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SOLVING OPTIMAL STOPPING PROBLEMS VIA EMPIRICAL DUAL OPTIMIZATION

BY DENIS BELOMESTNY[1]

Duisburg-Essen University

In this paper we consider a method of solving optimal stopping problems in discrete and continuous time based on their dual representation. A novel and generic simulation-based optimization algorithm not involving nested simulations is proposed and studied. The algorithm involves the optimization of a genuinely penalized dual objective functional over a class of adapted martingales. We prove the convergence of the proposed algorithm and demonstrate its efficiency for optimal stopping problems arising in option pricing.

1. Introduction. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ be a standard filtered probability space, and let Z_t be an adapted process satisfying

$$\mathrm{E} \sup_{t\in[0,T]} |Z_t|^2 < \infty.$$

Consider the following optimal stopping problem:

$$Y^* = \sup_{\tau\in\mathcal{T}[0,T]} \mathrm{E}[Z_\tau], \tag{1.1}$$

where $\mathcal{T}[0,T]$ is the set of stopping times taking values in $[0,T]$ for some $T > 0$. Solving the optimal stopping problem (2.1) is straightforward in low dimensions. However, many problems arising in practice have high dimensions, and these applications have forced the development of simulation-based algorithms for optimal stopping problems. There are basically two approaches toward solving optimal stopping problems: a primal approach and a dual approach. Solving high-dimensional optimal stopping problems by the primal approach and Monte Carlo is a challenging task because the determination of the optimal value function in the primal approach uses a backward dynamic programming principle that seems to be incompatible with the forward nature of Monte Carlo simulation. Much research was focused on the development of fast methods to compute approximations to

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the optimal value function. One of the most successful algorithms, and the one adopted most widely by practitioners, is the Longstaff–Schwartz algorithm. It is based on approximating the conditional expectations by the least-squares regression on a given basis of functions and hence boils down to solving a quadratic optimization problem. During the last century, the primal approach was, in effect, the only method available, but in recent years another quite different "dual" approach has been discovered by Rogers (2002) and Haugh and Kogan (2004) that is based on a dual representation for the optimal value function. The dual representation involves the minimization of the dual objective functional over the set of all adapted martingales $\mathcal{M}$, where the minimum is attained at some "optimal" martingale M^* that coincides with the martingale in the Doob–Meyer decomposition of the value process. In fact, finding such an optimal martingale is as difficult as solving the original stopping problem. The so-called martingale duality approach aims at approximating the "optimal" martingale and then uses this approximation to compute upper bounds by Monte Carlo. There are two types of algorithms toward approximating the "optimal" martingale M^*. The first one needs a preliminary estimate for the value process Y^* in order to approximate the Doob martingale M^*. The early paper of Andersen and Broadie (2004) uses, for example, the Longstaff–Schwartz algorithm to construct a pilot estimate for Y^* and then employs sub-simulation to approximate M^*. Another dual algorithm that does not involve sub-simulation, was suggested in Belomestny, Bender and Schoenmakers (2009), where an approximation for the martingale M^* was constructed using martingale representation theorem and an approximation of the value process. Let us note that the performance of the above two methods deteriorate sharply as the number of exercise dates increases. The second type of algorithms is based on the direct optimization of the dual objective functional over a parameterized set of martingales and does not require a preliminary estimate of Y^*. The recent work of Desai, Farias and Moallemi (2013) uses optimization and sub-simulation to approximate M^* and Y_t^* simultaneously in an efficient way. However, it becomes less efficient in the case of continuous optimal stopping problems, as it involves sub-simulations at each time step. Another "pure" dual algorithm was proposed in Rogers (2010) and further refined in Schoenmakers, Huang and Zhang (2011). Let us finally mention the recent work of Christensen (2011) where a quite different approach was proposed that uses neither the dual representation nor Monte Carlo. This approach is based on the excessive function characterization of the value function for continuous optimal stopping problems.

The contribution of the current paper is threefold. On the one hand, we propose a novel dual optimization-based algorithm for solving optimal stopping problem in discrete and continuous time which does not require nested Monte Carlo simulations. Our algorithm makes use of the martingale representation theorem to parametrize the set of martingales we optimize over. This allows us to obtain surprisingly good results in a number of benchmark option pricing problems using rather generic sets of basis functions (trigonometric polynomials) to approximate the integrand in the martingale representation theorem. In the previous literature one was able to obtain such bounds only by using either many sub-simulations or special basis functions [e.g., European deltas in Belomestny, Bender and Schoenmakers (2009) or excessive functions in Christensen (2011)]. On the other hand, we propose a novel approach toward variance reduction based on the genuine penalization of the dual objective functional. Last but not the least, we rigorously analyze the convergence of the proposed dual algorithm and derive the corresponding convergence rates. Note that as opposed to the Longstaff–Schwartz algorithm, the convergence of dual algorithms has not been yet rigorously studied. Even the convergence of the well-known primal-dual Andersen–Broadie algorithm is not an obvious issue, as the errors stemming from the Longstaff–Schwartz algorithm are to be taken into account in a proper way; see, for example, Belomestny (2011).

The paper is organized as follows. In Section 2 we formulate the main algorithm and address the convergence issue. In Section 3 we discuss how to build up a class of martingales with good approximation properties using the so-called martingale representation. Section 4 contains several numerical examples illustrating the efficiency of our approach. Section 5 concludes the paper. Finally, in Section 6 the proofs of the main results together with some auxiliary results are collected. In particular, we derive a novel concentration inequality for some empirical process over parameterized classes of martingales.

2. Main results.

2.1. *Empirical penalized dual algorithm.* Consider the following optimal stopping problem:

$$Y_t^* = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}[t,T]} \mathrm{E}[Z_\tau | \mathcal{F}_t], \qquad t \in [0, T], \tag{2.1}$$

where $\mathcal{T}[t, T]$ is the set of stopping times taking values in $[t, T]$ for some $T > 0$. Let $\mathcal{A}$ stand for the space of all adapted martingales starting at 0, then we have the following dual representation [see Rogers (2002)] for the value process Y_t^* :

$$Y_t^* = \inf_{M \in \mathcal{A}} \Bigl(M_t + \mathrm{E}\Bigl[\sup_{u \in [t,T]} (Z_u - M_u) \big| \mathcal{F}_t \Bigr] \Bigr). \tag{2.2}$$

The infimum is attained by taking $M = M^*$, where

$$Y_t^* = Y_0^* + M_t^* - A_t^*$$

is the Doob–Meyer decomposition of the supermartingale Y_t^*, M^* being a martingale and A^* being an increasing process with $A_0^* = 0$. Moreover, the identity

$$Y_t^* = M_t^* + \sup_{u \in [t,T]} \bigl(Z_u - M_u^* \bigr) \tag{2.3}$$

holds for all $t \in [0, T]$ with probability 1. Hence, for an arbitrarily chosen adapted martingale M with $M_0 = 0$, the value

$$\mathrm{E}\Big[\sup_{u\in[0,T]}(Z_u - M_u)\Big] \tag{2.4}$$

defines an upper bound for Y_0^*, and the upper bound will be tight if M minimizes (2.4). On the other hand, we are interested in martingales M leading to the random variable $\sup_{t\in[0,T]}(Z_t - M_t)$ with a low variance, since this would imply faster convergence of a Monte Carlo estimate for (2.4). By compromising both requirements, one ends up with the optimization problem

$$\inf_{M\in\mathcal{A}}\Big\{\mathrm{E}\Big[\sup_{t\in[0,T]}(Z_t - M_t)\Big] + \lambda\sqrt{\mathrm{Var}\Big[\sup_{t\in[0,T]}(Z_t - M_t)\Big]}\Big\}, \tag{2.5}$$

where λ is a nonnegative number determining the degree of penalization by the variance. Note that due to (2.3) the Doob martingale M^* is one solution of the optimization problem (2.5).

Fixing a set of martingales $\mathfrak{M} \subset \mathcal{A}$ and replacing the true quantities in (2.5) by their empirical counterparts, we arrive at the following empirical optimization problem:

$$M_n = \operatorname*{arg\,inf}_{M\in\mathfrak{M}}\left(\frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M) + \lambda\sqrt{V_n(M)}\right), \qquad \lambda > 0, \tag{2.6}$$

where $Z^{(j)}(M),\ j = 1, \dots, d$ are i.i.d random variables having the same distribution as

$$Z(M) = \sup_{s\in[0,T]}(Z_s - M_s) \tag{2.7}$$

and

$$V_n(M) = \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\big(Z^{(i)}(M) - Z^{(j)}(M)\big)^2. \tag{2.8}$$

The approach based on the empirical optimization problem (2.6) has several obvious advantages. First, it delivers "true" upper bound without use of sub-simulation, thus resulting in a nonnested Monte Carlo. Second, it does not exclusively focus on finding Doob martingale and takes advantage of the richness [see Schoenmakers, Huang and Zhang (2011)] of the class $\mathcal{A}^*$ of adapted martingales starting at 0 and satisfying

$$Y^* = \sup_{t\in[0,T]}(Z_t - M_t), \qquad \text{a.s.} \tag{2.9}$$

Another useful feature of our algorithm which will be proved in the next section is that the variance of the r.v. $Z(M_n) = \sup_{s\in[0,T]}(Z_s - M_{n,s})$ is, with high probability, bounded by a multiple of the r.v.

$$\inf_{M\in\mathfrak{M}, M'\in\mathcal{A}^*} d\big(M, M'\big),$$

where d is a deterministic metric on $\mathcal{A}$. The above property implies that the variance of $Z(M_n)$ can be made arbitrary small by considering classes of martingales $\mathfrak{M}$ with better approximation properties with respect to the solution class $\mathcal{A}^*$. Last but not least, our approach is applicable to the case of continuous optimal stopping problems, as it does not involve regression (or subsimulations) at each discretization step as in other approaches based on the dynamic programming formulation.

2.2. *Convergence.* Let (Ψ, ρ) be a metric space. Furthermore, let $\mathcal{M} = \{M(\psi) : \psi \in \Psi\}$ be a family of adapted continuous local martingales defined on $(\Omega, \mathcal{F}, \mathrm{P})$.

DEFINITION 2.1. A quadratic ρ-modulus $\|\mathcal{M}\|_\rho$ of a family $\mathcal{M} = \{M(\psi) : \psi \in \Psi\}$ of continuous local martingales is defined as an $\mathbb{R}_+ \cap \{\infty\}$-valued stochastic process $t \mapsto \|\mathcal{M}\|_{\rho,t}$ given by

$$\|\mathcal{M}\|_{\rho,t} = \operatorname*{ess\,sup}_{\substack{\psi,\phi \in \Psi \\ \psi \neq \phi}} \frac{\sqrt{\langle M(\psi) - M(\phi)\rangle_t}}{\rho(\psi, \phi)}, \qquad t \in [0, T],$$

where $\langle M \rangle$ stands for the quadratic variation process of the continuous local martingale M.

For a given subset $\widetilde{\Psi}$ of the metric space (Ψ, ρ) denote by $N(\varepsilon, \widetilde{\Psi}, \rho)$ the smallest number of closed balls, with ρ-radius $\varepsilon > 0$, which cover the set $\widetilde{\Psi}$ and define

$$J(\delta) = \int_0^\delta \sqrt{\log[1 + N(\varepsilon, \widetilde{\Psi}, \rho)]}\, d\varepsilon$$

for all $\delta > 0$. Denote also by $\mathcal{M}^* = \{M(\psi) : \psi \in \Psi^*\}$ a subset of $\mathcal{M}$ containing all martingales M that fulfill (2.3). In the sequel we shall assume that the family $\mathcal{M}$ is rich enough so that $\mathcal{M}^*$ is not empty. Let us now formulate the main result on the convergence of $\mathrm{E}[Z(M_n)]$ for M_n defined in (2.6).

