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HIGH LEVEL EXCURSION SET GEOMETRY FOR NON-GAUSSIAN INFINITELY DIVISIBLE RANDOM FIELDS

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We consider smooth, infinitely divisible random fields $(X(t), t \in M)$, $M \subset \mathbb{R}^d$, with regularly varying Lévy measure, and are interested in the geometric characteristics of the excursion sets

$$A_u = \{t \in M : X(t) > u\}$$

over high levels u.

For a large class of such random fields, we compute the $u \to \infty$ asymptotic joint distribution of the numbers of critical points, of various types, of X in A_u, conditional on A_u being nonempty. This allows us, for example, to obtain the asymptotic conditional distribution of the Euler characteristic of the excursion set.

In a significant departure from the Gaussian situation, the high level excursion sets for these random fields can have quite a complicated geometry. Whereas in the Gaussian case nonempty excursion sets are, with high probability, roughly ellipsoidal, in the more general infinitely divisible setting almost any shape is possible.

1. Introduction. Let $(X(t), t \in M)$, where M is a compact set in $\mathbb{R}^d$ of a kind to be specified later, be a smooth infinitely divisible random field. We shall assume, again in a sense that we shall make precise later, that X has regularly varying tails. Note that this means that the tails of X are heavier than exponential and, in particular, heavier than those of a Gaussian random field. Nevertheless, the model we are considering allows both heavy tails (e.g., infinite mean or variance) and light tails, in the sense of the existence of finite moments of arbitrary given order.

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We are interested in studying the excursions of the random field over levels $u > 0$, particularly when the level u becomes high. Writing

$$A_u \equiv A_u(X, M) \triangleq \{t \in M : X(t) > u\} \tag{1}$$

for the excursion set of X over the level u, we shall study the geometric characteristics of A_u under the condition that it is not empty, that is, under the condition that the level u is, in fact, exceeded. In particular, we shall be interested in computing the conditional limit distribution of the Euler characteristic of A_u as $u \to \infty$. We refer the reader to [1] for a recent detailed exposition of the geometric theory of the excursion sets of smooth Gaussian and related random fields, and to [2] for applications of the theory.

In a significant departure from the well-understood Gaussian situation, the excursion sets over high levels for the random fields in this paper can have quite a complicated geometry. In the Gaussian case excursion sets, unless they are empty, tend, with high probability, to contain a single component which is almost ellipsoidal in shape, and so have an Euler characteristic equal to one. In contrast, the Euler characteristics of the excursion sets in our fields can have highly nondegenerate conditional distributions. As a consequence, these models are sufficiently flexible to open the possibility of fitting empirically observed excursion sets with widely different geometric characteristics. This, more statistical, problem is something we plan to tackle in the future.

The main result of the paper is Theorem 3.1. While it is rather too technical to summarize here in full, here is the beginning of a special case. Suppose that $N_X(i, u)$ is the number of critical points of X in A_u of index i. Thus, if $d = 2$, $N_X(0, u)$ is the number of local minima of X above the level u in the interior of M, $N_X(1, u)$ the number of saddle points and $N_X(2, u)$ the number of local maxima, all above the level u. Then Theorem 3.1 gives an explicit expression for the limiting joint distribution

$$\lim_{u\to\infty} \mathbb{P}\{N_X(i, u) = n_i, i = 0, \dots, d, | A_u \neq \varnothing\}, \tag{2}$$

when M is the unit cube $I_d \triangleq [0, 1]^d$.

In fact, Theorem 3.1 goes far beyond this, since it includes not only these critical points, but also the critical points of X restricted to the various boundaries of I_d (i.e., faces, edges, etc.). The importance of this result lies in the fact that Morse theory shows how to use the full collection of these critical points to describe much of the geometry of A_u, whether this geometry be algebraic, integral, or differential.

Furthermore, Theorem 3.1 can also be exploited to describe a very simple stochastic model for high level excursion sets, as well as to develop a simple algorithm for simulating them.

An important point to note is that although Theorem 3.1 is stated only for M the unit cube, it is "obvious" from the proof that the result holds in much higher generality. For example, only trivial changes to the proof are needed to establish the result for convex polytopes. A little more effort will establish a version for convex M with smooth boundary. We also claim—without proof—that Theorem 3.1, properly reformulated, will continue to hold for locally convex, C^2, Whitney stratified manifolds of the kind treated in [1]. However, in this case the additional details that would need to be added to provide a complete proof would take more space than justified. Thus, while we shall continue to write M for our parameter set, indicating a level of generality, throughout the remainder of this paper we shall treat only the case $M = I_d$.

The remainder of the paper begins in Section 2, where we define our model, discuss the smoothness assumptions we are imposing, as well as those related to the regular variation of the tails. Section 3 contains the main result of the paper, on the joint distribution of the numbers of high level critical points of infinitely divisible random field's. This is followed with one of its main consequences, the distribution of the Euler characteristic of high level excursion sets, in Section 4. In Section 5, we introduce a class of moving average infinitely divisible random fields and derive conditions under which the main result of the Section 3 applies to them. We also provide examples to show that, by choosing appropriately the parameters of the model, one can make the geometric structure of the high level excursion sets either "Gaussian-like" or "non-Gaussian-like." Finally, Section 6 contains the proof of the main theorem.

Throughout the paper, C stands for finite positive constants whose precise value is irrelevant and which may change from line to line.

2. Smooth infinitely divisible random fields and regular variation. In this section, we shall define the random fields of interest to us, describe their distributional structure, and then specify the smoothness assumptions necessary for studying the geometry of their excursion sets.

A reader familiar with the theory of infinitely divisible processes will note that the route we take goes back to first principles to some extent (e.g., it would be more standard, nowadays, to start with the function space Lévy measure λ_X of Section 2.3 rather than invest a couple of pages in defining it). The need for this, as should become clear below, is to be able to carefully define random fields, along with their first and second order partial derivatives, on a common probability space.

2.1. *Probabilistic structure of infinitely divisible random fields.* As a first step, we shall need to define our random fields on a region slightly larger than the basic parameter space M, and so, in a notation that will remain fixed throughout the paper, we take $\widetilde{M}$ be a bounded open set in $\mathbb{R}^d$, with $M \subset \widetilde{M}$.

We now consider infinitely divisible random fields of the form

$$X(t) = \int_S f(s;t)\mu(ds), \qquad t \in \widetilde{M}, \tag{3}$$

where $(S, \mathcal{S})$ is a measurable space and μ is an infinitely divisible random measure on S with characteristics defined below. (We refer you to [11] for more information on infinitely divisible random measure s and stochastic integrals with respect to these measures.)

The infinitely divisible random measure μ, which we shall define in a moment, is characterized by its "generating triple" (γ, F, β). Here, γ is a σ-finite measure on $(S, \mathcal{S})$, and plays the role of the variance measure for the Gaussian part of μ. More important for us is the Lévy measure F, which is a σ-finite measure on $S \times (\mathbb{R} \setminus \{0\})$, equipped with the product σ-field. Finally, β is a signed measure on $(S, \mathcal{S})$, which plays the role of the shift measure for μ. Denote by $\mathcal{S}_0$ the collection of sets B in $\mathcal{S}$ for which

$$\gamma(B) + \|\beta\|(B) + \int_{\mathbb{R}\setminus\{0\}} [\![x]\!]^2 F(B, dx) < \infty,$$

where $\|\beta\|$ is the total variation norm of β and

$$[\![x]\!] = \begin{cases} x, & \text{if } |x| \le 1, \\ \operatorname{sign}(x), & \text{otherwise.} \end{cases}$$

With all elements of the triple defined, we can now define the infinitely divisible random measure $(\mu(B), B \in \mathcal{S}_0)$ as a stochastic process for which, for every sequence of disjoint $\mathcal{S}_0$-sets $B_1, B_2, \ldots$, the random variables $\mu(B_1), \mu(B_2), \ldots$ are independent (i.e., μ is independently scattered) and if, in addition, $\bigcup_n B_n \in \mathcal{S}_0$, then $\mu(\bigcup_n B_n) = \sum_n \mu(B_n)$ a.s. (i.e., μ is σ-additive). Finally, for every $B \in \mathcal{S}_0$, $\mu(B)$ is an infinitely divisible random variable with characteristic function given by

$$\mathbb{E}\{e^{i\theta\mu(B)}\} = \exp\left\{-\frac{1}{2}\gamma(B)\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta[\![x]\!]) F(B, dx) + i\theta\beta(B)\right\}$$

for $\theta \in \mathbb{R}$. The monograph [18] can be consulted for information on infinitely divisible random variable's.

We shall assume (without loss of generality) that the Lévy measure F has the form

$$F(A) = \int_S \rho(s; A_s) m(ds), \tag{4}$$

for each measurable $A \subset S \times (\mathbb{R} \setminus \{0\})$, where $A_s = \{x \in \mathbb{R} \setminus \{0\} : (s, x) \in A\}$ is the s-section of the set A. In (4), m is a σ-finite measure on $(S, \mathcal{S})$ (the *control measure* of μ), and the measures $(\rho(s; \cdot))$ (the *local Lévy measures*) form a family of Lévy measures on $\mathbb{R}$ such that for every Borel set $C \subset \mathbb{R} \setminus \{0\}$, $s \to \rho(s; C)$ is a measurable function on S. We can, and shall, choose the control measure m in (4) in such a way that $\|\beta\|$ is absolutely continuous with respect to m, and define the Radon–Nikodym derivative $b = d\beta/dm$. The local Lévy measures ρ, which, intuitively, control the Poisson structure of the random measure μ around different points of the space S, will play a central role in all that follows.

Note that while it is possible, and common, to choose m in with the added feature that γ is also absolutely continuous with respect to m, and that $\rho(s; \mathbb{R} \setminus$

$\{0\}) > 0$ on a set of $s \in S$ of full measure m, we shall not require this and so shall not do so.

Finally, we assume that the kernel $f(s;t), s \in S, t \in \widetilde{M}$, in (3) is deterministic and real, such that, for every $t \in \widetilde{M}$, the mapping $f(\cdot;t) : S \to \mathbb{R}$ is measurable, and that the following three inequalities hold:

$$\int_S f(s;t)^2 \gamma(ds) < \infty, \tag{5}$$

$$\int_S \int_{\mathbb{R}\setminus\{0\}} [\![xf(s;t)]\!]^2 F(ds, dx) < \infty \tag{6}$$

and

$$\int_S \left| b(s) f(s;t) + \int_{\mathbb{R}\setminus\{0\}} \big([\![xf(s;t)]\!] - [\![x]\!] f(s;t)\big) \rho(s;dx) \right| m(ds) < \infty. \tag{7}$$

These conditions guarantee that the random field $(X(t), t \in \widetilde{M})$ in (3) is well defined.

A particularly simple, but rather useful, example of this setup is studied in Section 5 below, when X is a moving average random field. In this example, both γ and β components of the generating triple vanish, so, in particular, the random field has no Gaussian component. Furthermore, $S = \mathbb{R}^d$, the control measure m is Lebesgue, and the local Lévy measures $\rho(s, \cdot)$ are independent of s. Finally, the kernel function f is of the form $f(s,t) = g(s+t)$ for some suitable g, and so the random field is given by

$$X(t) = \int_{\mathbb{R}^d} g(s+t) \mu(ds), \qquad t \in \widetilde{M} \subset \mathbb{R}^d. \tag{8}$$

The random measure μ has, in this case, the stationarity property $\mu(A) \stackrel{\mathcal{L}}{=} \mu(t + A)$ for all Borel A of a finite Lebesgue measure and $t \in \mathbb{R}^d$, which immediately implies that a moving average random field is stationary. An impatient reader, who already wants to see results without wading through technicalities, might want to now skip directly to Section 5.2 to see what our results have to say for moving averages.

Returning to the model (3), note that it has been defined in considerable generality, so as to allow for as wide a range of applications as possible. For example, we retain the Gaussian component of the random field X. However, the tail assumptions imposed below will have the effect of ensuring that the Gaussian component will not play a role in the geometric structure of high level excursion sets.

