Electronic Journal of Statistics
Vol. 5 (2011) 1765–1796
ISSN: 1935-7524
DOI: 10.1214/11-EJS655

Testing the structural stability of temporally dependent functional observations and application to climate projections*

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Abstract: We develop a self-normalization (SN) based test to test the structural stability of temporally dependent functional observations. Testing for a change point in the mean of functional data has been studied in Berkes, Gabrys, Horváth and Kokoszka [4], but their test was developed under the independence assumption. In many applications, functional observations are expected to be dependent, especially when the data is collected over time. Building on the SN-based change point test proposed in Shao and Zhang [23] for a univariate time series, we extend the SN-based test to the functional setup by testing the constant mean of the finite dimensional eigenvectors after performing functional principal component analysis. Asymptotic theories are derived under both the null and local alternatives. Through theory and extensive simulations, our SN-based test statistic proposed in the functional setting is shown to inherit some useful properties in the univariate setup: the test is asymptotically distribution free and its power is monotonic. Furthermore, we extend the SN-based test to identify potential change points in the dependence structure of functional observations. The method is then applied to central England temperature series to detect the warming trend and to gridded temperature fields generated by global climate models to test for changes in spatial bias structure over time.

AMS 2000 subject classifications: Primary 62G10; secondary 62G20.

*Supported in part by NSF Grant DMS-0724752. We are grateful to an associate editor and a referee for helpful comments. We thank Robertas Gabrys and Piotr Kokoszka for providing the central England temperature data used in this paper.

Keywords and phrases: Change point, CUSUM, functional data, time series, self-normalization.

Received March 2011.

Contents

1. Introduction

Major advances in technology are enabling data collection at increasingly high resolution. These advancements challenge state-of-the-art models and methods in statistics. It has long been recognized that functional data analysis (FDA), which deals with the analysis of curves and surfaces, is an effective tool for analyzing large high resolution data sets. Systematic methods and theory have been developed for FDA mainly under the independence assumption (Ramsay and Silverman [19, 20], Ferraty and Vieu [11]), with relatively little attention paid to the analysis of dependent functional data. However, for functional data observed over time, the independence assumption is rarely satisfied in practice. This paper aims to develop new tests to assess the structural stability of temporally dependent functional data. Our work is partially motivated by our ongoing research on the development of high-resolution climate projections through statistical downscaling, which by definition assumes a temporally stable relationship between observations and climate models. Climate change is one of the most urgent problems facing the world this century. To study climate change, scientists have relied primarily on climate projections from global/regional climate models, which are deterministic numerical models that involve systems of differential equations and produce outputs at a prespecified grid. As numerical model outputs are widely used in situations where real observations are not available, it is an important but still open question whether the properties of numerical model outputs remain stable relative to real observations over time.

To assess the structural stability, we view functional observations as a realization from a functional time series process, and test for a change point in the mean and autocovariance of the functional time series. The detection of one or multiple change points in the first or second order structure of a functional time series is itself an important problem, as failure to account for such change points could lead to invalid inference. There is a large literature and long history on change point testing in scalar or vector time series (see Csörgő and Horváth [9], Perron [18] and references therein), but research on change point testing for functional data is very recent. Berkes et al. [4] proposed a CUSUM-based (cumulative sum) approach to test the assumption of a common functional mean for independent functional data. Berkes et al.'s test (BGHK, hereafter) is invalid for functional time series since it does not take the temporal dependence into account. Hörmann and Kokoszka [13] recognized the limitation of the BGHK test and modified their test by introducing a consistent long run variance (LRV) estimator. Hörmann and Kokoszka's work has been extended recently by Aston and Kirch [3] for weak dependent functional data with a wide class of dependence structure and two types of alternatives, namely, at most one change point and epidemic changes. However, there is a bandwidth parameter involved in both Hörmann and Kokoszka's test (HK, hereafter) and Aston and Kirch's test. Its selection is not addressed and the finite sample performance of their test has not been examined. To avoid choosing the bandwidth parameter, Shao and Zhang [23] proposed a self-normalization (SN, hereafter) based test in the univariate time series setup, where an inconsistent normalization matrix is introduced to accommodate the dependence. The idea of using inconsistent normalization is not new, as it has been previously applied by Lobato [16] and Shao [22] to the inference in univariate time series. In this article, we extend the SN-based test in the univariate setup to test the structural stability of temporally dependent functional data. To our knowledge, this is the first attempt to generalize the SN idea to inference problems for functional data.

The extension of the SN concept to the functional setup is nontrivial since functional observations are collected on a space of infinite dimension and traditional methods developed for univariate/multivariate time series are not applicable in this case. To circumvent this difficulty, our method relies on the functional principal component analysis (PCA) which projects the functional data onto a space spanned by the first few principal components (PC's). The SN-based test statistic is constructed based on the principal component scores. To accommodate the dependence, we introduce a normalization matrix which is built by taking the single change point alternative into account. The normalization matrix is inconsistent but proportional to the unknown LRV matrix, which is canceled out in the limiting null distribution of the SN-based test. The proposed test is thus asymptotically pivotal with critical values tabulated in Shao and Zhang [23]. Compared to the methods in previous studies, the SN-based test is asymptotically distribution-free and is shown to enjoy the monotonic power property in the functional setup. In addition, the SN-based test can be easily extended to detect multiple change points in the mean function and to detect a change point in the lag one autocovariance operator, the latter of which is investigated in this paper.

To illustrate this method, the SN-based tests are used to examine the stationarity of biases in simulated spatio-temporal temperature data over a subregion of North America which consists of two sequences of functional surfaces (with spatial resolution 87×35) based on station observations and historical simulations from global climate models. Since spatially distributed temperature fields are usually viewed as smooth images by scientists, FDA is an appropriate tool for analyzing and revealing key characteristics of such a large dataset. Statistical analysis based on the SN-based test is shown to be helpful in addressing the scientific question of whether the bias between observations and model output remains stable over time.

The remainder of the paper is organized as follows: Section 2.1 describes the testing procedure for detecting changes in the mean; Section 2.2 presents an SN-based test for one change point in the autocovariance operator at lag one; we investigate the theoretical properties of SN-based tests in section 3; Section 4 and 5 are devoted to the finite sample performance of the SN-based tests and application to two climate datasets (curves and surfaces); and Section 6 summarizes our conclusions. Proofs of the theoretical results are presented in the supplemental materials.

2. Methodology

Mathematically, we consider functional observations $X_i(t)$, $t \in \mathcal{I}$, $i = 1, 2, \ldots, N$ defined on some compact set $\mathcal{I}$ of the Euclidian space, where $\mathcal{I}$ could be one dimensional (e.g. a curve) or multidimensional (e.g. a surface or manifold). For simplicity, we consider the Hilbert space $\mathbb{H}$ of square integrable functions defined on $\mathcal{I} = [0, 1]$ (and $\mathcal{I}^2 = [0, 1]^2$). For any $f, g \in \mathbb{H}$, the inner product between f and g is defined as $< f, g >= \int_{\mathcal{I}} f(t)g(t)dt$. We denote $||\cdot||$ as the corresponding norm, i.e., $||f|| =< f, f >^{1/2}$. Assuming the random elements all come from the same probability space $(\Omega, \mathcal{A}, \mathcal{P})$, we let L^p be the space of real valued random variables with finite L^p norm, i.e., $(\mathbf{E}|X|^p)^{1/p} < \infty$. We further define $L^p_{\mathbb{H}}$ as the space of $\mathbb{H}$-valued random variables X such that $e_p(X) := (\mathbf{E}||X||^p)^{1/p} < \infty$. We then let $D[0, 1]$ be the space of functions on $[0, 1]$ which are right-continuous and have left limits, endowed with the Skorokhod topology (see Billingsley [5]). Weak convergence in $D[0, 1]$ or more generally in the $\mathbb{R}^d$-valued function space $D^d[0, 1]$ is denoted by "$\Rightarrow$", where $d \in \mathbb{N}$. Finally "$\rightarrow^d$" denotes convergence in distribution.

2.1. Testing for a change point in the mean function

Given the functional observations $\{X_i(t)\}_{i=1}^N$, we are interested in testing whether the mean function remains constant over time, i.e.,

$$H_{0,1} : \mathbf{E}[X_1(t)] = \mathbf{E}[X_2(t)] = \cdots = \mathbf{E}[X_N(t)], \qquad t \in \mathcal{I}. \tag{2.1}$$

Under the null, we can write $X_i(t) = \mu_1(t) + Y_i(t)$ with $\mathbf{E}[Y_i(t)] = 0$, $i = 1, 2, \ldots, N$. Under the alternative $H_{a,1}$, we assume there is a change point in the mean function, i.e.,

$$X_i(t) = \begin{cases} \mu_1(t) + Y_i(t) & 1 \leq i \leq k^*; \\ \mu_2(t) + Y_i(t) & k^* < i \leq N, \end{cases} \tag{2.2}$$

where $k^* = \lfloor N\lambda \rfloor$ is an unknown change point for some $\lambda \in (0, 1)$, $\{Y_i(t)\}$ is a zero-mean functional sequence, and $\mu_1(t) \neq \mu_2(t)$ for some t. To describe our methodology, we first introduce some useful notation commonly adopted in the literature of functional data; see e.g. Berkes et al. [4]. For $X(\cdot) \in L^p_{\mathbb{H}}$ with $p \geq 2$, we define $c(t, s) = \text{cov}\{X(t), X(s)\}$, $t, s \in \mathcal{I}$ as the covariance function. By Mercer's Lemma (Riesz and Sz-Nagy [21]), $c(t, s)$ admits the spectral decomposition,

$$c(t, s) = \sum_{j=1}^{\infty} \lambda_j \phi_j(s) \phi_j(t), \tag{2.3}$$

where λ_j and ϕ_j are the eigenvalue and eigenfunction respectively. The eigenvalues are ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Based on the Karhunen-Loève expansion (Bosq [6]), we have $X_i(t) = \mathbf{E}[X_i(t)] + \sum_{j=1}^{\infty} \eta_{i,j} \phi_j(t)$, where $\{\eta_{i,j}\}$ are the principal components (scores) defined by $\eta_{i,j} = \int_{\mathcal{I}} \{X_i(t) - \mathbf{E}[X_i(t)]\} \phi_j(t) dt$. A natural estimator of the covariance function $c(t, s)$ is

$$\hat{c}(t, s) = \frac{1}{N} \sum_{i=1}^{N} \{X_i(t) - \bar{X}_N(t)\}\{X_i(s) - \bar{X}_N(s)\}, \tag{2.4}$$

where $\bar{X}_N(t) = \frac{1}{N} \sum_{i=1}^{N} X_i(t)$ is the sample mean function. The eigenfunctions and corresponding eigenvalues of $\hat{c}(t, s)$ are defined by

$$\int_{\mathcal{I}} \hat{c}(t, s) \hat{\phi}_j(s) ds = \hat{\lambda}_j \hat{\phi}_j(t). \tag{2.5}$$

Then the empirical scores are given by

$$\hat{\eta}_{i,j} = \int_{\mathcal{I}} \{X_i(t) - \bar{X}_N(t)\} \hat{\phi}_j(t) dt, \qquad i = 1, 2, \ldots N; \quad j = 1, 2, \ldots, K,$$

where K is the number of principal components we consider and is assumed to be fixed throughout. Under the null, the score vector $\eta_i = (\eta_{i,1}, \eta_{i,2}, \ldots, \eta_{i,K})'$, $i = 1, 2, \ldots, N$ has a constant mean, whereas the mean changes under the alternative. If we let $\hat{\eta}_{\mathbf{i}} = (\hat{\eta}_{i1}, \ldots, \hat{\eta}_{iK})'$ and $S_{N,\hat{\eta}}(t_1, t_2) = \sum_{i=t_1}^{t_2} \hat{\eta}_{\mathbf{i}}$, for $1 \leq t_1 \leq t_2 \leq N$, we can then define the so-called CUSUM process as

$$T_{N,\hat{\eta}}(k, K) := \frac{1}{\sqrt{N}} \left\{ S_{N,\hat{\eta}}(1, k) - \frac{k}{N} S_{N,\hat{\eta}}(1, N) \right\}, \quad k = 1, 2, \ldots, N. \tag{2.6}$$

To test the assumption of a common functional mean for independent and identically distributed (iid) functional data, Berkes et al. [4] introduced a CUSUM-based test statistic which takes the following form

$$H_{N,\hat{\eta}}(K) := \frac{1}{N^2}\sum_{j=1}^{K}\hat{\lambda}_j^{-1}\sum_{k=1}^{N}\left(\sum_{i=1}^{k}\hat{\eta}_{i,j} - \frac{k}{N}\sum_{i=1}^{N}\hat{\eta}_{i,j}\right)^2. \tag{2.7}$$

It can be rewritten as

$$H_{N,\hat{\eta}}(K) = \frac{1}{N}\sum_{k=1}^{N} T_{N,\hat{\eta}}(k,K)'\hat{\Sigma}_\eta^{-1}T_{N,\hat{\eta}}(k,K), \tag{2.8}$$

where $\hat{\Sigma}_\eta = \mathrm{diag}(\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_K)$. As pointed out by Hörmann and Kokoszka [13], the BGHK test is not applicable to functional time series because it does not take the temporal dependence of η_i's into account. In the dependent case, one usually needs to estimate the LRV matrix (i.e., the spectral density function evaluated at zero frequency) of η_i consistently. The commonly-used lag window type estimator can be used to obtain a consistent LRV estimator $\tilde{\Sigma}_\eta$ (see Hörmann and Kokoszka [13]). A test statistic can then be constructed by applying certain continuous functional $\mathcal{G}$ to the normalized CUSUM process $T_{N,\hat{\eta}}(\lfloor Nr\rfloor, K)'\tilde{\Sigma}_\eta^{-1}T_{N,\hat{\eta}}(\lfloor Nr\rfloor, K)$, $r \in [0,1]$. In the iid case, the LRV matrix of η_i is simply given by $\Sigma_\eta = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K)$, which can be consistently estimated by replacing each eigenvalue with its empirical estimate. From equation (2.8), it is easy to see that the BGHK test is basically a special case of this procedure with $\mathcal{G}(f) = \int_{\mathcal{I}}|f(x)|dx$. For temporally-dependent functional data, the HK test statistic is asymptotically valid, but it involves a truncation lag (bandwidth parameter) in the LRV estimator, the selection of which is not addressed. In fact, the choice of the bandwidth is a difficult task in the detection problem even in the univariate setup. The bandwidth that is a fixed function of the sample size (e.g., $N^{1/3}$, where N is the sample size) is not adaptive to the magnitude of the dependence in the series, whereas the data-dependent bandwidth could lead to nonmonotonic power (i.e., the power can decrease when the alternative gets farther away from the null) as shown in previous studies (Vogelsang [24]; Crainiceanu and Vogelsang [8]; Juhl and Xiao [15]). Recently, Shao and Zhang [23] proposed a SN-based test in the univariate time series setup, that is able to overcome the nonmonotonic power problem and has very accurate size and respectable power properties. Here we pursue an extension of the SN-based test to the functional setup.