THEOREM 2.2. *Let $\mathfrak{M} = \{M(\psi) : \psi \in \widetilde{\Psi}\}$ be a family of continuous local martingales satisfying $\|\mathfrak{M}\|_{\rho,T} \le \Theta$ almost surely, for some finite Θ. Let also ψ^* be an element of Ψ^* such that $\rho(\psi, \psi^*) \le \sigma$ for all $\psi \in \widetilde{\Psi}$ and some $\sigma < \infty$. Set*

$$\mathfrak{C} = \mathfrak{C}(\widetilde{\Psi}) = \int_0^\sigma \varepsilon^{-1} J(\varepsilon) \sqrt{\log[1 + N(\varepsilon, \widetilde{\Psi}, \rho)]}\, d\varepsilon,$$

and assume that $\mathfrak{C} < \infty$. Fix some $\varkappa > 0$ and $0 < \delta < 1$ with $J(1)\log(1/\delta) \le \sqrt{n}$, and define

$$M_n = \operatorname*{arg\,inf}_{M \in \mathfrak{M}} \left(\frac{1}{n} \sum_{j=1}^n Z^{(j)}(M) + \big(\varkappa + \lambda_n(\delta/4)\big)\sqrt{V_n(M)} \right), \tag{2.10}$$

where $Z^{(j)}(M)$, $j = 1, \dots, n$, *and* $V_n(M)$ *are defined in* (2.7) *and* (2.8), *respectively, and* $\lambda_n(\alpha) = 4(2\sqrt{2\log(2/\alpha)} + \mathfrak{C})/\sqrt{n}$ *for any* $\alpha > 0$. *Then it holds for some constant* $C > 0$ (*not depending on* δ, n *and* $\varkappa$) *with probability at least* $1 - \delta$,

$$0 \le Y(M_n) - Y^* \le C(\varkappa + 2\lambda_n(\delta/4)) \inf_{\psi \in \widetilde{\Psi}} \mathcal{R}(\psi, \psi^*), \tag{2.11}$$

$$\sqrt{V(M_n)} \le C\left(1 + \frac{2\lambda_n(\delta/4)}{\varkappa}\right) \inf_{\psi \in \widetilde{\Psi}} \mathcal{R}(\psi, \psi^*), \tag{2.12}$$

where $Y(M) = \mathrm{E}[\sup_{s\in[0,T]}(Z_s - M_s)]$, $V(M) = \mathrm{Var}[\sup_{s\in[0,T]}(Z_s - M_s)]$ *and*

$$\mathcal{R}(\psi, \psi^*) = \rho(\psi, \psi^*)\sqrt{1 \vee |\log(\rho(\psi, \psi^*))|}$$

for any $\psi \in \Psi$.

REMARK 2.3. Note that $Y(M_n)$ and $V(M_n)$ are random variables measurable w.r.t. the σ-algebra generated by the paths used to compute M_n.

REMARK 2.4. The condition $\mathfrak{C} < \infty$ roughly means that $J(\varepsilon) = O(\varepsilon^{1/2+\delta})$ as $\varepsilon \to 0$ for some $\delta > 0$.

Discussion. Theorem 2.2 shows that the martingale M_n delivered by our algorithm has a nice property that the corresponding approximation error $Y(M_n) - Y^*$ and the square root variance $\sqrt{V(M_n)}$ can be bounded from above with high probability by the quantities proportional to the smallest distance between the classes of martingales $\mathfrak{M}$ and $\mathcal{A}^*$ as measured by ρ. Hence, if the set $\mathfrak{M}$ contains at least one martingale solving (2.3) we get, as expected, $Y(M_n) = Y^*$ with probability $1 - \delta$. In general, the larger is the class $\mathfrak{M}$, the smaller is the above distance. However, if the class $\mathfrak{M}$ is infinite-dimensional, maximizing the empirical objective functional in (2.10) over $\mathfrak{M}$ may not be well defined or even if M_n exists, it might be difficult to compute. Instead, one can restrict the maximization to a sequence of finite-dimensional approximating spaces $\mathfrak{M}_n = \{M(\psi) : \psi \in \Psi_n\}$ such that $\bigcup_n \Psi_n$ is dense in Ψ^*. Such a sequence of approximating spaces is usually called a *sieve*. We are interested in sieves that are compact, nondecreasing ($\mathfrak{M}_n \subset \mathfrak{M}_{n+1} \subset \cdots \subset \mathfrak{M}$) and such that for any $n \in \mathbb{N}$ and some $\psi^* \in \Psi^*$ there exists an element $\pi_n\psi^*$ in Ψ_n satisfying $\rho(\psi^*, \pi_n\psi^*) \to 0$ as $n \to \infty$, where π_n can be regarded as a projection of ψ^* to Ψ_n. For such sieves Theorem 2.2 implies that

$$V(M_n) \xrightarrow{\mathrm{P}} 0, \qquad n \to \infty, \tag{2.13}$$

provided $\mathfrak{C}(\Psi_n)/\sqrt{n}$ remains bounded as $n \to \infty$. In the next section we discuss how to get the martingale sieves $\mathfrak{M}_n$ in a constructive way. The asymptotic relation (2.13) implies that the variance of the Monte Carlo estimate of $Y(M_n)$,

$$Y_m(M_n) = \frac{1}{m} \sum_{j=n+1}^{n+m} Z^{(j)}(M_n),$$

based on a new, independent sequence of r.v.

$$\left(Z^{(n+1)}(M_n), \dots, Z^{(n+m)}(M_n)\right)$$

has the standard deviation of order $o(1/\sqrt{m})$ as $m, n \to \infty$. Therefore one can speak about fast convergence rates in this situation. Let us mention at this place that the primal-dual algorithm of Andersen and Broadie (2004) has the same variance "self-reduction" property [see Chen and Glasserman (2007)]: nearer is the preliminary regression estimate Y_t of the value function to the true one, the lower variance has the r.v. $\sup_{t\in[0,T]}(Z_t - M_t)$ with M based on Y. However, the results on the speed of the variance decay in dependence on the number of basis functions and Monte Carlo paths used in regression step are not yet available in the literature.

REMARK 2.5. If the class $\widetilde{\Psi}$ is of Vapnik–Cervonenkis type, that is,

$$N(\varepsilon, \widetilde{\Psi}, \rho) \lesssim \varepsilon^{-\beta}, \qquad \varepsilon \to 0$$

for some $\beta > 0$, then the quantity $J(\delta)$ is finite for any $\delta > 0$.

REMARK 2.6. A natural question is whether the bounds of Theorem 2.2 can be achieved without using the penalization by empirical variance. The answer is, in general, no. To see this, let Z_t be an uniformly integrable submartingale. Then Z_t admits the so-called Doob–Meyer decomposition

$$Z_t = Z_0 + M_t + A_t,$$

where M_t with $M_0 = 0$ is a uniformly integrable martingale, and A_t is an increasing predictable process. Using the optional sampling theorem, we derive

$$Y^* = \sup_{\tau\in\mathcal{T}[0,T]} \mathrm{E}[Z_\tau] = \mathrm{E}[Z_T] = Z_0 + \mathrm{E}[A_T].$$

Define $M_t^* = M_t + \mathrm{E}[A_T|\mathcal{F}_t] - \mathrm{E}[A_T]$, then $Y^* = \sup_{t\in[0,T]}(Z_t - M_t^*)$ with probability 1. Furthermore, the martingale $\widetilde{M} = M$ fulfills

$$Y^* = \mathrm{E}\Big[\sup_{t\in[0,T]}(Z_t - \widetilde{M}_t)\Big],$$

and if A_T is not deterministic, then $Y^* \neq \sup_{t\in[0,T]}(Z_t - \widetilde{M}_t) = Z_0 + A_T$ with positive probability. Hence, $\widetilde{M}$ solves, along with M^*, the original dual problem (2.2), but does not have the almost sure property (2.3). Consider now the empirical optimization problem

$$M_n = \operatorname*{arg\,inf}_{M\in\{M^*,\widetilde{M}\}} \left(\frac{1}{n}\sum_{j=1}^n Z^{(j)}(M)\right)$$

with $Z(M) = \sup_{t\in[0,T]}(Z_t - M_t)$. Due to CLT, it obviously holds

$$\liminf_{n\to\infty} \mathrm{P}(M_n = \widetilde{M}) = \liminf_{n\to\infty} \mathrm{P}\left(\sum_{j=1}^n \xi_j < 0\right) > 0,$$

where $\xi_1, \dots, \xi_n$ are i.i.d. random variables distributed as $A_T - \mathrm{E}[A_T]$. Therefore

$$V(M_n) = V(\widetilde{M}) = \mathrm{Var}\Big[\sup_{t \in [0,T]} (Z_t - \widetilde{M}_t)\Big] = \mathrm{Var}[A_T] > 0$$

with positive probability for any natural number n and the bound (2.12) does not hold any longer.

3. Martingales via martingale representations. Suppose that $Z_t = G_t(X_t)$, where $G_t : \mathbb{R}^d \to \mathbb{R}$ is a Hölder function on $[0, T] \times \mathbb{R}$ and X_t is a d-dimensional Markov process solving the following system of SDE's:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \qquad X_0 = x. \tag{3.1}$$

The coefficient functions $\mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are supposed to be Lipschitz in space and $1/2$-Hölder continuous in time, with m denoting the dimension of the Brownian motion $W = (W^1, \dots, W^m)^\top$ under measure P. It is well known that under the assumption that a martingale M_t is square integrable and is adapted to the filtration generated by W_t, there is a square integrable (row vector valued) process $H_t = (H_t^1, \dots, H_t^m)$ satisfying

$$M_t = \int_0^t H_s\, dW_s. \tag{3.2}$$

It is not hard to see that in the Markovian setting $Y_t^* = V(t, X_t)$, it holds $H_s = \psi(s, X_s)$ for some vector function $\psi(s, x) = (\psi_1(s, x), \dots, \psi_m(s, x))$ satisfying

$$\int_0^T \mathrm{E}\big[|\psi(s, X_s)|^2\big]\, ds < \infty.$$

As a result,

$$M_t = M_t(\psi) = \int_0^t \psi(s, X_s)\, dW_s.$$

Thus, the set of adapted square-integrable martingales can be "parameterized" by the set $L_{2,P}([0, T] \times \mathbb{R}^d)$ of square-integrable m-dimensional vector functions ψ on $[0, T] \times \mathbb{R}^d$ that satisfy $\|\psi\|_{2,\mathrm{P}}^2 := \int_0^T \mathrm{E}[|\psi(s, X_s)|^2]\, ds < \infty$. Let Ψ^* be a set of $\psi \in L_{2,P}([0, T] \times \mathbb{R}^d)$ such that $M_t(\psi)$ solves (2.3). Choose a family of finite-dimensional linear models of functions, called sieves, with good approximation properties. We consider linear sieves of the form

$$\Psi_K = \{\beta_1 \phi_1 + \cdots + \beta_K \phi_K : \beta_1, \dots, \beta_K \in \mathcal{C}\}, \tag{3.3}$$

where $\phi_1, \dots, \phi_K$ are some given vector functions with components from the space of bounded continuous functions $C_b([0, T] \times \mathbb{R}^d)$, and $\mathcal{C}$ is a compact set in $\mathbb{R}$. Next define a class of adapted square-integrable martingales via

$$\mathfrak{M}_K = \{M_t(\psi) : \psi \in \Psi_K\}$$

and set

$$M_n := \operatorname*{arg\,inf}_{M\in\mathfrak{M}_{K_n}}\left(\frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M)+(\varkappa+\lambda_n)\sqrt{V_n(M)}\right), \tag{3.4}$$

where $K_n\to\infty$ as $n\to\infty$. As can be easily seen

$$\begin{aligned}\sqrt{\langle M-M'\rangle_T} &\le \sqrt{T}\cdot\sup_{x\in\mathbb{R}^d}\sup_{t\in[0,T]}\left|\psi(t,x)-\psi'(t,x)\right|\\ &=\sqrt{T}\cdot\rho(\psi,\psi')\end{aligned}$$

with $M_t=M_t(\psi)$ and $M'_t=M_t(\psi')$ for any $\psi,\psi'\in C_b([0,T]\times\mathbb{R}^d)\times\cdots\times C_b([0,T]\times\mathbb{R}^d)$. Hence the quadratic ρ -modulus of the family $\mathfrak{M}_K$ is bounded by $\sqrt{T}$ with probability 1. For many linear sieves of the form (3.3) and diffusion processes X, it holds that

$$\log\bigl[1+N(\varepsilon,\Psi_K,\rho)\bigr]\lesssim K^{d+1}\log(1/\varepsilon),\qquad \varepsilon\to 0$$

and in this situation we have with probability at least $1-\delta$

$$\sqrt{V(M_n)}=O(a_n),$$

where $a_n=\inf_{\psi\in\Psi_{K_n},\psi^*\in\Psi^*}\rho(\psi,\psi^*)$, provided $K_n^{d+1}/\sqrt{n}=O(1)$ for $n\to\infty$.