2.2. *Regularity properties.* We shall require that the sample paths of X satisfy a number of regularity properties for the theory we are developing to hold. The main assumption will be that the paths of X are a.s. C^2, for which good sufficient conditions exist. The secondary assumptions require a little more regularity, centered around the notion of *Morse functions*. For more details, including for the case of stratified manifolds, see Chapter 9 in [1].

We need a little notation. With $M = I_d$, we write $\partial_k M$ for the collection of the $2^{d-k}\binom{d}{k}$ k-dimensional open faces of M. Thus, for example, $\partial_d M$ is the interior of M, and $\partial_0 M$ the collection of 2^d vertices.

Next, recall that if $\widetilde{M}$ is an open neighborhood of M, a function $f : \widetilde{M} \to \mathbb{R}$ is called a Morse function on M if it satisfies the following two conditions on each $\partial_k M$, $k = 0, \ldots, d$:

(i) $f_{|\partial_k M}$ it is nondegenerate on $\partial_k M$, in the sense that the determinant of the Hessian of $f_{|\partial_k M}$ at its critical points does not vanish.

(ii) The restriction of f to $\overline{\partial_k M} = \bigsqcup_{j=0}^{k} \partial_j M$ has no critical points on $\bigsqcup_{j=0}^{k-1} \partial_j M$.

Here is our first, and henceforth ubiquitous, assumption.

ASSUMPTION 2.1. On an event of probability 1, the random field X has C^2 sample paths on $\widetilde{M}$ and is a Morse function on M.

Sufficient conditions for Assumption 2.1 to hold are not hard to come by. As far as the C^2 assumption is concerned, it suffices to treat the Gaussian and non-Gaussian components of X separately. For the Gaussian part, there is a rich and easy to apply theory, and Section 1.4.2 of [1] covers what is needed here.

Necessary and sufficient conditions for the C^2 assumption on the non-Gaussian component are not known, but a number of sufficient conditions exist. It is not our goal in this paper to develop the best possible conditions of this sort, so we restrict ourselves to one situation that covers, nonetheless, a wide range of random fields. Specifically, we shall assume that the γ and β components in the generating triple of the infinitely divisible random measure M vanish, and that the local Lévy measures ρ in (4) are symmetric; that is, $\rho(s; -A) = \rho(s; A)$ for each $s \in S$ and each Borel $A \in \mathbb{R} \setminus \{0\}$. That is, μ is a symmetric infinitely divisible random measure without a Gaussian component.

The following result gives sufficient conditions for a symmetric infinitely divisible random field without a Gaussian component to have sample functions in C^2. The proof is not difficult, and so is left to the reader. (The conditions are also necessary after a slight tightening of the assumptions on the null sets involved, cf. Theorem 5.1 of [4].)

THEOREM 2.2. *For a symmetric random field of the form* (3), *with μ an infinitely divisible random measure without a Gaussian component*, *suppose that the kernel $f : S \times \widetilde{M} \to \mathbb{R}$ is* (*product*)-*measurable. Assume that for every $s \in S$ outside of set of zero m-measure the function $f(s; \cdot) : \widetilde{M} \to \mathbb{R}$ is C^2. Furthermore*, *assume that the partial derivatives*

$$f_i(s;t) = \frac{\partial f}{\partial t_i}(s;t), \qquad i = 1, \dots, d,$$

$$f_{ij}(s;t) = \frac{\partial^2 f}{\partial t_i\, \partial t_j}(s;t), \qquad i, j = 1, \dots, d,$$

satisfy the following conditions:

(i) *The integrability condition* (6) *holds when the kernel* $f(s;t)$ *there is replaced by any of the* $f_i(s;t)$ *or* $f_{ij}(s;t)$.

(ii) *The random fields*

$$X_{ij}(t) = \int_S f_{ij}(s;t)\mu(ds), \qquad t \in \widetilde{M}, \tag{9}$$

$i, j = 1, \dots, d$, *are all sample continuous.*

Then the random field $(X(t), t \in \widetilde{M})$ *has* (*a version with*) *sample functions in* C^2.

Thus, in searching for sufficient conditions for the a.s. second order differentiability of X, it suffices to establish the continuity of the random fields of (9). While there are no known necessary and sufficient conditions for sample continuity of general infinitely divisible random field's, various sufficient conditions are available. See, for example, Chapter 10 of [17] for the special case of stable random fields, or [9] for some other classes of infinitely divisible random fields.

This is as far as we shall go at the moment discussing the issue of differentiability in Assumption 2.1. Conditions sufficient for X to be a Morse function, also required in this assumption, are, in principle, available as well. For example, it follows from the arguments of Section 11.3 of [1] (cf. Theorem 11.3.1 there) that a C^2 field X will also be, a.s., a Morse function on the unit cube I_d if the following two conditions are satisfied, for each face J of I_d, and for all $t \in J$:

(i) The marginal densities $p_t(x)$ of $\nabla X_{|J}(t)$ are continuous at 0, uniformly in t.

(ii) The conditional densities $p_t(z|x)$ of $Z = \det \nabla^2 X_{|J}(t)$ given $\nabla X_{|J}(t) = x$ are continuous in (z, x) in a neighbourhood of 0, uniformly in t.

It does not seem to be trivial to translate the above conditions into general conditions on the kernel f and the triple (γ, F, β), and we shall not attempt to do so in this paper. On the other hand, given a specific kernel and triple, they are generally not too hard to check. In the purely Gaussian case, simple sufficient conditions are provided by Corollary 11.3.2 of [1], but it is the more involved infinitely divisible case that is at the heart of the current paper. If the latter random field is, actually a so-called type-G random field (see [15]) (symmetric α-stable random fields, $0 < \alpha < 2$ are a special case of type-G random fields), then these fields can be represented as mixtures of centered Gaussian random fields, and Corollary 11.3.2 in [1] may be helpful once again.

We close this section with a remark and a further assumption.

REMARK 2.3. Unless X is Gaussian, Assumption 2.1 implies that it is possible to modify the kernel f in (3), without changing the finite-dimensional distributions of X, in such a way that $f(s,\cdot)$ is C^2 for every $s \in S$; see Theorem 4 of [13]. For simplicity, we shall therefore assume throughout that f has such C^2 sections. This ensures, in particular, measurability of functions of the type $\sup_{t\in M} |f(s,t)|$, $s \in S$, which we shall take as given in what follows.

ASSUMPTION 2.4. The kernel $f(s,t)$, $s \in S$, $t \in \widetilde{M}$, along with its first and second order spatial partial derivatives f_i and f_{ij} are (uniformly) bounded and, for for every $s \in S$, the function $f(s,\cdot)$ is a Morse function on M.

2.3. *The function space Lévy measure.* Although the infinitely divisible random field's we are studying in this paper were constructed above via stochastic integrals (3) and, as such, are characterised by the triple (γ, F, β) of the random measure μ and the kernel f, in what follows the most important characteristic of the infinitely divisible random field (3) will be its function space Lévy measure. This is a measure on the cylinder sets of $\mathbb{R}^{\widetilde{M}}$, related to the parameters in the integral representation of the field by the formula

$$\lambda_X = F \circ T_f^{-1}, \tag{10}$$

where F is the Lévy measure of the infinitely divisible random measure μ and $T_f : S \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^{\widetilde{M}}$ is given by

$$T_f(s,x) = xf(s,\cdot), \qquad s \in S, x \in \mathbb{R} \setminus \{0\}, \tag{11}$$

cf. [11]. Thus, the finite-dimensional distributions of X are given via the joint characteristic function

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left\{i\sum_{j=1}^{k}\gamma_j X(t_j)\right\}\right\} \\
&\quad = \exp\Bigg\{-Q(\gamma_1,\dots,\gamma_k) \\
&\qquad + \int_{\mathbb{R}^{\widetilde{M}}}\left[\exp\left(i\sum_{j=1}^{k}\gamma_j x(t_j)\right) - 1 - i\sum_{j=1}^{k}\gamma_j [\![x(t_j)]\!]\right]\lambda_X(dx) \\
&\qquad + iL(\gamma_1,\dots,\gamma_k)\Bigg\}
\end{aligned} \tag{12}$$

for $k \geq 1$, $t_1, \ldots, t_k \in \widetilde{M}$, and real numbers $\gamma_1, \ldots, \gamma_k$, where Q is a quadratic function (corresponding to the Gaussian part of X), and L is a linear function (corresponding to the shift). Their exact forms are not important for us at the moment.

Note that the Lévy measures of the first and second order partial derivatives X_i and X_{ij} are similarly (cf. Theorem 5.1, [4]) given by

$$\lambda_{X_i} = F \circ T_{f_i}^{-1}, \qquad \lambda_{X_{ij}} = F \circ T_{f_{ij}}^{-1}, \qquad i, j = 1, \ldots, d. \tag{13}$$

2.4. *Regular variation.* We now turn to the final set of technical assumptions on our infinitely divisible random field's, these being related to the regular variation of their Lévy measures, and which we formulate in terms of the local Lévy measures of (4). These are our final set of assumptions, and our main results hinge on them.

Recall that a function f is regularly varying at infinity, with exponent α, if

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha} \qquad \text{for all } \lambda > 0. \tag{14}$$

ASSUMPTION 2.5. There exists a $H : (0, \infty) \to (0, \infty)$ that is regularly varying at infinity with exponent $-\alpha$, $\alpha > 0$, and nonnegative measurable functions w_+ and w_- on S such that

$$\lim_{u \to \infty} \frac{\rho(s; (u, \infty))}{H(u)} = w_+(s), \qquad \lim_{u \to \infty} \frac{\rho(s; (-\infty, -u))}{H(u)} = w_-(s) \tag{15}$$

for all $s \in S$. Furthermore, the convergence is uniform in the sense there is $u_0 > 0$ such that, for all $u > u_0$ and all $s \in S$,

$$\frac{\rho(s; (u, \infty))}{H(u)} \leq 2w_+(s), \qquad \frac{\rho(s; (-\infty, -u))}{H(u)} \leq 2w_-(s).$$

The following simple lemma relates Assumption 2.5 to the corresponding behaviour of the Lévy measure λ_X on a set of crucial importance to us. We adopt the standard notation $a_+ = \max(a, 0)$ and $a_- = (-a)_+$ for the positive and negative parts of a real.

LEMMA 2.6. *Let Assumption 2.5 hold*:

(i) *Assume that the kernel $f(s, t), t \in \widetilde{M}$ is uniformly* (*in $s \in S$*) *bounded*, *and that for some $\epsilon > 0$*,

$$\int_S \bigl(w_+(s) + w_-(s)\bigr) \sup_{t \in M} |f(s, t)|^{\alpha - \epsilon} m(ds) < \infty. \tag{16}$$

Then

$$
\begin{aligned}
&\lim_{u\to\infty}\frac{\mathbb{P}\{\sup_{t\in M}X(t)>u\}}{H(u)}\\
&\quad=\lim_{u\to\infty}\frac{\lambda_X\{g:\sup_{t\in M}g(t)>u\}}{H(u)}\\
&\quad=\int_S\Big[w_+(s)\sup_{t\in M}f(s,t)_+^\alpha+w_-(s)\sup_{t\in M}f(s,t)_-^\alpha\Big]m(ds),
\end{aligned}
\tag{17}
$$

where M can be replaced with $\widetilde{M}$ throughout. Furthermore,

$$
\begin{aligned}
\lim_{u\to\infty}\frac{\mathbb{P}\{\sup_{t\in\widetilde{M}}|X(t)|>u\}}{H(u)}&=\lim_{u\to\infty}\frac{\lambda_X\{g:\sup_{t\in\widetilde{M}}|g(t)|>u\}}{H(u)}\\
&=\int_S\big(w_+(s)+w_-(s)\big)\sup_{t\in\widetilde{M}}|f(s,t)|^\alpha m(ds).
\end{aligned}
\tag{18}
$$

(ii) *Assume that the first order partial derivatives $f_i(s,t), t\in\widetilde{M}, i=1,\dots,d$, are uniformly (in $s\in S$) bounded, and that for some $\epsilon>0$,*

$$
\int_S\big(w_+(s)+w_-(s)\big)\sup_{t\in\widetilde{M}}|f_i(s,t)|^{\alpha-\epsilon}m(ds)<\infty.
\tag{19}
$$

