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REFLECTING RANDOM WALK IN FRACTAL DOMAINS[1]

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In this paper, we show that reflecting Brownian motion in any bounded domain D can be approximated, as $k \to \infty$, by simple random walks on "maximal connected" subsets of $(2^{-k}\mathbb{Z}^d) \cap D$ whose filled-in interiors are inside of D.

1. Introduction. We proved in a recent article [1] that reflecting Brownian motion in a domain D can be approximated by a sequence of random walks on subsets A_k of $(2^{-k}\mathbb{Z}^d) \cap D$. We chose A_k's in a "natural" way, to be described in a moment. Our main theorem in [1] was limited to only some domains D ("extension domains"). We also provided a counterexample showing that random walks on A_k's do not converge to the reflecting Brownian motion in D for some domains D. In this paper, we will show in Theorems 3.6 and 4.2 that reflecting Brownian motion on any domain can be approximated by a sequence of discrete-time, as well as continuous-time, random walks if the state spaces D_k for the random walks are constructed in a different "natural" way.

The sets A_k were constructed in [1] as follows. First, we found the maximal connected set consisting of line segments contained in D, joining neighboring vertices in $(2^{-k}\mathbb{Z}^d) \cap D$. Then we let A_k be the set of vertices in $(2^{-k}\mathbb{Z}^d) \cap D$ at the ends of these line segments. It turns out that the "correct" way (employed in the present article) to construct the state space for the random walk is to start with the maximal connected set consisting of cubes contained in D, with edge length 2^{-k} and vertices in $(2^{-k}\mathbb{Z}^d) \cap D$. Then we let D_k be the set of vertices in $(2^{-k}\mathbb{Z}^d) \cap D$ which belongs to these cubes. Intuitively speaking, A_k may penetrate very thin crevices in D. The simple random walk on A_k may spend a nonnegligible amount of time in such branches of A_k, but reflecting Brownian motion spends very little time in sets with very small volume. Replacing edges in the construction of A_k's with cubes eliminates the mismatch between the shapes of D and the approximating discrete set.

The technical essence of the paper is Theorem 2.1 which shows that, in a sense, the Dirichlet form for reflecting Brownian motion can be approximated from below

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by discrete Dirichlet forms. This theorem and the remaining part of the proof of the main result are challenging because "naive" discrete approximating schemes for the Dirichlet form of reflecting Brownian motion do not work; see Example 2.2.

In the rest of the Introduction, we will review some basic facts about reflecting Brownian motion in nonsmooth domains and elaborate on some of the points mentioned above.

Reflecting Brownian motion in a bounded domain D in $\mathbb{R}^d$ is a symmetric Markov process that behaves like Brownian motion inside D and is "pushed" back along the "inward normal" direction at the boundary ∂D of D. It is a prototype of diffusions with boundary condition and can be used to study heat equations with Neumann and Robin's boundary condition. It is also widely used in modeling, for example, in physics, in queuing theory and in financial mathematics. Reflecting Brownian motion has been studied by various authors using various methods; see [1, 2] and the references therein. When D is a bounded extension domain (see next paragraph for its definition), reflecting Brownian motion X can be constructed as a strong Markov process on $\overline{D}$ starting from every point in $\overline{D}$ except a polar set. Every bounded Lipschitz domain is an extension domain. When D is a general bounded domain, reflecting Brownian motion can still be constructed on $\overline{D}$, but typically it is no longer a strong Markov process. In a recent paper [1], we developed three discrete approximation schemes for reflecting Brownian motion in bounded domains, providing effective ways to simulate the process in practice. The first two approximation schemes are discrete-time and continuous-time simple random walks on grids $2^{-k}\mathbb{Z}^d \cap D$ inside D. For these two approximation schemes, we need to assume that D is a bounded extension domain. A counter example is given in [1], showing that these approximation schemes do not work for some bounded domains. However, the third approximation scheme developed in [1], called myopic conditioning, works for any bounded domain D. Myopic conditioning generates a continuous-time and continuous-space process and, therefore, it is not suited for computer simulations. The purpose of this paper is to develop discrete-time and continuous-time simple random walk approximations on grids inside D that work for every bounded domain D.

We now give a precise description of reflecting Brownian motion on bounded domains. Let $d \geq 1$ and D be a bounded domain in $\mathbb{R}^d$. The Sobolev space $W^{1,2}(D)$ of order $(1, 2)$ is the space of $L^2(D)$-functions on D whose distributional derivative ∇f is also $L^2(D)$-integrable. It is well known that $W^{1,2}(D)$ is a Hilbert space under norm $\|f\|_{1,2} := (\|f\|^2_{L^2(D)} + \|\nabla f\|^2_{L^2(D)})^{1/2}$. We define on $W^{1,2}(D)$ a bilinear form

$$\mathcal{E}(f, g) = \frac{1}{2}\int_D \nabla f(x) \cdot \nabla g(x)\, dx \qquad \text{for } f, g \in W^{1,2}(D).$$

It is known (see, e.g., [4]) that $(\mathcal{E}, W^{1,2}(D))$ is a Dirichlet form on $L^2(D; dx)$. When $C(\overline{D}) \cap W^{1,2}(D)$ is dense in both $(C(\overline{D}), \|\cdot\|_\infty)$ and in $(W^{1,2}(D), \|\cdot\|_{1,2})$,

$(\mathcal{E}, W^{1,2}(D))$ is a regular Dirichlet form on $L^2(\overline{D}; m)$, where m is the Lebesgue measure on D extended to $\overline{D}$ by setting $m(\partial D) = 0$. In this case, there is a continuous conservative strong Markov process X on $\overline{D}$ associated with $(\mathcal{E}, W^{1,2}(D))$, starting from quasi every point from $\overline{D}$. The process is called (normally) reflecting Brownian motion on $\overline{D}$. It is known (see, e.g., Theorems 1 and 2 on pages 13 and 14 of [13]) that $(\mathcal{E}, W^{1,2}(D))$ is a regular Dirichlet form on $L^2(\overline{D}; m)$ if D is star-shaped with respect to a point in D or if D has continuous boundary. Note that $(\mathcal{E}, W^{1,2}(\mathbb{R}^d))$ is a regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$. Hence $(\mathcal{E}, W^{1,2}(D))$ is a regular Dirichlet form on $L^2(\overline{D}; m)$ if D is an extension domain in the following sense: there is a linear continuous operator $T : W^{1,2}(D) \to W^{1,2}(\mathbb{R}^d)$ such that $Tf = f$ a.e. on D for every $f \in W^{1,2}(D)$. Recall that a domain D is called a *locally uniform domain* if there are $\delta \in (0, \infty]$ and $C > 0$ such that for every $x, y \in D$ with $|x - y| < \delta$, there is a rectifiable curve γ in D connecting x and y with $\text{length}(\gamma) \le C|x - y|$ and moreover,

$$\min\{|x - z|, |z - y|\} \le C \operatorname{dist}(z, D^c) \qquad \text{for every } z \in \gamma.$$

A domain is said to be a *uniform domain* if the above property holds with $\delta = \infty$. The above definition is taken from Väisälä [15], where various equivalent definitions are discussed. Uniform domain and locally uniform domain are also called (ε, ∞)-domain and (ε, δ)-domain, respectively, in [12]. For example, the classical van Koch snowflake domain in the conformal mapping theory is a uniform domain in $\mathbb{R}^2$. Note that every bounded Lipschitz domain is uniform, and every *nontangentially accessible domain* defined by Jerison and Kenig in [11] is a uniform domain (see (3.4) of [11]), while every Lipschitz domain is an (ε, δ)-domain. It is proved in [12] that every locally uniform domain is an extension domain. However, for general domain D, $(\mathcal{E}, W^{1,2}(D), \mathcal{E})$ does not need to be regular on $L^2(\overline{D}; dx)$. A unit disk in $\mathbb{R}^2$ with a slit removed is such an example. See page 14 of [13] for an example of D due to Kolsrud with $\partial D = \partial \overline{D}$ such that the Dirichlet form $(\mathcal{E}, W^{1,2}(D), \mathcal{E})$ is not regular on $L^2(\overline{D}; dx)$. Nevertheless, for any domain $D \subset \mathbb{R}^d$, one can always find a compact regularizing space $\widetilde{D}$ that contains D as a dense open subset such that $(\mathcal{E}, W^{1,2}(D))$ becomes a regular Dirichlet space on $L^2(\widetilde{D}; \widetilde{m})$, where $\widetilde{m}$ is the Lebesgue measure on D extended to $\widetilde{D}$ by setting $\widetilde{m}(\widetilde{D} \setminus D) = 0$; see [8] and [2]. Let $\widetilde{X}$ be the associated conservative strong Markov process on $\widetilde{D}$, which can also be called reflecting Brownian motion on D. Let X be the projection of $\widetilde{X}$ onto $\overline{D}$. Since for any given time $t > 0$, $\mathbb{P}_{\widetilde{m}}(\widetilde{X}_t \in \widetilde{D} \setminus D) = 0$, under the normalized Lebesgue measure on D, $\widetilde{X}$ and X have the same finite-dimensional distributions.

A key technical element of this paper is to show that, for any bounded domain D in $\mathbb{R}^d$, there exists a sequence $\{\varphi_j, j \ge 1\}$ of bounded smooth functions on D that is dense in the Sobolev space $W^{1,2}(D)$, separates points in D and satisfies the property (2.1) described below. We can deduce from its existence that there is a metric ρ on D ("refinement of the Euclidean metric") which induces the same Euclidean topology inside D and has the property that reflecting Brownian motion on

D can be lifted as a strong Markov process on the ρ-closure $\widetilde{D}$ of D. This enables us to show that the random walk approximation on grids whose filled-in interiors are inside D works for reflecting Brownian motions on arbitrary bounded domains. In this paper, we also provide a proof that any weak limit of random walks on grids inside D is a stationary symmetric Markov process (see Theorem 3.3). This is a key step in proving that the weak limit is indeed the stationary reflecting Brownian motion in D, using a Dirichlet form approach. This claim was made in [1] but regrettably no proof was given there.

The rest of the paper is organized as follows. In Section 2, we establish a result (Theorem 2.1) regarding the Sobolev space $W^{1,2}(D)$ that will play an important role in this paper. Though the result is purely analytic, we employ some probabilistic techniques in its proof. The proof that reflecting Brownian motion in any bounded domain D can be approximated by discrete-time random walk on grids inside D is given in Section 3. The corresponding result for continuous-time random walk approximation is presented in Section 4.

2. Energy form estimates. Let $D \subset \mathbb{R}^d$ be a domain (connected open set) that has finite Lebesgue measure. Fix an arbitrarily small $c_1 \in (0, 1)$ and a point $x_0 \in D$. For each integer k, let $\mathcal{A}_k$ be the family of all closed d-dimensional cubes $Q \subset D$ with edge length 2^{-k}, such that:

(i) the vertices of Q belong to $(2^{-k}\mathbb{Z})^d$;
(ii) the distance from Q to ∂D is greater than $c_1 2^{-k}$;
(iii) there exists a sequence of cubes $Q_1, Q_2, \dots, Q_m = Q$, satisfying (i) and (ii), and such that $x_0 \in Q_1$, and $Q_j \cap Q_{j+1}$ is a $(d-1)$-dimensional cube, for all $j = 1, 2, \dots, m-1$.

Since D has a finite volume, there is some $k_0 \in \mathbb{Z}$ such that $\mathcal{A}_k = \varnothing$ for every $k \le k_0$. Using scaling if necessary, we may and do assume that $\mathcal{A}_k = \varnothing$ for $k \le 0$. Let $D_k = \bigcup_{Q \in \mathcal{A}_k} Q$. Let $\mathcal{A}'_k$ be the family of all edges, and let $\mathcal{A}''_k$ be the family of all vertices of all cubes $Q \in \mathcal{A}_k$. We will write $\overline{xy}$ to denote the line segment with endpoints x and y. Note that if $\overline{xy} \in \mathcal{A}'_k$, then so is $\overline{yx}$. Thus in the summation on the left-hand side of (2.1), each line segment in $\mathcal{A}'_k$ is counted twice.

