : X_u(t) > \sqrt{2}t - (\tfrac{3}{2\sqrt{2}} + 2\sqrt{2}\varepsilon) \log t \big\}$$

and

$$B_t = \big\{ |N(\varepsilon \log t)| \ge t^{\varepsilon/2}, \ X_v(\log t) \ge -\sqrt{2}\varepsilon \log t \ \forall v \in N(\varepsilon \log t) \big\}.$$

Define $N(v; t)$ to be the set of descendants of particle v that are alive at time t. Let $l_t = t - \varepsilon \log t$. Then for all large t,

$$\mathbb{P}(A_t \cap B_t) \le \mathbb{E}\Bigg[\prod_{v \in N(\varepsilon \log t)} \mathbb{P}\Bigg(\nexists u \in N(v; t) :$$

$$X_u(t) > \sqrt{2}t - \bigg(\frac{3}{2\sqrt{2}} + 2\sqrt{2}\varepsilon \bigg) \log t \Big| \mathcal{F}_{\log t} \Bigg) \mathbb{1}_{B_t} \Bigg]$$

$$\le \mathbb{E}\Bigg[\prod_{v \in N(\log t)} \mathbb{P}\Bigg(\nexists u : X_u(l_t) > \sqrt{2}l_t - \frac{3}{2\sqrt{2}} \log l_t \bigg) \mathbb{1}_{B_t} \Bigg]$$

$$\le (1 - C_3)^{t^{\varepsilon/2}},$$

where $C_3 > 0$ is the constant from Proposition 9. Thus by Borel–Cantelli, for any lattice times $t_n \to \infty$, $\mathbb{P}(A_{t_n} \cap B_{t_n} \text{ infinitely often}) = 0$. But for all large t, $|N(\varepsilon \log t)| \ge e^{(\varepsilon \log t)/2} = t^{\varepsilon/2}$ and $X_v(\varepsilon \log t) \ge -\sqrt{2}\varepsilon \log t$ for all $v \in N(\log t)$, so we deduce that $\mathbb{P}(A_{t_n} \text{ infinitely often}) = 0$. Then it is again a simple task using the exponential tightness of Brownian motion to check that no particles move fur-

ther than $(3 - 2\sqrt{2})\varepsilon \log t$ between lattice times infinitely often (provided that we choose $t_n - t_{n-1}$ small enough). $\square$

PROPOSITION 15. *The lower bound in* (2) *holds*:

$$\limsup_{t \to \infty} \frac{M_t - \sqrt{2}t}{\log t} \geq -\frac{1}{2\sqrt{2}} \qquad \text{almost surely.}$$

PROOF. This is related to the proof of the lower bound in Theorem 1; the basic idea is similar to that in the proof given by Hu and Shi [11]. We let $\beta_t = \sqrt{2} - \frac{1}{2\sqrt{2}} \frac{\log t}{t}$ and

$$V(t) = \{v \in N(t) : X_v(r) < \beta_t r + 1 \ \forall r \leq t, \beta_t t - 1 \leq X_v(t) \leq \beta_t t\}$$

and define

$$I_n = \int_n^{2n} \mathbb{1}_{\{V(t) \neq \varnothing\}} \, dt.$$

We estimate the first two moments of I_n. From our earlier lower bound on $\mathbb{P}(H(y, t) \neq 0)$ (from the proof of Proposition 9, taking $y = \frac{1}{\sqrt{2}} \log t$) we get

$$\mathbb{E}[I_n] = \int_n^{2n} \mathbb{P}(V(t) \neq \varnothing) \, dt \geq c_1 \int_n^{2n} e^{-(\sqrt{2} \cdot \log t)/\sqrt{2}} \, dt = c_2.$$

Now,

$$\mathbb{E}[I_n^2] = \mathbb{E}\left[\int_n^{2n} \int_n^{2n} \mathbb{1}_{\{V(s) \neq \varnothing\}} \mathbb{1}_{\{V(t) \neq \varnothing\}} \, ds \, dt\right]$$

$$= 2 \int_n^{2n} \int_n^t \mathbb{P}(V(s) \neq \varnothing, V(t) \neq \varnothing) \, ds \, dt.$$