4. Numerical study. In this section we test our algorithm on several benchmark examples related to American/Bermudan option pricing problems arising in finance. Let us first give some general details on the implementation of our algorithm. First, we need to construct a set of approximating martingales. To this end we are going to use the martingale representation theorem as described in Section 3. It is known [see, e.g., Belomestny, Bender and Schoenmakers (2009)] that in the Markovian setting $Y_t^*=V(t,X_t)$ and under some rather general assumptions on the diffusion process X in (3.1) the Doob martingale M^* with $M_0^*=0$ has a representation

$$M_t^*=\int_0^t\sum_{i=1}^{d}\frac{\partial V(u,X_u)}{\partial X^i}\sigma^i(u,X_u)\,dW_u. \tag{4.1}$$

Fix now some linear space $\widetilde{\Psi}$ of functions $\psi:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$. The equality (4.1) motivates us to consider the following optimization problem:

$$\begin{aligned}\psi_{n,\lambda} := \operatorname*{arg\,inf}_{\psi\in\widetilde{\Psi}}\Bigg\{&\frac{1}{n}\sum_{j=1}^{n}Z^{(j)}(\psi)\\ &+\lambda\sqrt{\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\bigl(Z^{(i)}(\psi)-Z^{(j)}(\psi)\bigr)^2}\Bigg\}\end{aligned} \tag{4.2}$$

with

$$Z^{(j)}(\psi) := \sup_{t\in[0,T]} \bigl[G_t\bigl(X_t^{(j)}\bigr) - M_t^{(j)}(\psi)\bigr], \tag{4.3}$$

$$M_t^{(j)}(\psi) := \int_0^t \sum_{i=1}^d \sigma^i\bigl(u, X_u^{(j)}\bigr)\psi_i\bigl(u, X_u^{(j)}\bigr)\, dW_u^{(j)} \tag{4.4}$$

and some $\lambda > 0$, where $(W_t^{(j)}, X_t^{(j)}) \in \mathbb{R}^m \times \mathbb{R}^d$, $j = 1, \dots, n$, is the set of trajectories obtained, for example, by discretizing the system of SDEs (3.1).

REMARK 4.1. The construction of the class (4.4) of approximating martingales is based on some prior information on the underlying process in form of the matrix σ. Moreover, this construction implies that we are actually aiming at approximating the Doob martingale M^* in this case.

REMARK 4.2. Let us discuss the choice of the penalization parameter λ in more details. On the one side, the parameter λ can be chosen according to Theorem 2.2, that is, $\lambda = \varkappa + 4(2\sqrt{2\log(2/\alpha)} + \mathfrak{C})/\sqrt{n}$ for some $\varkappa > 0$ and $\alpha > 0$. This choice, however, requires knowledge of

$$\mathfrak{C} = \mathfrak{C}(\widetilde{\Psi}) = \int_0^\sigma \varepsilon^{-1} J(\varepsilon)\sqrt{\log[1 + N(\varepsilon, \widetilde{\Psi}, \rho)]}\, d\varepsilon,$$

which might be difficult to compute in concrete situations. On the other side, λ can be found empirically by minimizing the "out of sample" variance and mean of the r.v. $Z(M_n)$. This would require some additional computational efforts.

In all examples below we use the Euler scheme and $n_{\mathrm{disc}} = 200$ discretization points to approximate (3.1). The integral in (4.4) can be then easily approximated through the sum

$$\sum_{l=1}^{n_{\mathrm{disc}}} \sum_{i=1}^d \sigma^i\bigl(u_l, X_{u_l}^{(j)}\bigr)\psi_i\bigl(u_l, X_{u_l}^{(j)}\bigr)\bigl(W_{u_{l+1}}^{(j)} - W_{u_l}^{(j)}\bigr).$$

As to the choice of linear space $\widetilde{\Psi}$, we are striving for the most generic choice not involving special functions like European deltas as in Belomestny, Bender and Schoenmakers (2009). In all examples to follow we first make a basic variable transformation and then use trigonometric bases. Let us also comment on the optimization problem (4.2) which is convex (at least for n large enough), provided $\widetilde{\Psi}$ is a linear space. Note, however, that the objective functional in (4.2) is, in general, not smooth. In order to avoid computational problems related to the nonsmoothness of $Z(\psi)$, we smooth it [see Nesterov (2005) for some theoretical justification] and consider instead Z the functional

$$Z_p(\psi) = p^{-1}\log\biggl(\int_0^T \exp\bigl(p\bigl(G_s(X_s) - M_s(\psi)\bigr)\bigr)\, ds\biggr), \tag{4.5}$$

where $M_t(\psi) = \int_0^t \sum_{i=1}^d \sigma^i(u, X_u)\psi_i(u, X_u)\, dW_u^i$. An alternative expression for $Z_p(\psi)$ is

$$Z_p(\psi) = Z(\psi) + p^{-1} \log\left(\int_0^T \exp\bigl(p\bigl(Z_s - M_s(\psi) - Z(\psi)\bigr)\bigr)\, ds\right). \tag{4.6}$$

It follows from representation (4.6) that

$$0 \le Z_p(\psi) - Z(\psi) \le p^{-1} \log T.$$

Hence $Z_p(\psi) \to Z(\psi)$ as $p \to \infty$. The advantage of using $Z_p(\psi)$ instead of $Z(\psi)$ is that the standard gradient-based optimization routines can be used to compute $\psi_{n,\lambda}$.

4.1. *American put on a single asset*. We start with analyzing the continuously exercisable American put option on a single asset, the simplest American-type option. We assume the asset price follows the geometric Brownian motion process

$$dX_t = rX_t\, dt + \sigma X_t\, dW_t,$$

where $r = 0.06$, $\sigma = 0.4$, W_t is the standard Brownian motion, and the stock pays no dividends. The option has a strike price of $K = 100$ and a maturity of $T = 0.5$, and the payoff upon exercise at time t is $G(X_t) = e^{-rt}(K - X_t)^+$.

In our implementation of (4.2) we take $\widetilde{\Psi}_L$ to be a linear space of functions $\psi : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ such that

$$\psi(t, x) \in \operatorname{span}\{\zeta_k\bigl(y_t(x)\bigr), \xi_k\bigl(y_t(x)\bigr), k = 0, \dots, L\},$$

where $y_t(x) = \frac{1}{T-t} \log(x/K)$ and

$$\zeta_k(z) = \begin{cases} 0, & z < -0.5, \\ \sin(k \cdot z), & |z| \le 0.5, \\ 1, & z > 0.5, \end{cases}$$

$$\xi_k(z) = \begin{cases} 0, & z < -0.5, \\ \cos(k \cdot z), & |z| \le 0.5, \\ 1, & z > 0.5. \end{cases}$$

Table 1 is obtained using the following two-step procedure. First, we generate $n = 10{,}000$ "training" paths on which we solve optimization (4.2) to get $\psi_{n,\lambda}$. In the second step we use $N = 100{,}000$ new paths to test the martingale resulting from $\psi_{n,\lambda}$ and to get the final estimate

$$Y_{n,\lambda}(N) = \frac{1}{N} \sum_{j=n+1}^{n+N} Z^{(j)}(\psi_{n,\lambda}).$$

The values in Table 1 are reported together with the standard deviations obtained by repeating the "testing" step 100 times. The times in the last column of the

TABLE 1
Upper bounds for the standard one-dimensional American put with parameters $K = 100, r = 0.06, T = 0.5$ and $\sigma = 0.4$ obtained using the linear space $\widetilde{\Psi}_5$. Values for two different values of the parameter λ are presented

X_0	True value	Upper bound $Y_{10^4,0}(10^5)$	Upper bound $Y_{10^4,2}(10^5)$	Time (sec)
80	21.6059	21.63044 (0.04354)	21.64156 (0.01321)	53
90	14.9187	14.92159 (0.01750)	14.93001 (0.00576)	51
100	9.9458	9.93455 (0.01354)	9.94712 (0.00423)	47
110	6.4352	6.41561 (0.01329)	6.42911 (0.00479)	47
120	4.0611	4.03417 (0.01127)	4.04883 (0.00392)	43

table give the duration of the "training" step. By inspecting Table 1 one can draw several conclusions. First, the values of the upper bound $Y_{10^4,2}(10^5)$ are almost exact. Second, the penalization with the empirical variance ($\lambda = 2$) reduces the standard deviation by a factor of three. Finally, our approach is able to compete with the very powerful method of Christensen (2011) (perhaps with a little bit longer computational time).

4.2. *American puts on the cheapest of d assets.* In this section, we study the performance of our approach for multiasset American options, where traditional lattice techniques usually suffer from serious numerical constraints. Specifically, we price the American put option on the cheapest of d assets. This example was also studied by Rogers (2002). The risk-neutral dynamics for d-dimensional underlying process X is given by

$$dX_t^i = rX_t^i\,dt + \sigma_i X_t^i\,dW_t^i, \qquad i = 1, \dots, d,$$

where $W_t^1, \dots, W_t^d$ are d independent Brownian motions. The payoff at time t is equal to

$$G(X_t) = e^{-rt}\Bigl(K - \min_{k=1,\dots,d} X_t^k\Bigr)^+.$$

In our numerical experiment we take $d = 2$, $\sigma_i = \sigma = 0.4$, $r = 0.06$ and $K = 100$ and consider linear space $\widetilde{\Psi}_L$ of functions $\psi : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^2$ such that

$$\begin{aligned}\psi_1(t, x) \in \operatorname{span}\{&\zeta_k(y_t^1(x)), \xi_k(y_t^1(x)), \zeta_k(y_t^1(x))1(y_t^1 \le y_t^2),\\ &\xi_k(y_t^1(x))1(y_t^1 \le y_t^2),\\ &\zeta_k(y_t^1(x) + y_t^2(x)), \xi_k(y_t^1(x) + y_t^2(x)), k = 0, \dots, L\}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\psi_2(t, x) \in \operatorname{span}\{&\zeta_k(y_t^2(x)), \xi_k(y_t^2(x)), \zeta_k(y_t^2(x))1(y_t^2 \le y_t^1),\\ &\xi_k(y_t^2(x))1(y_t^2 \le y_t^1),\\ &\zeta_k(y_t^1(x) + y_t^2(x)), \xi_k(y_t^1(x) + y_t^2(x)), k = 0, \dots, L\}\end{aligned} \tag{4.8}$$

TABLE 2
Upper bounds (with standard deviations) for the 2-dimensional Bermudan min-puts with parameters $K = 100, r = 0.06, \sigma = 0.4$

X_0^1	X_0^2	True value (FD)	Upper bound $Y_{10^4,0}(10^5)$	Upper bound $Y_{10^4,2}(10^5)$	Times (sec)
80	80	37.30	37.65877 (0.02832)	37.65921 (0.00912)	67
100	100	25.06	25.16745 (0.02341)	25.17551 (0.00778)	63
120	120	15.92	15.93370 (0.01949)	15.94191 (0.00611)	61

with ζ_k, ξ_k defined in Section (4.1) and $y_t^1(x) = \frac{1}{T-t}\log(x^1/K)$, $y_t^2(x) = \frac{1}{T-t}\log(x^2/K)$.

Table 2 is again obtained using a two-step procedure as described in Section 4.1 and the linear space $\widetilde{\Psi}_7$. The results can be significantly improved by adding to $\widetilde{\Psi}_7$ some special functions, like European deltas or harmonic functions.