THEOREM 2.1. *Suppose that $D \subset \mathbb{R}^d$ is domain with finite volume and $c_1 \in (0, 1)$, $x_0 \in D$ and D_k's are defined as above. There exists a countable sequence of bounded functions $\{\varphi_j\}_{j \ge 1} \subset W^{1,2}(D) \cap C^\infty(D)$ such that*:

(i) *$\{\varphi_j\}_{j\ge1}$ is dense in $W^{1,2}(D)$*;
(ii) *$\{\varphi_j\}_{j\ge1}$ separates points in D*;
(iii) *for each $j \ge 1$,*

$$\limsup_{k\to\infty} 2^{k(2-d)} \sum_{\overline{xy}\in\mathcal{A}'_k} \big(\varphi_j(x) - \varphi_j(y)\big)^2 \le 2 \int_D |\nabla \varphi_j(x)|^2 \, dx. \tag{2.1}$$

PROOF. *Step* 1. First note that the Sobolev space $(W^{1,2}(D), \|\cdot\|_{1,2})$ is separable. This can be seen as follows. Let G_1 be the 1-resolvent for the Dirichlet form $(\mathcal{E}, W^{1,2}(D))$; that is, G_1 is the linear operator from $L^2(D;m)$ to $W^{1,2}(D)$ uniquely defined by

$$\mathcal{E}_1(G_1 f, g) = \int_D f(x)g(x)\,dx \qquad \text{for every } g \in W^{1,2}(D).$$

Here $\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + \int_D u(x)v(x)\,dx$. It follows that $G_1 L^2(D;dx)$ is dense in the space $(W^{1,2}(D), \|\cdot\|_{1,2})$ and that

$$\mathcal{E}_1(G_1 f, G_1 f) = \int_D f G_1 f(x)\,dx \le \int_D f(x)^2\,dx.$$

Since $L^2(D;dx)$ is separable, there is a sequence $\{f_k, k \ge 1\}$ of bounded functions that is dense in $L^2(D;dx)$. Consequently, $\{\eta^k := G_1 f_k, k \ge 1\}$ is a sequence of bounded functions that is dense in $(W^{1,2}(D), \|\cdot\|_{1,2})$.

Theorem 2 on page 251 of [7] implies that for every function η^k, there exists a sequence of functions $\{\eta^k_j, j \ge 1\} \subset W^{1,2}(D) \cap C^\infty(D)$ with the property that $\lim_{j\to\infty} \|\eta^k_j - \eta^k\|_{1,2} = 0$. Moreover, the proof given in [7] shows that we can choose η^k_j so that $\sup_{x\in D} |\eta^k_j(x)| \le 3 \sup_{x\in D} |\eta^k(x)|$.

Step 2. Constants $c_1, c_2, \ldots$ may change value from one "step" to another in this proof.

We will use a regularized version of the distance function defined in [14], Theorem 2, page 171. That theorem implies that there exist $0 < c_1, c_2, c_3, c_4 < \infty$ such that for every integer j there is a C^∞ function $d_j : D \to (0, 2^{-j}]$ with the following properties:

$$c_1\big(\mathrm{dist}(x,\partial D) \wedge 2^{-j}\big) \le d_j(x) \le c_2\big(\mathrm{dist}(x,\partial D) \wedge 2^{-j}\big), \tag{2.2}$$

$$\sup_{x\in D} \big|\nabla d_j(x)\big| \le c_3, \tag{2.3}$$

$$\sup_{x\in D} \left| d_j(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_m} d_j(x) \right| \le c_4 \qquad \text{for } 1 \le i, m \le d. \tag{2.4}$$

By dividing d_j by an appropriate constant, we may and will assume from now on that (2.2) and (2.3) hold with $c_2 = c_3 = 1$.

The existence of functions d_j follows essentially from [14], Theorem 2, page 171. The only difference between our claim and that in [14], Theorem 2, page 171, is that [14] is concerned with the condition

$$c_1 \,\mathrm{dist}(x,\partial D) \le d_j(x) \le c_2 \,\mathrm{dist}(x,\partial D),$$

in place of our condition (2.2). The method of proof given in [14] applies to (2.2) if we subdivide cubes constructed in [14], Section 1.2, page 167, with edges longer than 2^{-j} into cubes with edge length 2^{-j}.

Step 3. Let $\psi:\mathbb{R}^d \to [0,\infty)$ be a C^∞ "mollifier" with support in the ball $B(0,1/2)$ such that $\int_{B(0,1/2)} \psi(x)\,dx = 1$. For $r > 0$ let $\psi_r(x) = r^{-d}\psi(x/r)$, and note that $\sup_x \psi_r(x) = c_1 r^{-d}$.

The function $\psi_r'(y) := \frac{\partial}{\partial r}\psi_r(y)$ is C^∞. It is supported in $B(0,r/2)$ and satisfies the condition $\int_{B(0,r/2)} \psi_r'(y)\,dy = 0$. Let $\|\cdot\|_1$ denote the L^1 norm with respect to the Lebesgue measure restricted to D. Note that $\|\psi_r'\|_1 = c_2 r^{-1}$ and $\|\psi_r' \vee 0\|_1 = \|\psi_r' \wedge 0\|_1 = \|\psi_r'\|_1/2$. Consider $x \in D$, and let

$$a_x^+(\cdot) = \frac{\psi'_{d_j(x)}(\cdot) \vee 0}{\|\psi'_{d_j(x)} \vee 0\|_1} \quad \text{and} \quad a_x^-(\cdot) = -\frac{\psi'_{d_j(x)}(\cdot) \wedge 0}{\|\psi'_{d_j(x)} \wedge 0\|_1}.$$

The functions $a_x^+(\cdot)$ and $a_x^-(\cdot)$ are probability density functions. Let A_x^+ and A_x^- be independent $\mathbb{R}^d$-valued random variables with densities $a_x^+(\cdot)$ and $a_x^-(\cdot)$, respectively. Let $\int_{A_x^+}^{A_x^-} d\xi$ denote the integral with respect to the length measure on the line segment joining A_x^+ and A_x^-. Clearly, the measure $\mathbb{E}\int_{A_x^+}^{A_x^-} d\xi$ is supported on $\overline{B(0,d_j(x)/2)}$. We will now show that it has a density bounded above by $c_3 d_j(x)^{1-d}$. In other words, for every set $K \subset D$,

$$\mathbb{E}\int_{A_x^+}^{A_x^-} \mathbf{1}_K(\xi)\,d\xi \le c_3 d_j(x)^{1-d} m\big(K \cap B\big(0, d_j(x)/2\big)\big).$$

Clearly the functions $a_x^+(\cdot)$ and $a_x^-(\cdot)$ are bounded by $\alpha_x := c_4 d_j(x)^{-d}$. Consider any $z \in B(x, d_j(x)/2)$ and small $\delta > 0$. The probability that at least one of the random points A_x^+ or A_x^- belongs to $B(z,2\delta)$ is less than $c_5\delta^d\alpha_x \le c_6\delta^d d_j(x)^{-d}$.

For $k \ge 2$, the probability of the event $F_k = \{A_x^+ \in B(z,\delta 2^k) \setminus B(z,\delta 2^{k-1})\}$ is bounded by $c_7\delta^d 2^{dk}\alpha_x$. Let G be the intersection of $B(x,d_j(x)/2)$ and the smallest cone with vertex A_x^+ containing $B(z,\delta)$. The conditional probability, given F_k, of the event $\{A_x^- \in G\}$ is bounded by α_x times the volume of G; hence, it is bounded by $c_8\alpha_x d_j(x)(2^{-k}d_j(x))^{d-1} = c_9\alpha_x d_j(x)^d 2^{k(1-d)}$. Multiplying the two estimates and summing over $k \ge 2$, such that $B(z,\delta 2^{k-1})$ does not contain $B(x,d_j(x)/2)$, gives the bound

$$\sum_{\delta 2^k \le 2 d_j(x)} c_7\delta^d 2^{dk}\alpha_x c_9\alpha_x d_j(x)^d 2^{k(1-d)} \le c_{10}\delta^{d-1} d_j(x)^{1-d}.$$

Adding to this quantity $c_6\delta^d d_j(x)^{-d}$ [the estimate representing the case when at least one of the random points A_x^+ or A_x^- belongs to $B(z,2\delta)$] gives a similar bound $c_{11}\delta^{d-1}d_j(x)^{1-d}$. The last quantity is an upper bound for the probability that the line segment joining A_x^+ and A_x^- intersects $B(z,2\delta)$. Since $\int_{A_x^+}^{A_x^-} \mathbf{1}_{B(z,2\delta)}(\xi)\,d\xi \le 2\delta$ with probability 1, we obtain

$$\mathbb{E}\int_{A_x^+}^{A_x^-} \mathbf{1}_{B(z,2\delta)}(\xi)\,d\xi \le c_{11}\delta^{d-1}d_j(x)^{1-d}2\delta = c_{12}\delta^d d_j(x)^{1-d}.$$

This estimate holds for all $z \in B(x, d_j(x)/2)$ and all small $\delta > 0$ so the density of the measure $\mathbb{E}\int_{A_x^+}^{A_x^-} d\xi$ is bounded by $c_{13} d_j(x)^{1-d}$.

We will also need the following version of the above estimate. Let $\psi_r''(y) = \frac{\partial^2}{\partial r^2}\psi_r(y)$, $b_x^+(\cdot) = (\psi_{d_j(x)}''(\cdot) \vee 0)/\|\psi_{d_j(x)}'' \vee 0\|_1$ and $b_x^-(\cdot) = -(\psi_{d_j(x)}''(\cdot) \wedge 0)/\|\psi_{d_j(x)}'' \wedge 0\|_1$. Note that $\|\psi_r'\|_1 = c_{14} r^{-2}$, $\int_{B(0,r/2)} \psi_r''(y)\, dy = 0$ and so $\|\psi_r'' \vee 0\|_1 = \|\psi_r'' \wedge 0\|_1 = \|\psi_r''\|_1/2$. The functions $b_x^+(\cdot)$ and $b_x^-(\cdot)$ are probability density functions. Let B_x^+ and B_x^- be independent $\mathbb{R}^d$-valued random variables with densities $b_x^+(\cdot)$ and $b_x^-(\cdot)$. The measure $\mathbb{E}\int_{B_x^+}^{B_x^-} d\xi$ has a density bounded above by $c_{15} d_j(x)^{1-d}$. In other words, for every set $K \subset D$,

$$\mathbb{E}\int_{B_x^+}^{B_x^-} \mathbf{1}_K(\xi)\, d\xi \le c_{15} d_j(x)^{1-d} m\bigl(K \cap B\bigl(0, d_j(x)/2\bigr)\bigr).$$

We omit the proof because it is analogous to the one given above.

Step 4. Consider a function $\eta \in W^{1,2}(D) \cap C^\infty(D)$ and for integer $j \ge 1$ and $x \in D$, let

$$\eta_j(x) = \int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)\eta(x-y)\, dy. \tag{2.5}$$

We will show that $\eta_j \in W^{1,2}(D) \cap C^\infty(D)$ and $\eta_j \to \eta$ in $W^{1,2}(D)$ as $j \to \infty$. Since η and d_j are C^∞ functions, so is η_j.