Then

$$
\begin{aligned}
&\lim_{u\to\infty}\frac{\lambda_{X_i}\{g:\sup_{t\in\widetilde{M}}|g(t)|>u\}}{H(u)}\\
&\quad=\int_S\big(w_+(s)+w_-(s)\big)\sup_{t\in\widetilde{M}}|f_i(s,t)|^\alpha m(ds).
\end{aligned}
\tag{20}
$$

(iii) *Assume that the second order partial derivatives $f_{ij}(s,t), t\in\widetilde{M}, i,j=1,\dots,d$, are uniformly (in $s\in S$) bounded, and that for some $\epsilon>0$,*

$$
\int_S\big(w_+(s)+w_-(s)\big)\sup_{t\in\widetilde{M}}|f_{ij}(s,t)|^{\alpha-\epsilon}m(ds)<\infty.
\tag{21}
$$

Then

$$
\begin{aligned}
&\lim_{u\to\infty}\frac{\lambda_{X_{ij}}\{g:\sup_{t\in\widetilde{M}}|g(t)|>u\}}{H(u)}\\
&\quad=\int_S\big(w_+(s)+w_-(s)\big)\sup_{t\in\widetilde{M}}|f_{ij}(s,t)|^\alpha m(ds).
\end{aligned}
\tag{22}
$$

PROOF. The first equality in (17) follows from the second equality there by Theorem 2.1 in [16]. As for the second equality in (17), it follows from (10) and

(4) that

$$\lambda_X\Big\{g:\sup_{t\in\widetilde{M}} g(t)>u\Big\}=\int_S\Big[\rho\Big(s;\Big(\frac{u}{\sup_{t\in\widetilde{M}} f(s,t)_+},\infty\Big)\Big)\\ +\rho\Big(s;\Big(-\infty,\frac{-u}{\sup_{t\in\widetilde{M}} f(s,t)_-}\Big)\Big)\Big]m(ds).$$

Using the uniform boundedness of the kernel and Potter's bounds (cf. [10] or [3], Theorem 1.5.6) we see that for any $\epsilon>0$ there is $C>0$ such that for all $u>1$,

$$\frac{\rho(s;(u/\sup_{t\in\widetilde{M}} f(s,t)_+,\infty))}{H(u)}\le C\sup_{t\in\widetilde{M}} f(s,t)_+^{\alpha-\epsilon}$$

and

$$\frac{\rho(s;(-\infty,-u/\sup_{t\in\widetilde{M}} f(s,t)_-))}{H(u)}\le C\sup_{t\in\widetilde{M}} f(s,t)_-^{\alpha-\epsilon}.$$

The limit (17) now follows from Assumption 2.5 via (16), regular variation, and dominated convergence. The proof of (18) is identical, as are the proofs of (ii) and (iii). □

REMARK 2.7. The assumption of uniform boundedness of the kernel f in (3) and its partial derivatives will be kept throughout the paper (it is already a part of Assumption 2.4), but the only place it is used is in Lemma 2.6. It is not difficult to see that this assumption can be removed at the expense of appropriate assumptions on the behaviour near the origin of the local Lévy measures in (4) and of slightly modifying the integrability condition (16). Given that this paper is already rather heavy on notation, we shall continue to work with uniform integrability, which helps keep things comparatively tidy. Note that it is also clear that, for the purpose of proving (17) alone, the integrability assumption (16) could be relaxed.

3. Limiting distributions for critical points.

Our initial aim, as described in the Introduction, was to obtain information about the distribution of the Euler characteristic of the excursion sets of (1). As is known from Morse critical point theory, Euler characteristics of excursion sets are closely related to the critical points above fixed levels. We shall describe this connection in the following section and, for the moment, concentrate on the critical points of X, which are also of intrinsic interest.

Recall the partition of the cube M into collections $\partial_k M$ of facets of dimension k. Let J denote one such facet, of dimension $0\le k\le d$.

Let g be a C^2 function on $\widetilde{M}$, and and for $i=0,1,\dots,\dim(J)$, let $\mathcal{C}_g(J;i)$ be the set of *critical points of index* i of $g_{|J}$. These are the points for which $\nabla g(t)$ is normal to J at t, and for which the index of the Hessian of $g_{|J}$, computed with respect to the natural orthonormal basis of J and when considered as a matrix, has index i. (Recall that the index of a matrix is the number of its negative eigenvalues.) Let

$$N_g(J;i) = \text{Card}(\mathcal{C}_g(J;i))$$

and, for real u,

$$N_g(J;i:u) = \text{Card}\big(\mathcal{C}_g(J;i) \cap \{t : g(t) > u\}\big)$$

be the the overall number of the critical points of different types of $g_{|J}$, and the number of these critical points above the level u, correspondingly. Since g is a Morse function, it is standard fare that all of the above numbers are finite (e.g., [1]).

Just a little more notation is required for the main theorem. Let f be the kernel in the integral representation (3) of an infinitely divisible random field. For $k = 0, 1, \ldots, d$, a facet J and $i = 0, 1, \ldots, \dim(J)$, let

$$c_i(J;s) = N_{f(s;\cdot)}(J;i)$$

be the number of the critical points of the s-section of f of the appropriate type, well defined since by Assumption 2.4 the sections are Morse functions.

Furthermore, let $(t_l(J;i;s), l = 1, \ldots, c_i(J;s))$ be an enumeration of these critical points, and, for $1 \le m \le c_i(J;s)$ let

$$f_{[m]}^{(J;i:+)}(s), \qquad f_{[m]}^{(J;i:-)}(s)$$

be, correspondingly, the mth largest of the positive and negative parts of $f(s; t_l(J; i; s))$. [Both quantities are set to zero if $m > c_i(J;s)$.]

Finally, extend these definitions to $m = 0$ by setting

$$f_{[0]}^{(J;i:+)}(s) = \sup_{t \in M}(f(s;t))_+, \qquad f_{[0]}^{(J;i:-)}(s) = \sup_{t \in M}(f(s;t))_-.$$

The following theorem, proven in Section 6, is the main result of this paper. It describes the limiting, conditional, joint distribution of the number of critical points of all possible types of a infinitely divisible random field over the level u, as $u \to \infty$, given that the random field actually exceeds level u at some point. We recall that, in the theorem, M is the unit cube and $\widetilde{M}$ an open, bounded, neighborhood of M. However, extensions to more general M, as explained in the Introduction, can also be proven.

THEOREM 3.1. *Let $(X(t), t \in \widetilde{M})$ be an infinitely divisible random field with representation* (3), *satisfying Assumptions* 2.1, 2.4 *and* 2.5. *Assume that* (16) *holds for some $\epsilon > 0$. Then, for any collection $\mathcal{J}$ of facets in the various $\partial_k M$, $k \in \{0, 1, \ldots, d\}$, and any collection of nonnegative integers $\{n(J;i) = 0, 1, \ldots, i =$*

$0, 1, \dots, \dim(J)$, $J \in \mathcal{J}\}$, *as* $u \to \infty$,

$$
\mathbb{P}\Big\{N_X(J; i : u) \geq n(J; i),\ J \in \mathcal{J},\ i = 0, 1, \dots, \dim(J) \Big| \sup_{t \in M} X(t) > u\Big\}
$$

$$
\begin{aligned}
&\to \int_S \Big[w_+(s) \Big(\min_{J,i} f^{(J;i:+)}_{[n(J;i)]}(s)\Big)^{\alpha} + w_-(s) \Big(\min_{J,i} f^{(J;k-i:-)}_{[n(J;i)]}(s)\Big)^{\alpha}\Big] m(ds) \\
&\quad \times \Big(\int_S \Big[w_+(s) \sup_{t \in M} f(s,t)_+^{\alpha} + w_-(s) \sup_{t \in M} f(s,t)_-^{\alpha}\Big] m(ds)\Big)^{-1}.
\end{aligned} \tag{23}
$$

REMARK 3.2. While the structure of (23) might be rather forbidding at first sight, its meaning is actually rather simple. The main point of Theorem 3.1 is that, once the random field reaches a high level, its behavior above that level is very similar to that of the much simpler random field,

$$
Z(t) = V f(W, t), \qquad t \in \widetilde{M}, \tag{24}
$$

where $(V, W) \in (\mathbb{R} \setminus \{0\}) \times S$ is a random pair, the joint law of which is the finite restriction of the Lévy measure F to the set

$$
\Big\{(x, s) \in (\mathbb{R} \setminus \{0\}) \times S : \sup_{t \in \widetilde{M}} |x f(s;t)| > 1\Big\},
$$

normalized to be a probability measure on that set.

REMARK 3.3. In fact, one can go much further than in the previous remark, and interpret the limit (23) as showing that limiting conditional joint distribution of critical points is a mixture distribution, that can be described as follows. Set

$$
H = \int_S \Big[w_+(s) \sup_{t \in M} f(s,t)_+^{\alpha} + w_-(s) \sup_{t \in M} f(s,t)_-^{\alpha}\Big] m(ds).
$$

(1) Select a random point $W \in S$ with probability law η on S where

$$
\frac{d\eta}{dm}(s) = H^{-1}\Big[w_+(s) \sup_{t \in M} f(s,t)_+^{\alpha} + w_-(s) \sup_{t \in M} f(s,t)_-^{\alpha}\Big], \qquad s \in S.
$$

(2) Given $W = s$, select a random value $I \in \{-1, 1\}$ with the law

$$
P(I = 1 | W = s) = \frac{w_+(s) \sup_{t \in M} f(s,t)_+^{\alpha}}{w_+(s) \sup_{t \in M} f(s,t)_+^{\alpha} + w_-(s) \sup_{t \in M} f(s,t)_-^{\alpha}}.
$$

(3) Let V_α be a random variable independent of W and I, with $\mathbb{P}\{V_\alpha \leq x\} = x^\alpha$ for $0 \leq x \leq 1$. Then the numbers of critical points $(N_X(J; i : u), J \in \mathcal{J})$, given that $\sup X > u$, have, as $u \to \infty$, the same distribution as the numbers of critical points of the random field

$$
\Big(\frac{f(W,t)_+}{\sup_{r \in M} f(W,r)_+}, t \in M\Big) \tag{25}
$$

above the level V_α if $I = 1$, and the numbers of critical points of

$$\left(\frac{f(W,t)_-}{\sup_{r\in M} f(W,r)_-}, t \in M\right) \tag{26}$$

above the level V_α if $I = -1$.

REMARK 3.4. While Theorem 3.1 counts critical points classified by their indices, there are also other properties of critical points that are of topological importance. For example, in [1] considerable emphasis was laid on the so-called "extended outward critical points." These are the critical points $t \in M$ for which $\nabla f(t) \in N_t(M)$, where $N_t(M)$ is the normal cone of M at t.

Extended outward critical points play a major role in Morse theory, in terms of defining the Euler characteristics of excursion sets. It will be easy to see from the proof of Theorem 3.1 that its the statement remains true if one replaces critical points by extended outward critical points. This will be used in certain applications of Theorem 3.1 below.

4. The Euler characteristic of excursion sets. One application of Theorem 3.1 is to the Euler characteristic $\varphi(A_u)$ of the excursion set A_u over a high level u. We shall not define the Euler characteristic here, but rather send you to [1] for details. The Euler characteristic of an excursion set of a Morse function is equal to the alternating sum of the numbers of extended outward critical points of the function over the level. This leads to the following result, an immediate corollary of Theorem 3.1, (9.4.1) in [1], and Remarks 3.3 and 3.4 above.

COROLLARY 4.1. *Under the conditions of Theorem* 3.1, *the conditional distribution of the Euler characteristic of the excursion set of an infinitely divisible random field computed with its limiting conditional distribution given that the level is exceeded*, *is given by the mixture of the Euler characteristics of the random fields* (25) *and* (26), *with the mixing distribution as described in Remark* 3.3. *In particular*, *the expected Euler characteristic of the excursion set of the limiting* (*conditional*) *random field is given by*

$$H^{-1}\int_S \Big[w_+(s)\sup_{t\in M} f(s,t)_+^\alpha \mathbb{E}\{C_+(s)\} + w_-(s)\sup_{t\in M} f(s,t)_-^\alpha \mathbb{E}\{C_-(s)\}\Big] m(ds).$$

Here, *for* $s \in S$, $C_\pm(s)$ *is the Euler characteristic of the excursion set of the field* $(f(s,t)_\pm / \sup_{r\in M} f(s,r)_\pm, t \in M)$ *above the level* V_α.