To avoid choosing the bandwidth parameter, we define the following normalization matrix which takes the alternative into account. Let

$$\begin{aligned} V_{N,\hat{\eta}}(k,K) := \frac{1}{N^2}\Bigg[&\sum_{t=1}^{k}\left\{S_{N,\hat{\eta}}(1,t) - \frac{t}{k}S_{N,\hat{\eta}}(1,k)\right\}\left\{S_{N,\hat{\eta}}(1,t) - \frac{t}{k}S_{N,\hat{\eta}}(1,k)\right\}' \\ &+ \sum_{t=k+1}^{N}\left\{S_{N,\hat{\eta}}(t,N) - \frac{N-t+1}{N-k}S_{N,\hat{\eta}}(k+1,N)\right\} \\ &\left\{S_{N,\hat{\eta}}(t,N) - \frac{N-t+1}{N-k}S_{N,\hat{\eta}}(k+1,N)\right\}'\Bigg]. \end{aligned} \tag{2.9}$$

The SN-based statistic can thus be defined as

$$\begin{aligned} G_{N,\hat{\eta}}(K) &= \sup_{k=1,2,\ldots,N-1} \{T_{N,\hat{\eta}}(k,K)' V_{N,\hat{\eta}}^{-1}(k,K) T_{N,\hat{\eta}}(k,K)\} \\ &= \mathcal{C}(N^{-1/2} S_{N,\hat{\eta}}(1, \lfloor Nr \rfloor), r \in [0,1]), \end{aligned} \tag{2.10}$$

where $\mathcal{C}$ is the implicitly defined continuous mapping that corresponds to $G_{N,\hat{\eta}}(K)$. Here the self-normalizer $\{V_{N,\hat{\eta}}(k,K)\}_{k=1}^{N}$ plays two roles. On the one hand, it is able to absorb the dependence without consistent estimation of the LRV matrix. This means that the resulting limiting null distribution is nuisance parameter-free. On the other hand, it is specially designed for the change point testing problem, and it has been shown very effective in eliminating the nonmonotonic power problem in the univariate time series setting by Shao and Zhang [23]. Though there is no theoretical justification of the monotonic power property of the SN-based test even in the univariate setting, the empirical power of the SN-based test is seen to be monotonic in our simulation studies (see section 4.1). Let $\mathbf{B}_K(t)$ be a K dimensional vector with each component an independent standard Brownian motion. Under suitable assumptions (see section 3), we are able to show that

$$G_{N,\hat{\eta}}(K) \to^d G(K) := \sup_{r\in[0,1]} \{\mathbf{B}_K(r) - r\mathbf{B}_K(1)\}' \mathbf{V}_K^{-1}(r) \{\mathbf{B}_K(r) - r\mathbf{B}_K(1)\}, \tag{2.11}$$

where $\mathbf{V}_K(r) = \int_0^r \mathbf{W}_{1,K}(s,r)\mathbf{W}_{1,K}(s,r)' ds + \int_r^1 \mathbf{W}_{2,K}(s,r)\mathbf{W}_{2,K}(s,r)' ds$ with $\mathbf{W}_{1,K}(s,r) = \mathbf{B}_K(s) - \mathbf{B}_K(r)s/r$ for $s \in [0,r]$ and $\mathbf{W}_{2,K}(s,r) = [\{\mathbf{B}_K(1) - \mathbf{B}_K(s)\} - (1-s)/(1-r)\{\mathbf{B}_K(1) - \mathbf{B}_K(r)\}]$ for $s \in [r,1]$. The critical values of the nonstandard null distribution $G(K)$ have been tabulated by Shao and Zhang [23] for $K = 1, 2, \ldots, 10$ via simulations.

2.2. Testing for a change point in the Lag-1 autocovariance operator

As an extension of the above SN-based test, we consider the problem of testing the stability of the autocovariance operator at lag one, which partially describes the dependence structure of temporally dependent functional data. Recently, Horváth et al. [14] proposed a test for the constancy of the ARH(1) (functional autoregressive model of order one) operator against a one change point alternative. As pointed out in their paper, since the constancy of the ARH(1) operator implies the stability of the autocovariance operator at lag one, their test effectively checks whether the lag one autocovariance operator stays constancy over time. Our test differs from theirs in two aspects. First, we do not assume a parametric ARH(1) model in our theory and our test can be easily extended to test for the constancy of lag k autocovariance operator for $k = 1, 2, \ldots, m$, either separately or jointly. Second, our SN-based test is free of any bandwidth parameter, which is required in Horváth et al.'s work.

Without loss of generality, we assume that the functional observations $\{X_i(t)\}_{i=1}^{N}$ have a constant mean zero and admit the Karhunen-Loève expansion, $X_i(t) = \sum_{j=1}^{\infty} \eta_{i,j}\phi_j(t)$, $i = 1, 2, \ldots, N$. Let $R_i(\cdot) = \mathbf{E}[< X_i, \cdot > X_{i+1}]$ be the lag one autocovariance operator at time i. We are interested in testing the null hypothesis that

$$H_{0,2} : R_1 = \cdots = R_{N-1}$$

versus the alternative

$$H_{a,2} : R_1 = \cdots = R_{\tilde{k}^*} \neq R_{\tilde{k}^*+1} = \cdots = R_{N-1},$$

where the change point $\tilde{k}^* = \lfloor N\tilde{\lambda} \rfloor$ with $\tilde{\lambda} \in (0,1)$ and $\tilde{k}^* < N-1$ is unknown. Following Horváth et al. [14], we focus on the action of the lag one autocovariance operator R_i on the space spanned by $\{\phi_1(t), \phi_2(t), \ldots, \phi_K(t)\}$, and we test the constancy of $\{R_i\phi_j,\ j = 1, 2, \ldots, K\}$. Based on the representation that $R_i\phi_j = \sum_{l=1}^{\infty} < R_i\phi_j, \phi_l > \phi_l$, the constancy of R_i implies the stability of $< R_i\phi_j, \phi_l >$, $j, l = 1, 2, \ldots, K$, which motivates us to test the stability of the vector $(< R_i\phi_1, \phi_1 >, \ldots, < R_i\phi_1, \phi_K >, \ldots, < R_i\phi_K, \phi_1 >, \ldots, < R_i\phi_K, \phi_K >) \in \mathbb{R}^{K^2}$, for $i = 1, 2, \ldots, N-1$. Under $H_{0,2}$, we note that $< R_i\phi_j, \phi_l > = \mathbf{E}[< X_i, \phi_j >< X_{i+1}, \phi_l >] = \mathbf{E}[\eta_{i,j}\eta_{i+1,l}]$. Defining $\xi_i(j,l) = \eta_{i,j}\eta_{i+1,l}$ and its empirical counterpart by $\hat{\xi}_i(j,l) = \hat{\eta}_{i,j}\hat{\eta}_{i+1,l}$. We further define the vector $\xi_i = (\xi_i(1,1), \ldots, \xi_i(1,K), \ldots, \xi_i(K,1), \ldots, \xi_i(K,K))' \in \mathbb{R}^{K^2}$ and its sample counterpart $\hat{\xi}_i = (\hat{\xi}_i(1,1), \ldots, \hat{\xi}_i(1,K), \ldots, \hat{\xi}_i(K,1), \ldots, \hat{\xi}_i(K,K))'$. We aim to test the mean change of the vector ξ_i based on $\hat{\xi}_i, i = 1, 2, \ldots, N-1$. We define the empirical partial sum process by $S_{N,\hat{\xi}}(t_1, t_2) = \sum_{i=t_1}^{t_2} \hat{\xi}_i$. Analogous to $T_{N,\hat{\eta}}(k, K)$ and $V_{N,\hat{\eta}}(k, K)$, we define $T_{N,\hat{\xi}}(k, K^2)$ and $V_{N,\hat{\xi}}(k, K^2)$ with $S_{N,\hat{\eta}}(t_1, t_2)$ replaced by $S_{N,\hat{\xi}}(t_1, t_2)$. The test statistic is then given by

$$G_{N,\hat{\xi}}(K^2) = \sup_{k=1,2,\ldots,N-1} \{T_{N,\hat{\xi}}(k, K^2)' V_{N,\hat{\xi}}^{-1}(k, K^2) T_{N,\hat{\xi}}(k, K^2)\}. \tag{2.12}$$

We will show that $G_{N,\hat{\xi}}(K^2)$ has the limiting null distribution $G(K^2)$ in the next section.

3. Theoretical results

In this section, we justify the asymptotic validity of the SN-based test statistic by studying its asymptotic properties under both the null and local alternatives. To this end, we adopt the dependence measure proposed in Hörmann and Kokoszka [13], which is applicable to the temporally-dependent functional process.

Definition 3.1. Assume that $\{X_i\} \in L^p_{\mathbb{H}}$ with $p > 0$ admits the following representation

$$X_i = f(\varepsilon_i, \varepsilon_{i-1}, \ldots),\ i = 1, 2, \ldots, \tag{3.1}$$

where the ε_i's are iid elements taking values in a measurable space S and f is a measurable function $f : S^\infty \to \mathbb{H}$. For each $i \in \mathbb{N}$, let $\{\varepsilon_j^{(i)}\}_{j\in\mathbb{Z}}$ be an independent copy of $\{\varepsilon_j\}_{j\in\mathbb{Z}}$. The sequence $\{X_i\}$ is said to be L^p-m-approximable if

$$\sum_{m=1}^{\infty} e_p(X_m - X_m^{(m)}) < \infty, \tag{3.2}$$

where

$$X_i^{(m)} = f(\varepsilon_i, \varepsilon_{i-1}, \dots, \varepsilon_{i-m+1}, \varepsilon_{i-m}^{(i)}, \varepsilon_{i-m-1}^{(i)}, \dots). \tag{3.3}$$

It can be verified that a functional linear process is L^p-m-approximable when the operator coefficients satisfy certain summability conditions and the innovation sequence is in $L^p_{\mathbb{H}}$ (see Proposition 2.1 in Hörmann and Kokoszka [13]). Definition 3.1 is also applicable to other nonlinear functional time series models such as functional bilinear models and functional ARCH models (see Examples 2.3 and 2.4 in Hörmann and Kokoszka [13]). For the temporally dependent functional data, the PC's are temporally correlated. We denote by $\Sigma_{\eta,K}$ the LRV matrix of the first K PC's, i.e,

$$\Sigma_{\eta,K} = \sum_{h=-\infty}^{\infty} \mathbf{E}[\eta_\mathbf{0} \eta_\mathbf{h}'],$$

with $\eta_i = (\eta_{i,1}, \eta_{i,2}, \dots, \eta_{i,K})' \in \mathbb{R}^K$. Similarly we can define the LRV matrix Σ_{ξ,K^2} for $\{\xi_i\}$. To derive the asymptotic properties of the SN-based tests, we make the following assumption.

Assumption 3.2. *Assume that $Y_i(t) := X_i(t) - \boldsymbol{E}[X_i(t)] \in L^p_{\mathbb{H}}$ are L^p-m-approximable mean zero random elements. The eigenvalues of $c(t,s)$ satisfy that $\lambda_1 > \lambda_2 > \cdots > \lambda_K > \lambda_{K+1} > 0$.*

Theorem 3.3. *Suppose that $\boldsymbol{E}[X_i(t)] \in L^2_{\mathbb{H}}$ and Assumption 3.2 holds with $p = 4$. Assume that $\Sigma_{\eta,K}$ is positive definite. Then (2.11) holds under $H_{0,1}$.*

With a similar argument, we have the following result for $G_{N,\hat{\xi}}(K^2)$ under slightly stronger assumptions.