We have

$$\begin{aligned}
\eta_j(x)^2 &= \left(\int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)\eta(x-y)\, dy\right)^2 \\
&\le \int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)^2\, dy \int_{B(0,d_j(x)/2)} \eta(x-y)^2\, dy \\
&\le c_1\bigl(d_j(x)^{-d}\bigr)^2 d_j(x)^d \int_{B(0,d_j(x)/2)} \eta(x-y)^2\, dy \\
&\le c_2 d_j(x)^{-d} \int_{B(0,d_j(x)/2)} \eta(x-y)^2\, dy.
\end{aligned} \tag{2.6}$$

Suppose that $z \in B(x, d_j(x)/2)$. Then

$$\operatorname{dist}(z, \partial D) \ge \operatorname{dist}(x, \partial D) - |x-z| \ge d_j(x) - |x-z| \ge d_j(x)/2.$$

Hence

$$d_j(z) \ge c_3\bigl(\operatorname{dist}(z, \partial D) \wedge 2^{-j}\bigr) \ge c_3\bigl(d_j(x)/2 \wedge 2^{-j}\bigr) = c_3 d_j(x)/2.$$

Therefore, $c_4 d_j(z) \ge d_j(x)/2$ and $\mathbf{1}_{B(x,d_j(x)/2)}(z) \le \mathbf{1}_{B(z,c_4 d_j(z))}(x)$. Assuming again that $z \in B(x, d_j(x)/2)$, we obtain

$$\begin{aligned}
\operatorname{dist}(z, \partial D) &\le \operatorname{dist}(x, \partial D) + |x-z| \le \operatorname{dist}(x, \partial D) + d_j(x)/2 \\
&\le \operatorname{dist}(x, \partial D) + \operatorname{dist}(x, \partial D)/2 < 2\operatorname{dist}(x, \partial D).
\end{aligned}$$

Hence

$$
\begin{aligned}
d_j(x) &\geq c_5\big(\mathrm{dist}(x, \partial D) \wedge 2^{-j}\big) \geq c_5\big(\mathrm{dist}(z, \partial D)/2 \wedge 2^{-j}\big) \\
&\geq c_5\big(d_j(z)/2 \wedge 2^{-j}\big) = c_5 d_j(z)/2.
\end{aligned}
$$

This implies that

$$
d_j(x)^{-d}\mathbf{1}_{B(x,d_j(x)/2)}(z) \leq c_5^{-d}\big(d_j(z)/2\big)^{-d}\mathbf{1}_{B(z,c_4 d_j(z))}(x). \tag{2.7}
$$

For later reference we derive an inequality that is slightly more general than what is needed in this step. For a set $Q \subset D$, let $\widehat{Q} = \bigcup_{x\in Q} B(x, d_j(x)/2)$. We combine (2.6) and (2.7) to see that

$$
\begin{aligned}
\int_Q \eta_j(x)^2\,dx &\leq c_2 \int_Q d_j(x)^{-d} \int_{B(0,d_j(x)/2)} \eta(x-y)^2\,dy\,dx \\
&= c_2 \int_Q d_j(x)^{-d} \int_{B(x,d_j(x)/2)} \eta(z)^2\,dz\,dx \\
&= c_2 \int_Q \int_{\widehat{Q}} d_j(x)^{-d}\mathbf{1}_{B(x,d_j(x)/2)}(z)\eta(z)^2\,dz\,dx \\
&\leq c_2 \int_{\widehat{Q}} \int_Q c_5^{-d}\big(d_j(z)/2\big)^{-d}\mathbf{1}_{B(z,c_4 d_j(z))}(x)\,dx\,\eta(z)^2\,dz \\
&\leq c_6 \int_{\widehat{Q}} \eta(z)^2\,dz.
\end{aligned} \tag{2.8}
$$

In particular, the inequality applies to $Q = D = \widehat{Q}$. Hence

$$
\int_D \eta_j(x)^2\,dx \leq c_6 \int_D \eta(z)^2\,dz. \tag{2.9}
$$

For any $x \in D$, j and $1 \leq i \leq d$, note that $\psi_{d_j(x)}(y) = 0$ for $y \notin B(0, d_j(x)/2)$, and $\eta(x-y)$ is differentiable in $y \in \overline{B(0, d_j(x)/2)}$. So we have

$$
\begin{aligned}
\left(\frac{\partial}{\partial x_i}\eta_j(x)\right)^2 &= \left(\frac{\partial}{\partial x_i}\int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)\eta(x-y)\,dy\right)^2 \\
&= \bigg(\int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)\,\frac{\partial}{\partial x_i}\eta(x-y)\,dy \\
&\qquad + \int_{\mathbb{R}^d}\left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\right)\eta(x-y)\,dy\bigg)^2 \\
&= \bigg(\int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)\,\frac{\partial}{\partial x_i}\eta(x-y)\,dy \\
&\qquad + \int_{B(0,d_j(x)/2)}\left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\right)\eta(x-y)\,dy\bigg)^2
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&\le 2\biggl(\int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)\, \frac{\partial}{\partial x_i}\eta(x-y)\,dy\biggr)^2 \\
&\quad + 2\biggl(\int_{B(0,d_j(x)/2)} \biggl(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\biggr)\eta(x-y)\,dy\biggr)^2 \\
&\le 2\int_{B(0,d_j(x)/2)} \psi_{d_j(x)}(y)^2\,dy \int_{B(0,d_j(x)/2)} \biggl(\frac{\partial}{\partial x_i}\eta(x-y)\biggr)^2\,dy \\
&\quad + 2\biggl(\int_{B(0,d_j(x)/2)} \biggl(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\biggr)\eta(x-y)\,dy\biggr)^2 \\
&\le c_7 d_j(x)^{-d} \int_{B(0,d_j(x)/2)} \biggl(\frac{\partial}{\partial x_i}\eta(x-y)\biggr)^2\,dy \\
&\quad + 2\biggl(\int_{B(0,d_j(x)/2)} \biggl(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\biggr)\eta(x-y)\,dy\biggr)^2.
\end{aligned}
$$

Recall that $\psi_r'(y) := \frac{\partial}{\partial r}\psi_r(y)$ is a C^∞ function supported in $B(0,1/2)$ with $\int_{B(0,1/2)} \psi_r'(y)\,dy = 0$. It follows from (2.3) that $\sum_{i=1}^d |\frac{\partial}{\partial x_i} d_j(x)| \le d$. We have

$$
\begin{aligned}
\sum_{i=1}^d \biggl|\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\biggr| &= \sum_{i=1}^d \biggl|\frac{\partial}{\partial x_i} d_j(x)\biggr| |\psi'_{d_j(x)}(y)| \\
&\le d|\psi'_{d_j(x)}(y)| \le c_9 d_j(x)^{-d-1}.
\end{aligned}
\tag{2.11}
$$

Recall the definitions of the random variables A_x^+ and A_x^- from step 3. It follows from $\int_{B(0,d_j(x)/2)} \psi'_{d_j(x)}(y)\,dy = 0$ and $\int_{B(0,d_j(x)/2)} |\psi'_{d_j(x)}(y)|\,dy \le c_{10}/d_j(x)$ that

$$
\begin{aligned}
&\biggl|\int_{B(0,d_j(x)/2)} \biggl(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\biggr)\eta(x-y)\,dy\biggr| \\
&\qquad \le c_{10} d_j(x)^{-1} \bigl|\mathbb{E}\bigl(\eta(x-A_x^+) - \eta(x-A_x^-)\bigr)\bigr| \\
&\qquad \le c_{10} d_j(x)^{-1} \mathbb{E}\bigl|\eta(x-A_x^+) - \eta(x-A_x^-)\bigr| \\
&\qquad \le c_{10} d_j(x)^{-1} \mathbb{E}\int_{A_x^+}^{A_x^-} |\nabla \eta(x-z)|\,dz,
\end{aligned}
\tag{2.12}
$$

where the last integral is along a line segment from A_x^+ to A_x^-. By step 3, the measure $\mathbb{E}\int_{A_x^+}^{A_x^-} dz$ has a density that is bounded above by $c_{11} d_j(x)^{1-d}$ and vanishes outside of the ball $B(0, d_j(x)/2)$. In other words, for every set $K \subset D$,

$\mathbb{E}\int_{A_x^+}^{A_x^-} \mathbf{1}_K(z)\,dz \le c_{11} d_j(x)^{1-d} m(K \cap B(0, d_j(x)/2))$. It follows that

$$\begin{aligned}
&\left|\int_{B(0,d_j(x)/2)} \left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\right)\eta(x-y)\,dy\right| \\
&\qquad \le c_{10} d_j(x)^{-1}\mathbb{E}\int_{A_x^+}^{A_x^-} |\nabla\eta(x-z)|\,dz \\
&\qquad \le c_{10} d_j(x)^{-1}\int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)| c_{11} d_j(x)^{1-d}\,dz \\
&\qquad = c_{12} d_j(x)^{-d}\int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)|\,dz.
\end{aligned} \tag{2.13}$$

This implies that

$$\begin{aligned}
&\left(\int_{B(0,d_j(x)/2)} \left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(y)\right)\eta(x-y)\,dy\right)^2 \\
&\qquad \le \left(c_{12} d_j(x)^{-d}\int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)|\,dz\right)^2 \\
&\qquad \le c_{13} d_j(x)^{-2d} d_j(x)^d \int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)|^2\,dz \\
&\qquad = c_{13} d_j(x)^{-d}\int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)|^2\,dz.
\end{aligned}$$

We combine this estimate with (2.10) to obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_i}\eta_j(x)\right)^2 \\
&\qquad \le c_7 d_j(x)^{-d}\int_{B(0,d_j(x)/2)} \left(\frac{\partial}{\partial x_i}\eta(x-y)\right)^2 dy \\
&\qquad\qquad + c_{14} d_j(x)^{-d}\int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)|^2\,dz.
\end{aligned}$$

Summing over i yields

$$|\nabla\eta_j(x)|^2 \le c_{15} d_j(x)^{-d}\int_{B(0,d_j(x)/2)} |\nabla\eta(x-z)|^2\,dz.$$

Recall that we write $\widehat{Q} = \bigcup_{x\in Q} B(x, d_j(x)/2)$ for $Q \subset D$. The same argument that leads from (2.6) to (2.9) gives

$$\int_Q |\nabla\eta_j(x)|^2\,dx \le c_{16}\int_{\widehat{Q}} |\nabla\eta(x)|^2\,dx. \tag{2.14}$$

This formula and (2.9) show that $\eta_j \in W^{1,2}(D)$. We have pointed out earlier in the proof that $\eta_j \in C^\infty(D)$.

Let $K_\varepsilon = \{x \in D : \mathrm{dist}(x, D^c) > \varepsilon\}$. We will show that $\eta_j \to \eta$ in $W^{1,2}(D)$ as $j \to \infty$. To see this, fix an arbitrarily small $\delta > 0$ and find $\varepsilon > 0$ so small that

$$\int_{K_{2\varepsilon}^c} (\eta(x)^2 + |\nabla \eta(x)|^2)\, dx < \delta. \tag{2.15}$$

Note that the integral in the above formula is over the set $K_{2\varepsilon}^c$, not K_ε^c. Since $\overline{K}_\varepsilon \subset D$ and η is C^∞, we have

$$\lim_{j\to\infty} \int_{K_\varepsilon} ((\eta_j(x) - \eta(x))^2 + (|\nabla \eta_j(x)|^2 - |\nabla \eta(x)|^2))\, dx = 0, \tag{2.16}$$

because the integrand converges to 0 uniformly. It suffices to show that there exists a constant $c_{17} < \infty$, not depending on δ or ε, such that for large j, we have

$$\int_{K_\varepsilon^c} (\eta_j(x)^2 + |\nabla \eta_j(x)|^2)\, dx < c_{17}\delta. \tag{2.17}$$

By (2.8) applied to $Q = K_\varepsilon^c$,

$$\int_{K_\varepsilon^c} \eta_j(x)^2\, dx \le c_6 \int_{K_{2\varepsilon}^c} \eta(z)^2\, dz \le c_6\delta,$$

while (2.14) and (2.15) imply that

$$\int_{K_\varepsilon^c} |\nabla \eta_j(x)|^2\, dx \le c_{16}\delta.$$

The last two estimates yield (2.17) and complete the proof of the claim that $\eta_j \to \eta$ in $W^{1,2}(D)$ as $j \to \infty$.

Step 5. Recall the constant $c_1 \in (0, 1)$ and sets $\{D_k, k \ge 1\}$ from the beginning of this section. For each integer $k \ge 1$, let $\mathcal{B}_k$ be the family of all closed d-dimensional cubes $Q \subset D$ with edge length 2^{-k}, such that: (i) the vertices of Q belong to $(2^{-k}\mathbb{Z})^d$; (ii) the distance from Q to ∂D is greater than $c_1 2^{-k}$; and (iii) the interiors of Q and D_k are disjoint. Let $\mathcal{M}_1 = \mathcal{B}_1$ and let $\mathcal{M}_k \subset \mathcal{B}_k$ consist of those cubes in $\mathcal{B}_k$ that are not a subset of any cube in $\mathcal{B}_{k-1}$ for $k \ge 2$.