5. An example: Moving average fields. The power and variety of the results of the previous two sections can already be seen in a relatively simple but application rich class of random fields, the moving average fields with kernel g that were introduced at (8). Our basic assumptions, that will hold throughout this section, are:

(i) The function g is C^2 on $\mathbb{R}^d$ and satisfies (6) and (7).

(ii) μ is an infinitely divisible random measure on $\mathbb{R}^d$, for which the Gaussian and shift components in the generating triple, γ and β, vanish.

(iii) The control measure m in (4) is d-dimensional Lebesgue measure.

(iv) The local Lévy measures $\rho(s, \cdot) = \rho(\cdot)$ are independent of $s \in \mathbb{R}^d$.

By choosing different kernels g, we shall see that quite different types of high level excursion sets arise, as opposed to the Gaussian case, in which ellipsoidal sets are, with high probability, ubiquitous.

5.1. *Checking the conditions of Theorem 3.1 for type G moving averages.* In this subsection, we exhibit a broad family of moving average random fields (8) for which we shall verify the conditions required by the main result of Section 3. These are the so-called type G random fields. We emphasize that the applicability of our main results is not restricted to type G random fields. For the latter, we can use standard tools to check the assumptions of Theorem 3.1, which is why they are presented here. The main result of this subsection is the following theorem.

THEOREM 5.1. *A moving average infinitely divisible random field X satisfying Conditions 5.2 and 5.3 below also satisfies the assumptions of Theorem 3.1.*

CONDITION 5.2. The local Lévy measure ρ is a symmetric measure of the form

$$\rho(B) = \mathbb{E}\{\rho_0(Z^{-1}B)\}, \tag{27}$$

where B is a Borel set, Z is a standard normal random variable, and ρ_0 is a symmetric Lévy measure on $\mathbb{R}$. Furthermore, the function $\rho_0((u,\infty)), u > 0$ is regularly varying at infinity with exponent $-\alpha, \alpha > 1$, and there is $\beta \in [1, 2)$ such that

$$\rho_0((u,\infty)) \le au^{-\beta}$$

for all $0 < u < 1$, for some $0 < a < \infty$.

In fact, for *any* Lévy measure ρ_0 on $\mathbb{R}$, (27) defines a Lévy measure on $\mathbb{R}$; see, for example, Proposition 2.2 in [8]. Furthermore, it is simple to check that the behavior of the measures ρ and ρ_0 are similar at zero and infinity. Specifically,

$$\lim_{u\to\infty} \frac{\rho((u,\infty))}{\rho_0((u,\infty))} = \mathbb{E}\{Z_+^\alpha\} \tag{28}$$

and

$$\rho((u,\infty)) \le \mathbb{E}\{\max(|Z|^\beta, 1)au^{-\beta}\} \tag{29}$$

for $0 < u < 1$. In particular, a moving average infinitely divisible random field satisfying Condition 5.2 automatically also satisfies Assumption 2.5. It suffices to choose $H(u) = \rho_0((u,\infty))$, $u > 0$, and $w_+(s) = w_-(s) = \mathbb{E}\{Z_+^\alpha\}$.

It is Condition 5.2 that makes the random field a "type G random field." It implies that the random field X can be represented as a certain mixture of stationary Gaussian fields, cf. [8]. Under the conditions we impose, each one of the latter is a.s. a Morse function, which will tell us that the moving average itself has sample functions which are, with probability 1, Morse functions.

If it is known from other considerations that the sample functions of a specific infinitely divisible random field are, with probability 1, Morse functions, then (27) is not needed, and only the assumptions on the behavior of the tails of the Lévy measure $\rho((u,\infty))$ as $u \to 0$ or $u \to \infty$ are required. In the present form of Condition 5.2, these assumptions become the conclusions (28) and (29) from the corresponding assumptions on the Lévy measure ρ_0.

CONDITION 5.3. The kernel g is in C^3, and its restriction to any bounded hypercube is a Morse function. Assume that the first and the second derivatives $g_i, i = 1,\dots,d$, and $g_{ij}, i, j = 1,\dots,d$, satisfy (6) and (7). Assume, further, that for almost every $s \in \mathbb{R}^d$ there is no subspace of dimension strictly less than $(d^2 + 3d)/2$ to which the vectors $(g_i(s), i = 1,\dots,d, g_{ij}(s), i, j = 1,\dots,d, i \le j)$ belong.

Finally, assume that the function

$$T_g(s) = \sup_{t\in[-1,1]^d} |g(s+t)|, \qquad s \in \mathbb{R}^d,$$

satisfies $T_g \in L^{\alpha-\varepsilon}(\mathbb{R}^d)$, while the function

$$\widetilde{T}_g(s) = \max_{i,j\in 1,\dots,d} |g_{ij}(s)| + \sup_{t\in[-1,1]^d, i,j,k\in 1,\dots,d} |g_{ijk}(s+t)|, \qquad s \in \mathbb{R}^d,$$

satisfies $\widetilde{T}_g \in L^{\alpha-\varepsilon}(\mathbb{R}^d) \cap L^\beta(\mathbb{R}^d)$ for some $\varepsilon > 0$ and for the α and β for which Condition 5.2 holds.

Since these are assumptions on the kernel g in the integral representation (8) of the random field, and the kernel is often explicitly given, the above conditions are, generally, easy to apply. See the examples below.

Clearly, a moving average infinitely divisible random field satisfying Condition 5.3 will also satisfy Assumption 2.4. It also satisfies (16). Theorem 5.1 is then an immediate consequence of the following two lemmas, the first one of which follows in a straightforward fashion from the metric entropy condition in Remark 2.1 in [9].

LEMMA 5.4. *A moving average infinitely divisible random field satisfying Condition 5.3 has sample paths in C^2.*

To complete the proof of Theorem 5.1, we need to check that a moving average satisfying Conditions 5.2 and 5.3 has sample functions that are, with probability 1, Morse functions. As mentioned above, we shall accomplish this by representing the random field X as a mixture of zero mean Gaussian random fields, each one of which will have, with probability 1, sample functions that are Morse functions.

LEMMA 5.5. *A moving average infinitely divisible random field satisfying Conditions 5.2 and 5.3 has sample functions that are, with probability 1, Morse functions.*

PROOF. Let ν and $\widetilde{\nu}$ be probability measures on $\mathbb{R}$ and $\mathbb{R}^d$ absolutely continuous with respect to the Lévy measure ρ_0 and to d-dimensional Lebesgue measure λ_d, respectively. Let

$$\psi(x) = \frac{d\nu}{d\rho_0}(x), \qquad x \in \mathbb{R}, \qquad \varphi(s) = \frac{d\widetilde{\nu}}{d\lambda_d}(s), \qquad s \in \mathbb{R}^d.$$

Then the random field X has a representation as an infinite sum of the form

$$X(t) = \sum_{k=1}^{\infty} Z_k V_k g(t + H_k) \mathbb{1}\big(\psi(V_k)\varphi(H_k)\Gamma_k \le 1\big), \qquad t \in \mathbb{R}^d, \tag{30}$$

where $(Z_1, Z_2, \ldots)$ are i.i.d. standard normal random variables, $(V_1, V_2, \ldots)$ are i.i.d. random variables with a common law ν, $(H_1, H_2, \ldots)$ are i.i.d. random vectors in $\mathbb{R}^d$ with a common law $\widetilde{\nu}$, and $(\Gamma_1, \Gamma_2, \ldots)$ are the points of a unit rate Poisson process on $(0, \infty)$. All four sequences are independent. See [8], Section 5, for details. Furthermore, by Lemma 5.4 and Theorem 2.2, the first and second order partial derivatives of X are also moving average random fields, with corresponding series representations. In particular, all that needs changing is to replace g in (30) by an appropriate derivative.

We may assume, without loss of generality, that the standard Gaussian sequence $(Z_1, Z_2, \ldots)$ is defined on a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, and the remaining random variables on the right-hand side of (30) are defined on a different probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, so that the random fields defined by the series are defined on the product probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$. Thus, for every fixed $\omega_2 \in \Omega_2$, the conditional random field $X((\omega_1, \omega_2))$, $\omega_1 \in \Omega_1$, is a centered Gaussian random field. We now apply to this random field Corollary 11.3.2 in [1].

Firstly, we check the condition on the incremental variance of the second order partial derivatives there. In obvious notation, for every $i, j = 1, \ldots, d$, and $t, s \in M$,

$$\begin{aligned}
&\mathbb{E}_1\big\{\big(X_{ij}(t) - X_{ij}(s)\big)^2\big\} \\
&\qquad = \sum_{k=1}^{\infty} V_k^2 \big(g_{ij}(t + H_k) - g_{ij}(s + H_k)\big)^2 \mathbb{1}\big(\psi(V_k)\varphi(H_k)\Gamma_k \le 1\big).
\end{aligned}$$

Bounding the Hölder constant of a function by the largest value of its partial derivatives, as in the proof of Lemma 5.4, we obtain

$$\mathbb{E}_1\{(X_{ij}(t) - X_{ij}(s))^2\} \\ \leq d^2\|t-s\|^2 \sum_{k=1}^{\infty} V_k^2 \widetilde{T}_g^2(H_k)\mathbb{1}(\psi(V_k)\varphi(H_k)\Gamma_k \leq 1)$$

and, hence, the incremental variance condition will follow once we check that the infinite sum above converges. For this, we need to check (see [14]) that

$$\int_0^\infty \mathbb{E}_2\{\min[1, V^2\widetilde{T}_g(H)^2\mathbb{1}(\psi(V)\varphi(H)x \leq 1)]\}\,dx < \infty.$$

(The random variables without a subscript represent generic members of the appropriate sequences.) By the definition of the derivatives ψ and φ, this reduces to checking that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \min[1, y^2\widetilde{T}_g(s)^2]\,ds\,dy < \infty,$$

which is an elementary consequence of the integrability assumptions imposed on $\widetilde{T}_g$ in Condition 5.3, and of the assumptions imposed on the Lévy measure ρ_0 in Condition 5.2.

It remains to check that the joint distribution under $\mathbb{P}_1$ of the random vectors of partial derivatives (X_i, X_{ij}) is nondegenerate for P_2-almost every ω_2. This, however, follows from representing the derivatives as sums akin to (30), along with the part of Condition 5.3 that rules out the possibility that the derivatives of the kernel g belong to a lower dimensional subspace. □

5.2. *Examples*: *How the shape of the kernel can affect the geometry of excursion sets*. In the examples below, Condition 5.2 is a standing assumption, and will not be mentioned explicitly. Our first example is of an infinitely divisible moving average random field whose high level excursion sets have a similar geometric structure to those of Gaussian random fields.

EXAMPLE 5.6. Let g be a nonnegative kernel satisfying Condition 5.3, that is also rotationally invariant and radially decreasing; that is, $g(t) = g_r(\|t\|)$ for some nonnegative, decreasing g_r on $[0, \infty)$. An example is the Gaussian kernel $g(t) = \exp\{-a\|t\|^2\}$, $a > 0$, for which it is trivial to check that the restrictions in Condition 5.3 on the various partial derivatives of g hold.

Corollary 4.1 tells us that the Euler characteristic of the excursion set over a high level, given that the level is exceeded, is asymptotically that of the field

$$\left(\sup_{r\in I_d} g(s+r)\right)^{-1} g(s+t), \qquad t \in I_d, \tag{31}$$

with a randomly chosen $s \in \mathbb{R}^d$ and over a random level V_α.

The assumption of rotational invariance and radial monotonicity on the kernel g implies that, in this case, the excursion set of the random field is the intersection of a Euclidian ball centered at the point $-s$ and the cube I_d. This is a convex set and, hence, has Euler characteristic equal to 1, regardless of the point $s \in \mathbb{R}^d$ or the random level V_α.