Theorem 3.4. *Suppose that Assumption 3.2 holds with $p = 8$. Also assume that $\boldsymbol{E}[X_i(t)] = 0$ and Σ_{ξ,K^2} is positive definite. Then under $H_{0,2}$, we have*

$$G_{N,\hat{\xi}}(K^2) \to^d G(K^2). \tag{3.4}$$

We now turn to the consistency of the proposed tests. As mentioned before, we consider the one-time shift alternative that the mean function or the lag one autocovariance operator remains constant before the change point and then becomes another constant afterward. In the case of detecting the mean change, we consider the alternative (2.2). Let

$$\tilde{c}(t,s) = c(t,s) + \lambda(1-\lambda)\{\mu_1(t) - \mu_2(t)\}\{\mu_1(s) - \mu_2(s)\}.$$

It is not hard to see that $\tilde{c}(t,s)$ is a covariance operator since it is symmetric and positive definite. Let γ_i and $v_i(t)$ be the corresponding eigenvalues and eigenfunctions satisfying that $\int_{\mathcal{I}} \tilde{c}(t,s) v_i(s) ds = \gamma_i v_i(t)$ and $\gamma_1 > \gamma_2 > \cdots > \gamma_K > 0$. Set $\Delta(t) = \mu_1(t) - \mu_2(t)$ and $\boldsymbol{\Delta}_K = (<\Delta, v_1>, <\Delta, v_2>, \dots, <\Delta, v_K>)' \in \mathbb{R}^K$. To ensure that $\boldsymbol{\Delta}_K \neq \mathbf{0}$, we suppose $\Delta(t) \notin \text{span}\{v_1(t), v_2(t), \dots, v_K(t)\}^\perp$ which means that the difference of the two mean functions does not belong to the orthogonal complement of the space spanned by the first K eigenfunctions of $\tilde{c}$. Note that if $\boldsymbol{\Delta}_K = \mathbf{0}$, then

$$\int_{\mathcal{I}} \tilde{c}(t,s)v_i(s)ds = \int_{\mathcal{I}} c(t,s)v_i(s)ds = \gamma_i v_i(t), \quad i = 1, 2, \ldots, K,$$

which means γ_i and v_i are also the eigenvalues and eigenfunctions of $c(t,s)$. In this case, the proposed test only has trivial power against the alternative. When $\boldsymbol{\Delta}_K \neq \mathbf{0}$, the following proposition shows the consistency of the SN-based test.

Proposition 3.5. *Consider the alternative (2.2) with $\lambda \in (0,1)$ fixed and $\boldsymbol{\Delta}_K \neq \mathbf{0}$. Suppose that assumption 3.2 holds with $p = 4$, then we have $G_{N,\hat{\eta}}(K) \to \infty$ in probability.*

In what follows, we consider the local alternatives where the difference of the mean functions depends on the sample size N. In this case, we shall use the notation $\tilde{c}^{(N)}(t,s)$, $v_i^{(N)}$, $\Delta^{(N)}(t)$ and $\boldsymbol{\Delta}_K^{(N)}$ instead of $\tilde{c}(t,s)$, v_i, $\Delta(t)$ and $\boldsymbol{\Delta}_K$.

Proposition 3.6. *Consider the alternative (2.2) where $\lambda \in (0,1)$ is fixed, and $||\Delta^{(N)}|| = O(|\boldsymbol{\Delta}_K^{(N)}|)$ with $|\boldsymbol{\Delta}_K^{(N)}| = o(1)$ and $\liminf_{N\to\infty} \frac{N^{1/2}|\boldsymbol{\Delta}_K^{(N)}|}{\log\log N} > 0$. Here $|\boldsymbol{\Delta}_K^{(N)}|$ denotes the Euclidean norm of $\boldsymbol{\Delta}_K^{(N)}$. Further suppose that assumption 3.2 holds with $p = 4$, then we have $G_{N,\hat{\eta}}(K) \to \infty$ in probability.*

Here we allow $|\boldsymbol{\Delta}_K^{(N)}|$ to decay to zero at a rate $(\log\log N)/N^{1/2}$ in order to have nontrivial power. The condition $||\Delta^{(N)}|| = O(|\boldsymbol{\Delta}_K^{(N)}|)$ implies that the projection of the change on the first K PC's takes a nonzero proportion. As a by-product of our test, the change point can be naturally estimated by

$$\hat{k}^* = \text{argmax}_{k=1,2,\ldots,N-1}\{T_{N,\hat{\eta}}(k,K)' V_{N,\hat{\eta}}^{-1}(k,K) T_{N,\hat{\eta}}(k,K)\}. \tag{3.5}$$

When $K = 1$, we are able to show that the SN-based change point estimator is in fact consistent. However, we encountered some technical difficulty when proving the consistency result for $K > 1$. To study the power properties of the test statistic $G_{N,\hat{\xi}}(K^2)$, we may further assume that the functional sequence comes from two stationary subsequences $\{X_i^{(1)}(t)\}_{i=-\infty}^{k^*}$ and $\{X_i^{(2)}(t)\}_{i=k^*+1}^{\infty}$ under the $H_{a,2}$. Following the arguments presented in the proofs of Proposition 3.5, we can show (omitting the details) that $G_{N,\hat{\xi}}(K^2)$ is consistent under the alternative $H_{a,2}$ with $\tilde{\lambda} \in (0,1)$ fixed.

4. Numerical studies

To demonstrate the merits of the SN-based test statistics in a finite sample, we carried out several simulation studies to investigate the size and power properties of the proposed tests: for a change in the mean function in Section 4.1, for a change in the autocovariance operator at lag one in Section 4.2, and for a mean change in the 2-d functional data (a surface) in Section 4.3. Throughout the simulations, the number of Monte Carlo replications is set to be 1000.

4.1. Detecting the mean change for curves

Here we investigate the finite sample properties of the SN-based test for detecting the change of mean function. First, we consider independent functional observations. We follow the simulation setup in Berkes et al.[4], where the mean function $\mu_1(t)$ was chosen to be zero under the null hypothesis and two different cases of $Y_i(t)$ were considered, namely the trajectories of the standard Brownian motion (BM) and Brownian Bridge (BB). Under the alternative (2.2), let $\mu_2(t) = t$ or $\mu_2(t) = \sin(t)$. The change point k^* is set to be $N/2$. We generate data on a grid of 10^3 equispaced points in $[0, 1]$, and then convert discrete data to functional observations by using B-splines with 20 basis functions. We also tried 40 and 100 basis functions with sample size $N = 50, 100$ and $K = 1, 2, 3$, and found that the number of basis functions does not affect our results much. We compare the SN-based test (i.e., (2.10)) with the BGHK test; see (2.8). The empirical size and size-adjusted power of both the SN-based test and the BGHK test are summarized in Table 1. Size-adjusted power is computed using finite sample critical values based on the Monte Carlo simulation under the null hypothesis. It can be seen that the empirical size of the SN-based test is comparable with the BGHK test in all cases considered here. As the expense of accounting for dependence, the SN-based test loses some power, but the power loss is fairly moderate.

To further examine the effect of dependence on the tests, we generate a functional sequence $\{Y_i(t)\}_{i=1}^N$ from the ARH(1) model which is defined as

$$Y_i(t) = \int_{\mathcal{I}} \psi(t,s) Y_{i-1}(s) ds + \varepsilon_i(t), \quad t \in \mathcal{I}, \quad i = 1, 2, \ldots, N,$$

where $\psi(t, s)$ is the kernel function and $\{\varepsilon_i(t)\}$ is a functional innovation sequence. To ensure that the ARH(1) model has a stationary solution, we assume

$$||\psi||_{HS}^2 = \int_0^1 \int_0^1 \psi^2(t,s) dt ds < 1,$$

where $||\cdot||_{HS}$ denotes the so-called Hilbert-Schmidt norm. Following the setup in Gabrys and Kokoszka [12], we choose two kernel functions, the Gaussian kernel,

$$\psi(t,s) = C \exp\left(\frac{t^2 + s^2}{2}\right)$$

and the Wiener kernel,

$$\psi(t,s) = C \min(t,s),$$

in our simulations. We consider the null hypothesis (2.1) and alternative hypothesis (2.2) as in the independent case, except that $\{Y_i(t)\}$ is now generated from the ARH(1) model. We compare the SN-based test with the BGHK test and the HK test. To implement the latter test, we have to estimate the LRV matrix of the first K scores consistently. Given a p-dimensional multivariate series

TABLE 1

Empirical size (upper panel) and size-adjusted power (lower panel) in percentage for the SN-based test (in row (i)) and the BGHK test (in row (ii)) for independent functional data generated from BM or BB. The size-adjusted power is computed under the alternative (2.2) with $\mu_2(t) = t$ or $\mu_2(t) = \sin(t)$, and $k^* = N/2$. The sample size $N = 50, 100$, and the number of PCs $K = 1, 2, 3$. The number of Monte Carlo replications is 1000

		$K = 1$			$K = 2$			$K = 3$		
		10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 50$										
BM	(i)	10.7	5.7	0.7	9.6	3.7	0.7	10.8	5.2	1.4
	(ii)	10.0	5.3	1.2	10.3	5.0	0.8	10.9	5.5	1.0
BB	(i)	7.5	3.8	0.8	8.2	4.6	1.1	10.7	6.0	1.3
	(ii)	10.6	5.4	0.8	10.9	5.1	1.1	10.5	5.2	1.2
$N = 100$										
BM	(i)	9.9	5.1	1.1	9.2	4.3	0.5	9.1	4.6	0.7
	(ii)	10.4	5.4	0.5	10.3	4.5	0.6	9.5	3.8	0.6
BB	(i)	10.0	5.1	1.3	8.4	3.5	0.7	9.9	4.7	0.7
	(ii)	9.6	5.2	0.9	9.3	4.9	0.6	9.1	4.1	0.9
$N = 50$										
BM, t	(i)	77.6	64.5	44.9	71.7	58.4	39.4	67.4	51.7	23.8
	(ii)	89.5	79.8	48.9	83.6	73.7	48.9	77.8	65.4	38.8
BB, t	(i)	99.8	99.4	95.6	100	100	99.6	100	100	99.9
	(ii)	100	100	99.7	100	100	100	100	100	100
BM, $sin(t)$	(i)	70.0	57.7	38.9	62.1	48.3	29.1	56.0	41.4	17.0
	(ii)	82.1	71.9	39.4	74.4	61.4	36.4	66.9	52.4	28.7
BB, $sin(t)$	(i)	99.3	98.1	89.7	100	99.6	96.9	100	99.9	99.4
	(ii)	99.9	99.7	97.6	100	100	100	100	100	100
$N = 100$										
BM, t	(i)	96.9	89.9	70.8	92.9	87.4	73.0	90.9	84.0	66.7
	(ii)	99.3	98.4	95.5	99.1	97.9	94.0	98.5	96.8	91.2
BB, t	(i)	100	99.9	99.6	100	100	100	100	100	100
	(ii)	100	100	100	100	100	100	100	100	100
BM, $sin(t)$	(i)	92.7	84.2	62.7	87.1	78.7	59.2	83.9	73.8	52.1
	(ii)	98.4	95.8	89.6	96.3	93.5	86.6	95.2	90.9	78.0
BB, $sin(t)$	(i)	99.9	99.7	98.8	100	100	100	100	100	100
	(ii)	100	100	100	100	100	100	100	100	100

$\{u_i = (u_{i1}, \ldots, u_{ip})\}_{i=1}^N$, the LRV matrix can be estimated nonparametrically by

$$\hat{\Omega} = \sum_{|j| \leq b_N} K\left(\frac{j}{b_N}\right) \hat{\Gamma}_j,$$

where b_N is the bandwidth, $K(\cdot)$ is the kernel function and $\hat{\Gamma}_j$ is the sample autocovariance function at lag j. Here we use the Bartlett kernel, i.e, $k(x) = (1 - |x|)\mathbf{I}\{|x| \leq 1\}$, with the data-dependent truncation lag $b_N = 1.1447\{\hat{\alpha}(1)N\}^{1/3}$, where

$$\hat{\alpha}(1) = \left\{\sum_{i=1}^{p} \frac{4\hat{\sigma}_i^4 \hat{\rho}_i^2}{(1-\hat{\rho}_i)^6 (1+\hat{\rho}_i)^2}\right\} \left\{\sum_{i=1}^{p} \frac{\hat{\sigma}_i^4}{(1-\hat{\rho}_i)^4}\right\}^{-1}. \tag{4.1}$$

Here $\hat{\rho}_i$ is the least squares coefficient estimate by regression u_{ki} on $u_{(k-1)i}$ and $\hat{\sigma}_i^2$ is the estimate of the innovation variance. The plug-in bandwidth formula

TABLE 2
Empirical size in percentage of the SN-based test (in row (i)), the BGHK test (in row (ii)) and the HK test (in row (iii)) for temporally dependent functional data generated from ARH(1) process. The sample size $N = 50, 100$, and the number of PCs $K = 1, 2, 3$. The number of Monte Carlo replications is 1000

		$K=1$			$K=2$			$K=3$		
		10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 50$										
Gaussian, BM	(i)	15.2	10.3	3.9	15.2	8.4	2.4	14.5	8.0	2.2
	(ii)	44.1	32.2	16.2	37.5	25.0	12.4	32.7	23.0	11.4
	(iii)	17.7	9.2	0.6	11.1	3.2	0.3	4.9	1.1	0.0
Gaussian, BB	(i)	17.3	10.6	3.1	14.0	7.1	2.5	14.5	8.2	2.3
	(ii)	42.7	32.3	13.8	36.1	25.5	10.0	34.9	23.2	9.6
	(iii)	19.8	8.8	0.2	11.1	2.6	0.0	6.3	1.5	0.0
Wiener, BM	(i)	16.0	10.4	4.0	16.1	9.2	3.0	16.0	9.8	2.9
	(ii)	46.4	33.6	16.6	40.2	26.9	12.6	36.6	25.0	10.2
	(iii)	17.5	8.4	0.5	10.9	2.8	0.1	6.1	0.7	0.0
Wiener, BB	(i)	17.0	10.4	2.9	13.3	7.3	2.2	15.4	9.7	2.2
	(ii)	42.8	31.0	14.3	37.9	26.7	11.2	36.4	23.9	10.0
	(iii)	19.0	8.9	0.2	11.2	3.3	0.0	6.5	1.8	0.0
$N = 100$										
Gaussian, BM	(i)	13.3	7.8	2.0	11.7	5.7	1.2	11.7	6.1	1.2
	(ii)	51.2	35.9	16.4	39.7	27.9	11.6	34.9	24.1	9.7
	(iii)	15.2	7.4	0.4	11.6	3.9	0.2	7.2	2.1	0.0
Gaussian, BB	(i)	11.6	6.7	1.6	10.9	4.9	1.1	11.5	7.1	1.2
	(ii)	46.7	33.0	13.9	35.9	25.1	10.2	36.4	25.8	11.4
	(iii)	16.1	8.0	1.4	12.4	5.3	0.3	10.0	3.5	0.1
Wiener, BM	(i)	13.7	7.8	2.1	11.7	5.8	1.3	12.9	7.1	1.3
	(ii)	52.2	37.2	17.5	43.8	29.7	12.8	38.3	26.1	11.7
	(iii)	15.3	7.4	0.5	11.6	4.1	0.2	7.5	2.6	0.0
Wiener, BB	(i)	11.9	6.4	1.9	10.4	5.6	1.2	12.0	7.8	1.3
	(ii)	45.1	32.0	13.5	38.5	27.5	12.8	37.9	27.3	11.9
	(iii)	16.4	7.6	1.5	13.3	5.9	0.5	10.8	3.7	0.2

(4.1) is suggested by Andrew [2] and is shown to minimize the MSE of the LRV estimator when the true model is the vector autoregressive model of order one.