We recall that for a set $Q \subset D$, $\widehat{Q} = \bigcup_{x\in Q} B(x, d_j(x)/2)$. We claim that there exists $M < \infty$, independent of j, such that every point $x \in D$ belongs to at most M distinct sets of the form $\widehat{Q}$ where $Q \in \bigcup_k \mathcal{M}_k$. This claim can be proved in a way that is totally analogous to the proof of [14], Proposition 3, page 169, so we omit its proof.

Step 6. We have shown in step 1 that we can find a sequence of bounded functions $\{\eta^k\}_{k\ge1}$ in $W^{1,2}(D) \cap C^\infty(D)$ that is dense in $W^{1,2}(D)$. Let $\{\eta^k_j\}_{j\ge1}$ be a sequence constructed from η^k as in (2.5). The family $\{\eta^k_j\}_{k,j\ge1}$ is dense in $W^{1,2}(D)$ and consists of bounded C^∞ functions. Let us relabel the family

$\{\eta_j^k\}_{k,j\ge 1}$ as $\{\varphi_j\}_{j\ge 1}$. We see that the family $\{\varphi_j\}_{j\ge 1}$ consists of bounded functions in $W^{1,2}(D)\cap C^\infty(D)$ and satisfies part (i) of the theorem.

By adding an appropriate sequence of functions in $C_b^\infty(\overline{D})$, if necessary, we can assume that condition (ii) is satisfied by $\{\varphi_j\}_{j\ge 1}$.

We will show that (2.1) holds for φ_j for each fixed $j\ge 1$. Some functions φ_j belong to $C_b^\infty(\overline{D})$. It is easy to see that (2.1) holds for such functions. Hence, we will assume that φ_j belongs to the family $\{\eta_j^k\}_{k,j\ge 1}$. Then there exists a function φ in $W^{1,2}(D)\cap C^\infty(D)$ such that φ_j was constructed from φ as in (2.5).

Fix an arbitrarily small $\varepsilon>0$ and find an integer R so large that

$$\|\nabla\varphi\mathbf{1}_{D_R^c}\|_{L^2(D)}<\varepsilon.$$

Note that D_R is a compact set and choose an integer $S>R$ so large that D_R is a subset of the interior of D_S. Recall $\mathcal{A}_k$, $\mathcal{A}_k'$ and $\mathcal{A}_k''$ from the beginning of this section. Let e_i denote the unit vector in the positive direction of x_i-axis. Since φ_j is $C^\infty(D)$, we have by the mean value theorem for some $\theta_i^+(x)\in\overline{x,x+2^{-k}e_i}$ and $\theta_i^-(x)\in\overline{x,x-2^{-k}e_i}$ that

$$\begin{aligned}
&\limsup_{k\to\infty}2^{k(2-d)}\sum_{\overline{x,y}\in\mathcal{A}_k',\overline{xy}\subset D_S}\big(\varphi_j(x)-\varphi_j(y)\big)^2\\
&\quad=\limsup_{k\to\infty}2^{k(2-d)}\sum_{x\in\mathcal{A}_k''\cap D_S}\ \sum_{y\in\mathcal{A}_k''\cap D_S:\,\overline{xy}\in A_k'}\big(\varphi_j(x)-\varphi_j(y)\big)^2\\
&\quad\le\limsup_{k\to\infty}2^{k(2-d)}\sum_{x\in\mathcal{A}_k''\cap D_S}\sum_{i=1}^d\Big(\big(\varphi_j(x)-\varphi_j(x+2^{-k}e_i)\big)^2\\
&\qquad\qquad+\big(\varphi_j(x)-\varphi_j(x-2^{-k}e_i)\big)^2\Big)\\
&\quad=\limsup_{k\to\infty}2^{k(2-d)}\sum_{x\in\mathcal{A}_k''\cap D_S}\sum_{i=1}^d\bigg(\bigg|\frac{\partial\varphi_j(\theta_i^+(x))}{\partial x_i}\bigg|^2+\bigg|\frac{\partial\varphi_j(\theta_i^+(x))}{\partial x_i}\bigg|^2\bigg)2^{-2k}\\
&\quad\le 2\int_D|\nabla\varphi_j|^2.
\end{aligned}$$

It suffices to find $c_1<\infty$ independent of ε, S and R and such that

$$\limsup_{k\to\infty}2^{k(2-d)}\sum_{\overline{xy}\in\mathcal{A}_k',\overline{xy}\not\subset D_S}\big(\varphi_j(x)-\varphi_j(y)\big)^2\le c_1\varepsilon.$$

We assumed that D_R is a subset of the interior of D_S, so for large k, if $\overline{x,y}\in\mathcal{A}_k'$ and $\overline{xy}\not\subset D_S$, then $\overline{xy}\subset D_R^c$. Hence, it will suffice to find $c_1<\infty$ such that

$$\limsup_{k\to\infty}2^{k(2-d)}\sum_{\overline{xy}\in\mathcal{A}_k',\overline{xy}\subset D_R^c}\big(\varphi_j(x)-\varphi_j(y)\big)^2\le c_1\varepsilon.\tag{2.18}$$

Recall the notation from step 5. Consider a large integer k, $\ell \le k$ and $Q \in \mathcal{M}_\ell$. Suppose that

$$\sum_{x,y\in(2^{-k}\mathbb{Z})^d\cap Q, |x-y|=2^{-k}} \big(\varphi_j(x)-\varphi_j(y)\big)^2 = a. \tag{2.19}$$

Let $N = 2^{(k-\ell)d}$, and let $\{Q_1, Q_2, \dots, Q_N\}$ be the family of all cubes such that $Q_n \in \mathcal{B}_k$ and $Q_n \subset Q$. Let a_n be the maximum of $(\varphi_j(x)-\varphi_j(y))^2$ over all pairs $x, y \in (2^{-k}\mathbb{Z})^d \cap Q_n$ such that $|x-y| = 2^{-k}$. By the mean value theorem, there is some z in the line segment joining x and y in Q_n such that

$$\big|\nabla\varphi_j(z)\big| \ge a_n^{1/2} 2^k. \tag{2.20}$$

It is easy to check that

$$d2^{d-1}\sum_{n=1}^N a_n \ge a. \tag{2.21}$$

Step 7. In this step, we will prove (2.18). We have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_m}\varphi_j(x)\right) &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_m}\left(\int_{\mathbb{R}^d}\psi_{d_j(x)}(y)\varphi(x-y)\,dy\right)\right)\\
&= \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}^d}\psi_{d_j(x)}(y)\,\frac{\partial}{\partial x_m}\varphi(x-y)\,dy\right.\\
&\qquad\left. + \int_{\mathbb{R}^d}\left(\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right)\\
&= \frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}^d}\psi_{d_j(x)}(x-y)\,\frac{\partial}{\partial x_m}\varphi(y)\,dy\right.\\
&\qquad\left. + \int_{\mathbb{R}^d}\left(\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right)\\
&= \int_{\mathbb{R}^d}\left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(x-y)\right)\frac{\partial}{\partial x_m}\varphi(y)\,dy\\
&\quad + \int_{\mathbb{R}^d}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y)\right)\varphi(x-y)\,dy\\
&\quad + \int_{\mathbb{R}^d}\left(\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y)\right)\frac{\partial}{\partial x_i}\varphi(x-y)\,dy\\
&= 2\int_{B(x,d_j(x)/2)}\left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(x-y)\right)\frac{\partial}{\partial x_m}\varphi(y)\,dy\\
&\quad + \int_{B(0,d_j(x)/2)}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y)\right)\varphi(x-y)\,dy.
\end{aligned} \tag{2.22}$$

We estimate the first of the last two integrals, using (2.11) as follows:

$$
\begin{aligned}
&\left|\int_{B(x,d_j(x)/2)}\left(\frac{\partial}{\partial x_i}\psi_{d_j(x)}(x-y)\right)\frac{\partial}{\partial x_m}\varphi(y)\,dy\right| \\
&\quad\le \int_{B(x,d_j(x)/2)} c_1 d_j(x)^{-d-1}\left|\frac{\partial}{\partial x_m}\varphi(y)\right|dy \qquad (2.23)\\
&\quad\le c_2 d_j(x)^{-d-1}\int_{B(x,d_j(x)/2)}|\nabla\varphi(z)|\,dz.
\end{aligned}
$$

To estimate the second integral, we apply the same method as in the derivation of (2.13). Recall that $\psi_r'(y)=\frac{\partial}{\partial r}\psi_r(y)$ is a C^∞ function supported in $B(0,1/2)$ with $\int_{B(0,1/2)}\psi_r'(y)\,dy=0$. The function $\psi_r''(y)=\frac{\partial^2}{\partial r^2}\psi_r(y)$ is C^∞. It is supported in $B(0,1/2)$ and satisfies the condition $\int_{B(0,1/2)}\psi_r''(y)\,dy=0$. Note that $\|\psi_r'\|_1=c_3r^{-1}$, $\|\psi_r'\vee 0\|_1=\|\psi_r'\wedge 0\|_1=\|\psi_r'\|_1/2$, $\|\psi_r''\|_1=c_4r^{-2}$, $\|\psi_r''\vee 0\|_1=\|\psi_r''\wedge 0\|_1=\|\psi_r''\|_1/2$. We have

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y) &= \frac{\partial}{\partial x_i}\left(\left(\frac{\partial}{\partial x_m}d_j(x)\right)\psi'_{d_j(x)}(y)\right)\\
&=\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}d_j(x)\right)\psi'_{d_j(x)}(y)\\
&\quad+\left(\frac{\partial}{\partial x_m}d_j(x)\right)\left(\frac{\partial}{\partial x_i}d_j(x)\right)\psi''_{d_j(x)}(y).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left|\int_{B(0,d_j(x)/2)}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}\psi_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right|\\
&\quad\le\left|\int_{B(0,d_j(x)/2)}\left(\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}d_j(x)\right)\psi'_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right| \qquad (2.24)\\
&\qquad+\left|\int_{B(0,d_j(x)/2)}\left(\left(\frac{\partial}{\partial x_m}d_j(x)\right)\left(\frac{\partial}{\partial x_i}d_j(x)\right)\psi''_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right|.
\end{aligned}
$$

Recall that the random variables A_x^+ and A_x^- are defined in step 3. Recall also that $|\frac{\partial}{\partial x_i}d_j(x)|\le 1$ and $|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}d_j(x)|\le c'd_j(x)^{-1}$ for all i,m and x. We obtain

$$
\begin{aligned}
&\left|\int_{B(0,d_j(x)/2)}\left(\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}d_j(x)\right)\psi'_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right|\\
&\quad\le\left|\int_{B(0,d_j(x)/2)}(c'd_j(x)^{-1}\psi'_{d_j(x)}(y))\varphi(x-y)\,dy\right|\\
&\quad\le c_5 d_j(x)^{-2}\left|\mathbb{E}(\varphi(x-A_x^+)-\varphi(x-A_x^-))\right|.
\end{aligned}
$$

The same reasoning as in (2.12) and (2.13) yields

$$\begin{aligned}&\left|\int_{B(0,d_j(x)/2)}\left(\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}d_j(x)\right)\psi'_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right|\\&\qquad\le c_6 d_j(x)^{-d-1}\int_{B(0,d_j(x)/2)}\left|\nabla\varphi(x-z)\right|dz.\end{aligned}\tag{2.25}$$