4.3. *Bermudan max-calls on d assets.* This is a benchmark example studied in Broadie and Glasserman (1997), Haugh and Kogan (2004) and Rogers (2002) among others. Specifically, the model with d identically distributed assets is considered, where each underlying has dividend yield δ. The risk-neutral dynamic of assets is given by

$$\frac{dX_t^k}{X_t^k} = (r-\delta)\,dt + \sigma\,dW_t^k, \qquad k = 1, \dots, d,$$

where $W_t^k, k = 1, \dots, d$, are independent one-dimensional Brownian motions and r, δ, σ are constants. At any time $t \in \{t_0, \dots, t_{\mathcal{I}}\}$ the holder of the option may exercise it and receive the payoff

$$G(X_t) = e^{-rt}\bigl(\max(X_t^1, \dots, X_t^d) - K\bigr)^+.$$

We consider a two-dimensional example where $t_i = iT/\mathcal{I}, i = 0, \dots, \mathcal{I}$, with $T = 3, \mathcal{I} = 9$. In order to construct the linear space $\widetilde{\Psi}_L$ we again use the functions $\psi : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^2$ with coordinate functions defined in (4.7) and (4.8), respectively. Table 3 is obtained by setting $L = 7$. One can observe that the re-

TABLE 3
Bounds (with standard deviations) for 2-dimensional Bermudan max call with parameters $\kappa = 100, r = 0.05, \sigma = 0.2, \delta = 0.1$

X_0^1	X_0^2	Upper bound $Y_{10^4,0}(10^5)$	Upper bound $Y_{10^4,2}(10^5)$	A&B Price interval	Time (sec)
90	90	8.07742 (0.00832)	8.08012 (0.00313)	[8.053, 8.082]	58
100	100	14.01900 (0.01405)	14.02131 (0.00466)	[13.892, 13.934]	61
110	110	21.60967 (0.01798)	21.62144 (0.00521)	[21.316, 21.359]	64

TABLE 4
Bounds (with standard deviations) for d-dimensional Bermudan max-call with parameters $\kappa = 100, r = 0.05, \sigma = 0.2, \delta = 0.1$

d	Upper bound $Y_{10^4,0}(10^5)$	Upper bound	A&B price interval	Time (sec)
3	11.28986 (0.00939)	11.29100 (0.00326)	[11.265, 11.308]	73
5	16.68231 (0.01405)	16.69506 (0.00467)	[16.602, 16.655]	80

sults of Table 3 are especially good for small values of X_0. For example, the upper bound $Y_{10^4,2}(10^5)$ for $X_0 = (90, 90)$ almost coincides with the exact value Y_0^* and was previously obtained only by using either European deltas [see Belomestny, Bender and Schoenmakers (2009)] or many sub-simulations; see Andersen and Broadie (2004). As can be seen from Table 4, the upper bound [$X_0 = (90, \dots, 90)$] remains tight as the dimension d increases.

5. Conclusion. This paper proposes an efficient and self-contained dual algorithm for solving optimal stopping problems in discrete and continuous time which is based on the direct minimization of the penalized dual objective functional over a genuinely parameterized set of martingales. We analyze the asymptotic properties of the estimated value function and show that its variance can be made arbitrarily small by a proper choice of approximating martingales. From the methodological point of view, the probabilistic tools developed in the paper can be used to analyze the convergence of various types of empirical optimization problems arising in computational stochastics and finance.

6. Proofs of main results.

6.1. *Proof of Theorem* 2.2. Let us first sketch the main steps of the proof. Our main interest lies in estimating the quantities $Y(M_n) - Y^*$ and $V(M_n)$. In order to obtain these estimates we need a kind of uniform (over $M \in \mathfrak{M}$) concentration inequality for the empirical process

$$\mathcal{E}_n(M) = \frac{1}{n}\sum_{j=1}^{n}(Z^{(j)}(M) - \mathrm{E}[Z(M)]) = \frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M) - Y(M)$$

that gives probabilistic bounds for $\sqrt{n} \cdot \mathcal{E}_n(M)$ in terms of the empirical variance $V_n(M)$. Indeed, such an inequality would allow us to get an upper bound for the quantity $Y(M_n) + \varkappa\sqrt{V_n(M_n)}$ with $\varkappa > 0$ in terms of $\mathcal{Q}_n(M_n)$, where

$$\mathcal{Q}_n(M) = \frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M) + (\varkappa + \lambda_n(\delta/2))\sqrt{V_n(M)}.$$

Unfortunately, the usual concentration inequalities could not be used here, as they would provide us with the bounds in terms of the true variance $V(M)$ and not in terms of the empirical one $V_n(M)$. However, there is another, less-known type of concentration inequalities for self-normalized empirical processes [see Bercu, Gassiat and Rio (2002)], and this is exactly what we need. We extend the above inequalities to the case of general family of random variables. As a next step, in order to derive a bound for $V(M_n)$, we need a kind of uniform concentration inequality for the empirical process $\Delta_n(\psi) = (V(M(\psi)) - V_n(M(\psi)))$ that holds uniformly over the set $\widetilde{\Psi}$ and gives probabilistic bounds for $\sqrt{n} \cdot \Delta_n(\psi)$ in terms of $\rho(\psi, \psi^*)$ for any fixed $\psi^* \in \Psi^*$. The latter type of inequality cannot be derived from the well-known concentration inequalities for selfbounding random variables [see, e.g., Devroye and Lugosi (2008)], since variance $V(M)$ is a highly nonlinear function of M and the random variable $Z(M)$ is usually not bounded. The corresponding concentration inequality making use of the local subgaussianity of $V(M)$, is presented in Section 6 and can be interesting in its own right. Finally, using the inequality $\mathcal{Q}_n(M_n) \le \mathcal{Q}_n(M)$, that holds for any $M \in \mathfrak{M}$, we will arrive at (2.11) and (2.12).

Part 1: The following proposition allows us to derive uniform bounds for the empirical process $\sqrt{n} \cdot \mathcal{E}_n(M)$ in terms of the empirical variance $V_n(M)$.

PROPOSITION 6.1. *Let $\mathfrak{X}$ be a family of centered and normalized random variables on a common probability space* $(\Omega, \mathcal{F}, \mathrm{P})$ *with finite bracketing number in* $L_2(\mathrm{P})$ *such that*

$$\limsup_{n\to\infty} \mathrm{E}\Big[\sup_{X\in\mathfrak{X}} \max\big|\sqrt{n} \cdot \mathbb{E}_n[X]\big|\Big] \le \mathfrak{C} < \infty$$

for some positive constant $\mathfrak{C} = \mathfrak{C}(\mathfrak{X})$, *where*

$$\mathbb{E}_n[X] = \frac{1}{n}\sum_{j=1}^{n} X^{(j)}$$

and $X^{(1)}, \dots, X^{(n)}$ *are i.i.d. copies of the element* $X \in \mathfrak{X}$. *Define*

$$W_n(X) = \frac{\mathbb{E}_n[X]}{\sqrt{V_n(X)}}$$

with

$$V_n(X) = \frac{1}{n}\sum_{j=1}^{n} (X^{(j)})^2.$$

Then for any $\kappa > 0$ *and* $\alpha > \sqrt{2}$, *one can find some positive* θ *and* n_0 *depending on* $\mathfrak{X}$, α *and* κ *such that, for* $n \ge n_0$ *and for any* $x \in [0, \theta\sqrt{n}]$

$$\mathrm{P}\Big(\sup_{X\in\mathfrak{X}} \big|\sqrt{n} \cdot W_n(X)\big| \ge (x + \alpha\mathfrak{C})\Big) \le 2\exp\Big(-\frac{x^2}{4\alpha^2(1+\kappa)}\Big).$$

For the case of noncentered and nonnormalized random variables X, one can derive from Proposition 6.1 the following corollary.

COROLLARY 6.2. *Let $\mathfrak{X}$ be a family class of random variables on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with finite bracketing number in $L_2(\mathrm{P})$ such that*

$$\sup_{X \in \mathfrak{X}} \mathrm{E}|X|^2 < \infty$$

and

$$\limsup_{n \to \infty} \mathrm{E}\Big[\sup_{X \in \mathfrak{X}} \big|\sqrt{n} \cdot \mathbb{E}_n[X - \mathrm{E}X]\big|\Big] \le \mathfrak{C} < \infty$$

for some positive constant $\mathfrak{C} = \mathfrak{C}(\mathfrak{X})$. Define

$$W_n(X) = \frac{\mathbb{E}_n[X] - \mathrm{E}[X]}{\sqrt{V_n(X)}}$$

with

$$V_n(X) = \frac{1}{n} \sum_{j=1}^{n} \big(X^{(j)} - \mathbb{E}_n[X]\big)^2.$$

Then for any $\kappa > 0$ and $\alpha > \sqrt{2}$, one can find some positive θ and n_0 depending on $\mathfrak{X}$, α and κ such that, for $n \ge n_0$ and for any $x \in [0, \theta\sqrt{n}]$,

$$\mathrm{P}\bigg(\sup_{X \in \mathfrak{X}} \big|\sqrt{n} \cdot W_n(X)\big| \ge \frac{\sqrt{2}(x + \alpha\mathfrak{C})}{1 - \sqrt{2}(x + \alpha\mathfrak{C})/n}\bigg) \le 2\exp\bigg(-\frac{x^2}{4\alpha^2(1+\kappa)}\bigg),$$

provided $\sqrt{2}(x + \alpha\mathfrak{C}) < n$. As a result, by fixing some $\delta > 0$ with $\log(1/\delta) \le \sqrt{n}$ and taking $x = 2\alpha\sqrt{(1+\kappa)\log(4/\delta)}$, we get with probability at least $1 - \delta$

$$\sup_{X \in \mathfrak{X}} \big|\sqrt{n} \cdot W_n(X)\big| \ge 2\sqrt{2}\alpha \cdot \big(2\sqrt{(1+\kappa)\log(2/\delta)} + \mathfrak{C}\big)$$

for all $n > n_0$.

Part 2: Next we need the concentration inequality for the empirical process $\sqrt{n} \cdot (V_n(M) - V(M))$. The following proposition is proved in Section 6.5.

PROPOSITION 6.3. *Let $\mathfrak{M} = \{M(\psi) : \psi \in \widetilde{\Psi}\}$ be a family of continuous local martingales, where $\widetilde{\Psi}$ is a subspace of the metric space (Ψ, ρ). Suppose that $\|\mathfrak{M}\|_{\rho,T} \le \Theta$ a.s. for some finite Θ and*

$$J = \int_0^1 \sqrt{\log\big[1 + N(\varepsilon, \widetilde{\Psi}, \rho)\big]}\, d\varepsilon < \infty.$$

Denote $\Delta_n(\psi) = V_n(M(\psi)) - V(M(\psi))$ *for any* $\psi \in \Psi$, *then for any fixed* $\psi^* \in \mathcal{M}^*$ *such that* $\sup_{\psi \in \widetilde{\Psi}} \rho(\psi, \psi^*) < \infty$ *it holds*

$$\mathrm{P}\left(\sup_{\psi \in \widetilde{\Psi}} \left| \frac{\sqrt{n} \cdot \Delta_n(\psi)}{\mathcal{R}^2(\psi, \psi^*)} \right| > U\right) \le \exp\left(-\frac{D \cdot U}{J}\right)$$

for any $U > 0$ *and some constant* $D > 0$ *depending on* Θ, *where*

$$\mathcal{R}(\psi, \psi') = \rho(\psi, \psi')\sqrt{1 \vee |\log(\rho(\psi, \psi'))|}$$

for any $\psi, \psi' \in \Psi$.