We report the simulation results for $N = 50, 100$, $K = 1, 2, 3$, $||\psi||_{HS} = 0.5$ and BM and BB innovations in Table 2 and Table 3. Several other values of $||\psi||_{HS}$ (e.g. 0.3,0.8) were also considered, but the results are not reported here to save space. From Table 2, we see that the size distortion of the BGHK test is severely large compared to the other two tests. This is due to the fact that it is designed only for independent functional data and is invalid in the temporally-dependent case. For the HK test, the size distortion is less severe but seems sensitive to the choice of K. It tends to be oversized for small K but undersized for large K. For the SN-based test, size distortion is apparent for $N = 50$, but improves for $N = 100$. The size for the SN-based test seems quite robust to the choice of K. Table 3 presents selected results of the size-corrected powers from which several observations can be made. First, the BGHK test delivers the highest power among the three tests, which is largely due to its severe upward size distortion. Second, the power of the SN-based test is comparable

Table 3

Size-adjusted power in percentage of the SN-based test (in row (i)), the BGHK test (in row (ii)) and the HK test (in row (iii)) for temporally dependent functional data generated from ARH(1) process. The size-adjusted power is computed under the alternative (2.2) with $\mu_2(t) = t$ or $\mu_2(t) = \sin(t)$, and $k^* = N/2$. The sample size $N = 50, 100$, and the number of PCs $K = 1, 2, 3$. The number of Monte Carlo replications is 1000

		$K=1$			$K=2$			$K=3$		
		10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 50$										
Gaussian, BM, t	(i)	35.7	22.5	5.0	48.6	34.7	11.4	46.0	32.5	11.0
	(ii)	44.0	25.0	11.8	46.2	28.2	11.8	42.7	25.6	11.4
	(iii)	36.1	21.6	7.4	46.9	32.4	7.6	40.8	26.8	4.9
Gaussian, BB, t	(i)	75.3	66.8	33.2	99.3	96.9	84.9	99.1	97.1	88.2
	(ii)	89.9	83.0	69.4	99.8	99.7	97.6	100	99.9	98.1
	(iii)	65.1	40.6	11.0	79.6	53.1	23.3	61.2	37.3	11.8
Gaussian, BM, $sin(t)$	(i)	31.8	18.5	4.1	36.3	24.8	7.4	36.8	24.0	6.8
	(ii)	38.3	20.7	8.9	36.7	21.3	7.7	32.8	20.1	7.5
	(iii)	31.6	19.2	5.1	37.0	25.0	4.4	35.4	21.9	3.2
Gaussian, BB, $sin(t)$	(i)	67.5	60.0	26.4	96.0	89.8	69.6	96.3	91.3	73.6
	(ii)	83.5	76.7	57.2	99.0	96.5	91.3	99.6	98.3	88.6
	(iii)	61.6	40.6	11.6	79.0	55.1	23.7	63.6	39.3	13.7
Wiener, BM, t	(i)	34.3	20.0	4.7	35.5	25.1	7.8	34.2	19.9	7.5
	(ii)	42.6	23.8	11.0	42.1	26.2	13.3	43.7	30.0	13.1
	(iii)	32.6	21.4	6.0	33.8	21.4	6.0	29.3	15.6	5.2
Wiener, BB, t	(i)	76.3	68.1	39.2	95.4	89.5	73.1	98.3	96.6	84.0
	(ii)	89.4	83.0	71.6	99.4	98.8	94.9	100	99.6	98.0
	(iii)	63.5	41.3	13.1	72.8	45.5	9.8	55.4	28.8	8.0
Wiener, BM, $sin(t)$	(i)	30.1	17.7	3.7	29.7	21.1	5.0	27.8	16.3	5.4
	(ii)	37.1	19.6	8.3	34.5	20.8	9.2	35.7	22.8	8.9
	(iii)	29.4	18.5	4.7	28.3	16.8	2.7	23.5	11.7	3.8
Wiener, BB, $sin(t)$	(i)	69.2	59.7	32.5	88.4	77.9	58.4	93.6	88.7	66.0
	(ii)	83.8	76.6	59.9	96.6	92.5	84.1	99.0	98.1	88.5
	(iii)	61.1	41.4	13.7	70.3	44.6	11.5	57.4	31.5	8.6
$N = 100$										
Gaussian, BM, t	(i)	53.5	39.2	19.1	76.8	64.6	37.1	76.3	64.4	36.9
	(ii)	65.7	56.5	30.2	80.8	66.9	40.3	77.7	64.9	38.4
	(iii)	61.1	47.5	23.1	81.7	68.1	40.0	77.4	64.4	29.9
Gaussian, BB, t	(i)	94.4	89.9	71.0	100	100	99.6	99.9	99.9	99.7
	(ii)	99.1	98.0	91.6	100	100	100	100	100	100
	(iii)	96.6	86.3	42.2	99.8	96.0	59.7	97.4	85.9	25.6
Gaussian, BM, $sin(t)$	(i)	46.6	33.6	15.0	61.5	47.7	22.8	63.7	50.4	26.4
	(ii)	58.6	48.7	24.1	66.9	52.2	27.8	65.5	51.1	26.9
	(iii)	54.9	40.9	18.3	66.4	52.4	26.5	65.5	51.6	20.9
Gaussian, BB, $sin(t)$	(i)	91.4	84.9	62.5	100	99.8	96.4	100	99.7	97.7
	(ii)	98.0	95.0	84.7	100	100	99.9	100	100	99.7
	(iii)	93.6	83.0	43.3	99.8	95.5	63.6	97.4	89.2	31.0
Wiener, BM, t	(i)	51.4	36.8	17.4	55.1	43.3	18.1	54.5	42.5	20.8
	(ii)	63.6	53.4	29.8	69.9	57.9	35.4	70.3	59.0	33.1
	(iii)	57.9	44.8	19.6	58.0	45.0	20.0	57.1	39.4	13.7
Wiener, BB, t	(i)	95.1	89.7	70.5	100	99.7	95.8	100	99.9	99.9
	(ii)	99.2	98.2	91.4	100	100	100	100	100	100
	(iii)	96.6	87.6	38.1	99.0	87.7	44.9	95.4	80.5	22.7
Wiener, BM, $sin(t)$	(i)	44.3	31.2	13.7	44.5	32.9	12.3	42.9	29.7	13.1
	(ii)	56.9	45.2	24.4	58.3	45.1	26.0	57.7	45.7	22.8
	(iii)	52.1	37.3	15.9	47.1	33.2	12.7	43.7	27.0	7.3
Wiener, BB, $sin(t)$	(i)	92.2	85.1	60.2	99.6	97.2	89.1	99.7	98.9	94.9
	(ii)	98.3	95.8	84.8	100	99.7	97.4	100	100	99.3
	(iii)	93.9	84.2	40.7	97.8	86.1	47.3	95.9	84.0	27.2

to that of the HK test for $N = 50$ and BM innovations. Furthermore the SN-based test tends to have moderate power loss when sample size increases to 100. In the case of the BB innovations, the SN-based test is superior to the HK test in power. Overall, the severe size distortion of the BGHK test under weak dependence suggests its inability to accommodate dependence and thus is not recommended in testing for a change point for dependent functional data. The HK test is able to account for dependence but it is sensitive to the choice of bandwidth b_N and K. As shown below, the data-dependent bandwidth used in the HK test could lead to nonmonotonic power. Compared to the other two tests, the SN-based test tends to have more accurate size at the sacrifice of some power, which is consistent with the "better size but less power" phenomenon seen in the univariate setup (see Shao and Zhang [23]).

Furthermore, we examine the monotonic power property of the SN-based test in the functional setup through simulations. In the univariate setting, the change point test, which involves LRV estimation using the data-dependent bandwidth, can exhibit nonmonotonic power (see e.g. Vogelsang [24]; Crainiceanu and Vogelsang [8]; Altissimo and Corradi [1]). There are some recent studies aiming to overcome the nonmonotonic power problem in the univariate time series setup (see Juhl and Xiao [15]; Shao and Zhang [23]). To study the monotonic power property, we focus on the change of mean function and consider the data generating process

$$X_i(t) = Y_i(t) + \delta f(t) I\{i > N/2\}, \; i = 1, 2, \ldots, N, \tag{4.2}$$

where $\{Y_i(t)\}$ follows the ARH(1) model with Gaussian kernel, and BM or BB innovations. The constant δ here is used to control the magnitude of change and $f(t) = t$ or $\sin(t)$. The size-adjusted power for $K = 1$ and $N = 50$ is plotted as a function of δ in Figure 1. Qualitatively similar results were observed for $N = 100$, but are not reported to conserve space. We compare the performance of the SN-based test with the BGHK test and the HK test. Like the univariate case, the SN-based test shows monotonic power in all situations even though it could lose moderate power to the BGHK test. Not surprisingly, due to the upward bias of the data-dependent bandwidth, the HK test exhibits nonmonotonic power, with power going to zero for relatively large changes in the mean function. These results indicate that the nonmonotonic power issue still exists in the functional setting if one estimates the LRV matrix of scores nonparametrically using data-dependent bandwidth. In contrast, the SN-based test inherits the monotonic power property, that holds in the univariate case (Shao and Zhang [23]).

In the univariate setting, Crainiceanu and Vogelsang [8] and Juhl and Xiao [15] have proposed different methods to overcome the nonmonotonic power problem in testing for a change point in mean. However, their methods both involve bandwidth parameters and their finite sample performance is unsatisfactory as seen from the numerical comparison in Shao and Zhang [23]. For example, Crainiceanu and Vogelsang [8] proposed to estimate the long run variance using residuals obtained under the one-break model but the size distortion is large for time series with strong dependence (e.g., AR(1) model with AR(1) coefficient 0.8). Juhl and Xiao [15] used residuals from nonparametric regression to

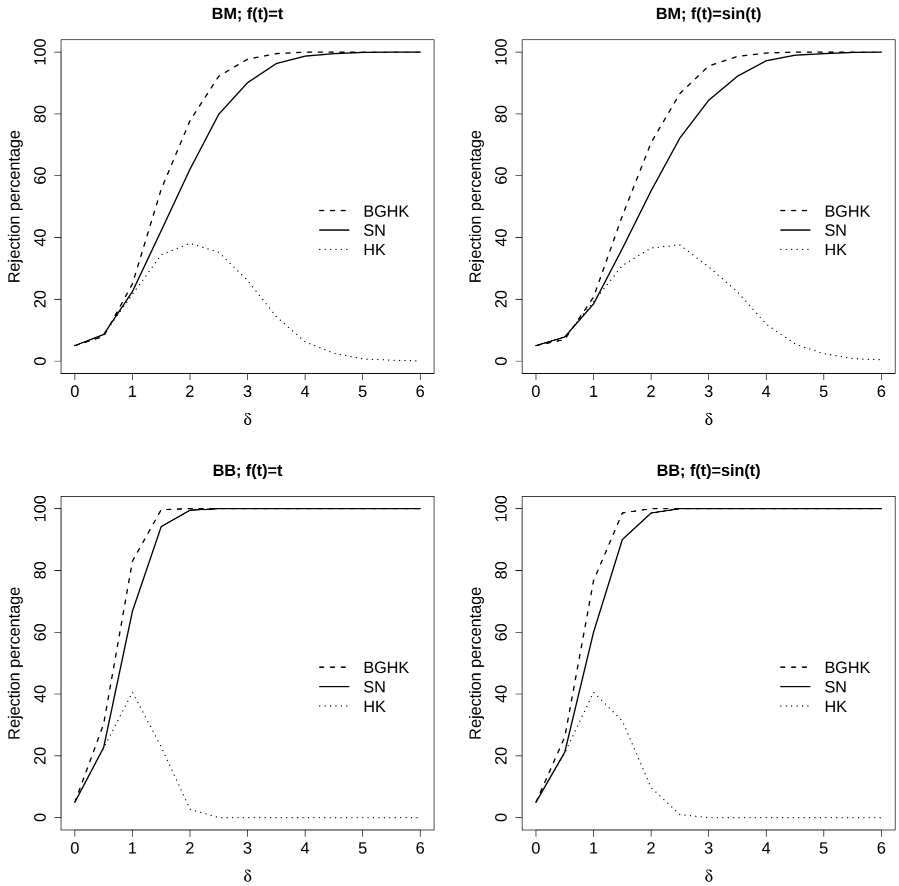

Fig 1. *Size-adjusted power for detecting the mean change with different magnitude of change. Sample size $N = 50$. Note: the quantity δ measures the magnitude of change; see equation (4.2).*

estimate long run variance, but they did not completely eliminate the nonmonotonic power problem (section 4.1 of Shao and Zhang [23]). In a sense, the two methods mentioned above were proposed to account for a possible change point in the LRV estimator. But they did not perform well in finite sample, so we expect that the extensions of these methods to functional setting will not work well, although a serious investigation is beyond the scope of the paper.