We apply the same argument with ψ'' in place of ψ'. Let $b_x^+(\cdot)=(\psi''_{d_j(x)}(\cdot)\vee 0)/\|\psi''_{d_j(x)}\vee 0\|_1$ and $b_x^-(\cdot)=-(\psi''_{d_j(x)}(\cdot)\wedge 0)/\|\psi''_{d_j(x)}\wedge 0\|_1$. The functions $b_x^+(\cdot)$ and $b_x^-(\cdot)$ are probability density functions that vanish outside the ball $B(0,d_j(x)/2)$. Let B_x^+ and B_x^- be independent $\mathbb{R}^d$-valued random variables with densities $b_x^+(\cdot)$ and $b_x^-(\cdot)$. We have

$$\begin{aligned}&\left|\int_{B(0,d_j(x)/2)}\left(\left(\frac{\partial}{\partial x_m}d_j(x)\right)\left(\frac{\partial}{\partial x_i}d_j(x)\right)\psi''_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right|\\&\qquad\le c_7 d_j(x)^{-2}\left|\mathbb{E}\left(\varphi\left(x-B_x^+\right)-\varphi\left(x-B_x^-\right)\right)\right|\\&\qquad\le c_7 d_j(x)^{-2}\mathbb{E}\left|\varphi\left(x-B_x^+\right)-\varphi\left(x-B_x^-\right)\right|\\&\qquad\le c_7 d_j(x)^{-2}\mathbb{E}\int_{B_x^+}^{B_x^-}\left|\nabla\varphi(x-z)\right|dz,\end{aligned}$$

where the last integral is along a line segment from B_x^+ to B_x^-. By step 3, the measure $\mathbb{E}\int_{B_x^+}^{B_x^-}dz$ has a density that is bounded above by $c_8 d_j(x)^{1-d}$ and vanishes outside the ball $B(0,d_j(x)/2)$. In other words, for every set $K\subset D$, $\mathbb{E}\int_{B_x^+}^{B_x^-}\mathbf{1}_K(z)\,dz\le c_8 d_j(x)^{1-d}m(K\cap B(0,d_j(x)/2))$. It follows that

$$\begin{aligned}&\left|\int_{B(0,d_j(x)/2)}\left(\left(\frac{\partial}{\partial x_m}d_j(x)\right)\left(\frac{\partial}{\partial x_i}d_j(x)\right)\psi''_{d_j(x)}(y)\right)\varphi(x-y)\,dy\right|\\&\qquad\le c_7 d_j(x)^{-2}\mathbb{E}\int_{B_x^+}^{B_x^-}\left|\nabla\varphi(x-z)\right|dz\\&\qquad\le c_7 d_j(x)^{-2}\int_{B(0,d_j(x)/2)}\left|\nabla\varphi(x-z)\right|c_8 d_j(x)^{1-d}\,dz\\&\qquad= c_9 d_j(x)^{-d-1}\int_{B(0,d_j(x)/2)}\left|\nabla\varphi(x-z)\right|dz.\end{aligned}\tag{2.26}$$

We combine (2.22), (2.23), (2.24), (2.25) and (2.26), and then we use Hölder's inequality to see that

$$\begin{aligned}&\left|\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_m}\varphi_j(x)\right)\right|\\&\qquad\le c_{10} d_j(x)^{-d-1}\int_{B(x,d_j(x)/2)}\left|\nabla\varphi(z)\right|dz\end{aligned}\tag{2.27}$$

$$\leq c_{11} d_j(x)^{-d-1} d_j(x)^{d/2} \left(\int_{B(x,d_j(x)/2)} \left| \nabla \varphi(z) \right|^2 dz \right)^{1/2}$$

$$= c_{11} d_j(x)^{-d/2-1} \left(\int_{B(x,d_j(x)/2)} \left| \nabla \varphi(z) \right|^2 dz \right)^{1/2}.$$

Recall that $\widehat{Q} = \bigcup_{x \in Q} B(x, d_j(x)/2)$. We will prove that

$$\int_{\widehat{Q}} \left| \nabla(\varphi) \right|^2 (x)\, dx \geq c_{12} 2^{k(2-d)} a \tag{2.28}$$

for some constant c_{12}, where a is the constant defined in (2.19). If the inequality holds with $c_{12} = 1$, then we are done. So let us suppose that

$$\int_{\widehat{Q}} \left| \nabla(\varphi) \right|^2 (x)\, dx \leq 2^{k(2-d)} a. \tag{2.29}$$

We combine this with (2.27) to see that for $x \in Q$,

$$\begin{aligned} \left| \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_m} \varphi_j(x) \right) \right| &\leq c_{11} d_j(x)^{-d/2-1} \left(\int_{B(x,d_j(x)/2)} \left| \nabla \varphi(z) \right|^2 dz \right)^{1/2} \\ &\leq c_{11} d_j(x)^{-d/2-1} \left(\int_{\widehat{Q}} \left| \nabla \varphi(z) \right|^2 dz \right)^{1/2} \\ &\leq c_{11} d_j(x)^{-d/2-1} \left(2^{k(2-d)} a \right)^{1/2}. \end{aligned}$$

It follows from this and (2.20) that the set of $x \in Q_n$ such that $|\nabla \varphi_j(x)| \geq a_n^{1/2} 2^k/2$ contains a ball with radius greater than

$$c_{13} a_n^{1/2} 2^k / \left(d_j(x)^{-d/2-1} \left(2^{k(2-d)} a \right)^{1/2} \right) = c_{13} a_n^{1/2} a^{-1/2} 2^{kd/2} d_j(x)^{d/2+1}$$

and, therefore, it has a volume greater than

$$\left(c_{13} a_n^{1/2} a^{-1/2} 2^{kd/2} d_j(x)^{d/2+1} \right)^d = c_{14} a_n^{d/2} a^{-d/2} 2^{kd^2/2} d_j(x)^{d^2/2+d}.$$

Hence

$$\begin{aligned} \int_{Q_n} \left| \nabla \varphi_j(x) \right|^2 dx &\geq \left(a_n^{1/2} 2^k/2 \right)^2 c_{14} a_n^{d/2} a^{-d/2} 2^{kd^2/2} d_j(x)^{d^2/2+d} \\ &= c_{15} a_n^{1+d/2} a^{-d/2} 2^{k(2+d^2/2)} d_j(x)^{d^2/2+d} \end{aligned}$$

and, therefore,

$$\begin{aligned} \int_Q \left| \nabla \varphi_j(x) \right|^2 dx &= \sum_{n=1}^N \int_{Q_n} \left| \nabla \varphi_j(x) \right|^2 dx \\ &\geq \sum_{n=1}^N c_{15} a_n^{1+d/2} a^{-d/2} 2^{k(2+d^2/2)} d_j(x)^{d^2/2+d}. \end{aligned} \tag{2.30}$$

By the Hölder inequality and (2.21),

$$\sum_{n=1}^{N} a_n^{1+d/2} \geq N^{-d/2}\left(\sum_{n=1}^{N} a_n\right)^{1+d/2} \geq N^{-d/2}\left(ad^{-1}2^{1-d}\right)^{1+d/2}$$
$$= c_{16} N^{-d/2} a^{1+d/2}.$$

This and (2.30) give

$$\begin{aligned}\int_Q |\nabla \varphi_j(x)|^2\,dx &\geq c_{17} a N^{-d/2} 2^{k(2+d^2/2)} d_j(x)^{d^2/2+d}\\ &= c_{17} a \left(2^{(k-\ell)d}\right)^{-d/2} 2^{k(2+d^2/2)} d_j(x)^{d^2/2+d}\\ &= c_{17} a 2^{2k} d_j(x)^d \left(d_j(x) 2^{\ell}\right)^{d^2/2}\\ &\geq c_{18} a 2^{2k} d_j(x)^d \geq c_{19} a 2^{k(2-d)}\\ &= c_{19} 2^{k(2-d)} \sum_{x,y\in(2^{-k}\mathbb{Z})^d\cap Q,|x-y|=2^{-k}} \left(\varphi_j(x)-\varphi_j(y)\right)^2.\end{aligned}$$

It follows from this and (2.8) that

$$\begin{aligned}2^{k(2-d)} \sum_{x,y\in(2^{-k}\mathbb{Z})^d\cap Q,|x-y|=2^{-k}} \left(\varphi_j(x)-\varphi_j(y)\right)^2 &\leq c_{20} \int_Q |\nabla \varphi_j(x)|^2\,dx\\ &\leq c_{21} \int_{\widehat{Q}} |\nabla \varphi(x)|^2\,dx.\end{aligned}$$

In view of (2.28) and (2.29), we conclude that the last inequality is always valid. Recall the constant M from step 5. Summing over all $Q \in \bigcup_{\ell \leq k} \mathcal{M}_\ell$, $Q \subset D_R^c$, we obtain

$$2^{k(2-d)} \sum_{\overline{xy}\in\mathcal{A}_k',\overline{xy}\subset D_R^c} \left(\varphi_j(x)-\varphi_j(y)\right)^2 \leq M c_{21} \int_{D_R^c} |\nabla \varphi(x)|^2\,dx \leq M c_{21} \varepsilon.$$

This shows that (2.18) holds and completes the proof of the theorem. □

EXAMPLE 2.2. Let $C_b^1(D)$ be the family of bounded continuous functions on D with continuous bounded first order derivatives. Using mean value theorem, it is easy to see that the inequality (2.1) holds for every $\varphi \in C_b^1(D)$ [in fact equality holds for such φ since $|\nabla\varphi|$ is bounded on D and so $\lim_{R\to\infty} \int_{D\setminus D_R} |\nabla\varphi(x)|^2\,dx = 0$]. However, we will sketch an example, without proof, of a domain D such that $C_b^1(D)$ is not dense in $W^{1,2}(D)$. The point of this example is to show that Theorem 2.1 cannot be strengthened by adding an extra property that the functions $\{\varphi_j\}_{j\geq 1}$ belong to $C_b^1(D)$.

Let

$$
\begin{aligned}
D_- &= \{(x_1, x_2) \in \mathbb{R}^2 : -1 < x_1 < 0, 0 < x_2 < 1\},\\
D_+ &= \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1\},\\
D_n &= \{(x_1, x_2) \in \mathbb{R}^2 : -1/n < x_1 < 1/n, 1/n < x_2 < 1/n + \delta_n\},\\
\partial D_n^+ &= \{(x_1, x_2) \in \mathbb{R}^2 : -1/n < x_1 < 1/n, x_2 = 1/n + \delta_n\},\\
\partial D_n^- &= \{(x_1, x_2) \in \mathbb{R}^2 : -1/n < x_1 < 1/n, x_2 = 1/n\},\\
D &= D_- \cup D_+ \cup \bigcup_{n=2}^{\infty} D_n \Big\backslash \bigcup_{n=2}^{\infty} (\partial D_n^+ \cup \partial D_n^-).
\end{aligned}
$$

We choose $\delta_n > 0$ so small that D_n's are disjoint. Consider a continuous function φ such that $\varphi(x) = 1$ for $x \in D_+ \setminus \bigcup_{n\geq 2} D_n$, $\varphi(x) = -1$ for $x \in D_+ \setminus \bigcup_{n\geq 2} D_n$ and φ is linear in every D_n. The widths δ_n of "channels" D_n can be chosen so small that $\varphi \in W^{1,2}(D)$ and, moreover, $\int_D |\nabla\varphi|^2$ can be made arbitrarily small.

We claim that the function φ described above cannot be approximated by functions $\eta \in C_b^1(D)$ with arbitrary accuracy. The reason is that for any such η, the oscillation of η in a set D_n is arbitrarily small, for large n. Hence, in a neighborhood of (0, 0), either $|\varphi - \eta|$ is nonnegligible on a nonnegligible set, or $|\nabla\varphi - \nabla\eta|$ is nonnegligible on a nonnegligible set. We leave the details to the reader because the claim made in this example is not needed for our main theorem.

3. Invariance principle for reflecting random walk. Let $\mathcal{C}$ be the algebra generated by functions $\{\varphi_j\}_{j\geq 1}$ from Theorem 2.1 over $\mathbb{Q}$. By the same proof as that for Lemma 2.2 in [2], we have the following.

LEMMA 3.1. *There exists a metric ρ on D which induces the same Euclidean topology inside D and such that the ρ-completion $\widetilde{D}$ of D is a regularizing space for Dirichlet form $(\mathcal{E}, W^{1,2}(D))$. Moreover, $\mathcal{C}$ is dense in $C_b(\widetilde{D}, \|\cdot\|_\infty)$.*

Let m be the Lebesgue measure on D extended to $\widetilde{D}$ by setting $m(\widetilde{D} \setminus D) = 0$. Then $(\mathcal{E}, W^{1,2}(D))$ is a strongly local regular Dirichlet form on $L^2(\widetilde{D}; m)$. Let $\widetilde{X}$ be the Hunt process on $\widetilde{D}$ associated with the regular Dirichlet form $(\mathcal{E}, W^{1,2}(D))$ on $L^2(\widetilde{D}; m)$, which is continuous and has infinite lifetime. Denote by j the projection map from $\widetilde{D}$ to $\overline{D}$. Then $X := j(\widetilde{X})$ is a continuous Markov process taking values on $\overline{D}$. In general X may not be a strong Markov process, as one can see from the example when D is the unit disk in $\mathbb{R}^2$ with a slit $(-1, 0) \times \{0\}$ removed. Both $\widetilde{X}$ and X can be called the reflecting Brownian motion on D.