In this case, the limiting conditional distribution of the Euler characteristic is degenerate at the point 1. Furthermore, the excursion set has, with high probability, a "ball-like shape," as is the case for smooth Gaussian random fields.

In spite of the "Gaussian-like" conclusion in the previous example, it is easy to modify it to make the high level excursion sets of an infinitely divisible random field behave quite differently. Here is a simple example.

EXAMPLE 5.7. We modify the kernel g of the previous example by adding to it oscillations, while preserving its smoothness and integrability properties. For example, for fixed $\theta \in \mathbb{R}^d$, take

$$g(t) = (1 + \cos\langle\theta, t\rangle)e^{-a\|t\|^2}, \qquad t \in \mathbb{R}^d.$$

Then, depending on the random choice of the point s in (31), the structure of the excursion sets in I_d could be quite varied, as it depends on the shape of g in the translated cube $I_d^{(-s)}$. Thus, depending on the random level V_α, the shape of the excursion set may be quite different from a ball-like shape. In particular, its Euler characteristic will have a nondegenerate distribution.

5.3. *The bottom line*. The bottom line, of course, is that the shape of the excursion sets is determined, to a large extent, by the shape of the kernel in the integral representation of the random field or, alternatively, by the geometric properties of the functions on which the Lévy measure of the random field is supported. By choosing appropriate parameters for the random field, one can generate quite different distributions for the Euler and other geometric characteristics of high level excursion sets.

Our hope is that this fact will generate greater flexibility in applications, allowing the practitioner to choose models with predetermined excursion set shapes. Furthermore, the description of the limiting conditional *distribution* (and not only the expected value) of the numbers of critical points and so the Euler characteristic should allow one to devise better statistical tests based on the observed excursion sets.

6. Proof of Theorem 3.1. The proof is rather long and rather technical, although the basic idea is not difficult.

The basic idea, which is common to many proofs involving infinitely divisible random field's X, is to write X as a sum of two parts, one which tends to be large and one which is made up of smaller perturbations. The large part, which, distributionally, behaves as a Poisson sum of deterministic functions with random multipliers, is comparatively simple to handle, and it is this part that actually accounts for the limit in Theorem 3.1. One then needs to show that the small pertubations can be ignored in the $u \to \infty$ limit. In the argument that follows this is somewhat more difficult than is usually the case, since even if the small part is small in magnitude it can, in principle, have a major effect on variables such as the number of critical points of the sum. [Think of any smooth function f to which is added $g(t) = \epsilon \cos(\langle \theta, t\rangle/\epsilon^2)$. No matter how large λ might be, nor how small ϵ might be, the critical points of $f + g$ are, effectively, determined by g, not f.]

Due to the length of the ensuing proof, we shall do our best to signpost it as it progresses.

(i) *Some notation for the parameter space and for critical points.* As mentioned earlier, in this section we shall take as our parameter space the cube I_d. The first step is to develop notation for describing its stratification.

Let $\mathcal{J}_k$ be the collection of the $2^{d-k}\binom{d}{k}$ faces of I_d of dimension k, $k = 0, \dots, d$, and let $\mathcal{J} = \bigcup_k \mathcal{J}_k$. For each face $J \in \mathcal{J}_k$, there is a corresponding set $\sigma(J) \subseteq \{1, \dots, d\}$ of cardinality k and a sequence $\epsilon(J) \in \{-1, 1\}^{\sigma(J)}$ such that

$$J = \{t = (t_1, \dots, t_d) \in I_d : t_j = \epsilon_j \text{ if } j \notin \sigma(J) \text{ and } 0 < t_j < 1 \text{ if } j \in \sigma(J)\}.$$

Let g be a C^2 function on an open set $\widetilde{M}$ containing I_d. For $J \in \mathcal{J}_k$ and $i = 0, 1, \dots, k$, let $\mathcal{C}_g(J; i)$ be the set of points $t \in J$ satisfying the following two conditions:

$$\frac{\partial g}{\partial t_j}(t) = 0 \qquad \text{for each } j \in \sigma(J), \tag{32}$$

$$\text{the matrix } \left(\frac{\partial^2 g(t)}{\partial t_m\, \partial t_n}\right)_{m,n\in\sigma(J)} \text{ has nonzero determinant} \tag{33}$$

and its index is equal to $k - i$.

Now define $N_g(J; i)$ and $N_g(J; i : u)$ in terms of $\mathcal{C}_g(J; i)$ as in Section 3.

(ii) *Splitting X into large and small components.* By Assumption 2.1, X and its first and second order partial derivatives are a.s. bounded on $\widetilde{M}$, and, by (10), the Lévy measure of X is concentrated on C^2 functions. Defining

$$S_L = \Big\{g \in C^2 : \max\Big[\sup_{t\in\widetilde{M}} |g(t)|, \sup_{t\in\widetilde{M}, i=1,\dots,d} |g_i(t)|, \sup_{t\in\widetilde{M}, i,j=1,\dots,d} |g_{ij}(t)|\Big] > 1\Big\},$$

the sample boundedness of X, along with (13) and general properties of Lévy measures on Banach spaces (e.g., [7]) imply that

$$\theta \triangleq \lambda_X\{S_L\} < \infty. \tag{34}$$

We are now ready to decompose the infinitely divisible random field X into a sum of two independent infinitely divisible components by writing

$$X(t) = X^L(t) + Y(t), \qquad t \in \widetilde{M}, \tag{35}$$

where X^L is a compound Poisson random field with characteristic functions, which, for $k \geq 1$, $t_1, \dots, t_k \in \widetilde{M}$, and real numbers $\gamma_1, \dots, \gamma_k$, are given by

$$\mathbb{E}\left\{\exp\left\{i \sum_{j=1}^{k} \gamma_j X_L(t_j)\right\}\right\} = \exp\left\{\int_{S_L} \left(\exp\left\{i \sum_{j=1}^{k} \gamma_j x(t_j)\right\} - 1\right)\lambda_X(dx)\right\}. \tag{36}$$

The second, or "residual," component Y has characteristic functions

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left\{i \sum_{j=1}^{k} \gamma_j Y(t_j)\right\}\right\} \\
&\quad = \exp\left\{-Q(\gamma_1, \dots, \gamma_k)\right. \\
&\qquad\qquad + \int_{\mathbb{R}^{\widetilde{M}} \setminus S_L} \left(\exp\left\{i \sum_{j=1}^{k} \gamma_j x(t_j)\right\} - 1 - i \sum_{j=1}^{k} \gamma_j [\![x(t_j)]\!]\right)\lambda_X(dx) \\
&\qquad\qquad \left. + i L_1(\gamma_1, \dots, \gamma_k)\right\},
\end{aligned}$$

where we are using the notation of (12), and

$$L_1(\gamma_1, \dots, \gamma_k) = L(\gamma_1, \dots, \gamma_k) - \int_{S_L} \sum_{j=1}^{k} \gamma_j [\![x(t_j)]\!]\lambda_X(dx).$$

We shall ultimately show that the limiting behaviour of the critical points of X depends only on the component X^L, so we study it first.

(iii) *A limit theorem for the critical points of* X^L. We start by noting that it follows from the form of the characteristic function (36) and the definition (10) that X^L can, in law, be written as

$$X^L(t) = \sum_{m=1}^{N} X_m f(S_k, t), \tag{37}$$

where N is a Poisson random variable with mean θ given by (34), independent of an i.i.d. sequence of random pairs $((X_m, S_m))$, $m = 1, 2, \dots)$ taking values in $(\mathbb{R} \setminus \{0\}) \times S$ with the common law $\theta^{-1}F$ restricted to the set

$$\left\{(s, x) \in (\mathbb{R} \setminus \{0\}) \times S : \sup_{t \in \widetilde{M}} |xf(s; t)| > 1\right\}.$$

Recall that F is the Lévy measure of the infinitely divisible random measure M in (3).

Since the sum in (37) is a.s. finite, and the kernel f has bounded C^2 sections $f(s;\cdot)$ for all $s \in S$, it follows that X^L is bounded and C^2 on $\widetilde{M}$.

We now decompose the compound Poisson term X^L itself into a sum of two independent pieces, the stochastically larger of which will be responsible for the limiting behavior of the critical points of X. For $u > 0$ and $1/2 < \beta < 1$, define the sequence of independent events

$$A_m(u) = \Big\{\max\Big[\sup_{t\in\widetilde{M}} |X_m f(S_m;t)|, \sup_{t\in\widetilde{M}, i=1,\dots,d} |X_m f_i(S_m;t)|, \sup_{t\in\widetilde{M}, i,j=1,\dots,d} |X_m f_{ij}(S_m;t)|\Big] > u^\beta\Big\}$$

and write

$$\begin{aligned} X^L(t) &= \sum_{m=1}^N X_m f(S_m;t)\mathbb{1}_{A_m(u)} + \sum_{m=1}^N X_m f(S_m;t)\mathbb{1}_{A_m(u)^c} \\ &\triangleq X^{(L,1)}(t) + X^{(L,2)}(t). \end{aligned} \tag{38}$$

In Lemma 6.1, we shall show that $X^{(L,2)}$ and its partial derivatives have suprema the tail probabilities of which decay faster than the function H, and so are unlikely to affect the critical points of X. We shall return to this point later.

Now, however, we shall concentrate on the critical points over high levels of $X^{(L,1)}$. Define two new events

$$B_1(u) = \Big\{\sum_{m=1}^N \mathbb{1}(A_m(u)) = 1\Big\}, \qquad B_2(u) = \Big\{\sum_{m=1}^N \mathbb{1}(A_m(u)) \geq 2\Big\}. \tag{39}$$

The first of these occurs when there is a single large term in the Poisson sum (37), the second when there are more. On the event $B_1(u)$, we define the random variable $K(u)$ to be the index of large term, and otherwise allow it to be arbitrarily.

In the notation of Section 3 in general and Theorem 3.1 in particular, it follows that, on the event $B_1(u)$, the following representation holds for the numbers of the critical points of $X^{(L,1)}$ over the level u. For $k = 0, 1, \dots, d$, a face $J \in \mathcal{J}_k$ and $i = 0, 1, \dots, k$,

$$\begin{aligned} &N_{X^{(L,1)}}(J; i : u) \\ &\quad = \mathbb{1}(X_{K(u)} > 0) \sum_{l=0}^{c_i(J;S_{K(u)})} \mathbb{1}\big(X_{K(u)} f\big(S_{K(u)}; t_l\big(J; i; S_{K(u)}\big)\big) > u\big) \\ &\qquad + \mathbb{1}(X_{K(u)} < 0) \sum_{l=0}^{c_{k-i}(J;S_{K(u)})} \mathbb{1}\big(X_{K(u)} f\big(S_{K(u)}; t_l\big(J; k-i; S_{K(u)}\big)\big) > u\big). \end{aligned}$$

Therefore, for any number $r = 1, 2, \dots$, on the event $B_1(u)$, we have $N_{X^{(L,1)}}(J; i : u) \ge r$ if, and only if,

$$X_{K(u)} > \bigl(f_{[r]}^{(J;i:+)}(S_{K(u)})\bigr)^{-1} u \quad \text{or} \quad X_{K(u)} < -\bigl(f_{[r]}^{(J;k-i:-)}(S_{K(u)})\bigr)^{-1} u.$$

We conclude that for any numbers $n(J; i) = 1, 2, \dots$, for all $J \in \mathcal{J}_k$, and for all $k = 0, 1, \dots, d$ and $i = 0, 1, \dots, k$,

$$\begin{aligned} &\mathbb{P}\bigl\{\bigl\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i) \text{ for all } J \text{ and } i\bigr\} \cap B_1(u)\bigr\} \\ &\quad = \mathbb{P}\Bigl\{\Bigl[X_{K(u)} > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \bigl(f_{[n(J;i)]}^{(J;i:+)}(S_{K(u)})\bigr)^{-1} u \text{ or} \\ &\qquad\qquad X_{K(u)} < -\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \bigl(f_{[n(J;i)]}^{(J;k-i:-)}(S_{K(u)})\bigr)^{-1} u\Bigr] \\ &\qquad\qquad \cap B_1(u)\Bigr\}. \end{aligned} \tag{40}$$