4.2. Detecting the changes in the lag-1 autocovariance operator

In this subsection, we study the finite sample performance of the SN-based test for detecting the change of the autocovariance operator at lag one. Under the null, we generate functional observations from the mean zero ARH(1) model with Gaussian kernel and $||\psi||_{HS} = 0.3$. Under the alternative, we consider the following data-generating process,

$$\begin{cases} Y_i(t) = \int_0^1 \psi_1(t,s)Y_{i-1}(s)ds + \varepsilon_i(t), & i = 1, 2, \dots, N/2; \\ Y_i(t) = \int_0^1 \psi_2(t,s)Y_{i-1}(s)ds + \varepsilon_i(t), & i = N/2+1, N/2+2, \dots, N, \end{cases} \tag{4.3}$$

where $\psi_1(s,t)$ and $\psi_2(s,t)$ are both Gaussian kernels with $||\psi_1||_{HS} = 0.3$ and $||\psi_2||_{HS} = 0.8$. The SN-based test is compared with the tests proposed in Horváth et al. [14] for detecting the stability of the ARH(1) model. Formally, Horváth et al.'s tests can be written as

$$I_{N,\hat{\xi}} = \frac{1}{N-1} \sum_{k=1}^{N-1} T_{N,\hat{\xi}}(k, K^2)' \tilde{\Sigma}_{\xi}^{-1}(k, K^2) T_{N,\hat{\xi}}(k, K^2), \tag{4.4}$$

where $\tilde{\Sigma}_\xi(k, K^2)$ is a consistent estimator of the LRV matrix of $\{\xi_i\}$. We define $\hat{D}_k$ as the nonparametric LRV matrix estimator computed from $\{\xi_1, \xi_2, \dots, \xi_k\}$ by using the Bartlett kernel with bandwidth given by (4.1). Similarly, we define the LRV matrix estimator $\hat{D}^*_{N-k}$ based on $\{\xi_{k+1}, \xi_{k+2}, \dots, \xi_{N-1}\}$. Following Horváth et al. [14], we consider two different ways of estimating the LRV matrix here: 1) $\tilde{\Sigma}_\xi(k, K^2) = \frac{k}{N-1}\hat{D}_k + \frac{N-k-1}{N-1}\hat{D}^*_{N-k}$; 2) $\tilde{\Sigma}_\xi(k, K^2) = \hat{D}_N$ for all $1 \le k \le N-1$, and we denote the resulting tests by HHK1 and HHK2. We present the empirical size and size-corrected power for $N = 50, 100, 200$, $K = 1, 2, 3$ and BM and BB innovations in Table 4. It can be clearly seen that the size distortion for HHK1 test is substantial, especially for $N = 50$ and $K = 2, 3$. The HHK2 test performs relatively well but tends to be undersized when K increases. Compared to HHK1 and HHK2 tests, the size performance of SN-based test is quite satisfactory. For the size-adjusted power, we find that the HHK1 test is the most powerful in all cases, presumably due to its upward size distortion. The HHK2 test has reasonable power for $K = 1$ while the power could drop dramatically as K increases for small sample size. The finding here agrees generally with the results in Horváth et al. [14] which shows that the HHK2 test is conservative for large K. The SN-based test delivers moderate power and the power seems robust to K. Overall, the simulation results clearly suggest a trade-off between the size distortion and power loss for the SN-based test, which has been found to be the case in the univariate setup.

4.3. Detecting the mean change for 2-d functional observations

Here, we perform a simulation study to demonstrate the validity of the SN-based test for detecting a mean function change in 2-d functional data. For simplicity, we focus on a rectangular region though our test can be applied to functional data on a region of irregular shape. Under the null, we generate 2-d functional observations $\{Y_i(s_1, s_2)\}_{i=1}^N$ in the following two ways:

1. $Y_i(s_1, s_2) = X_i^{(1)}(s_1)X_i^{(2)}(s_2)$, where $\{X_i^{(1)}\}$ and $\{X_i^{(2)}\}$ are mutually independent and contain possibly dependent continuous random processes respectively. Here we choose $\{X_i^{(j)}(s)\}, j = 1, 2$ to be BM, BB and ARH(1) process with Gaussian kernel with $||\psi||_{HS} = 0.5$ and BM innovations.

TABLE 4

Empirical size (upper panel) and size-adjusted power (lower panel) in percentage of the SN-based test (in row (i)) and the tests in Horváth et al. (2010) (in row (ii) for case 1 and (iii) for case 2) for detecting the change point in the lag one autocovariance operator. The size-adjusted power is computed under the alternative (4.3). The sample size $N = 50, 100$, and the number of PCs $K = 1, 2, 3$. The number of Monte Carlo replications is 1000

		$K=1$			$K=2$			$K=3$		
		10%	5%	1%	10%	5%	1%	10%	5%	1%
$N=50$										
BM	(i)	11.0	5.8	1.6	8.6	3.8	1.3	10.8	5.7	2.0
	(ii)	16.4	9.5	2.2	33.6	21.8	7.8	68.9	58.2	36.5
	(iii)	10.6	3.0	0.3	4.5	1.4	0.0	2.4	0.5	0.1
BB	(i)	8.5	4.1	0.8	8.4	4.4	0.7	10.5	5.6	1.1
	(ii)	15.8	9.3	2.2	32.2	21.6	7.6	68.9	59.6	37.8
	(iii)	9.4	3.8	0.4	4.5	1.3	0.1	2.2	0.7	0.1
$N=100$										
BM	(i)	9.2	5.4	1.4	8.5	4.4	1.0	7.3	3.9	0.7
	(ii)	14.1	7.1	1.5	16.9	10.6	2.5	36.4	27.1	11.3
	(iii)	10.0	3.8	0.7	6.4	2.7	0.5	3.1	1.3	0.1
BB	(i)	9.4	5.0	0.8	8.2	3.0	0.6	8.9	4.0	1.1
	(ii)	14.0	7.1	1.6	20.4	12.1	2.7	38.7	27.6	11.2
	(iii)	10.7	5.1	0.7	8.9	2.7	0.5	3.5	0.7	0.0
$N=200$										
BM	(i)	8.9	4.3	1.1	10.3	5.4	1.1	8.5	3.7	1.1
	(ii)	11.4	6.2	1.4	15.3	7.8	1.2	21.1	12.7	3.6
	(iii)	9.4	5.6	1.0	9.1	3.4	0.2	4.4	1.7	0.2
BB	(i)	10.0	5.4	1.3	8.6	4.5	0.9	9.8	6.0	0.9
	(ii)	12.3	5.2	1.3	14.7	7.5	1.4	21.1	12.1	0.4
	(iii)	10.4	3.9	0.7	8.3	3.0	0.7	6.3	2.1	0.1
$N=50$										
BM	(i)	27.3	17.2	5.2	22.5	14.5	6.4	22.1	14.1	4.8
	(ii)	37.8	26.6	12.5	36.7	26.0	12.5	45.7	34.0	20.1
	(iii)	27.0	15.6	2.6	11.8	5.4	1.2	5.9	2.8	1.2
BB	(i)	33.4	21.7	10.0	23.2	14.1	7.0	24.4	17.2	8.6
	(ii)	37.2	27.6	13.3	32.1	24.2	13.0	39.5	31.2	17.2
	(iii)	32.9	17.6	2.1	11.0	5.4	1.8	8.2	3.3	1.0
$N=100$										
BM	(i)	46.6	29.0	9.8	35.6	22.1	8.5	34.2	21.9	12.1
	(ii)	60.0	47.8	23.6	56.7	44.2	26.2	65.9	54.6	36.9
	(iii)	50.1	32.8	5.3	20.5	7.9	1.1	4.1	2.2	0.2
BB	(i)	44.7	30.7	12.9	32.9	24.6	12.6	28.9	20.4	7.0
	(ii)	55.5	42.2	20.8	46.8	36.9	18.6	57.6	48.1	32.0
	(iii)	45.5	26.2	6.7	14.2	5.3	1.5	4.5	1.8	0.6
$N=200$										
BM	(i)	70.2	57.8	26.8	54.0	38.0	17.2	54.6	40.8	15.6
	(ii)	87.9	74.8	46.4	78.9	69.0	47.2	85.6	79.3	65.3
	(iii)	80.5	54.7	21.5	39.8	23.9	5.8	11.7	3.7	0.3
BB	(i)	65.0	47.9	20.9	45.9	30.1	12.4	49.6	32.5	16.8
	(ii)	83.3	72.2	45.5	69.4	58.1	29.8	81.8	71.8	53.6
	(iii)	74.2	63.3	29.8	37.2	19.1	4.8	12.7	4.5	1.2

2. The sequence $\{Y_i(s,t)\}$ follows the $\mathrm{ARH}_2(1)$ model defined on $[0,1]^2$, that is,

$$Y_{i+1}(s_1,s_2) = \int_0^1 \int_0^1 \psi(s_1,s_2,u_1,u_2)Y_i(u_1,u_2)du_1du_2 + \varepsilon_{i+1}(s_1,s_2),$$

where $\psi(s_1,s_2,u_1,u_2) = C\exp\{(s_1^2+s_2^2+u_1^2+u_2^2)/2\}$ and $\varepsilon_i(s_1,s_2) = X_i^{(1)}(s_1)X_i^{(2)}(s_2)$ is the tensor product of independent BM or BB.

We suppose the data is collected on a grid of 20×20 equally spaced points on $[0,1]^2$. The discrete observations are smoothed by the thin plate spline (see Wahba [25]). In order to obtain the eigenvalues and eigenfunctions, the 2-d functional data is discretized to a fine grid of 30×30 equally spaced points. The data matrix corresponding to each observation is then concatenated into a single long vector. Hence the functional eigenanalysis problem for 2-d functional data is converted to an approximately equivalent matrix eigenanalysis task as in the one dimensional case (see Ramsay and Silverman [20] for more details). To illustrate the power properties of the SN-based test, we consider the following alternatives:

1. $Y_i(s_1,s_2) = \{X_i^{(1)}(s_1) + f(s_1)\}\{X_i^{(2)}(s_2) + f(s_2)\}$ with $f(s) = s$ or $f(s) = \sin(s)$.
2. $Y_i(s_1,s_2) = Z_i(s_1,s_2) + g(s_1,s_2)$, where $Z_i(s_1,s_2)$ is the aforementioned $\mathrm{ARH}_2(1)$ process, and $g(s_1,s_2) = s_1s_2$ or $g(s_1,s_2) = \sin(s_1)\sin(s_2)$.

The change point k^* is set to be $N/2$. The selected simulation results are summarized in Table 5. For the data generated by tensor product, the SN-based test is conservative when $N = 50$ and the size becomes closer to the nominal level as N increases to 100. For the $\mathrm{ARH}_2(1)$ process, the SN-based test is oversized, the size distortion diminishes and the power appreciates as sample size increases. It is also interesting to note that the special covariance structure of BB tends to give us more power, as we have seen before. In conclusion, the SN-based test delivers satisfactory size and reasonable power in the 2-d setting.

5. Applications

In this section, we consider two empirical datasets, namely, the single-point time series of central England temperature record and a spatio-temporal gridded dataset consisting of the bias between observed and model-simulated annual average temperature covering a subregion of North America (latitude: 34.25°N–51.25°N; longitude: 77.25°W–120.25°W) obtained from a coupled atmosphere-ocean general circulation model (AOGCM) and interpolated station observations.

5.1. Analysis of central England temperatures

We first apply the SN-based test to detect the change point in the functional mean of the central England temperature record that has been previously stud-

TABLE 5

(a) Empirical size (upper panel) and size-adjusted power (lower panel) in percentage of the SN-based test for detecting the mean change of 2-d functional observations generated by tensor product. (b) Empirical size (upper panel) and size-adjusted power (lower panel) in percentage of the SN-based test for detecting the mean change of 2-d functional observations generated from the $ARH_2(1)$ process. The sample size $N = 50, 100$, and the number of PCs $K = 1, 2, 3$. The number of Monte Carlo replications is 1000. The notation $(BM + t)^2$ denotes $(BM + t) \times (BM + t)$. Note: The Brownian motion is approximated by the partial sum of 1000 iid standard normal variables in (a) and it is approximated by the partial sum of 60 iid standard normal variables in (b)

	$K = 1$			$K = 2$			$K = 3$		
(a)	10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 50$									
BM × BM	6.8	3.6	0.7	6.4	2.8	0.5	8.6	3.2	0.6
BB × BB	8.4	4.4	0.9	7.2	3.3	0.0	8.1	3.5	0.2
ARH(1) × ARH(1)	10.2	4.4	0.4	9.1	4.0	0.7	10.8	5.4	0.8
$N = 100$									
BM × BM	10.2	4.9	1.0	9.0	3.8	1.0	9.9	4.9	1.3
BB × BB	7.6	3.7	1.0	7.3	3.5	0.5	9.4	4.4	0.3
ARH(1) × ARH(1)	9.6	5.0	0.8	8.0	3.9	1.0	10.1	4.3	0.9
$N = 50$									
$(\mathrm{BM} + t)^2$	62.4	47.0	20.7	53.3	38.6	16.4	45.8	33.5	14.0
$(\mathrm{BB} + t)^2$	99.8	99.1	95.5	100	99.7	98.8	99.8	99.8	99.7
$(\mathrm{BM} + \sin(t))^2$	52.2	37.8	14.8	42.8	25.9	11.3	32.8	22.9	7.7
$(\mathrm{BB} + \sin(t)^2$	99.2	97.1	89.0	99.0	98.7	96.3	99.9	99.8	98.0
$(\mathrm{ARH}(1) + t)^2$	31.1	22.1	10.1	34.0	25.1	9.2	30.6	20.8	8.2
$(\mathrm{ARH}(1) + \sin(t))^2$	26.0	17.4	7.2	26.1	17.1	6.2	23.1	14.0	4.5
$N = 100$									
$(\mathrm{BM} + t)^2$	77.6	67.8	41.6	72.7	60.5	32.1	69.0	53.0	25.0
$(\mathrm{BB} + t)^2$	100	100	99.6	100	100	99.9	100	100	100
$(\mathrm{BM} + \sin(t))^2$	67.9	55.4	28.5	55.8	45.2	20.5	51.8	36.5	14.2
$(\mathrm{BB} + \sin(t))^2$	100	99.9	98.6	100	100	99.9	100	100	100
$(\mathrm{ARH}(1) + t)^2$	43.7	31.1	15.3	47.4	36.3	13.4	47.1	36.2	16.6
$(\mathrm{ARH}(1) + \sin(t))^2$	35.5	25.2	10.5	37.1	24.2	8.4	33.4	23.6	8.7