We will now discuss the relationship between reflecting Brownian motion $\widetilde{X}$ on $\widetilde{D}$ and a better known construction of reflecting Brownian on an arbitrary domain D. For an arbitrary bounded domain D in $\mathbb{R}^d$, Fukushima [8] used the

Martin–Kuramochi compactification D^* of D to construct a conservative continuous Hunt process taking values in D^*. The process X^* is associated with the regular Dirichlet form $(\mathcal{E}, W^{1,2}(D))$ on $L^2(D^*; m^*)$, where m^* is Lebesgue measure on D extended to D^* by setting $m^*(D^* \setminus D) = 0$. Since each $f_i(x) \stackrel{\text{df}}{=} x_i$ is a function in $W^{1,2}(D)$, it admits a quasi-continuous extension to D^*, which we still denote by f_i. These functions induce a quasi-continuous projection map $j^* = (f_1, \ldots, f_d)$ from D^* to $\overline{D}$. Then $X' := j^*(X^*)$ is a continuous Markov process taking values on $\overline{D}$, which is called reflecting Brownian motion on $\overline{D}$ in [2]. Both $\widetilde{D}$ and D^* are regularizing spaces for the Dirichlet form $(\mathcal{E}, W^{1,2}(D))$ and so X and X' have the same finite-dimensional distributions under the initial distribution m. For $x \in D$, both X and X' starting from x behave like Brownian motion before they hit the boundary after a positive period of time. Consequently, X and X' have the same finite-dimensional distributions starting from any interior point in D. We can consider processes X and X' as maps from their underlying probability spaces into the space of continuous functions $C([0, \infty); \mathbb{R}^d)$. Then the distributions of X and X' in $C([0, \infty); \mathbb{R}^d)$ are identical, either with initial distribution m or with initial starting point in D. In this sense, convergence of reflecting random walks to X or X' is an equally strong result.

Without loss of generality, we assume that D contains the origin 0. Recall the definition of D_k from the previous section. We view $(2^{-k}\mathbb{Z})^d \cap D_k$ as a graph whose vertices are $(2^{-k}\mathbb{Z})^d \cap D_k$, and there is an edge between two vertices x and y if and only $|x - y| = 2^{-k}$ and the line segment connecting x and y is contained in D_k. By abuse of notation, in this section we will use D_k to denote the graph $(2^{-k}\mathbb{Z})^d \cap D_k$.

For $x \in D_k$, we use $v_k(x)$ to denote the degree of the vertex x in D_k. Let $\{X^k_{j2^{-2k}}, j = 0, 1, \ldots\}$ be the simple random walk on D_k that jumps every 2^{-2k} units of time. By definition, the random walk $\{X^k_{j2^{-2k}}, j = 0, 1, \ldots\}$ jumps to one of its nearest neighbors in D_k with equal probabilities. This discrete time Markov chain is symmetric with respect to measure m_k, where $m_k(x) = \frac{v_k(x)}{2d} 2^{-kd}$ for $x \in D_k$. Clearly m_k converge weakly to m on D. We now extend the time-parameter of $\{X^k_{j2^{-2k}}, j = 0, 1, \ldots\}$ to all nonnegative reals using linear interpolation over the intervals $((j - 1)2^{-2k}, j2^{-2k})$ for $j = 1, 2, \ldots$. We thus obtain a process $X^k = \{X^k_t, t \geq 0\}$. Its law with $X^k_0 = x$ will be denoted by $\mathbb{P}^k_x$.

Recall $\mathcal{A}'_k$ from the beginning of Section 2. Let $Q_k(x, dy)$ denote the one-step transition probability for the discrete time Markov chain $\{X^k_{j2^{-2k}}, j = 0, 1, \ldots\}$; that is, for $f \geq 0$ on D and $x \in D_k$,

$$Q_k f(x) := \int_D f(y) Q_k(x, dy) := \frac{1}{v_k(x)} \sum_{y \in D_k : \overline{xy} \in \mathcal{A}'_k} f(y).$$

For $f \in C^2(\overline{D})$, define

$$\mathcal{L}_k f(x) := \int_D \big(f(y) - f(x)\big) Q_k(x, dy) \\ = \frac{1}{v_k(x)} \sum_{y \in D_k : \overline{xy} \in \mathcal{A}'_k} \big(f(y) - f(x)\big), \qquad x \in D_k.$$

Then $\{f(X^k_{j2^{-2k}}) - \sum_{i=0}^{j-1} \mathcal{L}_k f(X^k_{i2^{-2k}}), \mathcal{G}^k_{j2^{-k}}, j = 0, 1, \dots\}$ is a martingale for every $f \in C^2(\overline{D})$, where $\mathcal{G}^k_t := \sigma(X^k_s, s \le t)$.

To study the weak limit of $\{X^k, k \ge 1\}$, we introduce an auxiliary process Y^k defined by $Y^k_t := X^k_{[2^{2k}t]2^{-2k}}$, where $[\alpha]$ denotes the largest integer that is less than or equal to α. Note that Y^k is a time-inhomogeneous Markov process. For every fixed $t > 0$, its transition probability operator is symmetric with respect to the measure m_k on D_k. Let $\mathcal{F}^k_t := \sigma(Y^k_s, s \le t)$. By abuse of notation, the law of Y^k starting from $x \in D_k$ will also be denoted by $\mathbb{P}^k_x$.

Note that $Y^k_t = X^k_t$ for every t of the form $t = j2^{-2k}$, where j is an integer. Moreover, $\sup_{t \ge 0} |X^k_t - Y^k_t| \le 2^{-k}$. It follows that if the laws of one of the sequences $\{X^k, k \ge 1\}$ or $\{Y^k, k \ge 1\}$ converge to a limit on $\mathbb{D}([0, T], \widetilde{D})$ for some T, then the same holds for the other sequence.

THEOREM 3.2. *Let D be a bounded domain in $\mathbb{R}^n$. Then the laws $\{\mathbb{P}^k_{m_k}, k \ge 1\}$ of $\{Y^k, k \ge 1\}$ are tight in the space $\mathbb{D}([0, T], \widetilde{D})$ for every $T > 0$.*

PROOF. Without loss of generality, we assume that $T = 1$. By [6], Theorem 3.9.1, and Lemma 3.1, it suffices to show that for every $g \in \mathcal{C}$, $\{g(X^k)\}_{k \ge 1}$ is relatively compact in $\mathbb{D}([0, 1], \mathbb{R})$ with the initial distribution $\mathbb{P}^k_{m_k}$.

For each fixed $k \ge 1$, we may assume, without loss of generality, that Ω is the canonical space $\mathbb{D}([0, \infty), \widetilde{D})$, and Y^k_t is the coordinate map on Ω. Given $t > 0$ and a path $\omega \in \Omega$, the time reversal operator r_t is defined by

$$r_t(\omega)(s) := \begin{cases} \omega\big((t - s)-\big), & \text{if } 0 \le s \le t, \\ \omega(0), & \text{if } s \ge t. \end{cases} \tag{3.1}$$

Here for $r > 0$, $\omega(r-) := \lim_{s \uparrow r} \omega(s)$ is the left limit at r, and we use the convention that $\omega(0-) := \omega(0)$. We note that

$$\lim_{s \downarrow 0} r_t(\omega)(s) = \omega(t-) = r_t(\omega)(0) \quad \text{and} \quad \lim_{s \uparrow t} r_t(\omega)(s) = \omega(0) = r_t(\omega)(t). \tag{3.2}$$

Observe that for every integer $T \ge 1$, $\mathbb{P}^k_{m_k}$ restricted to the time interval $[0, T)$ is invariant under the time-reversal operator r_T. Note that

$$M^{k,f}_t := f(Y^k_t) - f(Y^k_0) - \sum_{i=0}^{[2^{2k}t]-1} \mathcal{L}_k f(Y^k_{i2^{-2k}})$$

is an $\{\mathcal{F}_t^k, t \geq 0\}$-martingale for every $f \in \mathcal{C}$ (cf. [3]). We have

$$f(Y_t^k) - f(Y_0^k) = \frac{1}{2} M_t^{k,f} - \frac{1}{2}\big(M_{T-}^{k,f} - M_{(T-t)-}^{k,f}\big) \circ r_T \qquad \text{for } t \in [0, T). \tag{3.3}$$

For each $M^{k,f}$, there exists a continuous predictable quadratic variation process $\langle M^{k,f} \rangle_t$. Note that (e.g., see page 214 of [9])

$$\begin{aligned} \langle M^{k,f} \rangle_t - \langle M^{k,f} \rangle_s &= \int_s^t \sum_{y \in D_k} \big(f(Y_u^k) - f(y)\big)^2 Q_k(Y_u^k, y) m_k(y)\, du \\ &\leq 2d \|f\|_\infty^2 (t-s). \end{aligned}$$

Thus by Proposition VI.3.26 in [10], $\{\langle M^{k,f} \rangle_t\}_{k \geq 1}$ is $\mathcal{C}$-tight in $\mathbb{D}([0,1], \mathbb{R})$. As m_k converges weakly to m on $\widetilde{D}$, by [10], Theorem VI.4.13, the laws of $\{M^{k,f}\}_{k \geq 1}$ are tight in the sense of Skorokhod topology on $\mathbb{D}([0,1], \mathbb{R})$ with the initial distribution $\mathbb{P}_{m_k}^k$. Since the laws of $\{M^{k,f}, t \in [0,1], \mathbb{P}_{m_k}^k\}_{k \geq 1}$ are the same as the laws of $\{M_{(1-t)}^{k,f}, t \in [0,1], \mathbb{P}_{m_k}^k\}_{k \geq 1}$, it follows from (3.3) that $\{f(X^k)\}_{k \geq 1}$ and, consequently, $\{g(X^k)\}_{k \geq 1}$ is tight (and so relatively compact) in the sense of Skorokhod topology on $\mathbb{D}([0,1], \mathbb{R})$ with the initial distribution $\mathbb{P}_{m_k}^k$. □

Let $(\widetilde{X}, \mathbb{P})$ be a subsequential limit of $\{(Y^k, \mathbb{P}_{m_k}^k); k \geq 1\}$ on $\mathbb{D}([0, T], \widetilde{D})$.

THEOREM 3.3. $(\widetilde{X}, \mathbb{P})$ *is a stationary symmetric Markov process.*

PROOF. Let $(\widetilde{X}, \mathbb{P})$ be a subsequential limit of $\{(Y^k, \mathbb{P}_{m_k}^k); k \geq 1\}$ on $\mathbb{D}([0, T], \widetilde{D})$, say, along a subsequence $\{n_k, k \geq 1\}$. It suffices to show that the finite-dimensional distributions of $(\widetilde{X}, P)$ are determined by a semigroup. Clearly, $\widetilde{m}$ is an invariant measure for $\widetilde{X}$. For every $t \in [0, T]$, define a linear bounded operator on $L^2(D; m) = L^2(\widetilde{D}; \widetilde{m})$ by

$$P_t f \stackrel{\text{df}}{=} \mathbb{E}[f(\widetilde{X}_t) \mid \widetilde{X}_0], \qquad f \in L^2(D; m).$$

We are going to show that $\{P_t, t \geq 0\}$ is a strongly continuous symmetric semigroup on $L^2(D; m)$.