Part 3: Now we can begin with the proof of Theorem 2.2. By Corollary 6.2 it holds for any $\psi \in \widetilde{\Psi}$ with probability at least $1 - \delta/2$,

$$\begin{aligned} Y(M_n) + \varkappa\sqrt{V_n(M_n)} &\le \frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M_n) + (\varkappa + \lambda_n(\delta/4))\sqrt{V_n(M_n)} \\ &\le \frac{1}{n}\sum_{j=1}^{n} Z^{(j)}(M(\psi)) + (\varkappa + \lambda_n(\delta/4))\sqrt{V_n(M(\psi))} \\ &\le Y(M(\psi)) + (\varkappa + 2\lambda_n(\delta/4))\sqrt{V_n(M(\psi))}. \end{aligned}$$

Proposition 6.3 implies that with probability at least $1 - \delta/4$,

$$\begin{aligned} V_n(M(\psi)) &\le V(M(\psi)) + JD^{-1}\log(4/\delta)\frac{\mathcal{R}^2(\psi, \psi^*)}{\sqrt{n}} \\ &\le V(M(\psi)) + C\mathcal{R}^2(\psi, \psi^*) \end{aligned}$$

for some universal constant C, provided $J\log(1/\delta) \le \sqrt{n}$. Hence, using the elementary inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, we get

$$Y(M_n) + \varkappa\sqrt{V_n(M_n)} \le Y(M(\psi)) + (\varkappa + 2\lambda_n(\delta/4))\left[\sqrt{V(M(\psi))} + \sqrt{C}\mathcal{R}(\psi, \psi^*)\right]$$

with probability at least $1 - 3\delta/4$. By the Burkholder–Davis–Gundy inequality,

$$Y(M(\psi)) - Y^* \le \Theta\rho(\psi, \psi^*)$$

and $V(M(\psi)) \le \Theta^2\rho^2(\psi, \psi^*)$ for any $\psi \in \widetilde{\Psi}$. Therefore

$$Y(M_n) - Y^* \le 2\sqrt{C}\Theta(1 + \varkappa + 2\lambda_n(\delta/4))\mathcal{R}(\psi, \psi^*)$$

and

$$\sqrt{V_n(M_n)} \le 2\sqrt{C}\Theta\varkappa^{-1}(1 + \varkappa + 2\lambda_n(\delta/4))\mathcal{R}(\psi, \psi^*).$$

Using again Proposition 6.3, we get with probability at least $1-\delta$,

$$\begin{aligned}\sqrt{V(M_n)} &\le \sqrt{V_n(M_n)} + \sqrt{C}\mathcal{R}(\psi, \psi^*)\\ &\le 3\sqrt{C}\Theta\varkappa^{-1}(1+\varkappa+2\lambda_n(\delta/4))\mathcal{R}(\psi, \psi^*).\end{aligned}$$

Part 4: To finish the proof of Theorem 2.2, it suffices to prove the following proposition.

PROPOSITION 6.4. *Let $\widetilde{\Psi}$ be a subspace of the metric space (Ψ, ρ) such that $\rho(\psi, \psi^*) \le \sigma$ for some $\psi^* \in \Psi^*$, all $\psi \in \widetilde{\Psi}$ and some $\sigma > 0$. Define $\mathfrak{M} = \{M(\psi) : \psi \in \widetilde{\Psi}\}$ and set*

$$\mathfrak{C} = \int_0^\sigma \varepsilon^{-1} J(\varepsilon)\sqrt{\log[1 + N(\varepsilon, \widetilde{\Psi}, \rho)]}\, d\varepsilon.$$

If $\|\mathfrak{M}\|_{\rho,T} \le \Theta$ a.s. and $\mathfrak{C} < \infty$, then there is a constant A depending on Θ, such that

$$\limsup_{n\to\infty} \mathrm{E}\Big[\sup_{M\in\mathfrak{M}} |\mathbb{G}_n[Z(M)]|\Big] \le A\mathfrak{C}$$

with

$$\mathbb{G}_n[Z(M)] = \frac{1}{\sqrt{n}}\sum_{j=1}^n (Z^{(j)}(M) - \mathrm{E}[Z(M)]).$$

PROOF. We follow the proof of Lemma 19.34 in van der Vaart (1998) with some straightforward modifications. It holds

$$\begin{aligned}\mathbb{G}_n(M(\psi)) &= \frac{1}{\sqrt{n}}\sum_{j=1}^n (Z^{(j)}(M(\psi)) - \mathrm{E}[Z^{(j)}(M(\psi))])\\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^n (Z^{(j)}(M(\psi)) - Z^{(j)}(M(\psi^*)))\\ &\quad + \frac{1}{\sqrt{n}}\sum_{j=1}^n (\mathrm{E}[Z^{(j)}(M(\psi^*))] - \mathrm{E}[Z^{(j)}(M(\psi))]),\end{aligned}$$

since $\mathrm{Var}[Z(M(\psi^*))] = 0$. Setting

$$K_T = \sup_{\psi\in\widetilde{\Psi}} \sup_{t\in[0,T]} |M_t(\psi) - M_t(\psi^*)|,$$

we derive

$$|Z(M(\psi)) - Z(M(\psi^*))| \le K_T, \qquad |\mathrm{E}[Z(M(\psi))] - \mathrm{E}[Z(M(\psi^*))]| \le \mathrm{E}[K_T].$$

As a result,

$$\mathrm{E}\sup_{\psi\in\widetilde{\Psi}}\bigl|\mathbb{G}_n\bigl(M(\psi)\bigr)1\{K_T>a(\sigma)\sqrt{n}\}\bigr|\le\sqrt{n}\cdot\mathrm{E}\bigl[\bigl(K_T+\mathrm{E}[K_T]\bigr)1\{K_T>a(\sigma)\sqrt{n}\}\bigr]$$
$$\le 2\sqrt{n}\cdot\mathrm{E}\bigl[K_T\cdot 1\{K_T>a(\sigma)\sqrt{n}\}\bigr],$$

where we set

$$a(\sigma)=\frac{J(\sigma)}{\sqrt{\log[1+N(\sigma,\widetilde{\Psi},\rho)]}}.$$

Under the condition $\|\mathfrak{M}\|_{\rho,T}\le\Theta$ a.s. one can prove that

$$\mathrm{P}\Bigl(\sup_{\substack{\psi,\phi\in\widetilde{\Psi},\\ \rho(\psi,\phi)\le\delta}}\ \sup_{t\in[0,T]}\bigl|M_t(\psi)-M_t(\phi)\bigr|>x\Bigr)\le 2e^{-x^2/CJ^2(\delta)}\tag{6.1}$$

for all $x>0$, where C is a universal constant depending only on Θ. Inequality (6.1) implies

$$\mathrm{E}\bigl[K_T\cdot 1\{K_T>a(\sigma)\sqrt{n}\}\bigr]\le 2a(\sigma)\sqrt{n}e^{-na^2(\sigma)/CJ^2(\sigma)}+2\int_{a(\sigma)\sqrt{n}}^{\infty}e^{-x^2/CJ^2(\sigma)}\,dx.$$

Fix an integer q_0 such that $\sigma\le 2^{-q_0}\le 2\sigma$. For each natural number $q>q_0$, there exists a nested sequence of partitions $\widetilde{\Psi}=\bigcup_{i=1}^{N_q}\widetilde{\Psi}_{qi}$ of $\widetilde{\Psi}$ into N_q disjoint subsets such that $\rho(\psi,\phi)\le 2^{-q}$ for any $\psi,\phi\in\widetilde{\Psi}_{qi}$ and $N_q\le N(2^{-q+1},\widetilde{\Psi},\rho)$. Denote

$$\Delta_{qi}=\sup_{\psi,\phi\in\widetilde{\Psi}_{qi}}\ \sup_{t\in[0,T]}\bigl|M_t(\psi)-M_t(\phi)\bigr|,$$

and then (6.1) implies

$$\mathrm{E}\bigl[\Delta_{qi}^2\bigr]\le 2\int_0^{\infty}xe^{-x^2/CJ^2(2^{-q})}\,dx=CJ^2\bigl(2^{-q}\bigr).$$

Choose for each $q\ge q_0$ a fixed element ψ_{qi} from each partioning set $\widetilde{\Psi}_{qi}$, and set

$$\Pi_q\bigl[Z\bigl(M(\psi)\bigr)\bigr]=Z(\psi_{qi}),\qquad \Delta_q\bigl[Z\bigl(M(\psi)\bigr)\bigr]=\Delta_{qi}\qquad \text{if } \psi\in\widetilde{\Psi}_{qi}.$$

Then $\Pi_q[Z(M(\psi))]$ and $\Delta_q[Z(M(\psi))]$ run through a set of N_q functions if ψ runs through $\widetilde{\Psi}$. Define for each fixed n and $q\ge q_0$ numbers and indicator functions

$$a_q=J\bigl(2^{-q}\bigr)/\sqrt{\log[1+N_{q+1}]},$$
$$A_{q-1}\bigl[Z\bigl(M(\psi)\bigr)\bigr]=\mathbf{1}\bigl\{\Delta_{q_0}\bigl[Z\bigl(M(\psi)\bigr)\bigr]\le\sqrt{n}a_{q_0},\dots,\Delta_{q-1}\bigl[Z\bigl(M(\psi)\bigr)\bigr]\le\sqrt{n}a_{q-1}\bigr\},$$
$$B_q\bigl[Z\bigl(M(\psi)\bigr)\bigr]=\mathbf{1}\bigl\{\Delta_{q_0}\bigl[Z\bigl(M(\psi)\bigr)\bigr]\le\sqrt{n}a_{q_0},\dots,\Delta_{q-1}\bigl[Z\bigl(M(\psi)\bigr)\bigr]\le\sqrt{n}a_{q-1},\ \Delta_q\bigl[Z\bigl(M(\psi)\bigr)\bigr]>\sqrt{n}a_q\bigr\}.$$

Now decompose

$$
\begin{aligned}
&Z(M(\psi)) - \Pi_{q_0}[Z(M(\psi))] \\
&\quad = \sum_{q=q_0+1}^{\infty} (Z(M(\psi)) - \Pi_q[Z(M(\psi))]) B_q[Z(M(\psi))] \\
&\qquad + \sum_{q=q_0+1}^{\infty} (\Pi_q[Z(M(\psi))] - \Pi_{q-1}[Z(M(\psi))]) A_{q-1}[Z(M(\psi))].
\end{aligned}
$$

We observe that either all of the $B_q[Z(M(\psi))]$ are zero, in which case the $A_{q-1}[Z(M(\psi))]$ are 1, or alternatively, $B_{q_1}[Z(M(\psi))] = 1$ for some $q_1 > q_0$ (and zero for all other q), in which case $A_q[Z(M(\psi))] = 1$ for $q < q_1$ and $A_q[Z(M(\psi))] = 0$ for $q \geq q_1$. Our construction of partitions and choice of q_0 also ensure that

$$
a(\sigma) = \frac{J(\sigma)}{\sqrt{\log[1 + N(\sigma, \widetilde{\Psi}, \rho)]}} \leq \frac{J(2^{-q_0})}{\sqrt{\log[1 + N(2^{-q_0-1}, \widetilde{\Psi}, \rho)]}} \leq a_{q_0},
$$

whence $A_{q_0}[Z(M(\psi))] = 1$. Next we apply the empirical process $\mathbb{G}_n$ to both series on the right-hand side of separately, take absolute values, and next take suprema over $\psi \in \widetilde{\Psi}$. Because the partitions are nested, $\Delta_q[Z(M(\psi))] B_q[Z(M(\psi))] \leq \Delta_{q-1}[Z(M(\psi))] B_q[Z(M(\psi))] \leq \sqrt{n} a_{q-1}$. The last inequality holds if $B_q[Z(M(\psi))] = 0$ and also if $B_q[Z(M(\psi))] = 1$ by definition. Furthermore, as $B_q[Z(M(\psi))]$ is indicator of the event $\Delta_q[Z(M(\psi))] > \sqrt{n} a_q$, it follows

$$
\begin{aligned}
\sqrt{n} a_q \cdot \mathrm{E}[\Delta_q[Z(M(\psi))] B_q[Z(M(\psi))]] &\leq \mathrm{E}[(\Delta_q[Z(M(\psi))])^2 B_q[Z(M(\psi))]] \\
&\leq J^2(2^{-q})
\end{aligned}
$$

by the choice of $\Delta_q[Z(M(\psi))]$. Because $|\mathbb{G}_n[Z(M(\psi))]| \leq \mathbb{G}_n[Z'] + 2\sqrt{n} \cdot \mathrm{E}[Z']$ if $|Z(M(\psi))| \leq Z'$, we obtain by the triangle inequality and Lemma A.1 that the quantity

$$
\mathrm{E}\left\| \sum_{q=q_0+1}^{\infty} \mathbb{G}_n (Z(M(\psi)) - \Pi_q[Z(M(\psi))]) B_q[Z(M(\psi))] \right\|_{\widetilde{\Psi}}
$$

is bounded by

$$
\begin{aligned}
&\sum_{q=q_0+1} \mathrm{E}\| \mathbb{G}_n \Delta_q[Z(M(\psi))] B_q[Z(M(\psi))] \|_{\widetilde{\Psi}} \\
&\quad + 2\sqrt{n} \sum_{q=q_0+1}^{\infty} \| \mathrm{E}\{\Delta_q[Z(M(\psi))] B_q[Z(M(\psi))]\} \|_{\widetilde{\Psi}} \\
&\quad \lesssim \sum_{q=q_0+1}^{\infty} \left[a_{q-1} \log[1 + N_q] + C J(2^{-q}) \sqrt{\log[1 + N_q]} + \frac{J^2(2^{-q})}{a_q} \right].
\end{aligned}
$$