	$K = 1$			$K = 2$			$K = 3$		
(b)	10%	5%	1%	10%	5%	1%	10%	5%	1%
$N = 50$									
BM × BM	14.8	9.5	2.9	14.4	8.4	2.2	14.0	7.8	2.6
BB × BB	13.1	7.0	1.5	10.6	5.6	0.8	13.0	6.7	1.4
$N = 100$									
BM × BM	10.4	5.9	1.4	10.0	3.8	0.6	9.9	4.3	0.9
BB × BB	12.7	6.7	1.4	9.7	4.7	0.9	11.4	5.5	0.7
$N = 50$									
BM × BM, t	31.6	21.5	7.3	43.6	32.4	12.2	50.7	36.1	12.4
BB × BB, t	99.9	99.7	98.6	100	100	100	100	100	100
BM × BM, $\sin(t)$	25.3	16.4	5.1	29.7	20.1	5.6	33.2	20.8	5.5
BB × BB,$\sin(t)$	99.0	98.1	94.0	100	100	100	100	100	100
$N = 100$									
BM × BM, t	58.0	41.6	21.1	73.5	64.9	45.0	81.6	72.8	47.9
BB × BB, t	100	100	99.7	100	100	100	100	100	100
BM × BM,$\sin(t)$	46.4	30.6	12.8	53.3	42.6	24.1	57.8	47.9	24.6
BB × BB,$\sin(t)$	99.9	99.7	97.7	100	100	100	100	100	100

TABLE 6
Segmentation procedure of the central England temperature data into periods with constant mean function. Note: in each iteration, K is the smallest positive integer such that $\sum_{i=1}^{K} \hat{\lambda}_i / \sum_{i=1}^{12} \hat{\lambda}_i > 0.8$

Iteration	Segment	K	$G_{N,\hat{\eta}}(K)$	p-value	Estimated change-point $\hat{k}^*$
1	1780–2007	8	559.4	(0.001, 0.005)	1927
2	1780–1927	8	173.1	(0.1, 1)	—
3	1928–2007	8	323.9	(0.025, 0.05)	1993
4	1928–1993	7	49.2	(0.1, 1)	—
5	1994–2007	5	153.0	(0.05, 0.1)	—

ied in Berkes et al.[4]. This data set represents the longest continuous thermometer-based temperature record on earth, consisting of 228 years (1780-2007) of average daily temperatures in central England (see Parker et al. [17]). Following Berkes et al. [4], we view the data as 228 curves with 365 measurements on individual curve. The discrete observations were registered as functional data by using 12 B-spline basis functions. To compute $G_{N,\hat{\eta}}(K)$, we choose the smallest K such that $\sum_{i=1}^{K} \hat{\lambda}_i / \sum_{i=1}^{12} \hat{\lambda}_i > 0.8$ following Berkes et al. [4]. If the test indicates any change point, then it is estimated by $\hat{k}^*$ (see (3.5)). The procedure is repeated until each segment has a constant mean function. The results are summarized in Table 6. The two change points, 1927 and 1993, detected by the SN-based test are fairly close to the change points, 1925 and 1992, identified in Berkes et al. [4]. The SN-based test suggests the mean function is stable over the period from 1780 to 1927, whereas Berkes et al.'s test detected two more change points, 1807 and 1849. However, the change at 1849 is not as obvious relative to the changes at 1925 and 1992 according to Figure 2 in Berkes et al. [4]. Of course, since it is not known whether there is a change point at 1849, either our SN-based test fails to reject due to its relatively lower power or Berkes et al.'s test falsely rejects due to its large upward size distortion. Nevertheless, our results suggest that the evidence for supporting the one change point in 1849 is weak. Figure 2 plots the mean function in each partition segments suggested by the SN-based test. It clearly shows the warming trend of the central England temperature. As mentioned in Berkes et al. [4], although it is not realistic to believe that the change happens abruptly in one year, in practice this modeling assumption is useful in identifying a potential trend of change.

5.2. *Analysis of the bias between gridded observations and GCM simulations*

Next, we apply the SN-based test to a gridded spatio-temporal temperature data set covering a subregion of North America. The data set comes from two separate sources: gridded observations generated from interpolation of station records (HadCRU), and gridded simulations generated by an AOGCM (NOAA GFDL CM2.1). Both datasets provide daily average temperature for the same 19-year period, 1980-1998 (see Delworth et al. [10] and Brohan et al. [7]). Each surface is viewed as a 2-d functional datum. We aim to test whether the bias or difference

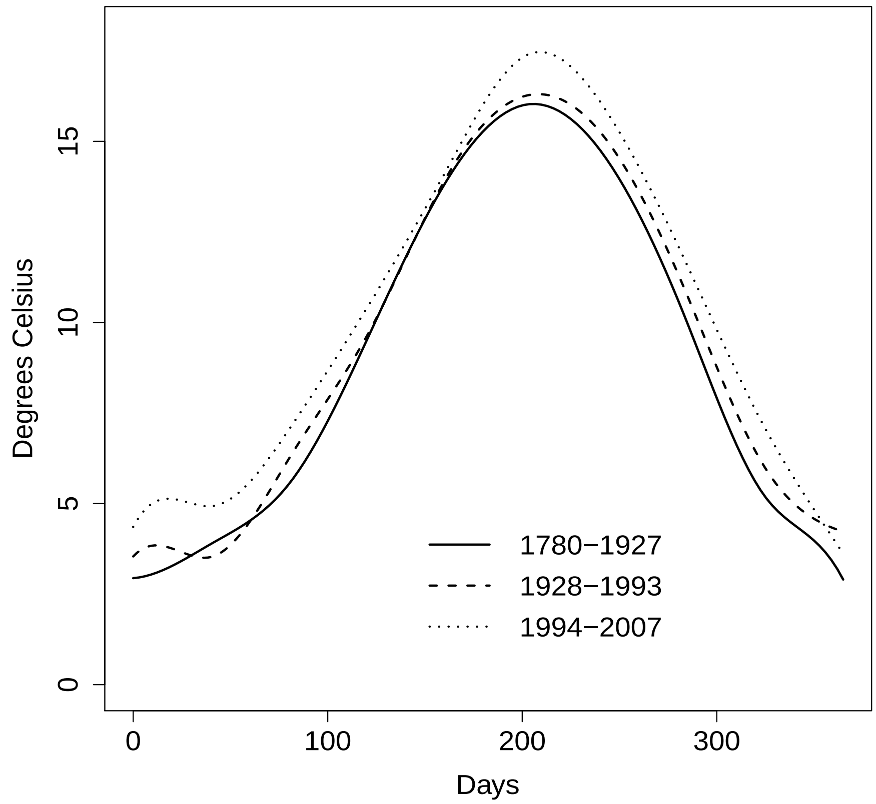

Fig 2. *Average daily temperature functions in the three estimated partition segments.*

between station observations and the model outputs is stable over the examined period (1980-1998). The data is first transformed to the same resolution (87×35) by bilinear interpolations. The bias is then computed by taking the difference of the two surfaces for each year and is converted to functional data through the thin plate spline. Figure 3 presents the first six PCs of the bias, which summarize the major patterns of the variability. The first PC which explains 60.4% of the total variation clearly dominates other types of variations. Although the first PC is negative over the whole region, it places more weight at the center of the domain than at the boundary. This indicates that a great amount of variability over a year will be found by the relatively heavy weights over the central region, which is relatively far from the ocean, with less contribution from the border area, which is close to the ocean. This is consistent with the physical reality that temperature variability is much higher over land than over ocean, and is much more sensitive to the land surface parameterization scheme used by the global model. Applying the SN-based test to test the mean change, we tabulate

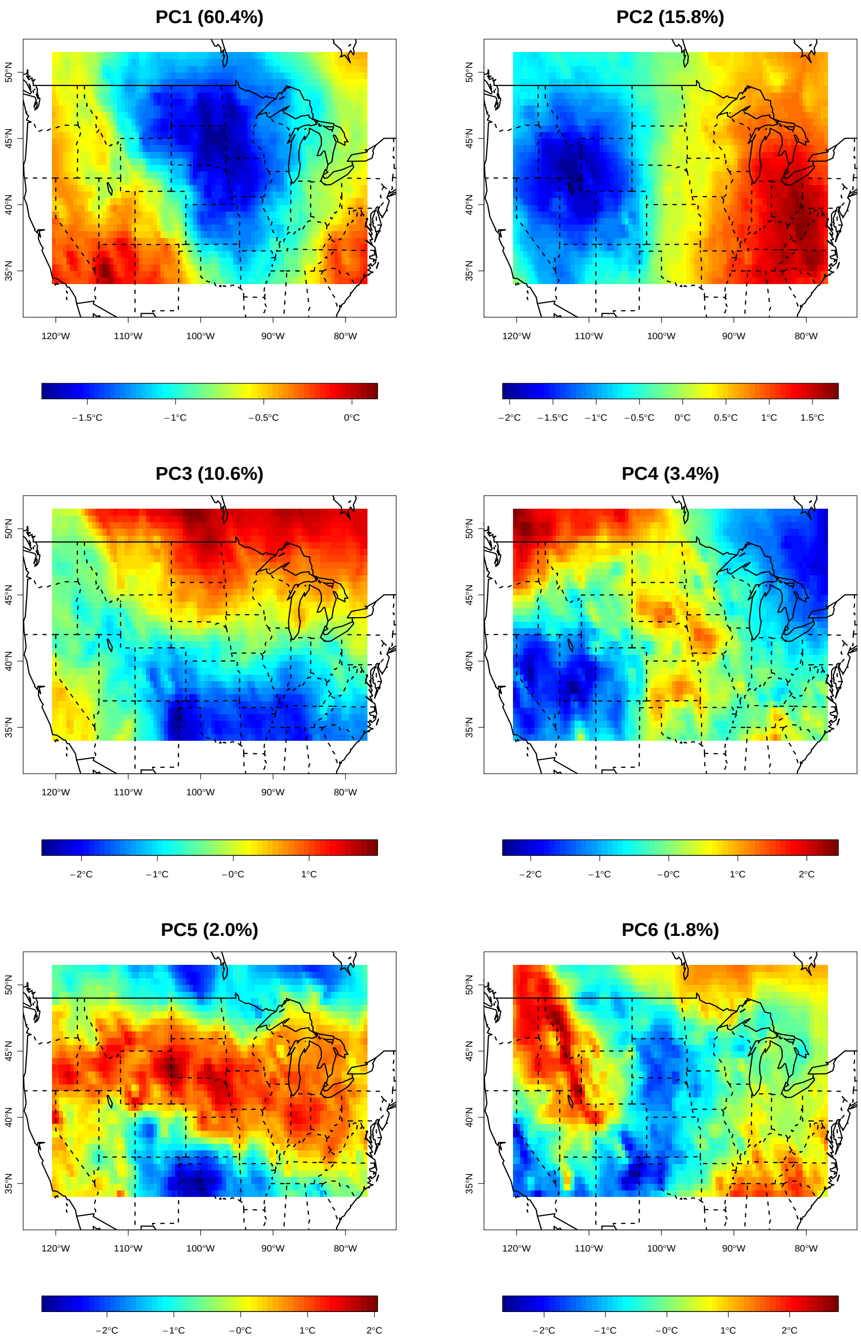

Fig 3. *The first six principal components of the bias between observations and model output. The number in the title of each figure stands for the percentage of variation explained by the corresponding principal component.*

Table 7

Test statistics and their p-values for the spatio-temporal temperature data covering a subregion of North America. Panel (a) shows the results of the tests based on the first K PCs; Panel (b) shows the results of the tests based on individual PC

(a)	K	$G_{N,\hat{\eta}}(K)$	p-value	Estimated change-point $\hat{k}^*$
	1	25.2	(0.1, 1)	—
	2	34.4	(0.1, 1)	—
	3	160.5	(0.005, 0.01)	1990
	4	182.7	(0.01, 0.025)	1991
	5	218.2	(0.025, 0.05)	1991
	6	221.9	(0.025, 0.05)	1991
(b)	PC	$G_{N,\hat{\eta}}$	p-value	Estimated change-point $\hat{k}^*$
	PC1	25.2	(0.1, 1)	—
	PC2	10.0	(0.1, 1)	—
	PC3	93.7	(0.001, 0.005)	1990
	PC4	3.1	(0.1, 1)	—
	PC5	2.5	(0.1, 1)	—
	PC6	3.6	(0.1, 1)	—

the results of the tests based on the first K PCs in Panel (a) of Table 7 and the results of the tests based on individual PCs in Panel (b) of Table 7. From Panel (a), we notice that when $K = 1$ or 2, the SN-based test does not detect any significant change points at the usual 5% significance level. When $K = 3$, the SN-based test with a p-value in the range $(0.005, 0.01)$ suggests that there is a change point at 1990. The results in Panel (b) are consistent with the finding from Panel (a) in that the test based on the third PC indicates a significant change point at 1990 but the tests based on other PCs do not detect any significant change points. The change in the bias is shown in Figure 4 by comparing the difference of the average biases in two periods (1980-1990; 1991-1998). It can be seen from Figure 4 that the bias tends to increase in the northern region while it decreases in the southern area. It is also worth nothing that the pattern of the change of the bias seems to be similar to that of the third PC as plotted in Figure 3.