But whenever $s \leq t$,

(4) $\quad \mathbb{P}(V(s) \neq \varnothing, V(t) \neq \varnothing) \leq \mathbb{E}[|V(s)||V(t)|] = \mathbb{E}[|V(s)|\mathbb{E}[|V(t)||\mathcal{F}_s]]$

and letting $N(u; t)$ be the set of descendants of particle u that are alive at time t,

$$\mathbb{E}[|V(t)||\mathcal{F}_s] = \sum_{u \in N(s)} \mathbb{E}\left[\sum_{v \in N(u;t)} \mathbb{1}_{\{v \in V(t)\}} \Big| \mathcal{F}_s\right].$$

Now for any $s, t > 0$, let

$$A_t(s) = \{u \in N(s) : X_u(r) < \beta_t r + 1 \ \forall r \leq s\}$$

and

$$B_t(s) = \{u \in N(s) : \beta_t s - 1 \le X_u(s) \le \beta_t s\}.$$

By the Markov property, and then applying the many-to-one lemma with $f(r) = \beta_t(r - s)$ and $\alpha = \beta_t s - x + 1$, we have

$$\mathbb{E}\left[\sum_{v \in N(u;t)} \mathbb{1}_{\{v \in V(t)\}} \bigg| \mathcal{F}_s\right]$$

$$= \mathbb{1}_{\{u \in A_t(s)\}}$$

$$\times \mathbb{E}\left[\sum_{v \in N(t-s)} \mathbb{1}_{\{X_v(r-s)+x < \beta_t r+1 \ \forall r \le t-s, \ \beta_t t-1 \le X_v(t-s)+x \le \beta_t t\}}\right]\bigg|_{x=X_u(s)}$$

$$= \mathbb{1}_{\{u \in A_t(s)\}} e^{t-s} \mathbb{Q}\left[\frac{(\beta_t s - x + 1)\mathbb{1}_{\{\beta_t t-x-1 \le \xi_{t-s} \le \beta_t t-x\}}}{(\beta_t t - x + 1 - \xi_{t-s})e^{\beta_t \xi_{t-s} - \beta_t^2 (t-s)/2}}\right]\bigg|_{x=X_u(s)}$$

$$\le \mathbb{1}_{\{u \in A_t(s)\}} e^{t-s} \frac{\beta_t s - X_u(s) + 1}{e^{\beta_t^2 t - \beta_t X_u(s) - \beta_t - \beta_t^2 (t-s)/2}} \mathbb{Q}[\mathbb{1}_{\{\beta_t t-x-1 \le \xi_{t-s} \le \beta_t t-x\}}]|_{x=X_u(s)}$$

$$\le c_3 e^{-2s} t^{1/2} e^{(s \log t)/(2t)}$$

$$\times \mathbb{1}_{\{u \in A_t(s)\}} \frac{\beta_t s - X_u(s) + 1}{e^{-\beta_t X_u(s)}} \mathbb{Q}(\xi_t \in B_t(t)|\xi_s = x)\big|_{x=X_u(s)},$$

where for the last equality we used the fact that Bessel processes satisfy the Markov property. Substituting back into (4) and applying the many-to-two lemma, we get

$$\mathbb{P}(V(s) \ne \varnothing, V(t) \ne \varnothing)$$

$$\le \mathbb{E}\left[\sum_{u,v \in N(s)} \mathbb{1}_{\{u \in V(s)\}} c_3 e^{-2s} t^{1/2} e^{(s \log t)/(2t)}\right.$$

$$\times \mathbb{1}_{\{v \in A_t(s)\}}(\beta_t s - X_v(s) + 1)e^{\beta_t X_v(s)}$$

$$\left.\times \mathbb{Q}(\xi_t \in B_t(t)|\xi_s = x)\big|_{x=X_v(s)}\right]$$

$$= e^{3s} \mathbb{Q}\left[\frac{\mathbb{1}_{\{T>s\}}}{\zeta^1(s)} \mathbb{1}_{\{\xi_s^1 \in B_s(s)\}} c_3 e^{-2s} t^{1/2} e^{(s \log t)/(2t)}\right.$$

$$\left.\times \zeta^1(s)e^{\beta_t^2 s/2} \mathbb{Q}(\xi_t^1 \in B_t(t)|\xi_s^1)\right]$$