Write $\mathcal{E}_m$ for the union of sets $(-\infty, -\max]$ and $[\max, \infty)$, where the "max" come from the preceding lines with $K(u)$ replaced by m. Then

$$\begin{aligned} &\mathbb{P}\bigl\{\bigl\{X_{K(u)} \in \mathcal{E}_{K(u)}\bigr\} \cap B_1(u)\bigr\} \\ &\quad = \mathbb{P}\Biggl\{\bigcup_{m=1}^{N}\Biggl(A_m(u) \cap \bigcap_{m_1 \ne m} A_{m_1}(u)^c \cap \{X_m \in \mathcal{E}_m\}\Biggr)\Biggr\} \\ &\quad = e^{-\theta} \sum_{n=0}^{\infty} \frac{\theta^n}{n!} n \mathbb{P}\Biggl\{\mathcal{E}_1(u) \cap \bigcap_{m_1=2}^{n} A_{m_1}(u)^c \cap \{X_1 \in \mathcal{E}_1\}\Biggr\} \\ &\quad = \theta \mathbb{P}\bigl\{A_1(u) \cap \{X_1 \in \mathcal{E}_1\}\bigr\} - \mathbb{P}\bigl\{\bigl\{X_{K(u)} \in \mathcal{E}_{K(u)}\bigr\} \cap B_2(u)\bigr\}. \end{aligned} \tag{41}$$

Applying this to the right-hand side of (40) and using part (iii) of Lemma 6.1 yields

$$\begin{aligned} &\mathbb{P}\bigl\{\bigl\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i) \text{ for all } J \text{ and } i\bigr\} \cap B_1(u)\bigr\} \\ &\quad = \theta \mathbb{P}\Bigl\{A_1(u) \cap \Bigl\{X_1 > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \bigl(f_{[n(J;i)]}^{(J;i:+)}(S_1)\bigr)^{-1} u\Bigr\}\Bigr\} \\ &\qquad + \theta \mathbb{P}\Bigl\{A_1(u) \cap \Bigl\{X_1 < -\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \bigl(f_{[n(J;i)]}^{(J;k-i:-)}(S_1)\bigr)^{-1} u\Bigr\}\Bigr\} \\ &\qquad - Q_{\text{small}}(u), \end{aligned} \tag{42}$$

where $Q_{\text{small}}(u)/H(u) \to 0$ as $u \to \infty$.

Assume for the moment that all the $n(J; i)$ are strictly positive. Since the parameter β in the definition of the event $A_1(u)$ is less than 1, it follows that, as

$u \to \infty$,

$$\mathbb{P}\Big\{A_1(u) \cap \Big\{X_1 > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;i:+)}_{[n(J;i)]}(S_1)\big)^{-1} u\Big\}\Big\}$$
$$\sim \mathbb{P}\Big\{X_1 > \max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;i:+)}_{[n(J;i)]}(S_1)\big)^{-1} u\Big\}$$
$$= \frac{1}{\theta} \int_S \rho\Big(s; \Big(\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;i:+)}_{[n(J;i)]}(s)\big)^{-1} u, \infty\Big)\Big) m(ds).$$

In the last step, we used the law of X_1 introduced after (37) and the decomposition (4) of the measure F, and in the middle one the asymptotic equivalence means that the two ratio of the two probabilities tends to 1 as $u \to \infty$. Since a similar asymptotic expression can be written for the second term in the right-hand side of (42), we obtain

$$\lim_{u \to \infty} \frac{\mathbb{P}\{\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i) \text{ for all } J \text{ and } i\} \cap B_1(u)\}}{H(u)}$$
$$= \lim_{u \to \infty} H(u)^{-1}$$
$$\times \int_S \Big[\rho\Big(s; \Big(\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;i:+)}_{[n(J;i)]}(s)\big)^{-1} u, \infty\Big)\Big)$$
$$+ \rho\Big(s; \Big(-\infty, -\max_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;k-i:-)}_{[n(J;i)]}(s)\big)^{-1} u\Big)\Big)\Big]$$
$$\times m(ds),$$

provided the last limit exists. Applying (16) and Potter's bounds, as in Lemma 2.6, to justify an interchange of limit and integration, and noting Assumption 2.5 relating ρ, ω and H, we have

$$\lim_{u \to \infty} \frac{\mathbb{P}\{\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i) \text{ for all } J \text{ and } i\} \cap B_1(u)\}}{H(u)}$$
$$= \int_S \Big[w_+(s) \min_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;i:+)}_{[n(J;i)]}(s)\big)^{\alpha}$$
$$+ w_-(s) \min_{J \in \mathcal{J}_k, k=0,1,\dots,d, i=0,1,\dots,k} \big(f^{(J;k-i:-)}_{[n(J;i)]}(s)\big)^{\alpha}\Big] m(ds)$$
$$\triangleq I_c.$$

Finally, since by part (iii) of Lemma 6.1, the event $B_2(u)$ has a probability of a smaller order, we can also conclude that

$$\lim_{u \to \infty} \frac{\mathbb{P}\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i) \text{ for all } J \text{ and } i\}}{H(u)} = I_c. \tag{43}$$

In view of (17), we can rewrite this as

$$\lim_{u\to\infty}\mathbb{P}\Big\{N_{X^{(L,1)}}(J;i:u)\ge n(J;i)\text{ for all }J\text{ and }i\,\Big|\,\sup_{t\in M}X_t\ge u\Big\}$$
$$=\frac{I_{\mathrm{c}}}{\int_S[w_+(s)\sup_{t\in M}f(s,t)_+^{\alpha}+w_-(s)\sup_{t\in M}f(s,t)_-^{\alpha}]m(ds)}. \tag{44}$$

This will complete the proof of the theorem, at least for the case of strictly positive $n(J;i)$, once we show that the lighter-tailed random fields Y of (35) and $X^{(L,2)}$ of (38) do not change the asymptotic distribution of the numbers of critical points of X. This will take us a while to show, and makes up the remainder of the proof.

Before we do this, note that handling situations in which some or all of the numbers $n(J;i)$ are zero is actually only an issue of semantics, once we recall our convention regarding the 0th order statistic introduced prior to the statement of the theorem. For example, in the case when *all* the $n(J;i)$ are zero, the event on the left-hand side of (44) should be interpreted as stating that $X^{(L,1)}$ has crossed the level u, given that it has done so. Not surprisingly, the resulting limit, and the right-hand side, turn out to be 1. Similar reductions work when only some of the $n(J;i)$ are zero.

(iv) *An outline of what remains to do.* It follows from what we have done so far that

$$X(t)=X^{(L,1)}(t)+X^{(L,2)}(t)+Y(t),\qquad t\in\widetilde{M}, \tag{45}$$

or, equivalently, that

$$X^{(L,1)}(t)=X(t)-X^{(L,2)}(t)-Y(t),\qquad t\in\widetilde{M}. \tag{46}$$

What we plan to show is that when when either X or $X^{(L,1)}$ reaches a high level u, then the lighter-tailed random fields Y and $X^{(L,2)}$ can be thought of as small perturbations, both in terms of their absolute values, and those of their first and second order partial derivatives. This will imply that the asymptotic conditional joint distributions of the number of the critical points of the random fields X and $X^{(L,1)}$ are not affected by the lighter tailed fields and, hence, coincide.

In fact, what we establish is that near every critical point of one of the random fields X and $X^{(L,1)}$ there is a critical point, of the same index, of the other. Equation (45) allows us to do this in one direction, and (46) will give us the other direction. The two equations are of the same type, and the fact that the terms in the right-hand side of (45) are independent, while the terms in the right-hand side of (46) are not, will play no role in the argument. Therefore, we shall treat in detail only one of the two directions, and describe only briefly the additional steps needed for the other. The first steps in this program involve collecting some probabilistic bounds on the closeness of critical points and the behavior of Hessians there.

(v) *Bounds on critical points and Hessians.* We start by introducing a function $D: S \to (0, \infty]$ that describes what we think of as the degree of nondegeneracy of the critical points of an s-section of the kernel f. This includes the minimal Euclidian distance between two distinct critical points of an s-section of the kernel f and the smallest absolute value of an eigenvalue of the Hessian matrices of the section evaluated at critical points. Specifically, starting with critical points, and recalling the definition of the $t_l(J; i; s)$ as the critical points of index i on the face J for the s-section of X, define

$$D_1(s) = \min\{\|t_{l_1}(J_1; i_1; s) - t_{l_2}(J_2; i_2; s)\| : J_j \in \mathcal{J}_{k_j}, \\ 0 \le k_1, k_2 \le d, 0 \le i_j \le k_j, 0 \le l_j \le c_{i_j}(J_j; s), j = 1, 2\},$$

where the minimum is taken over distinct points. Furthermore, define

$$D_2(s) = \min\{|\lambda| : \lambda \text{ is an eigenvalue of } (f_{mn}(s; t_l(J; i; s)))_{m,n \in \sigma(J)}; \\ J \in \mathcal{J}_k, 0 \le k \le d, 0 \le i \le k, 1 \le l \le c_i(J; s)\}.$$

As usual, both minima are defined to be equal to $+\infty$ if taken over an empty set.

Now set

$$D(s) = \min(D_1(s), D_2(s)). \tag{47}$$

Note that, by Assumption 2.4, D is a strictly positive function, so that for any any S-valued random variable W one has $\lim_{\tau \to 0} \mathbb{P}\{D(W) \le \tau\} = 0$. Choose W to have the law N_W given by

$$\frac{dN_W}{dm}(s) = c_*\big(w_+(s) + w_-(s)\big) \sup_{t \in I_d} |f(s,t)|^\alpha, \qquad s \in S, \tag{48}$$

where c_* is a normalising constant. That this is possible is a consequence of (16). For $\varepsilon > 0$, choose $\tau_0 > 0$ so small that $\mathbb{P}\{D(W) \le \tau_0\} \le \varepsilon$. With the random variable $K(u)$ as before, Lemma 6.2 gives us that

$$\limsup_{u \to \infty} \frac{\mathbb{P}\{\{D(S_{K(u)}) \le \tau_0, \sup_{t \in I_d} |X^{(L,1)}(t)| > u\} \cap B_1(u)\}}{H(u)} \le c_*^{-1} \varepsilon, \tag{49}$$

where $B_1(u)$ was defined at (39) and indicates that there was only one "large" component in the decomposition of X.

Note that, since the event $\{\sup_{t \in I_d} |X^{(L,1)}(t)| > u\} \cap B_1(u)$ is a subset of $B_1(u)$, on this event $X^{(L,1)}(t) = X_{K(u)} f(S_{K(u)}; t)$ for all $t \in \widetilde{M}$. Thus, again on this this event, since the supremum of this field over I_d exceeds u, while the kernel f is uniformly bounded, we conclude that $|X_{K(u)}| > u / \|f\|_\infty$. Therefore, on the event

$$\Big\{D\big(S_{K(u)}\big) > \tau_0, \sup_{t \in I_d} \big|X^{(L,1)}(t)\big| > u\Big\} \cap B_1(u)$$

the smallest eigenvalue length

$$D_{\min} \triangleq \min\{|\lambda| : \lambda \text{ is an eigenvalue of } \big(X^{(L,1)}_{mn}(t)\big)_{m,n\in\sigma(J)}; \quad J \in \mathcal{J}_k, 0 \le k \le d, t \text{ is a critical point on } J\} \tag{50}$$

satisfies $D_{\min} > (\tau_0/\|f\|_\infty)u$.

We now combine (49) with (43) as follows. Introduce the event $\widetilde{\Omega}_\tau(u)$ that occurs whenever the minimal Euclidian distance between two distinct critical points of the random field $(X^{(L,1)}(t), t \in I_d)$ is at least $\tau > 0$, while the smallest eigenvalue length of the Hessian evaluated at the critical points satisfies $D_{\min} > (\tau/\|f\|_\infty)u$. Thus, we have

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i)\ \forall J, i\} \cap \widetilde{\Omega}_{\tau_0}(u)\}}{H(u)} \ge I_{\mathrm{c}} - c_*^{-1}\varepsilon, \tag{51}$$

where I_{c} is as in (43). We can, furthermore, "sacrifice" another ε in the right-hand side of (51) to add to the event $\widetilde{\Omega}_\tau(u)$ a requirement that the *largest* eigenvalue of the Hessian evaluated at the critical points, which we denote by $D_{\max}$, satisfies $D_{\max} \le Mu$ for some positive $M = M(\epsilon)$. This is possible because $D_{\max}$ is bounded from above by the largest absolute value of the elements of the Hessian, which we bound from above by Mu with a large enough M. For the same reason, we can also bound from above the largest value of $\|\nabla X^{(L,1)}(t)\|$ over I_d by Mu.