6. Discussion and conclusion

In this article, SN-based tests have been developed to detect a change point in the functional mean and the lag-1 autocovariance operator of temporally dependent functional observations. The test statistic is constructed based on the estimated finite dimensional scores and the estimation effect turns out to be asymptotically negligible due to the special form of the SN-based statistic. The limiting null distribution of the SN-based statistic is nonstandard and its critical values have been tabulated by Shao and Zhang [23]. Compared to the existing tests developed for independent/dependent functional data, the SN-based test has some appealing features: 1) it is easy to implement and does not involve any bandwidth parameter; 2) it is shown to enjoy the monotonic power properties in the functional context; 3) our test, developed for temporally-dependent functional data, inherits the "better size but less power" property

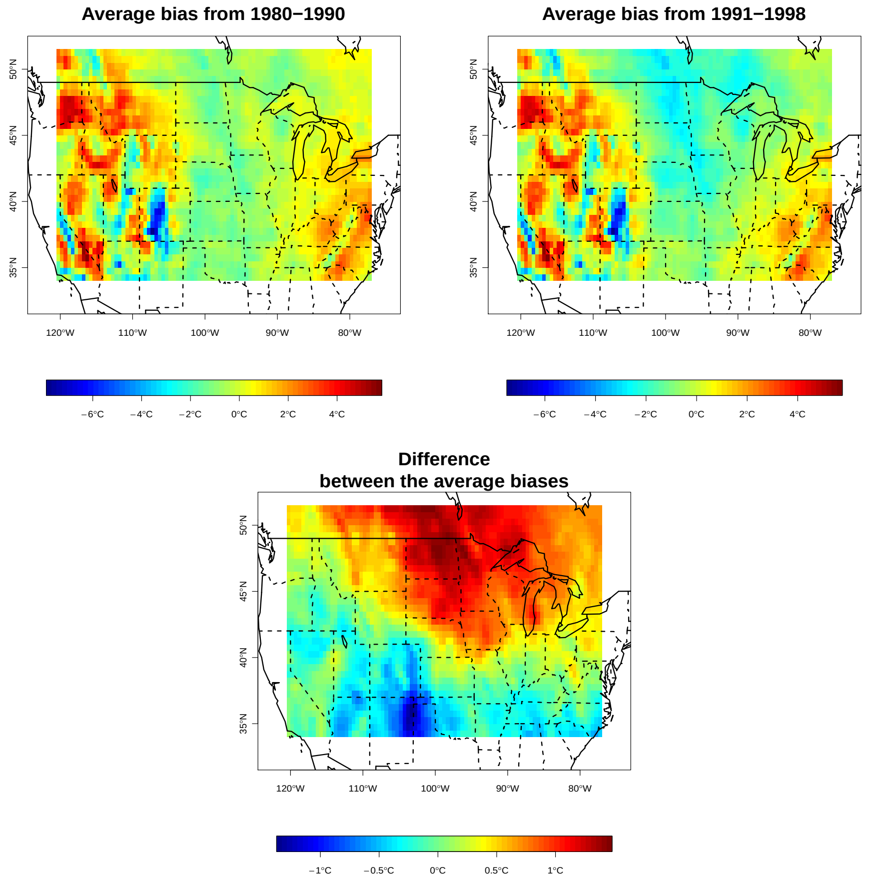

FIG 4. *Average biases in two periods (1980-1990; 1991-1998) and the change in the bias.*

of the SN-based test in the univariate setup. The finite sample performance is quite good and stable with respect to K. It can be readily applied to temporally-dependent functional curves and functional surfaces.

Our test statistic currently detects only one change point, but it can be further extended to the multiple change point alternative in a straightforward manner; see Shao and Zhang [23] section 2.3 for a discussion in the univariate setting. Furthermore, the SN-based test still requires a user-chosen parameter K which also appears in related work by Berkes et al. [4] and Hörmann and Kokoszka [13]. In practice, K can be chosen by $K = \inf\{J : \sum_{i=1}^{J} \hat{\lambda}_i / \sum_{i=1}^{m} \hat{\lambda}_i > \alpha\}$, where m is the number of basis functions in smoothing and α is a pre-specified number, say 85%. If the goal is to infer the low frequency behavior of the functional data, this ac-hoc method of choosing K should be practically useful and informative. On the other hand, if some high frequency behavior of the functional data is also of interest, then it is wise to let K dependent on the sample size N, which requires a modification of our asymptotic theory. This is beyond the scope of this article and will be investigated in the future.

7. Supplemental materials

We first introduce some useful notation. For $1 \le t_1 \le t_2 \le N$ and $A = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^{K\times N}$, let $S_N(t_1, t_2, A) = S_{N,A}(t_1, t_2) = \sum_{i=t_1}^{t_2} a_i$ with $a_i = (a_{i1}, a_{i2}, \ldots, a_{iK})' \in \mathbb{R}^K$. Let $\hat{\beta}_{ij} = \int_{\mathcal{I}} Y_i(t)\hat{\phi}_j(t)dt$, $\hat{\beta}_i = (\hat{\beta}_{i1}, \ldots, \hat{\beta}_{iK})'$ and $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_N) \in \mathbb{R}^{K\times N}$. Define $\eta = (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{K\times N}$ and its sample counterpart $\hat{\eta} = (\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_N) \in \mathbb{R}^{K\times N}$. Similarly, we can define $\xi \in \mathbb{R}^{K^2\times(N-1)}$ and its empirical counterparts $\hat{\xi}$ by replacing ξ_i with $\hat{\xi}_i$; see section 2.2. Let $|\cdot|$ be the Euclidian norm of a vector and $||\cdot||_M$ the matrix norm $||A||_M = \sup_{|x|\le 1}|Ax|$ for a matrix A. Denote C a generic constant which could be different from line to line.

Proof of Theorem 3.3. Define

$$G_{N,\eta}(K) = \mathcal{C}(N^{-1/2}S_N(1, \lfloor Nr\rfloor, \eta), r \in [0,1]).$$

Set $\hat{c}_i = \text{sign}(<\phi_i, \hat{\phi}_i>)$ and $\hat{\mathbf{C}} = \text{diag}(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_K)$. Following the arguments in Theorem 5.1 of Hörmann and Kokoszka [13], we can derive that $\{\eta_i\}$ is L^2-m-approximable. Hence by Theorem A.2 of Hörmann and Kokoszka [13], we have

$$N^{-1/2}S_N(1, \lfloor Nr\rfloor, \eta) \Rightarrow \Sigma_{\eta,K}^{1/2}\mathbf{B}_K(r).$$

Applying the continuous mapping theorem, we have that $G_{N,\eta}(K) \to^d G(K)$. Note that $\hat{\eta}_{i,j} = \hat{\beta}_{ij} - (1/N)\sum_{i=1}^N \hat{\beta}_{ij}$, for $j = 1, 2, \ldots, K$. Under the $H_{0,1}$, it is easy to see that both $T_{N,\hat{\eta}}(k, K)$ and $V_{N,\hat{\eta}}(k, K)$ remain the same if $\hat{\eta}_i$ is replaced by $\hat{\beta}_i$. Because of the quadratic form of $G_{N,\hat{\eta}}(K)$ and the simple fact that $\hat{\mathbf{C}}^2 = I_K$, the statistic $G_{N,\hat{\eta}}(K)$ does not change if $\hat{\eta}_i$ is replaced by $\hat{\mathbf{C}}\hat{\beta}_i$. Based on the result that

$$\sup_{r\in[0,1]} \frac{1}{\sqrt{N}} \left| S_N(1, \lfloor Nr\rfloor, \eta) - S_N(1, \lfloor Nr\rfloor, \hat{\mathbf{C}}\hat{\beta}) \right| = o_p(1),$$

which was stated in the proof of Theorem 5.1 of Hörmann and Kokoszka [13] (see equation A.9 therein), it is straightforward to see that the difference between $G_{N,\hat{\eta}}(K)$ and $G_{N,\eta}(K)$ is asymptotically negligible. Therefore the proof is complete. □

Proof of Theorem 3.4. Recall that $\xi_i = (\eta_{i,1}\eta_{i+1,1}, \ldots, \eta_{i,1}\eta_{i+1,K}, \ldots, \eta_{i,K}\eta_{i+1,1}, \ldots, \eta_{i,K}\eta_{i+1,K})'$. Define $\eta_{i,j}^{(m)} = \int_{\mathcal{I}} X_i^{(m)}(t)\phi_j(t)dt$, where $X_i^{(m)}$ is the m-dependent approximation of X_i (see Definition 3.1), and let $\xi_i^{(m)}$ be the counterpart of ξ_i by replacing $\eta_{i,j}$ with $\eta_{i,j}^{(m)}$ in ξ_i. Let $\gamma(t, s) = \mathbf{E}[X_1(t)X_2(s)]$ be the lag-1 autocovariance function. We first see that

$$\begin{aligned}
\mathbf{E}|\xi_1 - \xi_1^{(m)}|^2 &= \mathbf{E}\sum_{j=1}^{K}\sum_{l=1}^{K}(\eta_{1,j}\eta_{2,l} - \eta_{1,j}^{(m)}\eta_{2,l}^{(m)})^2 \\
&\le 2\left[\mathbf{E}\left\{\sum_{j=1}^{K}\sum_{l=1}^{K}(\eta_{1,j} - \eta_{1,j}^{(m)})^2\eta_{2,l}^2\right\}\right. \\
&\qquad \left. + \mathbf{E}\left\{\sum_{j=1}^{K}\sum_{l=1}^{K}(\eta_{2,l} - \eta_{2,l}^{(m)})^2(\eta_{1,j}^{(m)})^2\right\}\right] \\
&= 2(I_1 + I_2),
\end{aligned}$$

Note that $\mathbf{E}(\eta_{2,l}^4) = \mathbf{E}\left\{\int_{\mathcal{I}} X_2(t)\phi_l(t)dt\right\}^4 \le \mathbf{E}\{\int_{\mathcal{I}} X_2^2(t)dt\}^2 = \mathbf{E}||X_2||^4 < \infty$, where we have used the orthonormal property of $\phi_l(t)$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_1 &\le \left[\mathbf{E}\left\{\sum_{j=1}^{K}(\eta_{1,j} - \eta_{1,j}^{(m)})^2\right\}^2\right]^{1/2}\left\{\mathbf{E}\left(\sum_{l=1}^{K}\eta_{2,l}^2\right)^2\right\}^{1/2} \\
&\le C\left\{\sum_{j=1}^{K}\mathbf{E}(\eta_{1,j} - \eta_{1,j}^{(m)})^4\right\}^{1/2}\left(\sum_{l=1}^{K}\mathbf{E}\eta_{2,l}^4\right)^{1/2} \le C\left\{\sum_{j=1}^{K}\mathbf{E}(\eta_{1,j} - \eta_{1,j}^{(m)})^4\right\}^{1/2} \\
&= C\left[\sum_{j=1}^{K}\mathbf{E}\left\{\int_{\mathcal{I}}(X_1(t) - X_1^{(m)}(t))\phi_j(t)dt\right\}^4\right]^{1/2} = C\left(\mathbf{E}||X_1 - X_1^{(m)}||^4\right)^{1/2}
\end{aligned}$$

Similarly we get $I_2 \le C(\mathbf{E}||X_1 - X_1^{(m)}||^4)^{1/2}$. It follows that $(\mathbf{E}|\xi_1 - \xi_1^{(m)}|^2)^{1/2} \le Ce_4(X_1 - X_1^{(m)})$, which yields

$$\sum_{m=1}^{\infty}\left(\mathbf{E}|\xi_1 - \xi_1^{(m)}|^2\right)^{1/2} \le C\sum_{m=1}^{\infty}e_4(X_1 - X_1^{(m)}) < \infty.$$

Again using Theorem A.2 of Hörmann and Kokoszka [13], we know that $N^{-1/2}\{S_N(1,\lfloor Nr\rfloor,\xi) - \lfloor Nr\rfloor E[\xi_1]\} \Rightarrow \Sigma_{\xi,K^2}^{1/2}\mathbf{B}_{K^2}(r)$. Let

$$G_{N,\xi}(K^2) = \mathcal{C}(N^{-1/2}S_N(1,\lfloor Nr\rfloor,\xi), r \in [0,1]).$$