(i) We first show that each P_t is a bounded symmetric operator on $L^2(D; m)$. For every $f, g \in C_b(\widetilde{D}, \rho)$ and $t > 0$, it follows from the symmetry of $(Y^k, \mathbb{P}_{m_k}^k)$ that

$$\begin{aligned} \int_D f(x) P_t g(x) m(dx) &= \mathbb{E}[f(\widetilde{X}_0) g(\widetilde{X}_t)] = \lim_{k \to \infty} \mathbb{E}_{m_{n_k}}[f(Y_0^{n_k}) g(Y_t^{n_k})] \\ &= \lim_{k \to \infty} \mathbb{E}_{m_{n_k}}[g(Y_0^{n_k}) f(Y_t^{n_k})] = \mathbb{E}[g(\widetilde{X}_0) f(\widetilde{X}_t)] \\ &= \int_D g(x) P_t f(x) m(dx). \end{aligned} \tag{3.4}$$

In particular, by taking $g = 1$, we have

$$\int_D \mathbb{P}_t f(x) m(dx) = \int_D f(x) m(dx) \qquad \text{for } f \in C_b(\widetilde{D}, \rho). \tag{3.5}$$

Note that $C_c(D) \subset C_b(\widetilde{D}, \rho) \subset L^2(D; m)$ and $C_c(D)$ is dense in $L^2(D; m)$. Hence (3.5) holds for every $f \in L^2(D; m)$. Consequently, by the definition of $\mathbb{P}_t$ and Jensen's inequality for conditional expectation,

$$\int_D \big(P_t f(x)\big)^2 m(dx) \le \int_D P_t\big(f^2\big)(x) m(dx) = \int_D f(x)^2 m(dx). \tag{3.6}$$

Hence (3.4) holds for every $f, g \in L^2(D, m)$; in other words, for each $t > 0$, P_t is a symmetric contraction operator in $L^2(D; m)$.

(ii) Next we show that $\{P_t, t \ge 0\}$ is a semigroup on $L^2(D; m)$. For $x = (x_1, \dots, x_d) \in D_k$, let $U_k(x) := \prod_{i=1}^d [x_i - 2^{-k-1}, x_i + 2^{-k-1})$ be the half-closed, half-open cube centered at x. We define an extension operator $E_k : L^2(D_k, m_k) \to L^2(D, m)$ as follows: for $g \in L^2(D_k, m_k)$,

$$E_k g(z) := \begin{cases} g(x), & \text{for } z \in U_k(x) \text{ with } x \in D_k, \\ 0, & \text{elsewhere.} \end{cases} \tag{3.7}$$

For $f, g \in C_c(D)$ and t of the form $j2^{-2l}$, by the uniform continuity,

$$\begin{aligned} \lim_{k\to\infty} \int_D f(x) E_{n_k} P_t^{n_k} g(x) m(dx) &= \lim_{k\to\infty} \mathbb{E}_{m_k}\big[f\big(Y_0^k\big) g\big(Y_t^k\big)\big] = \mathbb{E}_m\big[f(\widetilde{X}_0) g(\widetilde{X}_t)\big] \\ &= \int_D f(x) P_t g(x) m(dx). \end{aligned}$$

Note that

$$\begin{aligned} \int_D \big(E_{n_k} P_t^{n_k} g(x)\big)^2 m(dx) &= \int_{D_{n_k}} \big(P_t^{n_k} g(x)\big)^2 m_{n_k}(dx) \\ &\le \int_{D_{n_k}} g(x)^2 m_{n_k}(dx) \end{aligned} \tag{3.8}$$

and that $\lim_{k\to\infty} \int_{D_{n_k}} g(x)^2 m_{n_k}(dx) = \int_D g(x)^2 m(dx)$ for $g \in C_c(D)$. Since every $f \in L^2(D; dx)$ can be approximated in L^2-norm by a sequence $\{f_k, k \ge 1\} \subset C_c(D)$, we deduce from the last three displays and the Cauchy–Schwarz inequality that

$$\lim_{k\to\infty} \int_D f(x) E_{n_k} P_t^{n_k} g(x) m(dx) = \int_D f(x) P_t g(x) m(dx) \tag{3.9}$$

for every $f \in L^2(D; m)$ and $g \in C_c(D)$.

We claim that for every $t = j2^{-2l}$,

$$\lim_{k\to\infty} \int_D \big|E_{n_k} P_t^{n_k} g(x) - P_t g(x)\big|^2 m(dx) = 0 \qquad \text{for every } g \in C_c(D). \tag{3.10}$$

For any fixed $x \in D$, there is $r > 0$ so that $B(x, 2r) \subset D$. When k is large enough, there is a unique $y_k \in D_k$ so that $x \in U_k(y_k)$. We denote this y_k by $\pi_k(x)$. Let $q^k(t, x, y)$ denote the transition density with respect to m_k of simple random walk on D_k killed upon leaving $B(x, r)$. It follows from Donsker's invariance principle and the uniform Hölder continuity ([5], Proposition 4.1) for the parabolic functions of the simple random walk on $2^{-k}\mathbb{Z}^d$ that $q^k(t, \pi_k(x), \pi_k(y))$ converge locally uniformly in $y \in B(x_0, r)$ to the transition density $q(t, x, y)$ of Brownian motion X on $\mathbb{R}^d$ with variance $1/(2d)$ killed upon leaving $B(x, r)$. For every $\varepsilon > 0$, there is $s > 0$ so that $\int_{B(x,r)} q(s, x, y)\, dy > 1 - (\varepsilon/2)$. Hence for k sufficiently large, we have

$$\mathbb{P}_{\pi_k(x)}\big(Y_s^k \in dy\big) = q(s, x, y) m_k(dy) + \mu_k(dy),$$

where μ_k is a signed measure with $|\mu_k|(D_k) < \varepsilon$. It follows from this and (3.9) that for $g \in C_c(D)$,

$$\begin{aligned}
&\limsup_{k\to\infty} \left| P_t^{n_k} g\big(\pi_{n_k}(x)\big) - \int_D q(s, x, y) P_{t-s} g(y)\, dy \right| \\
&\quad= \limsup_{k\to\infty} \left| P_s^{n_k}\big(P_{t-s}^{n_k} g\big)\big(\pi_{n_k}(x)\big) - \int_D q(s, x, y) P_{t-s} g(y)\, dy \right| \\
&\quad\le \limsup_{k\to\infty} \left| \int_{D_k} q(s, x, y) P_{t-s}^{n_k} g(y) m_k(dy) - \int_D q(s, x, y) P_{t-s} g(y)\, dy \right| \\
&\qquad + \varepsilon \|g\|_\infty \\
&\quad= \limsup_{k\to\infty} \left| \int_D q(s, x, y) E_{n_k} P_{t-s}^{n_k} g(y) m(dy) - \int_D q(s, x, y) P_{t-s} g(y)\, dy \right| \\
&\qquad + \varepsilon \|g\|_\infty \\
&\quad= \varepsilon \|g\|_\infty.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the above yields that $\{P_t^{n_k} g(\pi_{n_k}(x)); k \ge 1\}$ is a Cauchy sequence, and so it converges to some value $u(x)$. This convergence holds for every $x \in D$, so we have by (3.9) that $u = P_t g$ a.e.; that is, $E_{n_k} P_t^{n_k} g(x) = P_t^{n_k} g(\pi_{n_k}(x))$ converges to $P_t g(x)$ for a.e. $x \in D$. Hence by the bounded convergence theorem, (3.10) holds for every $g \in C_c(D)$.

Abusing the notation a little bit, for $f \in L^2(D; m)$, we define $\pi_k f$ as a function in $L^2(D_k, m_k)$ by

$$\pi_k f(x) = \frac{\int_{U_k(x)} f(y) m(dy)}{m(U_k(x))} \qquad \text{for } x \in D_k. \tag{3.11}$$

Clearly $\pi_k \circ E_k$ is an identity map on $L^2(D_k, m_k)$ and

$$\int_{D_k} \big|\pi_k f(x)\big|^2 m_k(dx) \le \int_D \big|f(x)\big|^2 m(dx).$$

Since $C_c(D)$ is dense in $L^2(D;m)$, we have from (3.10) that

$$\lim_{k\to\infty}\int_D \big|E_{n_k}P_t^{n_k}\pi_{n_k}g(x)-P_tg(x)\big|^2 m(dx)=0 \tag{3.12}$$
$$\text{for every } g\in L^2(D;m).$$

It follows then for $g\in L^2(D;m)$,

$$P_{t+s}g=\lim_{k\to\infty}E_{n_k}P_t^{n_k}P_s^{n_k}\pi_{n_k}g=\lim_{k\to\infty}\big(E_{n_k}P_t^{n_k}\pi_{n_k}\big)\big(E_{n_k}P_s^{n_k}\pi_{n_k}\big)g=P_tP_sg.$$

This establishes the semigroup property of $\{P_t,t\ge 0\}$.

We have now established that X is a stationary symmetric Markov process. □

The following result is needed in the proof of Theorem 3.6.

LEMMA 3.4. *In the above setting, for every $f\in C_c^\infty(D)$, the process $M_t^f := f(X_t)-f(X_0)-\frac{1}{2d}\int_0^t \Delta f(X_s)\,ds$ is a $\mathbb{P}$-square integrable martingale. This in particular implies that $\{X_t, t<\tau_D,\mathbb{P}\}$ is a Brownian motion killed upon leaving D, with initial distribution m_D and infinitesimal generator $\frac{1}{2d}\Delta$.*

PROOF. The proof is the same as that for [1], Lemma 2.2. □

The following is Lemma 2.3 of [1].

LEMMA 3.5. *Let D be a bounded domain in $\mathbb{R}^d$ and fix $k\ge 1$. Then for every $j\ge 1$ and $f\in L^2(D,m_k)$,*

$$(f-Q_k^{2j}f,f)_{L^2(D,m_k)}\le j(f-Q_k^2f,f)_{L^2(D,m_k)}\le 2j(f-Q_kf,f)_{L^2(D,m_k)}.$$

We will say that "Z_t is a Brownian motion running at speed $1/n$" if Z_{nt} is the standard Brownian motion, and we will apply the same phrase to other related process.

By an argument similar to that in [1], Section 2, but with Theorem 3.2 in place of [1], Lemma 2.2, and using Theorem 2.1 in the energy form argument in the proof of [1], Theorem 2.4, we can establish the following theorem.

THEOREM 3.6. *Let D be a bounded domain in $\mathbb{R}^d$ with $m(\partial D)=0$. Then for every $T>0$, the laws of $\{X^k,\mathbb{P}^k_{m_k}\}$ converge weakly in $C([0,T],\widetilde{D})$ to a stationary reflecting Brownian motion on $\widetilde{D}$ running at speed $1/d$ whose initial distribution is the Lebesgue measure in D. Consequently, for every $T>0$, the laws of $\{X^k,\mathbb{P}^k_{m_k}\}$ converge weakly in $C([0,T],\overline{D})$ to a stationary reflecting Brownian motion on $\overline{D}$ running at speed $1/d$ whose initial distribution is the Lebesgue measure in D.*

PROOF. Fix $T > 0$. We know from Theorem 3.2 that the laws of $(X^k, \mathbb{P}^k_{m_k})$ are tight in the space $\mathbb{D}([0,T], \widetilde{D})$. Let $(\widetilde{X}, \mathbb{P})$ be any of subsequential limits, say, along $(X^{k_j}, \mathbb{P}^{k_j}_{m_{k_j}})$. By Theorem 3.3 and its proof, $\widetilde{X}$ is a time-homogeneous Markov process on $\widetilde{D}$ with transition semigroup $\{P_t, t \geq 0\}$ that is symmetric in $L^2(\widetilde{D}, m)$. Let $\{P^k_t, t \in 2^{-k}\mathbb{Z}_+\}$ be defined by $P^k_t f(x) := \mathbb{E}^k_x[f(X^k_t)]$. For dyadic $t > 0$, say, $t = j_0/2^{2k_0}$ and $f \in \mathcal{C}$, we have by Theorem 2.1 and Lemma 3.5,

$$
\begin{aligned}
&\frac{1}{t}(f - P_t f, f)_{L^2(\widetilde{D};m)} \\
&\quad = \frac{1}{t}\lim_{j\to\infty}\bigl(f - P^{k_j}_t f, f\bigr)_{L^2(D;m_{k_j})} \\
&\quad = \frac{2^{2k_0}}{j_0}\lim_{j\to\infty}\bigl(f - Q^{j_0 2^{2k_j-2k_0}}_{k_j} f, f\bigr)_{L^2(D,m_{k_j})} \\
&\quad \leq \limsup_{j\to\infty}\frac{2^{2k_0}}{j_0} j_0 2^{2k_j-2k_0}(f - Q_{k_j} f, f)_{L^2(D,m_{k_j})} \\
&\quad = \limsup_{j\to\infty} 2^{(2-d)k_j}\frac{1}{2d}\sum_{x\in D_{k_j}}\sum_{y\in D_{k_j}:\overline{xy}\in\mathcal{A}'_{k_j}}\bigl(f(x)^2 - f(x)f(y)\bigr) \\
&\quad = \frac{1}{4d}\limsup_{j\to\infty} 2^{(2-d)k_j}\sum_{x,y\in D_{k_j}:\overline{xy}\in\mathcal{A}'_{k_j}}\bigl(f(x) - f(y)\bigr)^2 \\
&\quad \leq \frac{1}{2d}\int_D \bigl|\nabla f(x)\bigr|^2\,dx.
\end{aligned}
$$