Denoting the resulting event by $\Omega_\tau(u)$, we obtain

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{\{N_{X^{(L,1)}}(J; i : u) \ge n(J; i)\ \forall J, i\} \cap \Omega_{\tau_0}(u)\}}{H(u)} \ge I_{\mathrm{c}} - 2c_*^{-1}\varepsilon. \tag{52}$$

Now note that since, as stated above, X^L is bounded and C^2 on $\widetilde{M}$, and the same is true for X by Assumption 2.1, it follows that the "remainder" Y in (35) is also a.s. bounded and C^2. Furthermore, by construction, Y and its first and second order partial derivatives have Lévy measures that are supported on uniformly bounded functions. Consequently, the tail of their absolute suprema decays exponentially fast; see [5]. In particular, for $i, j = 1, \dots, d$,

$$\begin{aligned}\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |Y(t)| > u\}}{H(u)} &= \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |Y_i(t)| > u\}}{H(u)} \\ &= \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |Y_{ij}(t)| > u\}}{H(u)} \\ &= 0.\end{aligned}$$

It follows from this, part (ii) of Lemma 6.1, and the regular variation of H, that there is a function $l(u) \uparrow \infty$ such that $l(u)/u \to 0$ as $u \to \infty$ and

$$\begin{aligned}
\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |Y(t)| > l(u)\}}{H(u)} &= \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |Y_{ij}(t)| > l(u)\}}{H(u)} \\
&= \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |X^{L,2}(t)| > l(u)\}}{H(u)} \\
&= \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |X^{L,2}_{ij}(t)| > l(u)\}}{H(u)} \\
&= 0
\end{aligned} \tag{53}$$

for $i, j = 1, \dots, d$.

We now combine (52) and (53) in the following way. Let $\Omega_\tau^{(1)}(u)$ be the intersection of the event $\Omega_\tau(u)$ with the complements of all 4 events whose probabilities are displayed in (53) and set

$$\Omega_{cr}(u) = \{N_{X^{(L,1)}}(J; i : (1+\tau_2)u) \geq n(J; i)\ \forall J, i\} \cap \Omega_{\tau_1}^{(1)}(u).$$

Then, given $0 < \varepsilon_1 < 1$, and using the regular variation of H, we can find $\tau_1, \tau_2 > 0$ such that

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{\Omega_{cr}(u)\}}{H(u)} \geq (1-\varepsilon_1) I_{\mathrm{c}}. \tag{54}$$

(vi) *The (almost) end of the proof.* Continuing with the above notation, we now claim that, on the event $\Omega_{cr}(u)$, for u large enough so that

$$\frac{u}{l(u)} \geq \max\left(\frac{8k\|f\|_\infty}{\tau_1}, \frac{4}{\tau_2}\right), \tag{55}$$

we also have

$$N_X(J; i : u) \geq n(J; i), \qquad J \in \mathcal{J}_k, k = 0, 1, \dots, d, i = 0, 1, \dots, k. \tag{56}$$

Note that, once this is established, we shall have

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{N_X(J; i : u) \geq n(J; i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u)} \geq (1-\varepsilon_1) I_{\mathrm{c}}$$

and, since this holds for all $0 < \varepsilon_1 < 1$, we also have

$$\liminf_{u\to\infty} \frac{\mathbb{P}\{N_X(J; i : u) \geq n(J; i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u)} \geq I_{\mathrm{c}}. \tag{57}$$

Combining this with (17) gives Theorem 3.1, albeit with an inequality rather than an equality in (23).

To obtain the opposite inequality assume that, to the contrary, for some numbers $n(J;i)$,

$$\lim_{n\to\infty} \frac{\mathbb{P}\{N_X(J;i:u_n) \geq n(J;i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u_n)} > I_c \tag{58}$$

along some sequence $u_n \uparrow \infty$.

Now proceed by repeating the steps performed above and, this time using (46) rather than (45), and so demonstrate the existence of a critical point of $X^{(L,1)}$ near each one of X. Thus, (58) also holds with X replaced by $X^{(L,1)}$), viz.

$$\lim_{n\to\infty} \frac{\mathbb{P}\{N_{X^{(L,1)}}(J;i:u_n) \geq n(J;i), J \in \mathcal{J}, 0 \leq i \leq \dim J\}}{H(u_n)} > I_c.$$

Since this contradicts (43), (58) cannot be true, we have the required lower bound, and the proof of Theorem 3.1 is complete, modulo the need to establish the claim (56).

(vii) *Establishing* (56) *to finish the proof.* In order to establish (56), we shall show that, on the event $\Omega_{cr}(u)$, to every critical point above the level $(1+\tau_2)u$ of the random field $X^{(L,1)}$ we can associate a critical point above the level u of X which is in the same face and of the same type.

To this end, let t_0 be a critical point above the level $(1+\tau_2)u$ of $X^{(L,1)}$ that belongs to a face $J \in \mathcal{J}_k$ for some $0 \leq k \leq d$, and which is of the type i for some $0 \leq i \leq k$. Let $(e_1, \dots, e_k)$ be an orthonormal basis of $\mathbb{R}^k$ consisting of normalised eigenvectors of the Hessian matrix

$$\mathcal{H}^{(L,1)}(t_0) = \big(X^{(L,1)}_{mn}(t_0)\big)_{m,n\in\sigma(J)}, \tag{59}$$

and let $\lambda_1, \dots, \lambda_k$ be the corresponding eigenvalues. Note that, by the definition of the event $\Omega_{cr}(u)$, we have $|\lambda_n| > (\tau_1/\|f\|_\infty)u$ for $n = 1, \dots, k$. We naturally embed the vectors $(e_1, \dots, e_k)$ into the face J and make them d-dimensional vectors by appending to them the $d-k$ fixed coordinates of the face J. [We shall continue to denote these vectors by $(e_1, \dots, e_k)$.] Note that for small real numbers $\epsilon_1, \dots, \epsilon_k$ we have

$$\nabla X^{(L,1)}\left(t_0 + \sum_{j=1}^k \epsilon_j e_j\right) = \sum_{j=1}^k \epsilon_j \lambda_j e_j + o(\max(|\epsilon_1|, \dots, |\epsilon_k|)). \tag{60}$$

In particular, the directional derivatives

$$g_j^{(L,1)}(t) \triangleq \langle \nabla X^{(L,1)}(t), e_j \rangle, \qquad j = 1, \dots, k,$$

satisfy

$$g_j^{(L,1)}\left(t_0 + \sum_{j=1}^k \epsilon_j e_j\right) = \epsilon_j \lambda_j + o(\max(|\epsilon_1|, \dots, |\epsilon_k|)). \tag{61}$$

In what follows, we shall work with a small positive number $\epsilon > 0$, placing more and more conditions on it as we progress, to clarify precisely how small it will need to be. As a first step, take $\epsilon < \tau_1/2$, where τ_1 is as in (54).

Consider a k-dimensional cube (a subset of the face J) defined by

$$C_\epsilon = \left\{ t_0 + \sum_{j=1}^{k} \theta_j e_j, |\theta_j| \le \epsilon, j = 1, \dots, k \right\},$$

along with its $(k-1)$-dimensional faces

$$F_n^{\pm} = \left\{ t_0 + \sum_{j=1}^{k} \theta_j e_j, \theta_n = \pm\epsilon, |\theta_j| \le \epsilon, 1 \le j \le k, j \ne n \right\},$$

where $n = 1, \dots, k$. It follows from (61) that, for $\epsilon > 0$ small enough, $u > 1$, and, as above, M large enough, we have

$$2M\epsilon u \ge 2\epsilon|\lambda_n| \ge \left|g_n^{(L,1)}(t)\right| \ge \frac{\epsilon|\lambda_n|}{2} \ge \frac{\tau_1\epsilon}{2\|f\|_\infty}u \tag{62}$$

for all $t \in F_n^{\pm}$, $n = 1, \dots, k$. The assumption that ϵ be small enough now entails that (62) holds for all critical points and for all relevant n. Since the number of critical points is finite, this requirement is easy to satisfy. Similarly, the continuity of the eigenvalues of a quadratic matrix in its components (see, e.g., Section 7.2. and Corollary 2 in Section 7.4 of [6]) shows that, for all $\epsilon > 0$ small enough, the eigenvalues of the matrix of the second order partial derivatives $(X_{mn}(t)_{m,n\in\sigma(J)}$ have all absolute values satisfying $|\lambda_n| > (\tau/2\|f\|_\infty)u$ for $n = 1, \dots, k$ and $t \in C_\epsilon$. Finally, we require that ϵ be small enough that this lower bounds holds for all critical points t_0 considered above. In particular, this implies that the signs of these eigenvalues throughout C_ϵ are the same as those at the point t_0.

Next, for a nonempty $I \subset \{1, \dots, k\}$ and $p \in \{-1, 1\}^k$ consider the vector

$$x(I, p) = \sum_{i\in I} p_i e_i. \tag{63}$$

Consider a point t that belongs to the (relative to the face J) boundary of the cube C_ϵ and, more specifically, belongs to the face defined by

$$\left(\bigcap_{i\in I, p_i=1} F_i^{+} \right) \cap \left(\bigcap_{i\in I, p_i=-1} F_i^{-} \right) \tag{64}$$

and to no other $(k-1)$-dimensional face of C_ϵ. Define a function $h^{(L,1)}: C_\epsilon \to \mathbb{R}$ by

$$h^{(L,1)}(t) = \sum_{i=1}^{k} \left(g_i^{(L,1)}(t)\right)^2.$$

This is a C^1-function, and its gradient (within the face J) is given by

$$\nabla h^{(L,1)}(t) = 2\sum_{i=1}^{k} g_i^{(L,1)}(t)\nabla g_i^{(L,1)}(t) = 2\sum_{i=1}^{k} g_i^{(L,1)}(t)\mathcal{H}^{(L,1)}(t)e_i^T.$$

Note also that for all I and p as above,

$$\langle \nabla g_i^{(L,1)}(t_0), x(I,p)\rangle = \begin{cases} \lambda_i p_i, & \text{if } i \in I, \\ 0, & \text{if } i \notin I. \end{cases}$$

In particular, for any t belonging to the face of C_ϵ defined by (64),

$$\begin{aligned} &\langle \nabla h^{(L,1)}(t), x(I,p)\rangle \\ &\quad = 2\sum_{i=1}^{k} g_i^{(L,1)}(t)\langle \nabla g_i^{(L,1)}(t), x(I,p)\rangle \\ &\quad = 2\sum_{i\in I} \lambda_i p_i g_i^{(L,1)}(t) \\ &\qquad + 2\sum_{i=1}^{k} g_i^{(L,1)}(t)\langle (\nabla g_i^{(L,1)}(t) - \nabla g_i^{(L,1)}(t_0)), x(I,p)\rangle. \end{aligned} \tag{65}$$

It follows from (61) and (62) that

$$\begin{aligned} &g_n^{(L,1)}(t) > 0 \qquad \text{for } t \in F_n^+ \text{ if } \lambda_n > 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n < 0, \\ &g_n^{(L,1)}(t) < 0 \qquad \text{for } t \in F_n^+ \text{ if } \lambda_n < 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n > 0. \end{aligned} \tag{66}$$

Consequently, we can conclude, by (66) and (62), that the first term in the right-hand side of (65) is negative and, more specifically, does not exceed

$$-2\mathrm{Card}(I) D_{\min} \frac{\tau_1 \epsilon}{2\|f\|_\infty} u \le -(\tau_1/\|f\|_\infty)^2 \epsilon u^2.$$

We can bound the absolute value of the second term in the right-hand side of (65) from above by

$$2k \sum_{i=1}^{k} |g_i^{(L,1)}(t)| \cdot \|\nabla g_i^{(L,1)}(t) - \nabla g_i^{(L,1)}(t_0)\| \le 2k^2 M^2 \epsilon u^2$$

by the definition of the event $\Omega_{cr}(u)$. This, obviously, indicates that, for $\epsilon > 0$ small enough,

$$\langle \nabla h^{(L,1)}(t), x(I,p)\rangle \le -C\epsilon u^2,$$

where C is a finite positive constant determined by the parameters in the event $\Omega_{cr}(u)$. Writing $g_j(t) = \langle \nabla X(t), e_j\rangle$, $j = 1, \dots, k$, if we define

$$h(t) = \sum_{i=1}^{k} (g_i(t))^2, \tag{67}$$

then, on the event $\Omega_{cr}(u)$,

$$\langle \nabla h(t), x(I, p)\rangle \le -C\epsilon u^2 + kl(u)^2.$$

Taking into account that $l(u)/u \to 0$ as $u \to \infty$, where l is given by (53), we see that for u large enough it is possible to choose $\epsilon > 0$ small enough such that

$$\langle \nabla h(t), x(I, p)\rangle < 0 \tag{68}$$

for any t belonging to the face of C_ϵ defined by (64). The final requirement on ϵ is that (68) holds.