In view of the definition of a_q, the series on the right can be bounded by a multiple of the series $\sum_{q=q_0+1}^{\infty} J(2^{-q})\sqrt{\log[1+N_q]}$. To establish a similar bound for the second part of equation (6.2), note that there are at most N_q differences $\Pi_q[Z(M(\psi))] - \Pi_{q-1}[Z(M(\psi))]$ and at most N_{q-1} indicator functions $A_{q-1}[Z(M(\psi))]$. Because the partitions are nested, $(\Pi_q[Z(M(\psi))] - \Pi_{q-1}[Z(M(\psi))])A_{q-1}[Z(M(\psi))]$ is bounded by $\Delta_{q-1}[Z(M(\psi))]A_{q-1} \times [Z(M(\psi))] \le \sqrt{n}a_{q-1}$. Moreover, $\mathrm{E}[\Pi_q[Z(M(\psi))] - \Pi_{q-1}[Z(M(\psi))]]^2 \le CJ^2(2^{-q})$. Hence

$$\left\| \sum_{q_0+1}^{\infty} \mathbb{G}_n\big(\Pi_q\big[Z\big(M(\psi)\big)\big] - \Pi_{q-1}\big[Z\big(M(\psi)\big)\big]\big)A_{q-1}\big[Z\big(M(\psi)\big)\big] \right\|_{\widetilde{\Psi}}$$

$$\le \sum_{q=q_0+1}^{\infty} \big[a_{q-1}\log(1+N_q) + CJ\big(2^{-q}\big)\sqrt{\log[1+N_q]}\big].$$

Again this is bounded above by a multiple of the series $\sum_{q=q_0+1}^{\infty} J(2^{-q}) \times \sqrt{\log[1+N_q]}$. To conclude the proof it suffices to consider the terms $\Pi_{q_0}[Z(M(\psi))]$. Because $|\Pi_{q_0}[Z(M(\psi))]| \le K_T \le a(\delta)\sqrt{n} \le a_{q_0}\sqrt{n}$ and

$$\mathrm{E}\big(\Pi_{q_0}\big[Z\big(M(\psi)\big)\big]\big)^2 \le \mathrm{E}\Big[\sup_{t\in[0,T]}\big(M_t(\psi_{q_0 i}) - M_t\big(\psi^*\big)\big)\Big]^2 \le 4\Theta^2\sigma^2$$

by the Burkholder–Davis–Gundy inequality, we have

$$\mathrm{E}\|\mathbb{G}_n\Pi_{q_0}\big[Z\big(M(\psi)\big)\big]\|_{\widetilde{\Psi}} \lesssim a_{q_0}\log[1+N_{q_0}] + \sigma\sqrt{\log[1+N_{q_0}]}.$$

By the choice of q_0, this is bounded by a multiple of the first few items of the series

$$\sum_{q=q_0+1}^{\infty} J\big(2^{-q}\big)\sqrt{\log[1+N_q]}.$$

□

6.2. *Proof of Proposition* 6.1. The proof can be routinely carried out along with lines of Bercu, Gassiat and Rio (2002).

6.3. *Proof of Proposition* 6.3. In order to prove Proposition 6.3 we need the following lemma.

LEMMA 6.5. *Denote*

$$\mathcal{Q}(\psi,\psi') = \rho\big(\psi,\psi'\big)\sqrt{\log\log\big(\rho^2\big(\psi,\psi'\big)\vee e^2\big)}$$

for any $\psi,\psi' \in \Psi$. *There is* $\varepsilon > 0$ *such that for any* $\psi,\psi' \in \Psi$ *and* $\psi^* \in \Psi^*$, *it holds*

$$\mathrm{E}\left\{\exp\left(\theta\left[\frac{\sqrt{n}\cdot(\Delta_n(\psi) - \Delta_n(\psi'))}{\mathcal{Q}(\psi,\psi^*)\cdot\mathcal{Q}(\psi,\psi')}\right]\right) - 1\right\} \le C\theta^2$$

for some constant $C > 0$, provided $|\theta| \le \varepsilon$.

PROOF. Without loss of generality, we may, and do, assume that $\Theta = 1$. Fix a martingale $M^* = M(\psi^*) \in \mathcal{M}^*$. Since $Z(M^*) = \mathrm{E}[Z(M^*)]$ almost surely, we have for arbitrary $M = M(\psi)$, $M' = M(\psi') \in \mathcal{M}$

$$\begin{aligned}
&V_n(M) - V_n(M') \\
&\quad = \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \bigl(\widetilde{Z}^{(i)}(M) - \widetilde{Z}^{(j)}(M)\bigr)^2 \\
&\qquad - \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \bigl(\widetilde{Z}^{(i)}(M') - \widetilde{Z}^{(j)}(M')\bigr)^2 \\
&\quad = \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \bigl(\widetilde{Z}^{(i)}(M) - \widetilde{Z}^{(i)}(M') - \widetilde{Z}^{(j)}(M) + \widetilde{Z}^{(j)}(M')\bigr) \\
&\qquad \times \bigl(\widetilde{Z}^{(i)}(M) - \widetilde{Z}^{(j)}(M) + \widetilde{Z}^{(i)}(M') - \widetilde{Z}^{(j)}(M')\bigr) \\
&\quad = \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \bigl(Z^{(i)}(M) - Z^{(i)}(M') - Z^{(j)}(M) + Z^{(j)}(M')\bigr) \\
&\qquad\qquad\qquad \times \bigl(\widetilde{Z}^{(i)}(M) - \widetilde{Z}^{(j)}(M) + \widetilde{Z}^{(i)}(M') - \widetilde{Z}^{(j)}(M')\bigr)
\end{aligned}$$

with $\widetilde{Z}^{(i)} = Z^{(i)}(M) - Z^{(i)}(M^*)$, $i = 1, \dots, n$. Set

$$\xi_i = Z^{(i)}(M) - Z^{(i)}(M'), \qquad \zeta_i = \widetilde{Z}^{(i)}(M) + \widetilde{Z}^{(i)}(M'), \qquad i = 1, \dots, n,$$

then

$$\begin{aligned}
V_n(M) - V_n(M') &= \frac{1}{n(n-1)} \sum_{1\le i<j\le n} (\xi_i - \xi_j)(\zeta_i - \zeta_j) \\
&= \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \xi_i \zeta_i - \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \xi_i \zeta_j \\
&\quad - \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \xi_j \zeta_i + \frac{1}{n(n-1)} \sum_{1\le i<j\le n} \xi_j \zeta_j \\
&= \frac{2}{n} \sum_{i=1}^{n} \xi_i \zeta_i - \frac{1}{n(n-1)} \sum_{i\ne j} \xi_i \zeta_j.
\end{aligned}$$

Hence

$$\begin{aligned}
&V_n(M) - V(M) - \bigl(V_n(M') - V(M')\bigr) \\
&\quad = \frac{2}{n} \sum_{i=1}^{n} \bigl(\xi_i \zeta_i - \mathrm{E}[\xi_i \zeta_i]\bigr) - \frac{1}{n(n-1)} \sum_{i\ne j} \bigl(\xi_i \zeta_j - \mathrm{E}[\xi_i \zeta_j]\bigr).
\end{aligned} \tag{6.2}$$

Note that $(\xi_1, \zeta_1), \dots, (\xi_n, \zeta_n)$ is a family of i.i.d. random two-dimensional vectors such that

$$|\xi_i| \le \sup_{t\in[0,T]} \bigl|M_t^{(i)} - M_t^{\prime(i)}\bigr|, \qquad i = 1, \dots, n$$

and

$$|\zeta_i| \le 2 \sup_{t\in[0,T]} \bigl|M_t^{(i)} - M_t^{*(i)}\bigr|, \qquad i = 1, \dots, n.$$

Lemma A.3 implies that for any $x > 0$,

$$\mathrm{P}\biggl(\frac{|\xi_i|}{\sqrt{\langle M^{(i)} - M^{\prime(i)}\rangle_T \log\log(\langle M^{(i)} - M^{\prime(i)}\rangle_T \vee e^2)}} \ge x\biggr) \le C(\alpha)e^{-\alpha x^2}$$

and

$$\mathrm{P}\biggl(\frac{|\zeta_i|}{\sqrt{\langle M^{(i)} - M^{*(i)}\rangle_T \log\log(\langle M^{(i)} - M^{*(i)}\rangle_T \vee e^2)}} \ge x\biggr) \le C(\alpha)e^{-\alpha x^2/4}.$$

As a result,

$$\mathrm{P}\biggl(\frac{|\xi_i|}{\rho(\psi, \psi')\sqrt{\log\log(\rho^2(\psi, \psi') \vee e^2)}} \ge x\biggr) \le C(\alpha)e^{-\alpha x^2}$$

and

$$\mathrm{P}\biggl(\frac{|\zeta_i|}{\rho(\psi, \psi^*)\sqrt{\log\log(\rho^2(\psi, \psi^*) \vee e^2)}} \ge x\biggr) \le C(\alpha)e^{-\alpha x^2/4}$$

for $i = 1, \dots, n$. Using representation (6.2), we get

$$\begin{aligned}
&\frac{\sqrt{n}(\Delta_n(\psi) - \Delta_n(\psi'))}{\mathcal{R}(\psi, \psi^*)\cdot\mathcal{R}(\psi, \psi')} \\
&\qquad = \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\bigl(\widetilde{\xi}_i\widetilde{\zeta}_i - \mathrm{E}[\widetilde{\xi}_i\widetilde{\zeta}_i]\bigr) - \frac{1}{\sqrt{n}(n-1)}\sum_{i<j}\bigl(\widetilde{\xi}_i\widetilde{\zeta}_j - \mathrm{E}[\widetilde{\xi}_i\widetilde{\zeta}_j]\bigr) \\
&\qquad\qquad - \frac{1}{\sqrt{n}(n-1)}\sum_{j<i}\bigl(\widetilde{\xi}_i\widetilde{\zeta}_j - \mathrm{E}[\widetilde{\xi}_i\widetilde{\zeta}_j]\bigr) \\
&\quad = T_{1,n} + T_{2,n} + T_{3,n},
\end{aligned}$$

where the "normalized" random variables

$$\widetilde{\xi}_i = \frac{\xi_i}{\rho(\psi, \psi')\sqrt{\log\log(\rho^2(\psi, \psi') \vee e^2)}},$$

$$\widetilde{\zeta}_i = \frac{\zeta_i}{\rho(\psi, \psi^*)\sqrt{\log\log(\rho^2(\psi, \psi^*) \vee e^2)}}$$

satisfy

$$\mathrm{P}\big(|\widetilde{\zeta}_i| \vee |\widetilde{\xi}_i| \ge x\big) \le C(\alpha)e^{-\alpha x^2/4}, \qquad i = 1, \dots, n. \tag{6.3}$$

The inequalities in (6.3) immediately imply

$$\begin{aligned}\mathrm{P}\big(|\widetilde{\xi}_i \widetilde{\zeta}_i| > x\big) &\le \mathrm{P}\big(|\widetilde{\xi}_i|^2 + |\widetilde{\zeta}_i|^2 > 2x\big) \le \mathrm{P}\big(|\widetilde{\xi}_i| > \sqrt{2x}\big) + \mathrm{P}\big(|\widetilde{\zeta}_i| > \sqrt{2x}\big) \\ &\le 2C(\alpha)\exp(-\alpha x/2).\end{aligned}$$

Consider first the term $T_{1,n}$. For any $\theta \in \mathbb{R}$ we have

$$\mathrm{E}\big[\exp(\theta T_{1,n})\big] = \prod_{i=1}^{n} \mathrm{E}\big[\exp\big(\theta\big(\widetilde{\xi}_i \widetilde{\zeta}_i - \mathrm{E}[\widetilde{\xi}_i \widetilde{\zeta}_i]\big)/\sqrt{n}\big)\big]. \tag{6.4}$$

Since the random variables $\widetilde{\xi}_i \widetilde{\zeta}_i - \mathrm{E}[\widetilde{\xi}_i \widetilde{\zeta}_i]$, $i = 1, \dots, n$, possess finite moments of any order and have zero mean, it holds

$$\log \mathrm{E}\big[\exp\big(\varepsilon\big(\widetilde{\xi}_i \widetilde{\zeta}_i - \mathrm{E}[\widetilde{\xi}_i \widetilde{\zeta}_i]\big)\big)\big] = \tfrac{1}{2}\sigma^2\varepsilon^2 + o\big(\varepsilon^2\big), \qquad i = 1, \dots, n$$

as $\varepsilon \to 0$, where $\sigma^2 = \mathrm{E}(\widetilde{\xi}_i \widetilde{\zeta}_i - \mathrm{E}[\widetilde{\xi}_i \widetilde{\zeta}_i])^2$. Hence the inequality

$$\mathrm{E}\big[\exp\big(\varepsilon\big(\widetilde{\xi}_i \widetilde{\zeta}_i - \mathrm{E}[\widetilde{\xi}_i \widetilde{\zeta}_i]\big)\big)\big] \le e^{C_1\varepsilon^2} \tag{6.5}$$

holds for sufficiently small ε and any $C_1 > \sigma^2/2$. Combining (6.4) with (6.5), we get for all $n \in \mathbb{N}$ and sufficiently small $\theta > 0$,

$$\mathrm{E}\big[\exp(\theta T_{1,n}) - 1\big] \le e^{C_1\theta^2} - 1 \le C_2\theta^2.$$

Turn now to the terms $T_{2,n}$ and $T_{3,n}$. We need the following proposition to estimate $T_{2,n}$ and $T_{3,n}$.