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of $\widetilde{X}$, or equivalently, of semigroup $\{P_t, t \geq 0\}$. That is,

$$
\begin{aligned}
\mathcal{F} = \Bigl\{ & f \in L^2(\widetilde{D}; m): \\
& \sup_{t>0}\frac{1}{t}(f - P_t f, f)_{L^2(\widetilde{D},m)} = \lim_{t\to 0}\frac{1}{t}(f - P_t f, f)_{L^2(\widetilde{D},m)} < \infty \Bigr\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(f,f) &= \sup_{t>0}\frac{1}{t}(f - P_t f, f)_{L^2(\widetilde{D};m)} \\
&= \lim_{t\to 0}\frac{1}{t}(f - P_t f, f)_{L^2(\widetilde{D};m)} \qquad \text{for } f \in \mathcal{F}.
\end{aligned}
$$

Then for $f \in \mathcal{C}$,

$$
\mathcal{E}(f,f) = \sup_{t>0}\frac{1}{t}(f - P_t f, f)_{L^2(\widetilde{D};m)} \leq \frac{1}{2d}\int_D \bigl|\nabla f(x)\bigr|^2\,dx.
$$

This shows that $f \in \mathcal{F}$. As $\mathcal{C}$ is dense in $(W^{1,2}(D), \|\cdot\|_{1,2})$ in view of Theorem 2.1, we have $W^{1,2}(D) \subset \mathcal{F}$ and

$$\mathcal{E}(f,f) \le \frac{1}{2d}\int_D |\nabla f(x)|^2\,dx \qquad \text{for every } f \in W^{1,2}(D).$$

This, Lemma 3.4 and [1], Theorem 1.1 (or [4], Theorem 6.6.9) imply that $\mathcal{F} = W^{1,2}(D)$ and

$$\mathcal{E}(f,f) = \frac{1}{2d}\int_D |\nabla f(x)|^2\,dx \qquad \text{for } f \in W^{1,2}(D).$$

We deduce then that X is a stationary reflecting Brownian motion on D running at speed $1/d$. This proves that X^k converge weakly on $C([0,T],\widetilde{D})$ to the stationary reflecting Brownian motion on $\widetilde{D}$ running at speed $1/d$.

The last assertion comes from the fact that the projection map from $(\widetilde{D},\rho) \to \overline{D}$ is continuous. □

REMARK 3.7. Note that for every $x \in D$, $\widetilde{X}$ starting from x is a Brownian motion in D before hitting the boundary of D, and the mass of X spreads immediately across the whole set D immediately after the clock starts, while X^k starting from x runs like a simple random walk on $2^{-k}\mathbb{Z}$ before hitting the boundary of D_k. Using these properties, the weak convergence in Theorem 3.6 can be strengthened to show that $(X^k, \mathbb{P}^k_x)$ converges weakly to $(\widetilde{X}, \mathbb{P}_x)$ for every interior starting point $x \in D$. We leave the details to the reader.

4. Continuous-time reflected random walk. In this section, we show that reflected Brownian motion on $\overline{D}$ can be approximated by continuous-time random walks on grids.

Let D be a bounded domain in $\mathbb{R}^d$ and D_k be defined as in the beginning of Section 2. But in this section, X^k will be the continuous time simple random walk on D_k, making jumps at the rate 2^{-2k}. By definition, X^k jumps to one of its nearest neighbors with equal probabilities. This process is symmetric with respect to measure m_k, where $m_k(x) = \frac{v_k(x)}{2d}2^{-kd}$ for $x \in D_k$. Note that m_k converge weakly to the Lebesgue measure m on D, and recall $\mathcal{A}'_k$ from the beginning of Section 2. The Dirichlet form of X^k on $L^2(D_k; m_k)$ is given by

$$\mathcal{E}^k(f,f) = \frac{1}{4d}\sum_{x,y\in D_k:\,\overline{xy}\in\mathcal{A}'_k} 2^{-(d-2)k}\big(f(x)-f(y)\big)^2. \tag{4.1}$$

Let $\mathbb{P}^k_{m_k}$ denote the distribution of $\{X^k_t, t \ge 0\}$ with the initial distribution m_k.

LEMMA 4.1. *Assume that D is a bounded domain in $\mathbb{R}^d$. For every $T > 0$, the laws of stationary random walks $\{X^k, \mathbb{P}^k_{m_k}, k \ge 1\}$ are tight in the space $\mathbb{D}([0,T],\overline{D})$ equipped with the Skorokhod topology.*

PROOF. The proof is the same as that for [1], Lemma 3.2, so we omit it. □

THEOREM 4.2. *Let D be a bounded domain in $\mathbb{R}^d$. Then for every $T > 0$, the stationary random walks X^k on D_k converge weakly in the space $\mathbb{D}([0,T],\overline{D})$, as $k \to \infty$, to the stationary reflected Brownian motion on $\overline{D}$ running at speed $1/d$, whose initial distribution is the Lebesgue measure in D.*

PROOF. Let $(Z, \mathbb{P})$ be any of the subsequential limits of $(X^k, \mathbb{P}^k_{m_k})$ in the space $\mathbb{D}([0,T],\widetilde{D})$, say, along X^{k_j}. A similar argument as that for Theorem 3.3 shows that $(Z, \mathbb{P})$ is a time-homogeneous Markov process and its transition semigroup $\{P_t, t \geq 0\}$ is symmetric with respect to the measure m on D. Furthermore, by a similar argument as that in the proof of [1], Lemma 2.2, the process Z killed upon leaving D is a killed Brownian motion in D with speed $1/d$. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with Z, and let $\{P^k_t, t \geq 0\}$ be the transition semigroup for X^k. As X^{k_j} converge weakly to Z in $\mathbb{D}([0,T],\overline{D})$, we have for every $t > 0$ and every $f, g \in C(\overline{D})$,

$$\lim_{j\to\infty}(f, P^{k_j}_t g)_{L^2(D_k;m_k)} = (f, P_t g)_{L^2(D;m)}.$$

Let $E_k : L^2(D_k; m_k) \to L^2(D; m)$ and $\pi_k : L^2(D; m) \to L^2(D_k, m)$ be the extension operator and restriction operator defined by (3.7) and (3.11), respectively. Then the last display can be restated as

$$\lim_{j\to\infty}(f, E_{k_j} P^{k_j}_t \pi_{k_j} g)_{L^2(D;m)} = (f, P_t g)_{L^2(D;m)} \qquad \text{for } f, g \in C(\overline{D}). \tag{4.2}$$

Note that

$$\|E_k f\|_{L^2(D;m)} = \|f\|_{L^2(D_k;m_k)} \qquad \text{for } f \in L^2(D_k; m_k)$$

and

$$\|\pi_k g\|_{L^2(D_k;m_k)} \leq \|g\|_{L^2(D;m)} \qquad \text{for } g \in L^2(D; m).$$

Since $C(\overline{D})$ is dense in $L^2(D; m)$ and P^k_t and P_t are contraction operators on $L^2(D_k; m_k)$ and $L^2(D; m)$, respectively, we deduce from (4.2) that

$$\lim_{j\to\infty}(f, E_{k_j} P^{k_j}_t \pi_{k_j} g)_{L^2(D;m)} = (f, P_t g)_{L^2(D;m)} \qquad \text{for every } f, g \in L^2(D; m). \tag{4.3}$$

Recall from Theorem 2.1 that $\mathcal{C}$ is the algebra generated by functions $\{\varphi_j\}_{j\geq 1}$ over $\mathbb{Q}$. As a special case of (4.3), we obtain for every $t > 0$ and $f \in \mathcal{C}$,

$$\begin{aligned}(f, P_t f)_{L^2(D;m)} &= \lim_{j\to\infty}(f, E_{k_j} P^{k_j}_t \pi_{k_j} f)_{L^2(D;m)} \\ &= \lim_{j\to\infty}(\pi_{k_j} f, P^{k_j}_t \pi_{k_j} f)_{L^2(D_k;m_k)}.\end{aligned}$$

Since $\mathcal{C} \subset C_b(D)$ and D is bounded,

$$\lim_{k\to\infty} \int_{D_k} |f(x) - \pi_k f(x)|^2 m_k(dx) = 0 \qquad \text{for } f \in \mathcal{C}.$$

Hence we conclude that

$$(f, P_t f)_{L^2(D;m)} = \lim_{j\to\infty} (f, P_t^{k_j} f)_{L^2(D_k;m_k)} \qquad \text{for } f \in \mathcal{C}. \tag{4.4}$$

Thus we have for every $t > 0$ and $f \in \mathcal{C}$,

$$\begin{aligned}
\frac{1}{t}(f, f - P_t f)_{L^2(D;m)} &= \lim_{j\to\infty} \frac{1}{t}(f, f - P_t^{k_j} f)_{L^2(D_k;m_k)} \\
&\le \liminf_{j\to\infty} \sup_{s>0} \frac{1}{s}(f, f - P_s^{k_j} f)_{L^2(D_k;m_k)} \\
&= \liminf_{j\to\infty} \mathcal{E}^{k_j}(f, f) \\
&= \liminf_{j\to\infty} \frac{1}{4d} \sum_{x,y\in D_k : \overline{xy}\in\mathcal{A}'_k} 2^{-(d-2)k}(f(x) - f(y))^2 \\
&\le \frac{1}{2d} \int_D |\nabla f(x)|^2 m(dx),
\end{aligned}$$

where in the last inequality we used Theorem 2.1. Thus

$$\begin{aligned}
\mathcal{E}(f, f) &= \sup_{t>0} \frac{1}{t}(f - P_t f, f)_{L^2(D;m)} \\
&\le \frac{1}{2d} \int_D |\nabla f(x)|^2 m(dx) \qquad \text{for every } f \in \mathcal{C}.
\end{aligned}$$

Since $\mathcal{C}$ is dense in the Sobolev space $W^{1,2}(D)$ with respect to norm $\|\cdot\|_{1,2}$, it follows that $\mathcal{F} \supset W^{1,2}(D)$ and

$$\mathcal{E}(f, f) \le \frac{1}{2d} \int_D |\nabla f(x)|^2 m(dx) \qquad \text{for every } f \in W^{1,2}(D).$$

Define

$$\mathcal{E}^0(f, g) = \frac{1}{2d} \int_D \nabla f(x) \cdot \nabla g(x) m(dx) \qquad \text{for } f, g \in W^{1,2}(D).$$

Note that $(\mathcal{E}^0, W^{1,2}(D))$ is the Dirichlet form for the reflected Brownian motion on D running at speed $1/d$. On the other hand, as we have observed at the beginning of this proof, the process Z killed upon leaving D is a killed Brownian motion in D with speed $1/d$. Therefore according to [1], Theorem 1.1 (or [4], Theorem 6.6.9), $(\mathcal{E}, \mathcal{F}) = (\mathcal{E}^0, W^{1,2}(D))$. In other words, we have shown that every subsequential limit of X^k is reflected Brownian motion on D with initial distribution being the Lebesgue measure on D and with speed $1/d$. This shows that X^k converges weakly on the space $\mathbb{D}([0, \infty), \overline{D})$ to the stationary reflected Brownian motion X on D running at speed $1/d$. $\square$

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