Similarly, since by the definition of the event $\Omega_{cr}(u)$, the first order partial derivatives of $X^{(L,2)}$ and Y are bounded by $l(u) = o(u)$ in absolute value, we have that (66) and (55) also give us

$$\begin{aligned} g_n(t) > 0 &\qquad \text{for } t \in F_n^+ \text{ if } \lambda_n > 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n < 0, \\ g_n(t) < 0 &\qquad \text{for } t \in F_n^+ \text{ if } \lambda_n < 0 \text{ and for } t \in F_n^- \text{ if } \lambda_n > 0. \end{aligned} \tag{69}$$

In order to complete the proof and establish (56) it suffices to prove that, on $\Omega_{cr}(u)$, X has a critical point in the cube C_ϵ. If such a critical point exists, Lemma 6.3 below implies that it will be above the level u and of the same type as t_0. Furthermore, these critical points of X will all be distinct.

To establish the existence of this critical point, note that, by the continuity of ∇X and the compactness of C_ϵ, there is a point t_1 in C_ϵ at which the norm of the vector function $g(t) = (g_1(t), \dots, g_k(t))$ achieves its minimum over C_ϵ. We shall prove that, in fact, $g(t_1) = 0$. By the linear independence of the basis vectors $e_1, \dots, e_k$, this will imply that $g_j(t_1) = 0$ for $j = 1, \dots, k$, and so t_1 is, indeed, a critical point.

Suppose that, to the contrary, $g(t_1) \neq 0$, and consider firstly the possibility that the point t_1 belongs to the (relative to the face J) interior of C_ϵ. Note that the Jacobian of the transformation $g : C_\epsilon \to \mathbb{R}^k$ is given by $J_g(t) = \mathcal{E}\mathcal{H}(t)$, where $\mathcal{H}(t) = (X_{mn}(t))_{m,n\in\sigma(J)}$ is the Hessian of X, and $\mathcal{E}$ is a $k \times k$ matrix with rows $e_1, \dots, e_k$. We have already established above that, on the event $\Omega_{cr}(u)$, $\mathcal{H}$ is nondegenerate throughout C_ϵ. Since the vectors $e_1, \dots, e_k$ are linearly independent, we conclude that the matrix $\mathcal{E}$ is nondegenerate as well. Since the vector $g(t_1)$ does not vanish, it has a nonvanishing component. Without loss of generality, we can assume that $g_1(t_1) \neq 0$. Choose a vector $x \in \mathbb{R}^k$ for which $J_g(t_1)x' = (1, 0, \dots, 0)'$. Then for $\delta \in \mathbb{R}$, with $|\delta|$ small,

$$g(t_1 + \delta x) = g(t_1) + \delta J_g(t_1)x^T + o(|\delta|)$$

and so

$$\|g(t_1 + \delta x)\|^2 = \sum_{j=1}^k g_j(t_1)^2 + 2\delta g_1(t_1) + o(|\delta|) < \sum_{j=1}^k g_j(t_1)^2 = \|g(t_1)\|^2$$

for δ with $|\delta|$ small enough and such that $\delta g_1(t_1) < 0$. This contradicts the assumed minimality of $\|g(t_1)\|$ and so we must have $g(t_1) = 0$, as required, for this case.

It remains to consider the case $g(t_1) \neq 0$, but the point t_1 belongs to the boundary of the cube C_ϵ. Let $g(t_1)$ belong to the face of the cube defined by (64). With the function h defined in (67), we have, for $\delta > 0$ small,

$$h\big(t_1 + \delta x(I, p)\big) = h(t_1) + \delta(\nabla h(t), x(I, p)) + o(\delta).$$

By (68), this last expression is smaller than $h(t_1)$ if $\delta > 0$ is small enough. However, by the definition of the vector $x(I, p)$, the point $t_1 + \delta x(I, p)$ belongs to C_ϵ for $\delta > 0$ small. Once again, this contradicts the assumed minimality of $\|g(t_1)\|$.

Thus, we have established (56) and, therefore, (57), and so the theorem, modulo the need to prove the following three lemmas.

LEMMA 6.1. *The following three results hold*:

(i) *The random fields $X^{(L,1)}$ and $X^{(L,2)}$ on the right-hand side of the decomposition* (38) *are independent.*

(ii) *The random field $X^{(L,2)}$ has C^2 sample functions and satisfies*

$$\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |X^{(L,2)}(t)| > u\}}{H(u)} = \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |X_i^{(L,2)}(t)| > u\}}{H(u)}$$

$$= \lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |X_{ij}^{(L,2)}(t)| > u\}}{H(u)} = 0,$$

$i, j = 1, \dots, d$, *where H is the regularly varying function of Assumption* 2.5.

(iii) *The number of terms in the sum defining $X^{(L,1)}$ satisfies*

$$\lim_{u\to\infty} \frac{\mathbb{P}\{\sum_{m=1}^N \mathbb{1}(A_m(u)) \geq 2\}}{H(u)} = 0.$$

PROOF. The claim (i) follows from the fact that a Poisson random measure, when restricted to disjoint measurable sets, forms independent Poisson random measures on these sets (see, e.g., [12]). Since the sum defining the random field $X^{(L,2)}$ is a.s. finite, the fact that it has sample functions in C^2 follows from Assumption 2.1. Furthermore, for $\epsilon > 0$, choose $n_\epsilon > 0$ so large that $\mathbb{P}\{N > n_\epsilon\} \leq \epsilon$. The above discussion implies that the number $K(u, \epsilon)$ of the terms in the sum defining $X^{(L,2)}$ in (38) that satisfy

$$\sup_{t\in\widetilde{M}} |X_m f(S_m; t)| > \frac{u}{2n_\epsilon}$$

is Poisson with the mean no greater than

$$F\Big\{(s, x) \in (\mathbb{R} \setminus \{0\}) \times S : \sup_{t\in\widetilde{M}} |x f(s; t)| > u/(2n_\epsilon)\Big\}$$

$$= \lambda_X\Big\{g : \sup_{t\in\widetilde{M}} |g(t)| > u/(2n_\epsilon)\Big\} \sim C H(u)$$

as $u \to \infty$, where we have used (10) and Lemma 2.6. Therefore, for large u

$$\mathbb{P}\Big\{\sup_{t\in\widetilde{M}}\big|X^{(L,2)}(t)\big| > u\Big\} \le \mathbb{P}\{N > n_\epsilon\} + \mathbb{P}\{K(u,\epsilon) \ge 2\} \le \epsilon + CH(u)^2$$

and so

$$\limsup_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\widetilde{M}} |X^{(L,2)}(t)| > u\}}{H(u)} \le \epsilon.$$

Letting $\epsilon \to 0$ completes the proof of the first limit in part (ii) of the lemma, and the other limits are established in the same way. Part (iii) of the lemma can be proven similarly. □

LEMMA 6.2. *With the c_* of (48), the D_m in (47) satisfy*

$$\limsup_{u\to\infty} \frac{\mathbb{P}\{\{D(S_{K(u)}) \le \tau_0, \sup_{t\in I_d} |X^{(L,1)}(t)| > u\} \cap B_1(u)\}}{H(u)} \le c_*^{-1}\varepsilon.$$

PROOF. We use a decomposition as in (41) to obtain

$$\begin{aligned}
&\mathbb{P}\Big\{\Big\{D\big(S_{K(u)}\big) \le \tau_0, \sup_{t\in I_d}\big|X^{(L,1)}(t)\big| > u\Big\} \cap B_1(u)\Big\}\\
&\quad \le \theta\mathbb{P}\left\{A_1(u) \cap \bigcap_{m_1=2}^{n} A_{m_1}(u)^c \cap \Big\{D(S_1) \le \tau_0, \sup_{t\in I_d}\big|X^{(L,1)}(t)\big| > u\Big\}\right\}.
\end{aligned}$$

Since on the event $A_m(u)^c$ one has $\sup_{t\in\widetilde{M}} |X_m f(S_m;t)| \le u^\beta$, it follows that the latter probability can be asymptotically bounded by

$$\begin{aligned}
&\theta\mathbb{P}\Big\{D(S_1) \le \tau_0, \sup_{t\in I_d} |X_1 f(S_1;t)| > u\Big\}\\
&\quad = \int_S \int_{\mathbb{R}\setminus\{0\}} \mathbb{1}\Big(D(s) \le \tau_0, |x| \sup_{t\in I_d} |f(s;t)| > u\Big) F(ds,dx)\\
&\quad = \int_S \mathbb{1}\big(D(s) \le \tau_0\big)\bigg(\int_{\mathbb{R}\setminus\{0\}} \mathbb{1}\Big(|x| > u\Big(\sup_{t\in I_d} |f(s;t)|\Big)^{-1}\Big)\rho(s;dx)\bigg) m(ds)\\
&\quad \sim H(u) \int_S \mathbb{1}\big(D(s) \le \tau_0\big)\big(w_+(s) + w_-(s)\big) \sup_{t\in I_d} |f(s;t)|^\alpha m(ds),
\end{aligned}$$

where we used Assumptions 2.5, 2.4 and (16). The lemma now follows from the choice of τ_0. □

LEMMA 6.3. *Suppose that for every critical point t_0 of the random field $(X^{L,1}(t), t \in I_d)$, the random field $(X(t), t \in I_d)$ has, on the event $\Omega_{cr}(u)$, a critical point in the cube C_ϵ. Then the critical points of X in I_d correspond to distinct critical points of $X^{L,1}$, are themselves distinct, are all above the level u, and each of them is of the same type as the corresponding critical point of $X^{L,1}$.*

PROOF. The fact that the critical points of X corresponding to distinct critical points of the field $X^{L,1}$ are all distinct follows from the lower bound on the distance between two distinct critical points of $X^{L,1}$ in the definition of the event $\Omega_{cr}(u)$ and the choice of ϵ. The fact that all the critical points are above the level u follows from the lower bounds on the values of $X^{L,1}$ at its critical points in the definition of $\Omega_{cr}(u)$ and, once again, the choice of ϵ. It remains, therefore, to prove that a critical point in the cube C_ϵ of X is of the same type as the critical point t_0 of $X^{L,1}$.

To this end, note that the absolute values of the eigenvalues of the matrix of the second order partial derivatives $(X^{L,1}_{mn}(t) + Y_{mn}(t))_{m,n\in\sigma(J)}$ are, on the event $\Omega_{cr}(u)$, bounded from above by $2kl(u)$. Using continuity of the eigenvalues of a quadratic matrix in its components (see, once again, Section 7.2 and Corollary 2 in Section 7.4 of [6]), we see that the Euclidian distance between an eigenvalue of $(X_{mn}(t))_{m,n\in\sigma(J)}$ and the corresponding eigenvalue of $(X^{L,1}_{mn}(t))_{m,n\in\sigma(J)}$ is bounded from above by $2kl(u)$. Using the choice of ϵ then shows that the numbers of the negative eigenvalues of the two Hessians are identical, as required. □

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