PROPOSITION 6.6. *Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a sequence of i.i.d. centered random vectors in $\mathbb{R}^2$ such that $\mathrm{E}|X_i|^p < \infty$ and $\mathrm{E}|Y_i|^p < \infty$ for all $i = 1, \dots, n$, and some $p \ge 2$. Then*

$$\begin{aligned}&\mathrm{E}\Big|\sum_{1\le i<j\le n} X_i Y_j\Big|^p \\ &\le C^p \max\Bigg\{\sum_{1\le i<j\le n} \mathrm{E}|X_i|^p \mathrm{E}|Y_j|^p, \sum_{i=1}^{n-1} \mathrm{E}|X_i|^p\Bigg(\sum_{j=i+1}^{n} \mathrm{E}|Y_j|^2\Bigg)^{p/2}, \\ &\qquad \sum_{j=2}^{n} \mathrm{E}|Y_j|^p\Bigg(\sum_{i=1}^{j-1} \mathrm{E}|X_i|^2\Bigg)^{p/2}, \Bigg(\sum_{1\le i<j\le n} \mathrm{E}|X_i|^2 \mathrm{E}|Y_j|^2\Bigg)^{p/2}\Bigg\}\end{aligned} \tag{6.6}$$

for some constant $C > 0$ not depending on p.

PROOF. Denote $Q_n = \sum_{1\le i<j\le n} X_i Y_j$ and

$$V_j = \sum_{i=1}^{j-1} X_i Y_j, \qquad j = 2,\dots,n.$$

It is clear that $T_{2,n} = \sum_{j=2}^{n} V_j$ and $(V_j, j = 2,\dots,n)$ is a forward martingale-difference sequence (see the Appendix for definition) with respect to σ-algebras $\mathcal{F}_j = \sigma((X_1,Y_1),\dots,(X_j,Y_j))$, $j = 2,\dots,n$. By the martingale Rosenthal inequality (see Proposition A.2 in the Appendix),

$$\mathrm{E}\bigl[|Q_n|^p\bigr] \le B(p/\log p)\max\left\{\sum_{j=2}^{n}\mathrm{E}|V_j|^p, \mathrm{E}\left(\sum_{j=2}^{n}\mathrm{E}\bigl[V_j^2|\mathcal{F}_{j-1}\bigr]\right)^{p/2}\right\}$$

and

$$\mathrm{E}|V_j|^p \le B(p/\log p)\cdot \mathrm{E}|Y_j|^p \max\left\{\sum_{i=1}^{j-1}\mathrm{E}|X_i|^p, \left(\sum_{i=1}^{j-1}\mathrm{E}|X_i|^2\right)^{p/2}\right\} \tag{6.7}$$

for all $j = 2,\dots,n$. Then

$$\begin{aligned}
&\mathrm{E}\left(\sum_{j=2}^{n}\mathrm{E}\bigl[V_j^2|\mathcal{F}_{j-1}\bigr]\right)^{p/2}\\
&\quad = \mathrm{E}\left(\sum_{1\le i<j\le n}|X_i|^2\mathrm{E}|Y_j|^2 + 2\sum_{j=3}^{n}\sum_{1\le k<l\le j-1} X_k X_l \mathrm{E}|Y_j|^2\right)^{p/2}\\
&\quad \le 2^{p/2-1}\mathrm{E}\left(\sum_{1\le i<j\le n}|X_i|^2\mathrm{E}|Y_j|^2\right)^{p/2}\\
&\qquad + 2^{p-1}\mathrm{E}\left|\sum_{1\le k<l\le n-1} X_k X_l \sum_{j=l+1}^{n}\mathrm{E}|Y_j|^2\right|^{p/2}.
\end{aligned}$$

By the Rosenthal inequality,

$$\begin{aligned}
&\mathrm{E}\left(\sum_{1\le i<j\le n}|X_i|^2\mathrm{E}|Y_j|^2\right)^{p/2}\\
&\quad = \mathrm{E}\left(\sum_{i=1}^{n-1}|X_i|^2\sum_{j=i+1}^{n}\mathrm{E}|Y_j|^2\right)^{p/2}\\
&\quad \le B(p/2)\log^{-1}(p/2)\max\left\{\sum_{i=1}^{n-1}\mathrm{E}|X_i|^p\left[\sum_{j=i+1}^{n}\mathrm{E}|Y_j|^2\right]^{p/2},\right.\\
&\qquad\qquad \left.\left(\sum_{1\le i<j\le n}\mathrm{E}|X_i|^2\mathrm{E}|Y_j|^2\right)^{p/2}\right\}.
\end{aligned} \tag{6.8}$$

Using the Jensen inequality, we get for $2 \le p < 4$

$$\mathrm{E}\left|\sum_{1\le k<l\le n-1} X_k X_l \sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right|^{p/2} \le \left(\sum_{1\le k<l\le n-1} \mathrm{E}|X_k X_l|^2 \left(\sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right)^2\right)^{p/4}.$$

Moreover,

$$\begin{aligned}&\left(\sum_{1\le k<l\le n-1} \mathrm{E}|X_k X_l|^2 \left(\sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right)^2\right)^{p/4}\\ &\qquad \le \left(\sum_{1\le i<j\le n} \mathrm{E}|X_i|^2 \mathrm{E}|Y_j|^2\right)^{p/2}.\end{aligned} \tag{6.9}$$

Combining (6.7), (6.8) and (6.9), we arrive at the inequality (6.6). Thus Lemma 6.6 is proved for all $2 \le p < 4$. Suppose now that the inequality (6.6) holds for $p \le m-1$ with some $m > 4$. Let us prove it for $p = m$. It follows from the previous steps, that we only need to obtain an upper bound for the term

$$\mathrm{E}\left|\sum_{1\le k<l\le n-1} X_k X_l \sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right|^{m/2}.$$

Our induction hypothesis gives that the quantity

$$\mathrm{E}\left|\sum_{1\le k<l\le n-1} X_k X_l \sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right|^{m/2}$$

is bounded by

$$\begin{aligned} C^{m/2} \max\Bigg\{ &\sum_{1\le k<l\le n-1} \mathrm{E}\left|X_k X_l \sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right|^{m/2},\\ &\sum_{k=1}^{n-2} \mathrm{E}|X_k|^{m/2} \left(\sum_{l=k+1}^{n-1} |X_l|^2 \sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right)^{m/4},\\ &\sum_{l=2}^{n-1} |X_l|^{m/2} \left(\sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right)^{m/2} \left(\sum_{k=1}^{l-1} \mathrm{E}|X_k|^2\right)^{m/4},\\ &\left(\sum_{1\le k<l\le n-1} \mathrm{E}\big[|X_k|^2 |X_l|^2\big] \left(\sum_{j=l+1}^{n} \mathrm{E}|Y_j|^2\right)^2\right)^{m/4}\Bigg\}. \end{aligned} \tag{6.10}$$

Let us consider, for example, the first term in the above maximum. Using the inequality

$$\left(\mathrm{E}\sum_{k=1}^{n}|U_k|^p\right)^2 \le \max\left\{\sum_{k=1}^{n}\mathrm{E}|U_k|^{2p}, \left(\sum_{k=1}^{n}\mathrm{E}|U_k|\right)^{2p}\right\} \tag{6.11}$$

that holds for any $p > 1$ and any sequence of independent r.v. $U_1, \dots, U_n$ with $\mathrm{E}|U_k|^{2p} < \infty$, we get

$$\begin{aligned}\sum_{1\le k<l\le n-1}\mathrm{E}\left|X_k X_l \sum_{j=l+1}^{n}\mathrm{E}|Y_j|^2\right|^{m/2} &= \sum_{1\le k<l\le n-1}\mathrm{E}|X_k X_l|^{m/2}\left(\sum_{j=l+1}^{n}\mathrm{E}|Y_j|^2\right)^{m/2}\\ &\le \left[\sum_{i=1}^{n-1}\mathrm{E}|X_i|^{m/2}\left(\sum_{j=i+1}^{n}\mathrm{E}|Y_j|^2\right)^{m/4}\right]^2\end{aligned}$$

and

$$\begin{aligned}&\left[\sum_{i=1}^{n-1}\mathrm{E}|X_i|^{m/2}\left(\sum_{j=i+1}^{n}\mathrm{E}|Y_j|^2\right)^{m/4}\right]^2\\ &\qquad\le \max\left\{\sum_{i=1}^{n-1}\mathrm{E}|X_i|^{m}\left(\sum_{j=i+1}^{n}\mathrm{E}|Y_j|^2\right)^{m/2}, \left(\sum_{1\le i<j\le n}\mathrm{E}|X_i|^2\mathrm{E}|Y_j|^2\right)^{m/2}\right\}.\end{aligned}$$

To see that inequality (6.11) holds, just note that the function

$$h(t) = \log\left[\sum_{k=1}^{n}\mathrm{E}|U_k|^t\right]$$

is convex in the domain $t > 1$. Due to convexity of $h(t)$, we have

$$\left(\sum_{k=1}^{n}\mathrm{E}|U_k|^p\right)^{2p-1} \le \left(\sum_{k=1}^{n}\mathrm{E}|U_k|^{2p}\right)^{p-1}\left(\sum_{k=1}^{n}\mathrm{E}|U_k|\right)^{p}$$

for any $p > 1$. Hence

$$\begin{aligned}\left(\sum_{k=1}^{n}\mathrm{E}U_k^p\right)^2 &\le \left(\sum_{k=1}^{n}\mathrm{E}|U_k|^{2p}\right)^{2(p-1)/(2p-1)}\left(\sum_{k=1}^{n}\mathrm{E}|U_k|\right)^{2p/(2p-1)}\\ &\le \max\left\{\sum_{k=1}^{n}\mathrm{E}|U_k|^{2p}, \left(\sum_{k=1}^{n}\mathrm{E}|U_k|\right)^{2p}\right\}.\end{aligned}$$

Other terms on the right-hand side of (6.10) can be handled in a similar way. $\square$

Let us proceed with estimating the term $T_{2,n}$. Without loss of generality we may assume that $\mathrm{E}[\widetilde{\xi}] = \mathrm{E}[\widetilde{\zeta}] = 0$. Note that for any natural $p > 0$,

$$\begin{aligned}
\mathrm{E}\big[|\widetilde{\xi}|^p\big] &\le 2pC(\alpha)\int_0^\infty x^{p-1}\exp(-\alpha x^2)\,dx \\
&= \frac{2pC(\alpha)}{(2\alpha)^{p/2}}\int_0^\infty y^{p-1}\exp(-y^2/2)\,dy \\
&\le \frac{p\sqrt{2\pi}C(\alpha)}{(2\alpha)^{p/2}}\mathrm{E}\big[|Z|^p\big],
\end{aligned}$$

where $Z \sim N(0,1)$. Similarly

$$\mathrm{E}\big[|\widetilde{\zeta}|^p\big] \le \frac{2^{p/2}p\sqrt{2\pi}C(\alpha)}{\alpha^{p/2}}\mathrm{E}\big[|Z|^p\big].$$