$$+ e^{2s} \mathbb{Q}\left[\frac{e^T \zeta^1(T)\mathbb{1}_{\{T \le s\}}}{\zeta^1(s)\zeta^2(s)} \mathbb{1}_{\{\xi_s^1 \in B_s(s)\}} c_3 e^{-2s} t^{1/2} e^{(s \log t)/(2t)}\right.$$

$$\left.\times \zeta^2(s)e^{\beta_t^2 s/2} \mathbb{Q}(\xi_t^2 \in B_t(t)|\xi_s^2)\right]$$

$$\leq c_4 t^{1/2} \mathbb{Q}\big(\xi_s^1 \in B_s(s), \xi_t^1 \in B_t(t)\big)$$

$$+ c_4 t^{1/2} \mathbb{Q}\left[\frac{(\beta_t r - \xi_r^1 + 1)e^{T + \beta_t \xi_T^1 - \beta_t^2 T/2}}{(\beta_t s - \xi_s^1 + 1)e^{\beta_t \xi_s^1 - \beta_t^2 s/2}} \mathbb{1}_{\{T \leq s\}} e^s \mathbb{1}_{\{\xi_s^1 \in B_s(s), \xi_t^2 \in B_t(t)\}} \right]$$

$$\leq c_4 t^{1/2} \mathbb{Q}\big(\xi_s^1 \in B_s(s), \xi_t^1 \in B_t(t)\big)$$

$$+ c_5 t^{1/2} e^{(s \log t)/(2t)} \mathbb{Q}\big[(\beta_t T - \xi_T^1 + 1)e^{2T - (T \log t)/(2t) - \beta_t(\beta_t T - \xi_T^1 + 1)}$$

$$\times \mathbb{1}_{\{T \leq s\}} \mathbb{1}_{\{\xi_s^1 \in B_s(s), \xi_t^2 \in B_t(t)\}} \big].$$

We must now estimate the last line above. The $\mathbb{Q}(\cdot)$ part of the first term is the probability that a Bessel process is near the origin at time s, and then again at time t; so the first term is no bigger than a constant times $t^{1/2} s^{-3/2}(t - s + 1)^{-3/2}$. Then using notation from Section 2, the expectation $\mathbb{Q}[\cdot]$ in the second term is

$$\hat{P}\big[Y_\tau^1 e^{2\tau - (\tau \log t)/(2t) - \beta_t Y_\tau^1} \mathbb{1}_{\{\tau \leq s\}}$$

$$\times \mathbb{1}_{\{(\log s)/(2\sqrt{2}) - (s \log t)/(2\sqrt{2}t) + 1 \leq Y_s^1 \leq (\log s)/(2\sqrt{2}) - (s \log t)/(2\sqrt{2}t) + 2\}} \mathbb{1}_{\{1 \leq Y_t^2 \leq 2\}}\big].$$

Thus by Lemma 5,

$$\mathbb{P}\big(V(s) \neq \varnothing, V(t) \neq \varnothing\big) \leq c_6\big(t^{-2} + t^{-1}(t - s + 1)^{3/2}\big)$$

and hence

$$\mathbb{E}\big[I_n^2\big] \leq 2c_6 \int_n^{2n} \int_n^t \big(t^{-2} + t^{-1}(t - s + 1)^{3/2}\big) \, ds \, dt \leq c_7,$$

so

$$\mathbb{P}(I_n > 0) \geq \mathbb{P}\big(I_n \geq \mathbb{E}[I_n]/2\big) \geq \frac{\mathbb{E}[I_n]^2}{4\mathbb{E}[I_n^2]} \geq c_8 > 0.$$

When n is large, at time $2\delta \log n$ there are at least n^δ particles, all of which have position at least $-2\sqrt{2}\delta \log n$. By the above, the probability that none of these has a descendant that goes above $\sqrt{2}s - \frac{1}{2\sqrt{2}}\log s - 2\sqrt{2}\delta \log n$ for any s between $2\delta \log n + n$ and $2\delta \log n + 2n$ is no larger than

$$(1 - c_8)^{n^\delta}.$$

The result follows by the Borel–Cantelli lemma since $\sum_n (1 - c_8)^{n^\delta} < \infty$. $\quad\square$

PROOF OF THEOREM 2. The result is given by combining Lemmas 12, 13 and 14 and Proposition 15. $\quad\square$

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