By the continuous mapping theorem we have that $G_{N,\xi}(K^2) \to^d G(K^2)$. Because of the form of $G_{N,\hat{\xi}}(K^2)$, it is sufficient to show that

$$\begin{aligned}
\sup_{r\in[0,1]}\frac{1}{\sqrt{N}}\Big| & S_N(1,\lfloor Nr\rfloor,\xi) - S_N(1,\lfloor Nr\rfloor,\hat{\mathbf{C}}_\xi\hat{\xi}) \\
& - \left\{\mathbf{E}S_N(1,\lfloor Nr\rfloor,\xi) - \mathbf{E}S_N(1,\lfloor Nr\rfloor,\hat{\mathbf{C}}_\xi\hat{\xi})\right\}\Big| = o_p(1),
\end{aligned}$$

where $\hat{\mathbf{C}}_\xi = \text{diag}(\hat{c}_1\hat{c}_1, \ldots, \hat{c}_1\hat{c}_K, \hat{c}_2\hat{c}_1, \ldots, \hat{c}_K\hat{c}_K)$. The claim follows provided that for any $1 \le j, l \le K$,

$$I_{j,l} = \frac{1}{\sqrt{N}} \sup_{r\in[0,1]} \left| \sum_{i=1}^{\lfloor Nr \rfloor} \{\eta_{i,j}\eta_{i+1,l} - \mathbf{E}(\eta_{i,j}\eta_{i+1,l}) - \hat{c}_j\hat{c}_l\hat{\eta}_{i,j}\hat{\eta}_{i+1,l} + \mathbf{E}(\hat{c}_j\hat{c}_l\hat{\eta}_{i,j}\hat{\eta}_{i+1,l})\} \right|$$
$$= \frac{1}{\sqrt{N}} \sup_{r\in[0,1]} \left| \int_{\mathcal{I}}\int_{\mathcal{I}} \sum_{i=1}^{\lfloor Nr \rfloor} \{X_i(t)X_{i+1}(s) - \gamma(t,s)\}\hat{u}_{j,l}(t,s)dtds \right| = o_p(1),$$

where $\hat{u}_{j,l}(t,s) = \phi_j(t)\phi_l(s) - \hat{c}_j\hat{c}_l\hat{\phi}_j(t)\hat{\phi}_l(s)$. Observing that $||\hat{u}_{j,l}||^2 \le 2\{||\phi_j - \hat{c}_j\hat{\phi}_j||^2 + ||\phi_l - \hat{c}_l\hat{\phi}_l||^2\}$ and using the fact that $||\phi_j - \hat{c}_j\hat{\phi}_j|| = O_p(N^{-1/2})$ (see Theorem 3.2 in Hörmann and Kokoszka [13]), we derive $||\hat{u}_{j,l}|| = O_p(N^{-1/2})$. Since

$$I_{j,l} \le \frac{||\hat{u}_{j,l}||}{\sqrt{N}} \sup_{r\in[0,1]} \left[\int_{\mathcal{I}}\int_{\mathcal{I}} \left\{ \sum_{i=1}^{\lfloor Nr \rfloor} (X_i(t)X_{i+1}(s) - \gamma(t,s)) \right\}^2 dtds \right]^{1/2},$$

the conclusion follows from Lemma 7.1. □

Lemma 7.1. *Assume that $X_i \in L^8_{\mathbb{H}}$ and $\{X_i\}$ is L^8-m-approximable. Then under $H_{0,2}$, we have*

$$\frac{1}{N^2} \sup_{r\in[0,1]} \left[\int_{\mathcal{I}}\int_{\mathcal{I}} \left\{ \sum_{i=1}^{\lfloor Nr \rfloor} (X_i(t)X_{i+1}(s) - \gamma(t,s)) \right\}^2 dtds \right] = o_p(1).$$

Proof of Lemma 7.1. Let $Z_i(t,s) = X_i(t)X_{i+1}(s) - \gamma(t,s)$. We first show that the process $\{Z_i(t,s)\}$ is L^4-m-approximable in the Hilbert space of integrable functions defined on $[0,1]^2$. Note that

$$\begin{aligned}\mathbf{E}||Z_1 - Z_1^{(m)}||^4 &= \mathbf{E}\left\{ \int_{\mathcal{I}}\int_{\mathcal{I}} (X_1(t)X_2(s) - X_1^{(m)}(t)X_2^{(m)}(s))^2 dtds \right\}^2 \\ &\le C\mathbf{E}\left\{ ||X_1||^2||X_2 - X_2^{(m)}||^2 + ||X_2^{(m)}||^2||X_1 - X_1^{(m)}||^2 \right\}^2 \\ &\le C\Big\{ (\mathbf{E}||X_1||^8)^{1/2}(\mathbf{E}||X_2 - X_2^{(m)}||^8)^{1/2} \\ &\quad + (\mathbf{E}||X_2||^8)^{1/2}(\mathbf{E}||X_1 - X_1^{(m)}||^8)^{1/2} \Big\}.\end{aligned}$$

Thus we get $e_4(Z_1 - Z_1^{(m)}) \le C\{e_8(X_1 - X_1^{(m)}) + e_8(X_2 - X_2^{(m)})\}$, which, along with the assumption that $\{X_i\}$ is L^8-m-approximable, implies $\{Z_i\}$ is L^4-m-approximable. The rest of the proof essentially follows the argument in the proof of Theorem 5.1 in Hörmann and Kokoszka [13]. We omit the details here. □

Proof of Proposition 3.5. Define $\hat{\mathbf{\Delta}}_K = (< \Delta, \hat{\phi}_1 >, < \Delta, \hat{\phi}_2 >, \dots, < \Delta, \hat{\phi}_K >)'$. Let $\alpha_{ij} = \int_{\mathcal{I}} Y_i(t) v_j(t) dt$, $\alpha_i = (\alpha_{i1}, \dots, \alpha_{iK})'$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$. Following lemma A.1 in Berkes et al. [4], we have that under the local alternatives,

$$\hat{c}(t,s) = \frac{1}{N}\sum_{i=1}^{N}\{Y_i(t) - \bar{Y}_N(t)\}\{Y_i(s) - \bar{Y}_N(s)\} + \frac{k^*(N-k^*)}{N^2}\Delta(t)\Delta(s) + r_N(t,s),$$

where

$$\begin{aligned} r_N(t,s) = &\left(\frac{N-k^*}{N^2}\right)\sum_{i=1}^{k^*}\left[\{Y_i(t) - \bar{Y}_N(t)\}\Delta(t) + \{Y_i(s) - \bar{Y}_N(s)\}\Delta(s)\right] \\ &+ \frac{k^*}{N^2}\sum_{i=k^*+1}^{N}\left[\{Y_i(t) - \bar{Y}_N(t)\}\Delta(t) + \{Y_i(s) - \bar{Y}_N(s)\}\Delta(s)\right], \end{aligned}$$

is the remainder term. It can be shown that $||r_N(t,s)||^2 = \int_{\mathcal{I}}\int_{\mathcal{I}} r_N^2(t,s) dt ds = o_p(1)$ by using the ergodic theorem for L^p-m-approximable process $Y_i(t)$ (note that $Y_i(t)$ admits the ergodic representation $Y_i(t) = f(\varepsilon_i(t), \varepsilon_{i-1}(t), \dots))$. It follows that $||\hat{c}(t,s) - \tilde{c}(t,s)|| = o_p(1)$. By Lemma 4.3 in Bosq [6], we get $||\hat{c}_i\hat{\phi}_i - v_i|| = o_p(1)$ for all $1 \le i \le K$. Using this fact and similar arguments in the proof of Theorem 5.1 in Hörmann and Kokoszka [13], we get the following two results:

$$N^{-1/2} S_N(1, \lfloor Nr \rfloor, \alpha) \text{ satisfies the functional central limit theorem;} \tag{7.1}$$

$$\sup_{r\in[0,1]} \frac{1}{\sqrt{N}}\left|S_N(1, \lfloor Nr \rfloor, \hat{\mathbf{C}}\hat{\beta}) - S_N(1, \lfloor Nr \rfloor, \alpha)\right| = o_p(\log\log N), \tag{7.2}$$

which imply that

$$\sup_{r\in[0,1]} \frac{1}{\sqrt{N}}\left|S_N(1, \lfloor Nr \rfloor, \hat{\mathbf{C}}\hat{\beta})\right| = o_p(\log\log N). \tag{7.3}$$

Simple calculation shows that

$$T_{N,\hat{\eta}}(k^*, K) = \frac{1}{\sqrt{N}}\left(\sum_{i=1}^{k^*}\hat{\beta}_i - \frac{k^*}{N}\sum_{i=1}^{N}\hat{\beta}_i\right) + \frac{k^*(N-k^*)}{N^{3/2}}\hat{\mathbf{\Delta}}_K.$$

Notice that $|\hat{\mathbf{C}}\hat{\mathbf{\Delta}}_K - \mathbf{\Delta}_K| = o_p(1)$ provided that $|\hat{c}_i\hat{\phi}_i - v_i| = o_p(1)$. By (7.3), we get

$$\begin{aligned} |\hat{\mathbf{C}}T_{N,\hat{\eta}}(k^*, K)| = &\left|\frac{1}{\sqrt{N}}\left(\sum_{i=1}^{k^*}\hat{\beta}_i - \frac{k^*}{N}\sum_{i=1}^{N}\hat{\beta}_i\right) + \frac{k^*(N-k^*)}{N^{3/2}}(\hat{\mathbf{C}}\hat{\mathbf{\Delta}}_K - \mathbf{\Delta}_K)\right. \\ &\left. + \frac{k^*(N-k^*)}{N^{3/2}}\mathbf{\Delta}_K\right| = O_p(N^{1/2}). \end{aligned} \tag{7.4}$$

On the other hand, it is not hard to see that

$$\begin{aligned}
&\left\|\frac{1}{N^2}\sum_{t=1}^{k^*}\left\{\hat{\mathbf{C}}S_{N,\hat{\eta}}(1,t)-\frac{t}{k^*}\hat{\mathbf{C}}S_{N,\hat{\eta}}(1,k^*)\right\}\right.\\
&\qquad\left.\times\left\{\hat{\mathbf{C}}S_{N,\hat{\eta}}(1,t)-\frac{t}{k^*}\hat{\mathbf{C}}S_{N,\hat{\eta}}(1,k^*)\right\}'\right\|_M\\
=&\left\|\frac{1}{N^2}\sum_{t=1}^{k^*}\left\{S_N(1,t,\hat{\mathbf{C}}\hat{\beta})-\frac{t}{k^*}S_N(1,k^*,\hat{\mathbf{C}}\hat{\beta})\right\}\right.\\
&\qquad\left.\times\left\{S_N(1,t,\hat{\mathbf{C}}\hat{\beta})-\frac{t}{k^*}S_N(1,k^*,\hat{\mathbf{C}}\hat{\beta})\right\}'\right\|_M\\
\leq&\frac{K}{N^2}\sum_{t=1}^{k^*}\left|\left\{S_N(1,t,\hat{\mathbf{C}}\hat{\beta})-\frac{t}{k^*}S_N(1,k^*,\hat{\mathbf{C}}\hat{\beta})\right\}\right|^2=o_p((\log\log N)^2). \qquad (7.5)
\end{aligned}$$

Here we used the inequality that $||ab'||_M \leq K\max_{1\leq i,j\leq K}\{a_ib_j\} \leq K|a||b|$ for $a,b\in\mathbb{R}^K$. Similarly, we can prove the same result for the second term in $\hat{\mathbf{C}}V_{N,\hat{\eta}}(k^*,K)\hat{\mathbf{C}}'$ (see (2.9)). Therefore we have $||\hat{\mathbf{C}}V_{N,\hat{\eta}}(k^*,K)\hat{\mathbf{C}}'||= o_p((\log\log N)^2)$. Along with the fact that $||\hat{\mathbf{C}}T_{N,\hat{\eta}}(k^*,K)||=O_p(N^{1/2})$, we get

$$\begin{aligned}
G_{N,\hat{\eta}}(K) &\geq T_{N,\hat{\eta}}(k^*,K)'V^{-1}_{N,\hat{\eta}}(k^*,K)T_{N,\hat{\eta}}(k^*,K)\\
&\geq |T_{N,\hat{\eta}}(k^*,K)|^2/||V_{N,\hat{\eta}}(k^*,K)||_M,
\end{aligned}$$

where the right hand side diverges to infinity as $N\to+\infty$. Note the second inequality comes from the fact that $a'A^{-1}a\geq |a|^2/||A||_M$ for a vector $a\in\mathbb{R}^K$ and an invertible matrix $A\in\mathbb{R}^{K\times K}$. □

Proof of Proposition 3.6. Note that $||\tilde{c}^{(N)}(t,s)-c(t,s)||=o(1)$ under the assumption that $||\Delta^{(N)}||=o(1)$. Following the arguments in proof of Proposition 3.5, we can show that $||\hat{c}(t,s)-c(t,s)||=o_p(1)$ and hence $||\hat{c}_i\hat{\phi}_i-\phi_i||=o_p(1)$ for all $1\leq i\leq K$. Using similar arguments in the proof of Theorem 5.1 in Hörmann and Kokoszka [13], we get

$$\begin{aligned}
&\sup_{r\in[0,1]}\frac{1}{\sqrt{N}}\left|S_N(1,\lfloor Nr\rfloor,\hat{\mathbf{C}}\hat{\beta})\right|\\
\leq&\sup_{r\in[0,1]}\frac{1}{\sqrt{N}}\left|S_N(1,\lfloor Nr\rfloor,\hat{\mathbf{C}}\hat{\beta})-S_N(1,\lfloor Nr\rfloor,\eta)\right|\\
&+\sup_{r\in[0,1]}\frac{1}{\sqrt{N}}|S_N(1,\lfloor Nr\rfloor,\eta)|=o_p(\log\log N).
\end{aligned}$$

Note that $|\hat{\mathbf{C}}\hat{\mathbf{\Delta}}_K^{(N)}-\mathbf{\Delta}_K^{(N)}|=o_p(|\mathbf{\Delta}_K^{(N)}|)$ provided that $||\Delta^{(N)}||=O(|\mathbf{\Delta}_K^{(N)}|)$ and $||\hat{c}_i\hat{\phi}_i-v_i^{(N)}||\leq||\hat{c}_i\hat{\phi}_i-\phi_i||+||\phi_i-v_i^{(N)}||=o_p(1)$. Under the assumption that $\liminf_{N\to\infty}\frac{N^{1/2}|\mathbf{\Delta}_K^{(N)}|}{\log\log N}>0$, we have $|\hat{\mathbf{C}}T_{N,\hat{\eta}}(k^*,K)|=O_p(N^{1/2}|\mathbf{\Delta}_K^{(N)}|)$

and $||\hat{\mathbf{C}}V_{N,\hat{\eta}}(k^*,K)\hat{\mathbf{C}}'||_M = o_p((\log\log N)^2)$ (see (7.4) and (7.5)). Therefore we get

$$G_{N,\hat{\eta}}(K) \geq |T_{N,\hat{\eta}}(k^*,K)|^2/||V_{N,\hat{\eta}}(k^*,K)||_M,$$

where the right hand side diverges to infinity as $N \to +\infty$. □

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