As a result, we get from Proposition 6.6

$$\begin{aligned}
\mathrm{E}\big[|T_{2,n}|^p\big] \le C^p \max\big\{&n^{1-p/2}(n-1)^{1-p}\mathrm{E}\big[|Z|^{2p}\big], \\
&n^{-p/2}(n-1)^{1-p/2}\mathrm{E}\big[|Z|^p\big], (n-1)^{-p/2}\big\}
\end{aligned}$$

for some constant $C > 0$ and any $p > 1$. Hence for any $\theta \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{E}\big[\exp(\theta T_{2,n}) - 1\big] &= \sum_{k=2}^\infty \frac{\theta^k}{k!}\mathrm{E}\big[T_{2,n}^k\big] \\
&\le \sum_{k=2}^\infty \frac{|\theta|^k}{k!}\frac{B_1^k}{(n-1)^{k/2}}\mathrm{E}\big[Z^{2k}\big] \\
&= \mathrm{E}\big[\exp\big(B_1|\theta|Z^2/\sqrt{n-1}\big)\big] - 1 - B_1|\theta|\mathrm{E}\big[Z^2\big]/\sqrt{n-1} \qquad (6.12) \\
&= \frac{1}{\sqrt{1-2B_1|\theta|/\sqrt{n-1}}} - 1 - B_1|\theta|/\sqrt{n-1} \\
&\le B_2\theta^2,
\end{aligned}$$

provided $B_1|\theta|/\sqrt{n-1} < 1/2$, where B_1 and B_2 are two constants not depending on k and n. Analogously to (6.12), one can prove that

$$\mathrm{E}\big[\exp(\theta T_{3,n}) - 1\big] \le B_3\theta^2$$

for sufficiently small $|\theta|$. Hence by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mathrm{E}\big[e^{\theta(T_{1,n}+T_{2,n}+T_{3,n})} - 1\big] &\le \big[\mathrm{E}e^{2\theta T_{1,n}}\big]^{1/2}\big[\mathrm{E}e^{4\theta T_{2,n}}\big]^{1/4}\big[\mathrm{E}e^{4\theta T_{3,n}}\big]^{1/4} - 1 \\
&\le B_4\theta^2
\end{aligned}$$

for some constant $B_4 > 0$. Lemma 6.5 is proved. $\square$

Let us proceed with the proof of Proposition 6.3. Let $\{\widetilde{\Psi}^m\}_{m\in\mathbb{N}}$ be a sequence of finite subsets of $\widetilde{\Psi}$ such that $\widetilde{\Psi}^m \uparrow \widetilde{\Psi}$ as $m\to\infty$. Introduce the disjoint sets

$$H_p = \{\psi \in \widetilde{\Psi} : 2^{-p-1} < \rho(\psi,\psi^*) \le 2^{-p}\}$$

for any $p\in\mathbb{Z}$. Without loss of generality we may assume that H_p are empty for $p<0$. For every $m\in\mathbb{N}$, denote by $q(m,p)$ the smallest integer such that $q(m,p)>p$ and that each of the closed balls with centers in $\widetilde{\Psi}^m\cap H_p$ and ρ-radius $2\cdot 2^{-q(m,p)}$ contains exactly one point in $\widetilde{\Psi}^m\cap H_p$. Then it is clear that $\operatorname{Card}(\widetilde{\Psi}^m\cap H_p)\le N(2^{-q(m,p)},\widetilde{\Psi}\cap H_p,\rho)$. Next let us introduce some mappings $\pi_r^{m,p}:\widetilde{\Psi}^m\cap H_p\to\widetilde{\Psi}_r^{m,p}$, $p\le r\le q(m,p)$, defined by

$$\pi_r^{m,p}=\lambda_r^{m,p}\circ\lambda_{r+1}^{m,p}\circ\cdots\circ\lambda_{q(m,p)}^{m,p},$$

where the sets $\widetilde{\Psi}_r^{m,p}\subset\widetilde{\Psi}^m\cap H_p$ and the mappings $\lambda_r^{m,p}:\widetilde{\Psi}^m\cap H_p\to\widetilde{\Psi}_r^{m,p}$ are specified in the following way. For $p\le r<q(m,p)$, choose $\widetilde{\Psi}_r^{m,p}$ and define $\lambda_r^{m,p}$ such that they satisfy the following two conditions: $\operatorname{Card}(\widetilde{\Psi}_r^{m,p})\le N(2^{-r},\widetilde{\Psi}\cap H_p,\rho)$ and $\rho(\psi,\lambda_r^{m,p}(\psi))\le 2\cdot 2^{-r}$ for every $\psi\in\widetilde{\Psi}^m\cap H_p$. For $r=q(m,p)$, put $\widetilde{\Psi}_{q(m,p)}^{m,p}=\widetilde{\Psi}^m\cap H_p$ and denote by $\lambda_{q(m,p)}^{m,p}$ the identical mapping on $\widetilde{\Psi}^m\cap H_p$. In terms of the mappings $\pi_r^{m,p}$ which have been introduced, we consider the chaining given as follows: for every $n\in\mathbb{N}$ and $\psi\in\widetilde{\Psi}\cap H_p$,

$$|\Delta_n(\psi)|\le\sum_{r=p+1}^{q(m,p)}|\Delta_n(\pi_r^{m,p}(\psi))-\Delta_n(\pi_{r-1}^{m,p}(\psi))|+|\Delta_n(\pi_p^{m,p}(\psi))|.$$

Since $\rho(\pi_r^{m,p}(\psi),\pi_{r-1}^{m,p}(\psi))/\rho(\psi,\psi^*)\le 2^{-r+p+1}$ and $\rho(\pi_r^{m,p}(\psi),\psi^*)/\rho(\psi,\psi^*)\le 2$ on $\widetilde{\Psi}^m\cap H_p$, it follows from Lemma 6.5 and Lemma 8.2 in Kosorok (2008) that

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(\theta\sup_{\psi\in\widetilde{\Psi}^m\cap H_p}\left\{\frac{\sqrt{n}|\Delta_n(\pi_r^{m,p}(\psi))-\Delta_n(\pi_{r-1}^{m,p}(\psi))|}{\mathcal{R}^2(\psi,\psi^*)}\right\}\right)-1\right]\\
&\quad\le\mathrm{E}\left[\exp\left(\theta\sup_{\psi\in\widetilde{\Psi}^m\cap H_p}\left\{\frac{\mathcal{Q}(\pi_{r-1}^{m,p}(\psi),\pi_r^{m,p}(\psi))\mathcal{Q}(\pi_r^{m,p}(\psi),\psi^*)}{\mathcal{R}^2(\psi,\psi^*)}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\frac{\sqrt{n}|\Delta_n(\pi_r^{m,p}(\psi))-\Delta_n(\pi_{r-1}^{m,p}(\psi))|}{\mathcal{Q}(\pi_{r-1}^{m,p}(\psi),\pi_r^{m,p}(\psi))\mathcal{Q}(\pi_r^{m,p}(\psi),\psi^*)}\right\}\right)-1\right]\\
&\quad\le Kp^{-2}4^{-r+p+1}\log(1+N(2^{-r},\widetilde{\Psi}\cap H_p,\rho))
\end{aligned}$$

for all $|\theta|\le\varepsilon$, some $\delta>0$ and some constant $K>0$. Moreover note that $N(2^{-r},\widetilde{\Psi}\cap H_p,\rho)\le N(2^{-r+p+1},\widetilde{\Psi},\rho)$. Next

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(\theta\sup_{\psi\in\widetilde{\Psi}^m\cap H_p}\left\{\frac{|\sqrt{n}\cdot\Delta_n(\pi_p^{m,p}(\psi))|}{\mathcal{R}^2(\psi,\psi^*)}\right\}\right)-1\right]\\
&\quad\lesssim p^{-2}\log(1+N(2^{1+p},\widetilde{\Psi},\rho)).
\end{aligned}$$

Finally, we get for any $P > 0$,

$$
\begin{aligned}
&\mathrm{E}\left[\exp\left(\theta \sup_{\psi\in\widetilde{\Psi}^m\cap(H_1\cup\cdots\cup H_P)} \frac{|\sqrt{n}\cdot\Delta_n(\psi)|}{\mathcal{R}^2(\psi,\psi^*)}\right)-1\right]\\
&\quad\lesssim \sum_{p=1}^{P} p^{-2}\sum_{r=p+1}^{q(m,p)} 4^{-r+p+1}\log\bigl(1+N\bigl(2^{-r+p+1},\widetilde{\Psi},\rho\bigr)\bigr)\\
&\quad\lesssim \sum_{p=1}^{P} p^{-2}\int_0^1 \log\bigl(1+N(\sqrt{\varepsilon},\widetilde{\Psi},\rho)\bigr)\,d\varepsilon\\
&\quad\lesssim \int_0^1 \sqrt{\log\bigl(1+N(\varepsilon,\widetilde{\Psi},\rho)\bigr)}\,d\varepsilon.
\end{aligned}
$$

The proof of Proposition 6.3 is accomplished by letting $m\to\infty$ and $P\to\infty$.

APPENDIX

The following lemma is a straightforward generalization of Lemma 19.33 in van der Vaart (1998).

LEMMA A.1. *Let $\mathcal{X}$ be a finite collection of bounded real valued random variables defined on a common probability space* $(\Omega,\mathcal{F},\mathrm{P})$, *then*

$$
\mathrm{E}\|\mathbb{G}_n[X]\|_{\mathcal{X}} \lesssim \frac{\sup_{X\in\mathcal{X}}|X|}{\sqrt{n}}\log(1+|\mathcal{X}|)+\max_{X\in\mathcal{X}}\sqrt{\mathrm{E}[|X|^2]}\sqrt{\log(1+|\mathcal{X}|)},
$$

where $\mathbb{G}_n[X]=\frac{1}{n}\sum_{j=1}^n(X^{(j)}-\mathrm{E}[X])$ *and* $X^{(1)},\dots,X^{(n)}$ *are i.i.d. copies of* X.

Given a sequence of σ-algebras $(\mathcal{F}_n), n\ge 1$ on some probability space $(\Omega,\mathcal{F},\mathrm{P})$, we call a sequence of integrable r.v. Y_n to be a forward martingale-difference sequence w.r.t. $(\mathcal{F}_n)$ if:

- $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\cdots$;
- Y_n is $\mathcal{F}_n$-measurable;
- $\mathrm{E}[Y_n|\mathcal{F}_{n-1}]=0$ a.s. for any $n\ge 1$.

The following proposition can be found in Hitczenko (1990).

PROPOSITION A.2. *Let (X_k) be a forward martingale-difference sequence relative to $\mathcal{F}_k$ such that $\mathrm{E}|X_k|^p<\infty$ for some $p\ge 2$ and $k=1,\dots,n$; then*

$$
\mathrm{E}\left|\sum_{k=1}^n X_k\right|^p \le B\bigl(k\log^{-1}k\bigr)\max\left\{\sum_{k=1}^n \mathrm{E}|X_k|^p, \mathrm{E}\left[\sum_{k=1}^n \mathrm{E}\bigl[X_k^2|\mathcal{F}_{k-1}\bigr]\right]^{p/2}\right\}
$$

for some constant B not depending on k.

The next inequality can be found in de la Peña, Klass and Lai (2004).

LEMMA A.3. *For any continuous local martingale* $(M_t)_{t\in[0,T]}$ *with* $M_0 = 0$

$$\mathrm{P}\left(\frac{\sup_{0\le t\le T}|M_t|}{\sqrt{\langle M\rangle_T \log\log(\langle M\rangle_T \vee e^2)}} \ge x\right) \le C(\alpha)e^{-\alpha x^2},$$

where α *is a real number in* $(0, 1/2)$ *and* $C(\alpha)$ *is a positive constant.*

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DUISBURG-ESSEN UNIVERSITY
ALTENDORFERSTRASSE 11
45127 ESSEN
GERMANY
E-MAIL: denis.belomestny@uni-due.de